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AN APPROACH TO STABILIZATION FOR A CLASS OF CONTROL SYSTEMS WITH MIXED TIME-VARYING DELAYS

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Abstract. This paper studies the problem of stabilization for a class of control systems with discrete and distributed time-varying delays. The feedback control functions are restricted in such a way that the ratio between total cumulated output errors and total cumulated perturbations on whole time interval is upper bounded. The main result on solving the problem is derived from using a new approach, which is based on asymptotic behavior of evolution operators generated by matrix functions of homogeneous linear parts.

1. Introduction

Analysis of stability and stabilization of dynamical control systems are important both in theory and practice. The studies of this problem have been realized not only for the models described by ordinary differential equations, but also for the models described by delay differential equations. The presence of delayed dependencies gives us ability to describe the dynamic relationships between several variables in long term process. Each system may be affected by deterministic or random perturbations from the environmental. Consequence of these perturbations is uncertainty. The appearance of delays and uncertainties may be the source of instability and serious deterioration in the performance of the closed-loop systems. To deal with the trend away from the equilibrium position of systems, it is common to use the appropriate impact through the control functions. Typically, control functions are not arbitrary. Each of them must satisfy some certain restrictions. The restriction in this paper is mentioned as following: the ratio between total cumulated output errors and total cumulated perturbations on whole time interval is upper bounded by some certain positive constant. The magnitude of this ratio reflects the quality of the system. For linear systems, the observed value may be considered as the output error. The restriction of this type is main condition in the concept "Robust H_∞ control problem". This restriction should be emphasized, because for almost of dynamical systems there are the differences between theoretical behaviors and practical behaviors, between real states and observed states.

During the past decades, H_∞ control problem has been interested by many

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researchers (see [1], [6], [7]). As for some references, using the linear matrix inequalities (LMIs) seems to be popular approach method (see [6], [8]). On the one hand, this approach is very modern method, but on the other hand, this approach requires many too complicated assumptions on corresponding parameters and unknown variables ([1], [5]). Moreover, almost of these assumptions are given by the certain LMIs, and these LMIs can be solved just by few special softwares, for example by Matlab ([5], [8], [9]). Therefore, it is necessary to find some more simple conditions. In this paper, all assumptions are explicit. The main assumption is presented on the basic of the asymptotic behavior of evolution operator, generated by the homogeneous linear part of corresponding differential equations. These conditions involve the solving solutions of Riccati matrix equations ([2], [3]). Thus, instead of the implicit conditions given in terms of LMIs, the explicit conditions on coefficient matrices will be required.

Relating to the object under consideration, it is the need to recall, that there was much knowledge about the dynamical systems, described by ordinary differential equations:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t). \tag{1.1}$$

Ordinary differential equation (1.1) shows certain relationships between some various objects happening at the same time t. However, almost all modern systems operate with great speed. It is difficult to collect the instantaneous data about the state and the noise for the formulation of control strategies. Thus, in practices, the description by equation (1.1) may have too big approximation errors. In many situations, the states in the past of each system have often left their mark in the present: Inertia of the materials, heredity of organisms, drug resistance in the treatment of diseases, credibility with long term partners in the business,... are factors that should not let pass when modeling the corresponding activities. In other words, the rate of change of system at time t (which is characterized by $\dot{x}(t)$) depends not only on the state $x(t)$ at this time, but also on some or all previous states. But, the events happened too long in the past may have some insignificant influences on the present. So, the delays are often assumed not greater than a certain positive number, for example h. Thus, instead of equation (1.1), the object under the consideration in this paper is a class of uncertain control systems, described by the following differential equations with discrete and distributed time-varying delays:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + E(t)x(t-h(t)) + F(t)\int_{t-k(t)}^{t} x(s)ds + G(t)w(t). \tag{1.2}$$

