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TUBULAR SURFACES OF WEINGARTEN TYPES IN GALILEAN AND PSEUDO-GALILEAN

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MURAT KEMAL KARACAN[1] AND YILMAZ TUNCER[2]

Abstract. In this paper, we have defined canal surfaces in Galilean and Pseudo-Galilean 3-spaces.Then, we have studied Tubular surface in Galilean and Pseudo-Galilean 3-space satisfying some equations in terms of the Gaussian curvature and the mean curvature.We have discussed Weingarten,linear Weingarten conditions and $HK-$quadric type for this surface with respect to their curvatures

1. Introduction

A surface M in Euclidean space E^3 or Minkowski space E_1^3 is called a Weingarten surface if there is a smooth relation $U(k_1,k_2)=0$ between its two principal curvatures k_1 and k_2. If K and H denote respectively the Gauss curvature and the mean curvature of M, $U(k_1,k_2)=0$ implies a relation $\Phi(K,H)=0$. The existence of a non-trivial functional relation $\Phi(K,H)=0$ on a surface M parameterized by a patch $x(s,t)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely $\left|\frac{\partial(K,H)}{\partial(s,t)}\right|=0$ [10].

The simplest case when $U=ak_1+bk_2-c$ or $\Phi=aH+bK-c$ (a,b and c are constants with $a^2+b^2\neq 0$), the surfaces are called linear Weingarten surfaces. When the constant $b=0$, a linear Weingarten surface M reduces to a surface with constant Gaussian curvature. When the constant $a=0$, a linear Weingarten surface M reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [10].

Several geometers [1,10,14] have studied tubes in Euclidean 3-space and Minkowski 3-space satisfying some equation in terms of the Gaussian curvature, the mean curvature and the second Gaussian curvature.Following the Jacobi equation and the linear equation with respect to the Gaussian curvature K and the mean curvature H an interesting geometric question is raised: Classify all surfaces in Galilean and Pseudo-Galilean $3-$spaces satisfying the conditions

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$$\Phi(X,Y)=0$$
$$aX+bY=c$$

where $(X,Y)\in\{K,H\},\ X\neq Y$ and $(a,b,c)\neq(0,0,0)$.If a surface satisfies the equations

$$aH^2+2bHK+cK^2=constant, a\neq 0, \tag{1}$$

then a surface is said to be a $HK-$quadric surface [8].

In this paper, we have defined canal surfaces in Galilean and Pseudo-Galilean 3-spaces.Then, we have studied Weingarten and linear Weingarten tubular surfaces and also $HK-$quadric surface.We have obtained some conditions for that surfaces in Galilean and Pseudo Galilean 3-space.We show that tubular surfaces are not umbilical and minimal by using their principal curvatures.

2. Preliminaries

2.1. **Galilean 3-Space G_3.** The Galilean space is a three dimensional complex projective space, P_3, in which the absolute figure $\{w,f,I_1,I_2\}$ consists of a real plane w (the absolute plane), a real line $f\subset w$ (the absolute line) and two complex conjugate points, $I_1,I_2\in f$ (the absolute points).

We shall take, as a real model of the space G_3, a real projective space P_3, with the absolute $\{w,f\}$ consisting of a real plane $w\subset G_3$ and a real line $f\subset w$, on which an elliptic involution ϵ has been defined. Let ϵ be in homogeneous coordinates

$$w...x_0=0,\qquad f...x_0=x_1=0$$
$$\epsilon:(0:0:x_2:x_3)\to(0:0:x_3:-x_2).$$

In the nonhomogeneous coordinates, the similarity group H_8 has the form

$$\overline{x}=a_{11}+a_{12}x \tag{2}$$
$$\overline{y}=a_{21}+a_{22}x+a_{23}\cos\theta+a_{23}\sin\theta$$
$$\overline{z}=a_{31}+a_{32}x-a_{23}\sin\theta+a_{23}\cos\theta$$

where a_{ij} and θ are real numbers.For $a_{11}=a_{23}=1$,we have have the subgroup B_6 , the group of Galilean motions:

$$\overline{x}=a_{11}+a_{12}x$$
$$\overline{y}=b+cx+y\cos\theta+z\sin\theta$$
$$\overline{z}=d+ex-y\sin\theta+z\cos\theta.$$

In G_3, there are four classes of lines:

a) (proper) nonisotropic lines - they do not meet the absolute line f.

b) (proper) isotropic lines - lines that do not belong to the plane w but meet the absolute line f.

c) unproper nonisotropic lines - all lines of w but f.

d) the absolute line f.

Planes x =constant are Euclidean and so is the plane w. Other planes are isotropic.In what follows, the coefficients a_{12} and a_{23} a will play a special role. In particular, for $a_{12}=a_{23}=1$, (2) defines the group $B_6\subset H_8$ of isometries of the Galilean space G_3.

