Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 8 Issue 3(2016), Pages 78-110.

THE $\mathbf{D}_u$-CLASSICAL ORTHOGONAL POLYNOMIALS

(COMMUNICATED BY FRANCISCO MARCELLAN)

ATEF ALAYA

Abstract. The $\mathbf{D}_u$-classical orthogonal polynomial sequences are defined through the $\mathbf{D}_u$-Hahn's property: sequences that are orthogonal together with their $\mathbf{D}_u$-first derivative, where $\mathbf{D}_u(p) = p' + u\theta_0 p$, for all $p \in \mathbb{C}[X]$. We characterize them by means of a functional equation, a $\mathbf{D}_u$-second order linear differential equation, the first and the second structure relations. A $\mathbf{D}_u$-classical orthogonal sequence is especially a $\mathbf{D}$-Laguerre-Hahn sequence of class less than or equal to two. A complete classification of the $\mathbf{D}_u$-classical sequences is obtained. The functional equation coefficients, the structure relations coefficients, the three-term recurrence relation coefficients and the class are whenever given.

1. introduction

Lot of works dealing with orthogonal polynomials mention the very classical ones: continuous (Hermite, Laguerre, Bessel and Jacobi), discrete (Charlier, Meixner, Krawtchouk and Hahn) or their analogues with respect to lowering difference or differential operators (Hahn operator, Delta operator, Dunkl operator, etc..) [14], [18], [19], [32], [36], [47], [49], [51], [55], and [11], [13], [15], [35], [56].

Rodrigues formula, Hahn property, Bochner condition, first and second structure relations, and Pearson equation are important common tools to characterize and construct these polynomial sequences [1], [4], [6], [7], [8], [14], [18], [22], [26], [30], [32], [36], [37], [49], [50], [51], [57], [63], [20]. In fact, using these tools, different unified presentations of classical orthogonal polynomials are done in the literature either for continuous case, discrete case or their analogues: through an algebraic approach [47], [49], a functional approach [26], [39], a distributional approach [27], [53], a hypergeometrical approach [9], [33], a difference calculus approach [19], a matrix approach [63], etc...[44]

Besides, different natural procedures are used to build new orthogonal polynomial sequences (see [2], [5], [10], [13], [18], [21], [23], [24], [25], [38], [40], [46], [48]

2010 *Mathematics Subject Classification.* 33C45, 42C05.
Key words and phrases. linear functionals; Orthogonal polynomials; Laguerre-Hahn polynomials.

Submitted February 19, 2016. Published September 2, 2016.
A. Alaya was supported by Institute of Scientific Research and Revival of Islamic Heritage at Umm Al-Qura University (Project ID 43305032), Saudi Arabia.

and [54]). They start by using some classical polynomials and yield a lot of polynomial sequences but generally with no apparent link between them. In front of the accumulation of the obtained results of such procedures, the need of structure and classification of orthogonal polynomial sequences was natural. Among others, the papers [9], [12], [23], [25], [28], [31], [44], [34], [35], [41], [43], [45] and [46] provide sketches in this direction highlighting the so-called semiclassical polynomial sequences, and where classical orthogonal polynomials are seen as semiclassical ones of class zero. In the same context, in [33] the authors emphasize the hypergeometric character rather than the sequences therein. An algebraic approach is also presented in [47] for the so-called Laguerre-Hahn polynomials generalizing the semiclassical case. Some early tries worthy of being evoked too in the same object are [59], [60] and [61].

All these works usually give orthogonal polynomials generalizing (by their properties and their characterizations) the classical orthogonal polynomials, but generally it is difficult to explicitly construct them except under further assumptions as the symmetry [2], [3], [16], [17], [29], [52], [57], [58] and [62].

The aim of this work is to pick up orthogonal polynomial sequences under a lowering operator denoted by $\mathbf{D}_u$, generalizing the standard derivative $\mathbf{D} = \frac{d}{dx}$. This operator was first introduced to see all Laguerre-Hahn polynomials of class zero, build in [16], as the unique solutions of $\mathbf{D}_u$ Hahn's property. In [42], the authors define the $\mathbf{D}_u$-semiclassical polynomials and classify them using the notion of the class. Here, we expose the $\mathbf{D}_u$-classical orthogonal polynomials by means of Hahn's property with respect to the operator $\mathbf{D}_u$. In particular, through an algebraic approach, we state several characterizations of them as a natural extension of the corresponding properties for the very classical ones (generalizing the Hahn property, the Pearson equation, the second order linear differential equation and the structure relations). It is also shown that such polynomial sequences are $\mathbf{D}$-Laguerre-Hahn sequences of class at most two, and that are $\mathbf{D}_u$-semiclassical sequences of class zero. Finally, by solving a nonlinear system fulfilled by the corresponding three-term recurrence relation coefficients, we give explicitly all $\mathbf{D}_u$-classical orthogonal polynomial sequences.

2. BACKGROUND

Let $\mathbb{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathbb{P}'$ its dual. For $u \in \mathbb{P}'$, $\langle u, p\rangle$ means the action of the form (linear functional) u over the polynomial p. In particular, $(u)_n = \langle u, x^n\rangle,\ n \geq 0$, are the moments of u. When $(u)_0 = 1$, then u is said to be normalized.
We set $\mathbb{P}'_0 := \{u \in \mathbb{P}' \ \mid \ (u)_0 \neq -n,\ n \geq 1\}$.
Let us define the following operations on $\mathbb{P}'$:
For all $c \in \mathbb{C}$, p, $q \in \mathbb{P}$, and u, $v \in \mathbb{P}'$

$$\begin{aligned}
\langle qu, p\rangle &= \langle u, qp\rangle, \quad \langle u',\ p\rangle = -\langle u, p'\rangle,\\
\langle \delta_c,\ p\rangle &= p(c),\ \delta_c : \text{Dirac delta at c},\ (\delta := \delta_0),\\
\langle uv, p\rangle &= \langle v, up\rangle, \text{ where } \big(up\big)(x) = \langle u_y, \frac{xp(x) - yp(y)}{x - y}\rangle,\\
\langle (x-c)^{-1}u, p\rangle &= \langle u, \theta_c(p)\rangle = \langle u, \frac{p(x) - p(c)}{x - c}\rangle,
\end{aligned}$$

where $\langle u_y, .\rangle$ means the action of u over the polynomial on the y-variable.
Now, let $\{B_n\}_{n\geq 0}$ be a monic polynomial sequence (MPS), with $\deg B_n = n$, and let $\{w_n\}_{n\geq 0}$ be its dual sequence defined by $\langle w_n, B_m\rangle = \delta_{n,m},\ n,\ m \geq 0$, where $\delta_{n,m}$ is the Kronecker delta. Note that w_0 is normalized and it is said to be the canonical form of $\{B_n\}_{n\geq 0}$. The following Lemmas are helpful for the sequel.

Lemma 2.1. [47, 49] *Let $\{w_n\}_{n\geq 0}$ denotes the dual sequence of a given MPS $\{B_n\}_{n\geq 0}$. For any $u \in \mathbb{P}'$, and any integer $p \geq 1$, the following statements are equivalent*

(a) $\langle u, B_{p-1}\rangle \neq 0$, *and* $\langle u, B_n\rangle = 0,\ n \geq p$.

(b) $\exists\, \lambda_\mu \in \mathbb{C},\ 0 \leq \mu \leq p-1,\ \lambda_{p-1} \neq 0$ *such that* $u = \sum_{\mu=0}^{p-1} \lambda_\mu w_\mu$.

Lemma 2.2. [2, 46] *Let $u,\ v \in \mathbb{P}'$, $f, g \in \mathbb{P}$, and $(a, b, c) \in \mathbb{C}^3$ with $a \neq 0$, we have*

$$\delta u = u,\ \ uv = vu,\ \ f(uv) = (fu)v + x(u\theta_0 f)(x)v, \tag{2.1}$$

$$(fu)' = fu' + f'u,\ \ f(\tau_b u) = \tau_b\big((\tau_{-b}f)u\big),\ \ f(h_a u) = h_a\big((h_a f)\big)u,$$

$$x^{-1}(uv) = (x^{-1}u)v = u(x^{-1}v),\ \ (x-c)\big((x-c)^{-1}u\big) = u,$$

$$(x-c)^{-1}\big((x-c)u\big) = u - (u)_0\delta_c,$$

$$(x-c)^{-1}(fu) = f(c)\big((x-c)^{-1}u\big) + (\theta_c f)u - \langle u, \theta_c f\rangle\delta_c, \tag{2.2}$$

$$u\big(fg\big)(x) = \big((fu)g\big)(x) + xg(x)(u\theta_0 f)(x),$$

$$\big(\theta_0(fg)\big)(x) = \big((\theta_0 f)g\big)(x) + f(0)\big(\theta_0 g\big)(x). \tag{2.3}$$

Here the shift τ_b, and the dilation h_a are respectively defined by

$$\langle \tau_b u, f\rangle = \langle u, \tau_{-b} f\rangle = \langle u, f(x+b)\rangle,\ \ \textit{and}\ \ \langle h_a u, f\rangle = \langle u, h_a f\rangle = \langle u, f(ax)\rangle.$$

Lemma 2.3. [2, 46] *For all $u,\ v \in \mathbb{P}'$, and $f,\ g \in \mathbb{P}$, the following formulas hold*

$$\big(u\theta_0(fg)\big)(x) = g(x)(u\theta_0 f)(x) + \big((fu)\theta_0 g\big)(x), \tag{2.4}$$

$$f(x^{-1}u) = x^{-1}(fu) + \langle u, \theta_0 f\rangle\delta, \tag{2.5}$$

$$f\big(x^{-1}(uv)\big) = x^{-1}\big(u(fv)\big) + (v\theta_0 f)(x)u,\ \ f^2u^2 = (fu)^2 + 2xf(x)(u\theta_0 f)(x)u,$$

$$\langle u^2, \theta_0(fg)\rangle = \langle u, f(u\theta_0 g) + g(u\theta_0 f)\rangle. \tag{2.6}$$

Remark. [2, 16] *A form u has an inverse u^{-1} (i.e $uu^{-1} = \delta$), if and only if $(u)_0 \neq 0$.*

A MPS $\{B_n\}_{n\geq 0}$ is called orthogonal (MOPS) with respect to a form w, if $\langle w, B_n B_m\rangle = 0,\ n \neq m$, and $\langle w, B_n^2\rangle \neq 0,\ n \geq 0$. In this case, w is said to be regular (quasi-definite). Necessarily $w = (w)_0 w_0$, with $(w)_0 \neq 0$. In the sequel, we shall take any regular form w normalized. Hence $w = w_0$.

Definition 2.4. [18, 42]*A nonzero form w is said to be weakly regular, if for any polynomial A such that $Aw = 0$, then $A = 0$.*

Lemma 2.5. [42] *A regular form is weakly regular.*

Proposition 2.6. [46, 47, 49] *Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two MOPS with respect to u and v respectively, and A, B are two polynomials with* $\deg(A) = s$, *and* $\deg(B) = t$. *The following assertions are equivalent:*

(a) $Au = Bv$.

(b) $A(x)Q_n(x) = \sum_{\nu=n-t}^{n+s} \lambda_{n,\nu} P_\nu(x),\ n \geq t$ *with* $\lambda_{n,n-t} \neq 0,\ n \geq t.$

Proposition 2.7. [46, 47] *A MPS* $\{B_n\}_{n\geq 0}$ *with dual sequence* $\{w_n\}_{n\geq 0}$ *is orthogonal, if and only if one of the following statements holds:*

(a) $w_n = \langle w_0, B_n^2\rangle^{-1} B_n w_0,\ n \geq 0.$

(b) $\{B_n\}_{n\geq 0}$ *satisfies the three-term recurrence relation (TTRR)*

$$\begin{cases} B_0(x) = 1, \quad B_1(x) = x - \beta_0, \\ B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n \geq 0, \end{cases} \tag{2.7}$$

with $\beta_n \in \mathbb{C}$ *and* $\gamma_{n+1} \neq 0,\ n \geq 0.$

(c) *There exist two complex number sequences* $\{\beta_n\}_{n\geq 0}$ *and* $\{\gamma_{n+1}\}_{n\geq 0}$, *such that*

$$\gamma_{n+1}w_{n+1} = (x - \beta_n)w_n - w_{n-1},\ \ \gamma_{n+1} \neq 0,\ \ n \geq 0, \quad (w_{-1} = 0). \tag{2.8}$$

Definition 2.8. *Let* w *be a normalized form. For any MPS* $\{B_n\}_{n\geq 0}$, *we define the associated sequence of the first kind with respect to* w, *denoted by* $\{B_n^{(1)}(w)\}_{n\geq 0}$ *as follows (see* [1, 10]*)*

$$B_n^{(1)}(w)(x) = (w\theta_0 B_{n+1})(x) = \langle w, \frac{B_{n+1}(x) - B_{n+1}(\xi)}{x - \xi}\rangle,\ n \geq 0.$$

Once $\{B_n\}_{n\geq 0}$ is orthogonal with respect to w_0 and fulfills (2.7), then we have:

Proposition 2.9. [2, 47] *The sequence* $\{B_n^{(1)}(w)\}_{n\geq 0}$ *is orthogonal with respect to* ϖ, *if and only if:*

$$\begin{cases} \langle w, B_n\rangle = 0,\ n \geq 3, \\ \langle w, B_2\rangle \neq \gamma_2. \end{cases} \tag{2.9}$$

In this case, we have $w = Aw_0$ *with* $A(x) = \dfrac{\langle w, B_2\rangle}{\gamma_1\gamma_2}B_2(x) + \dfrac{\langle w, B_1\rangle}{\gamma_1}B_1(x) + 1.$

Besides, the sequence $\{B_n^{(1)}(u)\}_{n\geq 0}$ *verifies the following TTRR*

$$\begin{cases} B_0^{(1)}(w)(x) = 1 \quad , \quad B_1^{(1)}(w)(x) = x - \beta_0^{(1)}, \\ B_{n+2}^{(1)}(w)(x) = (x - \beta_{n+1}^{(1)})B_{n+1}^{(1)}(w)(x) - \gamma_{n+1}^{(1)}B_n^{(1)}(w)(x),\ n \geq 0, \end{cases} \tag{2.10}$$

where

$$\beta_0^{(1)} = \beta_1 - \langle w, B_1\rangle \quad , \quad \beta_n^{(1)} = \beta_{n+1},\ n \geq 1, \tag{2.11}$$

$$\gamma_1^{(1)} = \gamma_2 - \langle w, B_2\rangle \quad , \quad \gamma_{n+1}^{(1)} = \gamma_{n+2},\ n \geq 1. \tag{2.12}$$

Furthermore, the form ϖ *fulfills*

$$(Aw_0)\varpi = x\frac{B_1}{\gamma_1}w_0. \tag{2.13}$$

Proof. From (2.7), one has

$$B_{n+3}(x) - B_{n+3}(y) = (x - \beta_{n+2})\big(B_{n+2}(x) - B_{n+2}(y)\big) - \gamma_{n+2}\big(B_{n+1}(x) - B_{n+1}(y)\big) + (x - y)B_{n+2}(y),\ n \geq 0,$$

and

$$B_2(x) - B_2(y) = (x - \beta_1)\big(B_1(x) - B_1(y)\big) + (x - y)B_1(y).$$

Then $\{B_n^{(1)}(w)\}_{n\geq 0}$ fulfills

$$\left\{\begin{array}{l} B_0^{(1)}(w)(x)=1 \quad , B_1^{(1)}(w)(x)=x-\beta_1+\langle w, B_1\rangle, \\ B_{n+2}^{(1)}(w)(x)=(x-\beta_{n+2})B_{n+1}^{(1)}(w)(x)-\gamma_{n+2}B_n^{(1)}(w)(x)+ \\ \qquad\qquad \langle w, B_{n+2}\rangle,\ n\geq 0. \end{array}\right.$$

By virtue of *Favard*'s theorem [46, 56], $\{B_n^{(1)}(w)\}_{n\geq 0}$ is orthogonal if and only if

$$\langle w, B_n\rangle = 0,\ n\geq 3, \quad \text{and} \quad \langle w, B_2\rangle \neq \gamma_2. \tag{2.14}$$

The relations (2.10) − (2.12) hold.
Since we have (2.14), the expression of A is obtained by applying Lemma 2.1 to w and taking into account the assertion (a) of Proposition 2.7. On the other hand:

$$\begin{aligned} \langle \varpi, B_n^{(1)}(w)\rangle &= \delta_{n,0},\ n\geq 0 \\ \langle \varpi, w\theta_0 B_{n+1}\rangle &= \langle w_1, B_{n+1}\rangle,\ n\geq 0 \\ \langle x^{-1}(w\varpi)-w_1, B_n\rangle &= 0,\ \ n\geq 1. \end{aligned}$$

Besides $\langle x^{-1}(w\varpi)-w_1, B_0\rangle = 0$. Hence (2.13) holds. □

Remark. *When $w=w_0$, the conditions (2.9) are satisfied. Hence, if we set $B_n^{(1)} = B_n^{(1)}(w_0)$, then $\{B_n^{(1)}\}_{n\geq 0}$ verifies the following shifted TTRR*

$$\left\{\begin{array}{l} B_0^{(1)}(x)=\ 1, \quad B_1^{(1)}(x)=\ x-\beta_1, \\ B_{n+2}^{(1)}(x)=(x-\beta_{n+2})B_{n+1}^{(1)}(x)-\gamma_{n+2}B_n^{(1)}(x),\ n\geq 0. \end{array}\right.$$

Finally, if we denote $\{w_n^{(1)}\}_{n\geq 0}$ the dual sequence of $\{B_n^{(1)}\}_{n\geq 0}$, thus $\gamma_1 w_0^{(1)} = -x^2 w_0^{-1}$ [47].

The shifted MPS $\{\tilde{B}_n\}_{n\geq 0}$ of a given MPS $\{B_n\}_{n\geq 0}$ is given by:

$$\tilde{B}_n(x) = a^{-n}B_n(ax+b),\ n\geq 0, \qquad (a,b)\in \mathbb{C}^*\times\mathbb{C}. \tag{2.15}$$

A shift preserves the orthogonality. Precisely, if $\{B_n\}_{n\geq 0}$ is a MOPS with respect to w_0 and fulfills (2.7), then the sequence $\{\tilde{B}_n\}_{n\geq 0}$ defined by (2.15) is orthogonal with respect to

$$\tilde{w}_0 = (h_{a^{-1}}\circ\tau_{-b})w_0 \tag{2.16}$$

and one has

$$\left\{\begin{array}{l} \widetilde{B}_0(x)=1 \quad , \quad \widetilde{B}_1(x)=x-\widetilde{\beta}_0, \\ \widetilde{B}_{n+2}(x)=(x-\widetilde{\beta}_{n+1})\widetilde{B}_{n+1}(x)-\widetilde{\gamma}_{n+1}\widetilde{B}_n(x),\ n\geq 0, \end{array}\right.$$

with $\quad \widetilde{\beta}_n = \dfrac{\beta_n - b}{a},$ and $\widetilde{\gamma}_{n+1} = \dfrac{\gamma_{n+1}}{a^2},\ n\geq 0.$

Definition 2.10. *Let $v\in\mathbb{P}'$, and σ a nonnegative integer. A MPS $\{B_n\}_{n\geq 0}$ is said to be quasi-orthogonal of order σ with respect to v, if*

$$\langle v, x^m B_n\rangle = 0,\ 0\leq m\leq n-\sigma-1,\ n\geq \sigma+1, \tag{2.17}$$

$$\exists r\geq\sigma,\ \langle v, x^{r-\sigma}B_r\rangle \neq 0.$$

A MPS $\{B_n\}_{n\geq 0}$ is said to be strictly quasi-orthogonal of order σ, with respect to v, if it satisfies (2.17), and $\langle v, x^{n-\sigma}B_n\rangle \neq 0, n\geq\sigma$.

Remark.
1). The strict quasi-orthogonality of order zero is orthogonality.
2). A MOPS $\{B_n\}_{n\geq 0}$ with respect to its canonical form w_0 is (strictly) quasi-orthogonal of order σ with respect to a form v, if and only if there exists a (unique) polynomial ϕ of degree σ such that $v = \phi w_0$ [2, 23, 25].

From now onwards, the symbol $\boldsymbol{D}$ means the usual derivative operator; i.e $\boldsymbol{D} = \frac{d}{dx}$.

Definition 2.11. [3, 16, 23, 42] *Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to w_0. We say that w_0 is a $\boldsymbol{D}$-Laguerre-Hahn form if it fulfills a functional equation of type*

$$(\Phi w_0)' + (\Psi w_0) + B(x^{-1}w_0^2) = 0, \tag{2.18}$$

with $(\Phi, \Psi, B) \in \mathbb{P}^3$, Φ monic.
The sequence $\{B_n\}_{n\geq 0}$ is also called $\mathbf{D}$-Laguerre-Hahn. The minimum value of all integers $\max\big(\deg\Psi - 1, \max(\deg\Phi, \deg B) - 2\big)$, for each triplet (Φ, Ψ, B) satisfying (2.18), is called the class of w_0. If w_0 is of class s, then the sequence $\{B_n\}_{n\geq 0}$ is said to be of class s.

Between many characterizations of a $\mathbf{D}$-Laguerre-Hahn MOPS, we mention the following.

Proposition 2.12. [3, 16, 23, 42] *Let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to w_0. The following assertions are equivalent:*

(a) *$\{B_n\}_{n\geq 0}$ is $\boldsymbol{D}$-Laguerre-Hahn.*
(b) *$\{B_n\}_{n\geq 0}$ fulfills the so-called structure relation:*

$$\Phi(x)B'_{n+1}(x) - B(x)B_n^{(1)}(x) = \sum_{k=n-s}^{n+d} \theta_{n,k}B_k(x),\ n \geq s+1, \quad \theta_{n,n-s} \neq 0 \tag{2.19}$$

with $d = \sup(t, r)$ and $s = \sup(p-1, d-2)$, and where t, r and p are respectively the degrees of Φ, B and Ψ.

