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A GENERALIZED MEAN PROXIMAL ALGORITHM FOR SOLVING GENERALIZED MIXED EQUILIBRIUM PROBLEMS

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Abstract. In a recent paper [14], the fixed point and resolvent techniques have been used to give an iterative algorithm for solving generalized mixed equilibrium problem (GMEP). In this paper, an extended iterative algorithm, called generalized mean proximal algorithm (GMPA) is presented and its convergence to the solution of GMEP with θ-pseudomonotne and δ- Lipschitz continuous mappings in Hilbert spaces is proved. Our approach is based on the basic notions of generalized resolvents and fixed points. The results improve and extend important recent results.

1. Introduction

The generalized mixed equilibrium problem (GMEP) is one of the most general problems appearing in nonlinear analysis. It is well recognized that the GMEP includes variational inequalities, complementarity problems, saddle point problems, optimization, Nash equilibria problems, and numerous problems in physics, mechanics and economics as special cases. For details, refer to [19].

Very recently, Kazmi et al. [14] dealt with the analysis of iterative algorithms (which extend and improve the iterative methods given in [18], [23]) for solving the GMEP in real Hilbert spaces by using fixed point and resolvent methods. Further, under the θ-pseudomonotonicity and δ-Lipschitz continuity conditions for mappings, they proved the weak convergence of the sequences generated by these iterative algorithms to the solution of GMEP.

Inspired and motivated by recent research works in this filed, this paper introduces an extended iterative algorithm, called generalized mean proximal algorithm (GMPA); moreover, it presents the convergence of the iterative sequence generated by this algorithm to the solution of GMEP with θ-pseudomonotone and δ-Lipschitz continuous mappings in Hilbert spaces. The results obtained in this paper extend the recently corresponding results.

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2. Inequalities and basic concepts

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. Let K be a nonempty closed and convex set in H, and let 2^K denote the family of nonempty subsets of K. We denote by $\mathbb{R}$ the filed of real numbers and by $\mathbb{N}$ the set of all positive integers. Let $T : K \to K$ be a nonlinear mapping. The set of all fixed points of T is denoted by $\mathfrak{F}(T)$. Let $f : K \times K \to \mathbb{R}$ and $g : H \times H \to \mathbb{R}$ be nonlinear functions. We denote by $dom(f)$ (resp. $dom(g)$) the domain of f (resp. the domain of g). The interior of the domain of f is denoted by $int\ dom(f)$.
The GMEP (see, for example, [14]) for f, g and T is to find $u \in K$ such that

$$f(u, v) + \langle Tu, v - u \rangle + g(u, v) - g(u, u) \geq 0,\ \forall v \in K. \tag{2.1}$$

As special cases of the problem (2.1), we have the following problems:
1. If we consider $f(u, v) = 0$ and $g(u, v) \equiv g(v)$, $\forall u, v \in K$, then we get the following mixed variational inequality:
Find $u \in K$ such that

$$\langle Tu, v - u \rangle + g(v) - g(u) \geq 0,\ \forall v \in K, \tag{2.2}$$

which was originally considered and studied by Duvaut et al. [10].
2. If we consider $T = 0$ and $g = 0$, then we get the following equilibrium problem:
Find $u \in K$ such that

$$f(u, v) \geq 0,\ \forall v \in K, \tag{2.3}$$

which has been initialy introduced by Blum et al. [3].
3. If we take $T = 0$, then we get the following generalized equilibrium problem:
Find $u \in K$ such that

$$f(u, v) + g(u, v) - g(u, u) \geq 0,\ \forall v \in K, \tag{2.4}$$

which was introduced and studied by many authors (see, for example, [24]).
4. If we take $g = 0$, then we get the following mixed equilibrium problem:
Find $u \in K$ such that

$$f(u, v) + \langle Tu, v - u \rangle \geq 0,\ \forall v \in K, \tag{2.5}$$

which was studied by Moudafi et al. [20].
5. If $f = 0$ and $g = 0$, then we get the following classical variational inequality:
Find $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0,\ \forall v \in K, \tag{2.6}$$

which was first suggested and studied by Hartman et al. [12] in finite dimensional spaces.
Now, we present the basic concepts that will be needed in the sequel.
A multivalued mapping $A : K \to 2^K$ is said to be monotone mapping if for each $u, v \in K$ and $w \in A(u), w' \in A(v)$, we have

$$\langle w - w', u - v \rangle \geq 0.$$

A monotone mapping A is said to be maximal mapping if its graph $G(A) = \{(u, w) \in K \times K : w \in A(u)\}$ is not properly contained in the graph of any other monotone mapping $A' : K \to 2^K$ (see, for example, [25]).
For a given multivalued maximal monotone mapping $A : K \to 2^K$ and a constant

