Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 7 Issue 4(2015), Pages 17-23.

ROUGH CONVERGENCE OF A SEQUENCE OF FUZZY NUMBERS

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Abstract. We define the concept of rough limit set of a sequence of fuzzy numbers and obtain the relation between the set of rough limit and the extreme limit points of a sequence of fuzzy numbers. Finally, we investigate some properties of the rough limit set.

1. Introduction

The set of fuzzy numbers is denoted $L(\mathbb{R})$, and d denotes the supremum metric on $L(\mathbb{R})$. Now let r be a nonnegative real number. Then we say that a sequence $\{X_i\}$ of fuzzy numbers is $r-$*convergent* to a fuzzy number X_* and we write

$$X_i \xrightarrow{r} X_* \quad \text{as} \quad i \to \infty,$$

provided that for every $\varepsilon > 0$ there is an integer i_ε so that

$$d(X_i, X_*) < r + \varepsilon$$

whenever $i \geq i_\varepsilon$. The set

$$\mathrm{LIM}^r X_i := \{X_* \in L(\mathbb{R}) : X_i \xrightarrow{r} X_* \text{ , as } \ i \to \infty\}$$

is called the $r-$*limit set* of the sequence $\{X_i\}$.

According to this definition, a sequence of fuzzy numbers which is divergent can be convergent with a certain roughness degree. For instance, let us define

$$X_i(x) := \begin{cases} \eta(x) & \text{, if } i \text{ is odd integer} \\ \mu(x) & \text{, otherwise} \end{cases},$$

where

$$\eta(x) := \begin{cases} x & \text{, if } x \in [0,1] \\ -x+2 & \text{, if } x \in [1,2] \\ 0 & \text{, otherwise} \end{cases}$$

2000 *Mathematics Subject Classification.* 40A05, 03E72.
Key words and phrases. Fuzzy numbers; Analysis; Rough convergence.

Submitted Jul 14, 2015. Published November 11, 2015.
This paper was supported by grant SDU-BAP-2332-YL-10 from the Suleyman Demirel University, Isparta, Turkey.
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and

$$\mu(x) := \begin{cases} x-3 & , \text{ if } x \in [3,4] \\ -x+5 & , \text{ if } x \in [4,5] \\ 0 & , \text{ otherwise} \end{cases} .$$

Then we have

$$\mathrm{LIM}^r X_i = \begin{cases} \varnothing & , \text{ if } r < 3/2 \\ [\mu - r_1, \eta + r_1] & , \text{otherwise} \end{cases} ,$$

where r_1 is nonnegative real number with $[\mu - r_1, \eta + r_1] :=$ $\{X \in L(\mathbb{R}) : \mu - r_1 \preceq X \preceq \eta + r_1\}$.

The idea of rough convergence of a sequence can be interpreted as follows. Let $\{Y_i\}$ be a convergent sequence of fuzzy numbers. Assume that Y_i cannot be determined exactly for every $i \in \mathbb{N}$ (or some $i \in \mathbb{N}$). That is, Y_i cannot be calculated so we can use approximate value of Y_i for simplicity of calculation. We only know that Y_i belongs to the closed intervals $[\mu_i, \lambda_i]$, where $d(\mu_i, \lambda_i) \leq r$ for every $i \in \mathbb{N}$. We have to do with an approximated and known sequence $\{X_i\}$ satisfying $X_i \in [\mu_i, \lambda_i]$ for all i. Then the sequence $\{X_i\}$ may not be convergent, but the inequality

$$d(X_i, X_*) \leq d(X_i, Y_i) + d(Y_i, X_*) \leq r + d(Y_i, X_*)$$

implies that the sequence $\{X_i\}$ is $r-$convergent.

Phu [10] and Burgin [4] introduced the notion of rough convergence independently with different titles. Here we will adopt the definitions and notations in [10]. In [10], Phu showed that the set $\mathrm{LIM}^r x$ is bounded, closed and convex; and he also investigated the dependence of $\mathrm{LIM}^r x$ on the roughness degree r. In [11], he extended the results given in [10] to infinite dimensional normed spaces. Recently, Aytar [3] proved that the ordinary core of a sequence $x = (x_i)$ of real numbers is equal to its $2\overline{r}$-limit set, where $\overline{r} := \inf\{r \geq 0 : \mathrm{LIM}^r x \neq \varnothing\}$. Later, Aytar [2] defined the concept of rough statistical convergence. Defining the set of rough statistical limit points of a sequence, he obtained two statistical convergence criteria associated with this set. He also examined the relations between the set of all statistical cluster points and the set of all rough statistical limit points of a sequence. Pal et al. [9] and Dündar, Çakan [7] independently gave an extension of rough convergence at the same time, by using the notion of an ideal. They also stated some basic results related to the rough ideal limit set.