The scalar product in Galilean space G_3 is defined by

$$\langle X,Y\rangle_{G_3}=\begin{cases} x_1y_1 & , \ if \quad x_1\neq 0\ \vee y_1\neq 0\\ x_2y_2+x_3y_3\ , & if \quad x_1=0\ \wedge y_1=0\end{cases}$$

where $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$. The Galilean cross product is defined for $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ by

$$a \wedge_{G_3} b = \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The unit Galilean sphere is defined by [5]

$$S^2_{\pm} = \left\{ \alpha \in G_3 \middle| \ \langle \alpha, \alpha \rangle_{G_3} = \mp r^2 \right\} .$$

A curve $\alpha : I \subseteq R \to G_3$ of the class C^r $(r \geq 3)$ in the Galilean space G_3 is given defined by

$$\alpha(x) = (s, y(s), z(s)) ,$$

where s is a Galilean invariant and the arc length on α.The curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by

$$\kappa(s) = \sqrt{\left(y''(s)\right)^2 + \left(z''(s)\right)^2} , \quad \tau(x) = \frac{\det\left(\alpha'(s), \alpha''(s), \alpha'''(s)\right)}{\kappa^2(s)}.$$

The orthonormal frame in the sense of Galilean space G_3 is defined by

$$\begin{aligned} T &= \alpha'(s) = (1, y'(s), z'(s)) \\ N &= \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s)) \\ B &= \frac{1}{\kappa(s)}(0, -z''(s), y''(s)) \end{aligned} \tag{3}$$

The vectors T, N and B in (3) are called the vectors of the tangent, principal normal and the binormal line of α, respectively.They satisfy the following Frenet equations

$$\begin{aligned} T' &= \kappa N \\ N' &= \tau B \\ B' &= -\tau N. \end{aligned}$$

[9].For the treatment of Weingarten surfaces we need the Gaussian and the mean curvature of a surface. They are defined by

$$\begin{aligned} K &= \frac{L_{11}L_{22} - L_{12}^2}{W^2} = \frac{eg - f^2}{EG - F^2}, \\ 2H &= g_{11}L_{11} + 2g_{12}L_{12} + g_{22}L_{22} = \frac{Eg - 2Ff + Ge}{(EG - F^2)} \end{aligned} \tag{4}$$

where

$$g_{11} = \frac{x_2^2}{W^2} \ , \ g_{12} = -\frac{x_1 x_2}{W^2} \ , \ g_{12} = \frac{x_1^2}{W^2}$$

where

$$x_1 = \frac{\partial x}{\partial u} \ , \ x_2 = \frac{\partial x}{\partial v} \ , \ W = |x_1 x_2 - x_2 x_1| .$$

The coefficients of the second fundamental form (which are the normal components of $x_i = 1, 2$) can be determined from

$$L_{ij} = \left(\frac{x_1 x_{ij} - x_{ij} x_1}{x_1}\right) U = \left(\frac{x_2 x_{ij} - x_{ij} x_2}{x_2}\right) U.$$

The unit normal field U is an isotropic vector obtained by means of the Galilean cross product

$$U = \frac{x_1 \wedge_{G_3} x_2}{W}$$

[12]. The principal curvatures of the surface can be found by the following equations

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}.$$

2.2. **Pseudo-Galilean 3-Space G_3^1.** The geometry of the pseudo-Galilean space is similar (but not the same) to the Galilean space.The pseudo-Galilean space G_3^1 is a three-dimensional projective space in which the absolute consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and a hyperbolic involution on f. Projective transformations which presere the absolute form of a group H_8 and are in nonhomogeneous coordinates can be written in the form

$$\begin{aligned} \overline{x} &= a + bx \\ \overline{y} &= c + dx + r\cosh\theta.y + r\sinh\theta.z \\ \overline{z} &= e + fx + r\sinh\theta.y + r\cos\theta.z \end{aligned}$$

where a, b, c, d, e, f, r and θ are real numbers. Particularly, for $b = r = 1$, the group (4.1) becomes the group $B_6 \subset H_8$ of isometries (proper motions) of the pseudo-Galilean space G_3^1. The motion group leaves invariant the absolute figure and defines the other invariants of this geometry.It has the following form

$$\begin{aligned} \overline{x} &= a + x \\ \overline{y} &= c + dx + \cosh\theta.y + \sinh\theta.z \\ \overline{z} &= e + fx + \sinh\theta.y + \cos\theta.z. \end{aligned}$$

According to the motion group in the pseudo-Galilean space, there are nonisotropic vectors $X(x, y, z)$ (for which holds $x \neq 0$) and four types of isotropic vectors: spacelike $\left(x = 0, y^2 - z^2 > 0\right)$, timelike $\left(x = 0, y^2 - z^2 < 0\right)$ and two types of light-like vectors $(x = 0, y = \mp z)$.The scalar product of two vectors $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ in G_3^1 is defined by

$$\langle A, B\rangle_{G_3^1} = \begin{cases} a_1 b_1 & , \; if \quad a_1 \neq 0 \; \vee b_1 \neq 0 \\ a_2 b_2 - a_3 b_3 \, , & if \quad a_1 = 0 \; \wedge b_1 = 0. \end{cases}$$

