Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 5 Issue 3 (2013), Pages 18-26

# HYPERGEOMETRIC REPRESENTATION FOR BASKAKOV-DURRMEYER-STANCU TYPE OPERATORS

## (COMMUNICATED BY HÜSEYIN BOR)

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Abstract. In the present paper, we introduce and study the mixed summation-integral type operators having Baskakov and Beta basis functions in summation and integration, respectively. First, we estimate moments of these operators using hypergeometric series. Next, we obtain an error estimation in simultaneous approximation for Baskakov-Durrmeyer-Stancu operators.

## 1. Introduction

Khan [4] and Mishra [5] have proved some results dealing with the degree of approximation of functions in $L_p$- spaces using different types of operators. Baskakov-Durrmeyer operators were first considered by Sahai-Prasad [8] in 1985 . Sinha et al. [9] improved and corrected the results of [8]. In 2005, Finta [1], introduced a new type of Baskakov-Durrmeyer operator by taking the weight function of Beta operators on $L[0,\infty)$ as

$$D_n(f,x)=\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)f(t)dt+p_{n,0}(x)f(0), \tag{1}$$

where $p_{n,k}(x)=\frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}$ and $b_{n,k}(t)=\frac{(n+1)(n+2)_k}{k!}\frac{t^k}{(1+t)^{n+k+2}}$.
Wafi and Khatoon [11] have proved inverse theorem for generalized Baskakov operators. Recently, Gupta and Yadav [3] introduced the Baskakov -Beta- Stancu operator and invetigated like asymptotic formula, moments of these operators using hypergeometric series and errors estimation in simultaneous approximation. we

[0]2000 Mathematics Subject Classification: 41A25, 41A35.

Keywords and phrases. Hypergeometric series, Baskakov-Durrmeyer-Stancu operators, Estimation of errors.



Submitted December 19, 2012. Published June 28, 2013.

write the operators (1) as

$$\begin{aligned} D_n(f,x) &= (n+1)\sum_{k=1}^{\infty}\frac{(n)_k}{k!}\frac{x^k}{(1+x)^{n+k}}\int_0^{\infty}\frac{(n+2)_k}{k!}\frac{t^k}{(1+t)^{n+k+2}}f(t)dt+\frac{f(0)}{(1+x)^n} \\ &= (n+1)\int_0^{\infty}\frac{f(t)(1+x)^2}{[(1+x)(1+t)]^{n+2}}\sum_{k=1}^{\infty}\frac{(n)_k(n+2)_k}{(k!)^2}\frac{(xt)^k}{[(1+x)(1+t)]^k}dt+\frac{f(0)}{(1+x)^n}. \end{aligned}$$

By hypergeometric series ${}_2F_1(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k$ and the Pochhammer symbol $(n)_k$ is
$(n)_k=n(n+1)(n+2)(n+3)....(n+k-1)$, using the equality $(1)_k=k!$, we can write

$$D_n(f,x)=(n+1)\int_0^{\infty}\frac{f(t)(1+x)^2}{[(1+x)(1+t)]^{n+2}}\left[{}_2F_1\left(n,n+2;1;\frac{xt}{(1+x)(1+t)}\right)-1\right]dt+\frac{f(0)}{(1+x)^n}.$$

Now using ${}_2F_1(a,b;c;x)={}_2F_1(b,a;c;x)$ and applying Pfaff- Kummer transformation

$${}_2F_1(a,b;c;x)=(1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)$$

we have

$$D_n(f,x)=(n+1)\int_0^{\infty}f(t)(1+x)^2\left[\frac{{}_2F_1\left(n+2,1-n;1;\frac{-xt}{1+x+t}\right)}{(1+x+t)^{n+2}}-\frac{1}{[(1+x)(1+t)]^{n+2}}\right]dt+\frac{f(0)}{(1+x)^n}. \tag{2}$$

