A FRACTIONAL POWER FOR DUNKL TRANSFORMS

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ABSTRACT. A new fractional version of the Dunkl transform for real order α is obtained. An integral representation, a Bochner type identity and a Master formula for this transform are derived.

1. INTRODUCTION

Recently, various works have been published that develop the theory of fractional powers of operators. We mention particularly the fractional versions of classical integral transform such that Fourier transform and Hankel transform (see[15, 13, 14, 24]). Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^N$. It started twenty years ago with Dunkl's seminal work [4] and was further developed by several mathematicians (see[2, 6, 8, 17]). In this paper, we consider the Dunkl operators $T_i, i, \ldots, N$, associated with an arbitrary finite reflection group G and a nonnegative multiplicity function k. These operators are very important and they provide a useful tool in the study of special functions with root systems. The Dunkl kernel E_k has been introduced by C.F. Dunkl in [5]. It generalizes the usual exponential function in many respects, and can be characterized as the solution of a joint eigenvalue problem for the associated Dunkl operators. For a family of weight functions ω_k invariant under a reflection group G, Dunkl [6] introduced an integral transform associated with the kernel E_k and proved the Plancherel theorem. In [2], de Jeu studied the Dunkl transform by completely different method and proved the inversion formula and the Plancherel theorem. The Dunkl transform, initially defined on $L^1(\mathbb{R}^N, \omega_k(x)dx)$ by

$$D_k f(x) = \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} f(y) E_k(-ix, y) \omega_k(y) dy,$$

extends to an isometry of $L^2(\mathbb{R}^N, \omega_k(x)dx)$ and commutes with the reflection group G. In the setting of general Dunkl's theory Rösler [17] constructed systems of naturally associated multivariable generalized Hermite polynomials $\left\{ H_\nu; \ \nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}_+^N \right\}$

and Hermite functions $\{h_\nu; \; \nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}_+^N\}$. He proved that the generalized Hermite functions $\{h_\nu\}_{\nu \in \mathbb{Z}_+^N}$ form an orthonormal basis of eigenfunctions for the Dunkl operator on $L^2(\mathbb{R}^N, \omega_k(x)dx)$ with $D_k(h_\nu) = (-i)^{|\nu|} h_\nu$.

This paper deals with the constraction of a fractional power of the Dunkl transform called the fractional Dunkl transform (FDT), using the multivariable generalized Hermite function introduced by Rösler [17]. The resulting family of operators $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$ was proved to be a $\mathcal{C}_0$-group of unitary operators on $L^2\left(\mathbb{R}^N, \omega_k(x) \; dx\right)$, with infinitesimal generator T. The spectral properties of T is studied using the semigroup techniques. The FDT given in this paper has an integral representation which used with the analogue of the Funk-Hecke formula for k-spherical harmonics [23] to derive a Bochner type identity for the FDT. The Master formula of the FDT is proved and founded to be generalizing the one given by Rösler [17] in Proposition 3.10. This Master formula is used to develop a new proof of the statement (2) of the 3.4 Rösler's theorem [17].

The contents of the present paper are as follows. In section 2, some basic definitions and results about harmonic analysis associated with Dunkl operators are collected. In section 3, the fractional Dunkl transform definition is given and then some elementary properties of this transformation are listed. In section 4, the spectral properties of T is studied. In section 5, the integral representation of the fractional Dunkl transform as well as the Bochner type identity and Master formula are given. In section 6, we find a subspace of $L^2(\mathbb{R}^N, \; \omega_k(x)dx)$ in which we define T explicitly.

2. BACKGROUND: DUNKL THEORY

In this section, we recall some notations and results on Dunkl operators, Dunkl transform, and generalized Hermite functions (see, [4, 5, 2, 16, 19]).

Notation: We denote by $\mathbb{Z}_+$ the set of non-negative integers. For a multi-index $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}_+^N$, we write $|\nu| = \nu_1 + \cdots + \nu_N$. The $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^N$ is denoted by $\mathcal{P} = \mathbb{C}[\mathbb{R}^N]$. It has a natural grading

$$\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the subspace of homogenous polynomials of (total) degree n. $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^N$ and $\mathcal{C}_0(\mathbb{R}^N)$ is the space of continuous functions on $\mathbb{R}^N$ vanishing at infinity.

2.1. **Dunkl operators and Dunkl Kernel.** In $\mathbb{R}^N$, we consider the standard inner product

$$\langle x, \, y \rangle = \sum_{k=1}^{N} x_k y_k.$$

We shall use the same notation for its bilinear extension to $\mathbb{C}^N \times \mathbb{C}^N$. For $x \in \mathbb{R}^N$, denote $|x| = \sqrt{\langle x, \, x \rangle}$.

For $u \in \mathbb{R}^N \backslash \{0\}$, let σ_u be the reflection in the hyperplane $(\mathbb{R}u)^\perp$ orthogonal to u

$$\sigma_u(x) = x - 2\frac{\langle u, \, x \rangle}{|u|^2} u, \quad x \in \mathbb{R}^N. \tag{2.1}$$

A root system is a finite spanning set $\mathcal{R} \subset \mathbb{R}^N$ of nonzero vectors such that, for every $u \in \mathcal{R}$, σ_u preserves $\mathcal{R}$. We shall always assume that $\mathcal{R}$ is reduced, i.e. $\mathcal{R} \cap \mathbb{R}u = \pm u$, for all $u \in \mathcal{R}$. Each root system can be written as a disjoint union $\mathcal{R} = \mathcal{R}^+ \cup (-\mathcal{R}^+)$, where $\mathcal{R}^+$ and $(-\mathcal{R}^+)$ are separated by a hyperplane through the origin. The subgroup $G \subset O(N)$ generated by the reflections $\{\sigma_u; \; u \in \mathcal{R}\}$ is called the finite reflection group associated with $\mathcal{R}$. Henceforth, we shall normalize $\mathcal{R}$ so that $\langle u, u \rangle = 2$ for all $u \in \mathcal{R}$. This simplifies formulas, without loss of generality for our purposes. We refer to [12] for more details on the theory of root systems and reflection groups.

A multiplicity function on $\mathcal{R}$ is a G-invariant function $k : \mathcal{R} \to \mathbb{C}$, i.e. $k(\sigma u) = k(u)$, for fall $u \in \mathcal{R}$ and $\sigma \in G$. The $\mathbb{C}$-vector space of multiplicity functions on $\mathcal{R}$ is denoted by $\mathfrak{K}$. The dimension of $\mathfrak{K}$ is equal to the number of G-orbits in $\mathcal{R}$. We set $\mathfrak{K}^+$ to be the set of multiplicity functions k such that $k(u) \geq 0$ for all $u \in \mathcal{R}$.

For $\xi \in \mathbb{C}^N$ and $k \in \mathfrak{K}$, C. Dunkl [4] defined a family of first order differential-difference operators $T_\xi(k)$ that play the role of the usual partial differentiation. Dunkl's operators are defined by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\eta \in \mathcal{R}^+} k(\eta) <\eta, \xi> \frac{f(x) - f(\sigma_\eta x)}{\langle \eta, x \rangle}, \quad f \in \mathcal{C}^1(\mathbb{R}^N). \quad (2.2)$$

Here ∂_ξ denotes the derivative in the direction of ξ. Thanks to the G-invariance of the function k, this definition is independent of the choice of the positive subsystem $\mathcal{R}_+$. The operators $T_\xi(k)$ are homogeneous of degree (-1). Moreover, by the G-invariance of the multiplicity function k, the Dunkl operators satisfy

$$h \circ T_\xi(k) \circ h^{-1} = T_{h\xi}(k), \quad \forall \, h \in G,$$

where $h.f(x) = f(h^{-1}x)$. The most striking property of Dunkl operators $T_\xi(k)$, which is the foundation for rich analytic structures with them, is the following

Theorem 2.1. *For fixed k, $T_\xi(k) \circ T_\eta(k) = T_\eta(k) \circ T_\xi(k), \quad \forall \xi, \eta \in \mathbb{R}^N$.*

This result was obtained in [4] by a clever direct argumentation. An alternative proof, relying on Koszul complex ideas, is given in [7].

For $k \in \mathfrak{K}^+$, there exists a generalization of the usual exponential kernel $e^{\langle \cdot, \cdot \rangle}$ by means of the Dunkl system of differential equations.

Theorem 2.2. *Assume that $k \in \mathfrak{K}^+$.*
(i) (Cf. [5, 16].) There exists a unique holomorphic function E_k on $\mathbb{C}^N \times \mathbb{C}^N$ characterized by

$$\begin{cases} T_\xi(k)E_k(z, w) = \langle \xi, w \rangle E_k(z, w), & \forall \, \xi \in \mathbb{C}^N, \\ E_k(0, w) = 1, \end{cases} \quad (2.3)$$

Further, the Dunkl kernel E_k is symmetric in its arguments and satisfies

$$E_k(\lambda z, w) = E_k(z, \lambda w), \quad \overline{E_k(z, w)} = E_k(\bar{z}, \bar{w}) \quad and \quad E_k(gz, gw) = E_k(z, w) \quad (2.4)$$

for all $z, w \in \mathbb{C}^N$, $\lambda \in \mathbb{C}$ and $g \in G$.
(ii) (Cf. [20].) For all $x \in \mathbb{R}^N$, $y \in \mathbb{C}^N$ and all multi-indices $\nu \in \mathbb{Z}_+^N$,

$$|\partial_y^\nu E_k(x, y)| \leq |x|^{|\nu|} \max_{g \in G} e^{Re\langle gx, y \rangle}.$$

In particular,

$$|\partial_y^\nu E_k(x, y)| \leq |x|^{|\nu|} e^{|x||Rey|}, \quad (2.5)$$

and for all $x, y \in \mathbb{R}^N$:

$$|E_k(ix, y)| \leq 1. \tag{2.6}$$

Remark 2.1.
• *When* $k = 0$, *we have* $E_0(z, w) = e^{\langle z, w \rangle}$ *for* $z, w \in \mathbb{C}^N$.
• *For complex-valued* k, *there is a detailed investigation of (2.3) by Opdam* [16]. *Theorem 2.2 (i) is a weak version of Opdam's result.*
• *M. de Jeu had already an estimate on* E_k *with slightly weaker bounds in* [2], *differing by an additional factor* $\sqrt{|G|}$.

The counterpart of the usual Laplacian is the Dunkl-Laplacian operator defined by $\Delta_k := \sum_{i=0}^{N} T_{\xi_i}(k)^2$, where $\{\xi_1, \ldots, \xi_N\}$ is an arbitrary orthonormal basis of $(\mathbb{R}^N, \langle ., , \rangle)$. It is homogeneous of degree -2. By the normalization $\langle u, u \rangle = 2$, we can rewrite Δ_k as

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\eta \in \mathcal{R}^+} k(\eta) \left[\frac{\langle \nabla f(x), \eta \rangle}{\langle \eta, x \rangle} - \frac{f(x) - f(\sigma_\eta x)}{\langle \eta, x \rangle^2} \right], \tag{2.7}$$

where Δ and ∇ denote the usual Laplacian and gradient operators, respectively (cf.[4]).

