Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 5 Issue 4(2013), Pages 14-18.

LIGHTLIKE HYPERSURFACES WITH HARMONIC CURVATURE IN A LORENTZIAN SPACE FORM

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Abstract. In this paper, we study the conditions of having harmonic curvature of lightlike hypersurfaces in a Lorentzian space form and examine for the relationship between harmonic curvature and local symmetry.

1. Introduction

A semi-Riemannian manifold has harmonic curvature if the divergence of the curvature tensor vanishes. Another characterization of this case is that the Ricci tensor is a Codazzi tensor. Obviously, this condition is closely related with parallelity of the Ricci tensor. But the former one is a stronger condition than this one. The study of manifolds with harmonic curvature also motivated by the relationship with the Yang-Mills connections. A connection is a critical point of the Yang-Mills functional if the connection has a harmonic curvature [1]. The studies on manifolds having harmonic curvature are actually initiated by A. Derdziń ski [2], [3]. He gave examples of Riemannian manifolds with harmonic curvature and classified them. Then submanifolds with harmonic curvatures are examined at [5], [6], [7], [8].

The aim of this paper is to investigate the necessary conditions for a lightlike hypersurface to have harmonic curvature. For the preliminaries part, we will use the book [9].

2. Preliminaries

Let g be degenerate M. Then, there exists a vector field $\xi \neq 0$ on M such that $g(\xi, X) = 0,\ \forall X \in \Gamma(TM)$. The radical or the null space of TM is a subspace $RadTM$ defined by

$$RadTM = \{\xi \in TM : g(\xi, X) = 0, \forall X \in TM\}$$

and M is called a lightlike hypersurface if the rank of $RadTM$ is 1. The tangent space of M has the decomposition

$$TM = RadTM \perp S(TM) \tag{1}$$

[0]2010 Mathematics Subject Classification: Primary 53B30; Secondary 53B25.

Keywords and phrases. Lorentzian space form, Lightlike hypersurfaces, Local symmetry, Harmonic curvature.

Submitted June 23, 2013. Published September 23, 2013.

where the complementary vector bundle $S(TM)$ is called a screen distribution of M. So, a lightlike hypersurface of a semi-Riemannian manifold $\left(\bar{M}, \bar{g}\right)$ is shown by $(M, g, S(TM))$. For any null section $\xi \in RadTM$, on a coordinate neighborhood $U \subset M$, there exists a unique null section N of a unique vector bundle $tr\,(TM)$ in $S\,(TM)^{\perp}$ satisfying:

$$\bar{g}\,(N,\xi) = 1,\ \bar{g}\,(N,N) = \bar{g}\,(N,X) = 0,\ \forall X \in \Gamma\left(S\,(TM)\,|_{U}\right)$$

where $tr\,(TM)$ and N are called transversal vector bundle and the null transversal vector field of $S\,(TM)$ respectively. Then the tangent bundle TM of M is decomposed as follows:

$$T\bar{M}|_{M} = S(TM) \oplus_{orth} \left(TM^{\perp} \oplus tr\,(TM)\right) = TM \oplus tr\,(TM)\,.$$

Let ∇ be the Levi-Civita connection of M and P the projection morphism of $\Gamma\,(TM)$ on $\Gamma\,(S\,(TM))$ with respect to the decomposition (1). Then the local Gauss-Weingarten formulas are given by

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h\,(X,Y) \\ \bar{\nabla}_X N &= -A_N X + \tau\,(X)\,N \\ \nabla_X PY &= \nabla^{*}_X PY + h^{*}\,(X,PY) \\ \nabla_X \xi &= -A^{*}_{\xi} X - \tau\,(X)\,\xi \end{aligned} \tag{2}$$

for any $X, Y \in \Gamma\,(TM)$, where ∇ and ∇^{*} are the induced linear connections, h and h^{*} are the second fundamental forms, A_N and A^{*}_{ξ} are the shape operators on TM and $S\,(TM)$ respectively. τ is a 1-form on TM defined by $g\,(\nabla_X N,\xi)$. h is independent of the choice of $S\,(TM)$ and it satisfies

$$h\,(X,\xi) = 0,\ X \in \Gamma\,(TM)\,. \tag{3}$$

The linear connection ∇ of M is not metric and satisfies

$$\left(\nabla_X\ g\right)(Y,Z) = \bar{g}\,(h\,(X,Y)\,,Z) + \bar{g}\,(h\,(X,Z)\,,Y) \tag{4}$$

for any $X, Y, Z \in \Gamma\,(TM)$. But the connection ∇^{*} of $S\,(TM)$ is metric.

