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RELATED FIXED POINT THEOREMS OF CARISTI TYPE FOR TWO SET VALUED MAPPINGS

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Abstract. In this paper, we present some related Caristi type fixed point theorems for multivalued maps in complete metric spaces. As application, we establish a new version of the ε-variational principle. Examples are given to illustrate our results.

1. Introduction and preliminaries

In 1972, I. Ekeland (see [10]) obtained the following minimization theorem in a complete metric space.

Theorem 1.1. *Let (X, δ) be a complete metric space, φ a proper, bounded below and lower semicontinuous function of X into $(-\infty, \infty]$. For any $\varepsilon > 0$, we choose $u \in X$ such that*

$$\varphi(u) \leq \inf\{\varphi(x), x \in X\} + \varepsilon$$

Then there exists $v \in X$ such that :

(1) $\varphi(v) \leq \varphi(u)$
(2) $\delta(u, v) \leq 1$
(3) $\varphi(x) > \varphi(v) - \varepsilon\delta(x, v)$ *for all $x \in X$ and $x \neq v$.*

Further in 1976, J. Caristi [5] establish his famous fixed point theorem which is a generalization of Banach contraction principle (see [1]). Recall that this theorem states that :

Theorem 1.2. *Let T be a self mapping of a complete metric space (X, δ) and φ a lower semicontinuous function of X to $\mathbb{R}_+$. Assume that*

$$\delta(x, Tx) \leq \varphi(x) - \varphi(Tx) \tag{1.1}$$

for all $x \in X$. Then T has at least one fixed point in X.

On the other hand, many authors obtained some interesting generalizations of these two results for single and multivalued mappings, because of their important applications in applied mathematics : control theory, convex analysis, etc; (For

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more details see [16, 9, 13, 21, 8, 17, 15, 4, 18, 23, 11, 7]). It is proved that the above theorems are equivalent.

Recently, A. Latif and M.A. Khamsi (see [18]) obtained some fixed point theorems for multivalued mapping which generalize Caristi's theorem using the concept of ω-distance in complete metric space [14].

The following definitions will be needed throughout the paper :

Definition 1.3. A function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is said to be upper semi-continuous from the right at x_0 if for every $\varepsilon > 0$ there exists $\eta > 0$ such that $h(x) \leq h(x_0) + \varepsilon$ for each $x \in [x_0, x_0 + \eta[$.

Definition 1.4. Let T be a multivalued mapping. A sequence $\{x_n\}_n$ in X such that $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$ is called an orbit of T.

Here we give a definition which is slightly different of the standard continuity of multivalued mappings.

Definition 1.5. Let T be a multivalued mapping, we call T orbitally CS-continuous if, for every orbit $\{x_n\}_n$ of T which is a convergent sequence, we have $\lim x_n \in T(\lim x_n)$.

We recall the Brondsted principle (see [3]) that will be useful in the sequel.

Theorem 1.6. *Let (X, δ) be a metric space, and $\preccurlyeq$ a binary relation on X such that $(X, \preccurlyeq)$ is a partially ordered set , and $\varphi : X \longrightarrow \mathbb{R}_+$ a function . Assume that*
(a) φ is decreasing with respect to $\preccurlyeq$, i.e., $x \preccurlyeq y$ implies $\varphi(y) \preccurlyeq \varphi(x)$;
(b) for all $\varepsilon > 0$ there exists $\eta > 0$ such that $x \preccurlyeq y$ and $\varphi(x) - \varphi(y) < \eta$ implies $\delta(x, y) < \varepsilon$.
Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X (where x_0 may be taken arbitrary) and a point $\overline{x} \in X$ such that
(c) $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$, and $x \to \overline{x}$;
(d) $y_n \to \overline{x}$ for all sequences $\{y_n\}_{n\in\mathbb{N}}$, with $x_n \preccurlyeq y_n$,
Furthermore,
(e) if $x \preccurlyeq \overline{x}$ for all $n \in \mathbb{N}$, then $\overline{x}$ is maximal in $(X, \preccurlyeq)$.

In this paper we use a modified Caristi type inequality in complete metric space to prove some related fixed point theorems for multivalued mappings. As application, we give a new version of Ekeland minimization principle in complete product metric space with two distances. Examples are given to support the usability of our results and to distinguish them from the existing ones.