2.2. **Dunkl transform.** For fixed $k \in \mathfrak{K}^+$, let ω_k be the weight function on $\mathbb{R}^N$ defined by

$$\omega_k(x) = \prod_{\eta \in \mathcal{R}^+} |\langle \eta, x \rangle|^{2k(\eta)}.$$

It is G-invariant and homogeneous of degree 2γ, with the index

$$\gamma = \gamma(k) = \sum_{\eta \in \mathcal{R}^+} k(\eta).$$

Let dx be the Lebesgue measure corresponding to $\langle ., . \rangle$ and set $L_k^p(\mathbb{R}^N)$ the space of measurable functions on $\mathbb{R}^N$ such that

$$\|f\|_p = \left(\int_{\mathbb{R}^N} |f(x)|^p \omega_k(x)\, dx \right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty.$$

Following Dunkl [6], we define the Dunkl transform on the space $L_k^1(\mathbb{R}^N)$ by

$$D_k f(x) = \frac{c_k}{2^{\gamma + N/2}} \int_{\mathbb{R}^N} f(y) E_k(-ix, y) \omega_k(y) dy,$$

where c_k denotes the Mehta-type constant $c_k = \left(\int_{\mathbb{R}^N} e^{-|x|^2} w_k(x) dx \right)^{-1}$. Many properties of the Euclidean Fourier transform carry over to the Dunkl transform. In particular:

Theorem 2.3. *(Cf.* [6, 2]*.)*
a) (**Riemann-Lebesgue lemma**) *For all* $f \in L_k^1(\mathbb{R}^N)$, *the Dunkl transform* $D_k f$

belongs to $C_0(\mathbb{R}^N)$.

b) (L^1-*inversion*) For all $f \in L_k^1(\mathbb{R}^N)$ with $D_k f \in L_k^1(\mathbb{R}^N)$,

$$D_k^2 f = \check{f}, \ a.e, \ where \ \check{f}(x) = f(-x). \tag{2.8}$$

c) *The Dunkl transform* $f \to D_k f$ *is an automorphism of* $\mathcal{S}(\mathbb{R}^N)$.

d) (**Plancherel Theorem**)

i) *If* $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$, *then* $D_k f \in L_k^2(\mathbb{R}^N)$ *and* $\|D_k f\|_2 = \|f\|_2$.

ii) *The Dunkl transform has a unique extension to an isometric isomorphism of* $L_k^2(\mathbb{R}^N)$. *The extension is also denoted by* $f \to D_k f$.

We conclude this subsection with two important reproducing properties for the Dunkl kernel due to [6].

Theorem 2.4. *(Cf. [6].) For all* $p \in \mathcal{P}$ *and* $y, z \in \mathbb{C}^N$,

(1) $\frac{c_k}{2^{\gamma + N/2}} \int_{\mathbb{R}^N} e^{-\Delta_k/2} p(x) \ E_k(x, y) \ \omega_k(x) \ e^{-|x|^2/2} dx = e^{l(y)/2} p(y)$.

(2) $\frac{c_k}{2^{\gamma + N/2}} \int_{\mathbb{R}^N} E_k(x, y) E_k(x, z) \ \omega_k(x) \ e^{-|x|^2/2} dx = e^{(l(y)+l(z))/2} p(y)$.

2.3. **Generalized Hermite functions.** For an arbitrary finite reflection group G and for any non-negative multiplicity function k, Rösler [17] introduced a complete systems of orthogonal polynomials with respect to the weight function $\omega_k(x) \ e^{-|x|^2} dx$, called generalised Hermite polynomials. The key to their definition is the following bilinear form on $\mathcal{P}$, which was introduced in [5]:

$$[p, q]_k := (p(T)q)(0) \quad \text{for } p, q \in \mathcal{P}.$$

The homogeneity of the Dunkl operators implies that $\mathcal{P}_n \perp \mathcal{P}_m$ for $n \neq m$. Moreover, if $p, q \in \mathcal{P}_n$, then

$$[p, q]_k = 2^n c_k \int_{\mathbb{R}^N} e^{-\Delta_k/4} p(x) \ e^{-\Delta_k/4} q(x) \ \omega_k(x) \ e^{-|x|^2} dx.$$

This is obtained from Theorem 3.10 of [5] by rescaling, see lemma (2.1) in [17]. So in particular, $[., .]_k$ is a scalar product on the vector space $\mathcal{P}_\mathbb{R} = \mathbb{R}[x_1, \ldots, x_N]$.

Now let $\{\varphi_\nu, \ \nu \in \mathbb{Z}_+^N\}$ be an (arbitrary) orthonormal basis of $\mathcal{P}_\mathbb{R}$ with respect to $[., .]_k$ such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$ (For details concerning the construction and canonical choices of such a basis, we refer to [17]). Then the generalised Hermite polynomials $\{H_\nu, \nu \in \mathbb{Z}_+^N\}$ and the (normalised) generalised Hermite functions $\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ associated with G, k and $\{\varphi_\nu\}$ are defined by

$$H_\nu(x) := 2^{|\nu|} e^{-\Delta_k/4} \varphi_\nu(x) \quad \text{and} \quad h_\nu(x) := \frac{\sqrt{c_k}}{2^{|\nu|/2}} \ e^{-|x|^2/2} H_\nu(x) \quad (x \in \mathbb{R}^N). \tag{2.9}$$

We list some standard properties of generalised Hermite functions that we shall use in this article.

Theorem 2.5. *(Cf. [17].) Let* $\{H_\nu\}$ *and* $\{h_\nu\}$ *be the Hermite polynomials and Hermite functions associated with the basis* $\{\varphi_\nu\}$ *on* $\mathbb{R}^N$ *and let* $x, y \in \mathbb{R}^N$. *Then*

(1) *The* h_ν *satisfy* $h_\nu(-x) = (-1)^{|\nu|} h_\nu(x)$.

(2) $\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ *is an orthonormal basis of* $L_k^2(\mathbb{R}^N)$.

(3) *The* h_ν *are eigenfunctions of the Dunkl transform on* $L_k^2(\mathbb{R}^N)$, *with* $D_k h_\nu = (-i)^{|\nu|} h_\nu$.

(4) *(Mehler formula) For $r \in \mathbb{C}$ with $|r| < 1$,*

$$\sum_{\nu \in \mathbb{Z}_+^N} \frac{H_\nu(x) H_\nu(y)}{2^{|\nu|}} r^{|\nu|} = \frac{e^{-\frac{r^2(|x|^2+|y|^2)}{1-r^2}}}{(1-r^2)^{\gamma+(N/2)}} E_k \left(\frac{2zx}{1-z^2}, y \right).$$

Throughout this paper, $\mathcal{R}$ denotes a root system in $\mathbb{R}^N$, $\mathcal{R}^+$ a fixed positive subsystem of $\mathcal{R}$ and k a nonnegative multiplicity function defined on $\mathcal{R}$.

3. The fractional Dunkl transforms

If $A \in \mathbb{R}^{n \times n}$ is a square diagonalizable matrix A then we may write its eigenvalue decomposition $A = PDP^{-1}$. Clearly for any integer a it holds that

$$A^a = PD^a P^{-1}.$$

So it is a natural generalization to use the same formula as a definition if a is not integer. Exactly the same idea can be used for a linear operator A on a linear space if it has a sequence of eigenvectors that is complete in the whole space [14, 24]. Let $\{\lambda_k, e_k\}_{k=0}^\infty$ be the sequence of eigenvalues and corresponding eigenvectors. Since the set of eigenvectors is complete, we can associate with each element f in the Hilbert space a unique set of coordinates and conversely. These mappings are called the analysis and the synthesis operators respectively. They are adjoint operators. If $\mathcal{E}$ is the synthesis operator and $\mathcal{E}^*$ the analysis operator, which for a given set of basis vectors $\{e_k\}$ are defined by

$$\mathcal{E} : \{c_k\}_{k=0}^\infty \longmapsto f = \sum_{k=0}^\infty c_k e_k \quad \text{and} \quad \mathcal{E}^* : f \longmapsto \{c_k\}_{k=0}^\infty,$$

then we can write

$$A = \mathcal{E} \Lambda \mathcal{E}^*$$

where Λ is the simple diagonal scaling operator

$$\Lambda : \{c_k\}_{k=0}^\infty \longmapsto \{\lambda_k c_k\}_{k=0}^\infty.$$

Its fractional power is then clearly $A^a = \mathcal{E} \Lambda^a \mathcal{E}^*$.

3.1. Definition and properties.

In order to construct a fractional power of the Dunkl transform, we use the idea developed in the above by restricting ourselves to the Hilbert space $L_k^2(\mathbb{R}^N)$ with the inner product given by:

$$\langle f, g \rangle_k = \int_{\mathbb{R}^N} f(x) \overline{g(x)} \omega_k(x) dx.$$

Let $l^2(\mathbb{Z}_+^N)$ be the space of complex sequences $u = (u_\nu)_{\nu \in \mathbb{Z}_+^N}$ such that $\sum_{\nu \in \mathbb{Z}_+^N} |u_\nu|^2 < \infty$. This is a Hilbert space for the inner product

$$\langle u, v \rangle = \sum_{\nu \in \mathbb{Z}_+^N} u_\nu \bar{v}_\nu, \quad u = (u_\nu)_{\nu \in \mathbb{Z}_+^N}, \ v = (v_\nu)_{\nu \in \mathbb{Z}_+^N} \in l^2(\mathbb{Z}_+^N).$$

Define the analysis and the synthesis operators associated to the generalized Hermite functions $\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ respectively by

$$
\begin{array}{rcl}
\mathcal{E}: \quad l^2(\mathbb{Z}_+^N) & \longrightarrow & L_k^2(\mathbb{R}^N) \\
(u_\nu)_{\nu \in \mathbb{Z}_+^N} & \longmapsto & f = \displaystyle\sum_{\nu \in \mathbb{Z}_+^N} u_\nu h_\nu
\end{array}
$$

and

$$
\begin{array}{rcl}
\mathcal{E}^*: \quad L_k^2(\mathbb{R}^N) & \longrightarrow & l^2(\mathbb{Z}_+^N) \\
f & \longmapsto & (\langle f, h_\nu \rangle_k)_{\nu \in \mathbb{Z}_+^N}
\end{array} .
$$

As the generalized Hermite functions $\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ are a basis of eigenfunctions of the Dunkl transform D_k on $L_k^2(\mathbb{R}^N)$, satisfying $D_k(h_\nu) = e^{-i\pi|\nu|/2} h_\nu$, then we can write

$$D_k = \mathcal{E}\Lambda\mathcal{E}^*, \tag{3.1}$$

where Λ is the diagonal scaling operator

$$
\begin{array}{rcl}
\Lambda: \quad l^2(\mathbb{Z}_+^N) & \longrightarrow & l^2(\mathbb{Z}_+^N) \\
(u_\nu)_{\nu \in \mathbb{Z}_+^N} & \longmapsto & \left(e^{-i\pi|\nu|/2} u_\nu\right)_{\nu \in \mathbb{Z}_+^N}
\end{array} .
$$

Definition 3.1. *Let $\alpha \in \mathbb{R}$, we define the fractional Dunkl transform D_k^α on $L_k^2(\mathbb{R}^N)$ by*

$$D_k^\alpha = \mathcal{E}\Lambda^\alpha\mathcal{E}^*,$$

where Λ^α is a fractional power of the diagonal scaling operator given by

$$
\begin{array}{rcl}
\Lambda^\alpha: \quad l^2(\mathbb{Z}_+^N) & \longrightarrow & l^2(\mathbb{Z}_+^N) \\
(u_\nu)_{\nu \in \mathbb{Z}_+^N} & \longmapsto & \left(e^{i\alpha|\nu|} u_\nu\right)_{\nu \in \mathbb{Z}_+^N}
\end{array} .
$$

More explicitly, if $f \in L_k^2(\mathbb{R}^N)$, then

$$D_k^\alpha f = \sum_{\nu \in \mathbb{Z}_+^N} e^{i|\nu|\alpha} \langle f, h_\nu \rangle h_\nu. \tag{3.2}$$

We summarize the elementary properties of D_k^α in the next Proposition.