In this paper, we first define the concept of rough convergence of a sequence of fuzzy numbers. In the second step, we obtain the relation between the set of rough limit and the extreme limit points of a sequence of fuzzy numbers. When a sequence is convergent, we characterize the rough limit set of this sequence. Later, we prove that a necessary and sufficient condition for the rough limit set of a sequence to be nonempty is the boundedness of the sequence. Finally, we show that the rough limit set of a sequence is closed, bounded and convex.

2. Preliminary Concepts

In this section, we briefly recall some of the basic notations in the theory of fuzzy numbers and we refer to [1, 5, 6, 8, 12] for more details.

A *fuzzy number* X is a fuzzy subset of the real line $\mathbb{R}$, which is normal, fuzzy convex, upper semi-continuous, and the set X^0 is bounded where $X^0 := cl\{x \in \mathbb{R} : X(x) > 0\}$ and cl is the closure operator. These properties imply that for each

$\alpha \in (0,1]$, the $\alpha-$level set X^{α} defined by

$$X^{\alpha} := \{x \in \mathbb{R} : \ X(x) \geq \alpha\} = \left[\underline{X}^{\alpha}, \overline{X}^{\alpha}\right]$$

is a nonempty compact convex subset of $\mathbb{R}$.

The *supremum metric* d on the set $L(\mathbb{R})$ is defined by

$$d(X,Y) := \sup_{\alpha \in [0,1]} \max(\left|\underline{X}^{\alpha} - \underline{Y}^{\alpha}\right|, \left|\overline{X}^{\alpha} - \overline{Y}^{\alpha}\right|).$$

Now, given $X, Y \in L(\mathbb{R})$, we define

$$X \preceq Y \text{ if } \underline{X}^{\alpha} \leq \underline{Y}^{\alpha} \ \text{ and } \overline{X}^{\alpha} \leq \overline{Y}^{\alpha} \ \text{ for each } \ \alpha \in [0,1].$$

We write $X \prec Y$ if $X \preceq Y$ and there exists an $\alpha_0 \in [0,1]$ such that $\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}$.

A subset E of $L(\mathbb{R})$ is said to be *bounded above* if there exists a fuzzy number μ, called an *upper bound* of E, such that $X \preceq \mu$ for every $X \in E$. μ is called the *least upper bound* (sup) of E if μ is an upper bound and $\mu \preceq \mu'$ for all upper bounds μ'. A *lower bound* and the *greatest lower bound* (inf) are defined similarly. E is said to be *bounded* if it is both bounded above and below.

The notions of "sup" and "inf" have been defined only for bounded sets of fuzzy numbers. An important fact, proved by Wu and Wu [12], states that if the set $E \subset L(\mathbb{R})$ is bounded then its supremum and infimum exist (see also [8]).

The *limit infimum* and the *limit supremum* of a sequence $\{X_i\}$ is defined by

$$\begin{array}{c} \liminf\limits_{i \to \infty} X_i := \inf A_X \\ \limsup\limits_{i \to \infty} X_i := \sup B_X \end{array},$$

where

$$\begin{array}{c} A_X := \{\mu \in L(\mathbb{R}) : \text{The set } \ \{i \in \mathbb{N} : X_i \prec \mu\} \ \text{ is infinite}\} \\ B_X := \{\mu \in L(\mathbb{R}) : \text{The set } \ \{i \in \mathbb{N} : X_i \succ \mu\} \ \text{ is infinite}\}. \end{array}$$

Now, given two fuzzy numbers $X, Y \in L(\mathbb{R})$, we define their *sum* as $Z = X + Y$, where $\underline{Z}^{\alpha} := \underline{X}^{\alpha} + \underline{Y}^{\alpha}$ and $\overline{Z}^{\alpha} := \overline{X}^{\alpha} + \overline{Y}^{\alpha}$ for all $\alpha \in [0,1]$.