When $B = 0$ in (2.19), the sequence $\{B_n\}_{n\geq 0}$ (resp. w_0) is called $\mathbf{D}$-semiclassical [17]. A $\mathbf{D}$-semiclassical MOPS (resp. form) of class zero is known as $\mathbf{D}$-classical one.
We recall that for all $u \in \mathbb{P}'_0$, we define the linear derivative operator $\mathbf{D}_u$ (see [42])

$$\begin{aligned} \mathbf{D}_u : \mathbb{P} &\longrightarrow \mathbb{P} \\ p &\longmapsto \mathbf{D}_u(p) = p' + u\theta_0 p, \end{aligned}$$

or equivalently $\mathbf{D}_u(p)(x) = p'(x) + \langle u_y, \dfrac{p(x)-p(y)}{x-y}\rangle$, $p \in \mathbb{P}$. In particular, we have

$$\mathbf{D}_u(x^n) = \big(n + (u)_0\big)x^{n-1} + \sum_{\nu=0}^{n-2}(u)_{n-\nu-1}x^\nu,\ n \geq 2,\ \mathbf{D}_u(x) = (u)_0 + 1,\ \mathbf{D}_u(1) = 0$$

and

$$\mathbf{D}_u(w) = w' - x^{-1}(uw),\ w \in \mathbb{P}', \quad \langle \mathbf{D}_u(w), p\rangle = -\langle w, \mathbf{D}_u(p)\rangle,\ p \in \mathbb{P}. \tag{2.20}$$

The first $\mathbf{D}_u$-derivative MPS of a MPS $\{B_n\}_{n\geq 0}$ is denoted by $\{B_n^{[1]}(.; u)\}_{n\geq 0}$. Thus

$$B_n^{[1]}(x; u) = \big(n + (u)_0 + 1\big)^{-1}\mathbf{D}_u(B_{n+1})(x),\ n \geq 0. \tag{2.21}$$

If we denote by $\{w_n\}_{n\geq 0}$ (resp. $\{w_n^{[1]}(u)\}_{n\geq 0}$) the dual sequence of $\{B_n\}_{n\geq 0}$ (resp. $\{B_n^{[1]}(.;u)\}_{n\geq 0}$), then [42]:

$$\mathbf{D}_u(w_n^{[1]}(u)) = -\big(n+(u)_0+1\big)w_{n+1},\ n\geq 0. \tag{2.22}$$

In what follows, we need the following formulas [42]:

$$\begin{aligned}\mathbf{D}_u(fg)(x) =\ & \mathbf{D}_u(f)(x)g(x) + f(x)\mathbf{D}_u(g)(x) \\ & +\big(u\theta_0(fg)\big)(x) - \big(u\theta_0 f\big)(x)g(x) - \big(u\theta_0 g\big)(x)f(x),\end{aligned} \tag{2.23}$$

$$\mathbf{D}_u(fv) =\ f\mathbf{D}_u(v) + \mathbf{D}_u(f)v + \big(v\theta_0 f\big)(x)u - \big(u\theta_0 f\big)(x)v. \tag{2.24}$$

Besides, with some straightforward calculations, the following formulas hold.

Proposition 2.13. *For all $w\in\mathbb{P}'$, $p\in\mathbb{P}$, and $(a,b)\in\mathbb{C}\setminus\{0\}\times\mathbb{C}$, we have:*

$$\mathbf{D}_u(\tau_b p) = \tau_b \mathbf{D}_{\tau_{-b}u}(p),$$

$$\mathbf{D}_u(\tau_b w) = \tau_b \mathbf{D}_{\tau_{-b}u}(w), \tag{2.25}$$

$$\mathbf{D}_u(h_a p) = a h_a \mathbf{D}_{h_a u}(p), \tag{2.26}$$

$$\mathbf{D}_u(h_a w) = a^{-1} h_a \mathbf{D}_{h_{a^{-1}}u}(w). \tag{2.27}$$

In [42], the authors studied the so-called $\mathbf{D}_u$-semiclassical sequences. In particular, they showed that the defined sequences are special **D**-Laguerre Hahn ones. In this work we shall elaborate the $\mathbf{D}_u$-classical case. In section 3, we start by defining a $\mathbf{D}_u$-classical sequence through the $\mathbf{D}_u$-Hahn's property (see definition 3.1). Then, we prove four characterizations that generalize the standard ones in the usual **D**-classical case (Hermite, Laguerre, Bessel and Jacobi) ; the Pearson's equation, the second order linear differential equation, the first and the second structure relations. We show particulary that any $\mathbf{D}_u$-classical orthogonal polynomial sequence is, in sense of [42], a $\mathbf{D}_u$-semiclassical sequence of class zero. Hence, it is a **D**-Laguerre-Hahn's sequence of class s at most 2. In section 4, we establish and solve in detail the nonlinear system fulfilled by the corresponding three-term recurrence relation coefficients. This allows us to give explicitly the functional equation coefficients and precise the class s.

3. The $\mathbf{D}_u$-classical sequences

Definition 3.1. *Let $u\in\mathbb{P}_0'$. We say that a MPS $\{B_n\}_{n\geq 0}$ is $\boldsymbol{D}_u$-classical sequence, if it is orthogonal together with its first $\boldsymbol{D}_u$-derivative sequence $\{B_n^{[1]}(.;u)\}_{n\geq 0}$ given by (2.21) (The $\boldsymbol{D}_u$-Hahn's property).*
A regular form that its corresponding MPS is $\boldsymbol{D}_u$-classical is also said to be $\boldsymbol{D}_u$-classical. Thus, for any $\boldsymbol{D}_u$-classical MPS we have :

$$\begin{cases} B_0(x) = 1, \quad B_1(x) = x-\beta_0, \\ B_{n+2}(x) = (x-\beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x),\ n\geq 0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} B_0^{[1]}(x;u) = 1, \quad B_1^{[1]}(x;u) = x-\widehat{\beta}_0, \\ B_{n+2}^{[1]}(x;u) = (x-\widehat{\beta}_{n+1})B_{n+1}^{[1]}(x;u) - \widehat{\gamma}_{n+1}B_n^{[1]}(x;u),\ n\geq 0. \end{cases} \tag{3.2}$$

In the sequel, we denote by $\{w_n\}_{n\geq 0}$ (resp. $\{w_n^{[1]}(u)\}_{n\geq 0}$) the dual sequence of $\{B_n\}_{n\geq 0}$ (resp. $\{B_n^{[1]}(.;u)\}_{n\geq 0}$.). Also, the letter u will usually denote an element of $\mathbb{P}_0'$.

Unless otherwise stated, and in order to simplify the notation, we will write $w_n^{[1]}$ *instead of* $w_n^{[1]}(u)$ *and* $B_n^{[1]}$ *instead of* $B_n^{[1]}(x;u)$.

3.1. **The functional equation fulfilled by** w_0.

Lemma 3.2. *If* $\{B_n\}_{n\geq 0}$ *is a* $\mathbf{D_u}$*-classical sequence, then there exist two nonzero polynomials* Φ *and* $\widehat{B}$, Φ *monic,* $\deg\Phi\leq 2$, *and* $\deg\widehat{B}\leq 2$, *such that*

$$w_0^{[1]}=\kappa\Phi w_0,\quad \kappa \text{ is a normalizing factor,} \tag{3.3}$$

$$u=\widehat{B}w_0, \tag{3.4}$$

with

$$\kappa\Phi(x)=eB_2(x)+fB_1(x)+B_0(x), \tag{3.5}$$

where

$$e=\frac{1}{\gamma_1\gamma_2}\Big\{\gamma_2+\big((u)_0+2\big)\big(\gamma_3+\big(\beta_2-\widehat{\beta}_1\big)^2\big)-((u)_0+4)\widehat{\gamma}_2\Big\},$$

$$f=\frac{1}{\gamma_1}\Big[\beta_1+\big((u)_0+2\big)\beta_2-\big((u)_0+3\big)\widehat{\beta}_1\Big],$$

and

$$\widehat{B}(x)=aB_2(x)+bB_1(x)+(u)_0B_0(x), \tag{3.6}$$

with

$$a=\frac{1}{\gamma_1\gamma_2}\Big\{(u)_0\gamma_2-\big((u)_0+2\big)\big(\gamma_3+\widehat{\gamma}_1+\big(\beta_2-\widehat{\beta}_1\big)^2\big)+\big((u)_0+4\big)\widehat{\gamma}_2\Big\},$$

$$b=\frac{1}{\gamma_1}\Big[(u)_0\beta_1-\big((u)_0+2\big)\beta_2+\big((u)_0+3\big)\widehat{\beta}_1-\big((u)_0+1\big)\widehat{\beta}_0\Big].$$

Furthermore, we have two additional conditions:

$$\begin{cases}\big((u)_0+2\big)\gamma_3\gamma_4-2\big((u)_0+3\big)\gamma_4\widehat{\gamma}_2+\big((u)_0+4\big)\widehat{\gamma}_2\widehat{\gamma}_3=0,\\ \big((u)_0+2\big)\gamma_3(\beta_2+\beta_3-2\widehat{\beta}_1)-\big((u)_0+3\big)\widehat{\gamma}_2(2\beta_3-\widehat{\beta}_1-\widehat{\beta}_2)=0.\end{cases} \tag{3.7}$$

Proof. From (2.8), we have $\widehat{\gamma}_{n+1}w_{n+1}^{[1]}=(x-\widehat{\beta}_n)w_n^{[1]}-w_{n-1}^{[1]},\quad n\geq 0,\quad (w_{-1}^{[1]}=0)$. Applying $\mathbf{D}_u$ in both hand sides, we obtain thanks to (2.24) and (2.22)

$$w_n^{[1]}=-\big(n+(u)_0+2\big)\widehat{\gamma}_{n+1}w_{n+2}+\big(n+(u)_0+1\big)(x-\widehat{\beta}_n)w_{n+1}-\big(n+(u)_0\big)w_n, n\geq 1, \tag{3.8}$$

$$w_0^{[1]}+u-\big((u)_0+1\big)\big(x-\widehat{\beta}_0\big)w_1+\widehat{\gamma}_1\big((u)_0+2\big)w_2=0. \tag{3.9}$$

Taking into account assertion (a) of the Proposition 2.7, the relation (3.8) becomes

$$B_n^{[1]}w_0^{[1]}=Z_{n+2}w_0,\ \ n\geq 1, \tag{3.10}$$

with $Z_{n+2}(x)=\langle w_0^{[1]},\big(B_n^{[1]}\big)^2\rangle\Big[-\dfrac{(n+(u)_0+2)\widehat{\gamma}_{n+1}}{\langle w_0,B_{n+2}^2\rangle}B_{n+2}(x)$
$+\dfrac{(n+(u)_0+1)}{\langle w_0,B_{n+1}^2\rangle}(x-\widehat{\beta}_n)B_{n+1}(x)-\dfrac{(n+(u)_0)}{\langle w_0,B_n^2\rangle}B_n(x)\Big],\ \ n\geq 1.$

In particular, one has: $\begin{cases}B_1^{[1]}w_0^{[1]}=Z_3w_0,\\ B_2^{[1]}w_0^{[1]}=Z_4w_0.\end{cases}$

Since $B_2^{[1]}(x)=(x-\widehat{\beta}_1)B_1^{[1]}(x)-\widehat{\gamma}_1$, then $w_0^{[1]}=\kappa\Phi w_0$, where κ is a normalizing factor and $\kappa\Phi(x)=\widehat{\gamma}_1^{-1}\big((x-\widehat{\beta}_1)Z_3(x)-Z_4(x)\big)$. With help of the expressions of

$Z_2(x)$ and $Z_3(x)$, and thanks to (3.1), we get (3.5).
Substituting (3.3) in (3.10), we obtain thanks to Lemma 2.5

$$\kappa B_n^{[1]}(x)\Phi(x) = Z_{n+2}(x),\ n \geq 1. \tag{3.11}$$

The analysis of degrees in latter equation shows that $\deg \Phi \leq 2$.
Substuting (3.3) in (3.9), and taking into account the assertion (a) of Proposition 2.7, one has $u = \widehat{B}w_0$, where

$$\widehat{B}(x) = -\kappa\Phi(x) + ((u)_0 + 1)\gamma_1^{-1}(x - \widehat{\beta}_0)B_1(x) - ((u)_0 + 2)\widehat{\gamma}_1\gamma_1^{-1}\gamma_2^{-1}B_2(x).$$

Finally, the expression (3.6) is obtained using (3.5) and (3.1). □

Remark.
1). From (3.3) and Remark 2.10, any $\mathbf{D_u}$*-classical sequence* $\{B_n\}_{n\geq 0}$ *is (strictly) quasi-orthogonal of order equal to* $\deg\Phi$ *with respect to* $w_0^{[1]}$.
Besides, when $\widehat{B} \neq 0$, *it is also (strictly) quasi-orthogonal of order equal to* $\deg\widehat{B}$ *with respect to* u.
2). Assume $u \neq 0$. *From (3.4) and Lemma 2.5, the form* u *is weakly regular.*
3). From (3.3) and (3.4), one has $\kappa\Phi u = \widehat{B}w_0^{[1]}$. *Therefore, when* $\widehat{B} \neq 0$, *the orthogonal sequence* $\{B_n^{[1]}\}_{n\geq 0}$ *is quasi-orthogonal of order equal to* $\deg\widehat{B}$ *with respect to the weakly-regular form* $v = \Phi u$.

We conclude that for any nonzero $u \in \mathbb{P}_0'$, *a* $\mathbf{D_u}$*-classical MPS* $\{B_n\}_{n\geq 0}$ *is in particular a* $\mathbf{D_u}$*-semiclassical sequence (see* [42], *Definition 2.7, p7).*

Definition 3.3. *Let* w_0 *be a regular form with* $(w)_0 \neq -n,\ n \geq 1$, *and* $\Phi, \Psi, \widehat{B}$ *are three nonzero polynomials,* Φ *monic,* $\deg\Phi \leq 2$, $\deg\Psi = 1$ *and* $\deg\widehat{B} \leq 2$. *The triplet* $(\Phi, \Psi, \widehat{B})$ *is said to be admissible with respect to* w_0, *if we have*

$$\Psi'(0) - \frac{1}{2}\big\{\Phi''(0)n + \langle w_0, \Phi\rangle\widehat{B}''(0)\big\} \neq 0,\ n \geq 1.$$

Remark. *When* $\widehat{B} = 0$, *we recognize the usual notion of admissibility of a pair of polynomials* [49].

Proposition 3.4. *A MOPS* $\{B_n\}_{n\geq 0}$ *(with respect to* w_0*) is* $\mathbf{D_u}$*-classical, if and only if there exists a triplet* $(\Phi, \Psi, \widehat{B})$ *admissible with respect to* w_0, *such that* $u = \widehat{B}w_0$, *and* w_0 *is a solution of the functional equation*

$$\mathbf{D}_u(\Phi w_0) + \Psi w_0 = 0. \tag{3.12}$$

Proof. Applying $\mathbf{D}_u$ in both hand sides of (3.3) and using (2.22), we obtain the functional equation (3.12), with $\Psi(x) = \dfrac{((u)_0 + 1)}{\kappa\gamma_1}B_1(x)$. From Lemma 3.2, there exists a polynomial $\widehat{B}$ with $\deg\widehat{B} \leq 2$ such that $u = \widehat{B}w_0$.
On the other hand, using the relations $\langle w_0, B_n^2\rangle w_n = B_n w_0$ and $\langle w_0^{[1]}, (B_n^{[1]})^2\rangle w_n^{[1]} = B_n^{[1]}w_0^{[1]}$ in (2.22), we get:

$$-\kappa\Psi B_n^{[1]}w_0 + \mathbf{D}_u(B_n^{[1]})w_0^{[1]} + (w_0^{[1]}\theta_0 B_1^{[n]})u - (u\theta_0 B_n^{[1]})w_0^{[1]} =$$

$$-(n + (u)_0 + 1)\frac{\langle w_0^{[1]}, (B_n^{[1]})^2\rangle}{\langle w_0, B_{n+1}^2\rangle}B_{n+1}w_0,\ n \geq 0.$$

Taking into account Lemma 2.5, the fact that $\gamma_1 = \langle w_0, B_1^2\rangle$, $\kappa^{-1} = \langle w_0, \Phi\rangle$, and the regularity of w_0, we conclude that:

$$\Phi(x)\mathbf{D}_u^2(B_{n+1})(x) - \Psi(x)\mathbf{D}_u(B_{n+1})(x) = \tag{3.13}$$

$$-(n+(u)_0+1)^2\frac{\langle w_0,\Phi\rangle\langle w_0^{[1]},(B_n^{[1]})^2\rangle}{\langle w_0, B_{n+1}^2\rangle}B_{n+1}(x) - \Delta(B_{n+1};\Phi,\widehat{B},w_0)(x),\ n\geq 0,$$

with $\Delta(p\,;\Phi,\widehat{B},w_0) = \big[(\Phi w_0)\theta_0\mathbf{D}_{\widehat{B}w_0}(p)\big]\widehat{B} - \big[(\widehat{B}w_0)\theta_0\mathbf{D}_{\widehat{B}w_0}(p)\big]\Phi,\ \forall p\in\mathbb{P}$.
In fact (3.13) is a linear $\mathbf{D}_u$-differential equation of second order, hence we obtain a second necessary condition taking with each element of the sequence $\{B_n\}_{n\geq 0}$. We later prove that it is sufficient, too.
Examination of degrees in (3.13) proves that

$$\Psi'(0)-\frac{1}{2}\big\{\Phi''(0)n+\langle w_0,\Phi\rangle\widehat{B}''(0)\big\} = (n+(u)_0+1)\frac{\langle w_0,\Phi\rangle\langle w_0^{[1]},(B_n^{[1]})^2\rangle}{\langle w_0,B_{n+1}^2\rangle}\neq 0,\ n\geq 1.$$

Thus, the condition is necessary.
Let us prove that it is sufficient. We shall first establish that $w_0^{[1]} = \kappa\Phi w_0$. Indeed, we have $\mathbf{D}_u(\Phi w_0) + \Psi w_0 = 0$ and $\Psi(x) = \Psi'(0)B_1(x) + \langle w_0,\Psi\rangle$. But $\langle w_0,\Psi\rangle = \langle \Psi w_0, 1\rangle = -\langle \mathbf{D}_u(\Phi w_0), 1\rangle = 0$. Thus $\mathbf{D}_u(\Phi w_0) + \Psi'(0)\gamma_1 w_1 = 0$, or $\mathbf{D}_u(\Phi w_0) - \dfrac{\Psi'(0)\gamma_1}{(u)_0+1}\mathbf{D}_u(w_0^{[1]}) = 0$. Hence $w_0^{[1]} = \kappa\Phi w_0$, where $\kappa = \dfrac{(u)_0+1}{\Psi'(0)\gamma_1}$.
Now, we will prove that $\{B_n^{[1]}\}_{n\geq 0}$ is orthogonal with respect to $w_0^{[1]} = \kappa\Phi w_0$. Indeed, one has

$$\begin{aligned}
\langle w_0^{[1]}, x^m B_n^{[1]}\rangle &= (n+(u)_0+1)^{-1}\langle w_0^{[1]}, x^m\mathbf{D}_u(B_{n+1})\rangle\\
&= (n+(u)_0+1)^{-1}\langle w_0^{[1]}, \mathbf{D}_u(x^m B_{n+1}) - \mathbf{D}_u(x^m)B_{n+1}\\
&\qquad -\big\{\big((B_{n+1}u)\theta_0 x^m\big) - (u\theta_0 x^m)B_{n+1}\big\}\rangle\\
&= (n+(u)_0+1)^{-1}\langle w_0^{[1]}, \mathbf{D}_u(x^m B_{n+1})\\
&\qquad -mx^{m-1}B_{n+1} - \big((B_{n+1}u)\theta_0 x^m\big)\rangle\\
&= (n+(u)_0+1)^{-1}\langle -\mathbf{D}_u(w_0^{[1]}), x^m B_{n+1}\rangle\\
&\qquad -\kappa m\langle w_0, \Phi x^{m-1}B_{n+1}\rangle - \langle w_0^{[1]}, \big((B_{n+1}u)\theta_0 x^m\big)\rangle,
\end{aligned}$$

by virtue of (2.23), (3.3) and (2.3). But

$$\langle w_0^{[1]}, \big((B_{n+1}u)\theta_0 x^m\big)\rangle = \langle B_{n+1}u, (w_0^{[1]}\theta_0 x^m)\rangle = \langle w_0, \widehat{B}(w_0^{[1]}\theta_0 x^m)B_{n+1}\rangle.$$

Then,

$$\begin{aligned}
\langle w_0^{[1]}, x^m B_n^{[1]}\rangle &= (n+(u)_0+1)^{-1}\Big\{\langle -\mathbf{D}_u(w_0^{[1]}), x^m B_{n+1}\rangle\\
&\qquad -\langle w_0, \big\{m\kappa\Phi x^{m-1} + \widehat{B}(w_0^{[1]}\theta_0 x^m)\big\}B_{n+1}\rangle\Big\}\\
&= (n+(u)_0+1)^{-1}\langle w_0, \Big[\kappa\Psi x^m - \big\{m\kappa\Phi(x)x^{m-1}\\
&\qquad +\widehat{B}(w_0^{[1]}\theta_0 x^m)\big\}\Big]B_{n+1}\rangle.
\end{aligned}$$

Therefore, $\langle w_0^{[1]}, x^m B_n^{[1]}\rangle = 0,\quad 0\leq m\leq n-1,\ n\geq 1$. Besides, for $m = n$:

$$\langle w_0^{[1]}, x^n B_n^{[1]}\rangle = \frac{\Psi'(0) - \frac{1}{2}\big\{\Phi''(0)n + \langle w_0,\Phi\rangle\widehat{B}''(0)\big\}}{\langle w_0,\Phi\rangle(n+(u)_0+1)}\langle w_0, x^{n+1}B_{n+1}(x)\rangle\neq 0,$$

for all integer $n \geq 1$. □

As a consequence, a $\boldsymbol{D}_u$-classical form is a particular $\mathbf{D}_u$-Laguerre-Hahn one:

Theorem 3.5. *Let w_0 be a regular form, Φ be a monic polynomial, and $\widehat{B}$ a polynomial such that $\langle w_0, \widehat{B}\rangle \neq -n,\ n \geq 1$. We set $u = \widehat{B}w_0$, then the following assertions are equivalent:*

(a) *There exists a polynomial Ψ, $\deg \Psi \leq 1$ such that*

$$\mathbf{D}_u(\Phi w_0) + \Psi w_0 = 0. \tag{3.14}$$

(b) *There exists a polynomial Ψ_1, such that*

$$(\Phi w_0)' + \Psi_1 w_0 - (\Phi\widehat{B})(x^{-1}w_0^2) = 0. \tag{3.15}$$

In this case:

$$\Psi_1(x) = \Psi(x) + \Phi(x)(w_0\theta_0\widehat{B})(x) + \widehat{B}(x)(w_0\theta_0\Phi)(x). \tag{3.16}$$

Proof. $(a) \Rightarrow (b)$. From (2.20) one has $\mathbf{D}_u(\Phi w_0) = (\Phi w_0)' - x^{-1}\big(u(\Phi w_0)\big)$. But (2.1) and the hypothesis give

$$\begin{aligned} u(\Phi w_0) &= \Phi(uw_0) - x(w_0\theta_0\Phi)(x)u = \Phi\big((\widehat{B}w_0)w_0\big) - x(w_0\theta_0\Phi)(x)\widehat{B}w_0 \\ &= \Phi\big(\widehat{B}w_0^2 - x(w_0\theta_0\widehat{B})(x)w_0)\big) - x(w_0\theta_0\Phi)(x)\widehat{B}w_0 \\ &= \Phi\widehat{B}w_0^2 - xFw_0, \end{aligned}$$

where

$$F(x) = \Phi(x)(w_0\theta_0\widehat{B})(x) + \widehat{B}(x)(w_0\theta_0\Phi)(x). \tag{3.17}$$

Using (2.5) and (2.2), it follows that

$$x^{-1}\big(u(\Phi w_0)\big) = (\Phi\widehat{B})x^{-1}w_0^2 - Fw_0 - \Big[\langle w_0^2, \theta_0(\Phi\widehat{B})\rangle - \langle w_0, F\rangle\Big]\delta.$$

Thus, (3.14) becomes

$$(\Phi w_0)' + (F+\Psi)w_0 - (\Phi\widehat{B})(x^{-1}w_0^2) + \big(\langle w_0^2, \theta_0(\Phi\widehat{B})\rangle - \langle w_0, F\rangle\big)\delta = 0.$$

Hence (3.15) and (3.16) hold taking into account (2.6).

