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n-TUPLED COINCIDENCE AND COMMON FIXED POINT RESULTS FOR WEAKLY CONTRACTIVE MAPPINGS IN COMPLETE METRIC SPACES

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Abstract. In this paper, we prove results on n-tupled coincidence as well as n-tupled fixed point in partially ordered complete metric spaces for a pair of weakly contractive compatible mappings whenever n is even, wherein control functions are also employed. Our main theorem improves the corresponding results of Choudhury *et al.* (Ann. Univ. Ferrara 57: 1-16, 2011). We illustrate our main result with an example in arbitrary even order case which also substantiates the realized improvements.

1. Introduction

The enormous utility of Banach contraction principle is well known. This result is one of the pivotal results of metric fixed point theory. It has fruitful applications within as well as outside mathematics. Generalizations of this principle continues to be an active area of research. Many authors have extended this theorem employing relatively more general contractive conditions ensuring the existence of a fixed point. The investigation of fixed points in ordered metric spaces is a relatively new development which appears to have its origin (in 2004) in the paper of Ran and Reurings [21]. This paper was well complimented by the article of Nieto and López [20]. For similar other results in ordered metric spaces, one can be referred to [1]-[4],[14]-[16],[18],[19],[23].
In [9], Bhaskar and Lakshmikantham introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ wherein $(X, \preceq, d)$ be a partial metric space and also proved some coupled fixed point theorems in partially ordered complete metric spaces. Afterwards Berinde and Borcut [8] introduced the concept of tripled fixed point and proved some related theorems. Most recently, Imdad *et al.* [14] introduced the concepts of n-tupled coincidence as well as n-tupled fixed point and

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utilize these two definitions to obtain n-tupled coincidence as well as n-tupled common fixed point theorems for nonlinear mappings satisfying ϕ-contraction condition in partially ordered complete metric spaces.

The purpose of this paper is to prove some n-tupled coincidence as well as n-tupled fixed point theorems for a pair of weakly contractive compatible mappings enjoying mixed g-monotone property in a complete metric space equipped with a partial ordering.

2. Preliminaries

Definition 2.1. [9] Let $(X, \preceq)$ be a partially ordered set equipped with a metric d such that (X, d) is a metric space. We endow the product space $X \times X$ with the following partial ordering:

for $(x, y), (u, v) \in X \times X$, define $(u, v) \preceq (x, y) \Leftrightarrow u \preceq x, y \preceq v$.

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a mapping. Then T is said to be nondecreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $T(x_1) \preceq T(x_2)$ and nonincreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $T(x_2) \preceq T(x_1)$.

Definition 2.3. [9] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. Then F is said to have mixed monotone property if for any $x, y \in X$, $F(x, y)$ is monotonically nondecreasing in first argument and monotonically nonincreasing in second argument, that is, for

$$x_1, x_2 \in X, \; x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

$$y_1, y_2 \in X, \; y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq F(x, y_1).$$

Definition 2.4. [18] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then F is said to have mixed g-monotone property if for any $x, y \in X$, $F(x, y)$ is monotone g-nondecreasing in its first argument and monotone g-nonincreasing in its second argument, that is, for

$$x_1, x_2 \in X, \; g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

$$y_1, y_2 \in X, \; g(y_1) \preceq g(y_2) \Rightarrow F(x, y_2) \preceq F(x, y_1).$$

Definition 2.5. [9] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Definition 2.6. [18] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = gx \text{ and } F(y, x) = gy.$$

Furthermore, (x, y) is called the common coupled fixed point of F and g if $F(x, y) = gx = x, \; F(y, x) = gy = y$.

Remark 2.7. Definitions 2.4 and 2.6 for $g = I$ reduce to Definitions 2.3 and 2.5 respectively. Thus Definitions 2.4 and 2.6 generalize Definitions 2.3 and 2.5 respectively.

Definition 2.8. [10] Let $F : X \times X \to X$ and $g : X \to X$ be the two mappings. Then F is said to be g-compatible if

$$\begin{cases} \lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0 \\ \lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0, \end{cases}$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{cases} \lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x \\ \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y, \end{cases}$$

(for some $x, y \in X$) are satisfied.

Definition 2.9. [17] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied;
(a) ψ is monotonically increasing and continuous;
(b) $\psi(t) = 0$ if and only if $t = 0$.

Theorem 2.10. *[11] Let $(X, \preceq, d)$ be a complete partially ordered metric space. Let $\phi : [0,\infty) \to [0,\infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ while ψ be an altering distance function. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X and*

$$\psi(d(F(x,y), F(u,v))) \leq \psi(\max\{d(gx,gu), d(gy,gv)\}) - \phi(\max\{d(gx,gu), d(gy,gv)\})$$

for all $x, y, u, v \in X$ for which $gu \preceq gx$ and $gy \preceq gv$. Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and F is g-compatible. Also, suppose that
(a) F is continuous or
(b) X has the following properties:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $g(x_n) \preceq g(x)$ for all $n \geq 0$;
(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $g(y) \preceq g(y_n)$ for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$, then there exist $x, y \in X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$ i.e. F and g have a coupled coincidence point in X.

Recently, Berinde and Borcut [8] introduced the following partial order on the product space $X \times X \times X$:

$$(u,v,w) \preceq (x,y,z) \Leftrightarrow u \preceq x,\ y \preceq v,\ w \preceq z\ \forall\ (x,y,z), (u,v,w) \in X \times X \times X.$$

Definition 2.11. [8] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \times X \to X$ be a mapping. Then F is said to have mixed monotone property if F is monotone nodecreasing in first and third argument and monotone noincreasing in second argument, that is, for any $x, y, z \in X$

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z)$$
$$y_1, y_2 \in X,\ y_1 \preceq y_2 \Rightarrow F(x, y_2, z) \preceq F(x, y_1, z)$$
$$z_1, z_2 \in X,\ z_1 \preceq z_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2).$$

Definition 2.12. [8] An element $(x,y,z) \in X \times X \times X$ is called a tripled fixed point of the mapping $F : X \times X \times X \to X$ if $F(x,y,z) = x$, $F(y,x,y) = y$ and $F(z,y,x) = z$.

The following concept of n-fixed point was introduced by Gordji and Ramezani [12]. We suppose that the product space X^n is endowed with the following partial order, where n is the positive integer (odd or even): $(x^1, x^2, ..., x^n),\ (y^1, y^2, ..., y^n) \in X^n$

$$(x^1, x^2, ..., x^n) \preceq (y^1, y^2, ..., y^n) \Leftrightarrow x^{2i-1} \preceq y^{2i-1}\ \forall\ i \in \{1, 2, ..., [\frac{n+1}{2}]\}$$

$$(x^1, x^2, ..., x^n) \preceq (y^1, y^2, ..., y^n) \Leftrightarrow y^{2i} \preceq x^{2i}\ \forall\ i \in \{1, 2, ..., [\frac{n}{2}]\}.$$

Definition 2.13. [12] An element $(x^1, x^2, ..., x^n) \in X^n$ is called an n fixed point of the mapping $F : X^n \to X$ if

$$x^i = F(x^i, x^{i-1}, ..., x^2, x^1, x^2, ..., x^{n-i+1})\ \forall\ i \in \{1, 2, ..., n\}.$$

In this paper, we used the new definitions of n-tupled fixed point and n-tupled coincidence point given by Imdad *et al.* [14]. Throughout the paper, we consider n to be an even integer. We begin with the following definitions:

Definition 2.14. [14] Let $(X, \preceq)$ be a partially ordered set and $F : X^n \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is nondecreasing in its odd position arguments and nonincreasing in its even position arguments, that is, if,
(i) for all $x_1^1, x_2^1 \in X$, $x_1^1 \preceq x_2^1 \Rightarrow F(x_1^1, x^2, x^3, ..., x^n) \preceq F(x_2^1, x^2, x^3, ..., x^n)$
(ii) for all $x_1^2, x_2^2 \in X$, $x_1^2 \preceq x_2^2 \Rightarrow F(x^1, x_2^2, x^3, ..., x^n) \preceq F(x^1, x_1^2, x^3, ..., x^n)$
(iii) for all $x_1^3, x_2^3 \in X$, $x_1^3 \preceq x_2^3 \Rightarrow F(x^1, x^2, x_1^3, ..., x^n) \preceq F(x^1, x^2, x_2^3, ..., x^n)$

$$\vdots$$

for all $x_1^n, x_2^n \in X$, $x_1^n \preceq x_2^n \Rightarrow F(x^1, x^2, x^3, ..., x_2^n) \preceq F(x^1, x^2, x^3, ..., x_1^n)$.