$r > 0$, the resolvent mapping (or the proximal mapping) associated with A (see, for example, [11]) is the single-valued mapping $J_r^A : K \to K$ defined by

$$J_r^A(u) = (I + rA)^{-1}(u), \tag{2.7}$$

for any $u \in K$, where I is the identity mapping.
It is known (see, for example, [4]) that a monotone mapping A is maximal if and only if its resolvent mapping J_r^A is defined everywhere.
Moreover, it is well known (see, for example, [11]) that the resolvent is firmly nonexpansive, that is,

$$\langle J_r^A(u) - J_r^A(v), u - v\rangle \geq \|J_r^A(u) - J_r^A(v)\|^2, \ \forall u, v \in K.$$

Clearly, every firmly nonexpansive is nonexpansive, that is,

$$\|J_r^A(u) - J_r^A(v)\| \leq \|u - v\|, \ \forall u, v \in K.$$

For every $r > 0$ we set $A_r = \frac{1}{r}(I - J_r^A)$. It is called the Yosida approximation. It is known (see, for example, [1], [25]) that $A_r(u) \in AJ_r^A(u)$ for all $u \in K$ and $\mathfrak{F}(J_r^A) = A^{-1}0$.
Thus the problems of existence and approximation of zeros of maximal monotone mappings can be formulated as the corresponding problems of fixed points of firmly nonexpansive mappings.
For the bifunctions $f : K \times K \to \mathbb{R}$ and $g : H \times H \to \mathbb{R}$, let us assume that the following conditions hold.
(C_1) for each fixed $v \in K$, $u \longmapsto f(u, v)$ is a proper, convex and lower semicontinuous,
(C_2) for each fixed $v \in H$, $u \longmapsto g(u, v)$ is a proper, convex and lower semicontinuous,
(C_3) for all $u \in K$, $f(u, v) + f(v, u) \leq 0$,
(C_4) there exists $u \in dom\ f(\cdot, v) \cap dom\ g(\cdot, v)$ such that, either $f(\cdot, v)$ or $g(\cdot, v)$ is continuous at u.
(C_5) g is skew symmetric, i.e.,

$$g(u, u) - g(u, v) - g(v, u) + g(v, v) \geq 0, \ \forall u, v \in H.$$

Let $\partial(f + g)(\cdot, v) = \partial f(\cdot, v) + \partial g(\cdot, v)$ denote the subdifferential of function $(f + g)(\cdot, v)$.
It is well known (see, for example, [1]) that if $(int\ dom\ f(\cdot, v)) \cap\ dom\ g(\cdot, v) \neq \phi$, then the subdifferential $\partial(f + g)(\cdot, v)$ is a maximal monotone mapping with respect to the first argument, so we denote by

$$J_r^{f(\cdot,v)g(\cdot,v)}(u) = (I + r\partial(f + g)(\cdot, v))^{-1}(u), \ r > 0, \tag{2.8}$$

the resolvent mapping associated with $\partial(f + g)(\cdot, v)$ for any fixed $v \in K$.
On the other hand, Kazmi et al. [14] used the auxiliary principle technique to characterize the resolvents (defined by this technique) as firmly nonexpansive mappings.
Related to the GMEP (2.1), Kazmi et al. [14] considered the following, called the nonlinear resolvent equations (see, for example, [21], [22]):
Find $\omega \in K$ such that

$$TJ_r^{f(\cdot,v)g(\cdot,v)}\omega + A_r^{f(\cdot,v)g(\cdot,v)}\omega = 0, \tag{2.9}$$

where $T : K \to K$ is a given mapping, $A_r^{f(\cdot,v)g(\cdot,v)} = \frac{1}{r}(I - J_r^{f(\cdot,v)g(\cdot,v)}), r > 0.$
The equations (2.9) can be written as

$$\omega - J_r^{f(\cdot,v)g(\cdot,v)}\omega + rTJ_r^{f(\cdot,v)g(\cdot,v)}\omega = 0, \tag{2.10}$$

which can be modeled by the equation

$$u = \tilde{S}u, \tag{2.11}$$

where $\tilde{S}$ is a nonlinear mapping of K into itself defined by

$$\tilde{S}u = u - \gamma[\omega - J_r^{f(\cdot,v)g(\cdot,v)}\omega + rTJ_r^{f(\cdot,v)g(\cdot,v)}\omega], \ \gamma > 0. \tag{2.12}$$

Here $\omega = u - rTu$, that is, u is a fixed point of $\tilde{S}$.
It has been shown in [14] that the GMEP (2.1) and equations of the type (2.9) have the same set of solutions.
Let $\theta : K \times K \to \mathbb{R}$ be a given real-valued function.
According to [8], [15] the mapping $T : K \to K$ is said to be
(i) θ-pseudomonotone if for all $u, v \in K$, we have

$$\langle Tu, v - u\rangle + \theta(u + v) \geq 0$$

implies

$$\langle Tv, v - u\rangle + \theta(u + v) \geq 0;$$

(ii) δ-pseudo-contractive if for all $u, v \in K$, there exists a constant $\delta > 0$ such that

$$\langle Tu - Tv, u - v\rangle \leq \delta\|u - v\|^2;$$

(iii) δ-Lipschitz continuous if for all $u, v \in K$, there exists a constant $\delta > 0$ such that

$$\|Tu - Tv\| \leq \delta\|u - v\|.$$

It should be pointed out that the class of δ-pseudo-contractive (resp. pseudo-contractive, i.e., $\delta = 1$ in (ii)) mappings is larger that of δ-Lipschitz continuous (resp. nonexpansive, i.e., Lipschitz continuous with constant $\delta = 1$) mappings, the converse, however, is false (see, for example, [9]).