The second fundamental forms h and h^{*} are related to their shape operators as follows:

$$\bar{g}\,(h\,(X,Y)\,,\xi) = g\left(A^{*}_{\xi}X, Y\right),\ \bar{g}\left(A^{*}_{\xi}X, N\right) = 0, \tag{5}$$

$$\bar{g}\,(h^{*}\,(X,PY)\,,N) = g\,(A_N X, PY)\,,\ \bar{g}\,(A_N X, N) = 0. \tag{6}$$

From (5), A^{*}_{ξ} is $S\,(TM)$-valued and self-adjoint on TM such that

$$A^{*}_{\xi}\xi = 0. \tag{7}$$

The covariant derivatives of h and A_N with respect to the connection ∇ are defined as

$$\left(\nabla_X h\right)(Y,Z) = \nabla^{t}_X h\,(Y,Z) - h\,(\nabla_X Y, Z) - h\,(Y, \nabla_X Z)\,, \tag{8}$$

$$\nabla_X\,(A_N Y) = \left(\nabla_X A_N\right)Y + A_N\left(\nabla_X Y\right). \tag{9}$$

The Riemann curvature tensor of a lightlike hypersurface $(M, g, S\,(TM))$ of a semi-Riemannian manifold $\left(\bar{M}, \bar{g}\right)$ is given at [4] by

$$\bar{R}\,(X,Y)\,Z = R\,(X,Y)\,Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + \left(\nabla_X h\right)(Y,Z) - \left(\nabla_Y h\right)(X,Z) \tag{10}$$

and for a semi-Riemannian space form $\left(\bar{M}\left(c\right),\bar{g}\right)$ this becomes

$$R\left(X,Y\right)Z=c\left\{\bar{g}\left(Y,Z\right)X-\bar{g}\left(X,Z\right)Y\right\}-A_{h(X,Z)}Y+A_{h(Y,Z)}X. \quad (11)$$

The covariant derivative of R with respect to the connection ∇ is defined by

$$\begin{aligned}\left(\nabla_W R\right)\left(X,Y\right)Z &= \nabla_W R\left(X,Y\right)Z-R\left(\nabla_W X,Y\right)Z \\ &\quad -R\left(X,\nabla_W Y\right)Z-R\left(X,Y\right)\nabla_W Z. \end{aligned} \quad (12)$$

A lightlike hypersurface is called locally symmetric if its curvature tensor satisfies the equation $\nabla R=0$ with respect to the induced linear connection on it. Since semi-Riemannian space forms have constant sectional curvatures, they are locally symmetric. We know the following theorem for locally symmetric lightlike hypersurfaces [4]:

Theorem 2.1. *Let $\bar{M}$ be a locally symmetric semi-Riemannian manifold and M be a lightlike hypersurface of $\bar{M}$ such that $A_N\xi$ is not a null vector field. Then M is locally symmetric if and only if M is totally geodesic.*

3. Lightlike Hypersurfaces with Harmonic Curvature

For $n\geq 1$ the divergence of a tensor field T is defined by

$$divT\left(X_1,...,X_k\right)=tr\left[\left(Y,Z\right)\rightarrow\left(\nabla_Y T\right)\left(X_1,...,X_k,Z\right)\right].$$

Since the metric is degenerate for a lightlike hypersurface, the divergence is defined in a different manner that that. According to Theorem 3.5.1 of [9] we have

$$div^g X=\sum_{k=0}^{n}\varepsilon_k\tilde{g}\left(\nabla_{E_k}X,E_k\right)=\bar{g}\left(\nabla_\xi X,N\right)+\sum_{k=1}^{n}\varepsilon_k g\left(\nabla_{E_k}X,E_k\right)$$

where $\tilde{g}$ is the associate metric of g on M and a quasi-orthonormal frame field $\{E_0=\xi,E_k\}$ of M, $\varepsilon_k=g\left(E_k,E_k\right)$, $\varepsilon_0=1$.

A similar argument can be carried out for the curvature tensor R of a lightlike hypersurface $\left(M,g,S\left(TM\right)\right)$ of a Lorentzian manifold $\left(\bar{M},\bar{g}\right)$. That is, for any $X,Y,Z\in\Gamma\left(TM\right)$

$$\begin{aligned} divR\left(X,Y\right)Z &= \sum_{k=0}^{n}\varepsilon_k\tilde{g}\left(\left(\nabla_{E_k}R\right)\left(X,Y\right)Z,E_k\right) \\ &= \sum_{k=1}^{n}g\left(\left(\nabla_{E_k}R\right)\left(X,Y\right)Z,E_k\right)+\bar{g}\left(\left(\nabla_\xi R\right)\left(X,Y\right)Z,N\right). \end{aligned} \quad (13)$$

Here, for $k\in\{1,...,n\}$, $\{E_0=\xi,E_k\}$ is the induced quasi-orthonormal frame field of M by the frame field $\{E_0=\xi,E_k,N\}$ of $\bar{M}$ such that $S\left(TM\right)=span\left\{E_k\right\}$ and $RadTM=span\left\{\xi\right\}$. M is said to have harmonic curvature if the divergence of its curvature tensor R vanishes, that is $divR=0$ [2]. By the equation (13), it is clear that any locally symmetric space has harmonic curvature. But, in general the converse is not valid.