To any real number $a \in \mathbb{R}$, we can assign a fuzzy number $a_1 \in L(\mathbb{R})$, which is defined by

$$a_1(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}.$$

An *order interval* in $L(\mathbb{R})$ is defined as

$$[X, Y] := \{Z \in L(\mathbb{R}) : X \preceq Z \preceq Y\},$$

where $X, Y \in L(\mathbb{R})$.

A set E of fuzzy numbers is called *convex* if

$$\lambda \mu_1 + (1 - \lambda) \mu_2 \in E$$

for all $\lambda \in [0,1]$ and $\mu_1, \mu_2 \in E$.

3. Main Results

First, we give the relation between the set of rough limit and the extreme limit points of a sequence of fuzzy numbers.

Theorem 3.1. *If $LIM^r X_i \neq \varnothing$, then we have*

$$LIM^r X_i \subseteq [(\limsup X_i) - r_1, (\liminf X_i) + r_1].$$

In order to prove this theorem, we need the following lemma.

Lemma 3.2. *If $X_* \in LIM^r X_i$, then $d(\limsup X_i, X_*) \leq r$ and $d(\liminf X_i, X_*) \leq r$.*

Proof of Lemma 3.2. We assume that $d(\limsup X_i, X_*) > r$. Define $\widetilde{\varepsilon} := \frac{d(\limsup X_i, X_*) - r}{2}$. By definition of limit supremum, we have that given $i'_{\widetilde{\varepsilon}} \in \mathbb{N}$ there exists an $i \in \mathbb{N}$ with $i \geq i'_{\widetilde{\varepsilon}}$ such that $d(\limsup X_i, X_i) < \widetilde{\varepsilon}$.

Also, since $X_i \xrightarrow{r} X_*$ as $i \to \infty$, there is an integer $i''_{\widetilde{\varepsilon}}$ so that

$$d(X_i, X_*) < r + \widetilde{\varepsilon}$$

whenever $i \geq i''_{\widetilde{\varepsilon}}$. Let $i_{\widetilde{\varepsilon}} := \max\left\{i'_{\widetilde{\varepsilon}}, i''_{\widetilde{\varepsilon}}\right\}$. There exists $i \in \mathbb{N}$ such that $i \geq i_{\widetilde{\varepsilon}}$ and

$$\begin{aligned} d(\limsup X_i, X_*) &\leq d(\limsup X_i, X_i) + d(X_i, X_*) < r + 2\widetilde{\varepsilon} \\ &= r + d(\limsup X_i, X_*) - r = d(\limsup X_i, X_*). \end{aligned}$$

This contradiction proves the lemma. Similarly, $d(\liminf X_i, X_*) \leq r$ can be proved using defination of limit infumum. □

Now, we are ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1. We show that $X_* \in [(\limsup X_i) - r_1, (\liminf X_i) + r_1]$ for an arbitrary $X_* \in$LIM$^r X_i$, i.e., $(\limsup X_i) - r_1 \preceq X_* \preceq (\liminf X_i) + r_1$. First, assume that $(\limsup X_i) - r_1 \preceq X_*$ does not hold. Thus, there exists an $\alpha_0 \in [0,1]$ such that

$$(\underline{\limsup X_i}^{\alpha_0}) - r > \underline{X_*}^{\alpha_0} \qquad \text{or} \qquad (\overline{\limsup X_i}^{\alpha_0}) - r > \overline{X_*}^{\alpha_0}$$

holds which are equivalent to the inequalities

$$(\underline{\limsup X_i}^{\alpha_0}) - \underline{X_*}^{\alpha_0} > r \qquad \text{or} \qquad (\overline{\limsup X_i}^{\alpha_0}) - \overline{X_*}^{\alpha_0} > r.$$

On the other hand, according to Lemma 3.2, we have

$$\left|(\underline{\limsup X_i}^{\alpha_0}) - \underline{X_*}^{\alpha_0}\right| \leq r \;\; \text{and} \;\; \left|(\overline{\limsup X_i}^{\alpha_0}) - \overline{X_*}^{\alpha_0}\right| \leq r.$$

Thus we obtain a contradiction. Therefore, we get $(\limsup X_i) - r_1 \preceq X_*$.

The second part of this theorem can be proved using the similar arguments in the first part. □

Note that the converse inclusion in this theorem holds for sequences of real numbers, but it may not hold for sequences of fuzzy numbers, as can be seen in the following example.