3. Iterative algorithms

Let $T : K \to K$ be a self mapping and let $B = [a_{n,j}]$ be a lower triangular matrix with nonnegative entries, zero column limits, and row sums 1. For any $u_0 \in K$, the sequence $\{u_n\}$ given by

$$u_{n+1} = T\bar{u}_n, \ \bar{u}_n = \sum_{j=0}^{n} a_{n,j}u_j \tag{3.1}$$

is called the Mann iterative process (see, for example, [17]).
Especially, if $\bar{u}_n = u_n$ (i.e., $a_{n,j} = 1$ when $j = n$ and $a_{n,j} = 0$ for all $j \neq n$), then u_n is called the Picard iterative sequence.
Most of the research in this filed has focused on the following special Mann iterative sequence $\{u_n\}$:

$$u_0 \in K, \ u_{n+1} = (1 - \gamma_n)u_n + \gamma_nTu_n, \ n \geq 0, \tag{3.2}$$

where γ_n is a sequence in $(0,1)$ satisfying the conditions

$$(i)\ \gamma_0 = 1,\ (ii)\ 0 < \gamma_n < 1\ for\ n > 0,\ and\ (iii)\ \sum_{n=0}^{\infty} \gamma_n = \infty.$$

Recently, Combettes et al. [6] introduced the following so called generalized Mann iterative algorithm:

$$u_0 \in K,\ u_{n+1} = (1-\gamma_n)\bar{u}_n + \gamma_n(S_n\bar{u}_n + e_n),\ n \geq 0, \tag{3.3}$$

where $\{S_n\}$ is a sequence of mappings of K into itself, $e_n \in K$ is the error made in the computation of $S_n\bar{u}_n$, and $\gamma_n \in (0,2)$.
Motivated by (3.3), Saddeek [26] introduced the following so called mean proximal algorithm:

$$u_0 \in K,\ R_n u_{n+1} = (1-\gamma_n)R_n\bar{u}_n + \gamma_n(J_{r_n}^{R_n,A}(R_n(\bar{u}_n)) + e_n),\ n \geq 0, \tag{3.4}$$

where $J_{r_n}^{R_n,A} = (R_n + r_n A)^{-1}$ (i.e., R_n-resolvents (or generalized resolvents) of A), $A : K \to 2^K$ is a maximal monotone mapping, $\{R_n\}$ is a sequence of linear, positive definite and self-adjoint mappings of K into itself, $\{e_n\}$ is a bounded sequence of elements of K introduced to take into account possible inexact computation of $J_{r_n}^{R_n,A}(R_n(\bar{u}_n))$, $0 < r_n < \infty$, and $0 < a \leq \gamma_n \leq b < 2$.
If $R_n = I$ and $J_{r_n}^{A} = S_n$, then the algorithm (3.4) is reduced to the generalized Mann iterative algorithm (3.3).
If $A = \partial f$, where $f : K \to \mathbb{R}$ is a proper, convex and lower semicontinuous function on K, then algorithm (3.4) collapses to the following algorithm (see, for example, [13], [25]):

$$u_0 \in K,\ R_n u_{n+1} = (1-\gamma_n)R_n\bar{u}_n + \gamma_n(J_{r_n}^{R_n,f}(R_n(\bar{u}_n)) + e_n),\ n \geq 0, \tag{3.5}$$

where

$$J_{r_n}^{R_n,f}(R_n(\bar{u}_n)) = (R_n + r_n\partial f)^{-1}(R_n(\bar{u}_n)) = Arg\min_{z\in K}\ \{f(z) + \frac{1}{2r_n}\|z - \bar{u}_n\|^2\},$$

and $\{e_n\}$ is a bounded sequence of elements of K introduced to take into account possible inexact computation of $J_{r_n}^{R_n,f}(R_n(\bar{u}_n))$.
Based on the resolvent mapping (2.8) and the fixed point formulation (2.11), Kazmi et al. [14] extended the iterative methods given in ([18], [23]) and they introduced the following iterative algorithm for the GMEP (2.1):

$$u_0 \in K,\ u_{n+1} = \tilde{S}u_n,\ n \geq 0, \tag{3.6}$$

where $u \longmapsto \tilde{S}u$ is defined by (2.12).
This iterative approach requires the restrictive assumptions that T must be θ-pseudomonotone and δ- Lipschitz continuous to ensure the weak convergence.
We will now introduce and analyze the following GMPA for solving the GMEP (2.1) with $T = T_n R_n$.
Our generalized algorithm is based on the basic notions of fixed points and generalized resolvents of $f + g$.
Let $\{\gamma_n\}$ be a sequence in $(0,2)$ and $\{r_n\}$ a sequence in $(0,\infty)$. Given an initial $u_0 \in K$, define $\{u_n\}$ by

$$R_n u_{n+1} = R_n\bar{u}_n - \gamma_n[\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_n T_n R_n J_{r_n}^{R_n,f,g}\omega_n + e_n],\ n \geq 0, \tag{3.7}$$

where $\omega_n = \bar{u}_n - r_n T_n R_n \bar{u}_n$ and $J_{r_n}^{R_n,f,g} = (R_n + r_n\partial(f+g))^{-1}$.

