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WALKER OSSERMAN METRIC OF SIGNATURE $(3,3,)$

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Abstract. A Walker m-manifold is a pseudo-Riemannian manifold, which admits a field of parallel null r-planes, with $r \leq \frac{m}{2}$. The Riemann extension is an important method to produce Walker metric on the cotangent bundle T^*M of any affine manifold (M, ∇). In this paper, we investigate the torsion-free affine manifold (M, ∇) and their Riemann extension $(T^*M, \bar{g})$ as concerns heredity of the Osserman condition.

1. Introduction

The pseudo-Riemannian geometry is the study of the Levi-Civita connection, which is the unique torsion-free connection compatible the metric structure. The theory of affine connections is a classical topic in differential geometry, it was initially developped to solve pure geometrical problems. It provides an extremely important tool to study geometrical structures on manifolds and, as such, has been applied with great sources in many different setting. For affine connections, a survey of the development of the theory can be found in [19] and references therein. In [13], García-Rio *et al.* introduced the notion of the affine Osserman connections. Affine Osserman connections are well-understood in dimension two. For instance, in [6] and [13], the authors proved in a different way that an affine connection is Osserman if and only if its Ricci tensor is skew-symmetric. The situation is however more involved in higher dimensions where the skew-symmetry of the Ricci tensor is a necessary (but not a sufficient) condition for an affine connection to be Osserman. The concept of an affine Osserman connection has become a very active research subject. (See [7, 8, 9] for more details).

In this paper, we associate a pseudo-Riemannian structure of neutral signature to certain affine connections and use this correspondence to study both geometries. We examine affine Osserman connections, Riemann extensions and Walker structures. Our paper is organized as follows. Section 1 introduces this topic. The section 2 contains some definitions and basic results we shall need. In section 3, we study the Osserman condition on a family of affine connection (cf. Proposition 3.3). Finally in section 4, we construct an example of pseudo- Riemannian Walker Osserman metric of signature (3, 3), using the Riemann extensions. The Riemann extension provide

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a link between affine and pseudo-Riemannian geometries. It play an important role in various questions involving the spectral geometry of the curvature operator. (See for example [1, 2, 3, 7, 13] for more details).

2. Preliminaries

2.1. **Affine manifolds.** Let M be an m-dimensional smooth manifold and ∇ be an affine connection on M. Let us consider a system of coordinates $(u_1, \cdots, u_m)$ in a neighborhood $\mathcal{U}$ of a point p in M. In $\mathcal{U}$, the connection is given by

$$\nabla_{\partial_i}\partial_j = f_{ij}^k \partial_k, \tag{2.1}$$

where $\{\partial_i = \frac{\partial}{\partial_{u_i}}\}_{1\leq i\leq m}$ is a basis of the tangent space T_pM and the functions $f_{ij}^k(i, j, k = 1, \cdots, m)$ are called the coefficients of the affine connection. The pair (M, ∇) shall be called *affine manifold*.

We define a few tensor fields associated to a given affine connection ∇. The *torsion tensor field* T^∇, which is of type $(1, 2)$, is defined by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for any vector fields X and Y on M. The components of the torsion tensor T^∇ in local coordinates are

$$T_{ij}^k = f_{ij}^k - f_{ji}^k.$$

If the torsion tensor of a given affine connection ∇ vanishes, we say that ∇ is torsion-free.

The *curvature tensor field* $\mathcal{R}^\nabla$, which is of type $(1, 3)$, is defined by

$$\mathcal{R}^\nabla(X, Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for any vector fields X, Y and Z on M. The components in local coordinates are

$$\mathcal{R}^\nabla(\partial_k, \partial_l)\partial_j = \sum_i R_{jkl}^i \partial_i.$$

We shall assume that ∇ is torsion-free. If $\mathcal{R}^\nabla = 0$ on M, we say that ∇ is *flat affine connection*. It is known that ∇ is flat if and only if around a point there exists a local coordinates system such that $f_{ij}^k = 0$ for all i, j and k.

