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SOME RELATED FIXED POINT THEOREMS ON METRIC SPACES

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Abstract. We give some generalizations of the B. Fischer fixed point theorem (B. Fisher [7]) for two mappings on metric spaces by using a function α defined from $[0,+\infty[$ into $[0,1[$ and satisfies $\limsup_{t\to t_o^+}\alpha(t)<1$, for all $t_o\geq 0$. We study also the existence of solutions for a functional equation arising in dynamic programming.

1. Introduction

In 1981, B. Fisher presented the following related fixed point theorem on complete metric spaces:

Theorem 1.1 (B. Fischer [7]). *Let (X,d) and (Y,δ) two metric spaces; we assume that (X,d) is complete. Let $T:X\to Y$ and $S:Y\to X$ two mappings such that, for all $(x,y)\in X\times Y$,*

$$\begin{cases} d(Sy,STx)\leqslant c.Max\{d(x,Sy);\delta(y,Tx):d(x,STx)\}\\ \delta(Tx,TSy)\leqslant c.Max\{d(x,Sy);\delta(y,Tx);\delta(y,TSy)\},\end{cases}$$

where $c\in[0,1[$. Then there exists a unique pair $(x^*,y^*)\in X\times Y$ such that $Tx^*=y^*$ and $Sy^*=x^*$. And then $STx^*=x^*$ and $TSy^*=y^*$.

Recently, K. Chaira and El-Miloudi Marhrani proved the following results:

Theorem 1.2 (See [5]). *Let (X,d) and (Y,δ) two metric spaces; we assume that (X,d) is complete. Let $T:X\to Y$ and $S:Y\to X$ two mappings such that, for all $(x,y)\in X\times Y$,*

$$\begin{cases} d(Sy,STx)\leqslant \alpha(\delta(y,Tx))Max\{d(x,Sy);\delta(y,Tx)\}+\beta(\delta(y,Tx))d(x,STx)\\ \delta(Tx,TSy)\leqslant \alpha(d(x,Sy))Max\,\{d(x,Sy);\delta(y,Tx)\}+\beta(d(x,Sy))\delta(y,TSy),\end{cases}$$

where $\alpha,\beta:[0,+\infty[\to[0,1[$ are two functions satisfying

$$\limsup_{t\to t_0^+}(\alpha(t)+\beta(t))<1,\qquad \textit{for all } t_0\in[0,+\infty[.$$

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Then there exists a unique pair $(x^*, y^*) \in X \times Y$ *such that* $Tx^* = y^*$ *and* $Sy^* = x^*$; *and then,* $STx^* = x^*$ *and* $TSy^* = y^*$.

Theorem 1.3 (See [10])**.** *Let* X *be a non-empty set,* d *and* δ *two metrics on* X; *and* $T : X \to X$ *a mapping such that:*

(1) (X, d, δ) *is an (M)-space*
(2) *For all* $x, y \in X$, *one of the conditions:*
 (i) $d(x, Ty) \le \delta(x, y)$
 (ii) $\delta(x, Ty) \le d(x, y)$
 implies

$$\begin{cases} d(Tx, Ty) \le \alpha(\delta(x, y))\delta(x, y) \\ \delta(Tx, Ty) \le \alpha(d(x, y))d(x, y) \end{cases}$$

Then T *has a unique fixed point in* X.

In the present article, we give others generalizations of the B. Fischer results for related fixed point theorem on metric spaces. And, we study the existence of a solution for some functional equations arising in dynamic programming.

2. Main results

Let α be a function from $[0, +\infty[$ into $[0, 1[$ such that $\limsup_{s \to s_0^+} \alpha(s) < 1$ for all $s_0 \in [0, +\infty[$.