This is the form of the operators (1) in terms of hypergeometric functions.
Verma et al. [10] considered Baskakov -Durrmeyer- Stancu operators and studied some approximation properties of these operators. Very recently, Mishra and Patel [6] introduced a simple Stancu generalization of q-analogue of well known Durrmeyer operators. We first estimate moments of $q-$ Durrmeyer-Stancu operators. They also established the rate of convergence as well as Voronovskaja type asymptotic formula for $q-$ Durrmeyer-Stancu operators. Here, we introduce Baskakov -Durrmeyer- Stancu operators in terms of hypergeometric functions, for $0\le\alpha\le\beta$ as

$$D_{n,\alpha,\beta}(f,x)=(n+1)\int_0^{\infty}f\left(\frac{nt+\alpha}{n+\beta}\right)(1+x)^2\left[\frac{{}_2F_1\left(n+2,1-n;1;\frac{-xt}{1+x+t}\right)}{(1+x+t)^{n+2}}-\frac{1}{[(1+x)(1+t)]^{n+2}}\right]dt +\frac{f(0)}{(1+x)^n}. \tag{3}$$

For $\alpha=\beta=0$ the operators (3) reduces to the operators (1).
we know that

$$\sum_{k=0}^{\infty}p_{n,k}(x)=1,\ \int_0^{\infty}p_{n,k}(x)dx=\frac{1}{n-1},\ \sum_{k=1}^{\infty}b_{n,k}(t)=n+1,\ \int_0^{\infty}b_{n,k}(t)dt=1.$$

We take

$$C_\nu[0,\infty)=\{f\in C[0,\infty):f(t)=O(t)^\nu,\nu>0\}.$$

The operators $D_{n,\alpha,\beta}(f,x)$ are well defined for $f \in C[0,\infty)$. The behavior of these operators is very similar to the operators recently introduced in [7] by Mishra et al. In the present note, first, we estimate moments of Baskakov -Durrmeyer- Stancu operators with the help of hypergeometric series. Next, we give an error estimation in simultaneous approximation for the operators (3)

## 2. Auxiliary results

In the sequel we shall need several lemmas.

**Lemma 2.1.** For $n > 0$ and $s > -1$, we have

$$D_n(t^s,x) = \frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}\left[(1+x)^s {}_2F_1\left(1-n,-s;1;\frac{x}{1+x}\right) - (1+x)^{-n}\right]. \tag{4}$$

Moreover,

$$D_n(t^s,x) = \frac{(n+s-1)!(n-s)!}{n!(n-1)!}x^s + \frac{s(s-1)(n+s-2)!(n-s)!}{n!(n-1)!}x^{s-1} + O(n^{-2}). \tag{5}$$

*Proof.* Taking $f(t) = t^s$, $t = (1+x)u$ and using Pfaff-Kummer transformation the right-hand side of (2), we get

$$D_n(t^s,x) = (n+1)\int_0^\infty \frac{(1+x)^{s+3}u^s}{[(1+x)(1+u)]^{n+2}}\sum_{k=0}^\infty \frac{(1-n)_k(n+2)_k}{(k!)^2}\frac{(-x(1+x)u)^k}{[(1+x)(1+u)]^k}du$$
$$+\frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}(1+x)^{-n} = Q_1 + Q_2(say).$$

$$\begin{aligned}
Q_1 &= (n+1)\sum_{k=0}^\infty \frac{(n+2)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{s-n+1}\int_0^\infty \frac{u^{s+k}}{(1+u)^{n+k+2}}du \\
&= (n+1)\sum_{k=0}^\infty \frac{(n+2)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{s-n+1}B(s+k+1,n-s+1) \\
&= (n+1)\sum_{k=0}^\infty \frac{(n+2)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{s-n+1}\frac{\Gamma(s+k+1)\Gamma(n-s+1)}{\Gamma(n+k+2)}.
\end{aligned}$$