2. Common Caristi-type fixed point theorems

All multivalued maps throughout this paper have a nonempty values.

Theorem 2.1. *Let (X, δ) be a complete metric space, S and T two multivalued maps on X. Let $f, g : X \longrightarrow \mathbb{R}_+$ such that for some $\varepsilon > 0$,*

$$\begin{cases} \sup\{f(x) : x \in X, \ \varphi(x) \leq \inf_{z\in X} \varphi(z) + \varepsilon\} < \infty \\ \sup\{g(x) : x \in X, \ \varphi(x) \leq \inf_{z\in X} \varphi(z) + \varepsilon\} < \infty \end{cases} \tag{2.1}$$

where $\varphi : X \longrightarrow \mathbb{R}_+$ is lower semicontinuous.

Assume that for each $x \in X$ there exists $(u, v) \in Tx \times Sx$ such that :

$$\begin{cases} \delta(x, u) \leq f(x)(\varphi(x) - \varphi(v)) \\ \delta(x, v) \leq g(x)(\varphi(x) - \varphi(u)) \end{cases} \tag{2.2}$$

Then T and S have a common fixed point.

Proof. Define X_1 and α by

$$X_1 = \left\{ x \in X : \varphi(x) \leq \inf_{z \in X} \varphi(z) + \varepsilon \right\}$$

$$\alpha = \max\left\{ \sup_{z \in X_1} f(z), \sup_{z \in X_1} g(z) \right\} < \infty$$

If $f(x) = 0$, for all $x \in X$, we obtain $x = u \in Tx$. And consequently, $x = v$ by the second inequality. Then each element in X is a common fixed point of T and S. Assume that $\alpha > 0$ and define two single-valued mappings T_1 and S_1 as follows :

$$T_1 x = u \in Tx \quad \text{and} \quad S_1 x = v \in Sx \ .$$

We introduce the partial order "$\preccurlyeq$" in the nonempty set X_1 by

$$x \preccurlyeq y \Leftrightarrow \delta(x, y) \leq \alpha(\varphi(x) - \varphi(y))$$

Since φ is lower semi-continuous, the metric space (X_1, δ) is complete. So by theorem 1.6, $(X_1, \preccurlyeq)$ has a maximal element $\bar{x}$.

Note that $T_1 X_1 \subseteq X_1$ and $S_1 X_1 \subseteq X_1$. Indeed let $x \in X_1$ then $\varphi(x) \leq \inf_{z \in X} \varphi(z) + \varepsilon$ and by (2.2) we get

$$\begin{cases} 0 \leq \alpha(\varphi(x) - \varphi(S_1 x)) \\ 0 \leq \alpha(\varphi(x) - \varphi(T_1 x)) \end{cases} \tag{2.3}$$

and since $\alpha > 0$ we get

$$\begin{cases} \varphi(S_1 x) \leq \varphi(x) \\ \varphi(T_1 x) \leq \varphi(x) \end{cases} \tag{2.4}$$

which implies

$$\max\{\varphi(T_1 x), \varphi(S_1 x)\} \leq \inf_{z \in X} \varphi(z) + \varepsilon$$

then $T_1 x \in X_1$ and $S_1 x \in X_1$. Also, by hypothesis we get

$$\begin{cases} \delta(\bar{x}, T_1 \bar{x}) \leq \alpha(\varphi(\bar{x}) - \varphi(S_1 \bar{x})) \\ \delta(\bar{x}, S_1 \bar{x}) \leq \alpha(\varphi(\bar{x}) - \varphi(T_1 \bar{x})) \end{cases} \tag{2.5}$$

If $\varphi(S_1 \bar{x}) \leq \varphi(T_1 \bar{x})$ then $\delta(\bar{x}, S_1 \bar{x}) \leq \alpha(\varphi(\bar{x}) - \varphi(S_1 \bar{x}))$. Hence $\bar{x} \preccurlyeq S_1 \bar{x}$. Thus $\bar{x} = S_1 \bar{x}$. By the first inequality of (2.5), we obtain $\bar{x} = T_1 \bar{x}$.