$$x(t) = \phi(t), \forall t \in [-h, 0]. \tag{1.3}$$

To explain the elements in this relation, some notations have been mentioned as following: $\mathbb{R}^+$ is the set of all non-negative real numbers. For each $t \in \mathbb{R}^+$: $x(t)$ is the vector of the real n - dimensional linear space $\mathbb{R}^n$; $x(t)$ is state of considered system and $\dot{x}(t)$ is rate of the state change of this system at present time t. By the similar way: $u(t)$ is control function, $u(t) \in \mathbb{R}^m$; $w(t)(w(t) \in \mathbb{R}^l)$ is uncertainty; $z(t)(z(t) \in \mathbb{R}^r)$ is observation; $h(t)$ and $k(t)$ are delay functions satisfying the assumption $0 \leq h(t), k(t) \leq h, \forall t \geq 0$. Denoting $\mathcal{M}(m \times n)$ the set of all matrices having the size $m \times n$, we note that $A(t), E(t), F(t) \in \mathcal{M}(n \times n)$; $B(t) \in \mathcal{M}(m \times n)$; $G(t) \in \mathcal{M}(l \times n)$. The uncertainty $w(t)$ is said to be admissible if $w(.) \in L_2$, where $L_2 := L_2([0, \infty), \mathbb{R}^n)$ denotes the Banach space of all functions, mapping from $[0, +\infty)$ into $\mathbb{R}^n$ and being square integrable on $[0, +\infty)$. The norm of $w \in L_2$

is defined by $\|w\||_{L_2} := \Big(\int\limits_0^{+\infty} \|w(t)\|^2 dt\Big)^{\frac{1}{2}}$. For each u and w, the solution of (1.2), satisfying initial condition $x(t) = \phi(t), t \in [-h, 0]$ is denoted by $x(0, \phi, t)$ and in the sequel, shortly by $x(\phi, t)$ or $x(t)$. To move into infinite dimensional spaces, for the state function $x(t)$ we need the following notation x_t, which is defined by $x_t(s) := x(t+s), s \in [-h, 0]$. It means that x_t is the curve of $x = x(t)$ on interval $[t-h; t]$. By other word, x_t is an element of C, where $C := C([-h, 0], \mathbb{R}^n)(h > 0)$ is the Banach space of all continuous functions on $[-h, 0]$ with values belong to $\mathbb{R}^n$. The norm of $\phi \in C$ is defined by $\|\phi\||_C := \sup\limits_{t\in[-h,0]} \|\phi(t)\|$. However, for the aim of simplicity, through this paper the same symbol $\|.\|$ is used to denote the norm and the same symbol I is used to denote the identity operator for all different spaces.

Relating to the instruments of this study, there is a notice that: (1.1) is an ordinary differential equation, which may be effectively studied in the same finite dimensional spaces, while (1.2) is a functional differential equation, which may be effectively studied just in the infinite dimensional spaces (see [4], Krasovskii remark). Therefore, in the later stability study, instead of the Lyapunov functions operating in finite dimensional spaces the Lyapunov-Krasovskii functionals operating in infinite dimensional spaces will be used. These functionals are strong tools, by which the more general results may be obtained.

As a result, in this paper, the H_∞ control problem for a class of non-autonomous control systems is developed. The significant feature of this paper lies in three aspects. (i) The system is subjected the mixed discrete and distributed time-varying delays; (ii) More free-weighting matrices $C(t)$, $D(t)$ in the formula of observation variant $z(t)$ are adopted; (iii) Instead of using LMIs technique, the properties of evolution operators are used. As for the contents, this paper is constructed as follows: Beside the introduction part, the Section 2 presents some basic definitions and some technical propositions, which are needed for the proof of main result. The main result is proved in Section 3. The paper is ended by a numerical example and the conclusion.

2. Preliminaries

Usually, in addition to the control units, each control system also contains observer parts. In this paper, the observation variable is constructed on the basic of information about the states and about the control actions. For the convenience, it is given in the linear form:

$$z(t) = C(t)x(t) + D(t)u(t). \tag{2.1}$$

The feedback control function is assumed having linear form and being constructed just by the information of present state:

$$u(t) = K(t)x(t). \tag{2.2}$$

In these relations, it is worth to note that: $C(t) \in \mathcal{M}(r \times n)$; $K(t) \in \mathcal{M}(m \times n)$ and $D(t) \in \mathcal{M}(r \times m)$.