Using $\Gamma(n+k+2) = \Gamma(n+2)(n+2)_k$, we have

$$\begin{aligned}
Q_1 &= (n+1)\sum_{k=0}^\infty \frac{(n+2)_k(1-n)_k}{(k!)^2}(-x)^k(1+x)^{s-n+1}\frac{\Gamma(s+1)(s+1)_k\Gamma(n-s+1)}{\Gamma(n+2)(n+2)_k} \\
&= (1+x)^{s-n+1}\frac{\Gamma(s+1)\Gamma(n-s+1)}{\Gamma(n+1)}\sum_{k=0}^\infty \frac{(s+1)_k(1-n)_k}{(k!)^2}(-x)^k \\
&= (1+x)^{s-n+1}\frac{\Gamma(s+1)\Gamma(n-s+1)}{\Gamma(n+1)}{}_2F_1(1-n,1+s;1;-x).
\end{aligned}$$

Using ${}_2F_1(a,b;c;x) = (1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)$, we have

$$Q_1 = \frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}(1+x)^s {}_2F_1\left(1-n,-s;1;\frac{x}{1+x}\right).$$

Combining $Q_1$ and $Q_2$, we get

$$D_n(t^s,x) = \frac{\Gamma(n-s+1)\Gamma(s+1)}{\Gamma(n+1)}\left[(1+x)^s {}_2F_1\left(1-n,-s;1;\frac{x}{1+x}\right)-(1+x)^{-n}\right].$$

The other consequence (5) follows from the above equation by writting the expansion of hypergeometric series. □

**Lemma 2.2.** For $0 \le \alpha \le \beta$ and $m > 0$, we have

$$\begin{aligned} D_{n,\alpha,\beta}(t^s,x) &= x^s \frac{n^s}{(n+\beta)^s}\frac{(n+s-1)!(n-s)!}{n!(n-1)!} \\ &+ x^{s-1}\left\{s(s-1)\frac{n^s}{(n+\beta)^s}\frac{(n+s-2)!(n-s)!}{n!(n-1)!} + s\alpha\frac{n^{s-1}}{(n+\beta)^s}\frac{(n+s-2)!(n-s+1)!}{n!(n-1)!}\right\} \\ &+ x^{s-2}\left\{s(s-1)(s-2)\alpha\frac{n^{s-1}}{(n+\beta)^s}\frac{(n+s-3)!(n-s+1)!}{n!(n-1)!}\right. \\ &+ \left.\frac{s(s-1)}{2}\alpha^2\frac{n^{s-2}}{(n+\beta)^s}\frac{(n+s-3)!(n-s+2)!}{n!(n-1)!}\right\} + O(n^{-m}). \end{aligned}$$

*Proof.* Using binomial theorem, the relation between operators (2) and (3) can be defined as

$$\begin{aligned} D_{n,\alpha,\beta}(t^s,x) &= \sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}\right)^s dt + (1+x)^{-n}\left(\frac{\alpha}{n+\beta}\right)^s \\ &= \sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\sum_{j=0}^{\infty}\binom{s}{j}\frac{(nt)^j\alpha^{s-j}}{(n+\beta)^s}dt + (1+x)^{-n}\left(\frac{\alpha}{n+\beta}\right)^s \\ &= \sum_{j=0}^{\infty}\binom{s}{j}\frac{n^j\alpha^{s-j}}{(n+\beta)^s}\left\{D_n(t^j,x)-(1+x)^{-n}0\right\} + (1+x)^{-n}\left(\frac{\alpha}{n+\beta}\right)^s. \end{aligned}$$

Using (5), we get Lemma (2.2). □

**Lemma 2.3.** [2] For $m \in \mathbb{N}\bigcup\{0\}$, if

$$U_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)\left(\frac{k}{n}-x\right)^m,$$

then $U_{n,0}(x) = 1, U_{n,1}(x) = 0$ and we have the recurrence relation:

$$nU_{n,m+1}(x) = x(1+x)\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right].$$

Consequently, $U_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, where $[m]$ is integral part of $m$.