We define the *Ricci tensor* Ric^∇, of type $(0, 2)$ by

$$Ric^\nabla(Y, Z) = \mathrm{trace}\{X \mapsto \mathcal{R}^\nabla(X, Y)Z\}.$$

The components in local coordinates are given by

$$Ric^\nabla(\partial_j, \partial_k) = \sum_i R_{kij}^i.$$

It is known in Riemannian geometry that the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is, $Ric(Y, Z) = Ric(Z, Y)$. But this property is not true for an arbitrary affine connection which is torsion-free. In fact, the property is closely related to the concept of parallel volume element (cf. [19] for more details).

In an 2-dimensional manifold, the curvature tensor $\mathcal{R}^\nabla$ and the Ricci tensor Ric^∇ are related by

$$\mathcal{R}^\nabla(X, Y)Z = Ric^\nabla(Y, Z)X - Ric^\nabla(X, Z)Y. \tag{2.2}$$

For $X \in \Gamma(T_pM)$, we define the *affine Jacobi operator* $J_{\mathcal{R}^\nabla}$ with respect to X by $J_{\mathcal{R}^\nabla}(X) : T_pM \longrightarrow T_pM$ such that

$$J_{\mathcal{R}^\nabla}(X)Y := \mathcal{R}^\nabla(Y, X)X. \tag{2.3}$$

for any vector field Y. The affine Jacobi operator satisfies $J_{\mathcal{R}^\nabla}(X)X = 0$ and $J_{\mathcal{R}^\nabla}(\alpha X) = \alpha^2 J_{\mathcal{R}^\nabla}(X)Y$, for $\alpha \in \mathbb{R} - \{0\}$ and $X \in T_pM$. Let (M, ∇) be a three dimensional affine manifold and let $X = \sum_{i=1}^{3} \alpha_i \partial_i$ be a non-null vector on M, where $\{\partial_i\}$ denotes the coordinate basis and $\alpha_i \in \mathbb{R}^*$. Then the affine Jacobi operator is given by

$$\begin{aligned} J_{\mathcal{R}^\nabla}(X) &= \alpha_1^2 \mathcal{R}^\nabla(\cdot, \partial_1)\partial_1 + \alpha_1\alpha_2 \mathcal{R}^\nabla(\cdot, \partial_1)\partial_2 + \alpha_1\alpha_3 \mathcal{R}^\nabla(\cdot, \partial_1)\partial_3 \\ &+ \alpha_1\alpha_2 \mathcal{R}^\nabla(\cdot, \partial_2)\partial_1 + \alpha_2^2 \mathcal{R}^\nabla(\cdot, \partial_2)\partial_2 + \alpha_2\alpha_3 \mathcal{R}^\nabla(\cdot, \partial_2)\partial_3 \\ &+ \alpha_1\alpha_3 \mathcal{R}^\nabla(\cdot, \partial_3)\partial_1 + \alpha_2\alpha_3 \mathcal{R}^\nabla(\cdot, \partial_3)\partial_2 + \alpha_3^2 \mathcal{R}^\nabla(\cdot, \partial_3)\partial_3. \end{aligned}$$

2.2. **Affine Osserman manifolds.** Let (M, ∇) be an m-dimensional affine manifold, i.e., ∇ is a torsion free connection on the tangent bundle of a smooth manifold M of dimension m. Let $\mathcal{R}^\nabla(X, Y)$ be the curvature operator and $J_{\mathcal{R}^\nabla}(X)$ the Jacobi operator with respect to a vector $X \in T_pM$ associated.

Definition 2.1. [14] *One says that an affine manifold (M, ∇) is affine Osserman at $p \in M$ if the characteristic polynomial of $J_{\mathcal{R}^\nabla}(X)$ is independent of $X \in T_pM$. Also (M, ∇) is called affine Osserman if (M, ∇) is affine Osserman at each $p \in M$.*

Theorem 2.2. [14] *Let (M, ∇) be an m-dimensional affine manifold. Then (M, ∇) is called affine Osserman at $p \in M$ if and only if the characteristic polynomial of $J_{\mathcal{R}^\nabla}(X)$ is*

$$P_\lambda[J_{\mathcal{R}^\nabla}(X)] = \lambda^m$$

for every $X \in T_pM$.