In some papers (see, for example [1], [6]), for the sake of technical simplifications, the existence of matrix functions $C(t)$, $D(t)$ satisfying the following relation often has been assumed

$$D^T(t)D(t) = I \quad \text{and} \quad C^T(t)D(t) = 0, \forall t \geq 0. \tag{2.3}$$

In [1], it was shown that in some particular cases this assuming is correct. In general, the matrix functions satisfying relation (2.3) may be not exist. In this paper, for the observation variable z matrix functions $C(t), D(t)$ would be given in the general form, where relation (2.3) may be not fulfilled. To continue, we recall a definition, which is presented in [1]:

Definition 1. *System* (1.2), (1.3), (2.1), (2.2) *is said to be robustly L_2-stabilizable if there exists a matrix-valued function $K(t)$, such that the solution of the closed-loop system*

$$\dot{x}(t) = [A(t)+B(t)K(t)]x(t)+E(t)x(t-h(t))+F(t)\int_{t-k(t)}^{t} x(s)ds+G(t)w(t), \quad (2.4)$$

$$x(t) = \phi(t), \forall t \in [-h, 0], \phi \in C$$

belongs to $L_2([0,\infty),\mathbb{R}^n)$ for any admissible uncertainties $w \in L_2([0,\infty),\mathbb{R}^l)$.

We recall also the following definition, which is given in [7]:

Definition 2. *The robust H_∞ control problem for system* (1.2), (1.3), (2.1), (2.2) *is said to have a solution if for any given number $\gamma > 0$ there exist a matrix-valued function $K(t)$ and a number $C_0 > 0$ such that for any initial function $\phi \in C$, the corresponding solution of the closed-loop system* (2.4) *belongs to* $L_2([0,\infty),\mathbb{R}^n)$, *and the following inequality is satisfied:*

$$\sup_{\phi\in C} \frac{\int\limits_0^{+\infty} \|z(t)\|^2 dt}{C_0 + \int\limits_0^{+\infty} \|w(t)\|^2 dt} \le \gamma. \quad (2.5)$$

The main result will be established on the basic of the following conditions, which are named by (A_1), (A_2) and (A_3):

(A_1) The delay functions $h(t), k(t)$ are differentiable on $\mathbb{R}^+$ and there are positive numbers μ and λ such that

$$\dot{h}(t) \le \mu \le 1, \forall t \ge 0;\ \dot{k}(t) \le \lambda \le 1, \forall t \ge 0. \quad (2.6)$$

(A_2) All coefficient matrix functions are bounded on $\mathbb{R}^+$, for example:

$$\bar{a} := \sup_{t\ge0} \|A(t)\| < +\infty;\ \bar{b} := \sup_{t\ge0} \|B(t)\| < +\infty;$$

$$\bar{e} := \sup_{t\ge0} \|E(t)\| < +\infty;\ \bar{f} := \sup_{t\ge0} \|F(t)\| < +\infty;\ \bar{g} := \sup_{t\ge0} \|G(t)\| < +\infty. \quad (2.7)$$

As in the above, we assume that all uncertainties $w(t)$ are square integrable on $[0,+\infty)$ i.e. $w \in L_2([0,+\infty),\mathbb{R}^l)$, or

$$\|w\| := \Big(\int\limits_0^{\infty} \|w(t)\|^2 dt\Big)^{\frac{1}{2}} < +\infty. \quad (2.8)$$

Next, $U(t,s)$ denotes the evolution operator of the homogeneous equation

$$\dot{x}(t) = A(t)x(t),\ (t \in R^+).$$

The property of this operator plays central role in our checking the solutions of the following time-varying Riccati matrix equation

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t) = 0. \tag{2.9}$$

In this equation: $A(t)$ is a given $n \times n$ matrix function, bounded and continuous on R^+, $A^T(t)$ is transposition matrix of $A(t)$, $Q(t)$ is some given symmetric positive definite $n \times n$ matrix function and $P(t)$ is the unknown symmetric positive definite $n \times n$ matrix function.

(A_3) For evolution operator $U(t,s)$, generated by $A(t)$ there are positive constants N and δ, such that

$$\|U(t,s)\| \leq Ne^{-\delta(t-s)}, \forall t \geq s \geq 0. \tag{2.10}$$

The following proposition is proved in [2].

Proposition 2.1. *Suppose that the bounded on $\mathbb{R}^+$ $n \times n$ matrix function $A(t)$ generates an evolution operator $U(t,s)$, for which the relation* (2.10) *is satisfied. Then for any bounded on $\mathbb{R}^+$ symmetric positive definite $n \times n$ matrix function $Q(t)$ the matrix equation* (2.9) *has a solution $P(t)$, which is a symmetric positive definite matrix function bounded on $\mathbb{R}^+$. Moreover, $P(t)$ is found as follow*

$$P(t) = \int_t^\infty U^T(\tau,t)Q(\tau)U(\tau,t)d\tau. \tag{2.11}$$

We will need also the following proposition, its proof is simple.