Remark 3.1.
(i) If $r_n = r,\ \gamma_n = \gamma,\ R_n = I,\ T_n = T,\ e_n = 0$ and $\bar{u}_n = u_n$ for all $n \geq 0$, then the GMPA (3.7) reduces to algorithm 4.1 in Kazmi et al. [14].
(ii) If $T_n = 0,\ g = 0,$ and $I - J_{r_n}^{R_n,f}$ is replaced by $J_{r_n}^{R_n,f,}R_n$, then the GMPA collapses to algorithm (24) in Saddeek [26].
Henceforth, l^1 (resp. l^1_+) denotes the class of summable sequences in $\mathbb{R}$ (resp. $\mathbb{R}^+$).
We now recall the following definitions and lemma for our main results:
Definition 3.1. (see [6]) A matrix B is concentrating if every sequence $\{\mu_n\}$ in $\mathbb{R}^+$ such that

$$(\exists(\varepsilon_n) \in l^1_+)\ \mu_{n+1} \leq \bar{\mu}_n + \varepsilon_n\ \forall n \in \mathbb{N}$$

converges.
Definition 3.2. (see, for example, [7, p. 40]). A subset K of a Hilbert space H is said to be boundedly compact if every bounded sequence in K has a subsequence converging to a point in K.
Lemma 3.1. (see [16, Theorem 3.5.4]). Let $\{\mu_n\}$ be a sequence in $\mathbb{R}$. Then $\mu_n \to \mu \Rightarrow \bar{\mu}_n \to \mu$.

4. Convergence analysis

In this section, we first extend Theorem 4.1 of Kazmi et al. [14] to the case of two sequences of mappings. Then on a Hilbert space, we establish the convergence analysis of the GMPA (3.7) to the solution of the GMEP (2.1) with $T = T_nR_n$.
For this purpose, we define, as in Saddeek [26], $\|T_nu\|_{R_n^{-1}}$ by

$$\|T_nu\|_{R_n^{-1}} = \sup_{\eta \neq 0 \in K} \frac{|\langle T_nu, \eta\rangle|}{\|\eta\|_{R_n}},$$

where $\|\cdot\|^2_{R_n} = \langle \cdot,\cdot\rangle_{R_n} = \langle R_n\cdot,\cdot\rangle$.
If $\|\eta\| \leq \|\eta\|_{R_n}$, one can easily show that $\|T_nu\|_{R_n^{-1}} \leq \|T_nu\|$.
Theorem 4.1. Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $f : K \times K \to \mathbb{R}$ and $g : H \times H \to \mathbb{R}$ be nonlinear bifunctions such that $(int\ dom(f) \cap dom(g) \neq \phi)$ and the conditions $(C_1) - (C_5)$ hold. Let $\{R_n\}$ and $\{T_n\}$ be two sequences of mappings of K into itself such that for each $n \in \mathbb{N}$
(i) R_n is linear, positive definite and self adjoint,
(ii) T_nR_n is θ-pseudomonotone, where

$$\theta(u, v) = f(u, v) + g(u, v) - g(u, u)\ \forall\ u, v \in K,$$

(iii) T_nR_n is δ-pseudocontractive.
Let $\tilde{u} \in K$ be a solution of the following GMEP

$$f(u, v) + \langle T_nR_nu, v - u\rangle + g(u, v) - g(u, u) \geq 0,\ \forall v \in K. \tag{4.1}$$

Then

$$\langle u - \tilde{u}, \tilde{R_n}u - r_n(T_nR_nu - T_n\omega_n)\rangle \geq (1 - r_n\delta)\|\tilde{R_n}u\|^2_{R_n^{-1}},\ \forall u \in K, \tag{4.2}$$

where $\tilde{R_n}u = u - J_{r_n}^{R_n,f,g}(u - r_nT_nR_nu)$ and $\omega_n = u - r_nT_nR_nu$, provided $\|\eta\| \leq \|\eta\|_{R_n}\ \forall \eta \neq 0 \in K$.
Proof. Note that for each $n \in \mathbb{N}$, R_n is invertible and R_n^{-1} is linear because of condition (i).
Since for each $n \in \mathbb{N}$, the mapping T_nR_n is θ-pseudomonotone, then for all $\tilde{u}, \tilde{y} \in K$

$$\langle T_nR_n\tilde{u}, \tilde{y} - \tilde{u}\rangle + \theta(\tilde{u} + \tilde{y}) \geq 0$$

implies

$$\langle T_n R_n \tilde{y}, \tilde{y} - \tilde{u}\rangle + \theta(\tilde{u} + \tilde{y}) \geq 0,$$

where

$$\theta(\tilde{u} + \tilde{y}) = f(\tilde{u}, \tilde{y}) + g(\tilde{u}, \tilde{y}) - g(\tilde{u}, \tilde{u}),$$

i.e.,

$$f(\tilde{u}, \tilde{y}) + \langle T_n R_n \tilde{y}, \tilde{y} - \tilde{u}\rangle + g(\tilde{u}, \tilde{y}) - g(\tilde{u}, \tilde{u}) \geq 0,\ \forall \tilde{y} \in K.$$