$(b) \Rightarrow (a)$. Let us consider the polynomial F given by (3.17).
Setting $u = \widehat{B}w_0$ and $\Psi = \Psi_1 - F$, we have

$$\begin{aligned} \mathbf{D}_u(\Phi w_0) + \Psi w_0 &= (\Phi w_0)' + \Psi_1 w_0 - x^{-1}\big(u(\Phi w_0)\big) - Fw_0 \\ &= (\Phi\widehat{B})(x^{-1}w_0^2) - x^{-1}\big(u(\Phi w_0)\big) - Fw_0. \end{aligned}$$

But, calculations done above (in the proof of $(a) \Rightarrow (b)$) prove that $u(\Phi w_0) = \Phi\widehat{B}w_0^2 - xFw_0$. Therefore,

$$\begin{aligned} x^{-1}\big(u(\Phi w_0)\big) &= x^{-1}\Big(\Phi\widehat{B}w_0^2\Big) - Fw_0 + \langle w_0, F\rangle\delta, \\ &= x^{-1}\Big(\big(x\theta_0(\Phi\widehat{B}) + \Phi(0)\widehat{B}(0)\big)w_0^2\Big) - Fw_0 + \langle w_0, F\rangle\delta, \\ &= \theta_0(\Phi\widehat{B})w_0^2 - \langle w_0^2, \theta_0(\Phi\widehat{B})\rangle\delta + \Phi(0)\widehat{B}(0)x^{-1}w_0^2 - Fw_0 + \\ &\quad \langle w_0, F\rangle\delta \\ &= \theta_0(\Phi\widehat{B})w_0^2 + \Phi(0)\widehat{B}(0)x^{-1}w_0^2 - Fw_0 = (\Phi\widehat{B})\big(x^{-1}w_0^2\big) - Fw_0. \end{aligned}$$

Thus the desired result. □

Proposition 3.6. *Let $\{B_n\}_{n\geq 0}$ be a $\mathbf{D}_u$-classical sequence (with respect to w_0) fulfilling* (3.12) *with $u = \widehat{B}w_0$. The sequence $\{\widetilde{B}_n\}_{n\geq 0}$ defined by* (2.15) *is $\mathbf{D}_{\widetilde{u}}$-classical and it fulfills*

$$\mathbf{D}_{\widetilde{u}}(\widetilde{\Phi}\widetilde{w_0}) + \widetilde{\Psi}\widetilde{w_0} = 0, \quad \textit{and} \quad \widetilde{u} = \widetilde{\widehat{B}}\widetilde{w_0} = h_{a^{-1}} \circ \tau_{-b}u, \tag{3.18}$$

where $\widetilde{\widehat{B}}(x) = \widehat{B}(ax+b)$, $\tilde{w_0} = (h_{a^{-1}} \circ \tau_{-b})w_0$, $\widetilde{\Phi}(x) = a^{-t}\Phi(ax+b)$, $\widetilde{\Psi}(x) = a^{1-t}\Psi(ax+b)$, and $t = \deg(\Phi)$.

Proof.
Using (2.16), we have

$$\begin{aligned}\Phi(x)w_0 &= \Phi(x)(\tau_b \circ h_a)\widetilde{w}_0 = \tau_b\Big(\big(\tau_{-b}\Phi\big)(h_a\widetilde{w}_0)\Big)\\ &= \tau_b \circ h_a\big[\big((h_a \circ \tau_{-b})\Phi\big)\widetilde{w}_0)\big] = \tau_b \circ h_a\big[\Phi(ax+b)\widetilde{w}_0\big].\end{aligned}$$

Thanks to (2.25) and (2.27), we obtain

$$\mathbf{D}_u(\Phi w_0) = \tau_b\mathbf{D}_{\tau_{-b}u}\Big(h_a\big(\Phi(ax+b)\widetilde{w}_0\big)\Big) = a^{-1}\tau_b \circ h_a\mathbf{D}_{h_{a^{-1}}\circ\tau_{-b}u}\big(\Phi(ax+b)\widetilde{w}_0\big).$$

From (3.12), we get (3.18).
The triplet $(\widetilde{\Phi}, \widetilde{\Psi}, \widetilde{\widehat{B}})$ is admissible since for any integer $n \geq 1$, one has:

$$\widetilde{\Psi}'(0) - \frac{1}{2}\big\{\widetilde{\Phi}''(0)n + \langle\widetilde{w_0}, \widetilde{\Phi}\rangle\widetilde{\widehat{B}}''(0)\big\} = a^{2-t}\Big\{\Psi'(0) - \frac{1}{2}\big\{\Phi''(0)n + \langle w_0, \Phi\rangle\widehat{B}''(0)\big\}\Big\} \neq 0.$$

□

Proposition 3.7. *A $\mathbf{D}_u$-classical form ($u \neq 0$) fulfilling* (3.15) *is a $\mathbf{D}$-Laguerre-Hahn form of class* $\max(\deg(\Psi_1) - 1, \deg(\widehat{B}) + \deg(\Phi) - 2)$.

Proof.
Let w_0 be a $\mathbf{D}_u$-Laguerre-Hahn form of class s, thus

$$\begin{aligned}s &\leq \max\Big\{\deg(\Psi_1) - 1, \max\Big\{\deg(\Phi), \deg(\Phi) + \deg(\widehat{B})\Big\} - 2\Big\}\\ &\leq \max\Big\{\deg(\Psi_1) - 1, \deg(\Phi) + \deg(\widehat{B}) - 2\Big\}.\end{aligned}$$

Note that $\widehat{B} \neq 0$ since $u \neq 0$.
Necessarily $s = \max\Big\{\deg(\Psi_1) - 1, \deg(\Phi) + \deg(\widehat{B}) - 2\Big\}$. If not, the equation (3.15) will be simplified. Then, there exists a root c of Φ such that if we write $\Phi(x) = (x-c)\theta_c(\Phi)(x)$, we obtain

$$\Big(\theta_c(\Phi)w_0\Big)' + \Big(\theta_c(\Psi_1) + \theta_c^2(\Phi)\Big)w_0 - \theta_c(\Phi\widehat{B})(x^{-1}w_0^2) = 0.$$

From Theorem 3.5, the form w_0 fulfills

$$\mathbf{D}_u\Big(\theta_c(\Phi)w_0\Big) + \Psi_2 w_0 = 0, \tag{3.19}$$

where

$$\Psi_2(x) = \theta_c(\Psi_1) + \theta_c^2(\Phi) - \theta_c(\Phi)(x)(w_0\theta_0\widehat{B})(x) - \widehat{B}(x)\Big(w_0\theta_0\theta_c(\Phi)\Big)(x). \tag{3.20}$$

On the other hand, computing $\mathbf{D}_u\Big((x-c)\theta_c(\Phi)w_0\Big)$ gives, taking into account the regularity of w_0, and the relations (3.12), (3.19) and (3.20):

$$(x-c)\Psi_2(x)=\Psi(x)+\big((u)_0+1\big)\theta_c(\Phi)+\Big(\big(\theta_c(\Phi)w_0\big)\theta_0(x-c)\Big)\widehat{B}(x)$$
$$-\Big(u\theta_0(x-c)\Big)(x)\theta_c(\Phi)(x).$$

In particular, one has $\deg(\Psi_2)\leq 1$. So, we can write $\Psi_2(x)=aB_1(x)+b$, with $a,\ b\in\mathbb{C}$, and $a\neq 0$. Applying B_0 in both hand sides of (3.19) gives $b=0$. Therefore $\Psi_2(x)=aB_1(x)$. Similarly we get $\Psi(x)=\widetilde{a}B_1(x)$ with $\widetilde{a}\neq 0$. Hence, w_0 fulfills two functional equations (3.12) and (3.19). Therefore we have $\mathbf{D}_u\Big(\big(a\Phi-\widetilde{a}\theta_c(\Phi)\big)w_0\Big)=0$. So $a\Phi-\widetilde{a}\theta_c(\Phi)=0$. Examination of degrees proves that this is not possible. □

3.2. **The second order linear differential equation.**

Proposition 3.8. *A MOPS $\{B_n\}_{n\geq 0}$ with respect to w_0 is $\mathbf{D}_u$-classical if and only if there exist three polynomials Φ,Ψ and $\widehat{B}$, Φ monic,* $\deg\Phi\leq 2$, $\deg\Psi=1$ $\deg\widehat{B}\leq 2$, *and a complex number sequence $\{\lambda_n\}_{n\geq 0}$, $\lambda_n\neq 0$, $n\geq 0$, such that $u=\widehat{B}w_0$ and that $\{B_n\}_{n\geq 0}$ satisfies*

$$\Phi(x)\mathbf{D}_u^2(B_{n+1})(x)-\Psi(x)\mathbf{D}_u(B_{n+1})(x)= \tag{3.21}$$
$$\lambda_nB_{n+1}(x)-\Delta(B_{n+1};\Phi,\widehat{B},w_0)(x),\ n\geq 0,$$

with $\Delta(p;\Phi,\widehat{B},w_0)=\big[(\Phi w_0)\theta_0\mathbf{D}_{\widehat{B}w_0}(p)\big]\widehat{B}-\big[(\widehat{B}w_0)\theta_0\mathbf{D}_{\widehat{B}w_0}(p)\big]\Phi,\ \forall p\in\mathcal{P}$.

Proof.
We have seen that the condition is necessary (see (3.13)), where

$$\lambda_n=-\frac{\big(n+(u)_0+1\big)^2\langle w_0,\Phi\rangle\langle w_0^{[1]},(B_n^{[1]})^2\rangle}{\langle w_0,B_{n+1}^2\rangle},\ \ n\geq 0.$$

Conversely, the examination of leading coefficients in both hand sides of (3.21) gives

$$\frac{1}{2}\big\{\Phi''(0)n+\langle w_0,\Phi\rangle\widehat{B}''(0)\big\}-\Psi'(0)=\big(n+(u)_0+1\big)^{-1}\lambda_n\neq 0,\ \ n\geq 1.$$

Thus $(\Phi,\Psi,\widehat{B})$ is admissible. Besides, using (3.21) we get for $n\geq 0$:

$$\langle w_0,\Phi\mathbf{D}_u^2(B_{n+1})-\Psi\mathbf{D}_u(B_{n+1})+\Delta(B_{n+1};\Phi,\widehat{B},w_0)\rangle \ = \ \frac{\lambda_n\langle w_0,B_{n+1}\rangle}{n+(u)_0+1}=0.$$

But $\langle w_0,\Phi\mathbf{D}_u^2(B_{n+1})-\Psi\mathbf{D}_u(B_{n+1})+\Delta(B_{n+1};\Phi,\widehat{B},w_0)\rangle=$
$$\big(n+(u)_0+1\big)^{-1}\langle\mathbf{D}_u\Big[\mathbf{D}_u(\Phi w_0)+\Psi w_0\Big],B_{n+1}\rangle,\ \ n\geq 0.$$

Furthermore, it is easy to see that $\langle\mathbf{D}_u\Big[\mathbf{D}_u(\Phi w_0)+\Psi w_0\Big],B_0\rangle=0$. Hence $\langle\mathbf{D}_u\Big[\mathbf{D}_u(\Phi w_0)+\Psi w_0\Big],B_n\rangle=0,\ n\geq 0$. This implies that $\mathbf{D}_u\Big[\mathbf{D}_u(\Phi w_0)+\Psi w_0\Big]=0$. Then $\mathbf{D}_u(\Phi w_0)+\Psi w_0=0$. This ends the proof. □

3.3. **The first structure relation.**

Proposition 3.9. *A MOPS $\{B_n\}_{n\geq 0}$ is $\boldsymbol{D}_u$-classical if and only if there exists a monic polynomial Φ with* $\deg(\Phi)=t\leq 2$, *such that*

$$\Phi(x)B_n^{[1]}(x)=\sum_{\nu=n}^{n+t}\lambda_{n,\nu}B_\nu(x),\quad \lambda_{n,n}\neq 0,\ \ n\geq 0, \tag{3.22}$$

where $\{B_n^{[1]}\}_{n\geq 0}$ is defined by (2.21).

Proof. The condition is necessary. Indeed, the regular forms w_0 and $w_0^{[1]}$ verify $w_0^{[1]} = \kappa\Phi w_0$. Thanks to Proposition 2.6, this is equivalent to (3.22).
The condition is sufficient. Indeed, applying w_0 in both hand sides of (3.22) gives $\langle \Phi w_0, B_n^{[1]}\rangle = 0,\ \ n \geq 1$, and $\langle \Phi w_0, B_0^{[1]}\rangle = \lambda_{0,0} \neq 0$. Lemma 2.1 implies that $w_0^{[1]} = \kappa\Phi w_0$ where $\kappa = \langle w_0, \Phi\rangle^{-1} = \lambda_{0,0}^{-1}$. Then

$$\begin{aligned}\langle w_0^{[1]}, B_n^{[1]}B_m^{[1]}\rangle &= \kappa\langle w_0, \Phi B_n^{[1]}B_m^{[1]}\rangle \\ &= \kappa\sum_{\nu=n}^{n+t}\lambda_{n,\nu}\langle w_0, B_\nu B_m^{[1]}\rangle = \begin{cases} 0, & m < n,\ n \geq 1,\\ \kappa\lambda_{n,n}\langle w_0, B_n^2\rangle \neq 0, & m = n.\end{cases}\end{aligned}$$

The sequence $\{B_n^{[1]}\}_{n\geq 0}$ is orthogonal with respect to $w_0^{[1]}$. □

Remark.

(a) From (3.11), (3.5) and (3.1), we can write the structure relation (3.22) as given by Al-Salam and Chihara in the **D**-classical case [7]:

$$\Phi(x)B_n^{[1]}(x) = \big\{X_n + Y_n x\big\}B_{n+1}(x) - \gamma_{n+1}T_nB_n(x),\ \ n \geq 0, \tag{3.23}$$

where for $n \geq 1$:

$$\begin{aligned}X_n &= -\kappa^{-1}\langle w_0^{[1]}, (B_n^{[1]})^2\rangle\Big[\frac{\big(n+(u)_0+1\big)\widehat{\beta}_n}{\langle w_0, B_{n+1}^2\rangle} - \frac{\big(n+(u)_0+2\big)\beta_{n+1}\widehat{\gamma}_{n+1}}{\langle w_0, B_{n+2}^2\rangle}\Big],\\ X_0 &= \kappa^{-1}(f - \beta_1 e),\\ Y_n &= \kappa^{-1}\langle w_0^{[1]}, (B_n^{[1]})^2\rangle\Big[\frac{\big(n+(u)_0+1\big)}{\langle w_0, B_{n+1}^2\rangle} - \frac{\big(n+(u)_0+2\big)\widehat{\gamma}_{n+1}}{\langle w_0, B_{n+2}^2\rangle}\Big],\\ Y_0 &= \kappa^{-1}e,\\ T_n &= -\kappa^{-1}\langle w_0^{[1]}, (B_n^{[1]})^2\rangle\Big[\frac{\big(n+(u)_0\big)}{\langle w_0, B_{n+1}^2\rangle} + \frac{\big(n+(u)_0+2\big)\widehat{\gamma}_{n+1}}{\langle w_0, B_{n+2}^2\rangle}\Big],\\ T_0 &= \kappa^{-1}\big(e - \gamma_1^{-1}(2(u)_0+3)\big).\end{aligned}$$

Using the orthogonality of $\{B_n\}_{n\geq 0}$ in (3.23), we get (3.22). In particular:

$$\lambda_{n,n} = \gamma_{n+1}(Y_n - T_n) = \kappa^{-1}\frac{\langle w_0^{[1]}, (B_n^{[1]})^2\rangle}{\langle w_0, B_{n+1}^2\rangle}\big(2n + 2(u)_0 + 1\big) \neq 0,\ \ n \geq 1,$$

and $\lambda_{0,0} = \dfrac{\big(2(u)_0+3\big)}{\kappa\gamma_1} \neq 0$.

Thus, the condition $2n + 2(u)_0 + 1 \neq 0,\ \ n \geq 1$ holds.

(b) In fact, the condition $Y_n - T_n \neq 0,\ \ n \geq 0$ is a consequence of the regularity of w_0. Indeed, let $\{B_n\}_{n\geq 0}$ be a MOPS with respect to w_0, and fulfilling (3.23). Applying w_0 in both hand sides of (3.23) we get $\lambda w_0^{[1]} = \Phi w_0$, with $\lambda = \langle w_0, \Phi\rangle$. Since w_0 is regular, we have $\lambda \neq 0$.
Besides, from the orthogonality of $\{B_n\}_{n\geq 0}$, and thanks to (3.23), one has: For $n \geq 0$, and $0 \leq m \leq n$:

$$\begin{aligned}\langle w_0^{[1]}, B_n^{[1]}B_m^{[1]}\rangle &= \lambda^{-1}\langle \Phi w_0, B_n^{[1]}B_m^{[1]}\rangle\\ &= \lambda^{-1}\langle w_0, \Big\{\big\{X_n + Y_n x\big\}B_{n+1}(x) - \gamma_{n+1}T_nB_n(x)\Big\}B_m^{[1]}\rangle\\ &= \begin{cases} 0, & \text{if} \quad m < n,\ n \geq 1,\\ \gamma_{n+1}(Y_n - T_n)\langle w_0, B_n^2\rangle, & \text{if} \quad m = n.\end{cases}\end{aligned}$$

We must have $Y_n - T_n \neq 0,\ n \geq 0$. If not, there exists an integer $n_0 \geq 0$ such that $Y_{n_0} - T_{n_0} = 0$. Then $\langle w_0^{[1]}, (B_{n_0}^{[1]})^2\rangle = 0$. Thanks to (3.22), this implies $\langle B_{n_0}^{[1]} w_0^{[1]}, B_n^{[1]}\rangle = 0, n \geq 0$. Hence $B_{n_0}^{[1]} w_0^{[1]} = 0$. But $\lambda w_0^{[1]} = \Phi w_0$. Then $B_{n_0}^{[1]} \Phi w_0 = 0$. This contradicts the regularity of w_0.

