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A GENERALIZATION OF CONTRACTION PRINCIPLE IN QUASI-METRIC SPACES

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Abstract. We prove a fixed point theorem for some contraction mapping in complete quasi-metric space with w-distance, and a common fixed point theorem for two and three self mappings.

1. Introduction

The concept of w-distance has introduced by Kada, Suzuki and Takahashi in metric space [1]. Some authors used this concept in some results, Alegre, Romeguera and Tirado proved for multi-valued maps and w-distances on complete quasi-metric space [5], also Alegre, Marinard and Romeguera [2] obtained some results of fixed point theorem, they used w-distance and type function of Meir-Keeler and Jachymski type.

In [7] Azam and Shakeel proved the existence of common coincidence point and common fixed point for mapping satisfying a generalized weak contraction in metric space. Dutta and Choudhury [5]obtained the following generalization of some result obtained in[7].The authors in[8] have proved some fixed point theorems both for single-valued and multi-valued mapping in complete metric space and convex metric space.

The propose of this article is to study fixed point in quasi-metric space, we inspire our result from some result obtained in metric space[[4]-[8]], we avoid the concept of symmetry and we use the w-distance. We present also a common fixed point of maps satisfying some conditions, and we show a fixed point result for multi-valued mapping.

2. Preliminaries

Definition 1. *Let X be a nonempty set and let $d : X \times X \longrightarrow \mathbb{R}^+$ be a function satisfying following conditions :*

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(i) $d(x,y) = 0 \Leftrightarrow x = y$
(ii) $d(x,y) \leq d(x,z) + d(z,y)$
Then d is called a quasi-metric on X.

Definition 2. *Let (X,d) be a quasi-metric space and $q : X \times X \longrightarrow \mathbb{R}^+$ be a function satisfying following conditions :*
(w_1) *$q(x,y) \leq q(x,z) + q(z,y)$, for all $(x,y,z) \in X^3$,*
(w_2) *q is lower semi-continuous in its second variable,*
(w_3) *for each $\epsilon > 0$, there exists $\delta > 0$ such that $q(x,y) \leqslant \delta$ and $q(x,z) \leqslant \delta$ imply $d(y,z) \leq \epsilon$.*
Then q is called a w-distance on X.

Remark.
- *Any metric space is quasi-metric, but the converse is not true in general.*
- *Note that if d is a metric on X, then it is a w-distance on (X,d) unfortunately this does not hold for quasi-metric spaces.*
- *In general for $x,y \in X$, $q(x,y) \neq q(y,x)$ and not either of the implications $q(x,y) = 0 \Leftrightarrow x = y$ necessarily holds.*
- *$d^s(x,y) = \max\{d(x,y), d(y,x)\}$, for all $x,y \in X$, is a metric on X.*
- *The function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$, for all $x,y \in X$, is also a quasi-metric on X.*
- *If a quasi-metric d on X is also a w-distance on (X,d), then the topologies induced by d and by the metric d^s coincide, the base of the topology τ_d is open balls $\{B_d(x,r)\ ;\ x \in X,\ \epsilon > 0\}$, where $B_d(x,\epsilon) = \{y \in X\ ;\ d(x,y) < \epsilon\}$, for all $x \in X$ and $\epsilon > 0$.*

There exist many different notions of completeness for quasi-metric space(see[9]), In this paper we shall use the following general notion.

Definition 3. *Let (X,d) be a quasi-metric space.*
(X,d) is called complete if each Cauchy sequence in (X,d^s) converges with respect to the topology $\tau_{d^{-1}}$(there exists $z \in X$ such that $d(x_n,z) \to 0$)

Definition 4. *Let (X,d) be a quasi-metric space and q is a w-distance on X. If $q(x,y) = q(y,x)$, for all $x,y \in X$, we say that is a symmetric w-distance on (X,d).*

Definition 5. *(see[3])Let X be a non-empty set and $T,f : X \longrightarrow X$. be a self mappings on X.*

1. *A point $y \in X$ is called a point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. The point x is called coincidence point of T and f.*
2. *The mappings T and f are said to be weakly compatible if they commute at their coincidence point (that is, $Tfx = fTx$ whenever $Tx = fx$).*

Definition 6. *An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \longrightarrow 2^X$ if $x \in T(x)$.*

Lemma 2.1. *If q is a w-distance on a quasi-metric space (X,d), then for each $\epsilon > 0$, there exists $\delta > 0$ such that :*

$$\begin{cases} q(x,y) \leq \delta \\ q(x,z) \leq \delta \end{cases} \quad \textit{imply } d^s(y,z) \leq \epsilon$$

3. Main Results

We consider two functions $\phi,\ \psi\ : [0,+\infty[\rightarrow [0,+\infty[$ satisfied :

(1) ϕ is lower semi-continuous,
(2) ψ is monotone nondecreasing and continuous,
(3) $\psi(t) = 0$ (resp. $\phi(t) = 0$) if and only if $t = 0$.

Theorem 3.1. *Let (X, d) be a complete quasi-metric space. If there exist q w-distance and $T : X \rightarrow X$ be a self-mapping such that for all $x, y \in X$,*

$$\psi(q(Tx,Ty)) \leq \psi(q(x,y)) - \phi(q(x,y)), \tag{3.1}$$

then T has a unique fixed point $z \in X$. Moreover $q(z, z) = 0$.

Proof. For any $x_0 \in X$, we construct the sequence $(x_n)_{n\geq 0}$ by $x_n = Tx_{n-1}$, $n \in I\!N^*$.
First case : We show

$$q(x_{n+1}, x_n)\ \ and\ \ q(x_n, x_{n+1})\ \ \rightarrow 0\ as\ \ n \rightarrow \infty$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (3.1), we obtain :

$$\psi(q(x_n, x_{n+1})) \leq \psi(q(x_{n-1}, x_n)) - \phi(q(x_{n-1}, x_n)) \tag{3.2}$$

$$\psi(q(x_n, x_{n+1})) \leq \psi(q(x_{n-1}, x_n))$$

Which implies

$$q(x_n, x_{n+1}) \leq q(x_{n-1}, x_n)$$

the same $x = x_n$ and $y = x_{n-1}$ in (3.1), we obtain :

$$\psi(q(x_{n+1}, x_n)) \leq \psi(q(x_n, x_{n-1})) - \phi(q(x_n, x_{n-1})) \tag{3.3}$$