Corollary 2.3. *We say that (M, ∇) is affine Osserman if Spect$\{J_{\mathcal{R}^\nabla}(X)\} = \{0\}$ for any vector X*

Corollary 2.4. *If (M, ∇) is affine Osserman at $p \in M$ then the Ricci tensor is skew-symmetric at $p \in M$.*

The affine Osserman connections are of interest not only in the affine geometry, but also in the study of the pseudo-Riemannian Osserman metrics since they provide some nice examples of Osserman manifolds whose Jacobi operators have non-trivial Jordan normal form and which are not nilpotent. It has long been a task in this field to build examples of Osserman manifolds were not nilpotent and which exhibit non-trivial Jordan normal form. We will refer [1, 2] for more information.

2.3. **The Riemann extension construction.** Let $N := T^*M$ be the cotangent bundle of an m-dimensional manifold and let $\pi : T^*M \to M$ be the natural projection. A point ξ of the cotangent bundle is represented by an ordered pair (ω, p), where $p = \pi(\xi)$ is a point on M and ω is an 1-form on T_pM. If $u = (u_1, \cdots, u_m)$ are local coordinates on M, let $u' = (u_{1'}, \cdots, u_{m'})$ be the associated dual coordinates on the fiber where we expand an 1-form ω as $\omega = u_{i'} du_i$ $(i = 1, \cdots, m; i' = i + m)$; we shall adopt the Einstein convention and sum over repeated indices henceforth.

For each vector field $X = X^i\partial_i$ on M, the evaluation map $\iota X(p, \omega) = \omega(X_p)$ defines on function on N which, in local coordinates is given by

$$\iota X(u_i, u_{i'}) = u_{i'} X^i.$$

Vector fields on N are characterized by their action on function ιX; the complete lift X^C of a vector field X on M to N is characterized by the identity

$$X^C(\iota Z) = \iota[X, Z], \quad \text{for all} \quad Z \in \mathcal{C}^\infty(TM).$$

Moreover, since a $(0, s)$-tensor field on M is characterized by its evaluation on complete lifts of vectors fields on M, for each tensor field T of type $(1, 1)$ on M, we define a 1-form ιT on N which is characterized by the identity

$$\iota T(X^C) = \iota(TX).$$

Definition 2.5. *Let (M, ∇) be an affine manifold of dimension m. The Riemann extension $\bar{g}$ of (M, ∇) is the pseudo-Riemannian metric of neutral signature (m, m) on the cotangent bundle T^*M, which is characterized by the identity*

$$\bar{g}(X^C, Y^C) = -\iota(\nabla_X Y + \nabla_Y X).$$

In the system of induced coordinates $(u_i, u_{i'})$ on TM, the Riemann extension takes the form:

$$\bar{g} = \begin{pmatrix} -2u_{k'}\Gamma_{ij}^k & \mathrm{Id}_m \\ \mathrm{Id}_m & 0 \end{pmatrix},$$

with respect to $\{\partial_{u_1}, \ldots, \partial_{u_m}, \partial_{u'_1}, \ldots, \partial_{u'_m}\}$; here the indices i and j range from $1, \ldots, m$, $i' = i + m$, and Γ_{ij}^k are the Christoffel symbols of the connection ∇ with respect to the coordinates (u_i) on M. More explicitly:

$$\bar{g}(\partial_{u_i}, \partial_{u_j}) = -2u_{k'}\Gamma_{ij}^k, \quad \bar{g}(\partial_{u_i}, \partial_{u'_j}) = \delta_i^j, \quad \bar{g}(\partial_{u'_i}, \partial_{u'_j}) = 0.$$

Let (M, g) be a pseudo-Riemannian manifold. The Riemann extension of the Levi-Civita connection inherits many of the properties of the base manifold. For instance, (M, g) has constant sectional curvature if and only if $(TM, \bar{g})$ is locally conformally flat. However, the main applications of the Riemann extensions appear when considering affine connections are not the Levi-Civita connection of any metric. We have the following result:

Theorem 2.6. *([13]) Let $(T^*M, \bar{g})$ be the cotangent bundle of an affine manifold (M, ∇) equipped with the Riemann extension of the torsion free connection ∇. Then $(T^*M, \bar{g})$ is a pseudo-Riemannian globally Osserman manifold if and only if (M, ∇) is an affine Osserman manifold.*