If $\varphi(T_1 \bar{x}) \leq \varphi(S_1 \bar{x})$ then $\delta(\bar{x}, T_1 \bar{x}) \leq \alpha(\varphi(\bar{x}) - \varphi(T_1 \bar{x}))$, hence $\bar{x} \preccurlyeq T_1 \bar{x}$. Thus $T_1 \bar{x} = \bar{x}$. By the second inequality of (2.5), we obtain $\bar{x} = S_1 \bar{x}$. □

Corollary 2.2. *Let (X, δ) be a complete metric space, $\varphi : X \longrightarrow \mathbb{R}_+$ be a lower semicontinuous function and $T, S : X \to X$ two single-valued mappings such that for all $x \in X$,*

$$\begin{cases} \delta(x, Tx) \leq f(x)(\varphi(x) - \varphi(Sx)) \\ \delta(x, Sx) \leq g(x)(\varphi(x) - \varphi(Tx)) \end{cases}$$

where $f, g : X \longrightarrow \mathbb{R}_+$ satisfy conditions (2.1). Then, there exists an element $\bar{x} \in X$ such that $T\bar{x} = S\bar{x} = \bar{x}$.

Example 2.3. Choose $X = [0,1]$ with the usual distance, $f = g = 1$ and define T, S and φ as follows :

$$Tx = \begin{cases} 0 & \text{if} \quad x = \{0,1\} \\ 1 & \text{if} \quad x \in \left]0,1\right[\end{cases}$$

and

$$Sx = \begin{cases} 0 & \text{if} \quad x = \left\{0, \frac{1}{2}, 1\right\} \\ 1 & \text{if} \quad x \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[\end{cases}$$

and

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{x} & \text{if} \quad x \in \left]0,1\right] \end{cases}$$

it is clear that φ is lower semicontinuous,

(1) for all $x \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$

$$\begin{cases} \delta(x,Tx) & \leq & \varphi(x) - \varphi(Sx) \\ \delta(x,Sx) & \leq & \varphi(x) - \varphi(Tx) \end{cases} \Leftrightarrow \begin{cases} |x-1| = 1-x & \leq & \varphi(x) - \varphi(1) = \dfrac{1-x}{x} \\ |x-1| = 1-x & \leq & \varphi(x) - \varphi(1) = \dfrac{1-x}{x} \end{cases}$$

(2) and for $x \in \left\{0, \frac{1}{2}, 1\right\}$, we get also

$$\begin{cases} \delta(x,Tx) & \leq & \varphi(x) - \varphi(Sx) \\ \delta(x,Sx) & \leq & \varphi(x) - \varphi(Tx) \end{cases}$$

thus for all x in $[0,1]$

$$\begin{cases} \delta(x,Tx) \leq f(x)(\varphi(x) - \varphi(Sx)) \\ \delta(x,Sx) \leq g(x)(\varphi(x) - \varphi(Tx)) \end{cases}$$

so by theorem 2.1, T and S have a common fixed point. Note that $T0 = S0 = 0$.

Applying theorem 1.2 we give a generalized version of Caristi-Type result in a set endowed by two metrics.

Theorem 2.4. *Let (X, δ_i) be a complete metric space ($i = 1,2$), S and T two multivalued maps on X. Let $f, g : X \longrightarrow \mathbb{R}_+$ satisfy conditions (2.1). Assume that for each $x \in X$ there exists $u \in Tx$ and for each $y \in X$ there exists $v \in Sy$ such that :*

$$\begin{cases} \delta_1(x,u) \leq f(x)(\varphi(x) - \varphi(v)) \\ \delta_2(y,v) \leq g(y)(\varphi(y) - \varphi(u)) \end{cases} \tag{2.6}$$

Then there exists a common fixed point for T and S.