$$\psi(q(x_{n+1}, x_n)) \leq \psi(q(x_n, x_{n-1}))$$

which implies

$$q(x_{n+1}, x_n) \leq q(x_n, x_{n-1})$$

It follows that the sequence $(q(x_n, x_{n+1}))_n$ and $(q(x_{n+1}, x_n))_n$ is monotone decreasing and consequently there exists $r \geq 0$ and $r' \geq 0$ such that :

$$q(x_n, x_{n+1}) \rightarrow r \quad as\ \ n \rightarrow \ \ \infty$$

And

$$q(x_{n+1}, x_n) \rightarrow r' \quad as \quad n \rightarrow \infty$$

Letting $n \rightarrow \infty$ in (3.2) and (3.3), we obtain :

$$\psi(r) \leq \psi(r) - \liminf_{n\rightarrow+\infty} \phi(q(x_n, x_{n+1})) \leq \psi(r) - \phi(r)$$

$$\psi(r') \leq \psi(r') - \liminf_{n\rightarrow+\infty} \phi(q(x_{n+1}, x_n)) \leq \psi(r') - \phi(r')$$

Which is a contradiction unless $r = r' = 0$

Hence

$$q(x_{n+1}, x_n)\ \ and\ \ q(x_n, x_{n+1})\ \ \rightarrow 0\ as\ \ n \rightarrow \infty$$

Second case : We show that for each $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$ such that :

$$q(x_n, x_m) < \epsilon \text{ whenever } m > n \geq n_\epsilon.$$

Assume the contrary, then there exists $\epsilon_0 \in (0,1)$ such that, for each $k \in \mathbb{N}$, there exists $n(k), m(k) \in \mathbb{N}$ such that : $m(k) > n(k) > k$ and

$$q(x_{n(k)}, x_{m(k)}) \geq \epsilon_0 \tag{3.4}$$

Since $\lim\limits_{n\to+\infty} q(x_n, x_{n+1}) = 0$, there exists $n_{\epsilon_0} \in \mathbb{N}$ such that $q(x_n, x_{n+1}) < \epsilon_0$, for all $n \geq n_{\epsilon_0}$

We can choose $m(k)$ is the smallest integer with $m(k) > n(k) > k$ and satisfying (3.4) such that :

$$q(x_{n(k)}, x_{m(k)-1}) < \epsilon_0$$

We have :

$$\epsilon_0 \leq q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)})$$
$$\epsilon_0 \leq q(x_{n(k)}, x_{m(k)}) < \epsilon_0 + q(x_{m(k)-1}, x_{m(k)})$$

Then,

$$q(x_{n(k)}, x_{m(k)}) \to \epsilon_0 \quad as \quad k \to \infty$$

Again

$$q(x_{n(k)-1}, x_{m(k)-1}) \leq q(x_{n(k)-1}, x_{n(k)}) + q(x_{n(k)}, x_{m(k)}) + q(x_{m(k)}, x_{m(k)-1})$$

$$q(x_{n(k)}, x_{m(k)}) \leq q(x_{n(k)}, x_{n(k)-1}) + q(x_{n(k)-1}, x_{m(k)-1}) + q(x_{m(k)-1}, x_{m(k)})$$

Then,

$$q(x_{n(k)-1}, x_{m(k)-1}) \to \epsilon_0 \quad as \quad k \to \infty$$

Setting $x = x_{n(k)-1}$, $y = x_{m(k)-1}$in (3.1)

$$\psi(q(x_{n(k)}, x_{m(k)})) \leq \psi(q(x_{n(k)-1}, x_{m(k)-1})) - \phi(q(x_{n(k)-1}, x_{m(k)-1}))$$

We make k to $+\infty$, which gives :

$$\psi(\epsilon_0) \leq \psi(\epsilon_0) - \liminf_{k\to+\infty} \phi(q(x_{n(k)-1}, x_{m(k)-1})) \leq \psi(\epsilon_0) - \phi(\epsilon_0)$$

Thus, $\phi(\epsilon_0) \leq 0$, which is contradiction.

Third case : We show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s).

Let $\epsilon > 0$. From lemma 2.1), there exists $\delta = \delta(\epsilon) > 0$ such that :

$$\begin{cases} q(x,y) \leq \delta \\ q(x,z) \leq \delta \end{cases} \quad \text{imply } d^s(y,z) \leq \epsilon$$

For this δ, there exists $n_\delta \in I\!N$ such that, for all integers $n, m \geq n_\delta$,

$$\begin{cases} q(x_{n(\delta)}, x_n) < \delta \\ q(x_{n(\delta)}, x_m) < \delta \end{cases}$$

And then, $d^s(x_n, x_m) < \epsilon$.
Consequently $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Since (X, d) is complete, there exists $z \in X$ such that $\lim\limits_{n\to+\infty} d(x_n, z) = 0$.

Fourth case : Next we show that $\lim\limits_{n\to+\infty} q(x_n, z) = 0$.

Let $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such as, for each $n, m \geq n_\epsilon$, $q(x_n, x_m) < \epsilon$. Therefore, for each $n \geq n_\epsilon$,

$$\liminf_{m \to +\infty} q(x_n, x_m) \leq \epsilon$$

Since $\lim_{m\to\infty} d^s(x_m, z) = 0$ and q is lower semi-continuous in its second variable,

$$\forall n \geq n_\epsilon,\ q(x_n, z) \leq \liminf_{m\to+\infty} q(x_n, x_m) \leq \epsilon$$

Consequently $q(x_n, z) \to 0$ as $n \to \infty$

Substituting $x = x_n$ and $y = z$ in (3.1), we obtain :

$$\psi(q(x_{n+1}, Tz)) \leq \psi(q(x_n, z)) - \phi(q(x_n, z))$$

So $\lim_{n\to+\infty} q(x_{n+1}, Tz) = 0$.

Since $\begin{cases} q(x_{n+1}, z) \to 0 \\ q(x_{n+1}, Tz) \to 0 \end{cases}$, by using lemma 2.1), $d^s(Tz, z) = 0$ i.e. $z = Tz$

We have : $\psi(q(z, z)) \leq \psi(q(z, z)) - \phi(q(z, z))$, so $\phi(q(z, z)) \leq 0$. Thus, $q(z, z) = 0$.