Definition 2.15. [14] Let $(X, \preceq)$ be a partially ordered set. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. Then the mapping F is said to have the mixed g-monotone property if F is g-nondecreasing in its odd position arguments and g-nonincreasing in its even position arguments, that is, if,
(i) for all $x_1^1, x_2^1 \in X$, $gx_1^1 \preceq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, ..., x^n) \preceq F(x_2^1, x^2, x^3, ..., x^n$
(ii) for all $x_1^2, x_2^2 \in X$, $gx_1^2 \preceq gx_2^2 \Rightarrow F(x^1, x_2^2, x^3, ..., x^n) \preceq F(x^1, x_1^2, x^3, ..., x^n)$
(iii) for all $x_1^3, x_2^3 \in X$, $gx_1^3 \preceq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, ..., x^n) \preceq F(x^1, x^2, x_2^3, ..., x^n)$

$$\vdots$$

for all $x_1^n, x_2^n \in X$, $gx_1^n \preceq gx_2^n \Rightarrow F(x^1, x^2, x^3, ..., x_2^n) \preceq F(x^1, x^2, x^3, ..., x_1^n)$.

Definition 2.16. [14] An element $(x^1, x^2, ..., x^n) \in X^n$ is called an n-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = x^1 \\ F(x^2, x^3, ..., x^n, x^1) = x^2 \\ F(x^3, ..., x^n, x^1, x^2) = x^3 \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = x^n. \end{cases}$$

Example 2.17. Let (R, d) be a partially ordered metric space under natural setting and let $F : R^n \to R$ be a mapping defined by $F(x^1, x^2, x^3, ..., x^n) = \sin\ (x^1.x^2.x^3...x^n)$, for any $x^1, x^2, ..., x^n \in R$. Then $(0, 0, ..., 0)$ is an n-tupled fixed point of F.

Definition 2.18. [14] An element $(x^1, x^2, ..., x^n) \in X^n$ is called an n-tupled coincidence point of $F : X^n \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = g(x^1) \\ F(x^2, x^3, ..., x^n, x^1) = g(x^2) \\ F(x^3, ..., x^n, x^1, x^2) = g(x^3) \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = g(x^n). \end{cases}$$

Example 2.19. Let (R, d) be a partially ordered metric space under natural setting and let $F : R^n \to R$ be a mapping defined by $F(x^1, x^2, ..., x^n) = \frac{x^1+x^2+...+x^n}{n}$, for any $x^1, x^2, ..., x^n \in R$ while $g : R \to R$ is defined as $g(x) = \frac{x}{2}$. Then $(0, 0, ..., 0)$ is an n-tupled coincidence point of F and g.

Definition 2.20. An element $(x^1, x^2, ..., x^n) \in X^n$ is called an n-tupled common fixed point of $F : X^n \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = g(x^1) = x^1 \\ F(x^2, x^3, ..., x^n, x^1) = g(x^2) = x^2 \\ F(x^3, ..., x^n, x^1, x^2) = g(x^3) = x^3 \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = g(x^n) = x^n. \end{cases}$$

Remark 2.21. Definitions 2.16, 2.18 and 2.20 with $n = 2$ respectively yield the definitions of coupled fixed point, coupled coincidence point and common coupled fixed point.

Definition 2.22. Let $F : X^n \to X$ and $g : X \to X$ be the two mappings. Then F is said to be g-compatible if

$$\begin{cases} \lim\limits_{m\to\infty} d(g(F(x^1_m, x^2_m, x^3_m, ..., x^n_m)), F(gx^1_m, gx^2_m, gx^3_m, ..., gx^n_m)) = 0 \\ \lim\limits_{m\to\infty} d(g(F(x^2_m, x^3_m, ..., x^n_m, x^1_m)), F(gx^2_m, gx^3_m, ..., gx^n_m, x^1_m)) = 0 \\ \vdots \\ \lim\limits_{m\to\infty} d(g(F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m)), F(gx^n_m, gx^1_m, gx^2_m, ..., gx^{n-1}_m)) = 0, \end{cases}$$

where $\{x^1_m\}, \{x^2_m\}, ..., \{x^n_m\}$ are sequences in X such that

$$\begin{cases} \lim\limits_{m\to\infty} F(x^1_m, x^2_m, x^3_m, ..., x^n_m) = \lim\limits_{m\to\infty} g(x^1_m) = x^1 \\ \lim\limits_{m\to\infty} F(x^2_m, x^3_m, ..., x^n_m, x^1_m) = \lim\limits_{m\to\infty} g(x^2_m) = x^2 \\ \vdots \\ \lim\limits_{m\to\infty} F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m) = \lim\limits_{m\to\infty} g(x^n_m) = x^n, \end{cases}$$

for some $x^1, x^2, ..., x^n \in X$ are satisfied.

3. Main Results

Now, we prove our main result as follows:

Theorem 3.1. *Let $(X, \preceq, d)$ be a complete partially ordered metric space. Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and ψ be an altering distance function. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X and*

$$\psi(d(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n))) \leq \psi(max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\}) - \phi(max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\}) \tag{3.1}$$

for all $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$ for which $gy^1 \preceq gx^1,\ gx^2 \preceq gy^2,\ gy^3 \preceq gx^3, ..., gx^n \preceq gy^n$. Suppose that $F(X^n) \subseteq g(X)$, g is continuous and F is g-compatible. Also, suppose that

(a) F is continuous or
(b) X has the following properties:

(i) if nondecreasing sequence $\{x_m\} \to x$, then $g(x_m) \preceq g(x)$ for all $m \geq 0$;
(ii) if nonincreasing sequence $\{x_m\} \to x$, then $g(x) \preceq g(x_m)$ for all $m \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{cases} g(x_0^1) \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq g(x_0^2) \\ g(x_0^3) \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq g(x_0^n) \end{cases}$$

then F and g have an n-tupled coincidence point in X.

Proof. Let $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{cases} g(x_0^1) \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq g(x_0^2) \\ g(x_0^3) \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq g(x_0^n). \end{cases} \tag{3.2}$$

Since $F(X^n) \subseteq g(X)$, we can choose $x_1^1, x_1^2, x_1^3, ..., x_1^n \in X$ such that

$$\begin{cases} g(x_1^1) = F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ g(x_1^2) = F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \\ g(x_1^3) = F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ g(x_1^n) = F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}). \end{cases} \tag{3.3}$$

As earlier, one can also choose $x_2^1, x_2^2, x_2^3, ..., x_2^n \in X$ such that

$$\begin{cases} g(x_2^1) = F(x_1^1, x_1^2, x_1^3 ..., x_1^n) \\ g(x_2^2) = F(x_1^2, x_1^3, ..., x_1^n, x_1^1) \\ g(x_2^3) = F(x_1^3, ..., x_1^n, x_1^1, x_1^2) \\ \vdots \\ g(x_2^n) = F(x_1^n, x_1^1, x_1^2, ..., x_1^{n-1}). \end{cases}$$

Continuing this process, we can construct sequences $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\},\ (m \geq 0)$ such that

$$\begin{cases} g(x_{m+1}^1) = F(x_m^1, x_m^2, x_m^3, ..., x_m^n) \\ g(x_{m+1}^2) = F(x_m^2, x_m^3, ..., x_m^n, x_m^1) \\ \vdots \\ g(x_{m+1}^n) = F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1}). \end{cases} \tag{3.4}$$