Proposition 2.2. *For any bounded on $\mathbb{R}^+$ symmetric $n \times n$ matrix function $Q(t)$, there exists a positive number $\epsilon > 0$ being big enough, such that the matrix function $Q(t) + \epsilon I$ is symmetric positive definite.*

3. Main result

Let us consider control system (1.2), (1.3), (2.1), (2.2) with the acceptable uncertainty $w(t)$. Our interest will deal with the assumptions, under which the H_∞ control problem has a solution. These assumptions would be not presented through the LMIs, but through the appropriate conditions on the coefficient matrix functions and on the delay functions.

Theorem 3.1. *Suppose that conditions $(A_1), (A_2), (A_3)$ are fulfilled. Then, there exists a symmetric positive definite matrix function $Q(t)$ such that the feedback stabilizing control function for system* (1.2), (1.3), (2.1), (2.2) *may be chosen by $u(t) = -B^T(t)P(t)x(t)$, where $P(t)$ is a symmetric positive definite solution of Riccati equation* (2.9), *corresponding with $Q(t)$. Moreover, in this case, the robust H_∞ control problem has a solution.*

Proof. To prove the theorem, firstly we must show that for any initial function $\phi \in C$, the corresponding solution $x(\phi,t)$ of the closed-loop system belongs to L_2. Secondly, for each given positive number γ we can find a positive number C_0 such that the inequality (2.5) in Definition 2. is true. To continue the investigation, there is the need to use the auxiliary functionals. These auxiliary functionals are constructed through the solution $P(t)$ of the Riccati equation (2.9), corresponding a symmetric positive definite matrix function $Q(t)$, which will be chosen later. The auxiliary functionals are constructed as in following:

$$V(t, x_t) = V_1(t, x_t) + V_2(t, x_t) + V_3(t, x_t) + V_4(t, x_t),$$

where:

$$V_1(t, x_t) = x^T(t)P(t)x(t).$$

$$V_2(t, x_t) = \int\limits_{t-h(t)}^{t} \|x(s)\|^2 ds.$$

$$V_3(t, x_t) = \int\limits_{t-k(t)}^{t} \|x(s)\|^2 ds.$$

$$V_4(t, x_t) = \int\limits_{-h}^{0} ds \int\limits_{t-k(t)}^{t} \|x(\tau)\|^2 d\tau.$$

It is clear that, for all $t \geq 0$ every function $V_i(t, x_t), (i = 1, 2, 3, 4)$ is continuous, nonnegative and $V_i(t, 0) = 0$. Taking the derivative among equation (2.4) with $K(t) = -B^T(t)P(t)$, we have:

$$\begin{aligned}
\dot{V}_1(t, x_t) &= x^T(t)\dot{P}(t)x(t) + 2x^T(t)P(t)\dot{x}(t) \\
&= x^T(t)\Big[\dot{P}(t) + P(t)A(t) + A^T(t)P(t) - 2P(t)B(t)B^T(t)P(t)\Big]x(t) \\
&+ 2x^T(t)P(t)E(t)x(t-h(t)) + 2x^T(t)P(t)F(t) \int\limits_{t-k(t)}^{t} x(s)ds + 2x^T(t)P(t)G(t)w(t) \\
\dot{V}_2(t, x_t) &= \|x(t)\|^2 - (1 - \dot{h}(t))\|x(t-h(t))\|^2 \\
&\leq \|x(t)\|^2 - (1 - \mu)\|x(t-h(t))\|^2 \\
\dot{V}_3(t, x_t) &= \|x(t)\|^2 - (1 - \dot{k}(t))\|x(t-h(t))\|^2 \\
&\leq \|x(t)\|^2 - (1 - \lambda)\|x(t-h(t))\|^2 \\
\dot{V}_4(t, x_t) &= \int_{-h}^{0} \Big(\|x(t)\|^2 - (1 - \dot{k}(t))\|x(t+s)\|^2\Big)ds \\
&= h\|x(t)\|^2 - (1 - \dot{k}(t)) \int\limits_{-h}^{0} \|x(t+s)\|^2 ds \\
&\leq h\|x(t)\|^2 - (1 - \lambda) \int\limits_{-h}^{0} \|x(t+s)\|^2 ds.
\end{aligned}$$