Proposition 3.10. *A MOPS $\{B_n\}_{n\geq 0}$ is $\mathbf{D}_u$-classical if and only if there exist two polynomials Φ and $\widehat{B}$, Φ monic, $\deg(\Phi) \leq 2$, $\deg(\widehat{B}) \leq 2$ and $\langle w_0, \widehat{B}\rangle = 1$, such that*

$$\Phi(x)B'_{n+1}(x) - B(x)B_n^{(1)}(x) = q_3(x;n)B_{n+1}(x) + F_n B_n(x),\ n \geq 0, \tag{3.24}$$

where $q_3(.;n),\ n \geq 0$, is a polynomial $\deg\big(q_3(.;n)\big) \leq 3,\ n \geq 0$ and

$$B(x) = -\Phi(x)\widehat{B}(x). \tag{3.25}$$

In this case, we have:

$$q_3(x;n) = \Phi(x)\big(w_0\theta_0\widehat{B}(x)\big) + \big(n + (u)_0 + 1\big)\big\{X_n + Y_n x\big\},\ \ n \geq 0, \tag{3.26}$$

$$F_n(x) = -\big(n + (u)_0 + 1\big)\gamma_{n+1}T_n + \gamma_2^{-1}\langle w_0, \widehat{B}B_2\rangle\delta_{n,0},\ \ n \geq 0. \tag{3.27}$$

In particular, the sequence $\{B_n\}_{n\geq 0}$ is a $\mathbf{D}$-Laguerre-Hahn sequence.

Proof. The condition is necessary. Thanks to Remark 3.3 (a), and (3.1), we have :

$$\Phi(x)B'_{n+1}(x) + \Phi(x)(u\theta_0 B_{n+1}) = \big(n + (u)_0 + 1\big)\big\{X_n + Y_n x\big\}B_{n+1}(x) \tag{3.28}$$
$$-\big(n + (u)_0 + 1\big))\gamma_{n+1}T_n B_n(x),\ \ n \geq 0.$$

Since $u = \widehat{B}w_0$, we get:

$$\begin{aligned}(u\theta_0 B_{n+1})(x) &= \langle w_0,\ \frac{\widehat{B}(y)\big(B_{n+1}(x) - B_{n+1}(y)\big)}{x-y}\rangle \qquad (3.29)\\ &= \widehat{B}(x)B_n^{(1)}(x) - (w_0\theta_0\widehat{B}(x))B_{n+1}(x)\\ &\quad +\langle w_0, \Big(\frac{\widehat{B}(x) - \widehat{B}(y)}{x-y}\Big)B_{n+1}(y)\rangle,\\ &= \widehat{B}(x)B_n^{(1)}(x) - (w_0\theta_0\widehat{B}(x))B_{n+1}(x)\\ &\quad -{\gamma_2}^{-1}\langle w_0, \widehat{B}B_2\rangle\delta_{n,0},\ n \geq 0.\end{aligned}$$

Substituting (3.29) in (3.28), the equations (3.24)-(3.27) hold.
The condition is sufficient. Indeed, we set $u = \widehat{B}w_0$. Using (3.29) and (3.25) in (3.24), gives, for $n \geq 0$:

$$\Phi(x)B'_{n+1}(x) + \Phi(x)\big[(u\theta_0 B_{n+1})(x) - (w_0\theta_0\widehat{B})B_{n+1}(x)\big] = q_3(x,n)B_{n+1}(x) + F_n B_n(x),$$

which can be written as $\Phi(x)B_n^{[1]}(x) = A_1(x,n)B_{n+1}(x) - \gamma_{n+1}T_n B_n(x),\ n \geq 0$, where $T_n = -\big(n + (u)_0 + 1\big)^{-1}\gamma_{n+1}^{-1}F_n,\ n \geq 0$, and $A_1(x,n) = \big(n + (u)_0 + 1\big)^{-1}\big\{q_3(x,n) + \Phi(x)(u\theta_0\widehat{B})(x)\big\},\ n \geq 0$.Comparing the degrees, we get $\deg\big(A_1(.,n)\big) \leq 1,\ n \geq 0$. Thus, we can write $A_1(x,n) = X_n + Y_n x,\ n \geq 0$. From Remark 3.3 (b), necessarily $Y_n - T_n \neq 0,\ n \geq 0$. Taking into account the first part of Remark 3.3, and Proposition 3.9, the desired result holds.. □

3.4. **The second structure relation.**

Proposition 3.11. *A MOPS $\{B_n\}_{n\geq 0}$ is $\mathbf{D}_u$-classical if and only if there exist an integer $0 \leq t \leq 2$ and a polynomial $\widehat{B}$, $\deg(\widehat{B}) \leq 2$, such that $u = \widehat{B}w_0$ and:*

$$B_n(x) = \sum_{\nu=n-t}^{n} \lambda_{n,\nu} B_\nu^{[1]}(x), \quad \lambda_{n,n-t} \neq 0, \ \ n \geq t, \tag{3.30}$$

where $\{B_n^{[1]}\}_{n\geq 0}$ is defined by (2.21).

Proof. The condition is necessary. Indeed, we have $w_0^{[1]} = \kappa \Phi w_0$. From Proposition 2.6, this is equivalent to (3.30). Besides, from Lemma 3.2 one has $u = \widehat{B}w_0$.
The condition is also sufficient. Indeed, the relation (3.30) can be written as:

$$B_n(x) = B_n^{[1]}(x) + a_n B_{n-1}^{[1]}(x) + b_n B_{n-2}^{[1]}(x), \ n \geq 0, \tag{3.31}$$

with $\quad a_0 = b_0 = b_1 = 0.$
On the other hand, applying $\mathbf{D}_u$ in both hand sides of (3.1) gives

$$\begin{aligned}(n+(u_0)+2)B_{n+1}^{[1]}(x) = \big(n+(u)_0+1\big)(x-\beta_{n+1})B_n^{[1]}(x) - \big(n+(u)_0\big)\gamma_{n+1}B_{n-1}^{[1]}(x)\\ +\langle u,\, B_{n+1}\rangle + B_{n+1}(x),\ n \geq 0.\end{aligned}$$

By (3.31), the fact that $u = \widehat{B}w_0$, and the orthogonality of $\{B_n\}_{n\geq 0}$, we have:

$$\begin{cases} B_0^{[1]}(x) = 1,\ B_1^{[1]}(x) = x - \widehat{\beta}_0, \\ B_{n+2}^{[1]}(x) = (x-\widehat{\beta}_{n+1})B_{n+1}^{[1]}(x) - \widehat{\gamma}_{n+1}B_n^{[1]}(x),\ n \geq 0, \end{cases} \tag{3.32}$$

$$\text{with } \ \widehat{\beta}_0 = \beta_1 - \frac{a_1 + \langle u,\, B_1\rangle}{(u)_0+1},\ \widehat{\beta}_{n+1} = \beta_{n+2} - \big(n+(u)_0+2\big)^{-1}a_{n+2},\ n \geq 0$$

$$\widehat{\gamma}_1 = \frac{-b_2 + \big((u)_0+1\big)\gamma_2 - \langle u,\, B_2\rangle}{(u)_0+2},\ \widehat{\gamma}_{n+1} = \frac{-b_{n+2} + \big(n+(u)_0+1\big)\gamma_{n+2}}{n+(u)_0+2},\ n \geq 1.$$

Necessarily, $\widehat{\gamma}_n \neq 0,\ n \geq 1$, otherwise there exists an integer $N \geq 1$ such that $\widehat{\gamma}_N = 0$. Thanks to (3.32), there exists $c \in \mathbb{C}$ such that $B_N^{[1]}(c) = B_{N+1}^{[1]}(c) = 0$. Then, $B_n^{[1]}(c) = 0,\ n \geq N$. This is absurd because if not, the relation (3.31) implies that $B_n(c) = 0,\ n \geq N$. Thus the sequence $\{B_n\}_{n\geq 0}$ is not orthogonal. □

Proposition 3.12. *We suppose that $\{B_n\}_{n\geq 0}$ is $\mathbf{D}_u$-classical, and fulfills* (3.1) *and* (3.2). *Then $\{B_n\}_{n\geq 0}$ verifies:*

$$B_n(x) = B_n^{[1]}(x) + a_n B_{n-1}^{[1]}(x) + b_n B_{n-2}^{[1]}(x),\ n \geq 1, \tag{3.33}$$

with $\quad a_1 = \big((u)_0+1\big)(\beta_1 - \widehat{\beta}_0) - \langle u,\, B_1\rangle, \quad a_n = \big(n+(u)_0\big)(\beta_n - \widehat{\beta}_{n-1}),\ \ n \geq 2,$

and $\quad b_1 = 0, \quad b_2 = \big((u)_0+1\big)\gamma_2 - \big((u)_0+2\big)\widehat{\gamma}_1 - \langle u,\, B_2\rangle,$

$$b_n = \big(n+(u)_0-1\big)\gamma_n - \big(n+(u)_0\big)\widehat{\gamma}_{n-1},\ \ n \geq 3,$$

Proof. Taking the derivative in (3.1), then using (2.4), we get:

$$\begin{aligned}\big(n+(u)_0+2\big)B_{n+1}^{[1]}(x) = \big(n+(u)_0+1\big)(x-\beta_{n+1})B_n^{[1]}(x)\\ -\big(n+(u)_0\big)\gamma_{n+1}B_{n-1}^{[1]}(x) + \langle u,\, B_{n+1}\rangle + B_{n+1}(x),\ n \geq 0.\end{aligned} \tag{3.34}$$

But from (3.2), where $n \to n-1$, one has:

$$xB_n^{[1]}(x) = B_{n+1}^{[1]}(x) + \widehat{\beta}_n B_n^{[1]}(x) + \widehat{\gamma}_n B_{n-1}^{[1]}(x),\ n \geq 0. \tag{3.35}$$

Using (3.35) in (3.34), we obtain

$$B_{n+1}(x) = B_{n+1}^{[1]}(x) + \big(n+(u)_0+1\big)(\beta_{n+1}-\widehat{\beta}_n)B_n^{[1]}(x)$$
$$+\Big(\big(n+(u)_0\big)\gamma_{n+1} - \big(n+(u)_0+1\big)\widehat{\gamma}_n\Big)B_{n-1}^{[1]}(x) - \langle u,\, B_{n+1}\rangle,\ n\geq 0.$$

The relation (3.33) holds since we have $\langle u, B_n\rangle = \langle w_0, \widehat{B}B_n\rangle = 0,\ n\geq 3$, and $B_{-n}^{[1]} = 0,\ n\geq 1$. □

3.5. The functional equation fulfilled by $w_0^{[1]}$.

Proposition 3.13. *The form $w_0^{[1]}$ satisfies:*

$$(\Phi w_0^{[1]})' + \widetilde{\Psi}w_0^{[1]} - \kappa^{-1}\widehat{B}\Big(x^{-1}\big(w_0^{[1]}\big)^2\Big) + a\widehat{B}(0)\delta = 0,$$

where $\widetilde{\Psi}(x) = H - \Phi' - 2\widehat{B}(w_0\theta_0\Phi)(x),\ \ H := F-\Psi$, and $a = 2\langle w_0^{[1]}, (w_0\theta_0\Phi)\rangle - \kappa\langle w_0^2, \theta_0(\Phi^2)\rangle$.

Proof. Multiplying both hand sides of (3.15) by $\kappa\Phi$, and taking into account the identity $w_0^{[1]} = \kappa\Phi w_0$, we get

$$(\Phi w_0^{[1]})' + (H-\Phi')w_0^{[1]} - \kappa\widehat{B}\Phi^2(x^{-1}w_0^2) = 0. \tag{3.36}$$

On the other hand, we have

$$\Phi^2(x^{-1}w_0^2) = x^{-1}(\Phi^2 w_0^2) + \langle w_0^2, \theta_0(\Phi^2)\rangle\delta, \tag{3.37}$$

and $\Phi^2 w_0^2 = (\Phi w_0)^2 + 2x\Phi(x)(w_0\theta_0\Phi)(x)w_0$. Then (3.37) will be written, thanks to $w_0^{[1]} = \kappa\Phi w_0$, as follows

$$\Phi^2\big(x^{-1}w_0^2\big) = \kappa^{-2}x^{-1}\Big(w_0^{[1]}\Big)^2 + 2\kappa^{-1}x^{-1}\big(x(w_0\theta_0\Phi)(x)w_0^{[1]}\big) + \langle w_0^2, \theta_0(\Phi^2)\rangle\delta = 0.$$

With the help of the last equation, the relation (3.36) gives the desired result. □

Remark. *Another expression for a is: $a = \kappa\langle w_0, w_0(\theta_0\Phi)^2 - 2x(w_0\theta_0\Phi)(x)(\theta_0\Phi)(x)\rangle$.*

3.6. The nonlinear system fulfilled by $\beta_n,\ \gamma_{n+1},\ \widehat{\beta}_n$ and $\widehat{\gamma}_{n+1}$.

On the one hand, the comparison of constant terms in the equation $B_1^{[1]}(x) = \big((u)_0+2\big)^{-1}\big(B_2'(x) + (u\theta_0 B_2)(x)\big)$, gives

$$\big((u)_0+2\big)\widehat{\beta}_0 = \big((u)_0+1\big)\beta_1 + \beta_0 - \langle u,\, B_1\rangle. \tag{3.38}$$

On the other hand, multiplying both hand sides of (3.33) by x, and taking into account (3.2), we have

$$\begin{aligned} xB_{n+1}(x) = \ & B_{n+2}^{[1]}(x) + \Big\{\widehat{\beta}_{n+1} + a_{n+1}\Big\}B_{n+1}^{[1]}(x) + \\ & \Big\{\widehat{\gamma}_{n+1} + \widehat{\beta}_n a_{n+1} + b_{n+1}\Big\}B_n^{[1]}(x) + \Big\{a_{n+1}\widehat{\gamma}_n + b_{n+1}\widehat{\beta}_{n-1}\Big\}B_{n-1}^{[1]}(x) + \\ & b_{n+1}\widehat{\gamma}_{n-1}B_{n-2}^{[1]}(x) - \langle u,\, B_{n+1}\rangle B_1^{[1]}(x) - \langle u,\, B_{n+1}\rangle\widehat{\beta}_0,\ \ n\geq 1. \end{aligned} \tag{3.39}$$

Using (3.1) and (3.33), we get

$$\begin{aligned} xB_{n+1}(x) = B_{n+2}^{[1]}(x) + \Big\{a_{n+2} + \beta_{n+1}\Big\}B_{n+1}^{[1]}(x) + \\ \Big\{b_{n+2} + \beta_{n+1}a_{n+1} + \gamma_{n+1}\Big\}B_n^{[1]}(x) + \\ \Big\{b_{n+1}\beta_{n+1} + a_n\gamma_{n+1}\Big\}B_{n-1}^{[1]}(x) + b_n\gamma_{n+1}B_{n-2}^{[1]}(x) - \end{aligned} \tag{3.40}$$

$$\langle u,\, B_{n+2}\rangle - \beta_{n+1}\langle u,\, B_{n+1}\rangle - \gamma_{n+1}\langle u,\, B_n\rangle,\ \ n\geq 1.$$

Comparing (3.39) and (3.40) gives, thanks to (3.38), the following system:

$$\begin{aligned}
\big(n+(u)_0+2\big)\widehat{\beta}_n - \big(n+(u)_0\big)\widehat{\beta}_{n-1} &= \big(n+(u)_0+1\big)\beta_{n+1} - \big(n+(u)_0-1\big)\beta_n \\
&\quad +\langle u,\, B_1\rangle\delta_{n,1} + \big((u)_0\beta_0 - \langle u,\, B_1\rangle\big)\delta_{n,0},\ n\geq 0. \qquad (\beta_{-1}=0)\\
\big(n+(u)_0+3\big)\widehat{\gamma}_{n+1} - \big(n+(u)_0+1\big)\widehat{\gamma}_n &= \big(n+(u)_0+1\big)\gamma_{n+2} - \big(n+(u)_0-1\big)\gamma_{n+1} + \\
&\quad \big(n+(u)_0+1\big)(\beta_{n+1}-\widehat{\beta}_n)^2 + \langle u,\, B_2\rangle\delta_{n,1} \\
&\quad -\big[\langle u,\, B_2\rangle + \langle u,\, B_1\rangle(\beta_1-\widehat{\beta}_0) - (u)_0\gamma_1\big]\delta_{n,0},\ \ n\geq 0. \qquad (\widehat{\gamma}_0=0)\\
\big(n+(u)_0+2\big)\widehat{\gamma}_{n+1}\big[-2\beta_{n+2}+\widehat{\beta}_n+\widehat{\beta}_{n+1}\big] &+ \big(n+(u)_0+1\big)\gamma_{n+2}\big[-2\widehat{\beta}_n+\beta_{n+2}+\beta_{n+1}\big] \qquad (3.41)\\
&\quad -\big[\langle u,\, B_2\rangle(\beta_2-\widehat{\beta}_0) + \gamma_2\langle u,\, B_1\rangle\big]\delta_{n,0}=0,\ \ n\geq 0.\\
\big(n+(u)_0+3\big)\widehat{\gamma}_{n+2}\widehat{\gamma}_{n+1} - 2\big(n+(u)_0+2\big)\gamma_{n+3}\widehat{\gamma}_{n+1} &+ \big(n+(u)_0+1\big)\gamma_{n+3}\gamma_{n+2} \qquad (3.42)\\
&\quad -\gamma_3\langle u,\, B_2\rangle\delta_{n,0}=0,\ \ n\geq 0.
\end{aligned}$$

Remark. *The first additional condition (3.7) is none other than (3.42), with* $n=1$.

4. Canonical cases

If we set:

$$\widehat{\gamma}_n = \frac{(n+(u)_0)\vartheta_n}{n+(u)_0+1}\gamma_{n+1},\ \ \text{with}\ \ \vartheta_n\neq 0,\ n\geq 1, \tag{4.1}$$

then, (3.42) gives

$$\vartheta_{n+2}\vartheta_{n+1} - 2\vartheta_{n+1} + 1 = 0,\ n\geq 1, \tag{4.2}$$

$$\vartheta_2\vartheta_1 - 2\vartheta_1 + 1 - \frac{\langle u,\, B_2\rangle}{((u)_0+1)\gamma_2} = 0. \tag{4.3}$$

Since (4.2) is a Riccati equation, we consider $\vartheta_n = \dfrac{\xi_{n+1}}{\xi_n},\ n\geq 1,\ \xi_n\neq 0,\ n\geq 1$. Thus $\xi_{n+3}-2\xi_{n+2}+\xi_{n+1}=0,\ n\geq 1$. So, the general solution is $\xi_n = an+b,\ \ n\geq 2$, where $(a,b)\in\mathbb{C}^2\backslash\{(0,0)\}$. We deduce that $\vartheta_n = \dfrac{a(n+1)+b}{an+b},\ n\geq 2$. Therefore, the relation (4.3) gives $\Big[1-\dfrac{\langle u,\, B_2\rangle}{((u)_0+1)\gamma_2}\Big]\xi_1 = a+b$.

Two cases appear:

(A) $1-\dfrac{\langle u,\, B_2\rangle}{((u)_0+1)\gamma_2}=0$. Consequently, $\vartheta_n = \dfrac{n}{n-1},\ n\geq 2$ and ϑ_1 is arbitrary.

(B) $1-\dfrac{\langle u,\, B_2\rangle}{((u)_0+1)\gamma_2}\neq 0$. Here we have:

$$\vartheta_n = \frac{a(n+1)+b}{an+b},\ n\geq 2,\ \text{ and }\ \vartheta_1 = \frac{2a+b}{a+b}\Big[1-\frac{\langle u,\, B_2\rangle}{((u)_0+1)\gamma_2}\Big].$$

We have to distinguish two subcases:

$\mathbf{B}_1$. $a=0$. Then $\vartheta_n = 1,\ n\geq 2$, and $\vartheta_1 = 1-\dfrac{\langle u,\, B_2\rangle}{((u)_0+1)\gamma_2}$.

$\mathbf{B}_2$. $a\neq 0$. If we set $\rho=\dfrac{b}{a}$, we can write $\vartheta_n = \dfrac{n+\rho+1}{n+\rho},\ \ \rho\neq -n,\ \ n\geq 2$, and $\vartheta_1 = \dfrac{\rho+2}{\rho+1}\Big[1-\dfrac{\langle u,\, B_2\rangle}{((u)_0+1)\gamma_2}\Big],\ \ \rho\neq -1$.

Note that when $\rho\to\infty$, we reorganize the subcase $\mathbf{B}_1$.

Remark. *Taking into account (3.41), the second additional condition (3.7) becomes (4.1), with* $n = 1$.