In what follows, we shall prove that for all $m \geq 0$,

$$gx_m^1 \preceq gx_{m+1}^1,\ gx_{m+1}^2 \preceq gx_m^2,\ gx_m^3 \preceq gx_{m+1}^3, ..., gx_{m+1}^n \preceq gx_m^n. \tag{3.5}$$

Owing to (3.2) and (3.3), we have

$$gx_0^1 \preceq gx_1^1,\ gx_1^2 \preceq gx_0^2,\ gx_0^3 \preceq gx_1^3, ..., gx_1^n \preceq gx_0^n,$$

that is, (3.5) holds for $m = 0$. Suppose that (3.5) holds for some $m > 0$. As F has the mixed g-monotone property, we have from (3.4) that

$$\begin{aligned} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, ..., x_m^n) &\preceq F(x_{m+1}^1, x_m^2, x_m^3, ..., x_m^n) \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_m^3, ..., x_m^n) \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_{m+1}^3, ..., x_m^n) \\ &\preceq F(x_{m+1}^1, x_{m+1}^2, x_{m+1}^3, ..., x_{m+1}^n) \\ &= gx_{m+2}^1. \\ gx_{m+2}^2 = F(x_{m+1}^2, x_{m+1}^3, ..., x_{m+1}^n, x_{m+1}^1) & \\ &\preceq F(x_{m+1}^2, x_{m+1}^3, ..., x_{m+1}^n, x_m^1) \\ &\preceq F(x_{m+1}^2, x_{m+1}^3, ..., x_m^n, x_m^1) \\ &\preceq F(x_{m+1}^2, x_m^3, ..., x_m^n, x_m^1) \\ &\preceq F(x_m^2, x_m^3, ..., x_m^n, x_m^1) \\ &= gx_{m+1}^2. \end{aligned}$$

Also for the same reason,

$$gx_{m+1}^3 = F(x_m^3, ..., x_m^n, x_m^1, x_m^2) \preceq F(x_{m+1}^3, ..., x_{m+1}^n, x_{m+1}^1, x_{m+1}^2) = gx_{m+2}^3$$

$$\vdots$$

$$gx_{m+2}^n = F(x_{m+1}^n, x_{m+1}^1, x_{m+1}^2, ..., x_{m+1}^{n-1}) \preceq F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1}) = gx_{m+1}^n.$$

Hence by mathematical induction it follows that (3.5) holds for all $m \geq 0$. Therefore

$$\begin{cases} gx_0^1 \preceq gx_1^1 \preceq gx_2^1 \preceq ... \preceq gx_m^1 \preceq gx_{m+1}^1 \preceq ... \\ ... \preceq gx_{m+1}^2 \preceq gx_m^2 \preceq ... \preceq gx_2^2 \preceq gx_1^2 \preceq gx_0^2 \\ gx_0^3 \preceq gx_1^3 \preceq gx_2^3 \preceq ... \preceq gx_m^3 \preceq gx_{m+1}^3 \preceq ... \\ \vdots \\ ... \preceq gx_{m+1}^n \preceq gx_m^n \preceq ... \preceq gx_2^n \preceq gx_1^n \preceq gx_0^n. \end{cases} \tag{3.6}$$

Let

$$R_m = \max\{d(gx^1_{m+1}, gx^1_m), d(gx^2_{m+1}, gx^2_m), ..., d(gx^n_{m+1}, gx^n_m)\}.$$

Using (3.6) we have,

$$\psi(d(gx^1_m, gx^1_{m+1})) = \psi(d(F(x^1_{m-1}, x^2_{m-1}, ..., x^n_{m-1}), F(x^1_m, x^2_m, ..., x^n_m)))$$

$$\leq \psi(\max\{d(gx^1_{m-1}, gx^1_m), d(gx^2_{m-1}, gx^2_m), d(gx^3_{m-1}, gx^3_m), ..., d(gx^n_{m-1}, gx^n_m)\})$$

$$-\phi(\max\{d(gx^1_{m-1}, gx^1_m), d(gx^2_{m-1}, gx^2_m), d(gx^3_{m-1}, gx^3_m), ..., d(gx^n_{m-1}, gx^n_m)\}).$$

$$\psi(d(gx^2_m, gx^2_{m+1})) = \psi(d(F(x^2_{m-1}, ..., x^n_{m-1}, x^1_{m-1}), F(x^2_m, ..., x^n_m, x^1_m)))$$

$$\leq \psi(\max\{d(gx^2_{m-1}, gx^2_m), d(gx^3_{m-1}, gx^3_m), ..., d(gx^n_{m-1}, gx^n_m), d(gx^1_{m-1}, gx^1_m)\})$$

$$-\phi(\max\{d(gx^2_{m-1}, gx^2_m), d(gx^3_{m-1}, gx^3_m), ..., d(gx^n_{m-1}, gx^n_m), d(gx^1_{m-1}, gx^1_m)\}).$$

Similarly, we can inductively write

$$\psi(d(gx^n_m, gx^n_{m+1})) = \psi(d(F(x^n_{m-1}, x^1_{m-1}, ..., x^{n-1}_{m-1}), F(x^n_m, x^1_m, ..., x^{n-1}_m)))$$

$$\leq \psi(\max\{d(gx^n_{m-1}, gx^n_m), d(gx^1_{m-1}, gx^1_m), d(gx^2_{m-1}, gx^2_m), ..., d(gx^{n-1}_{m-1}, gx^{n-1}_m)\})$$

$$-\phi(\max\{d(gx^n_{m-1}, gx^n_m), d(gx^1_{m-1}, gx^1_m), d(gx^2_{m-1}, gx^2_m), ..., d(gx^{n-1}_{m-1}, gx^{n-1}_m)\}).$$

From above inequalities and the monotone property of ψ, we have

$$\begin{aligned}&\psi(\max\{d(gx^n_m, gx^n_{m+1}), d(gx^1_m, gx^1_{m+1}), ..., d(gx^{n-1}_m, gx^{n-1}_{m+1})\})\\ &= \max\{\psi(d(gx^n_m, gx^n_{m+1})), \psi(d(gx^1_m, gx^1_{m+1})), ..., \psi(d(gx^{n-1}_m, gx^{n-1}_{m+1}))\}\\ &\leq \psi(\max\{d(gx^n_{m-1}, gx^n_m), d(gx^1_{m-1}, gx^1_m), ..., d(gx^{n-1}_{m-1}, gx^{n-1}_m)\})\\ &\quad -\phi(\max\{d(gx^n_{m-1}, gx^n_m), d(gx^1_{m-1}, gx^1_m), ..., d(gx^{n-1}_{m-1}, gx^{n-1}_m)\}),\end{aligned}$$

that is,

$$\psi(R_m) \leq \psi(R_{m-1}) - \phi(R_{m-1}). \tag{3.7}$$

Using the property of ψ, we have

$$\psi(R_m) \leq \psi(R_{m-1})$$

which implies that

$$R_m \leq R_{m-1} \qquad \text{(by the property of } \psi).$$

Therefore $\{R_m\}$ is a monotonically decreasing sequence of nonnegative real numbers. Hence there exists $r \geq 0$ such that

$$R_m \to r \text{ as } m \to \infty.$$

Taking the limit as $m \to \infty$ in (3.7). Then by the continuities of ψ and ϕ, we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Therefore

$$R_m \to 0 \text{ as } m \to \infty, \tag{3.8}$$

so that

$$\lim_{m\to\infty} d(gx^1_m, gx^1_{m+1}) = 0,\ \lim_{m\to\infty} d(gx^2_m, gx^2_{m+1}) = 0, ..., \lim_{m\to\infty} d(gx^n_m, gx^n_{m+1}) = 0.$$