Combining all these inequalities, we have

$$\dot V(t,x_t) \le x^T(t)\Big(\dot P(t) + P(t)A(t) + A^T(t)P(t) - 2P(t)B(t)B^T(t)P(t) + 2I + hI\Big)x(t)$$
$$+ 2x^T(t)P(t)E(t)x(t-h(t)) + 2x^T(t)P(t)F(t)\int\limits_{t-k(t)}^{t} x(s)ds + 2x^T(t)P(t)G(t)w(t)$$
$$- (1-\mu)\|x(t-h(t))\|^2 - (1-\lambda)\|x(t-k(t))\|^2 - (1-\lambda)\int\limits_{-h}^{0}\|x(t+s)\|^2 ds$$

It is easy to see that for all $t \ge 0$ the following inequalities are true

$$2x^T(t)P(t)E(t)x(t-h(t)) - (1-\mu)\|x(t-h(t))\|^2$$
$$\le \frac{1}{1-\mu}x^T(t)P(t)E(t)E^T(t)P(t)x(t).$$

$$2x^T(t)P(t)F(t)\int\limits_{t-k(t)}^{t} x(s)ds - (1-\lambda)\int\limits_{-h}^{0}\|x(t+s)\|^2 ds$$
$$= 2x^T(t)P(t)F(t)\int\limits_{-k(t)}^{0} x(t+s)ds - (1-\lambda)\int\limits_{-h}^{0}\|x(t+s)\|^2 ds$$
$$\le 2x^T(t)P(t)F(t)\int\limits_{-k(t)}^{0} x(t+s)ds - (1-\lambda)\int\limits_{-k(t)}^{0}\|x(t+s)\|^2 ds$$
$$\le 2x^T(t)P(t)F(t)\int\limits_{-k(t)}^{0} x(t+s)ds - \frac{1-\lambda}{h}\Big(\int\limits_{-k(t)}^{0} x(t+s)ds\Big)^2$$
$$\le \frac{h}{1-\lambda}x^T(t)P(t)F(t)F^T(t)P(t)x(t).$$

Therefore,

$$\dot V(t,x_t) \le x^T(t)\Big(\dot P(t) + P(t)A(t) + A^T(t)P(t) - 2P(t)B(t)B^T(t)P(t) + (2+h+\epsilon)I$$
$$+ (C^T(t) - P(t)B(t)D^T(t))(C(t) - D(t)B^T(t)P(t)) + \frac{1}{1-\mu}P(t)E(t)E^T(t)P(t)$$
$$+ \frac{h}{1-\lambda}P(t)F(t)F^T(t)P(t)\Big)x(t) + 2x^T(t)P(t)G(t)w(t) - (1-\lambda)\|x(t-k(t))\|^2$$
$$- x^T(t)\Big(\epsilon I + (C^T(t) - P(t)B(t)D^T(t))(C(t) - D(t)B^T(t)P(t))\Big)x(t)$$
$$= \le x^T(t)\Big(\dot P(t) + P(t)A(t) + A^T(t)P(t) + Q(t)\Big)x(t) + 2x^T(t)P(t)G(t)w(t) - (1-\lambda)\|x(t-k(t))\|^2$$
$$- x^T(t)\Big(\epsilon I + (C^*(t) - P(t)B(t)D^*(t))(C(t) - D(t)B^*(t)P(t))\Big)x(t), \tag{3.1}$$

where $Q(t) = Q_1(t) + \epsilon I$, with

$$
\begin{aligned}
Q_1(t) &= -2P(t)B(t)B^T(t)P(t) + (2+h)I + (C^*(t) - P(t)B(tD^*(t))(C(t) - D(t)B^*(t)P(t)) \\
&\quad + \frac{1}{1-\mu}P(t)E(t)E^T(t)P(t) + \frac{h}{1-\lambda}P(t)F(t)F^T(t)P(t).
\end{aligned}
\tag{3.2}
$$

Since

$$-(1-\lambda)\|x(t-k(t))\|^2 - x^T(t)(C^T(t) - P(t)B(t)D^T(t))(C(t) - D(t)B^T(t)P(t))x(t) \leq 0,$$

then from (3.1) and (3.2) we have

$$
\begin{aligned}
\dot{V}(t, x_t) \leq\ & x^T(t)\Big(\dot{P}(t) + P(t)A(t) + A^T(t)P(t) + Q(t)\Big)x(t) \\
&+ 2x^T(t)P(t)G(t)w(t) - \epsilon\|x(t)\|^2.
\end{aligned}
\tag{3.3}
$$