Remark. *Before solving the above system, we shall give update expressions of polynomials* $\Phi(x)$ *and* $\widehat{B}(x)$. *Indeed, it is easy to see that:*

$$\kappa\Phi(x) = \frac{\langle w_0^{[1]}, B_2\rangle}{\gamma_1\gamma_2}B_2(x) + \frac{\langle w_0^{[1]}, B_1\rangle}{\gamma_1}B_1(x) + 1.$$

But from (3.33), one has $\langle w_0^{[1]}, B_2\rangle = b_2 = \big((u)_0+1\big)\gamma_2 - \big((u)_0+2\big)\widehat{\gamma}_1 - \langle u,\, B_2\rangle$. *Using (4.1), we obtain* $\langle w_0^{[1]}, B_2\rangle = \big((u)_0+1\big)\gamma_2(1-\vartheta_1) - \langle u,\, B_2\rangle$. *We also have* $\langle w_0^{[1]}, B_1\rangle = a_1$. *Using (3.38), gives* $\langle w_0^{[1]}, B_1\rangle = \frac{\big((u)_0+1\big)\beta_1 - \big((u)_0+1\big)\beta_0 - \langle u,\, B_1\rangle}{(u)_0+2}$.
For $\widehat{B}(x)$, *we should remark that from (3.5) and (3.6), we get* $e = \kappa$, $a + \kappa = \frac{((u)_0+1)(1-\vartheta_1)}{\gamma_1}$ *and* $b + f = \frac{((u)_0+1)(\beta_1-\widehat{\beta}_0)}{\gamma_1}$. *Finally, it is clear that:*

$$\widehat{B}(x) = \frac{\langle u, B_2\rangle}{\gamma_1\gamma_2}B_2(x) + \frac{\langle u, B_1\rangle}{\gamma_1}B_1(x) + (u)_0. \tag{4.4}$$

Case (**A**). The system (3.41) becomes:

$$\begin{aligned}(n+(u)_0+2)\widehat{\beta}_n - (n+(u)_0)\widehat{\beta}_{n-1} = (n+(u)_0+1)\beta_{n+1} - (n+(u)_0-1)\beta_n + \\ \langle u,\, B_1\rangle\delta_{n,1} + \big((u)_0\beta_0 - \langle u,\, B_1\rangle\big)\delta_{n,0},\ n \geq 0.\end{aligned} \tag{4.5}$$

$$\frac{(2n+(u)_0+3)}{n(n+(u)_0+2)}\gamma_{n+2} = \frac{(2n+(u)_0-1)}{(n-1)(n+(u)_0+1)}\gamma_{n+1} + (\beta_{n+1} - \widehat{\beta}_n)^2,\ n \geq 2. \tag{4.6}$$

$$\gamma_3 = \frac{((u)_0+3)}{((u)_0+2)((u)_0+5)}\Big[\big\{((u)_0+1)\vartheta_1 + 1\big\}\gamma_2 + ((u)_0+2)(\beta_2 - \widehat{\beta}_1)^2\Big].$$

$$\gamma_2 = \frac{((u)_0+2)}{((u)_0+1)((u)_0+3)\vartheta_1}\Big[\gamma_1 + ((u)_0+1)(\beta_1 - \widehat{\beta}_0)^2 - (\beta_1 - \widehat{\beta}_0)\langle u,\, B_1\rangle\Big].$$

$$\begin{aligned}(n+1)\widehat{\beta}_{n+1} - (n-1)\widehat{\beta}_n = (n+2)\beta_{n+2} - n\beta_{n+1} + \\ \Big\{(1-\vartheta_1)\widehat{\beta}_1 + (2-\vartheta_1)\widehat{\beta}_0 + 2(\vartheta_1-1)\beta_2 - \beta_1 + \frac{1}{(u)_0+1}\langle u,\, B_1\rangle\Big\}\delta_{n,0},\ n \geq 0.\end{aligned} \tag{4.7}$$

Subtracting (4.7) from (4.5), one has $\widehat{\beta}_n - \frac{(u)_0}{(u)_0+2}\beta_{n+1} = \widehat{\beta}_{n-1} - \frac{(u)_0}{(u)_0+2}\beta_n,\ n \geq 2$, then by iteration $\widehat{\beta}_n = \frac{(u)_0}{(u)_0+2}\beta_{n+1} + \widehat{\beta}_1 - \frac{(u)_0}{(u)_0+2}\beta_2,\ n \geq 1$. Inserting this equation in (4.5), we get

$$(2n+(u)_0+2)\beta_{n+1} - (2n+(u)_0-2)\beta_n = 2\Big(((u)_0+2)\widehat{\beta}_1 - (u)_0\beta_2\Big),\ n \geq 2.$$

Therefore, $\beta_n = d + \frac{((u)_0+2)\Big[((u)_0+2)d - \theta\Big]}{(2n+(u)_0)(2n+(u)_0-2)},\ n \geq 2$, where

$$d = \frac{1}{2((u)_0+2)}\Big\{((u)_0+2-\sigma)\beta_0 + ((u)_0+2+\sigma)\beta_1 - \frac{\sigma}{(u)_0+1}\langle u,\, B_1\rangle\Big\},\ \ \sigma = \vartheta_1^{-1},$$

and $\theta = \beta_1 + ((u)_0+1)\beta_0 + \langle u, B_1\rangle$.

Hence, $\widehat{\beta}_n = d + \dfrac{(u)_0\Big[((u)_0+2)d-\theta\Big]}{(2n+(u)_0)(2n+(u)_0+2)}$, $n \geq 1$.Thus, iterating (4.6) gives:

$$\gamma_{n+1} = \frac{(n-1)(n+(u)_0+1)}{(2n+(u)_0-1)(2n+(u)_0+1)}\Big\{ -\frac{\Big[((u)_0+2)d-\theta\Big]^2}{(2n+(u)_0)^2} + ((u)_0+5)\gamma_3 + \frac{\Big[((u)_0+2)d-\theta\Big]^2}{((u)_0+4)^2}\Big\},\ n \geq 2.$$

Moreover, we have $\gamma_3 = \dfrac{((u)_0+3)}{((u)_0+2)((u)_0+5)}\Big[\dfrac{((u)_0+\sigma+1)}{\sigma}\gamma_2 + \dfrac{4\Big[((u)_0+2)d-\theta\Big]^2}{((u)_0+2)((u)_0+4)^2}\Big]$,

$\gamma_2 = \dfrac{((u)_0+2)\sigma}{((u)_0+1)((u)_0+3)}\Big\{\gamma_1 + \dfrac{[\theta-((u)_0+2)\beta_0]}{(u)_0+2}\Big[\dfrac{((u)_0+1)[\theta-((u)_0+2)\beta_0]}{(u)_0+2} - \langle u,\, B_1\rangle\Big]\Big\}$.

The above system is completely solved.

Note that in this case, the polynomial $\Phi(x)$ is of degree 2 given by

$$\Phi(x) = (x-d)^2 - \frac{\mu}{4}, \quad \text{where} \quad \mu = 4\Big\{(\beta_0-d)^2 + \frac{((u)_0+\sigma+1)}{(u)_0+1}\gamma_1\Big\}. \tag{4.8}$$

Since $\kappa = -\dfrac{((u)_0+1)}{\sigma\gamma_1}$, we get $\Psi(x) = -\sigma(x-\beta_0)$, and $\widehat{B}(x) = \dfrac{((u)_0+1)}{\gamma_1}B_2(x) + \dfrac{\langle u, B_1\rangle}{\gamma_1}B_1(x) + (u)_0$.

Depending on the number of roots of $\Phi(x)$, two subcases appear:

- A_1. Φ has a double root ($\mu = 0$). In this case $\Phi(x) = (x-d)^2$. Through a shift, we can choose 0 as a root. Then we assume that $d = 0$. So $\Phi(x) = x^2$.
- A_2. Φ has two different roots ($\mu \neq 0$). Upper to an affine transformation, we can assume that $\Phi(x) = x^2 - 1$. This amounts to take $d = 0$ and $\mu = 4$.

In the subcase A_1, we have $d = \mu = 0$. Then

$$\begin{cases} \langle u,\, B_1\rangle = \dfrac{((u)_0+1)}{\sigma}\Big\{((u)_0+2-\sigma)\beta_0 + ((u)_0+2+\sigma)\beta_1\Big\}, \\ \beta_0^2 + \dfrac{((u)_0+\sigma+1)}{(u)_0+1}\gamma_1 = 0. \end{cases} \tag{4.9}$$

Necessarily, $\sigma \neq -((u)_0+1)$ and then $\beta_0 \neq 0$. Otherwise, (4.9) implies $\beta_1 = -\langle u, B_1\rangle$, and $\theta = 0$. Thus, $\gamma_3 = 0$ wihch is impossible. Consequently, $\gamma_1 = -\dfrac{((u)_0+1)\beta_0^2}{(u)_0+\sigma+1}$. If $\sigma = 2\alpha$, then a dilation allows us to assume that $\beta_0 = -\alpha^{-1}$. Hence, $\Psi(x) = -2(\alpha x+1)$ and $\gamma_1 = -\dfrac{((u)_0+1)}{\alpha^2((u)_0+2\alpha+1)}$. Taking into account (4.9), we get the results summarized in Table 1. In particular, thanks to Proposition 3.7, the sequence $\{B_n\}_{n\geq 0}$ is a $\mathbf{D}$-Laguerre-Hahn sequence of class $s = 2$.

For the subcase A_2, the relation (4.9) remains valid. Besides, we have $\dfrac{((u)_0+\sigma+1)}{(u)_0+1}\gamma_1 = 1 - \beta_0^2$. According to the value of β_0, we give explicitly the corresponding coefficients (See Table 1). In particular, thanks to Proposition 3.7, the sequence $\{B_n\}_{n\geq 0}$ is a $\mathbf{D}$-Laguerre-Hahn sequence of class $s = 2$.

TABLE 1. The $\mathbf{D}_u$-Classical Polynomials - **Case (A)**

Case (A): $\langle u, B_2\rangle = ((u)_0+1)\gamma_2 \quad , (u)_0 \neq -n,\ n \geq 1$		
$\mathbf{A_1}$: $\Phi(x) = x^2$	$\mathbf{A_2}$: $\Phi(x) = x^2-1$	
	$\beta_0 = \pm 1$	$\beta_0 \neq \pm 1$
$\Psi(x) = -2(\alpha x+1)$,	$\Psi(x) = ((u)_0+1)(x-\beta_0)$,	$\Psi(x) = -\sigma(x-\beta_0),\ \sigma \neq 0$,
$\beta_0 = -\frac{1}{\alpha},\ \alpha \neq 0$, $\beta_1 = \dfrac{((u)_0+1)((u)_0+2)+2\alpha^2\theta}{\alpha((u)_0+2)((u)_0+2\alpha+1)},\ (u)_0+2\alpha+1 \neq 0$, $\beta_n = \dfrac{-((u)_0+2)\theta}{(2n+(u)_0-2)(2n+(u)_0)},\ n \geq 2$, $\gamma_1 = \dfrac{-((u)_0+1)}{\alpha^2((u)_0+2\alpha+1)}$, $\gamma_2 = \dfrac{-2\alpha\theta^2}{((u)_0+2)((u)_0+3)((u)_0+2\alpha+1)},\ \theta \neq 0$, $\gamma_{n+1} = \dfrac{-(n-1)(n+(u)_0+1)\theta^2}{(2n+(u)_0-1)(2n+(u)_0)^2(2n+(u)_0+1)},\ n \geq 2$, $\widehat{\beta}_0 = \dfrac{-((u)_0+1)(\theta+2)-2}{((u)_0+2)((u)_0+2\alpha+1)}$, $\widehat{\beta}_n = \dfrac{-(u)_0\theta}{(2n+(u)_0)(2n+(u)_0+2)},\ n \geq 1$, $\widehat{\gamma}_1 = \dfrac{-((u)_0+1)\theta^2}{((u)_0+3)((u)_0+2)^2((u)_0+2\alpha+1)}$, $\widehat{\gamma}_n = \dfrac{-n(n+(u)_0)\theta^2}{(2n+(u)_0-1)(2n+(u)_0)^2(2n+(u)_0+1)},\ n \geq 2$,	$\beta_0 = \pm 1$, $\beta_1 \in \mathbb{C}$, $\beta_n = \dfrac{((u)_0+2)^2\beta_0}{(2n+(u)_0-2)(2n+(u)_0)},\quad n \geq 2$, $\gamma_1 \in \mathbb{C}\setminus\{0\},\ \gamma_1 - 2\beta_0\beta_1 - 2 \neq 0$, $\gamma_2 = \dfrac{-((u)_0+2)\big[\gamma_1-2\beta_0\beta_1-2\big]}{(u)_0+3}$, $\gamma_{n+1} = \dfrac{4(n-1)^2(n+(u)_0+1)^2}{(2n+(u)_0-1)(2n+(u)_0)^2(2n+(u)_0+1)},\ n \geq 2$, $\widehat{\beta}_0 = 2\beta_0+\beta_1$, $\widehat{\beta}_n = \dfrac{(u)_0((u)_0+2)\beta_0}{(2n+(u)_0)(2n+(u)_0+2)},\ n \geq 1$, $\widehat{\gamma}_1 = \dfrac{\gamma_1-2\beta_0\beta_1-2}{(u)_0+3}$, $\widehat{\gamma}_n = \dfrac{-4(n-1)n(n+(u)_0)(n+(u)_0+1)}{(2n+(u)_0-1)(2n+(u)_0)^2(2n+(u)_0+1)},\ n \geq 2$,	$\beta_0 \neq \pm 1$, $\beta_1 = \dfrac{\sigma\theta-((u)_0+1)((u)_0+2)\beta_0}{((u)_0+2)((u)_0+\sigma+1)},\ \sigma \neq -((u)_0+1)$, $\beta_n = \dfrac{-((u)_0+2)\theta}{(2n+(u)_0)(2n+(u)_0-2)},\ n \geq 2$, $\gamma_1 = \dfrac{(1-\beta_0^2)((u)_0+1)}{(u)_0+\sigma+1}$, $\gamma_2 = \dfrac{\sigma((u)_0+\theta+2)((u)_0-\theta+2)}{((u)_0+2)((u)_0+3)((u)_0+\sigma+1)},\ \theta \neq \pm((u)_0+2)$, $\gamma_{n+1} = \dfrac{(n-1)(n+(u)_0+1)[(2n+(u)_0+\theta)(2n+(u)_0-\theta)}{(2n+(u)_0-1)(2n+(u)_0)^2(2n+(u)_0+1)},\ \theta \neq \pm(2n+(u)_0),\ n \geq 2$, $\widehat{\beta}_0 = \dfrac{((u)_0+2)\sigma\beta_0-((u)_0+1)\theta}{((u)_0+2)((u)_0+\sigma+1)}$, $\widehat{\beta}_n = \dfrac{-((u)_0+2)\theta}{(2n+(u)_0)(2n+(u)_0+2)},\ n \geq 1$, $\widehat{\gamma}_1 = \dfrac{((u)_0+1)((u)_0+\theta+2)((u)_0-\theta+2)}{((u)_0+2)^2((u)_0+3)((u)_0+\sigma+1)}$, $\widehat{\gamma}_n = \dfrac{-(n-1)(n+(u)_0)(2n+(u)_0+\theta)(2n+(u)_0-\theta)}{(2n+(u)_0-1)(2n+(u)_0)^2(2n+(u)_0+1)},\ n \geq 2$,
$\widehat{B}(x) = -\alpha\Big\{\alpha((u)_0+2\alpha+1)x^2 + ((u)_0+\alpha(\theta+4))x+(\theta+2)\Big\}$, $\Psi_1(x) = -2\alpha^2((u)_0+2\alpha+1)x^3 - 2\alpha\big(\alpha(\theta+2)+1\big)x^2+(u)_0x+\theta$.	$\widehat{B}(x) = \frac{1}{\gamma_1}\Big\{((u)_0+1)x^2 - \big[(3(u)_0+4)\beta_0+((u)_0+2)\beta_1\big]x + 2(u)_0+3+((u)_0+2)\beta_0\beta_1-\gamma_1\Big\}$, $\Psi_1(x) = \frac{1}{\gamma_1}\Big\{2((u)_0+1)x^3 + -2\big[(2(u)_0+3)\beta_0+((u)_0+2)\beta_1\big]x^2 + \big[(u)_0\gamma_1-2((u)_0+1)\big]x + ((u)_0+2)\big[2\beta_1-\gamma_1\beta_0\big]+2(2(u)_0+3)\beta_0\Big\}$.	$\widehat{B}(x) = \frac{1}{1-\beta_0^2}\Big\{((u)_0+\sigma+1)x^2 + (\theta-(2\sigma+(u)_0)\beta_0)x + (\sigma{\beta_0}^2-\theta\beta_0-1)\Big\}$, $\Psi_1(x) = \frac{1}{1-\beta_0^2}\Big\{((u)_0+\sigma+1)x^3+2(\theta+(1-\sigma)\beta_0)x^2 - ((u)_0+2\sigma+2+(u)_0{\beta_0}^2)x - (\theta{\beta_0}^2+2(1-\sigma)\beta_0+\theta)\Big\}$.

Case (B).

The subcase $\mathbf{B_1}$. Here $\langle u,\, B_2\rangle = ((u)_0+1)(1-\vartheta_1)\gamma_2$. The following system holds

$$(n+(u)_0+2)\widehat{\beta}_n - (n+(u)_0)\widehat{\beta}_{n-1} = (n+(u)_0+1)\beta_{n+1} - (n+(u)_0-1)\beta_n,\ n\geq 2. \tag{4.10}$$

$$((u)_0+3)\widehat{\beta}_1 - ((u)_0+1)\widehat{\beta}_0 = ((u)_0+2)\beta_2 - (u)_0\beta_1 + \langle u,\, B_1\rangle. \tag{4.11}$$

$$\frac{\gamma_{n+2}}{n+(u)_0+2} - \frac{\gamma_{n+1}}{n+(u)_0+1} = (\beta_{n+1}-\widehat{\beta}_n)^2,\ \ n\geq 1.$$

$$\frac{((u)_0+1)\vartheta_1}{(u)_0+2}\,\gamma_2 - \gamma_1 = ((u)_0+1)(\beta_1-\widehat{\beta}_0)^2 - (\beta_1-\widehat{\beta}_0)\langle u,\, B_1\rangle.$$

$$\widehat{\beta}_{n+1} - \widehat{\beta}_n = \beta_{n+2} - \beta_{n+1},\ n\geq 1. \tag{4.12}$$

$$\vartheta_1\widehat{\beta}_1 - \widehat{\beta}_0 = \vartheta_1\beta_2 - \beta_1 + \frac{1}{(u)_0+1}\langle u,\, B_1\rangle. \tag{4.13}$$

Using (4.12) in (4.10), we get $\widehat{\beta}_n = -\frac{1}{2}\beta_{n+2} + \frac{3}{2}\beta_{n+1}, n\geq 1$. Thus $\beta_n = (\beta_3-\beta_2)(n-2)+\beta_2, n\geq 2$, so

$$\widehat{\beta}_n = \frac{(2n-3)}{2}(\beta_3-\beta_2) + \beta_2,\ \ n\geq 1. \tag{4.14}$$

Thanks to (4.11), one has

$$\widehat{\beta}_0 = -\frac{(u)_0+3}{2((u)_0+1)}\,\beta_3 + \frac{(u)_0+5}{2((u)_0+1)}\,\beta_2 + \frac{(u)_0}{(u)_0+1}\beta_1 - \frac{1}{(u)_0+1}\langle u, B_1\rangle, \tag{4.15}$$

comparing (3.38) and (4.15), we obtain

$$-\frac{((u)_0+2)((u)_0+3)}{2}\,\beta_3 + \frac{((u)_0+2)((u)_0+5)}{2}\,\beta_2 = \beta_1 + ((u)_0+1)\beta_0 + \langle u, B_1\rangle. \tag{4.16}$$

Thanks to (4.14), the relation (4.13) gives $\beta_3 = \beta_2 - \dfrac{2c}{\vartheta_1}$, where

$$c = \frac{1}{(u)_0+2}\Big\{\beta_0 - \beta_1 + \frac{1}{(u)_0+1}\langle u, B_1\rangle\Big\}. \tag{4.17}$$

Equation (4.16) becomes $\beta_2 = \beta_1 + \dfrac{c}{\vartheta_1}\Big[((u)_0+1)\vartheta_1 - ((u)_0+3)\Big]$. So $\beta_3 = \beta_1 + \dfrac{c}{\vartheta_1}\Big[((u)_0+1)\vartheta_1 - ((u)_0+5)\Big]$, which gives $\beta_n = -\dfrac{c}{\vartheta_1}[2n+(u)_0-1] - c + \beta_0 + \dfrac{1}{(u)_0+1}\langle u, B_1\rangle,\quad n\geq 2$. From (4.17), we have $\beta_1 = -((u)_0+2)c + \beta_0 + \dfrac{1}{(u)_0+1}\langle u, B_1\rangle$. Also, we have $\widehat{\beta}_n = -\dfrac{c}{\vartheta_1}\Big[2n+(u)_0\Big] - c + \beta_0 + \dfrac{1}{(u)_0+1}\langle u, B_1\rangle, n\geq 1$. From (3.38), we obtain $\widehat{\beta}_0 = -((u)_0+1)c + \beta_0$. Using the new expressions of β_n and $\widehat{\beta}_n$, we get for $n\geq 1$:

$$\begin{cases} \gamma_{n+1} = \frac{(n+(u)_0+1)}{((u)_0+1)\vartheta_1}\Big\{\gamma_1 + c((u)_0+1)\Big[c - \frac{1}{(u)_0+1}\langle u, B_1\rangle\Big] + \frac{c^2}{\vartheta_1}((u)_0+1)(n-1)\Big\}, \\ \widehat{\gamma}_{n+1} = \frac{(n+(u)_0+1)}{((u)_0+1)\vartheta_1}\Big\{\gamma_1 + c((u)_0+1)\Big[c - \frac{1}{(u)_0+1}\langle u, B_1\rangle\Big] + \frac{c^2}{\vartheta_1}((u)_0+1)n\Big\}, \\ \widehat{\gamma}_1 = \gamma_1 + ((u)_0+1)c\Big[c - \frac{1}{(u)_0+1}\langle u, B_1\rangle\Big]. \end{cases}$$

The system is completely solved. Note that in this case, $\deg\Phi\leq 1$ and

$$\kappa\Phi(x) = -\frac{((u)_0+1)c}{\gamma_1}x + \frac{((u)_0+1)c\beta_0}{\gamma_1} + 1.$$

If $\deg(\Phi) = 0$ (c=0), then $\Phi(x) = 1$. After a suitable affine transformation, we can suppose that $\beta_0 = -\dfrac{1}{(u)_0+1}\langle u, B_1\rangle := \lambda$ and $\gamma_1 = \dfrac{((u)_0+1)\rho}{2}$, where $\rho := \vartheta_1$. From (4.4) and (3.16), we get the expressions of $\widehat{B}(x)$ and $\Psi_1(x)$. In particular, we recognize here the **D**-Laguerre-Hahn sequence of class zero, nonsingular, of Hermite type [2, 16] (see first column in Table 2).