3. Example of affine Osserman connections

In the following M denotes a three-dimensional manifold and ∇ a smooth torsion-free affine connection. Choose a system (u_1, u_2, u_3) of local coordinates in a domaine $\mathcal{U} \subset M$ such that the affine connection ∇ is uniquely determined by six functions $f_1, \ldots, f_6$ given by the formulas

$$\left\{ \begin{array}{lcl} \nabla_{\partial_1}\partial_1 & = & f_1(u_1, u_2, u_3)\partial_2; \\ \nabla_{\partial_1}\partial_2 & = & f_2(u_1, u_2, u_3)\partial_2; \\ \nabla_{\partial_1}\partial_3 & = & f_3(u_1, u_2, u_3)\partial_2; \\ \nabla_{\partial_2}\partial_2 & = & f_4(u_1, u_2, u_3)\partial_2; \\ \nabla_{\partial_2}\partial_3 & = & f_5(u_1, u_2, u_3)\partial_2; \\ \nabla_{\partial_3}\partial_3 & = & f_6(u_1, u_2, u_3)\partial_2. \end{array} \right. \tag{3.1}$$

One can easily shown that the non-zero components of the Ricci tensor are given by

$$\begin{cases} Ric(\partial_1,\partial_1) &=& \partial_2 f_1 - \partial_1 f_2 + f_1 f_4 - f_2^2 \\ Ric(\partial_1,\partial_2) &=& \partial_2 f_2 - \partial_1 f_4 \\ Ric(\partial_1,\partial_3) &=& \partial_2 f_3 - \partial_1 f_5 + f_3 f_4 - f_2 f_5 \\ Ric(\partial_3,\partial_1) &=& \partial_2 f_3 - \partial_3 f_2 + f_3 f_4 - f_2 f_5 \\ Ric(\partial_3,\partial_2) &=& \partial_2 f_5 - \partial_3 f_4 \\ Ric(\partial_3,\partial_3) &=& \partial_2 f_6 - \partial_3 f_5 + f_4 f_6 - f_5^2. \end{cases} \tag{3.2}$$

The skew-symmetry of Ricci tensor means that, in any local coordinates, we have:

$$\begin{cases} Ric(\partial_1,\partial_1) = Ric(\partial_2,\partial_2) = Ric(\partial_3,\partial_3) &=& 0 \\ Ric(\partial_1,\partial_2) + Ric(\partial_2,\partial_1) &=& 0 \\ Ric(\partial_1,\partial_3) + Ric(\partial_3,\partial_1) &=& 0 \\ Ric(\partial_2,\partial_3) + Ric(\partial_3,\partial_2) &=& 0. \end{cases} \tag{3.3}$$

According (3.1) and (3.3), we have the following

Proposition 3.1. *The affine connection ∇ defined in (3.1) is skew-symmetric if the functions $f_i, i = 1,\dots,6$ satisfy the following partial differential equations:*

$$\begin{aligned} \partial_2 f_2 - \partial_1 f_4 = 0; \quad \partial_2 f_5 - \partial_3 f_4 &= 0 \\ \partial_2 f_1 - \partial_1 f_2 + f_1 f_4 - f_2^2 &= 0 \\ \partial_2 f_6 - \partial_3 f_5 + f_4 f_6 - f_5^2 &= 0 \\ 2\partial_2 f_3 - \partial_1 f_5 - \partial_3 f_2 + 2 f_3 f_4 - 2 f_2 f_5 &= 0. \end{aligned} \tag{3.4}$$

Proof. It follows from (3.1) and (3.3). □

Corollary 3.2. [8] *Let ∇ be as (3.1). Assume that $f_2 = f_3 = f_5 = 0$, then the affine connection (3.1) is skew-symmetric if and only if the coefficients of the connection (3.1) satisfy*

$$f_4(u_1,u_2,u_3) = f_1(u_2), \quad \partial_2 f_1 + f_1 f_4 = 0, \quad \text{and} \quad \partial_2 f_6 + f_4 f_6 = 0. \tag{3.5}$$

We have the following result:

Proposition 3.3. *Let (M,∇) be a 3-dimensional affine manifold with torsion free connection given by (3.1). Then (M,∇) is affine Osserman if and only if the Ricci tensor is skew-symmetric.*