Example 3.3. *Define*

$$X_i(x) := \begin{cases} -\frac{1}{2i}x + 1 & , \text{ if } x \in [0,1] \\ 0 & , \text{ otherwise} \end{cases}$$

and

$$X_*(x) := \begin{cases} 1 & , \text{ if } x \in [0,1] \\ 0 & , \text{ otherwise} \end{cases}.$$

Then we have $\left|\overline{X}_*^1 - \overline{X}_i^1\right| = |1-0| = 1$, *i.e.,* $d(X_i, X_*) \geq 1$ *for all* $i \in \mathbb{N}$. *Although the sequence* $\{X_i\}$ *is not convergent to* X_*, $\limsup X_i$ *and* $\liminf X_i$ *of this sequence are equal to* X_*. *Hence we get* $X_* \in [\limsup X_i - (1/2)_1, \liminf X_i + (1/2)_1]$, *but* $X_* \notin LIM^{1/2} X_i$.

Theorem 3.4. *If a sequence* $\{X_i\}$ *converges to the fuzzy number* X_*, *then* $LIM^r X_i = \overline{B}_r(X_*) := \{\mu \in L(\mathbb{R}) : d(\mu, X_*) \leq r\}$.

Proof. Let $\varepsilon > 0$. Since the sequence $\{X_i\}$ is convergent to X_*, there is an integer i_ε so that

$$d(X_i, X_*) < \varepsilon$$

whenever $i \geq i_\varepsilon$. Let $Y \in \overline{B}_r(X_*)$. We have

$$d(X_i, Y) \leq d(X_i, X_*) + d(X_*, Y) < \varepsilon + r$$

for every $i \geq i_\varepsilon$. Hence we have $Y \in \text{LIM}^r X_i$.

Now let $Y \in \text{LIM}^r X_i$. Hence there is an integer $i_\varepsilon^{'}$ so that

$$d(X_i, Y) < r + \varepsilon$$

whenever $i \geq i_\varepsilon^{'}$. Let $i_\varepsilon'' := \max\left\{i_\varepsilon, i_\varepsilon^{'}\right\}$. For all $i > i_\varepsilon''$, we obtain

$$d(Y, X_*) \leq d(X_i, Y) + d(X_i, X_*) < r + 2\varepsilon.$$

Since ε is arbitrary, we have $d(Y, X_*) \leq r$. Hence we get $Y \in \overline{B}_r(X_*)$. Thus, if the sequence $\{X_i\}$ converges to X_*, then $\text{LIM}^r X_i = \overline{B}_r(X_*)$. □

Theorem 3.5. *A sequence* $\{X_i\}$ *in* $L(\mathbb{R})$ *is* $r-$*convergent to* X_* *if there exists a sequence* $\{Y_i\}$ *in* $L(\mathbb{R})$ *such that*

$$Y_i \to X_* \quad \text{as } i \to \infty, \quad \text{and } d(X_i, Y_i) \leq r \quad \text{for every} \quad i \in \mathbb{N}.$$

Proof. Assume that $Y_i \to X_*$, as $i \to \infty$, and $d(X_i, Y_i) \leq r$ for every $i \in \mathbb{N}$. $Y_i \to X_*$, as $i \to \infty$ means that for every $\varepsilon > 0$ there exists an i_ε such that

$$d(Y_i, X_*) < \varepsilon \text{ for all } i \geq i_\varepsilon.$$

The inequality $d(X_i, Y_i) \leq r$ yields

$$d(X_i, X_*) \leq d(X_i, Y_i) + d(Y_i, X_*) < r + \varepsilon \text{ if } i \geq i_\varepsilon.$$

Hence the sequence $\{X_i\}$ is r-convergent to the fuzzy number X_*. □

Theorem 3.6. *The diameter of an* $r-$*limit set is not greater than* $2r$.

Proof. We have to show that $\sup\{d(Y, Z) : Y, Z \in \text{LIM}^r X_i\} \leq 2r$. Assume on the contrary that $\sup\{d(Y, Z) : Y, Z \in \text{LIM}^r X_i\} > 2r$. By this assumption, there exist $Y, Z \in \text{LIM}^r X_i$ satisfying $\lambda := d(Y, Z) > 2r$. For an arbitrary $\varepsilon \in (0, \lambda/2 - r)$, we have

$$\exists i_\varepsilon^{'} \in \mathbb{N} : \forall i \geq i_\varepsilon' \Rightarrow d(X_i, Y) < r + \varepsilon$$
$$\exists i_\varepsilon^{''} \in \mathbb{N} : \forall i \geq i_\varepsilon^{''} \Rightarrow d(X_i, Z) < r + \varepsilon.$$

Define $i_\varepsilon := \max\left\{i_\varepsilon^{'}, i_\varepsilon^{''}\right\}$. Thus we get

$$d(Y, Z) \leq d(X_i, Y) + d(X_i, Z) < 2(r + \varepsilon) < 2r + 2(\lambda/2 - r) = \lambda$$

for all $i \geq i_\varepsilon$ which contradicts to the fact that $\lambda = d(Y, Z)$. □

The rest of the paper contains some basic properties of the rough limit set.