If $\deg(\Phi) = 1$, $(c \neq 0)$, then $\kappa = -\dfrac{((u)_0+1)c}{\gamma_1}$, $\Phi(x) = x - \dfrac{\gamma_1}{((u)_0+1)c} - \beta_0$ and $\Psi(x) = -\dfrac{1}{c}(x-\beta_0)$. By means of a suitable affine transformation, we can choose β_0 and c such that $\dfrac{\gamma_1}{((u)_0+1)c} + \beta_0 = 0$, and $c = -\vartheta_1$. Thus, $\Phi(x) = x$ and $\Psi(x) = \dfrac{1}{\vartheta_1}(x-\beta_0)$. Putting $\alpha = \vartheta_1 - ((u)_0+2) + \beta_0 + \dfrac{1}{(u)_0+1}\langle u, B_1\rangle$, we get the expressions written in second column of Table 2.
In particular: If $\vartheta_1 = 1$, according to Proposition 3.7, the obtained sequence is the **D**-Laguerre-Hahn sequence of class zero, nonsingular, of Laguerre type [2, 16]. If $\vartheta_1 \neq 1$, we recognize the perturbated sequence of order one of the **D**-Laguerre-Hahn sequence of class zero, nonsingular, of Laguerre type [2, 16]. It is a **D**-Laguerre-Hahn sequence of class $s = 1$.

TABLE 2. The $\mathbf{D}_u$-Classical Polynomials - **Case** (**B**)

Subcase $\mathbf{B_1}$: $\vartheta_1 := 1 - \dfrac{\langle u,\ B_2\rangle}{((u)_0+1)\gamma_2} \neq 0$, $(u)_0 \neq -n,\ n \geq 1$	
$\Phi(x) = 1$	$\Phi(x) = x$
$\Psi(x) = \dfrac{2}{\rho}(x-\lambda)$, $\rho \neq 0$,	$\Psi(x) = \dfrac{1}{\vartheta_1}(x-\beta_0)$, $\beta_0 \neq 0$,
$\beta_0 = -\dfrac{1}{(u)_0+1}\langle u, B_1\rangle := \lambda$,	$\beta_0 := \alpha + 2(u)_0 + 1 + \lambda$,
	$\alpha := \vartheta_1 - ((u)_0+2) + \beta_0 + \dfrac{1}{(u)_0+1}\langle u, B_1\rangle$, $\lambda \in \mathbb{C}$,
$\beta_n = 0, n \geq 1$,	$\beta_1 = 2((u)_0+1) + \alpha + 1 + ((u)_0+1)(\vartheta_1 - 1)$,
$\gamma_1 = \dfrac{((u)_0+1)\rho}{2}$, $(\rho = \vartheta_1)$	$\beta_n = 2(n+(u)_0) + \alpha + 1,\ \ n \geq 2$,
$\gamma_{n+1} = \dfrac{n+(u)_0+1}{2}, n \geq 1$,	$\gamma_1 = \beta_0\vartheta_1((u)_0+1)$,
	$\gamma_{n+1} = (n+(u)_0+1)(n+\alpha+(u)_0+1),\ n \geq 1$,
$\widehat{\beta}_0 = \lambda$,	$, \alpha \neq -(n+(u)_0+1)$,
$\widehat{\beta}_n = 0, n \geq 1$,	$\widehat{\beta}_0 = 2(u)_0 + \alpha + 2 + ((u)_0+1)\vartheta_1 - 1 + \lambda$,
$\widehat{\gamma}_1 = \dfrac{((u)_0+1)\rho}{2}$, $\rho \neq 0$,	$\widehat{\beta}_n = 2(n+(u)_0) + \alpha + 2,\ \ n \geq 1$,
$\widehat{\gamma}_n = \dfrac{n+(u)_0}{2}, n \geq 2$,	$\widehat{\gamma}_1 = \vartheta_1((u)_0+1)(\alpha+(u)_0+2)$,
	$\widehat{\gamma}_n = (n+(u)_0)(n+\alpha+(u)_0+1), n \geq 2$,
	$\alpha \neq -(n+(u)_0+1), n \geq 2$,
$\widehat{B}(x) = \dfrac{2(1-\rho)}{\rho}x^2$	$\widehat{B}(x) = \dfrac{(1-\vartheta_1)}{(\alpha+2(u)_0+1+\lambda)\vartheta_1}x^2$
$+\dfrac{2\lambda(\rho-2)}{\rho}x$	$+\dfrac{((u)_0+1)(1-\vartheta_1)^2 + (\vartheta_1-1)(4(u)_0+2\alpha+3) - (u)_0 + (\vartheta_1-2)\lambda}{(\alpha+2(u)_0+1+\lambda)\vartheta_1}x$
$-1 + \dfrac{2\lambda^2}{\rho} + ((u)_0+1)\rho$,	$+\dfrac{(1-\vartheta_1)((u)_0+\alpha) + 2(u)_0 + \lambda}{\vartheta_1}$,
$\Psi_1(x) = \dfrac{2(2-\rho)}{\rho}x - \dfrac{4\lambda}{\rho}$.	$\Psi_1(x) = \dfrac{2(1-\vartheta_1)}{(\alpha+2(u)_0+1+\lambda)\vartheta_1}x^2$
	$+\dfrac{2((u)_0+1)(\vartheta_1-1)^2 + (\vartheta_1-1)(6(u)_0+3\alpha+\lambda+5) + \alpha + 1 - \lambda}{(\alpha+2(u)_0+1+\lambda)\vartheta_1}x$
	$+\dfrac{(1-\vartheta_1)((u)_0+\alpha) - \alpha - 1}{\vartheta_1}$.

The subcase $\mathbf{B_2}$. Here $\kappa = -\dfrac{((u)_0+1)\vartheta_1}{(\rho+2)\gamma_1}$, and the polynomial $\Phi(x)$ is of degree 2 given by (4.8), where

$$\begin{cases} d = \frac{1}{2}\big\{\beta_0+\beta_1+\frac{\rho+2}{((u)_0+2)\vartheta_1}[\beta_1-\beta_0-\frac{1}{(u)_0+1}\langle u,\, B_1\rangle]\big\}, \\ \mu = 4\big\{(\beta_0-d)^2+[\frac{(\rho+2)}{((u)_0+1)\vartheta_1}+1]\gamma_1\big\}. \end{cases} \tag{4.18}$$

As in the previous cases, we analyse the situations $\Phi(x)=x^2$, and $\Phi(x)=x^2-1$. Particulary, we usually suppose $d=0$. Note that here $\Psi(x) = -\dfrac{(\rho+2)}{\vartheta_1}B_1(x)$, and

$$\widehat{B}(x) = \frac{((u)_0+1)}{\gamma_1}\Big\{\Big[1-\frac{(\rho+1)\vartheta_1}{\rho+2}\Big]x^2+\Big[\Big[\frac{(\rho+1)\vartheta_1}{\rho+2}-1\Big](\beta_0+\beta_1)+\frac{\langle u,\, B_1\rangle}{(u)_0+1}\Big]x \tag{4.19}$$
$$+\Big[1-\frac{(\rho+1)\vartheta_1}{\rho+2}\Big](\beta_0\beta_1-\gamma_1)+\frac{(u)_0\gamma_1}{(u)_0+1}-\frac{\beta_0\langle u,\, B_1\rangle}{(u)_0+1}\Big\}.$$

In particular, the sequence $\{B_n\}_{n\geq 0}$ is a **D**-Laguerre-Hahn sequence of class $s\leq 2$. Besides, the system (3.41) becomes

$$(n+(u)_0+2)\widehat{\beta}_n-(n+(u)_0)\widehat{\beta}_{n-1} = (n+(u)_0+1)\beta_{n+1}-(n+(u)_0-1)\beta_n,\ n\geq 2. \tag{4.20}$$

$$((u)_0+3)\widehat{\beta}_1-((u)_0+1)\widehat{\beta}_0 = ((u)_0+2)\beta_2-(u)_0\beta_1+\langle u, B_1\rangle. \tag{4.21}$$

$$((u)_0+2)\widehat{\beta}_0 = ((u)_0+1)\beta_1+\beta_0-\langle u, B_1\rangle. \tag{4.22}$$

$$\frac{(2n+(u)_0+\rho+4)}{(n+(u)_0+2)(n+\rho+1)}\gamma_{n+2}-\frac{(2n+(u)_0+\rho)}{(n+(u)_0+1)(n+\rho)}\gamma_{n+1} = (\beta_{n+1}-\widehat{\beta}_n)^2,\ n\geq 2. \tag{4.23}$$

$$\frac{((u)_0+\rho+6)}{((u)_0+3)(\rho+2)}\gamma_3-\Big[\frac{((u)_0+1)\vartheta_1}{\rho+2}+1\Big]\frac{\gamma_2}{(u)_0+2} = (\beta_2-\widehat{\beta}_1)^2. \tag{4.24}$$

$$\frac{((u)_0+\rho+4)}{((u)_0+2)(\rho+2)}\vartheta_1\gamma_2-\frac{\gamma_1}{(u)_0+1} = (\beta_1-\widehat{\beta}_0)\big[\beta_1-\widehat{\beta}_0-\frac{1}{(u)_0+1}\langle u, B_1\rangle\big]. \tag{4.25}$$

$$(n+\rho+2)\widehat{\beta}_{n+1}-(n+\rho)\widehat{\beta}_n = (n+\rho+3)\beta_{n+2}-(n+\rho+1)\beta_{n+1},\ n\geq 1. \tag{4.26}$$

$$\vartheta_1\widehat{\beta}_1-\big(1-\frac{\vartheta_1}{\rho+2}\big)\widehat{\beta}_0 = \frac{(\rho+3)\vartheta_1}{\rho+2}\beta_2-\beta_1+\frac{1}{(u)_0+1}\langle u, B_1\rangle. \tag{4.27}$$

Subtracting (4.26) from (4.20), we get

$$((u)_0-\rho+1)(\widehat{\beta}_{n+1}-\widehat{\beta}_n) = ((u)_0-\rho-1)(\beta_{n+2}-\beta_{n+1}),\ n\geq 1. \tag{4.28}$$

Two situations arise:

•$\underline{B_{21}:}$ $\rho\neq(u)_0-1$

Here $\beta_{n+2}-\beta_{n+1} = \xi(\widehat{\beta}_{n+1}-\widehat{\beta}_n),\ n\geq 1$, where $\xi = \dfrac{(u)_0-\rho+1}{(u)_0-\rho-1}$. Then

$$\beta_{n+1} = \xi\widehat{\beta}_n+\beta_2-\xi\widehat{\beta}_1,\ n\geq 1.$$

Relation (4.20), where $n\to n+1$, gives

$$-\frac{(2n+(u)_0+\rho+5)}{((u)_0-\rho-1)}\widehat{\beta}_{n+1}+\frac{(2n+(u)_0+\rho+1)}{(u)_0-\rho-1}\widehat{\beta}_n = 2(\beta_2-\xi\widehat{\beta}_1),\ n\geq 1. \tag{4.29}$$

On the other hand, since $d=0$, we have

$$\langle u, B_1\rangle = \big(((u)_0+2)\ell-((u)_0+1)\big)\beta_0+\big(((u)_0+2)\ell+((u)_0+1)\big)\beta_1, \tag{4.30}$$

where $\ell := \frac{((u)_0+1)\vartheta_1}{\rho+2}$. Then, (4.22) becomes

$$\widehat{\beta}_0 = -\ell\beta_1 + (1-\ell)\beta_0. \tag{4.31}$$

Using (4.30) and (4.31), respectively in (4.21) and (4.27), gives

$$((u)_0+3)\widehat{\beta}_1 = ((u)_0+2)\beta_2 + \ell\beta_0 + (\ell+1)\beta_1 \tag{4.32}$$

$$(\rho+2)\widehat{\beta}_1 = (\rho+3)\beta_2 + \ell\beta_0 + (\ell+1)\beta_1 \tag{4.33}$$

The difference and the sum of the two last equations give respectively:

$$\beta_2 - \xi\widehat{\beta}_1 = 0, \tag{4.34}$$

$$((u)_0+\rho+5)(\widehat{\beta}_1 - \beta_2) = 2\ell\beta_0 + 2(1+\ell)\beta_1. \tag{4.35}$$

Therefore,

$$\widehat{\beta}_1 = -\frac{((u)_0-\rho-1)}{(u)_0+\rho+5}[\ell\beta_0 + 2(1+\ell)\beta_1], \tag{4.36}$$

$$\beta_2 = -\frac{((u)_0-\rho+1)}{(u)_0+\rho+5}[\ell\beta_0 + 2(1+\ell)\beta_1]. \tag{4.37}$$

Back to (4.29) and taking into account (4.34), we discuss two situations:

• $\underline{B_{211}:}$ $\quad 2n+(u)_0+\rho+1 \neq 0, \quad \forall n \geq 1.$

Iterating (4.29) gives

$$\begin{cases} \widehat{\beta}_n = -\frac{((u)_0+\rho+3)((u)_0-\rho-1)[\ell\beta_0+2(1+\ell)\beta_1]}{(2n+(u)_0+\rho+1)(2n+(u)_0+\rho+3)}, n \geq 1, \\ \beta_{n+1} = -\frac{((u)_0+\rho+3)((u)_0-\rho+1)[\ell\beta_0+2(1+\ell)\beta_1]}{(2n+(u)_0+\rho+1)(2n+(u)_0+\rho+3)}, n \geq 1. \end{cases} \tag{4.38}$$

Therefore,

$$\beta_{n+1} - \widehat{\beta}_n = -\frac{2((u)_0+\rho+3)[\ell\beta_0+2(1+\ell)\beta_1]}{(2n+(u)_0+\rho+1)(2n+(u)_0+\rho+3)}, n \geq 1. \tag{4.39}$$

In order to compute the coefficients γ_{n+1} and $\widehat{\gamma}_{n+1}$, we study two possibilities:

• $\Phi(x) = x^2$. Taking $\mu = 0$ and using (4.30) in (4.18), we have $\beta_0^2 + \gamma_1(1+\frac{1}{\ell}) = 0$. Besides, from (4.31) and (4.30) one has

$$\begin{cases} \beta_1 - \widehat{\beta}_0 = (\ell-1)\beta_0 + (\ell+1)\beta_1, \\ \beta_1 - \widehat{\beta}_0 - \frac{1}{(u)_0+1}\langle u, B_1\rangle = \frac{-\ell}{(u)_0+1}(\beta_0+\beta_1). \end{cases} \tag{4.40}$$

Necessarily $1+\frac{1}{\ell} \neq 0$. Otherwise $\beta_0 = 0$. Then, from (4.31) we obtain $\widehat{\beta}_0 - \beta_1 = 0$. Therefore, equation (4.35) leads to $\widehat{\beta}_1 - \beta_2 = 0$. Then (4.24) gives $(u)_0 + \rho + 6 = 0$. Taking $n = 3$ in (4.23), gives $\gamma_5 = 0$ which is absurd. We conclude that

$$\gamma_1 = \frac{-\ell\beta_0^2}{1+\ell}, \quad \beta_0 \neq 0. \tag{4.41}$$

In (4.25), necessarily $(u)_0+\rho+4 \neq 0$. If not, one has $\gamma_1 = -((u)_0+1)(\beta_1 - \widehat{\beta}_0)\big(\beta_1 - \widehat{\beta}_0 - \frac{1}{(u)_0+1}\langle u, B_1\rangle\big)$. Thanks to (4.40) and (4.41), we get $\big(\ell\beta_0 + (1+\ell)\beta_1\big)^2 = 0$, or equivalently $\ell\beta_0 + (1+\ell)\beta_1 = 0$. Thus, (4.35) proves that $\widehat{\beta}_1 - \beta_2 = 0$. Taking into account (4.34), we deduce $\widehat{\beta}_1 = \beta_2 = 0$. So, from (4.39) we obtain $\beta_{n+1} - \widehat{\beta}_n = 0$, $n \geq 1$. Equation (4.23), where $n = 2$, implies that $\gamma_4 = 0$ which is not possible. Once the expression of γ_2 is obtained, we easily deduce the rest of the three term recurrence coefficients (see first column in Table 3). In addition, we have

$\widehat{B}(x) = -\frac{1+\ell}{\ell\beta_0^2}\Big\{\big[(u)_0+1-(\rho+1)\ell\big]x^2 +$

$$\Big\{\big[((u)_0+\rho+3)\ell-2((u)_0+1)\big]\beta_0+((u)_0+\rho+3)\ell\beta_1\Big\}x+$$
$$\big[(u)_0+1-(\rho+1)\ell\big]\beta_0\beta_1-\tfrac{\ell\beta_0^2[(\rho+1)\ell-1]}{1+\ell}\Big\}.$$

$$\Psi_1(x)=-\tfrac{1+\ell}{\ell\beta_0^2}\Big\{\big[(u)_0+1-(\rho+1)\ell\big]x^3+$$
$$\big\{\big[((u)_0+2)\ell-((u)_0+1)\big]\beta_0+((u)_0+\rho+3)\ell\beta_1\big\}x^2+$$
$$\big\{\big[((u)_0+\rho+3)\ell-2((u)_0+1)\big]\beta_0^2+\big[(u)_0+1+((u)_0+2)\ell\big]\beta_0\beta_1-\tfrac{\beta_0^2[(\rho+1)\ell^2-\ell-(u)_0-1]}{1+\ell}\big\}x$$
$$+\big[(u)_0+1-(\rho+1)\ell\big]\beta_0^2\beta_1-\tfrac{\beta_0^3[(\rho+1)\ell^2-\ell-(u)_0-1]}{1+\ell}\Big\}.$$

• $\Phi(x)=x^2-1$. We note here that since too many subcases will appear later, we simply refer to (4.19) and (3.16) for computing the polynomials $\widehat{B}(x)$ and $\Psi_1(x)$. On the other hand, taking $\mu=4$ and using (4.30) in (4.18), we get $\gamma_1(1+\frac{1}{\ell})=1-\beta_0^2$. We distinguish two subcases:

If $\ell=-1$, relations (4.30), (4.31) and (4.40) are still valid. Besides, $\beta_0=\pm1$ and $\gamma_1\in\mathbb{C}\setminus\{0\}$. Thus, it is easy to obtain the corresponding coefficients given in the second column of Table 3.

If $\ell\neq-1$, we have $\gamma_1=\dfrac{\ell(1-\beta_0^2)}{\ell+1}$, $\beta_0\neq\pm1$ and relations (4.30), (4.31), (4.35) and (4.40). In order to explicit the coefficients γ_n and $\widehat{\gamma}_n$, we need to discus many situations when solving equations (4.23)-(4.25). The results are summarized in the last column of Table 3.

• $\underline{B_{212}:}$ $\exists N\geq1$, such that $2N+(u)_0+\rho+1=0$. From (4.34), we get

$$(N+(u)_0+1)\widehat{\beta}_1-(N+(u)_0)\beta_2=0. \tag{4.42}$$

Therefore, subtracting (4.20) from (4.26), we obtain $\widehat{\beta}_n=\dfrac{N+(u)_0}{N+(u)_0+1}\beta_{n+1}, n\geq1$. Then, from (4.20) we have $(n-N+2)\beta_{n+2}-(n-N)\beta_{n+1}=0, n\geq1$. Thus, the expressions of β_n and $\widehat{\beta}_n$ hold as shown in Table 4.

For $N\geq3$, we have $\beta_2=0$. Thanks to (4.42), one has $\widehat{\beta}_1=0$. Consequently, taking into account (4.30), (4.31) and (4.32), we get

$$\begin{cases}\widehat{\beta}_0=\beta_0+\beta_1, \beta_0=(1+\ell)\widehat{\beta}_0, \beta_1=-\ell\widehat{\beta}_0,\ \beta_1-\widehat{\beta}_0=-\beta_0,\\ \langle u,B_1\rangle=-[(u)_0\ell+(u)_0+1]\widehat{\beta}_0,\\ \beta_1-\widehat{\beta}_0-\frac{1}{(u)_0+1}\langle u,B_1\rangle=-\frac{\ell}{(u)_0+1}\widehat{\beta}_0,\end{cases} \tag{4.43}$$

where $\ell=-\dfrac{((u)_0+1)\vartheta_1}{2N+(u)_0-1}$. The coefficients γ_{n+1} satisfy the following system:

$$\begin{cases}\frac{(2n-2N+3)}{(n+(u)_0+2)(n-2N-(u)_0)}\gamma_{n+2}-\frac{(2n-2N-1)}{(n+(u)_0+1)(n-2N-(u)_0-1)}\gamma_{n+1}=0,\\ \text{where } n\geq2, n\notin\{N-1,N\},\\ \frac{1}{(N+(u)_0+1)^2}\gamma_{N+1}+\frac{3}{(N+(u)_0+1)(N+(u)_0+2)}\gamma_N=\frac{-\beta_N^2}{(N+(u)_0+1)^2},\\ \frac{3}{(N+(u)_0+2)(N+(u)_0)}\gamma_{N+2}+\frac{1}{(N+(u)_0+1)^2}\gamma_{N+1}=\frac{-\beta_N^2}{(N+(u)_0+1)^2},\\ \frac{(2N-5)}{((u)_0+3)(2N+(u)_0-1)}\gamma_3-\frac{1+\ell}{(u)_0+2}\gamma_2=0,\\ \frac{(2N-3)\ell}{(u)_0+2}\gamma_2+\gamma_1=-(1+\ell)\ell\widehat{\beta}_0^2.\end{cases}$$

Remark that $\ell\neq-1$, otherwise we obtain $\gamma_3=0$. If we assume that $\mu=0$, then thanks to (4.18) this is equivalent to $\gamma_1=-\frac{\beta_0^2}{1+\ell}$. So $\gamma_2=0$ which is impossible. We conclude that necessarily $\mu=4$, or equivalently $\gamma_1=\frac{\ell(1-\beta_0^2)}{1+\ell}$. The coefficients γ_n and $\widehat{\gamma}_n$ follow.