Next, we show that $\{gx^1_m\}, \{gx^2_m\}, ..., \{gx^n_m\}$ are Cauchy sequences. If possible suppose that atleast one of $\{gx^1_m\}, \{gx^2_m\}, ..., \{gx^n_m\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{t(k)\}$ such that for all positive integers k,

$$t(k) > m(k) > k,$$

$$D_k = \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\} \geq \epsilon$$

and

$$\max\{d(gx^1_{m(k)}, gx^1_{t(k)-1}), d(gx^2_{m(k)}, gx^2_{t(k)-1}), ..., d(gx^n_{m(k)}, gx^n_{t(k)-1})\} < \epsilon.$$

Now,

$$\begin{aligned}\epsilon \leq D_k &= \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\}\\ &\leq \max\{d(gx^1_{m(k)}, gx^1_{t(k)-1}), d(gx^2_{m(k)}, gx^2_{t(k)-1}), ..., d(gx^n_{m(k)}, gx^n_{t(k)-1})\}\\ &+\max\{d(gx^1_{t(k)-1}, gx^1_{t(k)}), d(gx^2_{t(k)-1}, gx^2_{t(k)}), ..., d(gx^n_{t(k)-1}, gx^n_{t(k)})\},\end{aligned}$$

that is,

$$\epsilon \leq D_k = \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\} \leq \epsilon + R_{t(k)-1}.$$

Letting $k \to \infty$ in above inequality and using (3.8), we have

$$\lim_{k\to\infty} D_k = \lim_{k\to\infty} \max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\} = \epsilon. \tag{3.9}$$

Again

$$\begin{aligned}D_{k+1} &= \max\{d(gx^1_{m(k)+1}, gx^1_{t(k)+1}), d(gx^2_{m(k)+1}, gx^2_{t(k)+1}), ..., d(gx^n_{m(k)+1}, gx^n_{t(k)+1})\}\\ &\leq \max\{d(gx^1_{m(k)+1},\ gx^1_{m(k)}), d(gx^2_{m(k)+1},\ gx^2_{m(k)}), ..., d(gx^n_{m(k)+1},\ gx^n_{m(k)})\}\\ &\quad +\max\{d(gx^1_{m(k)},\ gx^1_{t(k)}), d(gx^2_{m(k)},\ gx^2_{t(k)}), ..., d(gx^n_{m(k)},\ gx^n_{t(k)})\}\\ &\quad +\max\{d(gx^1_{t(k)},\ gx^1_{t(k)+1}), d(gx^2_{t(k)},\ gx^2_{t(k)+1}), ..., d(gx^n_{t(k)},\ gx^n_{t(k)+1}\})\\ &\quad = R_{m(k)} + D_k + R_{t(k)}\end{aligned}$$

and

$$D_k \leq R_{m(k)} + D_{k+1} + R_{t(k)}.$$

Letting $k \to \infty$ in the preceding inequality, using (3.8) and (3.9) we have

$$\begin{aligned}\lim_{k\to\infty} D_{k+1} = \lim_{k\to\infty} \max\{&d(gx^1_{m(k)+1}, gx^1_{t(k)+1}), d(gx^2_{m(k)+1}, gx^2_{t(k)+1}), ...,\\ &d(gx^n_{m(k)+1}, gx^n_{t(k)+1})\} = \epsilon.\end{aligned} \tag{3.10}$$

Since $t(k) > m(k)$ and

$$gx^1_{m(k)} \preceq gx^1_{t(k)},\ gx^2_{t(k)} \preceq gx^2_{m(k)},\ gx^3_{m(k)} \preceq gx^3_{t(k)}, ..., gx^n_{t(k)} \preceq gx^n_{m(k)},$$

therefore owing to (3.1) and (3.4), we have

$$\psi(d(gx^1_{m(k)+1}, gx^1_{t(k)+1})) = \psi(d(F(x^1_{m(k)}, x^2_{m(k)}, ..., x^n_{m(k)}), F(x^1_{t(k)}, x^2_{t(k)}, ..., x^n_{t(k)})))$$

$$\leq \psi(\max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), d(gx^3_{m(k)}, gx^3_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\})$$

$$-\phi(\max\{d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), d(gx^3_{m(k)}, gx^3_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)})\}),$$

that is,

$$\psi(d(gx^1_{m(k)+1}, gx^1_{t(k)+1})) \leq \psi(D_k) - \phi(D_k). \tag{3.11}$$

Also,

$$\psi(d(gx^2_{m(k)+1}, gx^2_{t(k)+1})) = \psi(d(F(x^2_{m(k)}, ..., x^n_{m(k)}, x^1_{m(k)}), F(x^2_{t(k)}, ..., x^n_{t(k)}, x^1_{t(k)})))$$

$$\leq \psi(\max\{d(gx^2_{m(k)}, gx^2_{t(k)}), d(gx^3_{m(k)}, gx^3_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)}), d(gx^1_{m(k)}, gx^1_{t(k)})\})$$

$$-\phi(\max\{d(gx^2_{m(k)}, gx^2_{t(k)}), d(gx^3_{m(k)}, gx^3_{t(k)}), ..., d(gx^n_{m(k)}, gx^n_{t(k)}), d(gx^1_{m(k)}, gx^1_{t(k)})\}),$$

that is,

$$\psi(d(gx^2_{m(k)+1}, gx^2_{t(k)+1})) \leq \psi(D_k) - \phi(D_k). \tag{3.12}$$

Similarly,

$$\psi(d(gx^n_{m(k)+1}, gx^n_{t(k)+1})) = \psi(d(F(x^n_{m(k)}, x^1_{m(k)}, ..., x^{n-1}_{m(k)}), F(x^n_{t(k)}, x^1_{t(k)}, ..., x^{n-1}_{t(k)})))$$

$$\leq \psi(\max\{d(gx^n_{m(k)}, gx^n_{t(k)}), d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^{n-1}_{m(k)}, gx^{n-1}_{t(k)})\})$$

$$-\phi(\max\{d(gx^n_{m(k)}, gx^n_{t(k)}), d(gx^1_{m(k)}, gx^1_{t(k)}), d(gx^2_{m(k)}, gx^2_{t(k)}), ..., d(gx^{n-1}_{m(k)}, gx^{n-1}_{t(k)})\}),$$

that is,

$$\psi(d(gx^n_{m(k)+1}, gx^n_{t(k)+1})) \leq \psi(D_k) - \phi(D_k). \tag{3.13}$$

Using (3.11), (3.12) and (3.13) along with monotone property of ψ, we have,

$$\psi(D_{k+1}) = \psi(\max\{(d(gx^n_{m(k)+1}, gx^n_{t(k)+1}), d(gx^1_{m(k)+1}, gx^1_{t(k)+1}), ..., d(gx^{n-1}_{m(k)+1}, gx^{n-1}_{t(k)+1})\})$$

$$= \max\{\psi(d(gx^n_{m(k)+1}, gx^n_{t(k)+1}), d(gx^1_{m(k)+1}, gx^1_{t(k)+1}), ..., d(gx^{n-1}_{m(k)+1}, gx^{n-1}_{t(k)+1}))\}$$

$$\leq \psi(D_k) - \phi(D_k).$$

Letting $k \to \infty$ in the above inequality, using (3.9), (3.10) and the continuities of ψ and ϕ we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction by virtue of a property of ϕ. Thus $\{gx^1_m\}, \{gx^2_m\}, ..., \{gx^n_m\}$ are Cauchy sequences in X. From the completeness of X, there exist $x^1, x^2, ..., x^n \in X$ such that

$$\begin{cases} \lim_{m\to\infty} F(x^1_m, x^2_m, x^3_m, ..., x^n_m) = \lim_{m\to\infty} g(x^1_m) = x^1 \\ \lim_{m\to\infty} F(x^2_m, x^3_m, ..., x^n_m, x^1_m) = \lim_{m\to\infty} g(x^2_m) = x^2 \\ \vdots \\ \lim_{m\to\infty} F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m) = \lim_{m\to\infty} g(x^n_m) = x^n. \end{cases} \tag{3.14}$$