The Pseudo-Galilean cross product is defined for $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ by

$$a \wedge_{G_3^1} b = \begin{vmatrix} 0 & -e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The unit Pseudo-Galilean sphere is defined by [2]

$$S_\mp^2 = \left\{ \alpha \in G_3^1 \middle| \; \langle \alpha, \alpha\rangle_{G_3^1} = \mp r^2 \right\}.$$

A curve $\alpha(t) = (x(t), y(t), z(t))$ is admissible if it has no inflection points, no isotropic tangents or tangents or normals whose projections on the absolute plane would be light-like vectors.For an admissible curve $\alpha : I \subseteq R \to G_3^1$ the curvature $\kappa(t)$ and the torsion $\tau(t)$ are defined by

$$\kappa(s) = \frac{\sqrt{\left(y''(t)\right)^2 - \left(z''(t)\right)^2}}{\left(x'(t)\right)^2}, \quad \tau(x) = \frac{y''(t)z'''(t) - y'''(t)z''(t)}{\left|x'(t)\right|^5 \kappa^2(t)}.$$

expressed in components.Hence, for an admissible curve $\alpha : I \subseteq R \to G_3^1$ parameterized by the arc length s with differential form $ds = dx$, given by

$$\alpha(t) = (x, y(s), z(s)), \tag{5}$$

the formulas (5) have the following form

$$\kappa(s) = \sqrt{\left|(y''(s))^2 - (z''(s))^2\right|}, \quad \tau(x) = \frac{y''(s)z'''(s) - y'''(s)z''(s)}{\kappa^2(s)}.$$

The associated trihedron is given by

$$\begin{aligned} T &= \alpha'(s) = (1, y'(s), z'(s)) \\ N &= \frac{1}{\kappa(s)}\alpha''(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s)) \\ B &= \frac{1}{\kappa(s)}(0, \epsilon z''(s), \epsilon y''(s)) \end{aligned}$$

where $\epsilon = \mp 1$, chosen by criterion $\det(T, N, B) = 1$, that means

$$\left|(y''(s))^2 - (z''(s))^2\right| = \epsilon\left((y''(s))^2 - (z''(s))^2\right).$$

We derive an important relation

$$\alpha'''(s) = \kappa'(s)N(s) + \kappa(s)\tau(s)B(s).$$

The curve α given by (5) is timelike (resp. spacelike) if $N(s)$ is a spacelike(resp. timelike) vector. The principal normal vector or simply normal is spacelike if $\epsilon = 1$ and timelike if $\epsilon = -1$. For derivatives of the tangent (vector) T, the normal N and the binormal B, respectively, the following Serret-Frenet formulas hold

$$\begin{aligned} T' &= \kappa N \\ N' &= \tau B \\ B' &= \tau N. \end{aligned}$$

[6].

A C^r-surface, $r \geq 2$, is a subset $\Phi \subset G_3^1$ for which there exists an open subset D of R^2 and C^r -mapping $M : D \to G_3^1$ satisfying $\Phi = x(D)$. A C^r surface $\Phi \subset G_3^1$ is called regular if M is an immersion, and simple if M is an embedding. It is admissible if it does not have pseudo-Euclidean tangent planes. If we denote

$$M = M(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)), x_i = \frac{\partial x}{\partial u_i}, y_i = \frac{\partial y}{\partial u_i}, z_i = \frac{\partial z}{\partial u_i}, i = 1, 2$$

then a surface is admissible if and only if $x_i \neq 0$, for some $i = 1, 2$.

Let $\Phi \subset G_3^1$ be a regular admissible surface. Then the unit normal vector field of a surface $M(u, v)$ is equal to

$$\begin{aligned} N(u, v) &= \frac{(0, x_1z_2 - x_2z_1, x_1y_2 - x_2y_1)}{W(u, v)}, \\ W(u, v) &= \sqrt{\left|(x_1y_2 - x_2y_1)^2 - (x_1z_2 - x_2z_1)^2\right|}. \end{aligned}$$

The function W is equal to the pseudo-Galilean norm the vector $x_1x_2 - x_2x_1$.Vector defined by

$$\sigma = \frac{(x_1x_2 - x_2x_1)}{W}$$

is called a side tangential vector.We will not consider surfaces with $W = 0$, i.e. surfaces having lightlike side tangential vector (lightlike surfaces).