Proposition 3.1. *Let $\alpha, \beta \in \mathbb{R}$. The fractional Dunkl transform D_k^α satisfies the following properties:*
1) $D_k^0 = I$, *which is the identity operator,*
2) $D_k^{-\pi/2} = D_k$,
3) $D_k^\alpha \circ D_k^\beta = D_k^{\alpha+\beta}$,
4) $D_k^{\alpha+2\pi} = D_k^\alpha$,
5) $D_k^\pi = \check{I}$, *where* $\check{I}f(x) = f(-x)$,
6) *For all f and $g \in L^2(\mathbb{R}^N, \omega_k(x)dx)$, $\langle D_k^\alpha f, g \rangle = \langle f, D_k^{-\alpha} g \rangle$.*

Proof 1), 2) and 4) follow immediately from (3.2).
3) From (3.2), we have

$$
\begin{aligned}
D_k^\alpha(D_k^\beta f) &= \sum_{\nu \in \mathbb{Z}_+^N} e^{i|\nu|\alpha} \langle D_k^\beta f, h_\nu \rangle h_\nu \\
&= \sum_{\nu \in \mathbb{Z}_+^N} e^{i|\nu|(\alpha+\beta)} \langle f, h_\nu \rangle h_\nu = D_k^{\alpha+\beta} f.
\end{aligned}
$$

5) By (3.2) and Theorem 2.5, 1) , we have

$$
\begin{aligned}
D_k^{-\pi} f &= \sum_{\nu \in Z_+^N} e^{-i|\nu|\pi} \langle f, h_\nu \rangle \, h_\nu \\
&= \sum_{\nu \in Z_+^N} (-1)^{|\nu|} \langle f, h_\nu \rangle \, h_\nu. \\
&= \hat{I} f.
\end{aligned}
$$

6) Let f and $g \in L^2(\mathbb{R}^N, \, \omega_k(x)dx)$. It is easy to check that

$$
\begin{aligned}
\langle D_k^\alpha f, g \rangle &= \sum_{\nu \in \mathbb{Z}_+^N} e^{i|\nu|\alpha} \langle f, h_\nu \rangle \, \overline{\langle g, h_\nu \rangle} = \sum_{\nu \in \mathbb{Z}_+^N} \langle f, h_\nu \rangle \, \overline{e^{-i|\nu|\alpha} \langle g, h_\nu \rangle} \\
&= \langle f, D_k^{-\alpha} g \rangle.
\end{aligned}
$$

Theorem 3.1. *The family of operators $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$ is a $\mathcal{C}_0$-group of unitary operators on $L_k^2(\mathbb{R}^N)$.*

Proof From Proposition 3.1, we deduce that the family $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$ satisfies the algebraic properties of a group:

$$
D_k^0 = I, \quad D_k^\alpha \circ D_k^\beta = D_k^{\alpha+\beta} = D_k^\beta \circ D_k^\alpha \; ; \quad \alpha, \, \beta \in \mathbb{R}.
$$

For the strong continuity, assume that $f \in L_k^2(\mathbb{R}^N)$. Then

$$
\|D_k^\alpha f - f\|_2^2 = \sum_{\nu \in \mathbb{Z}_+^N} |e^{i|\nu|\alpha} - 1|^2 \, |\langle f, h_\nu \rangle|^2 .
$$

For each $\nu \in \mathbb{Z}_+^N$, we have

$$
\begin{aligned}
\lim_{\alpha \to 0} |e^{i|\nu|\alpha} - 1|^2 \, |\langle f, h_\nu \rangle|^2 &= 0, \\
|e^{i|\nu|\alpha} - 1|^2 \, |\langle f, h_\nu \rangle|^2 &\leq 4 \, |\langle f, h_\nu \rangle|^2 .
\end{aligned}
$$

Since

$$
\sum_{\nu \in \mathbb{Z}_+^N} |\langle f, h_\nu \rangle|^2 = \|f\|_2^2 < \infty,
$$

then we can interchange limits and sum to get:

$$
\lim_{\alpha \to 0} \|D_k^\alpha f - f\|_2^2 = 0.
$$

Hence $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$ is a strongly continuous group of operators on $L_k^2(\mathbb{R}^N)$. In addition, by Proposition 3.1, we have for all $f, \, g \in L_k^2(\mathbb{R}^N)$,

$$
\langle D_k^\alpha f, \, g \rangle = \langle f, \, D_k^{-\alpha} g \rangle,
$$

and therefore $(D_k^\alpha)^* = D_k^{-\alpha} = (D_k^\alpha)^{-1}$, establishing that each D_k^α is unitary.

4. The infinitesimal generator of the $\mathcal{C}_0$-group $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$.

The infinitesimal generator T of the $\mathcal{C}_0$-group $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$ is defined by

$$
\begin{aligned}
T : L_k^2(\mathbb{R}^N) \supseteq D(T) &\longrightarrow L_k^2(\mathbb{R}^N), \\
f &\longmapsto Tf
\end{aligned}
$$

where

$$\begin{aligned}
D(T) &= \left\{ f \in L_k^2(\mathbb{R}^N) : \lim_{\alpha \to 0}(1/\alpha)[D_k^\alpha f - f] \in L_k^2(\mathbb{R}^N) \right\}, \\
Tf &= \lim_{\alpha \to 0}(1/\alpha)[D_k^\alpha f - f], \quad f \in D(T).
\end{aligned}$$

Our goal here is to study spectral properties of T. We indicate some necessary notation and definitions, needed in the sequel. We denote by $\mathcal{B}(L_k^2(\mathbb{R}^N))$, the set of all linear bounded operator in $L_k^2(\mathbb{R}^N)$. The resolvent set of T is the set $\rho(T)$ consisting of all scalars λ for which the linear operator $\lambda I - T$ is a 1-1 mapping from its domain $D(\lambda I - T) = D(T)$ on to the Hilbert space $L_k^2(\mathbb{R}^N)$ with $(\lambda I - T)^{-1} \in \mathcal{B}(L_k^2(\mathbb{R}^N))$. The spectrum of T is the set $\sigma(T)$ that is the complement of $\rho(T)$ in $\mathbb{C}$. The function $R(\lambda, T) = (\lambda I - T)^{-1}$ from $\rho(T)$ into $\mathcal{B}(L_k^2(\mathbb{R}^N))$ is the resolvent of T.

As T is the generator of the $\mathcal{C}_0$-group $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$, some elementary properties of T and D_k^α are listed in the following proposition (see [9], [10]).

Proposition 4.1. *Let $\alpha \in \mathbb{R}$. The following properties hold.*
i) If $f \in D(T)$, then $D_k^\alpha f \in D(T)$ and

$$\frac{d}{d\alpha}D_k^\alpha f = D_k^\alpha Tf = TD_k^\alpha f. \tag{4.1}$$

ii) For every $t \in \mathbb{R}$ and $f \in L_k^2(\mathbb{R}^N)$, one has

$$\int_0^t D_k^\alpha f \ d\alpha \in D(T).$$

iii) For every $\alpha \in \mathbb{R}$, one has

$$D_k^\alpha f - f \ = \ T \int_0^\alpha D_k^s f \ ds, \quad \text{if } f \in L_k^2(\mathbb{R}^N) \tag{4.2}$$

$$= \ \int_0^\alpha D_k^s Tf \ ds, \quad \text{if } f \in D(T). \tag{4.3}$$

Remark 4.1. *If we apply the Proposition 4.1, iii) to the rescaled semigroup*

$$S(\alpha) := e^{-\lambda \alpha} D_k^\alpha, \quad \alpha \in \mathbb{R}$$

whose generator is $B := T - \lambda I$ with domain $D(B) = D(T)$, we obtain for every $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}$,

$$-e^{-\lambda \alpha}D_k^\alpha f + f \ = \ (\lambda I - T)\int_0^\alpha e^{-\lambda s}D_k^s f \ ds; \quad f \in L_k^2(\mathbb{R}^N), \tag{4.4}$$

$$= \ \int_0^\alpha e^{-\lambda s}D_k^s(\lambda I - T)f \ ds; \quad f \in D(T). \tag{4.5}$$

Now we are interesting with the eigenvalues of T by giving an important formula relating the semigroup $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$, to the resolvent of its generator T.

Proposition 4.2. *For the operator T, the following properties hold:*
1) T is closed and densely defined.
2) The operator iT is self-adjoint.
3) $\sigma(T) = \sigma_p(T) \subset i\mathbb{Z}$, and for each $\lambda \in \mathbb{C} \backslash i\mathbb{Z}$ and for all $f \in L_k^2(\mathbb{R}^N)$,

$$R(\lambda, T)f = (1 - e^{-2\pi\lambda})^{-1}\int_0^{2\pi} e^{-\lambda s}D_k^s f \ ds. \tag{4.6}$$

Here $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ \lambda I - T \ is \ not \ injective\}$.

Proof 1) The fact that T is closed and densely defined follows from the Hille-Yosida Theorem (see[[10], p. 15]).

2) Since $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$ is unitary, it follows from Stone's Theorem [[10], p. 32] that T is skew-adjoint ($T^* = -T$) and therefore iT is self-adjoint.

3) If we replace α by 2π in (4.4) and (4.5) and we use the fact that $\{D_k^\alpha\}_{\alpha \in \mathbb{R}}$ is a periodic $\mathcal{C}_0$-group with period 2π, we get

$$(1 - e^{-2\pi\lambda})f \ = \ (\lambda I - T) \int_0^{2\pi} e^{-\lambda s} D_k^s f \ ds; \quad f \in L_k^2(\mathbb{R}^N), \qquad (4.7)$$

$$= \ \int_0^{2\pi} e^{-\lambda s} D_k^s (\lambda I - T)f \ ds; \quad f \in D(T). \qquad (4.8)$$

Let $\lambda \notin i\mathbb{Z}$. Then $1 - e^{-2\pi\lambda} \neq 0$. By the use of (4.7) and (4.8), $\lambda I - T$ is invertible ($\lambda \in \rho(T)$) and

$$(\lambda I - T)^{-1}f = R(\lambda, T)f = (1 - e^{-2\pi\lambda})^{-1} \int_0^{2\pi} e^{-\lambda s} D_k^s f \ ds.$$

The previous Proposition indicates that every point in the spectrum of T is an isolated point of the set $i\mathbb{Z}$. Let in be an element of the spectrum of T and

$$P_n = \frac{1}{2i\pi} \int_\Gamma R(\lambda, T) \ d\lambda,$$

the associated spectral projection, where Γ is a Jordan path in the complement of $i\mathbb{Z}\backslash\{in\}$ and enclosing in. The function $\lambda \longmapsto R(\lambda, T)$ can be expanded as a Laurent series

$$R(\lambda, T) = \sum_{k=-\infty}^{+\infty} (\lambda - in)^k B_k$$

for $0 < |\lambda - in| < \delta$ and some sufficiently small $\delta > 0$. The coefficients B_k of this series are bounded operators given by the formulas

$$B_k = \frac{1}{2i\pi} \int_\Gamma \frac{R(\lambda, T)}{(\lambda - in)^{k+1}} \ d\lambda, \quad k \in \mathbb{Z}.$$

The coefficient B_{-1} is exactly the spectral projection P_n corresponding to the decomposition $\sigma(T) = \{in\} \cup \{i\mathbb{Z}\backslash\{in\}\}$ of the spectrum of T. From (4.6), one deduces the identitie

$$P_n \ = \ B_{-1} = \lim_{\lambda \to in} (\lambda - in) R(\lambda, T)$$

$$= \ \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} D_k^s \ ds, \qquad (4.9)$$

which allows as to interpret P_n as the nth Fourier coefficient of the 2π-periodic function $s \longmapsto D_k^s$.

In the following Proposition we gather some properties of the operator P_n.

Proposition 4.3. *Let $n, m \in \mathbb{Z}$ such that $n \neq m$ and $f, g \in L_k^2(\mathbb{R}^N)$. Then*
i) $TP_n = inP_n$,
ii) $D_k^s P_n = e^{ins} P_n$,
iii) $P_n P_m = 0$,

iv) $\langle P_n f, g \rangle = \langle f, P_n g \rangle$. In particular $\langle P_n f, P_m g \rangle = 0$.

v) The linear span

$$\text{lin} \bigcup_{n \in \mathbb{Z}} P_n L_k^2(\mathbb{R}^N)$$

is dense in $L_k^2(\mathbb{R}^N)$.