Since F is g-compatible, we have from (3.14),

$$\begin{cases} \lim_{m\to\infty} d(g(F(x_m^1, x_m^2, x_m^3, ..., x_m^n)), F(gx_m^1, gx_m^2, gx_m^3, ..., gx_m^n)) = 0 \\ \lim_{m\to\infty} d(g(F(x_m^2, x_m^3, ..., x_m^n, x_m^1)), F(gx_m^2, gx_m^3, ..., gx_m^n, gx_m^1)) = 0 \\ \vdots \\ \lim_{m\to\infty} d(g(Fx_m^n, x_m^1, x_m^2, ..., x_m^{n-1}), F(gx_m^n, gx_m^1, gx_m^2, ..., gx_m^{n-1})) = 0. \end{cases} \tag{3.15}$$

Let condition (a) holds. Then for all $m \geq 0$, we have

$$\begin{aligned} d(gx^1, F(gx_m^1, gx_m^2, gx_m^3, ..., gx_m^n)) &\leq d(gx^1, g(F(x_m^1, x_m^2, x_m^3, ..., x_m^n))) \\ &\quad + d(g(F(x_m^1, x_m^2, x_m^3, ..., x_m^n), F(gx_m^1, gx_m^2, gx_m^3, ..., gx_m^n)). \end{aligned}$$

Taking $m \to \infty$ in above inequality, using (3.14), (3.15) and continuities of F and g, we have

$$d(gx^1, F(x^1, x^2, x^3, ..., x^n)) = 0; \text{ that is } gx^1 = F(x^1, x^2, x^3, ..., x^n).$$

Continuing this process, we obtain that

$$d(gx^2, F(x^2, x^3, ..., x^n, x^1)) = 0; \text{ that is } gx^2 = F(x^2, x^3, ..., x^n, x^1).$$

$$\vdots$$

$$d(gx^n, F(x^n, x^1, x^2, ..., x^{n-1})) = 0; \text{ that is } gx^n = F(x^n, x^1, x^2, ..., x^{n-1}).$$

Hence the element $(x^1, x^2, ..., x^n) \in X^n$ is an n-tupled coincidence point of the mappings $F : X^n \to X$ and $g : X \to X$. Next, we suppose that the condition (b) holds. From (3.6) and (3.14), we have

$$ggx_m^1 \preceq gx^1, \; gx^2 \preceq ggx_m^2, \; ggx_m^3 \preceq gx^3, ..., gx^n \preceq ggx_m^n. \tag{3.16}$$

Since F is g-compatible and g is continuous, by (3.14) and (3.15) we have

$$\begin{cases} \lim_{m\to\infty} ggx_m^1 = gx^1 = \lim_{m\to\infty} g(F(x_m^1, x_m^2, ..., x_m^n)) = \lim_{m\to\infty} F(gx_m^1, gx_m^2, ..., gx_m^n) \\ \lim_{m\to\infty} ggx_m^2 = gx^2 = \lim_{m\to\infty} g(F(x_m^2, ..., x_m^n, x_m^1)) = \lim_{m\to\infty} F(gx_m^2, ..., gx_m^n, gx_m^1) \\ \vdots \\ \lim_{m\to\infty} ggx_m^n = gx^n = \lim_{m\to\infty} g(F(x_m^n, x_m^1, ..., x_m^{n-1})) = \lim_{m\to\infty} F(gx_m^n, gx_m^1, ..., gx_m^{n-1}) \end{cases} \tag{3.17}$$

Now, using triangle inequality, we have

$$d(F(x^1, x^2, ..., x^n), gx^1) \leq d(F(x^1, x^2, ..., x^n), ggx_{m+1}^1) + d(ggx_{m+1}^1, gx^1),$$

that is,

$$d(F(x^1, x^2, ..., x^n), gx^1) \leq d(F(x^1, x^2, ..., x^n), g(F(x_m^1, x_m^2, ..., x_m^n)) + d(ggx_{m+1}^1, gx^1).$$

Taking $m \to \infty$ in the above inequality, using (3.17) we have

$$\begin{aligned} d(F(x^1, x^2, ..., x^n), gx^1) &\leq \lim_{m\to\infty} d(F(x^1, x^2, ..., x^n), g(F(x_m^1, x_m^2, ..., x_m^n)) \\ &\quad + \lim_{m\to\infty} d(ggx_{m+1}^1, gx^1) \\ &= \lim_{m\to\infty} d(F(x^1, x^2, ..., x^n), F(gx_m^1, gx_m^2, ..., gx_m^n)). \end{aligned}$$

Since ψ is continuous and monotonically increasing, from the above inequality we have

$$\begin{aligned}\psi(d(F(x^1,x^2,...,x^n),gx^1) &\leq \psi(\lim_{m\to\infty} d(F(x^1,x^2,...,x^n),F(gx_m^1,gx_m^2,...,gx_m^n)))\\ &= \lim_{m\to\infty}\psi(d(F(x^1,x^2,...,x^n),F(gx_m^1,gx_m^2,...,gx_m^n))).\end{aligned}$$

By (3.1) and (3.16), we have

$$\begin{aligned}\psi(d(F(x^1,x^2,...,x^n),gx^1)) \leq \lim_{m\to\infty}[\psi(\max\{&d(gx^1,ggx_m^1),d(gx^2,ggx_m^2),...,\\ &d(gx^n,ggx_m^n)\})-\phi(\max\{d(gx^1,ggx_m^1),\\ &d(gx^2,ggx_m^2),...,d(gx^n,ggx_m^n)\})].\end{aligned}$$

Using (3.17) and the properties of ψ and ϕ, we have $\psi(d(F(x^1,x^2,...,x^n),gx^1)) = 0$, which implies that

$$d(F(x^1,x^2,x^3,...,x^n),gx^1) = 0;\ \text{that is } gx^1 = F(x^1,x^2,x^3,...,x^n).$$

Again, we have

$$d(gx^2,F(x^2,x^3,...,x^n,x^1)) \leq d(gx^2,ggx_{m+1}^2) + d(ggx_{m+1}^2,F(x^2,x^3,...,x^n,x^1)),$$

that is,

$$d(gx^2,F(x^2,...,x^n,x^1) \leq d(gx^2,ggx_{m+1}^2)+d(g(F(x_m^2,...,x_m^n,x_m^1)),F(x^2,...,x^n,x^1)).$$

Taking $m\to\infty$ in the above inequality and using (3.17), we have

$$\begin{aligned}d(gx^2,F(x^2,x^3,...,x^n,x^1)) &\leq \lim_{m\to\infty} d(gx^2,ggx_{m+1}^2)\\ &+ \lim_{m\to\infty} d(g(F(x_m^2,...,x_m^n,x_m^1)),F(x^2,...,x^n,x^1))\\ &= \lim_{m\to\infty} d(F(gx_m^2,...,gx_m^n,gx_m^1)),F(x^2,...,x^n,x^1)).\end{aligned}$$

Since ψ is continuous and monotonically increasing, from the above inequality we have

$$\begin{aligned}\psi(d(gx^2,F(x^2,...,x^n,x^1))) &\leq \psi(\lim_{m\to\infty} d(F(gx_m^2,...,gx_m^n,gx_m^1)),F(x^2,...,x^n,x^1)))\\ &= \lim_{m\to\infty}\psi(d(F(gx_m^2,...,gx_m^n,gx_m^1)),F(x^2,...,x^n,x^1))).\end{aligned}$$

By (3.1) and (3.16), we have

$$\begin{aligned}\psi(d(gx^2,F(x^2,x^3,...,x^n,x^1))) \leq \lim_{m\to\infty}[\psi(\max\{&d(ggx_m^2,gx^2),d(ggx_m^3,gx^3),...,\\ &d(ggx_m^n,gx^n),d(ggx_m^1,gx^1)\})-\phi(\max\{d(ggx_m^2,gx^2),\\ &d(ggx_m^3,gx^3),...,d(ggx_m^n,gx^n),d(ggx_m^1,gx^1)\})].\end{aligned}$$