Theorem 2.1. *Let* (X, d) *and* (Y, δ) *be two metric spaces such that* (X, d) *is complete. Let* $T : X \to Y$ *and* $S : Y \to X$ *be two mappings such that for each* $(x, y) \in X \times Y$, *one of the conditions:*

(a): $d(x, STx) \le d(x, Sy)$
(b): $\delta(y, TSy) \le \delta(y, Tx)$

implies

$$\begin{cases} \delta(Tx, TSy) \le \alpha(d(x, Sy)) \max\{\delta(y, TSy), d(x, Sy), \delta(y, Tx)\} \\ d(Sy, STx) \le \alpha(\delta(y, Tx)) \max\{d(x, STx), \delta(y, Tx), d(x, Sy)\} \end{cases}$$

Then, there exists a unique pair $(x^*, y^*) \in X \times Y$ *such that* $Tx^* = y^*$ *and* $Sy^* = x^*$; *and then* $STx^* = x^*$ *and* $TSy^* = y^*$.

Proof. First step. Let $x_0 \in X$; for all $n \in \mathbb{N}$, we define the sequences $(x_n)_n$ and $(y_n)_n$ by $y_n = Tx_n$ and $x_{n+1} = Sy_n$. For all $n \in \mathbb{N}$, we have:

$$d(x_n, STx_n) = d(x_n, x_{n+1}) \le d(x_n, Sy_n)$$

thus

$$\begin{aligned} \delta(y_n, y_{n+1}) &= \delta(Tx_n, TSy_n) \\ &\le \alpha(d(x_n, Sy_n)) \max(\delta(y_n, TSy_n), \delta(y_n, Tx_n), d(x_n, Sy_n)) \\ &\le \alpha(d(x_n, x_{n+1})) \max(\delta(y_n, y_{n+1}), d(y_n, y_n), d(x_n, x_{n+1})) \end{aligned}$$

which gives

$$\delta(y_n, y_{n+1}) \le \alpha(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \tag{2.1}$$

For $x = x_{n+1}$ and $y = y_n$, we obtain

$$\delta(y_n, TSy_n) = \delta(y_n, y_{n+1}) \le \delta(y_n, Tx_{n+1})$$

as above, we obtain

$$d(x_{n+1}, x_{n+2}) \le \alpha(\delta(y_n, y_{n+1}))\delta(y_n, y_{n+1}) \tag{2.2}$$

It follows from (2.1) and (2.2) that the sequences $(d(x_n, x_{n+1}))_n$ and $(\delta(y_n, y_{n+1}))_n$ are decreasing and then convergent. By the hypothesis on α, we can deduce that there exists $k \in [0, 1[$ such that

$$\begin{cases} d(x_{n+1}, x_{n+2}) & \leq kd(x_n, x_{n+1}) \\ \delta(y_{n+1}, y_{n+2}) & \leq k\delta(y_n, y_{n+1}) \end{cases}$$

for large integers. Therefore $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences; then there exists $x^* \in X$ such that $\lim_{n\to+\infty} d(x_n, x^*) = 0$.
Second step. Let $y^* = Tx^*$; if $\lim_n \delta(y_n, y^*) \neq 0$, we obtain

$$\delta(y_n, TSy_n) \leq \delta(y_n, Tx^*)), \quad \text{for large integers } n;$$

which implies

$$\begin{aligned} \delta(Tx^*, TSy_n) & \leq \alpha(d(x^*, Sy_n) \max\{\delta(y_n, Tx^*), d(x^*, Sy_n)\} \\ & \leq h\delta(y_n, y^*) \end{aligned}$$

for some $h \in [0, 1[$ and large integers n; which is a contradiction. And then $\lim_n \delta(y_n, y^*) = 0$.
We have $x^* = Sy^*$. For this, assume that $x^* \neq Sy^*$. We have

$$d(x_n, TSx_n) \leq d(Sy^*, x_n), \quad \text{for large } n;$$

which implies

$$d(Sy^*, x_{n+1}) \leq \alpha(\delta(y^*, y_n) \max\{d(x_n, x_{n+1}), d(x_n, Sy^*), \delta(y^*, y_n)\}$$

for large integers. And since $\limsup_n \alpha(\delta(y_n, y^*)) < 1$, there exists $h \in [0, 1[$ such that

$$d(Sy^*, x^*) \leq hd(x^*, Sy^*);$$

which leads to $x^* = Sy^*$.
Third step. Uniqueness of x^* and y^*.
Assume that there exists $x \in X - \{x^*\}$ such that $STx = x$. We have $d(x, STx) \leq d(x, Sy^*)$; and then