Proof i) It follows directly from (4.7) applied to $\lambda = in$.

ii) Applying D_k^s to each member of (4.9) then according to Proposition 3.1, 3), we obtain

$$
\begin{aligned}
D_k^s P_n f &= \frac{1}{2\pi} \int_0^{2\pi} e^{-int} D_k^s D_k^t f \ dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{-int} D_k^{s+t} f \ dt.
\end{aligned}
$$

The change of variables $u = s + t$ gives the desired result.

iii) From (4.9) and ii), we have

$$
\begin{aligned}
P_n P_m f &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} D_k^s (P_m f) \ ds \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)s} \ ds \right) P_m f \\
&= 0.
\end{aligned}
$$

iii) Obvious.

iv) Assume that the linear span

$$\text{lin} \bigcup_{n \in \mathbb{Z}} P_n L_k^2(\mathbb{R}^N)$$

is not dense in $L_k^2(\mathbb{R}^N)$. By the Hahn-Banach theorem there exists a nonzero linear functional

$$\varphi : L_k^2(\mathbb{R}^N) \longrightarrow \mathbb{C}$$

vanishing on each $P_n L_k^2(\mathbb{R}^N)$, $n \in \mathbb{Z}$. By the Riesz representation theorem, there exists a unique vector $g \in L_k^2(\mathbb{R}^N) \backslash \{0\}$ such that

$$\varphi(f) = \langle f, g \rangle \ \text{for all} \ f \in L_k^2(\mathbb{R}^N).$$

Hence for all $n \in \mathbb{Z}$ and $f \in L_k^2(\mathbb{R}^N)$,

$$
\begin{aligned}
0 = \langle P_n f, g \rangle &= \left\langle \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} D_k^s f \ ds, g \right\rangle \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} \langle D_k^s f, g \rangle \ ds.
\end{aligned}
$$

For each $f \in L_k^2(\mathbb{R}^N)$, the function $s \longmapsto \langle D_k^s f, g \rangle$ has all its Fourier coefficients equal to zero, then it vanishes. This cannot be true, since if we take $f = g$ and $s = 0$,

$$\langle D_k^0 g, g \rangle = \|g\|_2^2 > 0.$$

Proposition 4.4. Let $f \in D(T)$. Then

$$f = \sum_{n=-\infty}^{+\infty} P_n f, \tag{4.10}$$

and therefore, if $f \in D(T^2)$

$$Tf \quad = \quad \sum_{n=-\infty}^{+\infty} inP_n f. \tag{4.11}$$

Proof We are going to show that the series $\sum_{n \in \mathbb{Z}} P_n f$ is summable for all $f \in D(T)$. For this, let $f \in D(T)$ and put $g = Tf$. The commutativity of T and P_n together with Proposition 4.3 gives:

$$P_n g = P_n Tf = TP_n f = inP_n f.$$

By the Cauchy-Schwartz inequality, it follows that

$$\left| \sum_{n \in H} \langle P_n f, h \rangle \right| \quad = \quad \left| \sum_{n \in H} (in)^{-1} \langle P_n g, h \rangle \right|$$

$$\leq \quad \left(\sum_{n \in H} n^{-2} \right)^{1/2} \left(\sum_{n \in H} |\langle P_n g, h \rangle|^2 \right)^{1/2},$$

where $h \in L_k^2(\mathbb{R}^N)$ and H be a finite subset of $\mathbb{Z} \backslash \{0\}$. The function $s \longmapsto \langle D_k^s g, h \rangle$ belongs to $L^2([0, 2\pi])$, then we obtain from Bessel's inequality

$$\sum_{n \in H} |\langle P_n g, h \rangle|^2 \quad \leq \quad \frac{1}{2\pi} \int_0^{2\pi} |\langle D_k^s g, h \rangle|^2 \, ds$$

$$\leq \quad \frac{\|h\|_2^2}{2\pi} \int_0^{2\pi} \|D_k^s g\|_2^2 \, ds = \|h\|_2^2 \, \|g\|_2^2.$$

Therefore, for any $h \in L_k^2(\mathbb{R}^N)$,

$$\left| \left\langle \sum_{n \in H} P_n f, h \right\rangle \right| = \left| \sum_{n \in H} \langle P_n f, h \rangle \right| \quad \leq \quad \|h\|_2 \, \|g\|_2 \left(\sum_{n \in H} n^{-2} \right)^{1/2}.$$

Taking supremum over $h \in L_k^2(\mathbb{R}^N)$ with $\|h\|_2 \leq 1$, we get

$$\left\| \sum_{n \in H} P_n f \right\|_2 \quad \leq \quad \|g\|_2 \left(\sum_{n \in H} n^{-2} \right)^{1/2},$$

which means that the series $\sum_{n \in \mathbb{Z}} P_n f$ converges converges in $L_k^2(\mathbb{R}^N)$.

Set

$$f_1 = \sum_{n=-\infty}^{+\infty} P_n f$$

and let $g \in L_k^2(\mathbb{R}^N)$. As the Fourier coefficients of the continuous, 2π-periodic functions

$$s \longmapsto \langle D_k^s f_1, g \rangle \quad \text{and} \quad s \longmapsto \langle D_k^s f, g \rangle$$

coincide. Then, for all $s \in \mathbb{R}$,

$$\langle D_k^s f_1, g \rangle = \langle D_k^s f, g \rangle.$$

In particular, for $s = 0$, $\langle f_1, g \rangle = \langle f, g \rangle$ and therefore $f_1 = f$.
Replacing f in (4.10) by Tf, then we get (4.11).

At the end of the section 4, we will show that $P_n = 0$, for any negative integer $n \neq 0$.

5. INTEGRAL REPRESENTATION.

In this section, we shall derive an integral representation for the fractional Dunkl transform D_k^α defined by (3.2), for suitable function f.

We define the operator $D_{k,r}^\alpha$ on $L_k^2(\mathbb{R}^N)$ by

$$D_{k,r}^\alpha f := \sum_{\nu \in \mathbb{Z}_+^N} r^{|\nu|} e^{i|\nu|\alpha} \langle f, h_\nu \rangle h_\nu, \tag{5.1}$$

where $0 < r \leq 1$ and so $D_k^\alpha = D_{k,1}^\alpha$.

In the next proposition, we collect some properties of $D_{k,r}^\alpha$.

Proposition 5.1. *Let $\alpha \in \mathbb{R}$ and $r \in]0,1]$. Then*
1) *$D_{k,r}^\alpha$ is a bounded operator on $L_k^2(\mathbb{R}^N)$ satisfying $\|D_{k,r}^\alpha f\|_2 \leq \|f\|_2$.*
2) *For all $f \in L_k^2(\mathbb{R}^N)$, $D_{k,r}^\alpha f \to D_k^\alpha f$ in $L_k^2(\mathbb{R}^N)$ as $r \to 1^-$.*

Proof Let $f \in L_k^2(\mathbb{R}^N)$. 1) According to Parseval's formula, we have

$$\|D_{k,r}^\alpha f\|_2^2 = \sum_{\nu \in \mathbb{Z}_+^N} r^{2|\nu|} |\langle f, h_\nu \rangle|^2$$

$$\leq \sum_{\nu \in \mathbb{Z}_+^N} |\langle f, h_\nu \rangle|^2 = \|f\|_2^2.$$

2) It is easy to see that

$$D_{k,r}^\alpha f - D_k^\alpha f = \sum_{\nu \in \mathbb{Z}_+^N} (r^{|\nu|} - 1) e^{i|\nu|\alpha} \langle f, h_\nu \rangle h_\nu.$$

Then

$$\left\| D_{k,r}^\alpha f - D_k^\alpha f \right\|_2^2 = \sum_{\nu \in \mathbb{Z}_+^N} |r^{|\nu|} - 1|^2 |\langle f, h_\nu \rangle|^2.$$

By the dominated convergence theorem it follows that $\lim_{r \to 1^-} \left\| D_{k,r}^\alpha f - D_k^\alpha f \right\|_2^2 = 0$.

Corollary 5.1. *For each fixed $f \in L_k^2(\mathbb{R}^N)$, there exists $\{r_j\}_{j=1}^\infty$, with $r_j \to 1^-$ as $j \to \infty$, such that*

$$D_k^\alpha f(x) = \lim_{j \to \infty} D_{k,r_j}^\alpha f(x)$$

for almost all $x \in \mathbb{R}^N$.

Proof This is a consequence of a standard result that if a sequence $\{f_n\}$ converges in $L_k^2(\mathbb{R}^N)$ to f, then there exists a subsequence $\{f_{n_k}\}$ that converges pointwise almost everywhere to f.

The operator $D_{k,r}^\alpha$ defined above have the integral representation given in the next lemma.

Lemma 5.1. *For $f \in L_k^2(\mathbb{R}^N)$ and $0 < r < 1$, we have*

$$D_{k,r}^\alpha f(x) = \int_{\mathbb{R}^N} K_\alpha(r,x,y) f(y) \omega_k(y) \, dy, \tag{5.2}$$

where

$$K_\alpha(r, x, y) = \sum_{\nu \in \mathbb{Z}_+^N} r^{|\nu|} e^{i|\nu|\alpha} h_\nu(x) \overline{h_\nu(y)}. \tag{5.3}$$

Proof Let $x \in \mathbb{R}^N$ and H be a finite subset of $\mathbb{Z}_+^N$. Then

$$\left\| \sum_{\nu \in H} h_\nu(x) h_\nu(y) (re^{i\alpha})^{|\nu|} \right\|_2^2 = \sum_{\nu \in H} |h_\nu(x)|^2 |r|^{2|\nu|}. \tag{5.4}$$

Since the series (see Theorem 3.12 in [17])

$$\sum_{\nu \in \mathbb{Z}_+^N} h_\nu(x) h_\nu(y) (re^{i\alpha})^{|\nu|}$$

converges absolutely for all $x, y \in \mathbb{R}^N$, then according to (5.4), the series

$$\sum_{\nu \in H} h_\nu(x) h_\nu(y) (re^{i\alpha})^{|\nu|}$$

converges in $L_k^2(\mathbb{R}^N)$ to a function denoted by $K_\alpha(r, x, .)$.
By the use of Cauchy-Schwartz inequalities, we obtain

$$
\begin{aligned}
D_{k,r}^\alpha f(x) &= \sum_{\nu \in \mathbb{Z}_+^N} (re^{i\alpha})^{|\nu|} h_\nu(x) \int_{\mathbb{R}^N} f(y) h_\nu(y) \omega_k(y) \, dy \\
&= \int_{\mathbb{R}^N} f(y) \sum_{\nu \in \mathbb{Z}_+^N} h_\nu(x) h_\nu(y) (re^{i\alpha})^{|\nu|} \omega_k(y) \, dy \\
&= \int_{\mathbb{R}^N} K_\alpha(r, x, y) f(y) \omega_k(y) \, dy.
\end{aligned}
$$

Now, we summarize some properties of the kernel $K_\alpha(r, x, y)$.

Proposition 5.2. *Let* $x, y \in \mathbb{R}^N$, $\alpha, r \in \mathbb{R}$ *such that* $0 < |\alpha| < \pi$ *and* $0 < r < 1$, *then we have*
1)

$$K_\alpha(r, x, y) = c_k \frac{e^{-\frac{1+r^2 e^{2i\alpha}}{2(1-r^2 e^{2i\alpha})}(|x|^2+|y|^2)}}{(1-r^2 e^{2i\alpha})^{\gamma+N/2}} E_k\left(\frac{2re^{i\alpha}x}{1-r^2 e^{2i\alpha}}, y\right), \tag{5.5}$$

2)

$$\lim_{r \to 1^-} K_\alpha(r, x, y) = A_\alpha K_\alpha(x, y), \tag{5.6}$$

where

$$K_\alpha(x, y) = e^{-\frac{i}{2}\cot(\alpha)(|x|^2+|y|^2)} E_k\left(\frac{ix}{\sin\alpha}, y\right), \tag{5.7}$$

$$A_\alpha = \frac{c_k e^{i(\gamma+N/2)(\hat{\alpha}\pi/2-\alpha)}}{(2|\sin\alpha|)^{\gamma+N/2}} \quad and \quad \hat{\alpha} = sgn(\sin\alpha). \tag{5.8}$$

3)

$$\left| e^{-\frac{(1+r^2 e^{2i\alpha})|y|^2}{2(1-r^2 e^{2i\alpha})}} E_k\left(\frac{2re^{i\alpha}x}{1-r^2 e^{2i\alpha}}, y\right) \right| \leq e^{\frac{2r^2(1-r^2)\cos^2(\alpha)|x|^2}{(r^4-2r^2\cos 2\alpha+1)(r^2+1)}}. \tag{5.9}$$

Proof 1) According to (2.9), we have

$$K_\alpha(r,x,y) = c_k e^{-(|x|^2+|y|^2)/2} \sum_{\nu \in \mathbb{Z}_+^N} (re^{i\alpha})^{|\nu|} \frac{H_\nu(x)H_\nu(y)}{2^{|\nu|}}.$$

Using Mehler's formula for the generalized Hermite polynomials (see Theorem 2.5, 4)) and setting $z = re^{i\alpha}$ with $|z| = r < 1$, we obtain the desired result.