It is easy to check that $Q_1(t)$ is symmetric and bounded on $\mathbb{R}^+$. Thus, by Proposition 2. we can choose $\epsilon > 0$ being big enough such that, for all $t \geq 0$ matrix function $Q(t) = Q_1(t) + \epsilon I$ is symmetric positive definite. In consequence, by Proposition 1., the corresponding Riccati equation

$$\dot{P}(t) + P(t)A(t) + A^T(t)P(t) + Q(t) = 0$$

has a symmetric positive definite solution $P(t)$. It is not difficult to check that, $Q(t)$ is bounded on $\mathbb{R}^+$. By Proposition 1., $P(t)$ is also bounded on $\mathbb{R}^+$. Denote $\bar{q} := \sup\limits_{t\geq 0}\|Q(t)\|, \bar{p} := \sup\limits_{t \geq 0}\|P(t)\|$. Therefore, estimating $\dot{V}(t, x_t)$, we have

$$\dot{V}(t, x_t) \leq -\epsilon\|x(t)\|^2 + 2x^T(t)P(t)G(t)w(t).$$

For any $s > 0$, taking integral from 0 to s both sides of last inequality, we have

$$\int_0^s \dot{V}(t, x_t)dt \leq -\epsilon\int_0^s \|x(t)\|^2 dt + 2\int_0^s x^T(t)P(t)G^T(t)w(t)dt,$$

or

$$
\begin{aligned}
V(s, x_s) - V(0, x_0) &\leq -\epsilon\int_0^s \|x(t)\|^2 dt + 2\bar{p}\bar{g}\Big(\int_0^s \|w(t)\|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^s \|x(t)\|^2 dt\Big)^{\frac{1}{2}} \\
&= -\epsilon\int_0^s \|x(t)\|^2 dt + 2\bar{p}\bar{g}\|w\|\Big(\int_0^s \|x(t)\|^2 dt\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Since $V(s, x_s) \geq 0, \forall t \geq 0$ and $\|w\| := \Big(\int\limits_0^s \|w(t)\|^2 dt\Big)^{\frac{1}{2}} < +\infty$, then we have

$$-V(0, x_0) \leq -\epsilon\int_0^s \|x(t)\|^2 dt + 2\bar{p}\bar{g}\|w\|\Big(\int_0^s \|x(t)\|^2 dt\Big)^{\frac{1}{2}}.$$

Last inequality implies that

$$\int_0^s \|x(t)\|^2 dt - 2\frac{\bar{p}\bar{g}\|w\|}{\epsilon}\Big(\int_0^s \|x(t)\|^2 dt\Big)^{\frac{1}{2}} - \frac{1}{\epsilon}V(0, x_0) \leq 0.$$

That is equivalent to

$$\Big(\int\limits_0^s \|x(t)\|^2 dt\Big)^{\frac{1}{2}} \leq \frac{\bar{p}\bar{g}\|w\|}{\epsilon} + \sqrt{\frac{\bar{p}^2\bar{g}^2\|w\|^2}{\epsilon^2} + \frac{1}{\epsilon}V(0,x_0)} < +\infty.$$

It means that $x \in Ł_2([0;+\infty),\mathbb{R}^n)$. The system is robustly L_2-stable.
To continue the proof, we note that (3.1) and (3.3) lead to

$$\dot{V}(t,x_t) \leq 2x^T(t)P(t)G(t)w(t) - (1-\lambda)\|x(t-k(t))\|^2 - \epsilon\|x(t)\|^2. \tag{3.4}$$

Therefore, we can made the estimation

$$\int\limits_0^s \Big(\|z(t)\|^2 - \gamma\|w(t)\|^2\Big)dt = \int\limits_0^s \Big(\|z(t)\|^2 - \gamma\|w(t)\|^2 + \dot{V}(t,x_t)\Big)dt - \int\limits_0^s \dot{V}(t,x_t)dt$$

$$\leq \int\limits_0^s \Big(- \epsilon\|x(t)\| - \gamma\|w(t)\|^2 + 2x^T(t)P(t)G(t)w(t) - (1-\lambda)\|x(t-k(t))\|^2\Big)dt$$

$$+ V(0,x_0) - V(t,x_t).$$

Since $V(t,x_t) \geq 0$ and $(1-\lambda)\|x(t-k(t))\|^2 \geq 0$, then we get

$$\int\limits_0^s \Big(\|z(t)\|^2 - \gamma\|w(t)\|^2\Big)dt$$

$$\leq \int\limits_0^s \Big(- \gamma\|w(t)\|^2 - \epsilon\|x(t)\|^2 + 2w^T(t)G^T(t)P(t)x(t)\Big)dt + V(0,x_0).$$