Proof. Since the Ricci tensor of any affine Osserman connection is skew-symmetric, it follow from previous expression that we have the following necessary conditions for the affine connections (3.1) to be Osserman

$$\begin{aligned} \partial_2 f_2 - \partial_1 f_4 = 0; \quad \partial_2 f_5 - \partial_3 f_4 &= 0 \\ \partial_2 f_1 - \partial_1 f_2 + f_1 f_4 - f_2^2 &= 0 \\ \partial_2 f_6 - \partial_3 f_5 + f_4 f_6 - f_5^2 &= 0 \end{aligned}$$

and

$$2\partial_2 f_3 - \partial_1 f_5 - \partial_3 f_2 + 2 f_3 f_4 - 2 f_2 f_5 = 0.$$

Then, the associated affine Jacobi operator can be expressed, with respect to the coordinate basis, as

$$(J_{\mathcal{R}^\nabla}(X)) = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & c \\ 0 & 0 & 0 \end{pmatrix},$$

with

$$\begin{aligned}
a &= \alpha_1\alpha_3\Big(\partial_1 f_3 - \partial_3 f_1 + f_2 f_3 - f_1 f_4\Big) \\
&\quad +\alpha_2\alpha_3\Big(2\partial_1 f_5 - \partial_2 f_3 - \partial_3 f_2 + f_2 f_5 - f_3 f_4\Big) \\
&\quad +\alpha_3^2\Big(\partial_1 f_6 - \partial_3 f_3 + f_2 f_6 - f_3 f_5\Big); \\
c &= -\alpha_1^2\Big(\partial_1 f_3 - \partial_3 f_1 + f_2 f_3 - f_1 f_5\Big) \\
&\quad -\alpha_1\alpha_2\Big(\partial_1 f_5 - \partial_2 f_3 - 2\partial_3 f_2 + f_3 f_4 - f_2 f_5\Big) \\
&\quad -\alpha_1\alpha_3\Big(\partial_1 f_6 - \partial_3 f_3 + f_2 f_6 - f_3 f_5\Big).
\end{aligned}$$

The characteristic polynomial of the affine Jacobi operator is now seen to be:

$$P_\lambda[J_{\mathcal{R}^\nabla}(X)] = -\lambda^3$$

which has zero eigenvalues. □

Example 3.4. *Follow Corollary 3.2, one can construct examples of affine Osserman connections. The following connection on $\mathbb{R}^3$ whose non-zero coefficients of the cofficients are given by*

$$\nabla_{\partial_1}\partial_1 = u_1 u_3 \partial_2 \quad \textit{and} \quad \nabla_{\partial_3}\partial_3 = (u_2 + u_3)\partial_2 \tag{3.6}$$

is nonflat affine Osserman

The concept of an affine Osserman connection has become a very active research subject. In [10], the authors give examples of affine Osserman connections which are locally symmetric but not flat on 3-dimensional manifolds. In [11], affine Osserman connections which are Ricci flat but not flat on 3-dimensional manifolds are given. In [12], examples of affine Osserman connections which are Ricci flat and which are not Ricci flat on 3-dimensional manifolds are exhibit.

4. Example of Walker Osserman metric

Let M be a pseudo-Riemannian manifold of signature (p, q). We suppose given a splitting of the tangent bundle in the form $TM = V_1 \oplus V_2$ where V_1 and V_2 are smooth subbundles which are called distribution. This defines two complementary projection π_1 and π_2 of TM onto V_1 and V_2. We say that V_1 is a parallel distribution if $\nabla\pi_1 = 0$. Equivalently this means that if X_1 is any smooth vector field taking values in V_1, then ∇X_1 again takes values in V_1. If M is Riemannian, we can take $V_2 = V_1^\perp$ to be the orthogonal complement of V_1 and in that case V_2 is again parallel. In the pseudo-Riemannian setting, of course, $V_1 \cap V_2$ need not be trivial. We say that V_1 is a null parallel distribution if V_1 is parallel and if the metric restricted to V_1 vanishes identically. Manifolds which admit null parallel distribution are called Walker manifolds. More precisely, a Walker manifold is a triple $(M, g, \mathcal{D})$ where M is an m-dimensional manifold, g an indefinite metric and $\mathcal{D}$ an r-dimensional parallel null distribution. of special interest are those manifolds admitting a field of null planes of maximum dimension $r = \frac{m}{2}$. In this particular case, it is convenient to use special coordinate systems associated with any Walker metric.