Uniqueness of the fixed point : Let $u \in X$ such that $u = Tu$ and $u \neq z$.
Suppose $q(u, z) > 0$. Putting $x = u$ and $y = z$, we have :

$$\psi(q(u, z)) = \psi(q(Tu, Tz)) \leq \psi(q(u, z))) - \phi(q(u, z)))$$

Then $\phi(q(u, z)) \leq 0$, which is contradiction. So $q(u, z) = 0$. And since $q(z, z) = 0$, we deduce from lemma 2.1), that $d^s(u, z) = 0$ i.e. $u = z$.
We conclude that z is the unique fixed point of T.

Example 3.2. *Let $X = \mathbb{R}_+$ and $d(x, y) = \max(y - x, 0)$, for all $(x, y) \in \mathbb{R}^2_+$.*
$(X, d)$is complete quasi-metric space.

Let $T : X \to X$ be defined as :

$$Tx = \begin{cases} x - \frac{x^2}{2} & if \quad 0 \leq x \leq 1 \\ \\ \sqrt{x} - 1 & if\ x > 1 \end{cases}$$

$\phi : [0, \infty) \to [0, \infty)$ be defined as :

$$\phi(t) = \begin{cases} t^2/2 & if \quad 0 \leq t \leq 1 \\ \\ \frac{1}{2} & if\ t > 1 \end{cases}$$

$\psi : [0, \infty) \to [0, \infty)$ be defined as :

$$\psi(t) = t$$

$q : [0, \infty) \times [0, \infty) \to [0, \infty)$ be defined as :

$$q(x, y) = y$$

Let $x \in \mathbb{R}$.
Case 1 : $y \in [0, 1]$

We have $q(Tx, Ty) = Ty = y - y^2/2$,

$$\psi(q(Tx, Ty)) = y - \frac{y^2}{2},\ \phi((q(x, y)) = \frac{y^2}{2} \text{ and } \psi(q(x, y)) = y$$

So,

$$\psi(q(Tx, Ty)) = \psi(q(x, y)) - \phi((q(x, y))$$

Case 2 : $y > 1$
We have :$q(Tx, Ty) = Ty = \sqrt{y} - 1$,

$$\psi(q(Tx, Ty)) = \sqrt{y} - 1,\ \phi((q(x, y)) = 1/2 \text{ and } \psi(q(x, y)) = y$$

So,

$$\psi(q(Tx, Ty)) = \sqrt{y} - 1 < y - 1/2 \Rightarrow \psi(q(Tx, Ty)) < \psi(q(x, y)) - \phi((q(x, y))$$

0 *is unique fixed point of* T.

Theorem 3.3. *Let* (X, d) *be a complete quasi-metric space and* q *be a symmetric* w*-distance. Let* $S, T : X \longrightarrow X$ *be a self mappings satisfying the inequality :*

$$\forall (x, y) \in X^2,\ \psi(q(Tx, Sy)) \le \psi(q(x, y)) - \phi(q(x, y)). \tag{3.5}$$

Then, there exists a unique point $z \in X$ *such that* $T(z) = z = S(z)$*. Moreover* $q(z, z) = 0$.

Proof. For any $x_0 \in X$, we construct the sequence $(x_n)_{n\in\mathbb{N}}$ in X by taking

$$\begin{cases} x_{2n+1}=Tx_{2n} \\ x_{2n+2}=Sx_{2n+1} \end{cases}$$

First case : We show

$$q(x_n, x_{n+1}) \ \to 0 \ as\ n \to \infty$$

Substituting $x = x_{2n}$ and $y = x_{2n+1}$ in(3.5), we obtain

$$\psi(q(x_{2n+1}, x_{2n+2})) \le \psi(q(x_{2n}, x_{2n+1})) - \phi(q(x_{2n}, x_{2n+1})) \tag{3.6}$$

$$\psi(q(x_{2n+1}, x_{2n+2})) \le \psi(q(x_{2n}, x_{2n+1}))$$

$$q(x_{2n+1}, x_{2n+2}) \le q(x_{2n}, x_{2n+1})$$

Then, $(q(x_n, x_{n+1}))_n$ is monotone decreasing. Consequently there exists $r \ge 0$ such that

$$q(x_n, x_{n+1}) \to r \quad as\ \ n \to \ \ \infty$$

Letting $n \to \infty$ in (3.6), we obtain :

$$\psi(r) \le \psi(r) - \liminf_{n\to+\infty} \phi(q(x_n, x_{n+1})) \le \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$

Second case : Now we show that for each $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$ such that :

$$q(x_{2n}, x_{2m}) < \epsilon \text{ whenever } m > n \ge n_\epsilon$$

Assume the contrary, then there exists $\epsilon_0 \in (0,1)$ such that, for each $k \in \mathbb{N}$, there exist two sequences of positives integers $(n(k))_k, (m(k))_k$ with $m(k) > n(k) > k$ and

$$q(x_{2n(k)}, x_{2m(k)}) \geq \epsilon_0 \tag{3.7}$$

We can choose $m(k)$ is the smallest integer with $m(k) > n(k) > k$ and satisfying (3.7) such that :

$$q(x_{2n(k)}, x_{2m(k)-2}) < \epsilon_0$$

We have :

$$q(x_{2n(k)}, x_{2m(k)}) \leq q(x_{2n(k)}, x_{2m(k)-2}) + q(x_{2m(k)-2}, x_{2m(k)-1}) + q(x_{2m(k)-1}, x_{2m(k)})$$

$$\epsilon_0 \leq q(x_{2n(k)}, x_{2m(k)}) < \epsilon_0 + q(x_{2m(k)-2}, x_{2m(k)-1}) + q(x_{2m(k)-1}, x_{2m(k)})$$

Then,

$$q(x_{2n(k)}, x_{2m(k)}) \to \epsilon_0$$

Again

$$q(x_{2n(k)}, x_{2m(k)+1}) \leq q(x_{2n(k)}, x_{2m(k)}) + q(x_{2m(k)}, x_{2m(k)+1})$$
$$q(x_{2n(k)}, x_{2m(k)}) \leq q(x_{2n(k)}, x_{2m(k)+1}) + q(x_{2m(k)+1}, x_{2m(k)})$$

Then,

$$q(x_{2n(k)}, x_{2m(k)+1}) \to \epsilon_0$$

We have :

$$q(x_{2n(k)+1}, x_{2m(k)+2}) \leq q(x_{2n(k)+1}, x_{2n(k)}) + q(x_{2n(k)}, x_{2m(k)+1}) + q(x_{2m(k)+1}, x_{2m(k)+2})$$
$$q(x_{2n(k)}, x_{2m(k)+1}) \leq q(x_{2n(k)}, x_{2n(k)+1}) + q(x_{2n(k)+1}, x_{2m(k)+2}) + q(x_{2m(k)+2}, x_{2m(k)+1})$$