**Lemma 2.4.** [10] For $m \in \mathbb{N}\bigcup\{0\}$, if

$$\begin{aligned} \mu_{n,m}(x) &= D_{n,\alpha,\beta}((t-x)^m,x) \\ &= \sum_{k=1}^{\infty} p_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt + p_{n,0}(x)\left(\frac{\alpha}{n+\beta}-x\right)^m \end{aligned}$$

then

$$\mu_{n,0}(x) = 1, \;\; \mu_{n,1}(x) = \frac{\alpha-\beta x}{n+\beta}$$

and for n>m we have recurrence relation:

$$\begin{aligned}(n-m)\left(\frac{n+\beta}{n}\right)\mu_{n,m+1}(x) &= x(1+x)\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right]\\ &+\left[(m+nx)+\left(\frac{n+\beta}{n}\right)\left(\frac{\alpha}{n+\beta}-x\right)(n-2m)\right]\mu_{n,m}(x)\\ &-\left(\frac{\alpha}{n+\beta}-x\right)\left[\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)-1\right]m\mu_{n,m-1}(x).\end{aligned}$$

From the recurrence relation, it easily verified that for all $x\in[0,\infty)$, we have

$$\mu_{n,m}(x)=O(n^{-[(m+1)/2]}).$$

**Lemma 2.5.** [2] There exist the polynomials $q_{i,j,s}(x)$ on $[0,\infty)$, independent of $n$ and $k$ such that

$$x^s(1+x)^s\frac{d^s}{dx^s}p_{n,k}(x)=\sum_{\substack{2i+j\le s\\ i,j\ge 0}} n^i(k-nx)^j q_{i,j,s}(x)p_{n,k}(x).$$

## 3. Main result

In this section, we give an estimate of the degree of approximation by $D^{(s)}_{n,\alpha,\beta}(f(t),x)$ for smooth functions.

**Theorem 3.1.** *Let $f\in C_\nu[0,\infty)$ for some $\nu>0$, $m>0$ and $s\le q\le s+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then for sufficiently large $n$*

$$||D^{(s)}_{n,\alpha,\beta}(f,x)-f^{(s)}(x)||_{C[a,b]}\le C_1 n^{-1}\sum_{i=s}^{q}||f^i||_{C[a,b]}+C_2 n^{1/2}\omega(f^{(q)},n^{1/2})+O(n^{-m}), \tag{6}$$

*where $C_1$, $C_2$ are constants independent of $f$ and $n$, $\omega(f,\delta)$ is the modulus of continuity of $f$ on $(a-\eta,b+\eta)$ and $||.||_{C[a,b]}$ denotes the sup-norm on $[a,b]$.*

*Proof.* By the Taylor's, expansion, we have

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(x)-f^{(q)}(\xi)}{q!}(t-x)^q\chi(t)+h(t,x)(1-\chi(t))$$

where $\xi$ lies between $t$ and $x$, and $\chi(t)$ is the characteristic function on interval $(a-\eta,b+\eta)$.
For $t\in(a-\eta,b+\eta)$ and $x\in[a,b]$, we have

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(x)-f^{(q)}(\xi)}{q!}(t-\xi)^q.$$

For $t\in[0,\infty)\backslash(a-\eta,b+\eta)$ and $x\in[a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$D_{n,\alpha,\beta}^{(s)}(f,x)-f^{(s)}(x)=\left\{\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}D_{n,\alpha,\beta}^{(s)}((t-x)^i,x)-f^{(s)}(x)\right\}+D_{n,\alpha,\beta}^{(s)}\Bigg(\frac{f^{(q)}(x)-f^{(q)}(\xi)}{q!}$$
$$(t-x)^q\chi(t),x\Bigg)+D_{n,\alpha,\beta}^{(s)}(h(t,x)(1-\chi(t)),x)=F_1+F_2+F_3.$$