2) Clearly

$$\lim_{r \to 1^-} \frac{1 + r^2 e^{2i\alpha}}{1 - r^2 e^{2i\alpha}} = i \cot \alpha,$$

$$\lim_{r \to 1^-} \frac{re^{i\alpha}}{1 - r^2 e^{2i\alpha}} = \frac{i}{2 \sin \alpha}$$

$$\lim_{r \to 1^-} (1 - r^2 e^{2i\alpha})^{-(\gamma + \frac{N}{2})} = (1 - e^{2i\alpha})^{-(\gamma + \frac{N}{2})}$$

$$= \frac{e^{i(\gamma + N/2)(\hat{\alpha}\pi/2 - \alpha)}}{(2|\sin \alpha|)^{\gamma + N/2}}, \quad \text{where } \hat{\alpha} = \text{sgn}(\sin \alpha).$$

Then, for $0 < |\alpha| < \pi$,

$$\lim_{r \to 1^-} K_\alpha(r,x,y) = A_\alpha K_\alpha(x,y),$$

where $K_\alpha(x,y)$ and A_α are defined respectively in (5.7) and (5.8).

3) It is straightforward to show that

$$a_r = \Re\left(\frac{1 + r^2 e^{2i\alpha}}{1 - r^2 e^{2i\alpha}}\right) = \frac{(1 - r^4)}{(1 + r^4) - 2r^2 \cos 2\alpha} > 0,$$

$$b_r = \Re\left(\frac{2re^{i\alpha}}{1 - r^2 e^{2i\alpha}}\right) = \frac{2(r - r^3)\cos\alpha}{1 + r^4 - 2r^2 \cos 2\alpha}. \tag{5.10}$$

From (2.5) and (5.10), we deduce the following majorization:

$$\left| E_k\left(\frac{2re^{i\alpha}x}{1 - r^2 e^{2i\alpha}}, y\right) \right| \le e^{|b_r| \, |x||y|}.$$

Hence,

$$\left| e^{-\frac{(1+r^2 e^{2i\alpha})|y|^2}{2(1-r^2 e^{2i\alpha})}} E_k\left(\frac{2re^{i\alpha}x}{1 - r^2 e^{2i\alpha}}, y\right) \right| \le e^{-a_r|y|^2 + |b_r| \, |x||y|}. \tag{5.11}$$

As $a_r > 0$, we deduce that

$$\sup_{s \ge 0}(-a_r s^2 + |b_r| \, |x|s) = -\frac{b_r^2|x|^2}{4a_r}. \tag{5.12}$$

Combining (5.11) and (5.12), we see that

$$\left| e^{-\frac{(1+r^2 e^{2i\alpha})|y|^2}{2(1-r^2 e^{2i\alpha})}} E_k\left(\frac{2re^{i\alpha}x}{1 - r^2 e^{2i\alpha}}, y\right) \right| \le e^{\frac{2r^2(1-r^2)\cos^2(\alpha)|x|^2}{(r^4 - 2r^2 \cos 2\alpha + 1)(r^2 + 1)}}.$$

Proposition 5.3. *Let $\alpha \in \mathbb{R} \setminus \pi\mathbb{Z}$ and $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$. Then the fractional Dunkl transform D_k^α have the following integral representation*

$$D_k^\alpha f(x) = A_\alpha \int_{\mathbb{R}^N} f(y)K_\alpha(x,y)\omega_k(y)dy, a.e, \tag{5.13}$$

where

$$K_\alpha(x,y) \;=\; e^{-\frac{i}{2}\cot(\alpha)(|x|^2+|y|^2)} E_k\left(\frac{ix}{\sin\alpha}, y\right)$$

and

$$A_\alpha = \frac{c_k e^{i(\gamma+N/2)(\hat\alpha\pi/2-\alpha)}}{(2|\sin\alpha|)^{\gamma+N/2}}.$$

Proof D_k^α is periodic in α with period 2π, we can assume that $0 < |\alpha| < \pi$. Let $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$. From Corollary 5.1,

$$D_k^\alpha f(x) = \lim_{j\to\infty} \int_{\mathbb{R}^N} K_\alpha(r_j, x, y) f(y) \omega_k(y)\, dy, a.e.$$

From Proposition 5.2, 2) we see that

$$\lim_{j\to\infty} K_\alpha(r_j, x, y) f(y) = A_\alpha K_\alpha(x, y) f(y).$$

Using again Proposition 5.2, 3), we obtain

$$\left| e^{-\frac{(1+r_j^2 e^{2i\alpha})|y|^2}{2(1-r_j^2 e^{2i\alpha})}} E_k\left(\frac{2r_j e^{i\alpha}x}{1-r_j^2 e^{2i\alpha}}, y\right) f(y) \right| \le M_x\, |f(y)|,$$

where $M_x = \sup\limits_{0 \le r < 1} e^{\frac{2r^2(1-r^2)\cos^2(\alpha)|x|^2}{(r^4-2r^2\cos 2\alpha+1)(r^2+1)}}$.

Hence, the dominated convergence theorem gives

$$D_k^\alpha f(x) = A_\alpha \int_{\mathbb{R}^N} f(y) K_\alpha(x, y) \omega_k(y) dy, a.e.$$

Definition 5.1. *We define the fractional Dunkl transform D_k^α for $f \in L_k^1(\mathbb{R}^N)$ by*

$$D_k^\alpha f(x) = A_\alpha \int_{\mathbb{R}^N} f(y) K_\alpha(x, y) \omega_k(y) dy.$$

Remark 5.1.
- For $\alpha = -\frac{\pi}{2}$, the fractional Dunkl transform D_k^α is reduces to the Dunkl transform D_k and when the multiplicity function $k \equiv 0$, D_k^α coincides with the fractional Fourier transform $\mathcal{F}^\alpha$ [1]

$$\mathcal{F}^\alpha f(x) = \frac{e^{(iN/2)(\hat\alpha\pi/2-\alpha)}}{(2\pi|\sin\alpha|)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{i}{2}(|x|^2+|y|^2)\cot\alpha + \frac{i}{\sin\alpha}\langle x,y\rangle} f(y)\, dy.$$

- In the one-dimensional case ($N = 1$), the corresponding reflection group W is $\mathbb{Z}_2$ and the multiplicity function k is equal to $\nu + 1/2 > 0$. The kernel $K_\alpha(x, y)$ defined by (5.7) becomes

$$K_\alpha(x,y) \;=\; e^{-\frac{i}{2}\cot\alpha(x^2+y^2)} E_\nu\left(\frac{ix}{\sin\alpha}, y\right), \tag{5.14}$$

where $E_\nu(x, y)$ is the Dunkl kernel of type A_2 given by (see [19])

$$K(ix, y) = j_\nu(xy) + \frac{ixy}{2(\nu+1)} j_{\nu+1}(xy),$$

and j_ν denotes the normalized spherical Bessel function

$$j_\nu(x) := 2^\nu \Gamma(\nu+1)\frac{J_\nu(x)}{x^\nu} = \Gamma(\nu+1)\sum_{n=0}^{+\infty}\frac{(-1)^n(x/2)^{2n}}{n!\Gamma(n+\nu+1)}.$$

Here J_ν is the classical Bessel function (see, Watson [21]). The related fractional Dunkl transform D_k^α in rank-one case takes the form

$$D_\nu^\alpha f(x) = \mathcal{B}_\nu \int_{-\infty}^{+\infty} K_\alpha(x,y)f(y)|y|^{2\nu+1}\ dy, \qquad (5.15)$$

where

$$\mathcal{B}_\nu = \frac{e^{i(\nu+1)(\hat\alpha\pi/2-\alpha)}}{\Gamma(\nu+1)(2|\sin(\alpha)|)^{\nu+1}}. \qquad (5.16)$$

Note that if f is an even function then, the fractional Dunkl transform (5.15) coincides with the fractional Hankel transform [13]

$$H_\nu^\alpha f(x) = 2\mathcal{B}_\nu \int_0^{+\infty} e^{-\frac{i}{2}(x^2+y^2)\cot\alpha}\ j_\nu\left(\frac{xy}{\sin\alpha}\right)f(y)y^{2\nu+1}\ dy.$$

• *More generally, for $W = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and the multiplicity function $k = (\nu_1,\ldots,\nu_N)$, the kernel $K_\alpha(x,y)$ defined by (5.7) is given explicitly by*

$$K_\alpha(x,y) = e^{-\frac{i}{2}\cot\alpha(|x|^2+|y|^2)}\prod_{j=1}^{N} E_{\nu_j}\left(\frac{ix_j}{\sin\alpha}, y_j\right),$$

where $x = (x_1,\ldots,x_N)$, $y = (y_1,\ldots,y_N) \in \mathbb{R}^N$ and $E_{\nu_j}(x_j, y_j)$ is the function defined by (5.14). In this case the fractional Dunkl transform will be

$$D_k^\alpha f(x) = A_\alpha \int_{\mathbb{R}^N} f(y)K_\alpha(x,y)\omega_k(y)\ dy,$$

where

$$A_\alpha = \frac{e^{i(\gamma+N/2)(\hat\alpha\pi/2-\alpha)}}{\Gamma(\nu_1+1)\ldots\Gamma(\nu_N+1)(2|\sin\alpha|)^{\gamma+N/2}}$$

and

$$\omega_k(y) = \prod_{j=1}^{N} |x_j|^{2\nu_j}.$$

5.1. **Bochner type identity for the fractional Dunkl transform.** In this section, we start with a brief summary on the theory of k-spherical harmonics. An introduction to this subject can be found in the monograph [8]. The space of k-spherical harmonics of degree $n \geq 0$ is defined by

$$\mathcal{H}_n^k = \mathrm{Ker}\Delta_k \cap \mathcal{P}_n.$$

Let $S^{N-1} = \left\{x \in \mathbb{R}^N;\ |x| = 1\right\}$ be the unit sphere in $\mathbb{R}^N$ with normalized Lebesgue surface mesure $d\sigma$ and $L^2(S^{N-1},\ \omega_k(x)\ d\sigma(x))$ be the Hilbert space with the following inner product given by

$$\langle f,g\rangle_k = \int_{S^{N-1}} f(\omega)\overline{g(\omega)}\omega_k(\omega)\ d\sigma(\omega).$$

As in the theory of ordinary spherical harmonics, the space $L^2(S^{N-1},\ \omega_k(x)\ d\sigma(x))$ decomposes as an orthogonal Hilbert space sum

$$L^2(S^{N-1},\ \omega_k(x)\ d\sigma(x)) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^k.$$

In [23], Y. Xu gives an analogue of the Funk-Hecke formula for k-spherical harmonics. The well-known special case of the Dunkl-type Funk-Hecke formula is the following (see [20]):

Proposition 5.4. *Let $N \geq 2$ and put $\lambda = \gamma + (N/2) - 1$. Then for all $Y \in \mathcal{H}_n^k$ and $x \in \mathbb{R}^N$,*

$$\frac{1}{d_k} \int_{S^{N-1}} K(ix,y)Y(y)\omega_k(y)\ d\sigma(y) = \frac{\Gamma(\lambda+1)}{2^n \Gamma(n+\lambda+1)} j_{n+\lambda}(|x|)Y(ix), \quad (5.17)$$

where

$$d_k = \int_{S^{N-1}} \omega_k(y)\ d\sigma(y).$$

In particular

$$\frac{1}{d_k} \int_{S^{N-1}} K(ix,y)\omega_k(y)\ d\sigma(y) = j_\lambda(|x|). \quad (5.18)$$

An application of the Dunkl-type Funk-Hecke formula is the following:

Theorem 5.1. *(Bochner type identity) If $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ is of the form $f(x) = p(x)\psi(|x|)$ for some $p \in \mathcal{H}_n^k$ and a one-variable ψ on $\mathbb{R}_+$, then*

$$D_k^\alpha f(x) = e^{in\alpha} p(x) H_{n+\gamma+(N/2)-1}^\alpha \psi(|x|). \quad (5.19)$$

In particular, if f is radial, then

$$D_k^\alpha f(x) = H_{n+\gamma+(N/2)-1}^\alpha \psi(|x|).$$

Proof Since D_k^α is periodic in α with period 2π, we can assume that $-\pi < \alpha \leq \pi$. We see immediately that

$$\begin{aligned} D_k^0 f(x) &= f(x), \\ D_k^\pi f(x) &= f(-x) \\ &= p(-x)\psi(-x) \\ &= (-1)^n p(x)\psi(x). \end{aligned}$$