Then,

$$q(x_{2n(k)+1}, x_{2m(k)+2}) \to \epsilon_0$$

Setting $x = x_{2n(k)}$, $y = y_{2m(k)+1}$in (3.5),

$$\psi(q(x_{2n(k)+1}, x_{2m(k)+2})) \leq \psi(q(x_{2n(k)}, x_{2m(k)+1})) - \phi(q(x_{2n(k)}, x_{2m(k)+1}))$$

We make k to $+\infty$, which gives :

$$\psi(\epsilon_0) \leq \psi(\epsilon_0) - \liminf_{k\to+\infty} \phi(q(x_{2n(k)}, x_{2m(k)-1})) \leq \psi(\epsilon_0) - \phi(\epsilon_0)$$

Then $\phi(\epsilon_0) \leq 0$, which is contradiction.

Third case : We show that $(x_{2n})_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s).

Let $\epsilon > 0$. From lemma 2.1), there exists $\delta = \delta(\epsilon) > 0$ such that :

$$\begin{cases} q(x,y) \leq \delta \\ q(x,z) \leq \delta \end{cases} \quad \text{imply } d^s(y,z) \leq \epsilon$$

For this δ, there exists $n_\delta \in I\!N$ such that, for all integers $n,\ m \geq n_\delta$,

$$\begin{cases} q(x_{2n(\delta)}, x_{2n}) < \delta \\ q(x_{2n(\delta)}, x_{2m}) < \delta \end{cases}$$

And then, $d^s(x_{2n}, x_{2m}) < \epsilon$.
Consequently $(x_{2n})_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Since (X, d) is complete, there exists $z \in X$ such that $\lim_{n\to+\infty} d(x_{2n}, z) = 0$.

Fourth case : Next we show that $\lim_{n\to+\infty} q(x_{2n}, z) = 0$.

Let $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such as, for each $n,\ m \geq n_\epsilon,\ q(x_{2n}, x_{2m}) < \epsilon$. Therefore, for each $n \geq n_\epsilon$,

$$\liminf_{m\to+\infty} q(x_{2n}, x_{2m}) \leq \epsilon$$

Since $\lim_{m\to\infty} d^s(x_{2m}, z) = 0$ and q is lower semi-continuous in its second variable,

$$\forall n \geq n_\epsilon,\ q(x_{2n}, z) \leq \liminf_{m\to+\infty} q(x_{2n}, x_{2m}) \leq \epsilon$$

Consequently $q(x_{2n}, z) \to 0$ as $n \to \infty$, and since $q(x_{2n}, x_{2n+1}) \ \to 0\ as\ \ n \to \infty$, we obtain : $q(x_{2n+1}, z) \ \to 0\ as\ \ n \to \infty$.
Substituting $x = x_{2n}$ and $y = z$ in (3.5), we obtain :

$$\psi(q(x_{2n+1}, Sz)) \leq \psi(q(x_{2n}, z)) - \phi(q(x_{2n}, z))$$

So $\lim_{n\to+\infty} q(x_{2n+1}, Sz) = 0$.
Since $\begin{cases} q(x_{2n+1}, z) \to 0 \\ q(x_{2n+1}, Sz) \to 0 \end{cases}$, by using lemma 2.1), $d^s(Sz, z) = 0$ i.e. $z = Sz$.
Substituting $x = z$ and $y = x_{2n+1}$ in (3.5), we obtain :

$$\psi(q(x_{2n+2}, Tz)) \leq \psi(q(x_{2n+1}, z)) - \phi(q(x_{2n+1}, z))$$

So $q(x_{2n+2}, Tz) \to 0$. Hence $d^s(Tz, z) = 0$ i.e. $z = Tz$.
Thus,

$$Tz = z = Sz$$

We have : $\psi(q(z, z)) \leq \psi(q(z, z)) - \phi(q(z, z))$, so $\phi(q(z, z)) \leq 0$. Thus, $q(z, z) = 0$.

Suppose there exists an point $v \in X$ such that $T(v) = v = S(v)$. We have :

$$\psi(q(z, v)) = \psi(q(T(z), S(v))) \leq \psi(q(z, v)) - \phi(q(z, v)) \Rightarrow \phi(q(z, v)) \leq 0$$

So $q(z, v) = 0$. And since $q(z, z) = 0$, we deduce from lemma 2.1), that $d^s(z, v) = 0$ i.e. $z = v$.
Thus, $z = v$.

Theorem 3.4. *Let (X, d) be a quasi-metric space. Let q be a w-distance on (X, d) and $T,\ f$ a self-mappings of X such that, for all $(x, y) \in X^2$,*

$$\psi(q(Tx, Ty)) \leq \psi(q(fx, fy)) - \phi(q(fx, fy)), \tag{3.8}$$

Assume that (fX, d) is a complete quasi-metric space and $TX \subseteq fX$.
Then T and f have a unique common coincidence point $z \in X$. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$. We define two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in X by

$$y_n = fx_{n+1} = Tx_n \quad n \in \{0, 1, 2, ...\}$$

This can be done, since $TX \subseteq fX$.
First case : We show

$$q(y_{n+1}, y_n) \ \ and \ \ q(y_n, y_{n+1}) \ \ \to 0 \ as \ \ n \to \infty$$

Substituting $x = x_n$ and $y = x_{n+1}$ in (3.8), for all $n \geq 1$ we obtain :

$$\psi(q(Tx_n, Tx_{n+1})) \leq \psi(q(fx_n, fx_{n+1})) - \phi(q(fx_n, fx_{n+1}))$$

$$\psi(q(y_n, y_{n+1})) \leq \psi(q(y_{n-1}, y_n)) - \phi(q(y_{n-1}, y_n)) \tag{3.9}$$