Using Lemma 2.2, we get

$$\begin{aligned}F_1 &= \sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^s}{dx^s}\Bigg[x^j\frac{n^j}{(n+\beta)^j}\frac{(n+j-1)!(n-j)!}{n!(n-1)!}\\
&+ x^{j-1}\Bigg(j(j-1)\frac{n^j}{(n+\beta)^j}\frac{(n+j-2)!(n-j)!}{n!(n-1)!}+j\alpha\frac{n^{j-1}}{(n+\beta)^j}\frac{(n+j-2)!(n-j+1)!}{n!(n-1)!}\Bigg)\\
&+ x^{j-2}\Bigg(j(j-1)(j-2)\alpha\frac{n^{j-1}}{(n+\beta)^j}\frac{(n+j-3)!(n-j+1)!}{n!(n-1)!}\\
&+ \frac{j(j-1)}{2}\alpha^2\frac{n^{j-2}}{(n+\beta)^j}\frac{(n+j-3)!(n-j+2)!}{n!(n-1)!}\Bigg)+O(n^{-m})\Bigg]-f^{(s)}(x).\end{aligned}$$

Hence

$$||F_1||_{C[a,b]}\leq C_1n^{-1}\sum_{i=s}^{q}||f^i||_{C[a,b]}+O(n^{-m}),\ uniformly\ on\ [a,b].$$

Next, we estimate $F_2$ as

$$\begin{aligned}|F_2| &\leq \sum_{k=1}^{\infty}|p_{n,k}^{(s)}(x)|\int_0^\infty b_{n,k}(t)\left\{\left|\frac{f^{(q)}(x)-f^{(q)}(\xi)}{q!}\right|\left|\frac{nt+\alpha}{n+\beta}-x\right|^q\chi(t)\right\}dt\\
&+ \frac{(n+s-1)!}{(n-1)!}(1+x)^{-n-s}\left|\frac{\alpha}{n+\beta}-x\right|^q\chi(t)\\
&\leq \frac{\omega(f^{(s)},\delta)}{q!}\Bigg[\sum_{k=1}^{\infty}|p_{n,k}^{(s)}(x)|\int_0^\infty b_{n,k}(t)\left(1+\frac{\left|\frac{nt+\alpha}{n+\beta}-x\right|}{\delta}\right)\left|\frac{nt+\alpha}{n+\beta}-x\right|^q dt\\
&+ \frac{(n+s-1)!}{(n-1)!}(1+x)^{-n-s}\left(1+\frac{\left|\frac{\alpha}{n+\beta}-x\right|}{\delta}\right)\left|\frac{\alpha}{n+\beta}-x\right|^q\Bigg]\\
&\leq \frac{\omega(f^{(s)},\delta)}{q!}\Bigg[\sum_{k=1}^{\infty}|p_{n,k}^{(s)}(x)|\int_0^\infty b_{n,k}(t)\left(\left|\frac{nt+\alpha}{n+\beta}-x\right|^q+\delta^{-1}\left|\frac{nt+\alpha}{n+\beta}-x\right|^{q+1}\right)dt\\
&+ \frac{(n+s-1)!}{(n-1)!}(1+x)^{-n-s}\left(\left|\frac{\alpha}{n+\beta}-x\right|^q+\delta^{-1}\left|\frac{\alpha}{n+\beta}-x\right|^{q+1}\right)\Bigg]\end{aligned}$$

Now, on application of Schwarz inequality for integration and then for summation, we get

$$\begin{aligned}\sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \int_0^\infty b_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^q &\leq \sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \left(\int_0^\infty b_{n,k}(t)dt\right)^{\frac{1}{2}} \\ &\left(\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2q}dt\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty} p_{n,k}(x)\right. \\ &\left.\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2s}dt\right)^{\frac{1}{2}}.\end{aligned}$$

Using Lemma 2.3, we get

$$\begin{aligned}\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j} &= n^{2j}\left\{\sum_{k=0}^{\infty} p_{n,k}(x)(k/n-x)^{2j}) - (1+x)^{-n}(-x)^{2j}\right\} \\ &= n^{2j}\left\{O(n^{-j})+O(n^{-r})\right\}\ (for\ any\ r>0). \\ &= O(n^j). \qquad (7)\end{aligned}$$