$$\begin{aligned} \delta(Tx, y^*) & = \delta(Tx, TSy^*) \\ & \leq \alpha(d(x, Sy^*) \max\{\delta(y^*, TSy^*), \delta(y^*, Tx), d(x, Sy^*)\} \\ & < d(x, Sy^*) \end{aligned}$$

and

$$\begin{aligned} d(Sy^*, x) & = d(Sy^*, STx) \\ & \leq \alpha(\delta(y^*, Tx) \max\{d(x, STx), d(x, Sy^*), \delta(y^*, Tx)\} \\ & < \delta(y^*, Tx) \end{aligned}$$

Thus, $x = x^*$.
In the same way, if there exists $y \in Y - \{y^*\}$ such that $TSy = y$ and since $0 = \delta(y, TSy) \leq \delta(y, Tx^*)$, we obtain

$$\begin{cases} \delta(Tx^*, y) = \delta(Tx^*, TSy) < d(x^*, Sy) \\ d(Sy, x^*) = d(Sy, STx^*) < \delta(y, Tx^*) \end{cases}$$

and we conclude that $y = Tx^* = y^*$.

Remark. *The theorem remain valid if we permute $\alpha(d(x, Sy))$ and $\alpha(\delta(y, Tx))$.*

Theorem 2.2. *Let (X, d) and (Y, δ) be two metric spaces such that (X, d) is complete; and let $T : X \to Y$, $S : Y \to X$ two mappings such that for all $(x, y) \in X \times Y$, one of the condition*

(a): $d(x, STx) \leq d(x, Sy)$
(b): $\delta(y, TSy) \leq \delta(y, Tx)$

implies

$$\begin{cases} \delta(Tx, TSy) \leq \alpha(\delta(y, Tx)) \max\{d(x, STx), \delta(y, TSy), d(x, Sy)\} \\ d(Sy, STx) \leq \alpha(d(x, Sy)) \max\{d(x, STx), \delta(y, TSy), \delta(y, Tx)\} \end{cases}$$

Then, there exists a unique pair $(x^*, y^*) \in X \times Y$ such that $Tx^* = y^*$ and $Sy^* = x^*$. And then $STx^* = x^*$ and $TSy^* = y^*$.

Proof. First step: For $x_0 \in X$, we define the sequences $(x_n)_n$ and $(y_n)_n$ by $y_n = Tx_n$ and $x_{n+1} = Sy_n$, for all $n \in I\!N$.
For $x = x_n$ and $y = y_n$, we have:

$$d(x_n, STx_n) = d(x_n, x_{n+1}) \leq \max\{d(x_n, Sy_n), \delta(y_n, Tx_n)\}$$

Then

$$\delta(Tx_n, TSy_n) \leq \alpha(\delta(y_n, Tx_n)) \max\{d(x_n, STx_n), \delta(y_n, TSy_n), d(x_n, Sy_n)\}$$

which implies

$$\delta(y_n, y_{n+1}) \leq \alpha(0) \max\{d(x_n, x_{n+1}), \delta(y_n, y_{n+1}), d(x_n, x_{n+1})\}$$

Therefore

$$\delta(y_n, y_{n+1}) \leq \alpha(0) d(x_n, x_{n+1})$$

For $x = x_{n+1}$ and $y = y_n$, we obtain:

$$\delta(y_n, TSy_n) = \delta(y_n, y_{n+1}) \leq \max\{d(x_{n+1}, Sy_n), \delta(y_n, Tx_{n+1})\}$$

which implies

$$d(x_{n+1}, x_{n+2}) \leq \alpha(d(x_{n+1}, Sy_n)) \max\{d(x_{n+1}, x_{n+2}), \delta(y_n, y_{n+1}), \delta(y_n, y_{n+1})\}$$

and then

$$d(x_{n+1}, x_{n+2}) \leq \alpha(0) \delta(y_n, y_{n+1}).$$

For $k = (\alpha(0))^2$, we obtain

$$\begin{cases} d(x_{n+1}, x_{n+2}) \leq k d(x_n, x_{n+1}) \\ \delta(y_{n+1}, y_{n+2}) \leq k \delta(y_n, y_{n+1}) \end{cases}$$

which shows that $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences. And, since (X, d) is complete, there exists $x^* \in X$ such that $\lim_{n \to +\infty} d(x_n, x^*) = 0$.
Second step. Let $y^* = Tx^*$; and assume that $\lim_n \delta(y_n, y^*) \neq 0$.