Which implies

$$q(y_n, y_{n+1}) \leq q(y_{n-1}, y_n)$$

The same $x = x_{n+1}$ and $y = x_n$ in (3.8),

$$\psi(q(Tx_{n+1}, Tx_n)) \leq \psi(q(fx_{n+1}, fx_n)) - \phi(q(fx_{n+1}, fx_n))$$

$$\psi(q(y_{n+1}, y_n)) \leq \psi(q(y_n, y_{n-1})) - \phi(q(y_n, y_{n-1})) \tag{3.10}$$

which implies

$$q(y_{n+1}, y_n) \leq q(y_n, y_{n-1})$$

It follows that the sequence $\{q(x_n, x_{n+1})\}$and $\{q(x_{n+1}, x_n)\}$is monotone decreasing and consequently there exists $r \geq 0$ and $r' \geq 0$ such that :

$$q(y_n, y_{n+1}) \to r \quad as \ \ n \to \quad \infty$$

and

$$q(y_{n+1}, y_n) \to r' \quad as \ \ n \to \quad \infty$$

Letting $n \to \infty$ in (3.9) and (3.10), we obtain :

$$\psi(r) \leq \psi(r) - \liminf_{n \to +\infty} \phi(q(y_n, y_{n+1})) \leq \psi(r) - \phi(r)$$

$$\psi(r') \leq \psi(r') - \liminf_{n \to +\infty} \phi(q(y_{n+1}, y_n)) \leq \psi(r') - \phi(r')$$

Which is a contradiction unless $r = r' = 0$.
Hence

$$q(y_{n+1}, y_n) \ \ and \ \ q(y_n, y_{n+1}) \ \ \to 0 \ as \ \ n \to \infty$$

Second case : We show that for each $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$ such chat :

$$q(y_n, y_m) < \epsilon \text{ whenever } m > n \geq n_\epsilon$$

Assume the contrary, then there exists $\epsilon_0 \in (0, 1)$ such that, for each $k \in \mathbb{N}$, there exists $(n(k), m(k)) \in \mathbb{N}^2$ such that $m(k) > n(k) > k$ and

$$q(y_{n(k)}, y_{m(k)}) \geq \epsilon_0 \tag{3.11}$$

We follow the same steps as in the proof of the previous theorem 3.1) to justify the

$$q(y_{n(k)}, y_{m(k)}) \to \epsilon_0$$

and

$$q(y_{n(k)-1}, y_{m(k)-1}) \to \epsilon_0$$

Setting $x = x_{n(k)}$, $y = x_{m(k)}$ in (3.8)

$$\psi(q(Tx_{n(k)}, Tx_{m(k)})) \leq \psi(q(fx_{n(k)}, fx_{m(k)})) - \varphi(q(fx_{n(k)}, fx_{m(k)}))$$

$$\psi(q(y_{n(k)}, y_{m(k)})) \leq \psi(q(y_{n(k)-1}, y_{m(k)-1})) - \varphi(q(y_{n(k)-1}, y_{m(k)-1}))$$

We make k to $+\infty$, which gives :

$$\psi(\epsilon_0) \leq \psi(\epsilon_0) - \liminf_{k\to+\infty} \phi(q(y_{n(k)-1}, y_{m(k)-1})) \leq \psi(\epsilon_0) - \phi(\epsilon_0)$$

Which is a contradiction.

Since (fX, d) is complete, there exists $z \in X$ such that $\lim_{n\to+\infty} d(y_n, fz) = 0$.

Third case : We follow the same steps as in the proof of the previous theorem 3.1) to justify the :

$$\lim_{n\to+\infty} q(y_n, fz) = 0$$

Substituting $x = x_{n+1}$ and $y = z$ in (3.8), we obtain :

$$\psi(q(Tx_{n+1}, Tz)) \leq \psi(q(fx_{n+1}, fz)) - \phi(q(fx_{n+1}, fz))$$

$$\psi(q(y_{n+1}, Tz)) \leq \psi(q(y_n, fz)) - \phi(q(y_n, fz))$$

We make n to $+\infty$, which gives :

$$\lim_{n\to+\infty} q(y_{n+1}, Tz) = 0$$

Since $\begin{cases} q(y_{n+1}, Tz) \to 0 \\ q(y_{n+1}, fz) \to 0 \end{cases}$, by using lemma 2.1), $d^s(Tz, fz) = 0$ i.e. $Tz = fz$, We put $w = Tz = fz$. Hence, we proved w is a point of coincidence of T and f.
Since $\psi(q(w, w)) \leq \psi(q(w, w)) - \phi(q(w, w))$, so $\phi(q(w, w)) \leq 0$. Thus, $q(w, w) = 0$.

Fourth case : Now we show that w is a unique point of coincidence.

Let w_1 be point of coincidence in X such that $w_1 = fv = Tv$, where $v \in X$. Suppose that $w \neq w_1$, then $fv \neq fw$. From (3.8), we have :

$$\psi(q(Tz, Tv)) \leq \psi(q(fz, fv)) - \phi(q(fz, fv))$$

$$\psi(q(w, w_1)) \leq \psi(q(w, w_1)) - \phi(q(w, w_1))$$

Then $\phi(q(w, w_1)) \leq 0$, which is contradiction. So $q(w, w_1) = 0$. And since $q(w, w) = 0$, we deduce from lemma 2.1), that $d^s(w, w_1) = 0$ i.e. $w = w_1$.
Thus we proved that T and f have a unique point of coincidence.

If T and f are weakly compatible, then from $fz = Tz = w$ we have $Tfz = fTz$, that is, $Tw = fw$.
Since w is a unique point of coincidence of T and f, then $w = Tw = fw$. Thus we proved that w is the unique common fixed point of T and f.

Example 3.5. *Let $X = \mathbb{R}_+$ and $d(x, y) = \max(y - x, 0)$, for all $(x, y) \in \mathbb{R}_+^2$. Let $T : X \to X$ be defined as :*

$$Tx = \begin{cases} \frac{\sin(x)}{4} & if \quad 0 \leq x \leq 1 \\ \\ \frac{\sin(1)}{8} & if\ x > 1 \end{cases}$$

$f : X \to X$ be defined as :

$$f(x) = \frac{x}{2}$$

$\phi : [0, \infty) \to [0, \infty)$ be defined as :

$$\phi(t) = \frac{t^2}{4}$$

$\psi : [0, \infty) \to [0, \infty)$ be defined as :

$$\psi(t) = t^2$$

$q : [0, \infty) \times [0, \infty) \to [0, \infty)$ be defined as :

$$q(x, y) = y$$

$TX = [0, \frac{\sin(1)}{4}]$ and $fX = \mathbb{R}_+$, so $TX \subseteq fX$

We have $(fX, d)$is complete quasi-metric space.

Let $x \in \mathbb{R}$.