This together with condition (C_3) implies that

$$-f(\tilde{y}, \tilde{u}) + \langle T_n R_n \tilde{y}, \tilde{y} - \tilde{u}\rangle + g(\tilde{u}, \tilde{y}) - g(\tilde{u}, \tilde{u}) \geq 0,\ \forall \tilde{y} \in K. \tag{4.3}$$

Now taking $\tilde{y} = u - \tilde{R_n} u$ in (4.3), we have

$$-f(u-\tilde{R_n}u, \tilde{u}) + \langle T_n R_n(u-\tilde{R_n}u), (u-\tilde{R_n}u)-\tilde{u}\rangle + g(\tilde{u}, u-\tilde{R_n}u) - g(\tilde{u}, \tilde{u}) \geq 0. \tag{4.4}$$

Now, let $\tilde{u}$ be a solution of GMEP (4.1), hence

$$f(\tilde{u}, v) + \langle T_n R_n \tilde{u}, v - \tilde{u}\rangle + g(\tilde{u}, v) - g(\tilde{u}, \tilde{u}) \geq 0. \tag{4.5}$$

In particular for $\tilde{u} = J_{r_n}^{R_n,f,g}u,\ u = J_{r_n}^{R_n,f,g}u - r_n T_n R_n J_{r_n}^{R_n,f,g}u$, we have

$$f(J_{r_n}^{R_n,f,g}u, v) + \frac{1}{r_n}\langle J_{r_n}^{R_n,f,g}u - u, v - J_{r_n}^{R_n,f,g}u\rangle + g(J_{r_n}^{R_n,f,g}u, v) - g(J_{r_n}^{R_n,f,g}u, J_{r_n}^{R_n,f,g}u) \geq 0. \tag{4.6}$$

Putting $v = J_{r_n}^{R_n,f,g}(u - r_n T_n R_n u) = u - \tilde{R_n} u$ in (4.5), we obtain

$$f(\tilde{u}, u - \tilde{R_n}u) + \langle T_n R_n \tilde{u}, (u - \tilde{R_n}u) - \tilde{u}\rangle + g(\tilde{u}, u - \tilde{R_n}u) - g(\tilde{u}, \tilde{u}) \geq 0. \tag{4.7}$$

Setting $u := u - r_n T_n R_n u,\ J_{r_n}^{R_n,f,g}(u) := J_{r_n}^{R_n,f,g}(u - r_n T_n R_n u) = u - \tilde{R_n} u$ and $v = \tilde{u}$ in (4.6), we obtain

$$f(u-\tilde{R_n}u, \tilde{u}) + \frac{1}{r_n}\langle u-\tilde{R_n}u-(u-r_nT_nR_nu), \tilde{u}-(u-\tilde{R_n}u)\rangle + g(u-\tilde{R_n}u, \tilde{u}) - g(u-\tilde{R_n}u, u-\tilde{R_n}u) \geq 0. \tag{4.8}$$

Now, adding (4.4) and (4.8) and using (C_5), we obtain

$$\langle \tilde{R_n}u - r_n T_n R_n u + r_n T_n R_n(u - \tilde{R_n}u), (u - \tilde{R_n}u) - \tilde{u}\rangle \geq 0. \tag{4.9}$$

This together with the δ-pseudocontractivity of T_n implies that

$$\begin{aligned}
\langle \tilde{R_n}u - r_n(T_nR_nu - T_nR_n(u - \tilde{R_n}u)), u - \tilde{u}\rangle &\geq \langle \tilde{R_n}u - r_n(T_nR_nu - T_nR_n(u - \tilde{R_n}u)), \tilde{R_n}u\rangle \\
&\geq \|\tilde{R_n}u\|_{R_n}^2 - r_n\langle T_nR_nu - T_nR_n(u - \tilde{R_n}u), \tilde{R_n}u\rangle \\
&\geq \|\tilde{R_n}u\|_{R_n^{-1}}^2 - r_n\langle T_nR_nu - T_nR_n(u - \tilde{R_n}u), \\
&u \quad -(u - \tilde{R_n}u)\rangle \\
&\geq (1 - r_n\delta)\|\tilde{R_n}u\|_{R_n^{-1}}^2,
\end{aligned}$$

the required result (4.2).

Theorem 4.2. Let all assumptions in Theorem 4.1 be satisfied, except for condition (iii) let it be replaced by the δ-Lipschitz continuous condition. Let $\tilde{u}$ be a solution of GMEP (4.1).