Proof. Put $X_1 = \{x \in X : \varphi(x) \leq \inf_{z \in X} \varphi(z) + \varepsilon\}$ and

$$\alpha = \max\left\{\sup_{z \in X_1} f(z), \sup_{z \in X_1} g(z)\right\} < \infty$$

By hypothesis corresponding to each $x \in X$ and $y \in X$ there exist $u \in Tx$ and $v \in Sy$ such that inequality (2.6) holds, we can define two single-valued mappings $T_1 : X \longrightarrow X$ and $S_1 : X \longrightarrow X$ by choosing $T_1(x) = u$ and $S_1(y) = v$. From (2.6), we obtain

$$\begin{cases} \delta_1(x, T_1x) \leq \alpha(\varphi(x) - \varphi(S_1y)) \\ \delta_2(y, S_1y) \leq \alpha(\varphi(y) - \varphi(T_1x)) \end{cases} \tag{2.7}$$

Note that X_1 is a nonempty set, since φ is lower semicontinuous function, X_1 is a closed subset of X, hence it is complete. Let us define

$$\begin{cases} \psi(x,y) & = \alpha(\varphi(x)+\varphi(y)) \\ L(x,y) & = (T_1 x, S_1 y) \\ \rho((x,y),(z,t)) & = \delta_1(x,z)+\delta_2(y,t) \end{cases}$$

for all x, y, z, t in X_1. By (2.7), we obtain

$$\rho((x,y), L(x,y)) \leq \psi(x,y) - \psi(L(x,y)) \tag{2.8}$$

If we take $X_2 = \left\{(x,y) \in X_1^2 : \psi(x,y) \leq \inf_{(z,t)\in X_1^2} \psi(z,t) + \varepsilon\right\}$, we obtain by the same arguments that (X_2, ρ) is a non-empty complete subset of X^2 since ψ is lower semicontinuous function and L is a self mapping of X_2. Indeed, by (2.8), we have

$$\psi(L(x,y)) \leq \psi(x,y) \leq \inf_{(z,t)\in X_1^2} \psi(z,t) + \varepsilon$$

for all $(x,y) \in X_2$ and thus $L(x,y) \in X_2$.
By theorem 1.2, there exist $(\bar{x}, \bar{y}) \in X_2$ such that

$$L(\bar{x},\bar{y}) = (\bar{x},\bar{y}) \Leftrightarrow T_1\bar{x} = \bar{x} \text{ and } S_1\bar{y} = \bar{y}$$

The second inequality of (2.7) leads to

$$\delta_2(\bar{x}, S_1\bar{x}) \leq \alpha(\varphi(\bar{x}) - \varphi(T_1\bar{x})) = 0$$

which ends the proof.

□

Remark 2.5. In theorem 2.4, we have $Fix(S) = Fix(T)$, where the $Fix(S)$ is the set of all fixed points of S.

Example 2.6. Let us choose $X = [0,1]$ with the usual metric, $f = g = 1$ and define T, S and φ as follows :

$$Tx = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{4} & \text{if} \quad x \in \left]0,\frac{1}{2}\right[\cup \left]\frac{1}{2},1\right[\\ \frac{1}{2} & \text{if} \quad x = \frac{1}{2} \\ 1 & \text{if} \quad x = 1 \end{cases}$$

and

$$Sx = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{4} & \text{if} \quad x = \frac{1}{4} \\ \frac{1}{2} & \text{if} \quad x \in \left]0,\frac{1}{4}\right[\cup \left]\frac{1}{4},1\right[\\ 1 & \text{if} \quad x = 1 \end{cases}$$

and

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad x = \left\{0,\frac{1}{4},\frac{1}{2},1\right\} \\ 1 & \text{if} \quad x \in \left]0,\frac{1}{4}\right[\cup \left]\frac{1}{4},\frac{1}{2}\right[\cup \left]\frac{1}{2},1\right[\end{cases}$$

then φ is lower semicontinuous and for all x in $[0,1]$, $\varphi(Tx) = \varphi(Sx) = 0$, so the following inequalities hold,

$$\begin{array}{lcl} \delta(x,Tx) \leq \varphi(x) - \varphi(Sy) & \Leftrightarrow & |x - Tx| \leq \varphi(x) \\ \delta(y,Sy) \leq \varphi(y) - \varphi(Tx) & \Leftrightarrow & |x - Sx| \leq \varphi(x) \end{array}$$

By theorem 2.4, we have $Fix(T) = Fix(S)$.