$$\delta(y_n, TSy_n) \leq \delta(y_n, Tx^*), \quad \text{for large integers } n.$$

$$\delta(Tx^*, TSy_n) \leq \alpha(\delta(y_n, Tx^*) \max\{d(x^*, STx^*), \delta(y_n, TSy_n), d(x^*, Sy_n)\} \quad (2.3)$$

$$d(Sy_n, STx^*) \leq \alpha(d(x^*, Sy_n) \max\{d(x^*, STx^*), \delta(y_n, TSy_n), \delta(y_n, Tx^*)\} \quad (2.4)$$

From (2.4), we obtain

$$d(x_{n+1}, Sy^*) \leq \alpha(d(x^*, x_{n+1}) \max\{d(x^*, STx^*), \delta(y_n, y^*)\}$$

for large integers.
Using the fact that $(d(x^*, x_{n+1}))_n$ is convergent, there exists $k \in [0, 1[$ such that

$$d(x_{n+1}, Sy^*) \leq k \max\{d(x^*, Sy^*), \delta(y_n, y^*)\}$$

which leads to

$$d(x_{n+1}, Sy^*) \leq k\delta(y_n, y^*)) \quad (2.5)$$

for large integers.
From (2.3), we obtain:

$$\delta(y^*, TSy_n) \leq \alpha(\delta(y_n, Tx^*)) \max\{d(x^*, STx^*), \delta(y_n, TSy_n), d(x^*, Sy_n)\}$$

we can deduce

$$\delta(y^*, y_{n+1}) \leq \alpha(\delta(y_n, y^*)) \max\{d(x^*, STx^*), \delta(y_n, y_{n+1}), d(x^*, x_{n+1})\} \qquad (2.6)$$

Using (2.5) and (2.6), we obtain

$$d(x_{n+1}, Sy^*) \leq k\delta(y_n, y^*)) \leq kd(x^*, Sy^*)$$

and then

$$d(x^*, Sy^*) \leq kd(x^*, Sy^*), \quad \text{for large intergers}$$

which gives $Sy^* = x^*$; and consequently $TSy^* = y^*$ and $STx^* = x^*$.

With the same arguments as in the proof of the theorem 2.2, we obtain:

Theorem 2.3. *Let (X, d) and (Y, δ) be two metric spaces such that (X, d) is complete; and let $T : X \to Y$, $S : Y \to X$ two mappings such that for all $(x, y) \in X \times Y$, one of the conditions:*

(a): $d(x, STx) \leq d(x, Sy)$
(b): $\delta(y, TSy) \leq \delta(y, Tx)$

implies

$$\begin{cases} \delta(Tx, TSy) \leq \alpha(\delta(y, Tx)) \max\{d(x, STx), \delta(y, TSy), \delta(y, Tx)\} \\ d(Sy, STx) \leq \alpha(d(x, Sy)) \max\{d(x, STx), \delta(y, TSy), d(x, Sy)\} \end{cases}$$

Then, there exists a unique pair $(x^*, y^*) \in X \times Y$ such that $Tx^* = y^*$ and $Sy^* = x^*$. And then $STx^* = x^*$ and $TSy^* = y^*$.

Corollary 2.4. *Let (X, d) and (Y, δ) be metric spaces such that (X, d) is complete, $T : X \to Y$ and $S : Y \to X$ two mappings. If there exists $r \in [0, 1[$ such that, for all $(x, y) \in X \times Y$, one of the conditions:*

(a): $d(x, STx) \leq d(x, Sy)$
(b): $\delta(y, TSy) \leq \delta(y, Tx)$

implies

$$\begin{cases} \delta(Tx, TSy) \leq r.max\{d(x, STx); d(x, Sy); \delta(y, Tx))\} \\ d(Sy, STx) \leq r.max\{\delta(y, TSy); \delta(y, Tx); d(x, Sy)\} \end{cases},$$

then, there exists a unique pair $(x^*, y^*) \in X \times Y$ such that $Tx^* = y^*$ and $Sy^* = x^*$. Consequently, $STx^* = x^*$ and $TSy^* = y^*$.