Then the iterative sequence $\{u_n\}$ generated by (3.7), where $e_n \in H$, $\{\|e_n\|_{R_n^{-1}}\} \in l^1$, $r_n\delta < 1$, $\lim_{n\to\infty} r_n = r > 0$, $0 < a \leq \gamma_n < \frac{2(1-r_n\delta)-a}{(1+r_n\delta)^2}$, $a \in (0, 1)$, $u_0 \in K$, and B is concentrating, converges weakly to $\tilde{u}$. If in addition *dom* $(R_n(I - r_nT_nR_n) + r_n\partial(f+g)(I - r_nT_nR_n))$ is boundedly compact, then $\{u_n\}$ converges strongly to $\tilde{u}$.

Proof. Let $\tilde{u}$ be a solution of GMEP (4.1). By using (3.7), and the convexity of $\|\cdot\|_{R_n}$, we get for $n \geq 0$

$$\begin{aligned}
\|u_{n+1}-\tilde{u}\|^2_{R_n} &= \|\bar{u}_n-\tilde{u}-\gamma_n R_n^{-1}(\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_n T_n R_n J_{r_n}^{R_n,f,g}\omega_n)+e_n)\|^2_{R_n}\\
&\leq (\|\bar{u}_n-\tilde{u}-\gamma_n R_n^{-1}(\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_n T_n R_n J_{r_n}^{R_n,f,g}\omega_n)\|_{R_n}+\gamma_n\|R_n^{-1}e_n\|_{R_n})^2\\
&= ([\|\bar{u}_n-\tilde{u}\|^2_{R_n} - 2\gamma_n\langle R_n^{-1}(\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_n T_n R_n J_{r_n}^{R_n,f,g}\omega_n), \bar{u}_n-\tilde{u}\rangle_{R_n}\\
&+ \gamma_n^2\|R_n^{-1}(\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_n T_n R_n J_{r_n}^{R_n,f,g}\omega_n)\|^2_{R_n}]^{\frac{1}{2}} + \gamma_n\|R_n^{-1}e_n\|_{R_n})^2\\
&= ([\|\bar{u}_n-\tilde{u}\|^2_{R_n} - 2\gamma_n\langle \omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_n T_n R_n J_{r_n}^{R_n,f,g}\omega_n, \bar{u}_n-\tilde{u}\rangle\\
&+ \gamma_n^2\|\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_n T_n R_n J_{r_n}^{R_n,f,g}\omega_n\|^2_{R_n^{-1}}]^{\frac{1}{2}} + \gamma_n\|e_n\|_{R_n^{-1}})^2\\
&= ([\|\bar{u}_n-\tilde{u}\|^2_{R_n} - 2\gamma_n\langle \tilde{R_n}\bar{u}_n - r_n(T_nR_n\bar{u}_n - T_nR_n(\bar{u}_n-\tilde{R_n}\bar{u}_n)), \bar{u}_n-\tilde{u}\rangle\\
&+ \gamma_n^2\|\tilde{R_n}\bar{u}_n - r_n(T_nR_n\tilde{u_n} - T_nR_n(\tilde{u_n}-\tilde{R_n}\bar{u}_n)\|^2_{R_n^{-1}}]^{\frac{1}{2}} + \gamma_n\|e_n\|_{R_n^{-1}})^2\\
&= ([\|\bar{u}_n-\tilde{u}\|^2_{R_n} - 2\gamma_n\langle \tilde{R_n}\bar{u}_n - r_n(T_nR_n\bar{u}_n - T_nR_n(\bar{u}_n-\tilde{R_n}\bar{u}_n)), \bar{u}_n-\tilde{u}\rangle\\
&+ \gamma_n^2[\|\tilde{R_n}\bar{u}_n\|^2_{R_n^{-1}} - 2r_n\langle \tilde{R_n}\bar{u}_n, T_nR_n\bar{u}_n - T_nR_n(\bar{u}_n-\tilde{R_n}\bar{u}_n)\rangle_{R_n^{-1}}\\
&+ r_n^2\|T_nR_n\tilde{u_n} - T_nR_n(\tilde{u_n}-\tilde{R_n}\bar{u}_n)\|^2_{R_n^{-1}}]^{\frac{1}{2}} + \gamma_n\|e_n\|_{R_n^{-1}})^2\\
&\leq ([\|\bar{u}_n-\tilde{u}\|^2_{R_n} - 2\gamma_n\langle \tilde{R_n}\bar{u}_n - r_n(T_nR_n\bar{u}_n - T_nR_n(\bar{u}_n-\tilde{R_n}\bar{u}_n)), \bar{u}_n-\tilde{u}\rangle\\
&+ \gamma_n^2[\|\tilde{R_n}\bar{u}_n\|^2_{R_n^{-1}} + 2r_n\|\tilde{R_n}\bar{u}_n\|_{R_n^{-1}}\|T_nR_n\bar{u}_n - T_nR_n(\bar{u}_n-\tilde{R_n}\bar{u}_n)\|_{R_n^{-1}}\\
&+ r_n^2\|T_nR_n\tilde{u_n} - T_nR_n(\tilde{u_n}-\tilde{R_n}\bar{u}_n)\|^2_{R_n^{-1}}]^{\frac{1}{2}} + \gamma_n\|e_n\|_{R_n^{-1}})^2. \qquad (4.10)
\end{aligned}$$