Theorem 3.1. *Let $\left(M,g,S\left(TM\right)\right)$ be a lightlike hypersurface of an $(n+2)-$dimensional Lorentzian space form $\left(\bar{M}\left(c\right),\bar{g}\right)$ and $\left\{E_k\right\}_{k=1,...,n}$ be an orthonormal frame field of the screen distribution $S(TM)$ of M. Having one of the inequalities $C\left(\xi,E_k\right)>0$ or $C\left(\xi,E_k\right)<0$ satisfied and $A_N\xi$ is a non-vanishing vector field, $\left(M,g,S\left(TM\right)\right)$ has harmonic curvature if and only if M is totally geodesic.*

Proof. At first, substituting (11) at (12), with the help of (4), (8) and (9) we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= c\{\bar{g}(h(W,Y),Z)X + \bar{g}(h(W,Z),Y)X \\ &\quad -\bar{g}(h(W,X),Z)Y - \bar{g}(h(W,Z),X)Y\} \\ &\quad + (\nabla_W A)_{h(Y,Z)}X + A_{(\nabla_W h)(Y,Z)}X \\ &\quad - (\nabla_W A)_{h(X,Z)}Y - A_{(\nabla_W h)(X,Z)}Y. \end{aligned}$$

Then putting this in (13) we obtain

$$\begin{aligned} divR(X,Y)Z &= \sum_{k=1}^{n} [c\{\bar{g}(h(E_k,Y),Z)g(X,E_k) + \bar{g}(h(E_k,Z),Y)g(X,E_k) \\ &\quad +\bar{g}(h(E_k,X),Z)g(Y,E_k) + \bar{g}(h(E_k,Z),X)g(Y,E_k)\} \\ &\quad +g\left((\nabla_{E_k}A)_{h(Y,Z)}X, E_k\right) + g\left(A_{(\nabla_{E_k}h)(Y,Z)}X, E_k\right) \\ &\quad -g\left((\nabla_{E_k}A)_{h(X,Z)}Y, E_k\right) - g\left(A_{(\nabla_{E_k}h)(X,Z)}Y, E_k\right)] \\ &\quad +\bar{g}\left((\nabla_\xi A)_{h(Y,Z)}X, N\right) + \bar{g}\left(A_{(\nabla_\xi h)(Y,Z)}X, N\right) \\ &\quad -\bar{g}\left((\nabla_\xi A)_{h(X,Z)}Y, N\right) - \bar{g}\left(A_{(\nabla_\xi h)(X,Z)}Y, N\right). \end{aligned}$$

If $h = 0$, it is obvious that $divR = 0$. Conversely, if $divR = 0$, in the equation above setting $Y = Z = \xi$ we find

$$-\sum_{k=1}^{n} g\left(A_{(\nabla_{E_k}h)(X,\xi)}\xi, E_k\right) - \bar{g}\left(A_{(\nabla_\xi h)(X,\xi)}\xi, N\right) = 0. \tag{14}$$

Now, from (2) and (7) we can write that $\nabla_\xi \xi = -\tau(\xi)\xi$ and using the equations (3) and (8) we get $(\nabla_\xi h)(X,\xi) = -h(X, \nabla_\xi \xi) = 0$. Then by substituting these equations in (14) we get

$$\sum_{k=1}^{n} g\left(A_{h\left(X,\nabla_{E_k}\xi\right)}\xi, E_k\right) = \sum_{k=1}^{n} B(X, \nabla_{E_k}\xi)\, g(A_N\xi, E_k) = 0.$$

In this equation, with (2) setting $X = \xi$ we obtain

$$\sum_{k=1}^{n} g\left(A_\xi^* E_k, A_\xi^* E_k\right) g(A_N\xi, E_k) = \sum_{k=1}^{n} \left|A_\xi^* E_k\right|^2 C(\xi, E_k) = 0.$$

Since one of the inequalities $C(\xi, E_k) > 0$ or $C(\xi, E_k) < 0$ is valid and $A_N\xi$ is non-vanishing, we have $A_\xi^* E_k = 0$. Hence $A_\xi^* = 0$, that is M is totally geodesic. □

Corollary 3.2. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)-$dimensional Lorentzian space form $\left(\bar{M}(c), \bar{g}\right)$ and $\{E_k\}_{k=1,\dots,n}$ be an orthonormal frame field of the screen distribution $S(TM)$ of M. Having one of the inequalities $C(\xi, E_k) > 0$ or $C(\xi, E_k) < 0$ satisfied and $A_N\xi$ is a non-vanishing vector field, M has harmonic curvature if and only if it is locally symmetric.*

Proof. By (13) it is obvious that any locally symmetric space has harmonic curvature. Conversely, if a lightlike hypersurface of a Lorentzian space form has harmonic curvature, then it is totally geodesic according to the previous theorem. Since any Lorentzian space form is locally symmetric, by Theorem 1 we can state that, any totally geodesic lightlike hypersurface of a Lorentzian space form is also locally symmetric. □

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