Corollary 2.5. *Let (X, d) be a complete metric space, δ a metric on X and $T : X \to X$ a mapping such that for each $(x, y) \in X^2$, one of the conditions:*

(a): $d(x, T^2x) \leq d(x, Ty)$
(b): $\delta(y, T^2y) \leq \delta(y, Tx)$

implies

$$\begin{cases} d(Tx, T^2y) \leq \alpha(d(y, Tx)) \max\{d(x, T^2x), d(y, T^2y), \delta(y, Tx)\} \\ d(Ty, T^2x) \leq \alpha(d(x, Ty)) \max\{d(x, T^2x), d(y, T^2y), d(x, Ty)\} \end{cases}$$

Then, there exists a unique element $x^* \in X$ such that $Tx^* = x^*$.

Proof. If we take $S = T$ in theorem2.2, we obtain a pair $(x^*, y^*) \in X \times X$ such that $Tx^* = y^*$, $Ty^* = x^*$ and then $T^2x^* = x^*$.
On the other hand, we have

$$d(x^*, T^2x^*) = 0 \leq d(x^*, Tx^*);$$

then

$$\begin{aligned} d(Tx^*, T^2x^*) &= d(Tx^*, x^*) \\ &\leq \alpha(d(x^*, Tx^*)) \max\{d(x^*, T^2x^*), \delta(x^*, T^2x^*), d(x^*, Tx^*)\} \\ &\leq \alpha(d(x^*, Tx^*)) d(x^*, Tx^*) \end{aligned}$$

which gives $Tx^* = x^*$.

Example 2.6. *Let $X = [0, 1]$ and define T and S by $Tx = \frac{1}{3}x^2$ and $Sy = 0$, for all $x, y \in X$. Let d the usual metric on X and α the function defined on $[0, +\infty[$ by $\alpha(t) = \frac{2}{3}e^{-t}$.*
For $\delta = d$, we obtain:

$$\delta(Tx, TSy) = \delta(\frac{1}{3}x^2, 0) = \frac{1}{3}x^2$$

and

$$\alpha(\delta(y, Tx)) \max\{d(x, STx), \delta(y, TSy), d(x, Sy\} = \frac{2}{3}e^{-|y-\frac{1}{3}x^2|} \max\{x, y\}$$

If $x \leq y$, we obtain:

$$\frac{1}{3}x^2 e^{-\frac{1}{3}x^2} \leq \frac{1}{3}y^2 e^{-\frac{1}{3}y^2} \leq \frac{2}{3}ye^{-y},$$

for all $y \in X$. And then

$$\delta(Tx, TSy) = \frac{1}{3}x^2 \leq \frac{2}{3}ye^{-(y-\frac{1}{3}x^2} = \frac{2}{3}e^{-|y-\frac{1}{3}x^2|} \max\{x, y\}$$

If $y \leq x$; we have

$$x^2 e^{-\frac{1}{3}x^2} \leq 2xe^{-x} \leq 2xe^{-y}$$

and then

$$\frac{1}{3}x^2 \leq \frac{2}{3}xe^{-(y-\frac{1}{3}x^2)} = \frac{2}{3}e^{-|y-\frac{1}{3}x^2|} \max\{x, y\}$$

The second inequality is obvious since $d(Sy, STx) = 0$.
Note that TS and ST have a unique fixed point $x^* = 0$.