Since the normal vector field satisfies $\langle N,\ N\rangle_{G_3^1} = \epsilon = \pm 1$, we distinguish two basic types of admissible surfaces: spacelike surfaces having timelike surface normals ($\epsilon = -1$) and timelike surfaces having spacelike normals ($\epsilon = 1$). A surface is spacelike if $(x_1 x_2 - x_2 x_1) = (x_1 y_2 - x_2 y_1)^2 - (x_1 z_2 - x_2 z_1)^2 > 0$ in all of its points, timelike otherwise.

The first fundamental form of a surface is induced from the metric of the ambient space G_3^1

$$ds^2 = (x_1 du_1 + x_2 du_2)^2 + \delta \left(\widetilde{M}_1 du_1 + \widetilde{M}_2 du_2 \right)^2,$$

where

$$\delta = \begin{cases} 0; & \text{if direction } du_1 : du_2 \text{ is non-isotropic} \\ 1; & \text{if direction } du_1 : du_2 \text{ is isotropic.} \end{cases}$$

By~above a vector, the projection of a vector on the pseudo-Euclidean $yz-$plane is denoted.The Gaussian curvature of a surface is defined by means of the coefficients of the second fundamental form

$$K = -\epsilon \frac{L_{11} L_{22} - L_{12}^2}{W^2} = \frac{eg - f^2}{EG - F^2}, \tag{6}$$

where $\langle N,\ N\rangle_{G_3^1} = \epsilon = -1$ for spacelike surfaces and $\langle N,\ N\rangle_{G_3^1} = \epsilon = 1$ for timelike surfaces. The second fundamental form II is given by

$$II = L_{11}\ du_1^2 + 2L_{12}\ du_1 du_2 + L_{22} du_2^2$$

where L_{ij} , $i; j = 1; 2$, are the normal components of $M_{11}; M_{12}; M_{22}$, respectively. It holds

$$L_{ij} = \epsilon \left(\frac{x_1 \widetilde{M_{ij}} - x_{ij} \widetilde{M_1}}{x_1} \right) \widetilde{U} = \epsilon \left(\frac{x_2 \widetilde{M_{ij}} - x_{ij} \widetilde{M_2}}{x_2} \right) \widetilde{U}.$$

The mean curvature of a surface is defined by

$$H = -\epsilon \frac{x_2^2 L_{11} - 2x_1 x_2 L_{12} + x_1^2 L_{22}}{2W^2}\ . \tag{7}$$

The unit normal field U is an isotropic vector obtained by means of the Galilean cross product

$$U = \frac{x_1 \wedge_{G_3^1} x_2}{W}$$

[3,4,13].

3. Weingarten Tubular Surface in Galilean 3-Space

Our purpose in this section, we will obtain the tubular surface from the canal surface in Galilean 3-space.If we find the canal surface with taking variable radius $r(s)$ as constant, then the tubular surface can be found, since the canal surface is a general case of the tubular surface.

An envelope of a 1-parameter family of surfaces is constructed in the same way that we constructed a 1−parameter family of curves.The family is described by a differentiable function $F(x, y, z, \lambda) = 0$,where λ is a parameter.When λ can be eliminated from the equations

$$F(x, y, z, \lambda) = 0$$

and

$$\frac{\partial F(x,y,z,\lambda)}{\partial \lambda}=0,$$

we get the envelope,which is a surface described implicitly as $G(x,y,z) = 0$.For example, for a 1-parameter family of planes we get a develople surface [11].

Definition 3.1. The envelope of a 1-parameter family $s \to S^2_{\pm}(s)$ of the spheres in G_3 is called a canal surface in Galilean 3-space.The curve formed by the centers of the Galilean spheres is called center curve of the canal surface.The radius of the canal surface is the function r such that $r(s)$ is the radius of the Galilean sphere $S^2_{\pm}(s)$.

Definition 3.2. Let $\alpha : (a,b) \to G_3$ be a unit speed curve whose curvature does not vanish.Consider a tube of radius r around α. Since the normal N and binormal B are perpendicular to α, the Galilean circle is perpendicular α and $\alpha(s)$.As this Galilean circle moves along α, it traces out a surface about α which will be the tube about α , provided r is not too large.

Theorem 3.3. *Let $\alpha : I \to G_3$ be a curve in Galilean 3-space. Suppose the center curve of a canal surface is a unit speed curve α with nonzero curvature . Then the canal surface can be parametrized*

$$C(s,t) = \alpha(s) + r(s)\left(N(s)\cos t + B(s)\cos t\right),$$

where T, N and B denote the tangent, principal normal and binormal of the curve α.