Theorem 2.7. *Let (X, δ_i) be a complete metric space ($i = 1, 2$), X' a non-empty closed subset of X^2 and L a multivalued map on X'.*

Let $f, g : X \longrightarrow \mathbb{R}_+$ satisfy conditions (2.1). Assume that for each $(x, y) \in X'$ there exists $(u, v) \in L(x, y)$ satisfy condition (2.6). Then there exists $(u^\star, v^\star) \in L(u^\star, v^\star)$ and $\varphi(u^\star) = \varphi(v^\star)$.

Proof. We use the same notations as theorem 2.4, i.e. for all $(x, y), (t, s) \in X'$ put :

$$\begin{aligned} \rho((x, y), (t, s)) &= \delta_1(x, t) + \delta_2(y, s) \\ \psi(x, y) &= \alpha(\varphi(x) + \varphi(y)). \end{aligned}$$

X' is a closed subset of X^2, then it is complete. As before, corresponding to each $z = (x, y) \in X'$ there exists $w = (u, v) \in Lz$, hence we can define a single-valued map $L_1 : X' \longrightarrow X'$ by choosing $L_1 z = w$ for all $z \in X'$.
As in the proof of theorem 2.4, inequalities (2.6) shows that

$$\rho((x, y), L_1(x, y)) \leq \psi(x, y) - \psi(L_1(x, y))$$

then by theorem 1.2 there exists $(u^\star, v^\star) = L_1(u^\star, v^\star)$. Using the inequalities (2.6), we claim that $\varphi(u^\star) = \varphi(v^\star)$. Indeed,

$$\begin{aligned} 0 = \delta_1(u^\star, u^\star) &\leq f(u^\star)(\varphi(u^\star) - \varphi(v^\star)) \\ 0 = \delta_2(v^\star, v^\star) &\leq g(v^\star)(\varphi(v^\star) - \varphi(u^\star)) \end{aligned}$$

It follows that $\varphi(v^\star) \leq \varphi(u^\star) \leq \varphi(v^\star)$. And then $\varphi(u^\star) = \varphi(v^\star)$, which proves the theorem. □

Applying theorem 2.4, we obtain the following result.

Theorem 2.8. *Let (X, δ_i) be a complete metric space ($i = 1, 2$), S and T two multivalued maps on X. Let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an upper semicontinuous function from the right and assume that for each $x \in X$ there exists $u \in Tx$ and for each $y \in X$ there exists $v \in Sy$ such that:*

$$\begin{cases} \delta_1(x, u) \leq \max\{h(\varphi(x)), h(\varphi(u))\}(\varphi(x) - \varphi(v)) \\ \delta_2(y, v) \leq \max\{h(\varphi(y)), h(\varphi(v))\}(\varphi(y) - \varphi(u)) \end{cases} \tag{2.9}$$

where $\varphi : X \longrightarrow \mathbb{R}_+$ is lower semicontinuous, then there exists a common fixed point for T and S.

Proof. Put $\varphi_0 = \inf_{x \in X} \varphi(x)$, by the upper semicontinuity from the right of h, there exist $r, \varepsilon \geqslant 0$ such that

$$h(t) \leq r \text{ for all } \quad t \in [\varphi_0, \varphi_0 + \varepsilon]$$

For all x, y in X, we define two mappings f and g as follows :

$$\begin{aligned} f(x) = \max\{h(\varphi(x)), h(\varphi(u))\} \\ g(y) = \max\{h(\varphi(y)), h(\varphi(v))\} \end{aligned}$$

it is clear that f and g maps X into $\mathbb{R}_+$. Inequalities (2.9) show that for all x, y in X,

$$\begin{aligned} \varphi(v) \leq \varphi(x) \\ \varphi(u) \leq \varphi(y) \end{aligned}$$

and thus for any x and y in X with $\max\{\varphi(x), \varphi(y)\} \leq \varphi_0 + \varepsilon$ we have

$$\max\{\varphi(u), \varphi(v)\} \leq \varphi_0 + \varepsilon,$$

clearly we get $f(x) \leq r$ and $g(x) \leq r$, and from this we obtain

$$\begin{cases} \sup\{f(x) : x \in X,\ \varphi(x) \leq \inf_{z\in X} \varphi(z) + \varepsilon\} < \infty \\ \sup\{g(x) : x \in X,\ \varphi(x) \leq \inf_{z\in X} \varphi(z) + \varepsilon\} < \infty \end{cases}$$