Example 2.7. *Let $X = [0, 1] \cup \{5\}$ with the usual metric; and S, T two mapping on X defined by*

$$Tx = \begin{cases} \frac{x+1}{2} & \text{if} \quad x \in [0, 1] \\ \frac{7}{8} & \text{if} \quad x = 5 \end{cases}$$

$$Sy = \begin{cases} \frac{y}{2} & \text{if} \quad x \in [0, 1] \\ \frac{1}{2} & \text{if} \quad y = 5 \end{cases}$$

Define α on $[0, +\infty[$ by

$$\alpha(t) = \begin{cases} 0 & \text{if} \quad t = \frac{33}{8} \\ \frac{1}{2}\left(1 + \frac{1}{2}\sin^2(t)\right) & \textit{otherwise} \end{cases}$$

We have $\delta(y, Tx) = \frac{33}{8}$ if and only if $(x, y) = (5, 5)$ or $(x, y) = (\frac{3}{4}, 5)$ and $d(x, Sy) \neq \frac{33}{8}$, for all $(x, y) \in X \times X$.
For $x = y = 5$, we have

$$d(x, STx) = \frac{73}{16} \quad \text{and} \quad d(x, Sy) = \frac{9}{2}$$

$$d(y,TSy) = \frac{17}{4} \quad \text{and} \quad d(y,Tx) = \frac{33}{8}$$

Note that

$$d(x,STx) > d(x,Sy) \quad \textit{and} \quad d(y,TSy) > d(y,Tx)$$

For $x = \frac{3}{4}$ *and* $y = 5$*, we have*

$$d(x,STx) = \frac{5}{16} > d(x,Sy) = \frac{1}{4}$$

and

$$\delta(y,TSy) = \frac{17}{4} > \delta(y,Tx) = \frac{33}{8}$$

For the other cases, we have

$$\min\{\alpha(d(y,Tx)),\alpha(d(x,Sy)\} \geq \frac{1}{2}$$

Case of the theorem2.1:
If $x, y \in [0,1]$*, we have*

$$\begin{cases} |x - \frac{y}{2}| & \leq \max\{|y - \frac{y+2}{4}|; |x - \frac{y}{2}|; |y - \frac{x+1}{2}|\} \\ |y - \frac{x+1}{2}| & \leq \max\{|x - \frac{x+1}{4}|; |x - \frac{y}{2}|; |y - \frac{x+1}{2}|\} \end{cases}$$

For $x = 5$ *and* $y \in [0,1]$*, we have*

$$\begin{cases} |\frac{3}{4} - \frac{y}{2}| & \leq \max\{|y - \frac{y+2}{4}|; |5 - \frac{y}{2}|; |y - \frac{7}{8}|\} \\ |y - \frac{7}{8}| & \leq \max\{|5 - ST5|; |5 - \frac{y}{2}|; |y - \frac{7}{8}|\} \end{cases}$$

For $x \in [0,1] - \{\frac{3}{4}\}$ *and* $y = 5$*, we have*

$$\begin{cases} |\frac{x+1}{2} - \frac{3}{4}| & \leq \max\{|5 - ST5|; |x - \frac{1}{2}|; |5 - \frac{x+1}{2}|\} \\ |\frac{1}{2} - \frac{x+1}{4}| & \leq \max\{|x - \frac{x+1}{4}|; |x - \frac{1}{2}|; |5 - \frac{x+1}{2}|\} \end{cases}$$

Then for all $(x,y) \in X^2 - \{(5,5),(\frac{3}{4},5)\}$*, we have*

$$\begin{cases} \delta(Tx,TSy) & \leq \alpha(d(x,Sy))\max\{\delta(y,TSy), d(x,Sy), \delta(y,Tx)\} \\ d(Sy,STx) & \leq \alpha(\delta(y,Tx))\max\{d(x,STx), \delta(y,Tx), d(x,Sy)\} \end{cases}$$

In the case of theorem2.2, we have:

$$\begin{cases} |x - \frac{y}{2}| & \leq \max\{|x - \frac{x+1}{4}|, |y - \frac{y+2}{4}|, |x - \frac{y}{2}|\} \\ |y - \frac{x+1}{2}| & \leq \max\{|x - \frac{x+1}{4}|, |y - \frac{y+2}{4}|, |y - \frac{x+1}{2}|\} \end{cases}$$

for all $(x,y) \in [0,1] \times [0,1]$.
For $x = 5$ *and* $y \in [0,1]$*, we have* $|\frac{3}{4} - \frac{y}{2}| \leq |5 - \frac{y}{2}|$*; therefore*

$$\begin{cases} |\frac{3}{4} - \frac{y}{2}| & \leq \max\{|5 - ST5|, |y - TSy|, |5 - Sy|\} \\ |y - \frac{7}{8}| & \leq \max\{|x - \frac{x+1}{4}|, |y - \frac{y+2}{4}|, |y - \frac{7}{8}|\} \end{cases}$$