As any δ-Lipschitz continuous mapping is δ -pseudocontractive, it results by (4.2), and (4.10) that

$$\begin{aligned}
\|u_{n+1}-\tilde{u}\|^2_{R_n} &\leq (\|\bar{u}_n-\tilde{u}\|_{R_n} - \gamma_n[2(1-r_n\delta)-\gamma_n(1+r_n\delta)^2]\|\tilde{R_n}\bar{u}_n\|^2_{R_n^{-1}}]^{\frac{1}{2}}\\
&+ \gamma_n\|e_n\|_{R_n^{-1}})^2. \qquad (4.11)
\end{aligned}$$

This, together with (3.1) (taking into account $\gamma_n < \frac{2(1-r_n\delta)}{(1+r_n\delta)^2}$ and $r_n\delta < 1$, implies that

$$\|u_{n+1}-\tilde{u}\|_{R_n} \leq \sum_{j=0}^{n} a_{n,j}\|u_j-\tilde{u}\|_{R_n} + 2\|e_n\|_{R_n^{-1}}. \qquad (4.12)$$

Since B is concentrating and $\{\|e_n\|_{R_n^{-1}} \in l^1\}$, it results that $\lim_{n\to\infty}\|u_n-\tilde{u}\|_{R_n} = \rho(\tilde{u})$.

Then, it follows from Lemma 3.1 and (4.12) that $\lim_{n\to\infty}\|\bar{u}_n-\tilde{u}\|_{R_n} = \rho(\tilde{u}) < \infty$.
Moreover, the sequence $\{\epsilon_n\}$ defined by $\epsilon_n = \sigma\|e_n\|_{R_n^{-1}} + 4\|e_n\|^2_{R_n^{-1}}$, where $\sigma = 4\sup_{n\geq 0}\|u_{n+1}-\tilde{u}\|_{R_n} < \infty$ lies in l^1.
Using the convexity of $\|\cdot\|^2_{R_n}$ and the restrictions on γ_n and r_n, from (4.11) we obtain

$$\begin{aligned}
\|\tilde{R_n}\bar{u}_n\|^2_{R_n^{-1}} &\leq \frac{1}{a^2}[\|\bar{u}_n-\tilde{u}\|^2_{R_n} - [\|u_{n+1}-\tilde{u}\|_{R_n} - \gamma_n\|e_n\|_{R_n^{-1}}]^2] \qquad (4.13)\\
&\leq \frac{1}{a^2}[\|\bar{u}_n-\tilde{u}\|^2_{R_n} - \|u_{n+1}-\tilde{u}\|^2_{R_n} + 2\gamma_n\|u_{n+1}-\tilde{u}\|_{R_n}\|e_n\|_{R_n^{-1}} + \gamma_n^2\|e_n\|^2_{R_n^{-1}}]\\
&\leq \frac{1}{a^2}[\sum_{j=0}^{n} a_{n,j}\|u_j-\tilde{u}\|^2_{R_n} - \|u_{n+1}-\tilde{u}\|^2_{R_n} + \epsilon_n].
\end{aligned}$$

Since $\{\|\bar{u}_n - \tilde{u}\|_{R_n}^2\}$ converges, then it follows from Lemma 3.1 that

$$\sum_{j=0}^{n} a_{n,j}\|u_j - \tilde{u}\|_{R_n}^2 - \|u_{n+1} - \tilde{u}\|_{R_n}^2 \to 0.$$

It therefore follows from (4.13) that

$$\lim_{n\to\infty} \|\tilde{R_n}\bar{u}_n\|_{R_n^{-1}} = 0. \tag{4.14}$$

Further, since T_nR_n is δ-Lipschitz continuous and

$$\begin{aligned}
\|u_{n+1} - \bar{u}_n\|_{R_n} &= \gamma_n\|R_n^{-1}(\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_nT_nR_nJ_{r_n}^{R_n,f,g}\omega_n + e_n)\|_{R_n}\\
&\le 2(\|R_n^{-1}(\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_nT_nR_nJ_{r_n}^{R_n,f,g}\omega_n)\|_{R_n} + \|R_n^{-1}e_n\|_{R_n})\\
&= 2(\|\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_nT_nR_nJ_{r_n}^{R_n,f,g}\omega_n\|_{R_n^{-1}}\|_{R_n^{-1}} + \|e_n\|_{R_n^{-1}})\\
&= 2(\|\omega_n - J_{r_n}^{R_n,f,g}\omega_n + r_nT_nR_nJ_{r_n}^{R_n,f,g}\omega_n\|_{R_n^{-1}} + \|e_n\|_{R_n^{-1}})\\
&= 2(\|\tilde{R_n}\bar{u}_n + r_n(T_nR_nJ_{r_n}^{R_n,f,g}\omega_n - T_nR_n\bar{u}_n)\|_{R_n^{-1}} + \|e_n\|_{R_n^{-1}})\\
&\le 2(\|\tilde{R_n}\bar{u}_n\|_{R_n^{-1}} + r_n\|\tilde{R_n}\bar{u}_n\|_{R_n^{-1}} + \|e_n\|_{R_n^{-1}}),
\end{aligned}$$