Using (3.17) and the properties of ψ and ϕ, we have $\psi(d(gx^2, F(x^2, x^3, ..., x^n, x^1))) = 0$, which implies that

$$d(gx^2, F(x^2, x^3, ..., x^n, x^1)) = 0; \text{ that is } F(x^2, x^3, ..., x^n, x^1) = gx^2.$$

Continuing in this way, we get

$$d(gx^n, F(x^n, x^1, x^2, ..., x^{n-1})) = 0; \text{ that is } gx^n = F(x^n, x^1, x^2, ..., x^{n-1}).$$

Hence the element $(x^1, x^2, ..., x^n) \in X^n$ is n-tupled coincidence point of the mappings $F : X^n \to X$ and $g : X \to X$. This completes the proof of the theorem. □

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that for real* $(x^1, x^2, ..., x^n)$, $(y^1, y^2, ..., y^n) \in X^n$ *there exists,* $(z^1, z^2, ..., z^n) \in X^n$ *such that* $(F(z^1, z^2, ..., z^n), F(z^2, ..., z^n, z^1), ..., F(z^n, z^1, ..., z^{n-1}))$ *is comparable to* $(F(x^1, x^2, ..., x^n), F(x^2, ..., x^n, x^1), ..., F(x^n, x^1, ..., x^{n-1}))$ *and* $(F(y^1, y^2, ..., y^n), F(y^2, ..., y^n, y^1), ..., F(y^n, y^1, ..., y^{n-1}))$*. Then* F *and* g *have a unique* n*-tupled common fixed point.*

Proof. The set of n-tupled coincidence points of F and g is non empty due to Theorem 3.1. Assume now, $(x^1, x^2, x^3, ..., x^n)$ and $(y^1, y^2, y^3, ..., y^n)$ are two n-tupled coincidence points, that is,

$$\begin{gathered} F(x^1, x^2, x^3, ..., x^n) = g(x^1), \; F(y^1, y^2, y^3, ..., y^n) = g(y^1) \\ F(x^2, x^3, ..., x^n, x^1) = g(x^2), \; F(y^2, y^3, ..., y^n, y^1) = g(y^2) \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = g(x^n), \; F(y^n, y^1, y^2, ..., y^{n-1}) = g(y^n). \end{gathered}$$

Now, we show that

$$g(x^1) = g(y^1), g(x^2) = g(y^2), ..., g(x^n) = g(y^n).$$

By assumption, there exists $(z^1, z^2, z^3, ..., z^n) \in X^n$ such that

$$(F(z^1, z^2, z^3, ..., z^n), F(z^2, z^3, ..., z^n, z^1), ..., F(z^n, z^1, z^2, ..., z^{n-1}))$$

is comparable to

$$(F(x^1, x^2, x^3, ..., x^n), F(x^2, x^3, ..., x^n, x^1), ..., F(x^n, x^1, x^2, ..., x^{n-1}))$$

and

$$(F(y^1, y^2, y^3, ..., y^n), F(y^2, y^3, ..., y^n, y^1), ..., F(y^n, y^1, y^2, ..., y^{n-1})).$$

Put $z_0^1 = z^1, z_0^2 = z^2, ..., z_0^n = z^n$ and choose $z_1^1, z_1^2, ..., z_1^n \in X$ such that

$$\begin{gathered} g(z_1^1) = F(z_0^1, z_0^2, z_0^3, ..., z_0^n) \\ g(z_1^2) = F(z_0^2, z_0^3, ..., z_0^n, z_0^1) \\ \vdots \\ g(z_1^n) = F(z_0^n, z_0^1, z_0^2, ..., z_0^{n-1}). \end{gathered}$$

Further define sequences $\{g(z_m^1)\}, \{g(z_m^2)\}, ..., \{g(z_m^n)\}$ such that

$$\begin{gathered} g(z_{m+1}^1) = F(z_m^1, z_m^2, z_m^3, ..., z_m^n) \\ g(z_{m+1}^2) = F(z_m^2, z_m^3, ..., z_m^n, z_m^1) \\ \vdots \\ g(z_{m+1}^n) = F(z_m^n, z_m^1, z_m^2, ..., z_m^{n-1}). \end{gathered}$$

Further set $x_0^1 = x^1, x_0^2 = x^2, ..., x_0^n = x^n$ and $y_0^1 = y^1, y_0^2 = y^2, ..., y_0^n = y^n$. In the same way, define the sequences $\{g(x_m^1)\}, \{g(x_m^2)\}, ..., \{g(x_m^n)\}$ and $\{g(y_m^1)\}, \{g(y_m^2)\}, ..., \{g(y_m^n)\}$. Then it is easy to show that

$$g(x_{m+1}^1) = F(x_m^1, x_m^2, x_m^3, ..., x_m^n),\ \ g(y_{m+1}^1) = F(y_m^1, y_m^2, y_m^3, ..., y_m^n)$$
$$g(x_{m+1}^2) = F(x_m^2, x_m^3, ..., x_m^n, x_m^1),\ \ g(y_{m+1}^2) = F(y_m^2, y_m^3, ..., y_m^n, y_m^1)$$
$$\vdots$$
$$g(x_{m+1}^n) = F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1}),\ \ g(y_{m+1}^n) = F(y_m^n, y_m^1, y_m^2, ..., y_m^{n-1}).$$

Since

$$(F(x^1, x^2, x^3, ..., x^n), F(x^2, x^3, ..., x^n, x^1), ..., F(x^n, x^1, x^2, ..., x^{n-1}))$$
$$= (g(x_1^1), g(x_1^2), ..., g(x_1^n)) = (g(x^1), g(x^2), ..., g(x^n))$$

and

$$(F(z^1, z^2, z^3, ..., z^n), F(z^2, z^3, ..., z^n, z^1), ..., F(z^n, z^1, z^2, ..., z^{n-1}))$$
$$= (g(z_1^1), g(z_1^2), ..., g(z_1^n))$$

are comparable, we have

$$g(x^1) \preceq g(z_1^1), g(z_1^2) \preceq g(x^2), g(x^3) \preceq g(z_1^3), ..., g(z_1^n) \preceq g(x^n).$$

It is easy to show that $(g(x^1), g(x^2), ..., g(x^n))$ and $(g(z_m^1), g(z_m^2), ..., g(z_m^n))$ are comparable, that is, for all $m \geq 1$,

$$g(x^1) \preceq g(z_m^1), g(z_m^2) \preceq g(x^2), g(x^3) \preceq g(z_m^3), ..., g(z_m^n) \preceq g(x^n).$$

Thus from (3.1) we have

$$\psi(d(g(x^1), g(z_{m+1}^1))) = \psi(d(F(x^1, x^2, x^3, ..., x^n), F(z_m^1, z_m^2, z_m^3, ..., z_m^n)))$$
$$\leq \psi(\max\{d(gx^1, gz_m^1), d(gx^2, gz_m^2), d(gx^3, gz_m^3), ..., d(gx^n, gz_m^n)\})$$
$$-\phi(\max\{d(gx^1, gz_m^1), d(gx^2, gz_m^2), d(gx^3, gz_m^3), ..., d(gx^n, gz_m^n)\}),$$
$$\psi(d(g(x^2), g(z_{m+1}^2))) = \psi(d(F(x^2, x^3, ..., x^n, x^1), F(z_m^2, z_m^3, ..., z_m^n, z_m^1)))$$
$$\leq \psi(\max\{d(gx^2, gz_m^2), d(gx^3, gz_m^3), ..., d(gx^n, gz_m^n), d(gx^1, gz_m^1)\})$$
$$-\phi(\max\{d(gx^2, gz_m^2), d(gx^3, gz_m^3), ..., d(gx^n, gz_m^n), d(gx^1, gz_m^1)\}),$$
$$\vdots$$
$$\psi(d(g(x^n), g(z_{m+1}^n))) = \psi(d(F(x^n, x^1, x^2 ..., x^{n-1}), F(z_m^n, z_m^1, z_m^2, ..., z_m^{n-1})))$$
$$\leq \psi(\max\{d(gx^n, gz_m^n), d(gx^1, gz_m^1), d(gx^2, gz_m^2), ..., d(gx^{n-1}, gz_m^{n-1})\})$$
$$-\phi(\max\{d(gx^n, gz_m^n), d(gx^1, gz_m^1), d(gx^2, gz_m^2), ..., d(gx^{n-1}, gz_m^{n-1})\}).$$