Proof. Let C denote a patch that parametrizes the envelope of the Galilean spheres defining the canal surface. Since the curvature of α is nonzero, the Frenet- Serret frame $\{T, N, B\}$ is well-defined, and we can write

$$C(s,t) - \alpha(s) = a(s,t)T + b(s,t)N + c(s,t)B \tag{8}$$

where a, b and c are differentiable on the interval on which α is defined. We must have

$$\|C(s,t){-}\alpha(s)\|^2_{G_3} = \begin{cases} a^2 = r^2(s), \text{If } \ a(s,t) \neq 0 \\ b^2 + c^2 = r^2(s), \text{If } \ a(s,t) = 0 \end{cases} \tag{9}$$

Equation (9) expresses analytically the geometric fact that $C(s,t)$ lies on a Galilean sphere $S^2_{\pm}(s)$ of radius $r(s)$ centered at $\alpha(s)$. Furthermore, $C(s,t){-}\alpha(s)$ is a normal vector to the canal surface; this fact implies that

$$\langle C(s,t){-}\alpha(s), C_s\rangle_{G_3} = 0, \tag{10}$$

$$\langle C(s,t){-}\alpha(s), C_t\rangle_{G_3} = 0. \tag{11}$$

Case 1: Let $a(s,t) \neq 0$. Equations (9) and (10) say that the vectors C_s and C_t are tangent to $S^2_{\pm}(s)$. From (8) and (9) we get

$$\begin{cases} a^2 = r^2 \\ aa_s = rr' \end{cases} \tag{12}$$

When we differentiate (8) with respect to s and use the Frenet-Serret formulas, we obtain

$$C_s = (1{+}a_s)\,T + (a\kappa{+}b_s{-}c\tau)\,N + (c_s{+}b\tau)\,B. \tag{13}$$

Then (9),(12), (13), and (10) imply that

$$(1 + a_s)a = 0. \tag{14}$$

From (12) and (14), we get

$$r = \pm(c_1 - s). \tag{15}$$

Thus (8) is not a surface.

Case 2: Let $a(s,t) = 0$.Equations (9) and (10) say that the vectors C_s and C_t are tangent to $S^2_\pm(s)$. From (8) and (9) we get

$$\begin{cases} b^2 + c^2 = r^2 \\ bb_s + cc_s = rr_s. \end{cases} \tag{16}$$

When we differentiate (8) with respect to t and use the Frenet-Serret formulas, we obtain

$$C_t = b_t N + c_t B. \tag{17}$$

Then (11), (16), (17) imply that

$$bb_t + cc_t = 0 \ (r_t = 0). \tag{18}$$

From (16) and (18), we get

$$\begin{aligned} b &= r(s)\cos t, \\ c &= r(s)\sin t\ . \end{aligned} \tag{19}$$

Thus (8) becomes

$$C\,(s,t) = \alpha(s) + r(s)\,(\,N(s)\,\mathrm{cost}{+}B(s)\,\mathrm{sint})\,. \tag{20}$$

The equation (20) means that, the surface is galilean sphere (cylinder) at the point $\alpha(s)$ with radius in $r(s)$.

It is easy to see that when the radius function $r(s)$ is constant ,the definition of canal surface reduces to the definition of a tube.In fact, we can characterize tubes among all canal surfaces.With the Frenet-Serret system in hand, we can construct a "tubular surface" of radius $r = const.$ about the curve by defining a surface with parameters s and t:

$$Tube\,(s,t) = \alpha(s) + r\ (N(s)\,\mathrm{cost}{+}B(s)\,\mathrm{sint})\,. \tag{21}$$

□

Definition 3.4. Let $\alpha : I \subset R \to G^3$ be a admissible curve in Galilean 3-space. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization

$$M(s,\theta) = \alpha(s) + \lambda\,[N(s)\cos\theta + B(s)\sin\theta]\,. \tag{22}$$

Theorem 3.5. *Let $(X,Y) \in \{(K,H)\}$ and let M a tubular surface defined by (3.15). If M is a (X,Y) -Weingarten surface, then the curvature of α is a non-zero constant. If α has non-zero constant torsion, then M is generated by a circular helix α in Galilean 3-space.*

Proof. We have the natural frame $\{M_s, M_\theta\}$ of the surface M given by

$$\begin{aligned} M_s &= T - (\lambda\tau\sin\theta)\,N + (\lambda\tau\cos\theta)\,B, \\ M_\theta &= -(\lambda\sin\theta)\,N + (\lambda\cos\theta)\,B. \end{aligned}$$

The components of the curvatures K and H are

$$\begin{aligned} E &= 1, F = 0, G = \lambda^2, \\ e &= \lambda\tau^2 - \kappa\cos\theta,\ f = \lambda\tau,\ g = \lambda. \end{aligned}$$

On the other hand, the unit normal vector field U is obtained by

$$U = -\left(N\cos\theta + B\sin\theta\right).$$

According to (4), the curvatures K and H are

$$K = \frac{-\kappa\cos\theta}{\lambda}\quad,\quad H = -\frac{1 - \lambda\kappa\cos\theta + \lambda^2\tau^2}{2\lambda}, \tag{23}$$

respectively. Differentiating K and H with respect to s and θ, we get

$$K_s = -\frac{\kappa'\cos\theta}{\lambda},\qquad K_\theta = \frac{\kappa\sin\theta}{\lambda}, \tag{24}$$

$$H_s = \frac{-\kappa'\cos\theta + 2\lambda\tau\tau'}{2},\qquad H_\theta = \frac{\kappa\sin\theta}{2}, \tag{25}$$