Case 1 : $y \in [0, 1]$
We have $q(Tx, Ty) = Ty = \frac{\sin(y)}{4}$,

$$\psi(q(Tx, Ty)) = \frac{\sin(y)^2}{16},\ \phi((q(fx, fy)) = \frac{y^2}{16} \text{ and } \psi(q(fx, fy)) = (fy)^2 = \frac{y^2}{4}$$

So,

$$\psi(q(Tx, Ty)) \leq \psi(q(fx, fy)) - \phi((q(fx, fy))$$

Case 2 : $y > 1$
We have $q(Tx, Ty) = Ty = \frac{\sin(1)}{8}$,

$$\psi(q(Tx, Ty)) = \frac{\sin(1)^2}{64},\ \phi((q(fx, fy)) = \frac{y^2}{16} \text{ and } \psi(q(fx, fy)) = \frac{y^2}{4}$$

Since $\psi(q(Tx, Ty)) = \frac{\sin(1)^2}{64} < \frac{y^2}{4} - \frac{y^2}{16}$, so

$$\psi(q(Tx, Ty)) < \psi(q(fx, fy)) - \phi((q(fx, fy))$$

0 is unique common fixed point of T and f.

Theorem 3.6. *Let (X, d) be quasi-metric space and q be a symmetric w-distance. Let $S, T, f : X \longrightarrow X$ be a self mappings satisfying the inequality :*

$$\forall (x, y) \in X^2,\ \psi(q(Tx, Sy)) \leq \psi(q(fx, fy)) - \phi(q(fx, fy)). \tag{3.12}$$

Assume that (fX, d) is a complete quasi-metric space and $TX \cup SX \subseteq fX$. Then T, f, S have a unique common coincidence point $z \in X$. Moreover, if (T, f) and (S, f) are weakly compatible, then T, S and f have a unique common fixed point.

Proof. Let $x_0 \in X$. We define two sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in X by taking

$$\begin{cases} y_{2n+1} = Tx_{2n} = fx_{2n+1} \\ \\ y_{2n+2} = Sx_{2n+1} = fx_{2n+2} \end{cases}$$

First case :

$$q(x_n, x_{n+1}) \ \to 0 \text{ as } n \to \infty$$

Substituting $x = x_{2n}$ and $y = x_{2n+1}$ in (3.12), we obtain :

$$\psi(q(Tx_{2n}, Sx_{2n+1})) \leq \psi(q(fx_{2n}, fx_{2n+1})) - \phi(q(fx_{2n}, fx_{2n+1}))$$

$$\psi(q(y_{2n+1}, y_{2n+2})) \leq \psi(q(y_{2n}, y_{2n+1})) - \phi(q(y_{2n}, y_{2n+1})) \tag{3.13}$$

Which implies

$$q(y_{2n+1}, x_{2n+2}) \leq q(y_{2n}, y_{2n+1})$$

Then, $(q(y_n, y_{n+1}))_n$ is monotone decreasing. Consequently there exists $r \geq 0$ such that

$$q(y_n, y_{n+1}) \to r \quad as \quad n \to \quad \infty$$

Letting $n \to \infty$ in (3.13), we obtain :

$$\psi(r) \leq \psi(r) - \liminf_{n\to+\infty} \phi(q(y_{2n}, y_{2n+1})) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$

Second case : We show that, for each $\epsilon \in (0, 1)$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$q(y_{2n}, y_{2m}) < \epsilon \text{ whenever } 2m > 2n \geq n_\epsilon$$

Assume the contrary, then there exists $\epsilon_0 \in (0, 1)$ such that, for each $k \in \mathbb{N}$, there exist two sequences of positives integers $(n(k))_n, (m(k))_n$ with $2m(k) > 2n(k) > k$ and

$$q(y_{2n(k)}, y_{2m(k)}) \geq \epsilon_0 \tag{3.14}$$

We follow the same steps as in the proof of the previous theorem 3.3) to justify the:

$$q(y_{2n(k)}, y_{2m(k)}) \quad q(y_{2n(k)}, y_{2m(k)+1}) \quad and \quad q(y_{2n(k)+1}, y_{2m(k)+2}) \to \epsilon_0 \quad as \quad k \to \infty$$

Setting $x = x_{2n(k)}$ and $y = x_{2m(k)+1}$ in (3.12), we obtain :

$$\psi(q(Tx_{2n(k)}, Sx_{2m(k)+1})) \leq \psi(q(fx_{2n(k)}, fx_{2m(k)+1})) - \phi(q(fx_{2n(k)}, fx_{2m(k)+1}))$$

$$\psi(q(y_{2n(k)+1}, y_{2m(k)+2})) \leq \psi(q(y_{2n(k)}, y_{2m(k)+1})) - \phi(q(y_{2n(k)}, y_{2m(k)+1}))$$

We make k to $+\infty$,

$$\psi(\epsilon_0) \leq \psi(\epsilon_0) - \liminf_{k\to+\infty} \phi(q(y_{2n(k)}, y_{2m(k)+1})) \leq \psi(\epsilon_0) - \phi(\epsilon_0)$$

Then $\phi(\epsilon_0) \leq 0$, which is contradiction.

Since (fX, d) is complete, there exists $z \in X$ such that $\lim_{n\to+\infty} d(y_{2n}, fz) = 0$.

Third case : We follow the same steps as in the proof of the previous theorem 3.3) to justify the :

$$\lim_{n\to+\infty} q(y_{2n}, fz) = 0$$

Substituting $y = z$ and $x = x_{2n}$ in (3.12), we obtain :

$$\psi(q(Tx_{2n}, Sz)) \leq \psi(q(fx_{2n}, fz)) - \phi(q(fx_{2n}, fz))$$

$$\psi(q(y_{2n+1}, Sz)) \leq \psi(q(y_{2n}, fz)) - \phi(q(y_{2n}, fz))$$

Imply $\lim_{n\to+\infty} q(y_{2n+1}, Sz) = 0$

Since $\begin{cases} q(y_{2n+1}, fz) \to 0 \\ q(y_{2n+1}, Sz) \to 0 \end{cases}$, by using lemma 2.1), $d^s(Sz, fz) = 0$ i.e. $fz = Sz$.

Substituting $x = z$ and $y = x_{2n+1}$ in (3.12), we obtain :

$$\psi(q(Sx_{2n+1}, Tz)) \leq \psi(q(fx_{2n+1}, fz)) - \phi(q(fx_{2n+1}, fz))$$
$$\psi(q(y_{2n+2}, Tz)) \leq \psi(q(y_{2n+1}, fz)) - \phi(q(y_{2n+1}, fz))$$

Imply $\lim_{n\to+\infty} q(y_{2n+2}, Tz) = 0$

Since $\begin{cases} q(y_{2n+2}, fz) \to 0 \\ q(y_{2n+2}, Tz) \to 0 \end{cases}$, by using lemma 2.1), $d^s(Tz, fz) = 0$ i.e. $fz = Tz$.