For $x \in [0,1] - \{\frac{3}{4}\}$ *and* $y = 5$*, we have*

$$|\frac{x+1}{2} - \frac{3}{4}| \leq |x - \frac{1}{2}| \quad \text{and} \quad |\frac{1}{2} - \frac{x+1}{4}| \leq |x - \frac{x+1}{4}|;$$

therefore

$$\begin{cases} |\frac{x+1}{2} - \frac{3}{4}| & \leq \max\{|x - Tx|, |5 - TS5|, |x - \frac{1}{2}|\} \\ |\frac{1}{2} - \frac{x+1}{4}| & \leq \max\{|x - \frac{x+1}{4}|, |5 - TS5|, |y - \frac{x+1}{2}|\} \end{cases}$$

which leads to

$$\begin{cases} \delta(Tx, TSy) & \leq \alpha(\delta(y, Tx)) \max\{d(x, STx), \delta(y, TSy), d(x, Sy)\} \\ d(Sy, STx) & \leq \alpha(d(x, Sy)) \max\{d(x, STx), \delta(y, TSy), \delta(y, Tx)\} \end{cases}$$

for all $(x, y) \in X \times X - \{(5,5), (\frac{3}{4}, 5)\}$.
We conclude that the hypothesis of the theorem 2.1 and theorem 2.2 are satisfied. And we have $T(\frac{1}{3}) = \frac{2}{3}$, $S(\frac{2}{3}) = \frac{1}{3}$, $ST(\frac{1}{3}) = \frac{1}{3}$ *and* $TS(\frac{2}{3}) = \frac{2}{3}$.

2.1. **Application.** Let E and F be two Banach spaces, W and D non empty subset of E and D respectively. And consider two bounded mapping $g : W \times D \to I\!R$ and $G : W \times D \times I\!R \to I\!R$. ($W$ and D are the state and decision spaces respectively). Some problems of dynamic programming implies the problem of solving the functional equations:

$$\begin{cases} p(x) = \sup_{d \in D}\{g(x, d) + G(x, d, q(d))\}, \\ q(y) = \sup_{w \in W}\{g(w, y) + G(w, y, p(w))\} \end{cases} \tag{2.7}$$

for all $(x, y) \in W \times D$
Denote by $\mathbb{B}(W)$ and $\mathbb{B}(D)$ the spaces of all real bounded functions on W and D respectively, provided with the uniform metrics $d_{\infty,W}$ and $d_{\infty,D}$ respectively. And define the functionals

$$A : \mathbb{B}(W) \longrightarrow \mathbb{B}(D) \quad \text{and} \quad B : \mathbb{B}(W) \longrightarrow \mathbb{B}(W)$$

by:

$$Ah(y) = \sup_{w \in W}\{g(w, y) + G(w, y, h(w))\} \quad \text{and} \quad Bk(x) = \sup_{d \in D}\{g(x, d) + G(x, d, k(d))\}$$

for all $(h, k) \in \mathbb{B}(W) \times \mathbb{B}(D)$ and $(x, y) \in W \times D$.
Assume that for all $(w, x_1, x_2) \in W^3$ and $(d, y_1, y_2) \in D^3$, one of the conditions
(i) $|h(w) - BAh(w)| \leq d_{\infty,W}(h, Bk)$
(ii) $|k(d) - ABk(d)| \leq d_{\infty,D}(k, Ah)$
implies

$$\begin{cases} |(g(w, y_1) - g(w, y_2)) + (G(w, y_1, Ah(y_1)) - G(w, y_2, k(y_2))| \\ \leqslant r \max\{|h(w) - BAh(w)|, |h(w) - Bk(w)|, |k(d) - Ah(d)|\} \\ |(g(x_1, d) - g(x_2, d)) + (G(x_1, d, Bk(x_1)) - G(x_2, d, h(x_2))| \\ \leqslant r \max\{k(d) - ABk(d), |h(w) - Bk(w)|, |k(d) - Ah(d)|\} \end{cases}$$