Now, we investigate a tubular surface M in G_3 satisfying the Jacobi equation $\Phi(X,Y) = 0$.We consider tubular surface M in G_3 satisfying $\Phi(K,H) = 0$, by using (23) and (24), we have

$$K_sH_\theta - K_\theta H_s = \kappa\tau\tau'\sin\theta = 0,$$

for every θ.Therefore we conclude that $\tau' = 0$. □

Theorem 3.6. *Let M be a tubular surface satisfying the linear equation $aK+bH = c$. If $\left(a + a\lambda^2\tau^2 + \lambda^2 c\right) \neq 0$, then it is an open part of a circular cylinder in Galilean 3-space. Let $(X,Y) \in \{(K,H)\}$. Then there are no (X,Y)-tubular linear Weingarten surface M in Galilean 3-space.*

Proof. We suppose that tubular surface M in G_3 is a linear Weingarten surface, that is, it satisfies the equation

$$aK + bH = c. \tag{26}$$

Then, by (23) , we have

$$(-2a\kappa - \lambda b\kappa)\cos\theta + \left(b + b\lambda^2\tau^2 - 2\lambda c\right) = 0.$$

According to the definition of the linear independent of vectors, we have

$$2a\kappa + b\lambda\kappa = 0$$
$$b + b\lambda^2\tau^2 - 2\lambda c = 0$$

which imply

$$\kappa\left(a + a\lambda^2\tau^2 + \lambda^2 c\right) = 0.$$

If $\left(a + a\lambda^2\tau^2 + \lambda^2 c\right) \neq 0$, then $\kappa = 0$ and $\tau = 0$. Thus, M is an open part of a circular cylinder in Galilean 3-space. □

Theorem 3.7. *The tubular surface $M(s,\theta)$ is not umblical and minimal.*

Proof. Let k_1, k_2 be the principal curvatures of $M(s,\theta)$ in G_3 . The principal curvatures are obtained as follows;

$$\begin{aligned} k_1 &= -\frac{1-\lambda\kappa\cos\theta+\lambda^2\tau^2}{2\lambda}+\sqrt{\left(\frac{1-\lambda\kappa\cos\theta+\lambda^2\tau^2}{2\lambda}\right)^2+\frac{\kappa\cos\theta}{\lambda}}, \\ k_2 &= -\frac{1-\lambda\kappa\cos\theta+\lambda^2\tau^2}{2\lambda}-\sqrt{\left(\frac{1-\lambda\kappa\cos\theta+\lambda^2\tau^2}{2\lambda}\right)^2+\frac{\kappa\cos\theta}{\lambda}}. \end{aligned} \quad (27)$$

Since M has not a curvature diagram such that $k_1 - k_2 = 0$ and $k_1 + k_2 = 0$ then M is not umblical and minimal. □

Theorem 3.8. *Let M be a tubular surface in Galilean 3-space. Then M is a $HK-$quadric surface if and only if M is a circular cylinder.*

At this point, we conclude that, only $HK-$quadratic tubular surface in Galilean 3-space is circular cylinder

Proof. Suppose that the surface M is $HK-$quadric.Then the equation (1) implies

$$aHH_s + b(H_sK + HK_s) + cKK_s = 0. \quad (28)$$

Then, by substituting (23), (24) and (25) into (28) it follows that

$$\left(8b\lambda\kappa - 2a\lambda^2\kappa^2 - 8c\kappa^2\right)\cos\theta\sin\theta + \left(2a\lambda\kappa - 4b\lambda^2\kappa\tau^2 + 2a\lambda^3\kappa\tau^2 - 4b\kappa\right)\sin\theta = 0.$$

Since the identity holds for every θ, all the coefficients must be zero. Therefore, we have

$$\begin{aligned} \left(8b\lambda\kappa - 2a\lambda^2\kappa^2 - 8c\kappa^2\right) &= 0, \\ \left(2a\lambda\kappa - 4b\lambda^2\kappa\tau^2 + 2a\lambda^3\kappa\tau^2 - 4b\kappa\right) &= 0. \end{aligned}$$

Thus, we get $\kappa = 0$.Then M is a circular cylinder. □

Now, we illustrated an example of Tubular surface about a admissible curve in Galilean 3-space.Let us consider a curve:

$$\alpha(s) = (as, b\cos s, b\sin s).$$

In this case, one cancalculate its Frenet-Serret trihedra as

$$\begin{aligned} T(s) &= (a, -b\sin s, b\cos s) \\ N(s) &= (0, -\cos s, -\sin s) \\ B(s) &= (0, \sin s, -\cos s). \end{aligned}$$