According to theorem 2.4, we conclude that T and S have a common fixed point. □

Corollary 2.9. *Under the hypotheses of theorem 2.8, with "upper semicontinuous function" replaced by "nondecreasing function" and inequalities (2.9) replaced by*

$$\begin{cases} \delta_1(x,u) \leq h(\varphi(x))(\varphi(x) - \varphi(v)) \\ \delta_2(y,v) \leq h(\varphi(y))(\varphi(y) - \varphi(u)) \end{cases} \tag{2.10}$$

T and S have a common fixed point.

Proof. For each x in X, define $f : X \longrightarrow \mathbb{R}_+$ by

$$f(x) = h(\varphi(x))$$

choose $X_1 = \{x \in X : \varphi(x) \leq \inf_{z\in X} \varphi(z) + \varepsilon\}$ for some $\varepsilon > 0$. Since h is a nondecreasing function, we have for all x in X_1

$$h(\varphi(x)) \leq h\left(\inf_{z\in X} \varphi(z) + \varepsilon\right) < \infty$$

then

$$\sup\left\{f(x) : x \in X,\ \varphi(x) \leq \inf_{z\in X} \varphi(z) + \varepsilon\right\} < \infty$$

And we conclude by theorem 2.4.

□

The next result can be derived directly from corollary 2.9,

Corollary 2.10. *Let (X, δ_i) be a complete metric space ($i = 1, 2$), S and T two multivalued maps on X. Let $\eta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an upper semicontinuous function. Assume that for each $x \in X$ there exists $u \in Tx$ and for each $y \in X$ there exists $v \in Sy$ such that $\delta_1(x,u) \leq \varphi(x)$ and $\delta_2(y,v) \leq \varphi(y)$ and :*

$$\begin{cases} \delta_1(x,u) \leq \eta(\delta_1(x,u))(\varphi(x) - \varphi(v)) \\ \delta_2(y,v) \leq \eta(\delta_2(y,v))(\varphi(y) - \varphi(u)) \end{cases} \tag{2.11}$$

where $\varphi : X \longrightarrow \mathbb{R}_+$ is lower semicontinuous. Then there exists a common fixed point for T and S.

Proof. Let define a function h from $\mathbb{R}_+$ to $\mathbb{R}_+$ by

$$h(t) = \sup\{\eta(r), 0 \leq r \leq t\}$$

since η is an upper semicontinuous function, h is well defined. It is evident that h is a nondecreasing function and by the assumptions $\delta_1(x,u) \leq \varphi(x)$ and $\delta_2(y,v) \leq \varphi(y)$, we have

$$\eta(\delta_1(x,u)) \leq \eta(\varphi(x))$$
$$\eta(\delta_2(y,v)) \leq \eta(\varphi(y))$$

And we conclude by corollary 2.9.

□

Theorem 2.11. *Let (X,δ) be a complete metric space and S, T two orbitally CS-continuous multivalued mappings on X. Let $f,g : X \longrightarrow \mathbb{R}_+$ be two upper bounded functions.*

Assume that for each $x \in X$ there exists $u \in Tx$ and for each $y \in X$ there exists $v \in Sy$ such that :

$$\delta(u,v) \leq f(x)(\varphi(x) - \varphi(v)) + g(y)(\varphi(y) - \varphi(u)) \tag{2.12}$$

where $\varphi : X \longrightarrow \mathbb{R}_+$ is continuous function. Then T and S have a common fixed point.

Proof. For each $x \in X$ and $y \in X$ there exist $u \in Tx$ and $v \in Sy$ such that inequality (2.12) holds; we can define two single-valued mappings $T_1 : X \longrightarrow X$ and $S_1 : X \longrightarrow X$ by choosing $T_1(x) = u$ and $S_1(y) = v$.
Let x_0 and y_0 be two arbitrary points of X, we Consider the following sequences

$$x_n = T_1^n x_0 \quad \text{and} \quad y_n = S_1^n y_0 \qquad \text{for } n = 1, 2, \ldots$$

it is clear that $x_{n+1} \in Tx_n$ and $y_{n+1} \in Sy_n$ and since f, g are both upper bounded, there exists $\alpha \geqslant 0$ such that for all $x \in X$,

$$f(x) \leq \alpha \quad \text{and} \quad g(x) \leq \alpha$$

From (2.12), we have for $i = 1, 2, \ldots$

$$\delta(x_i, y_i) \leq \alpha(\varphi(x_{i-1}) - \varphi(x_i) + \varphi(y_{i-1}) - \varphi(y_i))$$