Thanks to (4.42), it is easy to see that equations (4.43) are still valid for $N=2$. Besides, $\beta_3-\widehat{\beta}_2=\widehat{\beta}_1-\beta_2=-\frac{1}{(u)_0+3}\beta_2$ and

$$\begin{cases} \frac{(2n-1)}{(n+(u)_0+2)(n-(u)_0-4)}\gamma_{n+2}-\frac{(2n-5)}{(n+(u)_0+1)(n-(u)_0-5)}\gamma_{n+1}=0, n\geq 3,\\ \frac{3}{((u)_0+2)((u)_0+4)}\gamma_4+\frac{1}{((u)_0+3)^2}\gamma_3=-(\beta_2-\widehat{\beta}_1)^2,\\ \frac{1}{((u)_0+3)^2}\gamma_3+\frac{1+\ell}{(u)_0+2}\gamma_2=-(\beta_2-\widehat{\beta}_1)^2,\\ \frac{\ell}{(u)_0+2}\gamma_2+\gamma_1=-(1+\ell)\ell\widehat{\beta}_0^2. \end{cases}$$

As in the case where $N\geq 3$, we prove that $\ell\neq -1$, and so $\mu=4$. This leads to the expressions found in Table 4.

Finally, if $N=1$, the relations (4.30)-(4.37), give thanks to (4.42): $\widehat{\beta}_1-\beta_2=\ell\beta_0+(\ell+1)\beta_1$, $\beta_1-\widehat{\beta}_0=(\ell+1)\beta_1+(\ell-1)\beta_0$, $\beta_1-\widehat{\beta}_0-\frac{1}{(u)_0+1}\langle u,B_1\rangle=-\frac{\ell}{((u)_0+1)}(\beta_0+\beta_1)$. In particular, as done above, we obtain:

$\gamma_{n+1}=\frac{(n+(u)_0+1)(n-(u)_0-3)}{(2n-1)(2n-3)}, n\geq 2$, and $\widehat{\gamma}_n=\frac{(n+(u)_0)(n-(u)_0-2)}{(2n-1)(2n-3)}, n\geq 2$.

We can also prove that necessarily $\mu=4$. Hence, $(1+\ell)\gamma_1=\ell(1-\beta_0^2)$. Discussing the cases $\ell=-1$ and $\ell\neq -1$, ends the computations.

• $\underline{B_{22}:}$ $\rho=(u)_0-1$. Thanks to (4.28), we have $\widehat{\beta}_{n+1}=\widehat{\beta}_1, n\geq 1$. So, from (4.26) we get

$\beta_{n+1}=\widehat{\beta}_1+\frac{((u)_0+1)((u)_0+2)(\beta_2-\widehat{\beta}_1)}{(n+(u)_0+1)(n+(u)_0)}, n\geq 1$. Using (3.38) in (4.21) gives

$\widehat{\beta}_1=\frac{((u)_0+2)^2\beta_2+\beta_1+((u)_0+1)\beta_0+\langle u,B_1\rangle}{((u)_0+2)((u)_0+3)}$. Hence, (4.27) leads to

$$\beta_2=\frac{\big[((u)_0+1)((u)_0+3+((u)_0+4)\vartheta_1)\big]\beta_1-\big[((u)_0+1)((u)_0+3)-((u)_0^2+3(u)_0+4)\vartheta_1\big]\beta_0-((u)_0+3+2\vartheta_1)\langle u,B_1\rangle}{2\vartheta_1((u)_0+2)^2}.$$

therefore, $\widehat{\beta}_1=\frac{\big[(u)_0+1+((u)_0+2)\vartheta_1)\big]\beta_1-\big[(u)_0+1-((u)_0+2)\vartheta_1)\big]\beta_0-\langle u,B_1\rangle}{2((u)_0+2)\vartheta_1}$.

It is mentioned previously that $d=0$, thus

$$\langle u,B_1\rangle=\big[((u)_0+2)\vartheta_1-((u)_0+1)\big]\beta_0+\big[((u)_0+2)\vartheta_1+((u)_0+1)\big]\beta_1. \quad (4.44)$$

This implies $\widehat{\beta}_1=0$ and $\widehat{\beta}_n=0, n\geq 1$. Always with the help of (4.44), we obtain: $\widehat{\beta}_0=(1-\vartheta_1)\beta_0-\vartheta_1\beta_1$, $\beta_{n+1}=-\frac{((u)_0+1)\big[\vartheta_1\beta_0+(1+\vartheta_1)\beta_1\big]}{(n+(u)_0+1)(n+(u)_0)}, n\geq 1$, $\beta_1-\widehat{\beta}_0=(\vartheta_1-1)\beta_0+(1+\vartheta_1)\beta_1$, and $\beta_1-\widehat{\beta}_0-\frac{1}{(u)_0+1}\langle u,B_1\rangle=-\frac{\vartheta_1}{(u)_0+1}(\beta_0+\beta_1)$.

Then, for $n\geq 2$, we get

$$\begin{cases} \frac{(2n+2(u)_0+3)}{(n+(u)_0+2)(n+(u)_0)}\gamma_{n+2}-\frac{(2n+2(u)_0-1)}{(n+(u)_0+1)(n+(u)_0-1)}\gamma_{n+1}=\frac{((u)_0+1)^2\big[\vartheta_1\beta_0+(1+\vartheta_1)\beta_1\big]^2}{(n+(u)_0+1)^2(n+(u)_0)^2},\\ \frac{(2(u)_0+5)}{((u)_0+3)((u)_0+1)}\gamma_3-\frac{(\vartheta_1+1)}{(u)_0+2}\gamma_2=\frac{\big[\vartheta_1\beta_0+(1+\vartheta_1)\beta_1\big]^2}{((u)_0+2)^2},\\ \frac{(2(u)_0+3)\vartheta_1}{(u)_0+2}\gamma_2-\gamma_1=-\vartheta_1(\beta_0+\beta_1)\big[(\vartheta_1-1)\beta_0+(1+\vartheta_1)\beta_1\big]. \end{cases}$$

Discussing the situations $\Phi(x)=x^2$ and $\Phi(x)=x^2-1$, as done above, we find the expressions summarized in Table 5. In particular, for $\Phi(x)=x^2-1$, we have:

$$\begin{cases} \gamma_{n+1}=\frac{(n+(u)_0+1)(n+\rho)[(2n+(u)_0+\rho+1)^2-((u)_0+\rho+3)^2\omega^2]}{(2n+(u)_0+\rho+2)(2n+(u)_0+\rho+1)^2(2n+(u)_0+\rho)}, n\geq 3,\\ \text{with } (2n+(u)_0+\rho+1)^2-((u)_0+\rho+3)^2\omega^2\neq 0, n\geq 3,\\ \widehat{\gamma}_n=\frac{(n+(u)_0)(n+\rho+1)[(2n+(u)_0+\rho+1)^2-((u)_0+\rho+3)^2\omega^2]}{(2n+(u)_0+\rho+2)(2n+(u)_0+\rho+1)^2(2n+(u)_0+\rho)}, n\geq 3. \end{cases} \quad (4.45)$$

TABLE 3. The $\mathbf{D}_u$-Classical Polynomials - **Case** $(\mathbf{B})$

Subcase $\mathbf{B_{21}}$: $1-\frac{\langle u,\,B_2\rangle}{((u)_0+1)\gamma_2}\neq 0$, $(u)_0\neq -n,\ n\geq 1$,

$\vartheta_1:=\frac{\rho+2}{\rho+1}\left[1-\frac{\langle u,\,B_2\rangle}{((u)_0+1)\gamma_2}\right]$, $\rho\neq -n, n\geq 1$, $\ell:=\frac{((u)_0+1)\vartheta_1}{\rho+2}$, $\omega:=\ell\beta_0+(\ell+1)\beta_1$

$\mathbf{B_{211}}:\ \mathbf{2n+(u)_0+\rho+1\neq 0},\ \ \forall \mathbf{n\geq 1}$,

$\Phi(x)=x^2$,

$\Psi(x)=-\frac{(u)_0+1}{\ell}(x-\beta_0)$,

$\beta_0\in\mathbb{C}\setminus\{0\},\ \beta_1\in\mathbb{C}$,

$\beta_n=-\frac{((u)_0+\rho+3)((u)_0-\rho+1)\omega}{(2n+(u)_0-1)(2n+(u)_0+\rho+1)},\ n\geq 2$,

$\widehat{\beta}_0=-\ell\beta_1+(1-\ell)\beta_0$,

$\widehat{\beta}_n=-\frac{((u)_0+\rho+3)((u)_0-\rho-1)\omega}{(2n+(u)_0+\rho+1)(2n+(u)_0+\rho+3)},\ n\geq 1$,

$\gamma_1=\frac{-\ell\beta_0^2}{1+\ell},\ \beta_0\neq 0$,

$\gamma_2=\frac{-((u)_0+2)\omega^2}{(1+\ell)((u)_0+\rho+4)},\ \omega\neq 0$, $\widehat{\gamma}_1=\frac{-(\rho+2)\ell\omega^2}{(1+\ell)((u)_0+\rho+4)}$,

$\gamma_{n+1}=\frac{-(n+(u)_0+1)(n+\rho)((u)_0+\rho+3)^2\omega^2}{(2n+(u)_0+\rho+2)(2n+(u)_0+\rho)^2(2n+(u)_0+\rho+1)^2},\ n\geq 2$,

$\widehat{\gamma}_n=\frac{-(n+\rho+1)(n+(u)_0)((u)_0+\rho+3)^2\omega^2}{(2n+(u)_0+\rho+2)(2n+(u)_0+\rho)(2n+(u)_0+\rho+1)^2},\ n\geq 2$,

$\Phi(x)=x^2-1$

$\ell=-1$,

$\beta_0=\pm 1,\ \beta_1\in\mathbb{C}$, $\widehat{\beta}_0=\beta_1+2\beta_0$,

$\beta_n=\frac{((u)_0+1-\rho)((u)_0+\rho+3)\beta_0}{(2n+(u)_0+\rho+1)},\ n\geq 2$,

$\widehat{\beta}_n=\frac{((u)_0-\rho-1)(2n+(u)_0+\rho+3)\beta_0}{(2n+(u)_0+\rho+1)(2n+(u)_0+\rho+3)},\ n\geq 1$,

$\gamma_1\in\mathbb{C}\setminus\{0\}$, if $(u)_0+\rho+4=0$,

$\gamma_2\in\mathbb{C}\setminus\left\{-\frac{\gamma_1+2\beta_0(\beta_0+\beta_1)}{(u)_0+\rho+4}\right\}$, if not,

$\gamma_1+2\beta_0(\beta_0+\beta_1)\neq 0$,

$\gamma_3=\frac{4((u)_0+3)(\rho+2)}{((u)_0+\rho+6)((u)_0+\rho+5)^2}$,

$\widehat{\gamma}_1\in\mathbb{C}\setminus\{0\}$, if $(u)_0+\rho+4=0$,

$\widehat{\gamma}_1=-\frac{(\rho+2)\ell\left[\gamma_1+2\beta_0(\beta_0+\beta_1)\right]}{4((u)_0+2)((u)_0+\rho+4)}$, if not,

$\widehat{\gamma}_2=\frac{((u)_0+2)((u)_0+\rho+3)}{((u)_0+\rho+6)((u)_0+\rho+5)^2}$,

$\gamma_{n+1}=\frac{4(n-1)(n+(u)_0+\rho+1)(n+(u)_0+1)(n+\rho)}{(2n+(u)_0+\rho+1)^2(2n+(u)_0+\rho)},\ n\geq 3$,

$\widehat{\gamma}_n=\frac{4(n-1)(n+(u)_0+\rho+1)(n+(u)_0)(n+\rho+1)}{(2n+(u)_0+\rho+2)(2n+(u)_0+\rho+1)^2(2n+(u)_0+\rho)},\ n\geq 3$.

$\Psi(x)=-\frac{(\rho+2)}{\vartheta_1}(x-\beta_0)$

$\ell\neq -1$,

$\gamma_1=\frac{\ell(1-\beta_0^2)}{\ell+1},\ \beta_0\neq\pm 1,\ \beta_1\in\mathbb{C}$,

If $(u)_0+\rho+4=0:$ $\omega=\pm 1$,

$\gamma_2\in\mathbb{C}\setminus\{0\}, \widehat{\beta}_0=2\beta_0-\omega$,

$\beta_n=\frac{(2(u)_0+5)\omega}{(2n-3)(2n-5)},\ n\geq 2$,

$\widehat{\beta}_n=\frac{(2(u)_0+3)\omega}{(2n-1)(2n-3)},\ n\geq 1$,

$\gamma_3=-\frac{1}{2}((u)_0+3)[(\ell+1)\gamma_2+4((u)_0+2)]$,

$\gamma_{n+1}=\frac{(n+(u)_0+1)(n-(u)_0-4)}{(2n-3)^2},\ n\geq 3$,

$(\ell+1)\gamma_2+4((u)_0+2)\neq 0, \widehat{\gamma}_1\in\mathbb{C}\setminus\{0\}$,

$\widehat{\gamma}_2=-\frac{((u)_0+1)}{2}[(\ell+1)\gamma_2+4((u)_0+2)]$,

$\widehat{\gamma}_n=\frac{(n+(u)_0)(n-(u)_0-3)}{(2n-3)^2},\ n\geq 3$,

If $(u)_0+\rho+6=0:$

$\omega=\pm\frac{1}{3}$,

$\gamma_2=-\frac{4((u)_0+2)}{9(\ell+1)}$,

$\gamma_4=\gamma_3-2, \gamma_3\in\mathbb{C}\setminus\{0,2\}$,

$\beta_n=\frac{3\omega(2(u)_0+7)}{(2n-5)(2n-7)},\ n\geq 2$,

$\widehat{\beta}_0=-\ell\beta_1+(1-\ell)\beta_0$,

$\widehat{\beta}_n=\frac{3\omega(2(u)_0+5)}{(2n-5)(2n-3)},\ n\geq 1$,

$\gamma_{n+1}=\frac{(n+(u)_0+1)(n-(u)_0-6)}{(2n-5)^2},\ n\geq 4$,

$\widehat{\gamma}_1=\frac{4((u)_0+4)\ell}{9(\ell+1)},\ \ell\neq 0$

$\widehat{\gamma}_2=\frac{((u)_0+2)}{(u)_0+4}\gamma_3$,

$\widehat{\gamma}_3=\frac{(u)_0+2}{(u)_0+4}(\gamma_3-2)$,

$\widehat{\gamma}_n=\frac{(n+(u)_0)(n-(u)_0-5)}{(2n-5)^2},\ n\geq 4$,

If $\exists N\geq 4, 2N+(u)_0+\rho=0:$*

β_n and $\widehat{\beta}_n$are given by(4.38)

$\gamma_N,\gamma_{N+2}\in\mathbb{C}\setminus\{0\}$,

$\gamma_{n+1}=\frac{(n+(u)_0+1)(n-2N-(u)_0)}{(2n-2N-1)^2}$,

$n\neq\{N-1,N+1\}$ *and* $n\geq 3$,

$\widehat{\gamma}_{N-1},\widehat{\gamma}_{N+1}\in\mathbb{C}\setminus\{0\}$,

$\widehat{\gamma}_n=\frac{(n+(u)_0)(n-2N-(u)_0+1)}{(2n-2N-1)^2}$,

$n\neq\{N-1,N+1\}$ *and* $n\geq 3$,

If $\forall n\geq 2,\ 2n+(u)_0+\rho\neq 0:$

β_n and $\widehat{\beta}_n$are given by(4.38) for $n\geq 1$,

$\gamma_2=-\frac{((u)_0+2)(\omega^2-1)}{(\ell+1)((u)_0+\rho+4)},\ \omega\neq\pm 1$,

$\widehat{\gamma}_1=-\frac{(\rho+2)\ell(\omega^2-1)}{(\ell+1)((u)_0+\rho+4)}$,

γ_{n+1} and $\widehat{\gamma}_n$are given by(4.45) for $n\geq 3$, .

TABLE 4. The $\mathbf{D}_u$-Classical Polynomials - **Case (B)**

Subcase $\mathbf{B_{21}}$: $1-\frac{\langle u,\,B_2\rangle}{((u)_0+1)\gamma_2}\neq 0\quad,\ (u)_0\neq -n,\ n\geq 1,$

$\mathbf{B_{212}}:\quad \exists \mathbf{N}\geq \mathbf{1},\mathbf{2N}+\mathbf{(u)_0}+\rho+\mathbf{1}=\mathbf{0}.$

$\vartheta_1:=\frac{2N+(u)_0-1}{2N+(u)_0}\Big[1-\frac{\langle u,\,B_2\rangle}{((u)_0+1)\gamma_2}\Big],\ 2N+(u)_0+1\neq n, n\geq 1,\ \ell:=-\frac{((u)_0+1)\vartheta_1}{2N+(u)_0-1},\ \omega:=\ell\beta_0+(\ell+1)\beta_1$

$\Phi(x)=x^2-1,\ \Psi(x)=-\frac{(u)_0+1}{\ell}(x-\beta_0),$

$\beta_{n+1}=0, n\notin\{N-1,N\},\ N\geq 2,\ n\geq 1,$
$\beta_{N+1}+\beta_N=0,\ N\geq 2,$
$\beta_{n+1}=0, N=1,\ n\geq 2.$
$\beta_N\in\mathbb{C},\ N\geq 2,$
$\beta_{N+1}\in\mathbb{C},\ N=1,$
$\beta_1\in\mathbb{C},$

$\widehat{\beta}_n=0, n\notin\{N-1,N\},\ N\geq 2,\ n\geq 1,$
$\widehat{\beta}_N+\widehat{\beta}_{N-1}=0,\ N\geq 2,$
$\widehat{\beta}_n=0, N=1,\ n\geq 2.$
$\widehat{\beta}_{N-1}\in\mathbb{C},\ N\geq 2,$
$\widehat{\beta}_N\in\mathbb{C},\ N=1,$
$\widehat{\beta}_0\in\mathbb{C},$

$N=1$	$N=2$	$N\geq 3$
If $\ell=-1$: $\beta_0=\pm 1, \gamma_1\in\mathbb{C}\setminus\{0\},$ $\gamma_2=-((u)_0+2)(\gamma_1-2-2\beta_0\beta_1),$ $\widehat{\gamma}_1=-((u)_0+1)(\gamma_1-2-2\beta_0\beta_1),$ $\gamma_1-2-2\beta_0\beta_1\neq 0,$ If $\ell\neq -1$: $\gamma_1=\frac{\ell}{1+\ell}(1-\beta_0^2), \beta_0\neq\pm 1,$ $\gamma_2=-\frac{(u)_0+2}{1+\ell}\big[\omega^2-1\big],\ \omega\neq\pm 1,$ $\widehat{\gamma}_1=\frac{\ell((u)_0+1)}{1+\ell}\big[\omega^2-1\big].$ $\gamma_{n+1}=\frac{(n+(u)_0+1)(n-(u)_0-3)}{(2n-1)(2n-3)}, n\geq 2,$ $\widehat{\gamma}_n=\frac{(n+(u)_0)(n-(u)_0-2)}{(2n-1)(2n-3)}, n\geq 2.$	$\gamma_1=\frac{\ell(1-\beta_0^2)}{1+\ell}, \beta_0\neq\pm 1, \ell\neq 0,$ $\gamma_2=-\frac{(u)_0+2}{\ell+1}, \ell\neq -1,$ $\gamma_3=((u)_0+3)^2-\beta_2^2,$ $\beta_2\neq\pm((u)_0+3),$ $\gamma_{n+1}=\frac{(n+(u)_0+1)(n-(u)_0-5)}{(2n-3)(2n-5)}, n\geq 3,$ $\widehat{\gamma}_1=-\frac{\ell((u)_0+3)}{\ell+1},$ $\widehat{\gamma}_2=\Big(\frac{(u)_0+2}{(u)_0+3}\Big)^2\big[((u)_0+3)^2-\beta_2^2\big],$ $\widehat{\gamma}_n=\frac{(n+(u)_0)(n-(u)_0-4)}{(2n-3)(2n-5)}, n\geq 3.$	$\gamma_1=\frac{\ell(1-\beta_0^2)}{1+\ell}, \beta_0\neq\pm 1, \ell\neq 0, \ell\neq -1,$ $\gamma_2=\frac{-((u)_0+2)}{(2N-3)(\ell+1)},$ $\gamma_3=\frac{-((u)_0+3)(2N+(u)_0-1)}{(2N-3)(2N-5)},$ $\gamma_{n+1}=\frac{(n+(u)_0+1)(n-2N-(u)_0-1)}{(2n-2N-1)(2n-2N+1)},\ 2\leq n\leq N-1,$ $\gamma_N\in\mathbb{C}\setminus\{0\},$ $\gamma_{N+1}=-\Big[\frac{3(N+(u)_0+1)}{N+(u)_0+2}\gamma_N+\beta_N^2\Big],$ $\frac{3(N+(u)_0+1)}{N+(u)_0+2}\gamma_N+\beta_N^2\neq 0,$ $\gamma_{n+1}=$ $-\frac{3(n+(u)_0+1)(n-2N-(u)_0-1)}{\big((2n-2N)^2-1\big)(N+(u)_0+1)(N+(u)_0+2)}\gamma_N,\ n\geq N+1,$ $\widehat{\gamma}_1=\frac{(2N+(u)_0-1)\ell}{(\ell+1)(2N-3)},$ $\widehat{\gamma}_2=\frac{-((u)_0+2)(2N+(u)_0-2)}{(2N-3)(2N-5)},$ $\widehat{\gamma}_n=\frac{(n+(u)_0)(n-2N-(u)_0)}{(2n-2N-1)(2n-2N+1)},\ 2\leq n\leq N-1,$ $\gamma_N\in\mathbb{C}\setminus\{0\},$ $\widehat{\gamma}_N=-\Big(\frac{N+(u)_0}{N+(u)_0+1}\Big)^2\Big[\frac{3(N+(u)_0+1)}{N+(u)_0+2}\gamma_N+\beta_N^2\Big],$ $\frac{3(N+(u)_0+1)}{N+(u)_0+2}\gamma_N+\beta_N^2\neq 0,$ $\gamma_{n+1}=$ $-\frac{3(n+(u)_0)(n-2N-(u)_0)}{\big((2n-2N)^2-1\big)(N+(u)_0+1)(N+(u)_0+2)}\gamma_N,\ n\geq N+1,$