Thus, we obtained tubular surface as follow:

$$M(s,\theta) = (as, b\cos s - \lambda\cos\theta\cos s + \lambda b\sin\theta\sin s, b\sin s - \lambda\cos\theta\sin s - \lambda b\sin\theta\cos s).$$

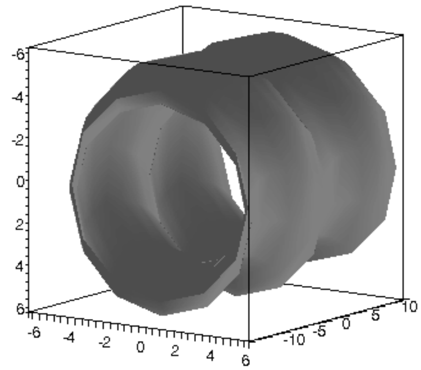

FIGURE 1. Tubes in Galilean 3-space

4. Weingarten Tubular Surface in Pseudo-Galilean 3-Space

Theorem 4.1. *Let $\alpha : I \to G_3$ be a curve in Pseudo-Galilean 3-space. Suppose the center curve of a canal surface is a unit speed curve α with nonzero curvature . Then the canal surface can be parametrized*

$$C(s,t) = \alpha(s) + r(s)\left(N(s)\cosh t + B(s)\sinh t\right),$$

where T, N and B denote the tangent, principal normal and binormal of the curve α.

Definition 4.2. Let $\alpha : I \subset R \to G_3^1$ be a admissible curve in Pseudo-Galilean 3-space. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization

$$M(s,\theta) = \alpha(s) + \lambda\left[N(s)\cosh\theta + B(s)\sinh\theta\right]. \tag{29}$$

Theorem 4.3. *Let $(X,Y) \in \{(K,H)\}$ and let M a tubular surface defined by (29). If M is a (X,Y) -Weingarten surface, then the curvature of α is a non-zero constant. If α has non-zero constant torsion, then M is generated by a circular helix α in Pseudo-Galilean 3-space.*

Proof. We have the natural frame $\{M_s, M_\theta\}$ of the surface M given by

$$\begin{aligned} M_s &= T + (\lambda\tau\sinh\theta)N + (\lambda\tau\cosh\theta)B, \\ M_\theta &= (\lambda\sinh\theta)N + (\lambda\cosh\theta)B. \end{aligned}$$

The components of the curvatures K and H are

$$\begin{aligned} E &= 1, F = 0, G = -\lambda^2, \\ e &= \lambda\tau^2 + \kappa\cosh\theta,\ f = \lambda\tau,\ g = \lambda. \end{aligned}$$

On the other hand, the unit normal vector field U is obtained by

$$U = (N\cosh\theta + B\sinh\theta).$$

Since $\langle U, U\rangle_{G_3^1} = 1$, then the surface M is timelike. On the other hand, according to (6) and (7), the curvatures K and H are

$$K = \frac{-\kappa\cosh\theta}{\lambda}, \quad H = \frac{-1 + \lambda\kappa\cosh\theta + \lambda^2\tau^2}{2\lambda}, \tag{30}$$

respectively. Differentiating K and H with respect to s and θ, we get

$$K_s = -\frac{\kappa' \cosh\theta}{\lambda}, \quad K_\theta = -\frac{\kappa \sinh\theta}{\lambda}, \tag{31}$$

$$H_s = \frac{\kappa' \cosh\theta + 2\lambda\tau\tau'}{2}, \quad H_\theta = \frac{\kappa \sinh\theta}{2}, \tag{32}$$

Now, we investigate a tubular surface M in G_3 satisfying the Jacobi equation $\Phi(X,Y) = 0$.We consider tubular surface M in G_3 satisfying $\Phi(K,H) = 0$, by using (31) and (32), we have

$$K_s H_\theta - K_\theta H_s = \kappa\tau\tau' \sinh\theta = 0.$$

for every θ.Therefore we conclude that $\tau' = 0$. □

Theorem 4.4. *Let M be a tubular surface satisfying the linear equation $aK+bH = c$. If $\left(-2b + a\lambda^2\tau^2 + 2\lambda c\right) \neq 0$, then it is an open part of a circular cylinder in Pseudo-Galilean 3-space. Let $(X,Y) \in \{(K,H)\}$. Then there are no (X,Y)-tubular linear Weingarten surface M.*

Proof. We suppose that tubular surface M in G_3^1 is a linear Weingarten surface, that is, it satisfies the equation

$$aK + bH = c.$$

Then, by (30) , we have

$$(-2b\kappa + \lambda a\kappa)\cosh\theta + \left(-a - 2\lambda c + a\lambda^2\tau^2\right) = 0.$$