Let (u_1, u_2, u_3) be the local coordinates on a 3-dimensional affine manifold (M, ∇). We expand $\nabla_{\partial_i}\partial_j = \sum_k f_{ij}^k \partial_k$ for $i, j, k = 1, 2, 3$ to define the Christoffel symbols of ∇. Let $\omega = u_4 du_1 + u_5 du_2 + u_6 du_3 \in T^*M$: (u_4, u_5, u_6) are the dual fiber coordinates. The *Riemann extension* is the pseudo-Riemannian metric $\bar{g}$ on the cotangent bundle T^*M of neutral signature $(3,3)$ defined by setting

$$
\begin{aligned}
\bar{g}(\partial_1, \partial_4) &= \bar{g}(\partial_2, \partial_5) = \bar{g}(\partial_3, \partial_6) = 1,\\
\bar{g}(\partial_1, \partial_1) &= -2u_4 f_{11}^1 - 2u_5 f_{11}^2 - 2u_6 f_{11}^3,\\
\bar{g}(\partial_1, \partial_2) &= -2u_4 f_{12}^1 - 2u_5 f_{12}^2 - 2u_6 f_{12}^3,\\
\bar{g}(\partial_1, \partial_3) &= -2u_4 f_{13}^1 - 2u_5 f_{13}^2 - 2u_6 f_{13}^3,\\
\bar{g}(\partial_2, \partial_2) &= -2u_4 f_{22}^1 - 2u_5 f_{22}^2 - 2u_6 f_{22}^3,\\
\bar{g}(\partial_2, \partial_3) &= -2u_4 f_{23}^1 - 2u_5 f_{23}^2 - 2u_6 f_{23}^3,\\
\bar{g}(\partial_3, \partial_3) &= -2u_4 f_{33}^1 - 2u_5 f_{33}^2 - 2u_6 f_{33}^3.
\end{aligned}
$$

Let consider the affine Osserman connection given (3.6). The Riemann extenxion $\bar{g}$ on $\mathbb{R}^6$ of the connection (3.6) has the form

$$
\bar{g} = \begin{pmatrix}
-2u_5 u_1 u_3 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & -2u_5(u_1+u_3) & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

The nonvanishing covariant derivatives of $\bar{g}$ are given by

$$
\begin{aligned}
\bar{\nabla}_{\partial_1}\partial_1 &= u_1 u_3 \partial_2 - u_3 u_5 \partial_4 + u_1 u_5 \partial_6, \quad \bar{\nabla}_{\partial_1}\partial_3 = -u_1 u_5 \partial_4 - u_5 \partial_6,\\
\bar{\nabla}_{\partial_1}\partial_5 &= -u_1 u_3 \partial_4, \bar{\nabla}_{\partial_3}\partial_3 = (u_1 + u_3)\partial_2 + u_5 \partial_4 - u_5 \partial_6,\\
\bar{\nabla}_{\partial_3}\partial_5 &= -(u_1 + u_3)\partial_6.
\end{aligned}
$$

The nonvanishing components of the curvature tensor of $(\mathbb{R}^6, \bar{g})$ are given by

$$
\begin{aligned}
R(\partial_1, \partial_3)\partial_1 &= -u_1 \partial_2; \quad R(\partial_1, \partial_3)\partial_3 = \partial_2; \quad R(\partial_1, \partial_3)\partial_5 = u_1 \partial_4 - \partial_6;\\
R(\partial_1, \partial_5)\partial_1 &= -u_1 \partial_6; \quad R(\partial_1, \partial_5)\partial_3 = u_1 \partial_4; \quad R(\partial_3, \partial_5)\partial_1 = \partial_6;\\
R(\partial_3, \partial_5)\partial_3 &= -\partial_4.
\end{aligned}
$$