From above inequalities and monotone property of ψ, we have

$$\psi(\max\{d(gx^n, gz_{m+1}^n), d(gx^1, gz_{m+1}^1), d(gx^2, gz_{m+1}^2), ..., d(gx^{n-1}, gz_{m+1}^{n-1})\})$$

$$= \max\{\psi(d(gx^n, gz^n_{m+1})), \psi(d(gx^1, gz^1_{m+1})), ..., \psi(d(gx^{n-1}, gz^{n-1}_{m+1}))\}$$

$$\leq \psi(\max\{d(gx^n, gz^n_m), d(gx^1, gz^1_m), d(gx^2, gz^2_m), ..., d(gx^{n-1}, gz^{n-1}_m)\})$$

$$-\phi(\max\{d(gx^n, gz^n_m), d(gx^1, gz^1_m), d(gx^2, gz^2_m), ..., d(gx^{n-1}, gz^{n-1}_m)\}). \tag{3.18}$$

Let

$$R_m = \max\{d(gx^1, gz^1_{m+1}), d(gx^2, gz^2_{m+1}), ..., d(gx^n, gz^n_{m+1})\}.$$

Then

$$\psi(R_m) \leq \psi(R_{m-1}) - \phi(R_{m-1}). \tag{3.19}$$

Using the property of ψ, we have

$$\psi(R_m) \leq \psi(R_{m-1}) \Rightarrow R_m \leq R_{m-1}.$$

Therefore $\{R_m\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists $r \geq 0$ such that

$$R_m \to r \text{ as } m \to \infty.$$

Taking the limit as $m \to \infty$ in (3.19), we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Therefore

$$R_m \to 0 \text{ as } m \to \infty.$$

Then

$$\lim_{m\to\infty} d(gx^1, gz^1_{m+1}) = 0, \ \lim_{m\to\infty} d(gx^2, gz^2_{m+1}) = 0, ..., \lim_{m\to\infty} d(gx^n, gz^n_{m+1}) = 0.$$

Similarly, we can prove that

$$\lim_{m\to\infty} d(gy^1, gz^1_{m+1}) = 0, \ \lim_{m\to\infty} d(gy^2, gz^2_{m+1}) = 0, ..., \lim_{m\to\infty} d(gy^n, gz^n_{m+1}) = 0.$$

On using the triangle inequality, we have

$$d(gx^1, gy^1) \leq d(gx^1, gz^1_{m+1}) + d(gz^1_{m+1}, gy^1) \to 0 \text{ as } m \to \infty$$

$$d(gx^2, gy^2) \leq d(gx^2, gz^2_{m+1}) + d(gz^2_{m+1}, gy^2) \to 0 \text{ as } m \to \infty$$

$$\vdots$$

$$d(gx^n, gy^n) \leq d(gx^n, gz^n_{m+1}) + d(gz^n_{m+1}, gy^n) \to 0 \text{ as } m \to \infty,$$

so that

$$g(x^1) = g(y^1), g(x^2) = g(y^2), ..., g(x^n) = g(y^n). \tag{3.20}$$

Since

$$g(x^1) = F(x^1, x^2, ..., x^n), g(x^2) = F(x^2, ..., x^n, x^1), ..., g(x^n) = F(x^n, x^1, ..., x^{n-1}),$$

and F is g-compatible, we have

$$gg(x^1) = F(gx^1, gx^2, gx^3, ..., gx^n)$$

$$gg(x^2) = F(gx^2, gx^3, ..., gx^n, gx^1)$$

$$\vdots$$

$$gg(x^n) = F(gx^n, gx^1, gx^2, ..., gx^{n-1}).$$

Write $g(x^1) = a^1, g(x^2) = a^2, ..., g(x^n) = a^n$, then we have

$$\begin{cases} g(a^1) = F(a^1, a^2, a^3, ..., a^n) \\ g(a^2) = F(a^2, a^3, ..., a^n, a^1) \\ \vdots \\ g(a^n) = F(a^n, a^1, a^2, ..., a^{n-1}). \end{cases} \tag{3.21}$$

Thus $(a^1, a^2, ..., a^n)$ is an n-tupled coincidence point of F and g. Owing to (3.20) with $y^1 = a^1, y^2 = a^2, ..., y^n = a^n$, it follows that

$$g(x^1) = g(a^1), g(x^2) = g(a^2), ..., g(x^n) = g(a^n),$$

that is,

$$g(a^1) = a^1, g(a^2) = a^2, ..., g(a^n) = a^n. \tag{3.22}$$

Using (3.21) and (3.22), we have

$$\begin{cases} a^1 = g(a^1) = F(a^1, a^2, a^3, ..., a^n) \\ a^2 = g(a^2) = F(a^2, a^3, ..., a^n, a^1) \\ \vdots \\ a^n = g(a^n) = F(a^n, a^1, a^2, ..., a^{n-1}). \end{cases} \tag{3.23}$$

Thus $(a^1, a^2, ..., a^n)$ is an n-tupled common fixed point of F and g. To prove the uniqueness, assume that $(b^1, b^2, ..., b^n)$ is another n-tupled common fixed point of F and g. In view of (3.20), we have

$$\begin{aligned} b^1 &= g(b^1) = g(a^1) = a^1 \\ b^2 &= g(b^2) = g(a^2) = a^2 \\ &\vdots \\ b^n &= g(b^n) = g(a^n) = a^n. \end{aligned}$$

This completes the proof of the theorem. □

Considering g to be an identity mapping in Theorem 3.1, we have the following corollary:

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set. Suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and ψ be an altering distance function. Let $F : X^n \to X$ be a mapping having the mixed monotone property on X and*

$$\begin{aligned} \psi(d(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n))) &\leq \psi(\max\{d(x^1, y^1), d(x^2, y^2), ..., d(x^n, y^n)\}) \\ &- \phi(\max\{d(x^1, y^1), d(x^2, y^2), ..., d(x^n, y^n)\}) \end{aligned}$$

for all $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$ for which $y^1 \preceq x^1,\ x^2 \preceq y^2,\ y^3 \preceq x^3, ..., x^n \preceq y^n$. Suppose that

(a) F is continuous or
(b) X has the following properties:

(i) if nondecreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$ for all $m \geq 0$;
(ii) if nonincreasing sequence $\{x_m\} \to x$, then $x \preceq x_m$ for all $m \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq x_0^2 \\ x_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq x_0^n \end{cases} \tag{3.24}$$

then F has an n-tupled fixed point in X.

Considering ψ and g to be identity mappings in Theorem 3.1, we have the following corollary:

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set. Suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Let $F : X^n \to X$ be a mapping having the mixed monotone property on X and*

$$\begin{aligned} d(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n)) \quad &\leq \quad \max\{d(x^1, y^1), d(x^2, y^2), ..., d(x^n, y^n)\} \\ &- \quad \phi(\max\{d(x^1, y^1), d(x^2, y^2), ..., d(x^n, y^n)\}) \end{aligned}$$

for all $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$ for which $y^1 \preceq x^1$, $x^2 \preceq y^2$, $y^3 \preceq x^3, ..., x^n \preceq y^n$.

Also in view of conditions (a) and (b) of Corollary 3.3, if (3.24) is satisfied, then F has an n-tupled fixed point in X.