Thus,

$$Tz = fz = Sz = w$$

Hence, we proved w is a point of coincidence of T,S and f.
Since $\psi(q(w, w)) \leq \psi(q(w, w)) - \phi(q(w, w))$, so $\phi(q(w, w)) \leq 0$. Thus, $q(w, w) = 0$.
Fourth case : We proved w is a unique point of coincidence
If there exists an other point $k \in X$ such that $k = T(v) = f(v) = S(v)$, we have :

$$\psi(q(w, k)) = q(T(z), S(v))) \leq \psi(q(w, k)) - \phi(q(w, k))$$
$$\phi(q(w, k)) \leq 0$$

Which is a contradiction.
So $q(w, k) = 0$. And since $q(w, w) = 0$, we deduce from lemma 2.1), that $d^s(w, k) = 0$ i.e. $k = w$
Thus we proved that T,S and f have a unique point of coincidence.
T and f are weakly compatible, then from $fz = Tz = w$ we have $Tfz = fTz$, that is, $Tw = fw$.
also S and f are weakly compatible, then from $fz = Sz = w$ we have $Sfz = fSz$, that is, $Sw = fw$.
Since w is a unique point of coincidence of T, $f$and S, then $w = Sw = Tw = fw$.
Thus we proved that w is the unique common fixed point of T, S and f.
Now, we prove theorem 3.1 for T is a multi-valued mapping in (X, d) with a symmetric w-distance.

Theorem 3.7. *Let (X, d) be a complete quasi-metric space, and $T : X \to 2^X$ be a multi-valued map such that for all $x \in X$, $T(x)$ is a nonempty τ^s-closed subset of X.*
If there exists q symmetric w-distance on X such that, for all $(x, y) \in X^2$ and for all $u \in T(x)$, there exists $v \in T(y)$ such that :

$$\psi(q(u, v)) \leq \psi(q(x, y)) - \phi(q(x, y)),$$

Then T has a fixed point $z \in X$. Moreover $q(z, z) = 0$.

Proof. Fix x_0 and let $x_1 \in Tx_0$. Then, there exists $x_2 \in Tx_1$ such that

$$\psi(q(x_1, x_2)) \leq \psi(q(x_0, x_1)) - \phi(q(x_0, x_1))$$

Following this process, we obtain a sequence $(x_n)_{n\geq 0}$ with $x_n \in Tx_{n-1}$, for all $n \in I\!N^*$, and

$$\psi(q(x_n, x_{n+1})) \leq \psi(q(x_{n-1}, x_n)) - \phi(q(x_{n-1}, x_n))$$

As in previous theorem $q(x_n, x_{n+1}) \to 0 \quad as \quad k \to \infty$.
Now, we show that for each $\epsilon \in (0,1)$, there exists $n_\epsilon \in \mathbb{N}$ such that $q(x_n, x_m) < \epsilon$ whenever $m > n > n_\epsilon$.
Assume the contrary, then there exists $\epsilon_0 \in (0,1)$ such that, for each $k \in \mathbb{N}$, there exists $n(k), m(k) \in \mathbb{N}$ such that : $m(k) > n(k) > k$ and

$$q(x_{n(k)}, x_{m(k)}) \geq \epsilon_0 \tag{3.15}$$

We have :

$$q(x_{n(k)}, x_{m(k)}) \to \epsilon_0 \quad as \quad k \to \infty$$

and

$$q(x_{n(k)-1}, x_{m(k)-1}) \to \epsilon_0 \quad as \quad k \to \infty$$

Since $x_{n(k)} \in Tx_{n(k)-1}$, $x_{m(k)} \in Tx_{m(k)-1}$,

$$\psi(q(x_{n(k)}, x_{m(k)})) \leq \psi(q(x_{n(k)-1}, x_{m(k)-1})) - \phi(q(x_{n(k)-1}, x_{m(k)-1}))$$

We make k to $+\infty$, which gives :

$$\psi(\epsilon_0) \leq \psi(\epsilon_0) - \liminf_{k\to+\infty} \phi(q(x_{n(k)-1}, x_{m(k)-1})) \leq \psi(\epsilon_0) - \phi(\epsilon_0)$$

Thus, $\phi(\epsilon_0) \leq 0$, which is contradiction.

From lemma 2.1), $(x_n)_{n\geq 0}$ is a Cauchy sequence in (X, d^s) (see theorem 3.1) so there exists $z \in X$ such that $d(x_n, z) \to 0$ and thus $q(x_n, z) \to 0$.
For each $n \in \mathbb{N}$ there exists $v_{n+1} \in T(z)$ such that :

$$\psi(q(x_{n+1}, v_{n+1})) \leq \psi(q(x_n, z)) - \phi(q(x_n, z))$$

Since $q(x_n, z) \to 0$ we have $q(x_{n+1}, v_{n+1}) \to 0$, so $\lim_{n\to+\infty} d^s(z, v_n) = 0$ from lemma 2.1). Hence, $z \in T(z)$, because Tz is closed in (X, d^s).

Now we prove that $q(z, z) = 0$ where $z \in T(z)$.
For such $y_o = z$, there exists $y_1 \in T(z)$ such that :

$$\psi(q(z, y_1)) \leq \psi(q(z, z)) - \phi(q(z, z))$$

As above we obtain a sequence $(y_n)_{n\geq 0}$ in X such that $y_{n+1} \in T(y_n)$, for all $n \in I\!N$, and

$$\psi(q(z, y_{n+1})) \leq \psi(q(z, y_n)) - \phi(q(z, y_n))$$

Hence $(q(z, y_n))_{n\geq 0}$ is non-increasing sequence in $(0, \infty)$ that converge to 0. Then $(y_n)_{n\geq 0}$ is a Cauchy sequence in (X, d^s) (using lemma 2.1)); there exists $u \in X$ such that $\lim_{n\to+\infty} d(y_n, u) = 0$.
From w_2, we have : $q(z, u) \leq \liminf_{n\to+\infty} q(z, y_n) = 0$, so $q(z, u) = 0$.
From w_1, we have : $q(x_n, u) \leq q(x_n, z) + q(z, u)$, for all $n \in I\!N$, and since $q(x_n, z) \to 0$, so $q(x_n, u) \to 0$; by the lemma 2.1), we obtain $d^s(u, z) = 0$. Hence, $u = z$ and $q(z, z) = 0$.