Theorem 3.7. *A sequence $\{X_i\}$ is bounded if and only if there exists an $r \geq 0$ such that LIM$^r X_i \neq \varnothing$.*

Proof. ($\Rightarrow$) Let $\{X_i\}$ be a bounded sequence and $s := \sup\{d(X_i, 0_1) : i \in \mathbb{N}\} < \infty$. Then we have $0_1 \in$LIM$^s X_i$, i.e., LIM$^r X_i \neq \varnothing$, where $r = s$.

($\Leftarrow$) If LIM$^r X_i \neq \varnothing$ for some $r \geq 0$, then there exists $X_* \in$LIM$^r X_i$. By definition, for every $\varepsilon > 0$ there is an integer i_ε so that

$$d(X_i, X_*) < r + \varepsilon$$

whenever $i \geq i_\varepsilon$. Define $t = t(\varepsilon) := \max\{d(X_*, 0_1), d(X_1, 0_1), d(X_2, 0_1), ..., d(X_{i_\varepsilon}, 0_1), r+\varepsilon\}$**.** Then we have $X_i \in \{\mu \in L(\mathbb{R}) : d(\mu, 0_1) \leq t + r + \varepsilon\}$ for every $i \in \mathbb{N}$, which proves the boundedness of the sequence $\{X_i\}$. □

The next theorem gives the inclusion relation between the rough limit sets of a sequence and its subsequence. Its proof is straightforward.

Theorem 3.8. *If $\{X_{k_i}\}$ is a subsequence of $\{X_i\}$, then LIM$^r X_i \subset$LIM$^r X_{k_i}$.*

Theorem 3.9. *For all $r \geq 0$, the $r-$limit set LIM$^r X_i$ of an arbitrary sequence $\{X_i\}$ is closed.*

Proof. Let $\{Y_i\} \subset$ LIM$^r X_i$ and $Y_i \to Y_*$ as $i \to \infty$. Let $\varepsilon > 0$. Since the sequence $\{Y_i\}$ converges to Y_*, there is an integer j_ε so that

$$d(Y_i, Y_*) < \frac{\varepsilon}{2}$$

whenever $i \geq j_\varepsilon$. Since $Y_{j_\varepsilon} \in$LIM$^r X_i$, there is an integer i_ε so that

$$d(X_i, Y_{j_\varepsilon}) < r + \frac{\varepsilon}{2}$$

whenever $i \geq i_\varepsilon$. Therefore, we have

$$\begin{array}{rcl} d(X_i, Y_*) & \leq & d(X_i, Y_{j_\varepsilon}) + d(Y_{j_\varepsilon}, Y_*) \\ & < & r + \varepsilon/2 + \varepsilon/2 = r + \varepsilon \end{array}$$

for every $i \geq i_\varepsilon$. Hence $Y_* \in$LIM$^r X_i$ implies that the set LIM$^r X_i$ is closed. □

4. Discussion

In this section, we would like to give a general picture for would-be applications. Let the sequence of fuzzy numbers that we have obtained by fuzzification from some real data be denoted by $\{X_i\}$ (For sequences having finitely many terms, we can give similar results). It is possible that to operate with the terms of this sequence and to defuzzify it are difficult. Instead, we may use the terms of a sequence $\{Y_i\}$ for approximating to the terms of the sequence $\{X_i\}$, where each Y_i is a triangular or trapezoidal fuzzy number, and $d(X_i, Y_i) \leq r$ for each $i \in \mathbb{N}$. Thus it will be much easier to apply the fuzzy process to the sequence $\{Y_i\}$ because it consists of triangular or trapezoidal fuzzy numbers. It should not be forgotten that there is always an error rate if r is positive in the operations done with the sequence $\{Y_i\}$. However, such an error rate can mostly be tolerated, compared to the difficulty of the operations.

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