Considering ψ and g to be identity mappings and $\phi(t) = (1 - k)t$, where $0 \leq k < 1$ in Theorem 3.1, we have the following corollary:

Corollary 3.5. *Let $(X, \preceq)$ be a partially ordered set. Suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^n \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n)) \leq k\ \max\{d(x^1, y^1), d(x^2, y^2), ..., d(x^n, y^n)\}$$

for all $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$ for which $y^1 \preceq x^1$, $x^2 \preceq y^2$, $y^3 \preceq x^3, ..., x^n \preceq y^n$.

Also in view of conditions (a) and (b) of Corollary 3.3, if (3.24) is satisfied, then F has an n-tupled fixed point in X.

Remark 3.6. With $n = 2$, Theorem 3.1 and Corollaries 3.3-3.5 respectively yield the results of Choudhury *et al.* [11]. However, from Theorem 3.2, we can deduce a unique coupled common fixed point theorem.

Example 3.7. Let $X = [0, 1]$. Then $(X, \preceq)$ is a partially ordered set with the natural ordering of real numbers. Let $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space with the required properties of Theorem 3.1. Define $g : X \to X$ by $g(x) = x^2$ for all $x \in X$ and $F : X^n \to X$ (wherein n is fixed) by

$$F(x^1, x^2, ..., x^n) = \begin{cases} \frac{(x^1)^2 - (x^2)^2 + (x^3)^2 - ..... + (x^{n-1})^2 - (x^n)^2}{n+1}, & \text{if } x^{i+1} \preceq x^i, i = 1, 3, ..., n-1 \\ 0 & \text{otherwise,} \end{cases}$$

for all $x^1, x^2, ..., x^n \in X$. Then F obeys the mixed g-monotone property. Let $\psi : [0,\infty) \to [0,\infty)$ and $\phi : [0,\infty) \to [0,\infty)$ be defined respectively as follows:

$$\psi(t) = t^2 \text{ and } \phi(t) = \frac{2n+1}{(n+1)^2} t^2, \text{ for } t \in [0,\infty).$$

Then ψ and ϕ have the properties mentioned in Theorem 3.1. Also F is g-compatible in X. Now choose $(x_0^1, x_0^2, \ldots, x_0^n) = (0, c, 0, c, ..., c)$ $(c > 0)$. Then

$$\begin{cases} g(x_0^1) = g(0) = 0 = F(x_0^1, x_0^2, x_0^3, ..., x_0^n) = g(x_1^1) \\ g(x_1^2) = F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq c^2 = g(c) = g(x_0^2) \\ g(x_0^3) = g(0) = 0 = F(x_0^3, ..., x_0^n, x_0^1, x_0^2) = g(x_1^3) \\ \vdots \\ g(x_1^n) = F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq c^2 = g(c) = g(x_0^n). \end{cases}$$

We next verify inequality (3.1) (of Theorem 3.1). We take $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$ such that

$$gy^1 \preceq gx^1,\ gx^2 \preceq gy^2,\ gy^3 \preceq gx^3, ..., gx^n \preceq gy^n.$$

Let

$$\begin{aligned} M &= \max\{d(gx^1, gy^1), d(gx^2, gy^2), d(gx^3, gy^3), ..., d(gx^n, gy^n)\} \\ &= \max\{|(x^1)^2 - (y^1)^2|, |(x^2)^2 - (y^2)^2|, |(x^3)^2 - (y^3)^2|, ..., |(x^n)^2 - (y^n)^2|\}. \end{aligned}$$

Then

$$M \geq |(x^1)^2 - (y^1)^2|,\ M \geq |(x^2)^2 - (y^2)^2|, M \geq |(x^3)^2 - (y^3)^2|, ..., M \geq |(x^n)^2 - (y^n)^2|.$$

The following four cases arise:

Case I: Let $x^1, x^2, x^3, ..., x^n,\ y^1, y^2, y^3, ..., y^n \in X$ such that $x^{i+1} \preceq x^i, y^{i+1} \preceq y^i$ for $i = 1, 3, ..., n-1$. Then

$$\begin{aligned} & d(F(x^1, x^2, x^3, ..., x^n), F(y^1, y^2, y^3, ..., y^n)) \\ &= d\Big(\frac{(x^1)^2 - (x^2)^2 + (x^3)^2 - .... - (x^n)^2}{n+1}, \frac{(y^1)^2 - (y^2)^2 + (y^3)^2 - .... - (y^n)^2}{n+1}\Big) \\ &= \Big|\frac{(x^1)^2 - (x^2)^2 + (x^3)^2 - .... - (x^n)^2}{n+1} - \frac{(y^1)^2 - (y^2)^2 + (y^3)^2 - .... - (y^n)^2}{n+1}\Big| \\ &= \Big|\frac{((x^1)^2 - (y^1)^2) - ((x^2)^2 - (y^2)^2) + ((x^3)^2 - (y^3)^2) - .... - ((x^n)^2 - (y^n)^2)}{n+1}\Big| \\ &\leq \frac{|(x^1)^2 - (y^1)^2| + |(x^2)^2 - (y^2)^2| + |(x^3)^2 - (y^3)^2| + ... + |(x^n)^2 - (y^n)^2|}{n+1} \\ &\leq \frac{n}{n+1} M. \end{aligned}$$

Case II: Let $x^1, x^2, x^3, ..., x^n,\ y^1, y^2, y^3, ..., y^n \in X$ such that $x^{i+1} \preceq x^i$ for $i = 1, 3, ..., n-1$ and $y^i \preceq y^{i+1}$ for atleast one i. Then (for $y^1 \preceq y^2$),

$$d(F(x^1, x^2, x^3, ..., x^n), F(y^1, y^2, y^3, ..., y^n))$$

$$= d\left(\frac{(x^1)^2 - (x^2)^2 + (x^3)^2 - .... - (x^n)^2}{n+1}, 0\right)$$
$$\leq \left|\frac{(x^1)^2 - (x^2)^2 + (x^3)^2 - .... - (x^n)^2 + (y^2)^2 - (y^1)^2}{n+1}\right|$$
$$= \left|\frac{((x^1)^2 - (y^1)^2) - ((x^2)^2 - (y^2)^2) + (x^3)^2 - (x^4)^2 + ... - (x^n)^2}{n+1}\right|$$
$$\vdots$$
$$\leq \frac{|(x^1)^2 - (y^1)^2| + |(x^2)^2 - (y^2)^2| + |(x^3)^2 - (y^3)^2| + ... + |(x^n)^2 - (y^n)^2|}{n+1}$$
$$\leq \frac{n}{n+1}M.$$

Case III: Let $x^1, x^2, x^3, ..., x^n,\ y^1, y^2, y^3, ..., y^n \in X$ such that $x^i \preceq x^{i+1}$ for atleast one i and $y^{i+1} \preceq y^i$ for $i = 1, 3, ..., n-1$. Then arguing as in Case II, one verify inequality (3.1).

Case IV: Let $x^1, x^2, x^3, ..., x^n,\ y^1, y^2, y^3, ..., y^n \in X$ such that $x^i \preceq x^{i+1}, y^i \preceq y^{i+1}$ for atleast one i. Then

$$d(F(x^1, x^2, x^3, ..., x^n), F(y^1, y^2, y^3, ..., y^n)) = d(0,0) \leq \frac{n}{n+1}M.$$

In all above cases

$$\begin{aligned}
&\psi(d(F(x^1, x^2, x^3, ..., x^n), F(y^1, y^2, y^3, ..., y^n))) \\
&\leq \frac{n^2}{(n+1)^2}M^2 = M^2 - \frac{2n+1}{(n+1)^2}M^2 \\
&\quad = \psi(\max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\}) \\
&\quad\ -\phi(\max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\}).
\end{aligned}$$

Hence all the conditions of Theorem 3.1 are satisfied and $(0, 0, 0, ..., 0)$ is an n-tupled coincidence point of F and g.

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