TABLE 5. The $\mathbf{D}_u$-Classical Polynomials - **Case (B)**

<table>
<tr><td colspan="3">Subcase $\mathbf{B_{22}}$: $1-\frac{\langle u,\,B_2\rangle}{((u)_0+1)\gamma_2}\neq 0$, $(u)_0\neq -n,\ n\geq 0,\quad \vartheta_1:=\frac{((u)_0+1)}{(u)_0}\Big[1-\frac{\langle u,\,B_2\rangle}{((u)_0+1)\gamma_2}\Big]$, $\omega:=\vartheta_1\beta_0+(1+\vartheta_1)\beta_1$</td></tr>
<tr><td>$\Phi(x)=x^2$,</td><td>$\Phi(x)=x^2-1$ and $\vartheta_1=-1$</td><td>$\Phi(x)=x^2-1$ and $\vartheta_1\neq -1$</td></tr>
<tr><td>$\beta_0\neq 0$,</td><td>$\beta_0=\pm 1$,</td><td>$\beta_0\neq\pm 1$,</td></tr>
<tr><td colspan="3">$\Psi(x)=-\frac{1}{\vartheta_1}((u)_0+1)(x-\beta_0)$, $\beta_1\in\mathbb{C}$, $\beta_n=-\frac{((u)_0+1)\omega}{(n+(u)_0)(n+(u)_0-1)}, n\geq 2$, $\widehat{\beta}_0=(1-\vartheta_1)\beta_0-\vartheta_1\beta_1$, $\widehat{\beta}_n=0, n\geq 1$,</td></tr>
<tr><td>$\gamma_1=-\frac{\vartheta_1\beta_0^2}{1+\vartheta_1}$, $\vartheta_1\neq -1$,
$\gamma_2=-\frac{((u)_0+2)\omega^2}{(2(u)_0+3)(\vartheta_1+1)}$, $\omega\neq 0$,
$\gamma_{n+1}=-\frac{((u)_0+1)^2\big((n+(u)_0)^2-1\big)\omega^2}{\big((2n+2(u)_0)^2-1\big)(n+(u)_0)^2}$,
$2n+(u)_0\neq\pm 1,\ n\geq 2$,
$\widehat{\gamma}_1=-\frac{((u)_0+1)\vartheta_1\omega^2}{(2(u)_0+3)(\vartheta_1+1)}$,
$\widehat{\gamma}_n=-\frac{((u)_0+1)^2\omega^2}{(2n+2(u)_0+1)(2n+2(u)_0-1)}, n\geq 2$,</td><td>$\gamma_1\in\mathbb{C}\setminus\{0\}$,
$\gamma_{n+1}=\frac{(n-1)(n+2(u)_0+1)\big((n+(u)_0)^2-1\big)}{(2n+2(u)_0+1)(2n+2(u)_0-1)(n+(u)_0)^2}$,
$n\geq 2,\quad \gamma_1+2\beta_0(\beta_0+\beta_1)\neq 0$,
$\widehat{\gamma}_n=\frac{(n-1)(n+2(u)_0+1)}{(2n+2(u)_0+1)(2n+2(u)_0-1)}, n\geq 2$,
• If $2(u)_0+3=0$: $\gamma_2,\widehat{\gamma}_1\in\mathbb{C}\setminus\{0\}$.
• If $2(u)_0+3\neq 0$:
$\begin{cases}\widehat{\gamma}_1=\frac{((u)_0+1)\vartheta_1}{2(u)_0+3}\{\gamma_1+2\beta_0(\beta_0+\beta_1)\},\\ \gamma_2=-\frac{(u)_0+2}{2(u)_0+3}\{\gamma_1+2\beta_0(\beta_0+\beta_1)\},\end{cases}$</td><td>$\gamma_1=\frac{\vartheta_1(1-\beta_0^2)}{1+\vartheta_1}$,
• If $2(u)_0+3=0$:
$\begin{cases}\gamma_2\in\mathbb{C}\setminus\{0\},\omega=\pm 1,\quad (\vartheta_1+1)\gamma_2+2\neq 0,\\ \gamma_3=-\frac{3}{4}\Big[(\vartheta_1+1)\gamma_2+2\Big],\ \gamma_{n+1}=\frac{(n-\frac{1}{2})(n-\frac{5}{2})}{(2n-3)^2}, n\geq 3,\\ \widehat{\gamma}_1\in\mathbb{C}\setminus\{0\},\ \widehat{\gamma}_2=-\frac{1}{4}\Big[(\vartheta_1+1)\gamma_2+2\Big],\ \widehat{\gamma}_n=\frac{1}{4}, n\geq 3,\end{cases}$
• If $2(u)_0+5=0$:
$\begin{cases}\gamma_2=\frac{2}{9(1+\vartheta_1)},\ \gamma_3\in\mathbb{C}\setminus\{0\},\ \omega=\pm\frac{1}{3},\\ \gamma_4=-(\gamma_3+\frac{3}{2}),\ \gamma_3+\frac{3}{2}\neq 0,\ \gamma_{n+1}=\frac{(n-\frac{3}{2})(n-\frac{7}{2})}{(2n-5)^2}, n\geq 4,\\ \widehat{\gamma}_1=\frac{2\vartheta_1}{3(1+\vartheta_1)},\ \widehat{\gamma}_2\in\mathbb{C}\setminus\{0\},\ \widehat{\gamma}_3=\frac{1}{3}(\gamma_3+\frac{3}{2}),\ \widehat{\gamma}_n=\frac{1}{4}, n\geq 4,\end{cases}$
• If $\exists n_0\geq 3,\ 2n_0+2(u)_0+1=0$:
$\begin{cases}\gamma_{n+1}=\frac{(n-n_0+\frac{1}{2})(n-n_0-\frac{3}{2})(n_0-\frac{1}{2})^2\omega^2}{(n-n_0-\frac{1}{2})^2}, n\geq n_0+2,\\ \gamma_{n_0+2}=-\Big\{\gamma_{n_0+1}+6(n_0-\frac{1}{2})^2\omega^2\Big\},\ \gamma_{n_0}=-\frac{5}{6}(n_0-\frac{1}{2})^2\omega^2,\\ \gamma_{n+1}=\frac{(n-n_0+\frac{1}{2})(n-n_0-\frac{3}{2})\Big\{(n-n_0+\frac{1}{2})^2-(n_0-\frac{1}{2})^2\omega^2\Big\}}{(n-n_0)(n-n_0-1)(2n-2n_0-1)^2},\\ 2\leq n\leq n_0-2,\ \widehat{\gamma}_n=(n_0-\frac{1}{2})^2\omega^2, n\geq n_0+2,\\ \widehat{\gamma}_{n_0+1}=-\frac{1}{3}\Big\{\gamma_{n_0+1}+6(n_0-\frac{1}{2})^2\omega^2\Big\},\ \widehat{\gamma}_{n_0}\in\mathbb{C}\setminus\{0\},\\ \widehat{\gamma}_{n_0-1}=-\frac{1}{2}(n_0-\frac{1}{2})^2\omega^2,\\ \widehat{\gamma}_n=\frac{(n-n_0+\frac{1}{2})^2-(n_0-\frac{1}{2})^2\omega^2}{4(n-n_0)(n-n_0-1)}, 2\leq n\leq n_0-2,\end{cases}$
• If $\forall n\geq 1,\ 2n+2(u)_0+1\neq 0$:
$\begin{cases}\gamma_2=\frac{((u)_0+2)}{(2(u)_0+3)(1+\vartheta_1)}(1-\omega^2),\\ \gamma_3=\frac{((u)_0+1)((u)_0+3)\Big(((u)_0+2)^2-((u)_0+1)^2\omega^2\Big)}{(2(u)_0+3)(2(u)_0+5)((u)_0+2)^2},\\ \gamma_{n+1}=\frac{\big((n+(u)_0)^2-1\big)\big[(n+(u)_0+1)^2-((u)_0+1)^2\omega^2\big]}{(2n+2(u)_0+1)(2n+2(u)_0-1)(n+(u)_0)^2}, n\geq 3,\\ (n+(u)_0+1)^2-((u)_0+1)^2\omega^2\neq 0, n\geq 1,\\ \widehat{\gamma}_1=\frac{((u)_0+1)\vartheta_1\big[1-\omega^2\big]}{(2(u)_0+3)(1+\vartheta_1)}, \widehat{\gamma}_n=\frac{(n+(u)_0)^2-((u)_0+1)^2\omega^2}{(2n+2(u)_0+1)(2n+2(u)_0-1)}, n\geq 2.\end{cases}$</td></tr>
</table>

Finally, note that if $u = 0$, we recognize the very classical orthogonal polynomial sequences. Precisely, we find Hermite and Laguerre polynomials respectively in the first and the second column of Table 2. The Bessel polynomials are in the first column of Table 3 and the Jacobi polynomials are obtained in the last case of the third column of Table 3.

Acknowledgments. The author would like to thank the anonymous referee for his/her comments that helped us to improve this article.
The author would like to thank Institute of Scientific Research and Revival of Islamic Heritage at Umm Al-Qura University (Project ID 43305032) for the financial support.

References

[1] Abdelkarim, F.; Maroni, P. *The D_w-classical orthogonal polynomials*. Results Math. 32 (1997), no. 1–2, 1-28.
[2] Alaya, J. *Quelques résultats nouveaux dans la théorie des polynômes de Laguerre-Hahn*. Thèse de doctorat d'état ès-sciences, Univ. Tunis II, Faculté des Sciences de Tunis. Tunisie, 1996.
[3] Alaya, J.; Maroni, P. *Symmetric Laguerre-Hahn forms of class $s = 1$*. Integral Transforms Spec. Funct. 4 (1996), no. 4, 301-320.
[4] Alfaro, M.; Álvarez-Nodarse, R. *A characterization of the classical orthogonal discrete and q-polynomials*. J. Comput. Appl. Math. 201 (2007), no. 1, 48-54.
[5] Alfaro, M.; Marcellán, F.; Peña, A.; Rezola, M. L. *When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials?* J. Comput. Appl. Math. 233 (2010), no. 6, 1446-1452.
[6] Al-Salam, W. A. *Characterization theorems for orthogonal polynomials*. Orthogonal polynomials, in: P. Nevai, Ed, Orthogonal Polynomials:Theory and Practice (Kluwer Academic Publishers, Dordrecht, 1990), 1-24.
[7] Al-Salam, W. A.; Chihara, T. S. *Another characterization of the classical orthogonal polynomials*. SIAM J. Math. Anal. 3 (1972), 65-70.
[8] Álvarez-Nodarse, R. *On characterizations of classical polynomials*. J. Comput. Appl. Math. 196 (2006), no. 1, 320-337.
[9] Álvarez-Nodarse, R.; Medem, J. C. *The q-classical polynomials and the q-Askey and Nikiforov-Uvarov Tableaus*. J. Comput. Appl. Math. 135 (2001), no. 2, 197-223.
[10] Álvarez-Nodarse, R.; Durán, A. J. *Using $\mathcal{D}$-operators to construct orthogonal polynomials satisfying higher order q-difference equations*. J. Math. Anal. Appl. 424 (2015), no. 1, 304-320.
[11] Area, I.; Godoy, I.; Ronveaux, A.; Zarzo, A. *Classical symmetric orthogonal polynomials of a discrete variable*. Integral Transforms Spec. Funct. 15 (2004), no. 1, 1-12.
[12] Askey, R. *Continuous Hahn polynomials*. J. Phys. A 18 (1985), no. 16, L1017–L1019.
[13] Askey, R. *Divided difference operators and classical orthogonal polynomials*. Rocky Mount. J. Math. 19 (1989), no. 1, 33-37.
[14] Ben Cheikh, Y.; Gaied, M. *Characterization of the Dunkl-classical symmetric orthogonal polynomials*. Appl. Math. Comput. 187 (2007), no. 1, 105-114.
[15] Berg, C.; Ismail, M. E. H. *q-Hermite polynomials and classical orthogonal polynomials*. Canad. J. Math. 48 (1996), no. 1, 43-63.
[16] Bouakkaz, H. *Les polynômes orthogonaux de Laguerre-Hahn de classe zéro*. Thèse de Doctorat, Université Pierre et Marie Curie, Paris (1990).
[17] Bouras, B.; Alaya, A. *A large family of semi-classical polynomials of class one*. Integral Transforms Spec. Funct. 18 (2007), no. 11–12, 913-931.
[18] Chihara, T. S. *An introduction to orthogonal polynomials*. Mathematics and its Applications, Vol. 13. Gordon and Breach Science Publishers, New York-London-Paris, 1978.
[19] Costas-Santos, R. S.; Marcellán, F. *q-classical orthogonal polynomials: a general difference calculus approach*. Acta Appl. Math. 111 (2010), no. 1, 107-128.
[20] Costas-Santos, R. S.; Marcellán, F. *Second structure relation for q-semiclassical polynomials of the Hahn Tableau*. J. Math. Anal. Appl. 329 (2007), no. 1, 206-228.

[21] De Jesus, M. N.; Petronilho, J. *On linearly related sequences of derivatives of orthogonal polynomials*. J. Math. Anal. Appl. 347 (2008), no. 2, 482-492.
[22] Dette, H.; Studden, W. J. *On a new characterization of the classical orthogonal polynomials*. J. Approx. Theory 71 (1992), no. 1, 3-17.
[23] Dini, J.; *Sur les formes linéaires et les polynômes orthogonaux de Laguerre-Hahn*. Thèse de l'Université Pierre et Marie Curie, Paris (1988).
[24] Dini, J.; Maroni, P.; Ronveaux, A. *Sur une perturbation de la récurrence vérifiée par une suite de polynômes orthogonaux*. Portugal. Math. 46 (1989), no. 3, 269-282.
[25] Dzoumba, J. *Sur les polynômes de Laguerre-Hahn*. Thèse de Troisième cycle, Université Pierre et Marie Curie, Paris (1985).
[26] Foupouagnigni, M.; Kenfack Nangho, M.; Mboutngam, S. *Characterization theorem for classical orthogonal polynomials on non-uniform lattices: the functional approach*. Integral Transforms Spec. Funct. 22 (2011), no. 10, 739-758.
[27] García, A. G.; Marcellán, F.; Salto, L. *A distributional study of discrete classical orthogonal polynomials*. J. Comput. Appl. Math. 57 (1995), no. 1–2, 147-162.
[28] Garza, L. G.; Garza, L. E.; Marcellán, F; Pinzón-Cortés, N. C. *A matrix approach for the semiclassical and coherent orthogonal polynomials*. Appl. Math. Comput., 256 (2015), 459–471.
[29] Ghressi, A.; Khériji, L. *The Symmetrical H_q-semiclassical orthogonal polynomials of class one*. SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009), Paper 076, 22 pp.
[30] Ismail, M. E. H. *Classical and quantum orthogonal polynomials in one variable*. Encyclopedia of Mathematics and its Applications, 98. Cambridge University Press, Cambridge, 2005.
[31] Khériji, L. *An introduction to the H_q-semiclassical orthogonal polynomials*. Methods Appl. Anal. 10 (2003), no. 3, 387-411.
[32] Khériji, L.; Maroni, P. *The H_q-Classical orthogonal polynomials*. Acta Appl. Math. 71 (2002), no. 1, 49-115.
[33] Koekoek, R.; Lesky, P. A.; Swarttouw, R. F. *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.
[34] Koepf, W.; Schmersau, D. *Recurrence equations and their classical orthogonal polynomial solutions*. Appl. Math. Comput. 128 (2002), no. 2–3, 303-327.
[35] Kwon, K. H.; Lee, D. W.; Park, S. B. *On δ-semiclassical orthogonal polynomials*. Bull. Korean Math. Soc. 34 (1997), no. 1, 63-79.
[36] Kwon, K. H.; Littlejohn, L. L. *Classification of classical orthogonal polynomials*. J. Korean Math. Soc. 34 (1997), no. 4, 973-1008.
[37] Kwon, K. H.; Littlejohn, L. L.; Yoo, B. H. *New characterizations of classical orthogonal polynomials*. Indag. Math. (N.S.) 7 (1996), no. 2, 199-213.
[38] Loureiro, A. F.; Maroni, P.; da Rocha, Z. *The generalized Bochner condition about classical orthogonal polynomials revisited*. J. Math. Anal. Appl. 322 (2006), no. 2, 645-667.
[39] Marcellán, F.; Branquinho, A.; Petronilho, J. *Classical orthogonal polynomials: a functional approach*. Acta Appl. Math. 34 (1994), no. 3, 283-303.
[40] Marcellán, F.; Pérez, T. E.; Piñar, M. A.; Ronveaux, A. *General Sobolev orthogonal polynomials*. J. Math. Anal. Appl. 200 (1996), no. 3, 614-634.
[41] Marcellán, F.; Salto, L. *Discrete semi-classical orthogonal polynomials*. J. Differ. Equations Appl. 4 (1998), no. 5, 463-496.
[42] Marcellán, F.; Sfaxi, R. *Lowering operators associated with D-Laguerre-Hahn polynomials*. Integral Transforms Spec. Funct. 22 (2011), 879–893.
[43] Marcellán, F.; Pérez, T. E.; Piñar, M. A. *Orthogonal polynomials on weighted Sobolev spaces: the semiclassical case*. Ann. Numer. Math. 2 (1995), no. 1–4, 93-122.
[44] Marcellán, F. ; Xu, Y. *On Sobolev orthogonal polynomials*. Expositiones Mathematicae, In Press.
[45] Maroni, P. *Prolégomènes à l' étude des polynômes orthogonaux semi-classiques*. Ann. Mat. Pura Appl. (4) 149 (1987), 165-184.
[46] Maroni, P. *Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques*. In: M. Alfaro, F. Marcellán; J. L. Rubio de Francia; J. Vinuesa (Eds.), Orthogonal polynomials and their applications (Segovia, 1986). Lecture Notes in Math., 1329, Springer, Berlin, 1988, 279-290.
[47] Maroni, P. *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*. In: C. Brezinski, L. Gori, A. Ronveaux (Eds.), Orthogonal

polynomials and their applications (Erice, 1990). IMACS Ann. Comput. Appl. Math., 9, Baltzer, Basel, 1991, 95-130.
[48] Maroni, P. *Variations around classical orthogonal polynomials. Connected problems*. J. Comput. Appl. Math. 48 (1993), no. 1–2, 133-155.
[49] Maroni, P. *Fonctions eulériennes. Polynômes orthogonaux classiques*. Technique de l'ingénieur, Paris, A 154 (1994), 1–30.
[50] Maroni, P.; da Rocha, Z. *A new characterization of classical forms*. Commun. Appl. Anal. 5 (2001), no. 3, 351-362.
[51] Maroni, P.; Mejri, M. *The $I_{(q,w)}$ classical orthogonal polynomials*. Appl. Numer. Math. 43 (2002), no. 4, 423-458.
[52] Maroni, P.; Mejri, M. *The symmetric D_w-semi-classical orthogonal polynomials of class one*. Numer. Algorithms 49 (2008), no. 1–4, 251-282.
[53] Medem, J. C.; Álvarez-Nodarse, R.; Marcellán, F. *On the q-polynomials: a distributional study*. J. Comput. Appl. Math. 135 (2001), no. 2, 157-196.
[54] Mejri, M. *q-extension of some symmetrical and semi-classical orthogonal polynomials of class one*. Appl. Anal. Discrete Math. 3 (2009), no. 1, 78-87.
[55] Nikiforov, A. F.; Suslov, S. K.; Uvarov, V. B. *Classical orthogonal polynomials of a discrete variable*. Springer Series in Computationnal Physics. Springer-Verlag, Berlin (1991).
[56] Ronveaux, A. *Polynômes orthogonaux dont les polynômes dérivés sont quasi-orthogonaux*. C. R. Acad. Sci. Paris Sr. A-B 289 (1979), no. 7, A433-A436.
[57] Sghaier, M. *Rodrigues formula for the Dunkl-classical symmetric orthogonal polynomials*. Filomat 27 (2013), no. 7, 1285-1290.
[58] Sghaier, M.; Alaya, J. *Semiclassical forms of class $s = 2$: the symmetric case, when $\Phi(0) = 0$*. Methods Appl. Anal. 13 (2006), no. 4, 387-409.
[59] Shrivastava, P. N. *Classical polynomials-a unified presentation*. Publ. Inst. Math. (Beograd) (N.S.) 23(37) (1978), 169-177.
[60] Srivastava, H. M.; Panda, R. *On the unified presentation of certain classical polynomials*. Boll. Un. Mat. Ital. (4) 12 (1975), no. 3, 306-314.
[61] Srivastava, H. M.; Singhal, J. P. *A unified presentation of certain classical polynomials*. Math. Comp. 26 (1972), 969-975.
[62] Tounsi, M. I.; Ben Salah, I.; Khériji, L. *On the Symmetric H_q-Semiclassical Polynomial Sequences of Even Class. Some Examples from the Class Two*. Mediterr. J. Math. 10 (2013), no. 3, 1293-1316.
[63] Verde-Star, L. *Characterization and construction of classical orthogonal polynomials using a matrix approach*. Linear Algebra Appl. 438 (2013), no. 9, 3635-3648.

Atef Alaya
Department of Mathematics, College of Applied Sciences, Umm Al-Qura University, P.O Box(715), Makkah, Saudi Arabia.
E-mail address: aaalaya@uqu.edu.sa

Secondary: Atef Alaya
département de mathématiques, Faculté des Sciences de Gabès, Université de Gabès, Cité Erriadh 6072, Zrig, Gabès, Tunisie.
E-mail address: Atef.Alaya@fsg.rnu.tn