Theorem 2.8. *Under the above conditions, there exists a unique pair* (h^*, k^*) *in* $\mathbb{B}(W) \times \mathbb{B}(D)$ *such that*

$$\begin{cases} h^*(x) = \sup_{d \in D}\{g(x, d) + G(x, d, k^*(d)\} \\ k^*(y) = \sup_{w \in W}\{g(w, y) + G(w, y, h^*(w)\} \end{cases}$$

for $(x, y) \in W \times D$; *and then the functional equation (2.7) has a unique solution.*

Proof. For $\varepsilon > 0$, $(h, k) \in \mathbb{B}(W) \times \mathbb{B}(D)$ and $(x, y) \in W \times D$, we have

$$\begin{cases} BAh(x) = \sup_{d \in D}\{g(x, d) + G(x, d, Ah(d))\} \\ Bk(x) = \sup_{d \in D}\{g(x, d) + G(x, d, k(d))\}, \end{cases}$$

Then there exists $(d_1, d_2) \in D^2$ such that:

$$\begin{cases} BAh(x) - \varepsilon < g(x, d_1) + G(x, d_1, Ah(d_1)) \leqslant BAh(x) \\ Bk(x) - \varepsilon < g(x, d_2) + G(x, d_2, k(d_2)) \leqslant Bk(x). \end{cases}$$

And then,

$$\begin{cases} g(x, d_1) - g(x, d_2) + G(x, d_1, Ah(d_1)) - G(x, d_2, k(d_2)) - \varepsilon < BAh(x) - Bk(x) \\ BAh(x) - Bk(x) < g(x, d_1) - g(x, d_2) + G(x, d_1, Ah(d_1)) - G(x, d_2, k(d_2)) + \varepsilon \end{cases}$$

Which gives,

$$\begin{aligned} |BAh(x) - Bk(x)| < \quad & |(g(x, d_1) - g(x, d_2)) + (G(x, d_1, Ah(d_1)) \\ & -G(x, d_2, k(d_2)))| + \varepsilon \end{aligned}$$

and there exists $(w_1, w_2) \in W^2$ such that :

$$\begin{aligned} |ABk(y) - Ah(y)| < \quad & |(g(w_1, y) - g(w_2, y)) + (G(w_1, y, Bk(w_1)) \\ & -G(w_2, y, h(w_2)))| + \varepsilon \end{aligned}$$

Therefore, one of the conditions
(i) $|h(x) - BAh(x)| \leq d_{\infty,W}(h, Bk)$
(ii) $|k(y) - ABk(y)| \leq d_{\infty,D}(k, Ah)$
implies

$$\begin{cases} |BAh(x) - Bk(x)| < r \max\{|h(x) - BAh(x)|, |h(x) - Bk(x)|, |k(y) - Ah(y)|\} + \varepsilon \\ |ABk(y) - Ah(y)| < r \max\{k(y) - ABk(y), |h(x) - Bk(x)|, |k(y) - Ah(y)|\} + \varepsilon, \end{cases}$$

And then, for any $(x, y) \in W \times D$, and an arbitrary $\varepsilon > 0$, one of the following conditions

(i): $d_{\infty,W}(h, BAh) \leqslant d_{\infty,W}(h, Bk)$
(ii): $d_{\infty,D}(k, ABk)\} \leqslant d_{\infty,D}(k, Ah)$

implies

$$\begin{cases} d_{\infty,D}(Bk, BAh) \leqslant r \max\{d_{\infty,W}(h, BAh)|, d_{\infty,W}(h, Bk), d_{\infty,D}(k, Ah)\} \\ d_{\infty,W}(Ah, ABk) \leqslant r \max\{d_{\infty,D}(k, ABk)|, d_{\infty,W}(h, Bk), d_{\infty,D}(k, Ah)\} \end{cases}$$

Therefore, there exists $(h^*, k^*) \in \mathbb{B}(W) \times \mathbb{B}(D)$ such that $Ah^* = k^*$ and $Bk^* = h^*$. And then (h^*, k^*) is the unique bounded solution of the functional equation (7).

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