Now, If $X = \sum_{i=1}^6 \alpha_i \partial_i$ is a vector field on $\mathbb{R}^6$, then the matrix associated to the Jacobi operator $J_{\mathcal{R}}(X) = \mathcal{R}(\cdot, X)X$ is given by

$$
(J_{\mathcal{R}}(X)) = \begin{pmatrix} A & 0 \\ B & A^t \end{pmatrix},
$$

where A is the 3×3 matrix given by

$$
A = \begin{pmatrix} 0 & 0 & 0 \\ 1 - u_1 & 0 & u_1 - 1 \\ 0 & 0 & 0 \end{pmatrix};
$$

and B is the 3×3 matrix given by

$$B = \begin{pmatrix} 2u_1 & 0 & -u_1 \\ 0 & 0 & 0 \\ -1-u_1 & 0 & 1 \end{pmatrix}.$$

Then we have the following

Proposition 4.1. $(\mathbb{R}^6, \bar{g})$ *is a Walker Osserman metric of signature* $(3, 3)$.

Walker geometry is intimately related with many questions in mathematical physics. Note that the Riemann extension is necessarily a Walker metric. It is a remarkable fact that Walker metrics satisfying some natural curvature conditions are locally Riemann extensions, thus leading the corresponding classification problem to a task in affine geometry as shown in [3].

Chaichi et al. [4] have studied conditions for a Walker metric to be Einstein, Osserman, or locally conformally flat and obtained thereby exact solutions to the Einstein equations for a restricted Walker manifold.

Appendix 1. Components of the curvature tensor

The non-zero components of the curvature tensor of the affine connection (3.1) are given by

$$\begin{aligned}
R(\partial_1,\partial_2)\partial_1 &= (\partial_1 f_2 - \partial_2 f_1 + f_2^2 - f_1 f_4)\partial_2\\
R(\partial_1,\partial_2)\partial_2 &= (\partial_1 f_4 - \partial_2 f_2)\partial_2\\
R(\partial_1,\partial_2)\partial_3 &= (\partial_1 f_5 - \partial_2 f_3 + f_2 f_5 - f_3 f_4)\partial_2\\
R(\partial_1,\partial_3)\partial_1 &= (\partial_1 f_3 - \partial_3 f_1 + f_2 f_3 - f_1 f_5)\partial_2\\
R(\partial_1,\partial_3)\partial_2 &= (\partial_1 f_5 - \partial_3 f_2)\partial_2\\
R(\partial_1,\partial_3)\partial_3 &= (\partial_1 f_6 - \partial_3 f_3 + f_2 f_6 - f_3 f_5)\partial_2\\
R(\partial_2,\partial_3)\partial_1 &= (\partial_2 f_3 - \partial_3 f_2 + f_3 f_4 - f_2 f_5)\partial_2\\
R(\partial_2,\partial_3)\partial_2 &= (\partial_2 f_5 - \partial_3 f_4)\partial_2\\
R(\partial_2,\partial_3)\partial_3 &= (\partial_2 f_6 - \partial_3 f_5 + f_4 f_6 - f_5^2)\partial_2.
\end{aligned}$$

Appendix 2. Osserman geometry

Let R be the curvature operator of a Riemannian manifold (M, g) of dimension m. The Jacobi operator $\mathcal{J}(x) : y \mapsto R(y, x)x$ is the self-adjoint endomorphism of the tangent bundle. Following the seminal work of Osserman [20], one says that (M, g) is *Osserman* if the eigenvalues of $\mathcal{J}$ are constant on the unit sphere bundle

$$S(M, g) := \{X \in TM : g(X, X) = 1\}.$$

Work of Chi [5], of Gilkey et al. [15] and of Nikolayevsky [16, 17] show that any complete and simply connected Osserman manifold of dimension $m \neq 16$ is a rank-one symmetric space; the 16-dimensional setting is exceptional and the situation is still not clear in that setting although there are some partial result due, again, to Nikolayevsky [18].

Suppose (M, g) is a pseudo-Riemannian manifold of signature (p, g) for $p > 0$ and $q > 0$. The pseudo-sphere bundles are defined by setting

$$S^{\pm}(M, g) := \{X \in TM : g(X, X) = \pm 1\}.$$

One says that (M, g) is spacelike (resp. timelike) Osserman if the eigenvalues of $\mathcal{J}$ are constant on $S^+(M, g)$ (resp. $S^-(M, g)$). The situation is rather different here as the Jacobi operator is no longer diagonalizable and can have nontrivial Jordan normal form as shown by García-Río et al. [13]. We refer to [14] for more information on Osserman manifolds.

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