Using inequality

$$-\gamma\|w(t)\|^2 + 2w^T(t)G^T(t)P(t)x(t) \leq \frac{1}{\gamma}x^T(t)P(t)G(t)G^T(t)P(t)x(t)$$

and

$$2x^T(t)P(t)G(t)G^T(t)P(t)x(t) \leq 2\bar{p}^2\bar{g}^2\|x(t)\|^2,$$

we have

$$\int\limits_0^s \Big(\|z(t)\|^2 - \gamma\|w(t)\|^2\Big)dt \leq \int\limits_0^s \Big(- \epsilon\|x(t)\|^2 + \frac{\bar{p}^2\bar{g}^2}{\gamma}\|x(t)\|^2\Big)dt + V(0,x_0).$$

As said above, ϵ is selected large enough such that matrix $Q_1(t) + \epsilon I$ is positive definite. Extra choosing $\epsilon > \dfrac{\bar{p}^2\bar{g}^2}{\gamma}$, from the last inequality we have

$$\int\limits_0^s \Big(\|z(t)\|^2 - \gamma\|w(t)\|^2\Big)dt \leq V(0,x_0) \leq (\bar{p} + 2h + h^2)\|\phi\|^2.$$

Denoting $C_0 := \dfrac{1}{\gamma}(\bar{p} + 2h + h^2)$ and letting $s \to +\infty$, we get

$$\int\limits_0^{+\infty} \Big(\|z(t)\|^2 - \gamma\|w(t)\|^2\Big)dt \leq \gamma C_0\|\phi\|^2.$$

That is equivalent to

$$\frac{\int\limits_0^{+\infty} \|z(t)\|^2 dt}{C_0\|\phi\|^2 + \int\limits_0^{+\infty} \|w(t)\|^2 dt} \leq \gamma.$$

Thus, the proof of theorem is completed. $\square$

Example. Consider system (1.2) where:

$$A(t) = \begin{pmatrix} -2 & 0 \\ e^{-3t} & -4 \end{pmatrix}; B(t) = \begin{pmatrix} 2e^{-t} \\ 1 \end{pmatrix}; E(t) = \begin{pmatrix} 1 & 0 \\ \cos t & -1 \end{pmatrix}; F(t) = \begin{pmatrix} 2 & \sin t \\ 0 & 1 \end{pmatrix};$$

$$G(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}; C(t) = \begin{pmatrix} 2 & 1 \\ 0 & 2e^{-t} \end{pmatrix}; D(t) = \begin{pmatrix} \cos t \\ 1 \end{pmatrix}; h(t) = \frac{1}{2}\sin^2 t; k(t) = \frac{1}{3}\cos^2 t.$$

It is not difficult to check that:

$$\bar{a} \leq 5; \quad \bar{b} \leq 3; \quad \bar{e} \leq 2; \quad \bar{f} \leq 3; \quad \bar{g} \leq 2; \quad \dot{h}(t) \leq \frac{1}{2}; \quad \dot{k}(t) \leq \frac{1}{3}; \quad h = \frac{1}{2};$$

$$\|U(t,s)\| \leq \frac{9}{2}e^{-6(t-s)}.$$

Thus, all assumptions of the main theorem are satisfied. The robust H_∞ control problem for this system has a solution. There is a notice that in this case, relation (2.3) is not satisfied.

4. Conclusion

In this paper, a new sufficient condition on stabilization of non-autonomous control systems with discrete and distributed time-varying delays is established. The presence of these delays allows us access to the real models with dynamic dependencies on the past as well as with diversity causal relationships. Using the auxiliary functionals operating in infinite dimensional spaces as a powerful tool, this research has received an extensive result on qualitative study for a class of functional differential equations. In addition, since the observation matrices are given freely, then the collected data is more adequate, and in consequence, the process of formulating control strategies is more advantage. The restriction on the upper boundedness of the ratio of the total cumulated errors and the total cumulated uncertainties is an actual condition. Ignoring it, the formulated model would be unrealistic. Finally, in the paper, the main assumptions are given directly on the coefficient matrices without the need of using linear matrix inequalities. It allows avoiding plenty of purely technical complicated calculations.

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