Now, let $0 < |\alpha| < \pi$. By spherical polar coordinates, we have

$$\begin{aligned} D_k^\alpha f(x) &= A_\alpha \int_{\mathbb{R}^N} f(y)K_\alpha(x,y)\omega_k(y)dy \\ &= A_\alpha \int_0^{+\infty} r^{N-1} F(r,x)\ dr, \end{aligned} \quad (5.20)$$

where

$$F(r,x) = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_{S^{N-1}} K_\alpha(x,ry)p(ry)\psi(r|y|)\omega_k(ry)\ d\sigma(y).$$

From (5.7) and the homogeneity of ω_k and p, we obtain

$$F(r,x) = \frac{2\pi^{N/2}}{\Gamma(N/2)} e^{-\frac{i}{2}(|x|^2+r^2)\cot(\alpha)} \psi(r) r^{2\gamma+n} \int_{S^{N-1}} p(y) E_k\left(\frac{irx}{\sin(\alpha)}, y\right) \omega_k(y)\, d\sigma(y).$$

Using (5.17), we get

$$\begin{aligned} F(r,x) &= \frac{2\pi^{N/2} d_k}{\Gamma(N/2)} \frac{\Gamma(\lambda+1)}{2^n \Gamma(\lambda+n+1)} \\ &\times \; e^{-\frac{i}{2}(|x|^2+r^2)\cot(\alpha)} \psi(r) r^{2\gamma+n} p\left(\frac{irx}{\sin(\alpha)}\right) j_{\lambda+n}\left(\frac{r|x|}{\sin(\alpha)}\right), \end{aligned}$$

where

$$\lambda = \gamma + (N/2) - 1.$$

Using again the homogeneity of p, we get

$$\begin{aligned} F(r,x) &= \frac{2\pi^{N/2} d_k}{\Gamma(N/2)} \frac{\Gamma(\lambda+1)}{2^n \Gamma(\lambda+n+1)} \left(\frac{i}{2\sin\alpha}\right)^n \\ &\times \; e^{-\frac{i}{2}(|x|^2+r^2)\cot(\alpha)} \psi(r) r^{2\gamma+2n} p(x) j_{\lambda+n}\left(\frac{r|x|}{\sin(\alpha)}\right). \end{aligned}$$

Now we can express a relationship between d_k and c_k. In fact

$$\begin{aligned} c_k^{-1} &= \int_{\mathbb{R}^N} e^{-|y|^2} \omega_k(y)\, dy \\ &= \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^{+\infty} r^{N-1} e^{-r^2} \int_{S^{N-1}} \omega_k(ry)\, d\sigma(y)\, dr \\ &= \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^{+\infty} r^{2\gamma+N-1} e^{-r^2} \int_{S^{N-1}} \omega_k(y)\, d\sigma(y)\, dr \\ &= \frac{\pi^{N/2} \Gamma(\lambda+1) d_k}{\Gamma(N/2)}. \end{aligned} \tag{5.21}$$

Recall that

$$A_\alpha = c_k \left(\frac{ie^{-i\alpha}}{2\sin\alpha}\right)^{\gamma+(N/2)},$$

then by the use of (5.21), we obtain

$$\begin{aligned} A_\alpha \frac{2\pi^{N/2} d_k}{\Gamma(N/2)} \frac{\Gamma(\lambda+1)}{2^n \Gamma(\lambda+n+1)} \left(\frac{i}{2\sin\alpha}\right)^n &= \frac{2\left(\frac{ie^{-i\alpha}}{2\sin\alpha}\right)^{\lambda+n+1}}{\Gamma(\lambda+n+1)} e^{in\alpha} \\ &= 2\mathcal{B}_\nu e^{in\alpha}. \end{aligned}$$

Hence

$$F(r,x) = 2\mathcal{B}_\nu e^{in\alpha} e^{-\frac{i}{2}(|x|^2+r^2)\cot(\alpha)} \psi(r) r^{2\gamma+2n} p(x) j_{\lambda+n}\left(\frac{r|x|}{\sin(\alpha)}\right). \tag{5.22}$$

Substituting (5.22) in (5.20) to get

$$
\begin{aligned}
D_k^\alpha f(x) &= 2\mathcal{B}_\nu e^{in\alpha} p(x) \\
&\times \int_0^{+\infty} e^{-\frac{i}{2}(|x|^2+r^2)\cot(\alpha)} \psi(r) r^{2(\lambda+n)+1} j_{\lambda+n}\left(\frac{r|x|}{\sin(\alpha)}\right) \, dr \\
&= e^{in\alpha} p(x) H_{n+\lambda}^\alpha \psi(|x|) \\
&= e^{in\alpha} p(x) H_{n+\gamma+(N/2)-1}^\alpha \psi(|x|).
\end{aligned}
$$

Application

Now, we give the material needed for an application of Bochner type identity. Let $\{p_{n,j}\}_{j\in J_n}$ be an orthonormal basis of $\mathcal{H}_n^k$. Let m, n be non-negative integers and $j \in J_n$. Define

$$
c_{m,n} = \left(\frac{m!\,\Gamma(N/2)}{\pi^{N/2}\Gamma((N/2)+\gamma+n+m)}\right)^{1/2}
$$

and

$$
\psi_{m,n,j}(x) = c_{m,n}\, p_{n,j}(x)\, L_m^{(n+\gamma+N/2-1)}(|x|^2)\, e^{-|x|^2/2}, \tag{5.23}
$$

where $L_m^{(a)}$ denote the Laguerre polynomial defined by

$$
L_m^{(a)}(x) = \frac{x^{-a}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+a}\right).
$$

It follows from Proposition 2.4 and Theorem 2.5 of Dunkl [6] that

$$
\{\psi_{m,n,j} : m,\ n = 0,\ 1,\ 2,\ldots,\ j \in J_n\}
$$

forms an orthonormal basis of $L_k^2(\mathbb{R}^N)$.

Theorem 5.2. *The family* $\{\psi_{m,n,j} : m,\ n = 0,\ 1,\ 2,\ldots,\ j \in J_n\}$ *are a basis of eigenfunctions of the fractional Dunkl transform* D_k^α *on* $L^2\left(\mathbb{R}^N,\ \omega_k(x)\,dx\right)$*, satisfying*

$$
D_k^\alpha \psi_{m,n,j} = e^{i\alpha(n+2m)}\psi_{m,n,j}. \tag{5.24}
$$

Proof We need only to prove (5.24). Applying Theorem 5.3 with p replaced by $p_{n,j}$ and with $\psi(r) = L_m^{(n+\gamma+N/2-1)}(r^2)\,e^{-r^2/2}$, we obtain

$$
D_k^\alpha \psi_{m,n,j}(x) = c_{m,n} e^{in\alpha} p_{n,j}(x) H_\nu^\alpha \psi(|x|),
$$

where

$$
\nu = n + \gamma + (N/2) - 1,
$$

and

$$
\mathcal{H}_\nu^\alpha \psi(|x|) = 2\mathcal{B}_\nu \int_0^{+\infty} e^{-\frac{i}{2}\cot(\alpha)(|x|^2+r^2)} j_\nu\left(\frac{r|x|}{\sin\alpha}\right) L_m^{(\nu)}(r^2) e^{-\frac{r^2}{2}} r^{2\nu+1} \, dr.
$$

Observe that

$$
H_\nu^\alpha \psi(|x|) = 2\mathcal{B}_\nu e^{-\frac{i}{2}\cot(\alpha)|x|^2} I_\nu,
$$

where

$$
\begin{aligned}
I_\nu &= \int_0^{+\infty} r^{2\nu+1} L_m^{(\nu)}(r^2) e^{-(\frac{1}{2}+\frac{i}{2}\cot(\alpha))r^2} j_\nu\left(\frac{r|x|}{\sin\alpha}\right) \, dr \\
&= 2^\nu \Gamma(\nu+1)\left(\frac{\sin\alpha}{|x|}\right)^\nu \int_0^{+\infty} r^{\nu+1} L_m^{(\nu)}(r^2) e^{-(\frac{1}{2}+\frac{i}{2}\cot(\alpha))r^2} J_\nu\left(\frac{r|x|}{\sin\alpha}\right) \, dr.
\end{aligned}
$$

To compute I_ν, we need the following formulas (see 7.4.21 (4) in [11])

$$\int_0^{+\infty} y^{\nu+1} e^{-\beta y^2} L_m^\nu(ay^2) J_\nu(zy) \, dy = d_m z^\nu e^{-z^2/(4\beta)} L_m^\nu \left[\frac{az^2}{4\beta(a-\beta)} \right]$$

where $d_m = ((\beta - a)^m/(2^{\nu+1}\beta^{\nu+m+1}))$, a, $\Re\beta > 0$, $\Re\nu > -1$.

Let us take $\beta = \frac{1}{2} + \frac{i}{2}\cot(\alpha) = \frac{ie^{-i\alpha}}{2\sin\alpha}$, $a = 1$ and $z = \frac{|x|}{\sin\alpha}$, then

$$d_m = \frac{e^{2i\alpha m}}{2^{\nu+1}\mathcal{A}_\alpha \Gamma(\nu+1)},$$

$$\frac{az^2}{4\beta(a-\beta)} = |x|^2,$$

$$-\frac{z^2}{4\beta} = -\frac{|x|^2}{2} + \frac{i}{2}\cot(\alpha)|x|^2.$$

Hence

$$\int_0^{+\infty} r^{\nu+1} L_m^{(\nu)}(r^2) e^{-(\frac{1}{2}+\frac{i}{2}\cot(\alpha))r^2} J_\nu\left(\frac{r|x|}{\sin\alpha}\right) \, dr = \frac{e^{2i\alpha m} e^{-\frac{|x|^2}{2}+\frac{i}{2}\cot(\alpha)|x|^2}}{2^{\nu+1}\mathcal{A}_\alpha\Gamma(\nu+1)}\left(\frac{|x|}{\sin\alpha}\right)^\nu L_m^{(\nu)}(|x|^2),$$

and therefore

$$H_\nu^\alpha \psi(|x|) = e^{2i\alpha m} L_m^{(\nu)}(|x|^2) \, e^{-|x|^2/2},$$

which finishes the proof.

5.2. **Master Formula for the fractional Dunkl transform.** In this section, we are interesting with a master formula for the fractional Dunkl transform. For this we need the following lemma

Lemma 5.2. *Let $p \in \mathcal{P}_n$ and $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$. Then for $\omega \in \mathbb{C}$ and $\Re(\omega) > 0$,*

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-\omega|y|^2} \omega_k(y) \, dy = \frac{e^{\frac{l(x)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega}{4}\Delta_k} p(x), \qquad (5.25)$$

where $l(x) = \sum_{j=1}^N x_j$.

   Proof First compute the above integral when $\omega > 0$.

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-\omega|y|^2} \omega_k(y) \, dy = c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-|\sqrt{\omega}y|^2} \omega_k(y) \, dy.$$

By the change of variables $u = \sqrt{\omega}y$ and the homogeneity of ω_k and p, we obtain

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-\omega|y|^2} \omega_k(y) \, dy$$

$$= \frac{c_k}{\omega^{\gamma+(n+N)/2}} \int_{\mathbb{R}^N} p(y) E_k\left(\frac{x}{\sqrt{\omega}}, 2y\right) e^{-|y|^2} \omega_k(y) \, dy. \qquad (5.26)$$

Using Theorem 2.4,1), we deduce the following identity:

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-|y|^2} \omega_k(y) \, dy = e^{l(x)} e^{\frac{\Delta_k}{4}} p(x). \qquad (5.27)$$

Combining (5.26) and (5.27) to get

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-\omega|y|^2} \omega_k(y) \, dy = \frac{e^{\frac{l(x)}{\omega}}}{\omega^{\gamma+(n+N)/2}} e^{\frac{\Delta_k}{4}} p\left(\frac{x}{\sqrt{\omega}}\right).$$

Now use Lemma 2.1 from [17] to obtain

$$e^{\frac{\Delta_k}{4}} p\left(\frac{x}{\sqrt{\omega}}\right) = \frac{1}{\omega^{n/2}} e^{\frac{\omega}{4}\Delta_k} p(x).$$

Hence, we find the equality (5.25) for $\omega > 0$. By analytic continuation, this holds for $\{\omega \in \mathbb{C} : \Re(\omega) > 0\}$.

We are now in a position to give the master formula.