Marin, Romaguera and Tirado showed the version of Boyd-Wong's in T_0 quasi-pseudo metric space (see [[6],theorem 2.2]). The authors had used the notion of Q-function instead the distance (Q-function satisfying w_1, w_3 in definition 2 and

if $x \in X$, $M > 0$, and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X that τ^{-1} converges to a point $y \in X$ and satisfies $q(x, y_n) \leq M$ for all $n \in \mathbb{N}$, then $q(x, y) \leq M$)
Now, we extend this version to quasi-metric space, we change the distance by w-distance and we obtain :

Theorem 3.8. *Let (X, d) be a a complete quasi-metric space. If there exist a w-distance q on (X, d) and a self-mapping T of X such that, for all $(x, y) \in X^2$,*

$$q(Tx, Ty) \leq \Phi(q(x, y)) \tag{3.16}$$

Where $\Phi : [0, +\infty[\to [0, +\infty[$ Φ is right upper semi-continuous function,and $\Phi(0) = 0$ and $\Phi(t) < t$, for all $t > 0$. Then, T has a unique fixed point $z \in X$. Moreover $q(z, z) = 0$.

In [2] the authors also proved theorem 3.8 (see[[2],Corollary3]), But they used another concept in the proof (function of Meir-Keeler and Jachymski type).

Theorem 3.9. *Let (X, d) be a a complete quasi-metric space. If there exist a symmetric w-distance q on (X, d) and a self-mappings T and S of X such that, for all $(x, y) \in X^2$,*

$$q(Tx, Sy) \leq \Phi(q(x, y)) \tag{3.17}$$

Where $\Phi : [0, +\infty[\to [0, +\infty[$ Φ is right upper semi-continuous function,and $\Phi(0) = 0$ and $\Phi(t) < t$, for all $t > 0$. Then, there exists a unique point $z \in X$ such that $T(z) = z = S(z)$. Moreover $q(z, z) = 0$.

Proof. For any $x_0 \in X$, we construct the sequence $(x_n)_{n\in\mathbb{N}}$ in X by taking

$$\begin{cases} x_{2n+1}=Tx_{2n} \\ x_{2n+2}=Sx_{2n+1} \end{cases}$$

First case :

$$q(x_n, x_{n+1}) \ \to 0 \ as \ \ n \to \infty$$

Substituting $x = x_{2n}$ and $y = x_{2n+1}$ in (3.17), we obtain :

$$q(x_{2n+1}, x_{2n+2}) \leq \Phi(q(x_{2n}, x_{2n+1})) \leq q(x_{2n}, x_{2n+1}) \tag{3.18}$$

$$q(x_{2n+1}, x_{2n+2}) \leq q(x_{2n}, x_{2n+1})$$

Then, $(q(x_n, x_{n+1}))_n$ is monotone decreasing. Consequently there exists $r \geq 0$ such that

$$q(x_n, x_{n+1}) \to r \quad as \ \ n \to \ \ \infty$$

Letting $n \to \infty$ in (3.18), we obtain :

$$r = \limsup_{n\to+\infty} q(x_{2n+1}, x_{2n+2}) \leq \limsup_{n\to+\infty} \Phi(q(x_{2n}, x_{2n+1})) \leq \Phi(r)$$

Which is a contradiction unless $r = 0$

Second case : We show that, for each $\epsilon \in (0,1)$, there exists $n_\epsilon \in \mathbb{N}$ such that:

$$q(x_{2n}, x_{2m}) < \epsilon \text{ whenever } 2m > 2n \geq n_\epsilon$$

Assume the contrary, then there exists $\epsilon_0 \in (0,1)$ such that, for each $k \in \mathbb{N}$, there exist two sequences of positives integers $(n(k))_n, (m(k))_n$ with $2m(k) > 2n(k) > k$ and

$$q(x_{2n(k)}, x_{2m(k)}) \geq \epsilon_0 \tag{3.19}$$

We follow the same steps as in the proof of the previous theorem 3.3) to justify the:

$$q(x_{2n(k)+1}, x_{2m(k)+2}) \to \epsilon_0$$

and

$$q(x_{2n(k)+1}, x_{2m(k)+2}) \leq \Phi(q(x_{2n(k)}, x_{2m(k)+1}))$$

We make k to $+\infty$,

$$\epsilon_0 \leq \phi(\epsilon_0)$$

Which is a contradiction.

Since (X,d) is complete, there exists $z \in X$ such that $\lim_{n\to+\infty} d(x_{2n}, z) = 0$.

Third case : We follow the same steps as in the proof of the previous theorem 3.3) to justify the :

$$\lim_{n\to+\infty} q(x_{2n}, z) = 0$$

Substituting $x = x_{2n}$ and $y = z$ in (3.17), we obtain :

$$q(x_{2n+1}, Sz) \leq \Phi(q(x_{2n}, z))$$

So $\lim_{n\to+\infty} q(x_{2n+1}, Sz) = 0$.

Since $\begin{cases} q(x_{2n+1}, z) \to 0 \\ q(x_{2n+1}, Sz) \to 0 \end{cases}$, by using lemma 2.1), $d^s(Sz, z) = 0$ i.e. $z = Sz$.

Substituting $x = z$ and $y = x_{2n+1}$ in (3.17), we obtain :

$$q(x_{2n+2}, Tz) \leq \Phi(q(x_{2n+1}, z))$$

So $q(x_{2n+2}, Tz) \to 0$. Hence $d^s(Tz, z) = 0$ i.e. $z = Tz$.

Thus,

$$Tz = z = Sz$$

If $q(z,z) \neq 0$, then $q(z,z) \leq \Phi(q(z,z)) < q(z,z)$, which is contradiction.

If there exists an other point $v \in X$ such that $T(v) = v = S(v)$, we have :

$$q(z,v) = q(T(z), S(v))) \leq \Phi(q(z,v)) < q(z,v)$$

Which is a contradiction.

So $q(z,v) = 0$. And since $q(z,z) = 0$, we deduce from lemma 2.1), that $d^s(z,v) = 0$ i.e. $z = v$ Thus, there exists a unique point $z \in X$ such that $T(z) = z = S(z)$.

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