By summing the above inequalities from $1, \ldots, n$ we obtain

$$\sum_{i=1}^{n} \delta(x_i, y_i) \leq \alpha(\varphi(x_0) + \varphi(y_0))$$

The same arguments applied to

$$\delta(x_{i+1}, y_i) \leq \alpha(\varphi(x_i) - \varphi(x_{i+1}) + \varphi(y_{i-1}) - \varphi(y_i))$$

give

$$\sum_{i=1}^{n} \delta(x_{i+1}, y_i) \leq \alpha(\varphi(x_1) + \varphi(y_0))$$

and since, $\delta(x_i, x_{i+1}) \leq \delta(x_i, y_i) + \delta(y_i, x_{i+1})$ for each $i = 1, 2, \ldots$, the sum

$$\sum_{i=1}^{\infty} \delta(x_{i+1}, x_i) < \infty$$

Hence $\{x_n\}_n$ it is a Cauchy sequence. We proceed analogously to show that $\{y_n\}_n$ is a Cauchy sequence and since X is a complete metric space, the sequences are convergent. Let

$$\lim_{n\to\infty} x_n = \bar{x}$$
$$\lim_{n\to\infty} y_n = \bar{y}$$

since T and S are orbitally CS-continuous we have $\bar{x} \in T\bar{x}$ and $\bar{y} \in S\bar{y}$. Using (2.12), we get for all $i = 1, 2, \ldots$

$$\begin{aligned} \delta(x_{i+1}, y_{i+1}) &\leq f(x_i)(\varphi(x_i) - \varphi(y_{i+1})) + g(y_i)(\varphi(y_i) - \varphi(x_{i+1})) \\ &\leq \alpha(\varphi(x_i) - \varphi(x_{i+1})) + \alpha(\varphi(y_i) - \varphi(y_{i+1})) \end{aligned}$$

Taking the limit with respect to i yields to

$$\delta(\bar{x}, \bar{y}) \leq \alpha(\varphi(\bar{x}) - \varphi(\bar{x})) + \alpha(\varphi(\bar{y}) - \varphi(\bar{y})) = 0$$

since φ is continuous, then T and S have a common fixed point.

$\square$

The next theorem yields information about existence of minimum in complete product metric space with two distances and it is analogous to ε-variational principle of Ekeland.

Theorem 2.12. *(ε-variational Principle).*

Let X be a complete metric space with two metrics δ_1 and δ_2 and let φ be a proper and lower semicontinuous function of X into $(-\infty,\infty]$. For any $\varepsilon>0$, choose $u,v\in X$ such that

$$\max\{\varphi(u),\varphi(v)\}\leq\inf\{\varphi(x):\,x\in X\}+\varepsilon$$

Then there exist $u^\star,v^\star\in X$ such that

(1) $\varphi(v^\star)=\varphi(u^\star)\leq\dfrac{\varphi(u)+\varphi(v)}{2}$
(2) $\delta_1(u,u^\star)+\delta_2(v,v^\star)\leq 2,$
(3) $\varphi(y)>\varphi(v^\star)-\varepsilon\delta_1(x,u^\star)$ or $\varphi(x)>\varphi(v^\star)-\varepsilon\delta_2(y,v^\star)$ for all $x,y\in X$ and $(x,y)\neq(u^\star,v^\star)$.