Theorem 5.3. *Let $p \in \mathcal{P}_n$ and $x \in \mathbb{R}^N$. Then*

$$D_k^\alpha \left[e^{-\frac{|y|^2}{2}} e^{-\frac{\Delta_k}{4}} p(y) \right](x) = e^{in\alpha} e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} p(x). \qquad (5.28)$$

Proof It follows easily from (5.13) that

$$D_k^\alpha \left[e^{-\frac{|y|^2}{2}} e^{-\frac{\Delta_k}{4}} p(y) \right](x) = A_\alpha e^{-\frac{i}{2}\cot(\alpha)|x|^2} \int_{\mathbb{R}^N} e^{-\frac{\Delta_k}{4}} p(y) E_k\left(\frac{ix}{\sin\alpha}, y\right) e^{-\omega|y|^2} \omega_k(y) \, dy,$$

where

$$\omega = \frac{1}{2} + \frac{i}{2}\cot(\alpha) = \frac{ie^{-i\alpha}}{2\sin\alpha}. \qquad (5.29)$$

Since

$$e^{-\frac{\Delta_k}{4}} p(y) = \sum_{s=0}^{[\frac{n}{2}]} \frac{(-1)^s}{s!4^s} \Delta_k^s p(y),$$

we conclude that

$$\int_{\mathbb{R}^N} e^{-\frac{\Delta_k}{4}} p(y) E_k\left(\frac{ix}{\sin\alpha}, y\right) e^{-\omega|y|^2} \omega_k(y) \, dy = \sum_{s=0}^{[\frac{n}{2}]} \frac{(-1)^s}{s!4^s} \int_{\mathbb{R}^N} \Delta_k^s p(y) E_k\left(\frac{ix}{\sin\alpha}, y\right) e^{-\omega|y|^2} \omega_k(y) \, dy. \qquad (5.30)$$

For $s \in \mathbb{Z}_+$ with $2s \le n$, the polynomial $\Delta_k^s p$ is homogeneous of degree $n - 2s$. Hence by the previous Lemma, we obtain

$$c_k \int_{\mathbb{R}^N} \Delta_k^s p(y) E_k\left(\frac{ix}{\sin\alpha}, y\right) e^{-\omega|y|^2} \omega_k(y) \, dy = \frac{e^{\frac{l(X_\alpha)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega}{4}\Delta_k} \left[\omega^{2s} \Delta_k^s p\right](X_\alpha) \qquad (5.31)$$

where

$$X_\alpha = \frac{ix}{2\sin\alpha}. \qquad (5.32)$$

Substituting (5.31) in (5.30) to get

$$\begin{aligned}
c_k \int_{\mathbb{R}^N} e^{-\frac{\Delta_k}{4}} p(y) E_k\left(\frac{ix}{\sin\alpha}, y\right) e^{-\omega|y|^2} \omega_k(y) \, dy &= \frac{e^{\frac{l(X_\alpha)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega}{4}\Delta_k} \sum_{s=0}^{[\frac{n}{2}]} \frac{(-1)^s \omega^{2s}}{s!4^s} \Delta_k^s p(X_\alpha) \\
&= \frac{e^{\frac{l(X_\alpha)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega}{4}\Delta_k} e^{-\frac{\omega^2}{4}\Delta_k} p(X_\alpha) \\
&= \frac{e^{\frac{l(X_\alpha)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega-\omega^2}{4}\Delta_k} p(X_\alpha).
\end{aligned}$$

Replacing ω and X_α by their values given in (5.29) and (5.32) and use Lemma 2.1 in [17], we obtain

$$
\begin{aligned}
e^{\frac{\omega-\omega^2}{4}\Delta_k}p(X_\alpha) &= \frac{i^n}{2^n \sin^n \alpha}e^{-\sin^2(\alpha)(\omega-\omega^2)\Delta_k}p(x) \\
&= \frac{i^n}{2^n \sin^n \alpha}e^{-\frac{\Delta_k}{4}}p(x).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\omega^{n+\gamma+(N/2)} &= \left(\frac{ie^{-i\alpha}}{2\sin\alpha}\right)^{n+\gamma+(N/2)} = \frac{i^n e^{-in\alpha}}{2^n \sin^n \alpha}\frac{e^{i(\gamma+N/2)(\hat\alpha\pi/2-\alpha)}}{(2|\sin\alpha|)^{\gamma+(N/2)}} \\
e^{\frac{l(X_\alpha)}{\omega}} &= e^{\frac{ie^{i\alpha}}{2\sin\alpha}|x|^2}.
\end{aligned}
$$

Then

$$
\int_{\mathbb{R}^N} e^{-\frac{\Delta_k}{4}}p(y)E_k\left(\frac{ix}{\sin\alpha},y\right)e^{-\omega|y|^2}\omega_k(y)\,dy = A_\alpha^{-1}e^{in\alpha}e^{\frac{ie^{i\alpha}}{2\sin\alpha}|x|^2}e^{-\frac{\Delta_k}{4}}p(x). \quad (5.33)
$$

Finally, if we multiply equation (5.33) by $A_\alpha e^{-\frac{i}{2}\cot(\alpha)|x|^2}$, we obtain the desired result.

A consequence of the Master formula (5.28) is

Corollary 5.2. *(Hecke type identity) If in addition to the assumption in Theorem 5.3, the polynomial $p \in \mathcal{H}_n^k$, then (5.28) becomes*

$$
D_k^\alpha\left[e^{-\frac{|\cdot|^2}{2}}p\right](x) = e^{in\alpha}e^{-\frac{|x|^2}{2}}p(x). \quad (5.34)
$$

Now, we are interesting to complete the spectral study of T started in proposition 4.4 by means of the Master formula. In fact we have the following

Corollary 5.3.
$L^2\left(\mathbb{R}^N,\ \omega_k(x)\ dx\right)$ *decomposes as an orthogonal Hilbert space sum according to*

$$
L^2\left(\mathbb{R}^N,\ \omega_k(x)\ dx\right) = \bigoplus_{n\in\mathbb{Z}_+}V_n,
$$

where

$$
V_n = \left\{e^{-\frac{|x|^2}{2}}e^{-\frac{\Delta_k}{4}}p(x);\quad p\in\mathcal{P}_n\right\}
$$

is the eigenspace of T corresponding to the eigenvalue in. In particular, T is essentially self-adjoint. The spectrum of its closure is purely discrete and given by

$$
\sigma(\overline{T}) = i\mathbb{Z}_+.
$$

Proof Let f be an element of the subspace V_n defined by

$$
f(x) = e^{-\frac{|x|^2}{2}}e^{-\frac{\Delta_k}{4}}p(x),
$$

where $p \in \mathcal{P}_n$. From (5.28), the limits

$$
\lim_{\alpha\to0}\frac{D_k^\alpha f - f}{\alpha} = \lim_{\alpha\to0}\frac{e^{in\alpha}-1}{\alpha}f
$$

exists in $L_k^2(\mathbb{R}^N)$ and equals inf. Then

$$
f \in D(T)\quad\text{and}\quad T(f) = inf. \quad (5.35)
$$

Hence, V_n is the eigenspace of T corresponding to the eigenvalue in.

6. Realization of the operator T.

The aim of the following is to find a subspace $\mathcal{W} \subset D(T)$ of $L_k^2(\mathbb{R}^N)$ in which we define T explicitly.

Lemma 6.1. *For $z \in \mathbb{C}^N$ set $l(z) = \sum_{i=1}^{N} z_i^2$. Then for all $z, \omega \in \mathbb{C}^N$,*

$$c_k \int_{\mathbb{R}^N} E_k(2z,x) E_k(2\omega,x) e^{-A|x|^2} \omega_k(x) \ dx = \frac{e^{\frac{l(z)+l(\omega)}{A}}}{A^{\gamma+N/2}} E_k(2z/A, \omega), \qquad (6.1)$$

where A is a complex number such that $\Re(A) > 0$.

Proof The result is obtained by means of a similar technic used in the proof of Lemma 5.2 and the following formula (see [6])

$$c_k \int_{\mathbb{R}^N} E_k(2z,x) E_k(2\omega,x) e^{-|x|^2} \omega_k(x) \ dx = e^{l(z)+l(\omega)} E_k(2z, \omega).$$

Theorem 6.1. *Let $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ such that $D_k f \in L_k^1(\mathbb{R}^N)$ and $\alpha \notin \left\{ \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z} \right\}$. Then*

$$D_k^{\alpha} f(x) = c_k \left(\frac{e^{-i\alpha}}{2\cos\alpha} \right)^{\gamma+\frac{N}{2}} \int_{\mathbb{R}^N} e^{\frac{i}{2}\tan(\alpha)(|x|^2+|y|^2)} E_k(\frac{ix}{\cos\alpha}, y) D_k f(y) \omega_k(y) dy. (6.2)$$

Proof Let $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ such that $D_k f \in L_k^1(\mathbb{R}^N)$. Let ϵ be an arbitrary positive number and put

$$F_\epsilon(x) = \int_{\mathbb{R}^N} f(y) g_\epsilon(y) \omega_k(y) \ dy,$$

where $g_\epsilon(y) = e^{-(\epsilon+\frac{i}{2}\cot\alpha)|y|^2} E_k\left(\frac{ix}{\sin\alpha}, y \right)$.
From (2.6), we deduce that

$$|f(y) g_\epsilon(y)| \leq |f(y)|,$$

so the dominated convergence theorem can be invoked again to give

$$\lim_{\epsilon \to 0} F_\epsilon(x) \quad = \quad \frac{e^{\frac{i}{2}|x|^2 \cot\alpha}}{A_\alpha} D_k^{\alpha} f(x). \qquad (6.3)$$

Using Lemma 6.1, we can show

$$D_k g_\epsilon(\xi) \quad = \quad \frac{e^{-\frac{|x|^2}{4\epsilon\sin^2\alpha+i\sin 2\alpha}}}{(2\epsilon+i\cot\alpha)^{\gamma+N/2}} \quad e^{-\frac{|\xi|^2}{4\epsilon+2i\cot\alpha}} \quad E_k\left(\frac{x}{2\epsilon\sin\alpha+i\cos\alpha}, \xi \right). (6.4)$$

Now applying the Parseval formula for the Dunkl transform (see Lemma 4.25, [2]) and using (6.4), we obtain

$$F_\epsilon(x) \quad = \quad \frac{e^{-\frac{|x|^2}{4\epsilon\sin^2\alpha+i\sin 2\alpha}}}{(2\epsilon+i\cot\alpha)^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{-\frac{|\xi|^2}{4\epsilon+2i\cot\alpha}} \quad E_k\left(\frac{x}{2\epsilon\sin\alpha+i\cos\alpha}, \xi \right) D_k f(-\xi) \omega_k(\xi) \ d\xi.$$

(2.5) gives again the following majorization:

$$\left| E_k\left(\frac{x}{2\epsilon\sin\alpha+i\cos\alpha}, \xi \right) \right| \leq e^{\frac{2\epsilon\sin\alpha}{4\epsilon^2\sin^2(\alpha)+\cos^2(\alpha)}|x||\xi|}.$$

Hence,

$$\left| e^{-\frac{|\xi|^2}{4\epsilon + 2i\cot\alpha}} \, E_k\left(\frac{x}{2\epsilon\sin\alpha + i\cos\alpha}, \xi\right) \right| \le e^{-p_\epsilon|\xi|^2 + q_\xi|\xi|}, \tag{6.5}$$

where

$$p_\epsilon = \frac{\epsilon}{4\epsilon^2 + \cot^2\alpha}$$

and

$$q_\epsilon = \frac{2\epsilon\sin(\alpha)|x|}{4\epsilon^2\sin^2(\alpha) + \cos^2(\alpha)}.$$

As $p_\epsilon > 0$, we deduce that

$$\sup_{s\ge 0}(-p_\epsilon s^2 + q_\epsilon s) = -\frac{q_\epsilon^2}{4p_\epsilon}. \tag{6.6}$$

Applying formula (6.5) and (6.6), we obtain

$$\left| e^{-\frac{|\xi|^2}{4\epsilon + 2i\cot\alpha}} \, E_k\left(\frac{x}{2\epsilon\sin\alpha + i\cos\alpha}, \xi\right) D_k f(-\xi) \right| \le e^{-\frac{4\epsilon|x|^2}{4\epsilon^2\sin^2(\alpha) + \cos^2(\alpha)}} |D_k f(-\xi)|$$

$$\le B_x|D_k f(-\xi)|,$$

where $B_x = \sup_{\epsilon\in]0,1]} e^{-\frac{4\epsilon|x|^2}{4\epsilon^2\sin^2(\alpha) + \cos^2(\alpha)}}$. The function $\xi \mapsto D_k f(-\xi)$ is in $L_k^1(\mathbb{R}^N)$, then the dominated convergence theorem implies

$$\lim_{\epsilon\to 0} F_\epsilon(x) = \frac{e^{\frac{i|x|^2}{\sin 2\alpha}}}{(i\cot\alpha)^{\gamma + N/2}} \int_{\mathbb{R}^N} e^{\frac{i|\xi|^2\tan\alpha}{2}} E_k\left(-\frac{ix}{\cos\alpha}, \xi\right) D_k f(-\xi)\omega_k(\xi)d\xi$$

Hence, (6.3) and (6.7) gives after simplification

$$D_k^\alpha f(x) = c_k\left(\frac{e^{-i\alpha}}{2\cos\alpha}\right)^{\gamma + \frac{N}{2}} e^{\frac{i}{2}|x|^2\tan\alpha} \int_{\mathbb{R}^N} e^{\frac{i}{2}|\xi|^2\tan\alpha} E_k\left(-\frac{ix}{\cos\alpha}, \xi\right) D_k f(-\xi)\omega_k(\xi)d\xi$$

Finally, if we make the change of variables $u = -y$ in (6.8), then we find (6.2).