According to the definition of the linear independent of vectors, we have

$$-2b\kappa + \lambda a\kappa = 0$$
$$-a - 2\lambda c + a\lambda^2\tau^2 = 0$$

which imply

$$\kappa\left(-2b + a\lambda^2\tau^2 + 2\lambda c\right) = 0.$$

If $\left(-2b + a\lambda^2\tau^2 + 2\lambda c\right) \neq 0$, then $\kappa = 0$ and $\tau = 0$. Thus, M is an open part of a circular cylinder in Pseudo-Galilean 3-space. □

Theorem 4.5. *The tubular surface $M(s,\theta)$ is not umblical and minimal.*

Proof. Let k_1, k_2 be the principal curvatures of $M(s,\theta)$ in G_3 . The principal curvatures are obtained as follows;

$$k_1 = \frac{-1 + \lambda\kappa\cosh\theta + \lambda^2\tau^2}{2\lambda} + \sqrt{\left(\frac{-1 + \lambda\kappa\cosh\theta + \lambda^2\tau^2}{2\lambda}\right)^2 + \frac{\kappa\cosh\theta}{\lambda}}$$

$$k_2 = \frac{-1 + \lambda\kappa\cosh\theta + \lambda^2\tau^2}{2\lambda} - \sqrt{\left(\frac{-1 + \lambda\kappa\cosh\theta + \lambda^2\tau^2}{2\lambda}\right)^2 + \frac{\kappa\cosh\theta}{\lambda}}.$$

Since M has not a curvature diagram such that $k_1 - k_2 = 0$ and $k_1 + k_2 = 0$ then M is not umblical and minimal.

□

Theorem 4.6. *Let M be a tubular surface in Pseudo-Galilean 3-space. Then M is a $HK-$ quadric surface if and only if M is a circular cylinder.*

At this point, we conclude that, only $HK-$quadric tubular surface is circular cylinder in Pseudo-Galilean 3-space.

Proof. Suppose that the surface M is $HK-$quadric. Then the equation (1) implies

$$aHH_\theta + b(H_\theta K + HK_\theta) + cKK_\theta = 0. \tag{33}$$

Then, by substituting (30), (31) and (32) into (33) it follows that

$$(\kappa \sinh\theta)\left[\left(-a\lambda + a\lambda^3\tau^2 + 2b - 2b\lambda^2\tau^2\right) + \left(a\lambda^2\kappa\cosh\theta - 4b\lambda\kappa\cosh\theta + 4c\kappa\cosh\theta\right)\right] = 0.$$

Since the identity holds for every θ, all the coefficients must be zero. Therefore, we have

$$\begin{aligned} (\kappa\sinh\theta)\left(-a\lambda + a\lambda^3\tau^2 + 2b - 2b\lambda^2\tau^2\right) &= 0, \\ \left(\kappa^2\sinh\theta\cosh\theta\right)\left(a\lambda^2 - 4b\lambda + 4c\right) &= 0. \end{aligned}$$

Thus, we get $\kappa = 0$.Then M is a circular cylinder. □

Example 4.7. Now, we illustrated an example of Tubular surface about a admissible curve in Pseudo-Galilean 3-space.Let us consider a curve:

$$\alpha(s) = \left(as, a\cosh\frac{s}{b}, a\sinh\frac{s}{b}\right).$$

In this case, one cancalculate its Frenet-Serret trihedra as

$$\begin{aligned} T(s) &= \left(a, \frac{a}{b}\sinh\frac{s}{b}, \frac{a}{b}\cosh\frac{s}{b}\right) \\ N(s) &= \left(0, \cosh\frac{s}{b}, \sinh\frac{s}{b}\right) \\ B(s) &= \left(0, \sinh\frac{s}{b}, \cosh\frac{s}{b}\right). \end{aligned}$$

Thus, we obtained tubular surface as follow:

$$M(s,\theta) = \left(as, a\cosh\frac{s}{b} + \lambda\cosh\theta\cosh\frac{s}{b} + \lambda\sinh\theta\sinh\frac{s}{b}, a\sinh\frac{s}{b} + \lambda\cosh\theta\sinh\frac{s}{b} + \lambda\sinh\theta\cosh\frac{s}{b}\right).$$

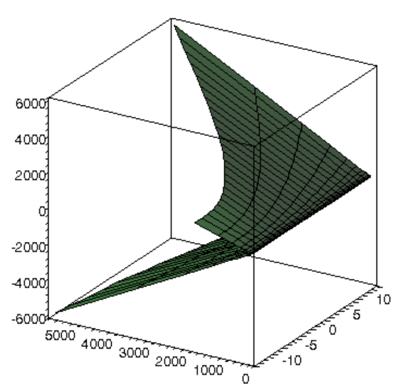

FIGURE 2. Tubes in Pseudo-Galilean 3-space

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[1] Department of Mathematics,
Uşak University University,
Uşak, Turkey
E-mail address: murat.karacan@usak.edu.tr

[2] Department of Mathematics,
Uşak University University,
Uşak, Turkey
E-mail address: yilmaz.tuncer@usak.edu.tr