Similarly, using Lemma 2.4, we get

$$\begin{aligned}\sum_{k=1}^{\infty} p_{n,k}(x)\int_0^\infty b_{n,k}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{2q}dt &= O(n^q)-(1+x)^{-n}(-x)^{2q} \\ &= O(n^{-q})+O(n^{-r})\ (for\ any\ r>0). \\ &= O(n^{-s}). \qquad (8)\end{aligned}$$

Hence

$$\sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \int_0^\infty b_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^q = O(n^{j/2})O(n^{-q/2}) = O(n^{(j-q)/2}),\ uniformly\ on\ [a,b]. \quad (9)$$

Therefore, by Lemma 2.5 and (9), we get

$$\begin{aligned}\sum_{k=1}^{\infty} |p_{n,k}^{(s)}(x)|\int_0^\infty b_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^q &\leq \sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq s\\ i,j\geq 0}} \frac{n^i q_{i,j,s}(x)}{x^s(1+x)^s} p_{n,k}(x)|k-nx|^j\int_0^\infty b_{n,k}(t) \\ &\left(\frac{nt+\alpha}{n+\beta}-x\right)^q dt \\ &\leq K\sum_{\substack{2i+j\leq s\\ i,j\geq 0}} n^i\left(\sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j\int_0^\infty b_{n,k}(t)\left|\frac{nt+\alpha}{n+\beta}-x\right|^q dt\right) \\ &= K\sum_{\substack{2i+j\leq s\\ i,j\geq 0}} n^i O(n^{(j-q)/2}) = O(n^{(s-q)/2}),\ uniformly\ on\ [a,b], (10)\end{aligned}$$

where $K = sup_{\substack{2i+j\leq s \\ i,j\geq 0}} sup_{x\in[a,b]} \frac{q_{i,j,s}(x)}{x^s(1+x)^s}$. Choosing $\delta = n^{-1/2}$ and making use of (10), we get for any $m > 0$,

$$||F_2||_{C[a,b]} \leq \frac{\omega(f^{(q)}, n^{-1/2})}{q!}[O(n^{(s-q)/2}) + n^{1/2}O(n^{(s-q-1)/2}) + O(n^{-m})] \leq C_2(n^{-(q-s)/2})\omega(f^{(q)}, n^{-1/2}).$$

For $t \in [0,\infty)\backslash(a-\eta, b+\eta)$, we can choose $\delta$ such that $|t-x| \geq \delta$ for all $x \in [a,b]$. Thus by Lemma 2.5, we get

$$|F_3| \leq \sum_{\substack{2i+j\leq s \\ i,j\geq 0}} \frac{n^i q_{i,j,s}(x)}{x^s(1+x)^s} \sum_{k=1}^{\infty} p_{n,k}(x)|k-nx|^j \int_{|t-x|\geq\delta} b_{n,k}(t)|h(t,x)|dt$$
$$+\frac{(n+s-1)!}{(n-1)!}(1+x)^{-n-s}\left|h\left(\frac{\alpha}{n+\beta}, x\right)\right|.$$

We can find a constant $M_1$ such that

$$|h(t,x)| \leq M_1\left|\frac{nt+\alpha}{n+\beta} - x\right|^{\beta} \; for \; |t-x| \geq \delta,$$

where $\beta \geq (\nu, q)$. Hence applying Schwarz inequality, (7) and (8), it is easy to see that $F_3 = O(n^{-r})$ for any $r > 0$ uniformly on $[a,b]$. Combining the estimates of $F_1, F_2$ and $F_3$, the required result follows. □

**Acknowledgement.** Authors are grateful to the anonymous learned referees for their valuable suggestions and comments for improvement of this paper. Authors mention their sense of gratitude to Professor Hüseyin Bor, editorial board of BMAA, Professor Faton Z. Merovci, secretary of BMAA for supporting this work. Kejal Khatri, one of the author, is thankful to the financial support of "Ministry of Human Resource Development", New Delhi, Govt. of India, for encouragement to this work.

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