Remark 6.1. *Using (6.2) together with the dominated convergence theorem, we get*

$$\lim_{\alpha\to 0^+} D_k^\alpha f(x) = \lim_{\alpha\to 0^-} D_k^\alpha f(x) = D_k^2 f(-x) = f(x), a.e,$$

$$\lim_{\alpha\to\pi^-} D_k^\alpha f(x) = \lim_{\alpha\to -\pi^+} D_k^\alpha f(x) = D_k^2 f(x) = f(-x)\ a.e.$$

Corollary 6.1. *Under the assumptions of Theorem 6.1, we have*

$$\frac{D_k^\alpha f(x) - f(x)}{\alpha} = r_1(\alpha)\frac{c_k}{2^{\gamma + (N/2)}} \int_{\mathbb{R}^N} e^{\frac{i}{2}\tan(\alpha)(|x|^2 + |y|^2)} E_k\left(\frac{ix}{\cos\alpha}, y\right) D_k f(y)\omega_k(y)dy$$

$$+ \frac{c_k}{2^{\gamma + \frac{N}{2}}} \int_{\mathbb{R}^N} r_2(\alpha, x, y) D_k f(y)\omega_k(y)dy, a.e, \tag{6.9}$$

where

$$r_1(\alpha) = \frac{\left(\frac{e^{-i\alpha}}{\cos\alpha}\right)^{\gamma + \frac{N}{2}} - 1}{\alpha} \quad and \quad r_2(\alpha, x, y) = \frac{e^{\frac{i}{2}\tan(\alpha)(|x|^2 + |y|^2)} E_k(\frac{ix}{\cos\alpha}, y) - E_k(ix, y)}{\alpha}.$$

Proof The result is consequence of (6.2) and (2.8).

Lemma 6.2. *Let $\alpha_0 \in]0, \frac{\pi}{2}[$ and $x, y \in \mathbb{R}^N$. Then*

$$|r_2(\alpha, x, y)| \leq \frac{1}{2}(1 + \tan^2 \alpha_0)(|x|^2 + |y|^2) + \frac{|\sin(\alpha_0)|}{\cos^2(\alpha_0)} \sqrt{N} \, |x||y|, \qquad (6.10)$$

where $\alpha \in]0, \alpha_0]$.

Proof By the mean value theorem, we have

$$|r_2(\alpha, x, y)| \leq \sup_{\alpha \in [0, \alpha_0]} \left| \frac{\partial}{\partial \alpha} r_3(\alpha, x, y) \right|,$$

where

$$r_3(\alpha, x, y) = e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} E_k \left(\frac{ix}{\cos \alpha}, y \right).$$

From (2.4), we get

$$E_k \left(\frac{ix}{\cos \alpha}, y \right) = E_k \left(x, \frac{iy}{\cos \alpha} \right).$$

Therefore,

$$r_3(\alpha, x, y) = e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} E_k \left(x, \frac{iy}{\cos \alpha} \right).$$

A simple calculations shows that

$$\begin{aligned}
\frac{\partial}{\partial \alpha} r_3(\alpha, x, y) &= \frac{i}{2}(1 + \tan^2 \alpha)(|x|^2 + |y|^2) r_3(\alpha, x, y) \\
&+ \frac{i \sin(\alpha)}{\cos^2(\alpha)} e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} \sum_{j=1}^{N} y_j \frac{\partial}{\partial y_j} E_k \left(x, \frac{iy}{\cos \alpha} \right). \quad (6.11)
\end{aligned}$$

From (2.5), the inequality

$$\left| \frac{\partial}{\partial y_j} E_k \left(x, \frac{iy}{\cos \alpha} \right) \right| \leq |x|$$

holds and hence

$$\begin{aligned}
\left| \frac{\partial}{\partial \alpha} r_3(\alpha, x, y) \right| &\leq \frac{1}{2}(1 + \tan^2 \alpha)(|x|^2 + |y|^2) + \frac{|\sin(\alpha)|}{\cos^2(\alpha)} |x| \sum_{j=1}^{N} |y_j| \\
&\leq \frac{1}{2}(1 + \tan^2 \alpha)(|x|^2 + |y|^2) + \frac{|\sin(\alpha)|}{\cos^2(\alpha)} \sqrt{N} \, |x||y| \\
&\leq \frac{1}{2}(1 + \tan^2 \alpha_0)(|x|^2 + |y|^2) + \frac{|\sin(\alpha_0)|}{\cos^2(\alpha_0)} \sqrt{N} \, |x||y|.
\end{aligned}$$

Which finishes the proof.

Theorem 6.2. *Let*

$$\mathcal{W} = \left\{ f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N) \; ; \; |y|^2 f \in L_k^2(\mathbb{R}^N) \quad and \quad |y|^2 D_k f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N) \right\}.$$

Then for all $f \in \mathcal{W}$,

$$Tf(x) = -i(\gamma + (N/2))f(x) + \frac{i}{2}|x|^2 f(x) + \frac{i}{2} D_k \left[|y|^2 D_k(y) \right] (-x) \; a.e. \quad (6.12)$$

Proof It is clear that

$$\lim_{\alpha \to 0} r_1(\alpha) = -i(\gamma + (N/2)).$$

In view of (2.6), we deduce

$$\left| E_k \left(\frac{ix}{\cos \alpha}, y \right) \right| \le 1.$$

Then

$$\left| e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} E_k \left(\frac{ix}{\cos \alpha}, y \right) D_k f(y) \right| \le |D_k f(y)|.$$

Let $y \in \mathbb{R}^N$ such that $|y| > 1$. Then

$$|D_k f(y)| \le |y|^2 |D_k f(y)|.$$

Since $y \longmapsto |y|^2 D_k f \in L_k^1(\mathbb{R}^N)$, it follows that $D_k f \in L_k^1(\mathbb{R}^N)$ and the dominated convergence theorem implies

$$
\begin{aligned}
&\lim_{\alpha \to 0} r_1(\alpha) \ \frac{c_k}{2^{\gamma + (N/2)}} \int_{\mathbb{R}^N} e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} E_k \left(\frac{ix}{\cos \alpha}, y \right) D_k f(y) \omega_k(y) dy \\
&= \quad -i(\gamma + (N/2)) \frac{c_k}{2^{\gamma + (N/2)}} \int_{\mathbb{R}^N} E_k(ix, y) D_k f(y) \omega_k(y) dy \\
&= \quad -i(\gamma + (N/2)) D_k^2 f(-x) \\
&= \quad -i(\gamma + (N/2)) f(x), \ \ a.e.
\end{aligned}
$$

From (6.11), we deduce

$$\lim_{\alpha \to 0} r_2(\alpha, x, y) = \frac{i}{2} (|x|^2 + |y|^2) K(ix, y).$$

By the previous Lemma, we have the following majorization:

$$|r_2(\alpha, x, y) D_k f(y)| \le f_1(y) + f_2(y) + f_3(y),$$

where

$$
\begin{aligned}
f_1(y) &= \frac{1}{2} (1 + \tan^2 \alpha_0) |x|^2 |D_k f(y)|, \\
f_2(y) &= \frac{1}{2} (1 + \tan^2 \alpha_0) |y|^2 |D_k f(y)|, \\
f_3(y) &= \frac{|\sin(\alpha_0)|}{\cos^2(\alpha_0)} \sqrt{N} \ |x||y||D_k f(y)|.
\end{aligned}
$$

Since $y \longmapsto |y|^2 D_k f \in L_k^1(\mathbb{R}^N)$, it follows that $f_1, f_2, f_3 \in L_k^1(\mathbb{R}^N)$ and therefore $f_1 + f_2 + f_3 \in L_k^1(\mathbb{R}^N)$. By virtue of the dominated convergence theorem, we have

$$\lim_{\alpha \to 0} \frac{c_k}{2^{\gamma + \frac{N}{2}}} \int_{\mathbb{R}^N} r_2(\alpha, x, y) D_k f(y) \omega_k(y) dy$$

$$= \frac{i}{2} \frac{c_k}{2^{\gamma + \frac{N}{2}}} \int_{\mathbb{R}^N} (|x|^2 + |y|^2) E_k(ix, y) D_k f(y) \omega_k(y) dy$$

$$= \frac{i|x|^2}{2} \frac{c_k}{2^{\gamma + \frac{N}{2}}} \int_{\mathbb{R}^N} E_k(ix, y) D_k f(y) \omega_k(y) dy$$

$$+ \frac{i}{2} \frac{c_k}{2^{\gamma + \frac{N}{2}}} \int_{\mathbb{R}^N} |y|^2 E_k(ix, y) D_k f(y) \omega_k(y) dy$$

$$= \frac{i}{2} |x|^2 f(x) + \frac{i}{2} \frac{c_k}{2^{\gamma + \frac{N}{2}}} \int_{\mathbb{R}^N} |y|^2 E_k(ix, y) D_k f(y) \omega_k(y) dy, a.e,$$

$$= \frac{i}{2} |x|^2 f(x) + \frac{i}{2} D_k \left[|y|^2 D_k(y) \right] (-x) \ a.e.$$

Corollary 6.2.
1) $\mathcal{S}(\mathbb{R}^N) \subset \mathcal{W} \subset D(T)$.
2) *For all $f \in \mathcal{S}(\mathbb{R}^N)$,*

$$-iTf = -(\gamma + (N/2))f + \frac{1}{2}(|x|^2 - \Delta_k)f$$

Proof 1) Obvious.
2) Let $f \in \mathcal{S}(\mathbb{R}^N)$. From Corollary 2.11 in [6], we deduce

$$-y_j^2 D_k f(y) = D_k[T_j^2 f](y),$$

where $j \in \{1, 2, \dots, N\}$. Then

$$-|y|^2 D_k f(y) = D_k[\Delta_k f](y).$$

Therefore

$$-D_k \left[|y|^2 D_k(y) \right] (-x) = D_k^2[\Delta_k f(y)](-x)$$
$$= \Delta_k f(x). \qquad (6.13)$$

Finally, from (6.12) and (6.13) we obtain the desired result.

Remark 6.2. *It is clear that the operator $2iT - (2\gamma + N)$ is an extension on $\mathcal{W}$ of the Hermite operator $\mathcal{H}_k = \Delta_k - |x|^2$ studied by Rösler [17] where it used another approach based on the notion of Lie algebra.*
In the same context, we give a new proof of the following result established in [17]

Corollary 6.3. *For $n \in \mathbb{N}$ and $p \in \mathcal{P}_n$, the function $f = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} p(x)$ satisfies*

$$(\Delta_k - |x|^2)f = -(2n + 2\gamma + N)f. \qquad (6.14)$$

In particular

$$(\Delta_k - |x|^2)h_\nu = -(2|\nu| + 2\gamma + N)h_\nu. \qquad (6.15)$$

Proof Since $f \in \mathcal{S}(\mathbb{R}^N)$, (6.14) is obtained by the use of the previous Corollary and (5.35)

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