then from (4.14) we obtain

$$\lim_{n\to\infty} \|u_{n+1} - \bar{u}_n\|_{R_n} = 0. \tag{4.15}$$

As $\{u_n\}$ is bounded, it results that there exists a subsequence of $\{u_n\}$, denoted by $\{u_{n_k}\}$, which converges to some $\tilde{v} \in K$.
Following the arguments in the proof of ([2, Corollary 6.1]), we can show that

$$\lim_{k\to\infty} \tilde{R}_{n_k}u_{n_k} = 0. \tag{4.16}$$

Therefore $\tilde{v} \in \mathfrak{F}(J_{r_n}^{R_n,f,g}(I - r_nT_nR_n))$ (i.e., $\tilde{v}$ is a solution of GMEP (4.1) (see [14, Lemma 3.3]).
Suppose there are two weak limit points of $\{\bar{u}_n\}$, say $\tilde{v}$ and $\tilde{\omega}$. As above, we have $\tilde{v}$ and $\tilde{\omega}$ are solutions of GMEP (4.1) and that

$$\lim_{n\to\infty} \|\bar{v}_n - \tilde{v}\|_{R_n} = \rho(\tilde{v}),\ \lim_{n\to\infty} \|\bar{v}_n - \tilde{\omega}\|_{R_n} = \rho(\tilde{\omega}). \tag{4.17}$$

Similar to the proof of [6, Theorem 3.5], we can show that the sequences $\{\|\bar{v}_n\|_{R_n}^2 - 2\langle\bar{v}_n, \tilde{v}\rangle_{R_n}\}$ and $\{\|\bar{v}_n\|_{R_n}^2 - 2\langle\bar{v}_n, \tilde{\omega}\rangle_{R_n}\}$ converge and therefore so does $\{\langle\bar{v}_n, \tilde{v} - \tilde{\omega}\rangle_{R_n}\}$.
This and the weak convergence of $\{\bar{v}_n\}$, imply that $\|\tilde{v} - \tilde{\omega}\|_{R_n}^2 = 0$ and, hence, $\tilde{v} = \tilde{\omega}$.
Thus, all the weak limit points of $\{\bar{v}_n\}$ coincide and hence, the sequence $\{\bar{v}_n\}$ converges weakly to the solution $\tilde{u}$ of GMEP (4.1). This together with (4.15) yields

$$u_n \rightharpoonup \tilde{u}. \tag{4.18}$$

Since *dom* $(R_n(I - r_nT_nR_n) + r_n\partial(f + g)(I - r_nT_nR_n))$ is boundedly compact, it follows by (4.16) and [5, Theorem 6.9] that the set of strong cluster points of the sequence $\{\bar{u}_n\}$ is nonempty.
The rest of the argument now follows exactly as in the proof of second assertion of Theorem 3.5 in [6] to yield that $\{u_n\}$ converges strongly to $\tilde{u}$, completing the proof of the Theorem.
Remark 4.1. As was mentioned in Remark 3.1, our results improve and extend many important recent results (for example, [6, 14,18 23,26]).

5. Application to minimization problem

We consider the problem of finding a minimizer of the sum of two proper convex lower semicontinuous bifunctions.

Theorem 5.1. Let $f : K \times K \to \mathbb{R}$ and $g : H \times H \to \mathbb{R}$ be two proper convex lower semicontinuous bifunctions such that $(int\ dom(f) \cap dom(g) \neq \phi)$ and the conditions $(C_3) - (C_5)$ hold. Let $\{R_n\}$ be a sequence of linear, positive definite and self adjoint mappings of K into itself. Let $u_0 \in K$ and let $\{u_n\}$ be a sequence generated by

$$R_n u_{n+1} = R_n \bar{u}_n - \gamma_n[\bar{u}_n - Arg \min_{v \in K} \{(f+g)(\bar{u}_n, v) + \frac{1}{2r_n}\|v - \bar{u}_n\|^2_{R_n}\} + e_n], \ n \geq 0, \tag{5.1}$$

where $r_n \subset (0, \infty)$, $\gamma_n \subset (0, 2)$, and $\{\|e_n\|_{R_n^{-1}}\}$ satisfy $\lim_{n\to\infty} r_n = r > 0$, $0 < a \leq \gamma_n < 2$ and $\{\|e_n\|_{R_n^{-1}}\} \in l^1$. If B is concentrating, dom $(R_n + r_n \partial(f+g))$ is boundedly compact and $(\partial(f+g))^{-1}(0) \neq \phi$, then $\{u_n\}$ converges strongly to $v \in K$, which is the minimizer of $f + g$.

Proof. Putting $T_n = 0$, $n \in \mathbb{N}$ and taking

$$J^{R_n,f,g}_{r_n}(\bar{u}_n) = Arg \min\nolimits_{v \in K} \{(f+g)(\bar{u}_n, v) + \tfrac{1}{2r_n}\|v - \bar{u}_n\|^2_{R_n}\}$$

in Theorem 4.2. The desired conclusion follows immediately.

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