Proof. Let $\varepsilon>0$ and choose $u,v\in X$ such that

$$\max\{\varphi(u),\varphi(v)\}\leq\varphi_0+\varepsilon$$

where $\varphi_0=\inf\{\varphi(x):\,x\in X\}$. Putting

$$X'=\left\{(x,y)\in X^2:\,\varphi(x)+\varphi(y)\leq\varphi(u)+\varphi(v)-\varepsilon\left[\delta_1(x,u)+\delta_2(y,v)\right]\right\}$$

it is a nonempty set and by lower semi-continuity of φ, X' is a closed so it is a complete metric space. For each $(x,y)\in X'$, let

$$H(x,y)=\left\{(t,z)\in X^2\;:\;(t,z)\neq(x,y),\left\{\begin{array}{l}\varphi(z)\leq\varphi(x)-\varepsilon\delta_1(x,t)\\ \varphi(t)\leq\varphi(y)-\varepsilon\delta_2(y,z)\end{array}\right.\right\}$$

and define a multivalued mapping L from X' to $2^{X'}$ by

$$L(x,y)=\left\{\begin{array}{l}\{(x,y)\}\text{ if }H(x,y)=\emptyset\\ H(x,y)\text{ if }H(x,y)\neq\emptyset\end{array}\right.$$

Indeed, if $H(x,y)=\emptyset$, $L(x,y)=\{(x,y)\}\in 2^{X'}$ and if $L(x,y)=H(x,y)$, we have for all $(t,z)\in L(x,y)$

$$\begin{array}{rcl}\varepsilon\delta_1(t,u)+\varepsilon\delta_2(z,v)&\leq&\varepsilon\delta_1(t,x)+\varepsilon\delta_1(x,u)+\varepsilon\delta_2(z,y)+\varepsilon\delta_2(y,v)\\ &\leq&\varphi(x)-\varphi(z)+\varphi(y)-\varphi(t)+\varepsilon\delta_1(x,u)+\varepsilon\delta_2(y,v)\\ &\leq&\varphi(x)-\varphi(z)+\varphi(y)-\varphi(t)+\varphi(u)-\varphi(x)+\varphi(v)-\varphi(y)\\ &\leq&\varphi(u)+\varphi(v)-\varphi(t)-\varphi(z)\end{array}$$

and hence $(t,z)\in X'$.
Note that for all $(x,y)\in X'$ and $(t,z)\in L(x,y)$

$$\left\{\begin{array}{l}\varphi(z)\leq\varphi(x)-\varepsilon\delta_1(x,t)\\ \varphi(t)\leq\varphi(y)-\varepsilon\delta_2(y,z)\end{array}\right.\Leftrightarrow\left\{\begin{array}{l}\delta_1(x,t)\leq\dfrac{1}{\varepsilon}\varphi(x)-\dfrac{1}{\varepsilon}\varphi(z)\\ \delta_2(y,z)\leq\dfrac{1}{\varepsilon}\varphi(y)-\dfrac{1}{\varepsilon}\varphi(t)\end{array}\right.$$

So from theorem 2.7 there exists $(u^\star,v^\star)\in X'$ such that

$$(u^\star,v^\star)\in L(u^\star,v^\star)\qquad\text{and}\qquad\varphi(u^\star)=\varphi(v^\star)$$

. Therefore $H(u^\star, v^\star) = \emptyset$, then

$$\varphi(y) > \varphi(u^\star) - \varepsilon\delta_1(x, u^\star) \quad \text{or} \quad \varphi(x) > \varphi(v^\star) - \varepsilon\delta_2(y, v^\star)$$

for all $x, y \in X$ and $(x, y) \neq (u^\star, v^\star)$.
Since, $(u^\star, v^\star) \in X'$ we obtain

$$2\varphi(u^\star) \leq \varphi(u) + \varphi(v) - \varepsilon[\delta_1(u^\star, u) + \delta_2(v^\star, v)] \leq \varphi(u) + \varphi(v)$$

hence

$$\varphi(u^\star) \leq \frac{\varphi(u) + \varphi(v)}{2}$$

Further we have

$$\begin{aligned} \varepsilon\delta_1(u^\star, u) + \varepsilon\delta_2(v^\star, v) &\leq \varphi(u) - \varphi(u^\star) + \varphi(v) - \varphi(v^\star) \\ &\leq \varphi(u) - \varphi_0 + \varphi(v) - \varphi_0 \\ &\leq 2\varepsilon \end{aligned}$$

and hence $\delta_1(u^\star, u) + \delta_2(v^\star, v) \leq 2$.

□

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