FP-Gorenstein Cotorsion Modules

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Abstract

Let R be a ring. In this paper, FP-Gorenstein cotorsion modules are introduced and studied. An R-module N is said to be FP-Gorenstein cotorsion if $\mathrm{Ext}^1_R(F, N) = 0$ for any finitely presented Gorenstein flat R-module F. We prove that the class of FP-Gorenstein cotorsion modules is covering and preenveloping over coherent rings. FP-Gorenstein cotorsion dimension of modules and rings are also studied. Some properties of FP-Gorenstein cotorsion modules are given.

Key Words: FP-Gorenstein cotorsion module; FP-Gorenstein cotorsion preenvelope; FP-Gorenstein cotorsion cover.
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1. **Introduction and preliminaries**

Throughout this paper, R will denote an associative ring with identity and all modules will be unitary. Unless otherwise stated, R-modules always denote left R-modules. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+; $\mathrm{fd}(M)$, $\mathrm{id}(M)$, $\mathrm{pd}(M)$ and FP-$\mathrm{id}(M)$ stand for the flat, injective, projective and FP-injective dimensions of M respectively. As usual, we use ${}_R\mathfrak{M}$ to denote the class of left R-modules, $\mathrm{wD}(R)$ the weakly global dimension of R and $\mathrm{D}(R)$ the left global dimension of R. For unexplained concepts, notions and facts, we refer the reader to [3, 7, 8, 9, 17, 19, 20, 21].

We first recall some notions and facts which we need in the later sections.

(1) Let M be an R-module and $\mathcal{X}$ a class of R-modules. A homomorphism $\phi : M \to X$ with $X \in \mathcal{X}$ is called an $\mathcal{X}$*−preenvelope* [7, 16, 17, 20] of M if for any homomorphism $f : M \to X'$ with $X' \in \mathcal{X}$, there is a homomorphism $g : X \to X'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of X when $X = X'$ and $f = \phi$, the $\mathcal{X}$-preenvelope ϕ is called an $\mathcal{X}$*-envelope* of M. $\mathcal{X}$ is a (pre)enveloping class provided

that each module has an $\mathcal{X}$-(pre)envelope. Dually, $\mathcal{X}$-precovers, $\mathcal{X}$-covers, and covering classes of modules can be defined.

(2) Let $\mathcal{X}, \mathcal{Y}$ be two classes of R-modules. $\mathcal{X}^{\perp} = \{N \in {}_R\mathfrak{M} | \operatorname{Ext}_R^1(X, N) = 0$ for all $X \in \mathcal{X}\}$ and ${}^{\perp}\mathcal{Y} = \{M \in {}_R\mathfrak{M} | \operatorname{Ext}_R^1(M, Y) = 0$ for all $Y \in \mathcal{Y}\}$. A module M is said to have a *sepcial $\mathcal{X}$-precover* [7] if there is an exact sequence $0 \to K \to X \to M \to 0$ with $X \in \mathcal{X}$ and $K \in \mathcal{X}^{\perp}$. Dually, M is said to be have a *special $\mathcal{Y}$-preenvelope* if there is an exact sequence $0 \to M \to Y \to L \to 0$ with $Y \in \mathcal{Y}$ and $L \in {}^{\perp}\mathcal{Y}$.

(3) Let $\mathcal{X}, \mathcal{Y}$ be two classes of R-modules. The pair $(\mathcal{X}, \mathcal{Y})$ is called a *cotorsion pair* (or *cotorsion theory*) [7, 8, 9] if $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. Let $\mathcal{S}$ be a class of R-modules. $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ is called the cotorsion pair *cogenerated* by $\mathcal{S}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *complete* if each module has a special $\mathcal{Y}$-preenvelope and *hereditary* if $\operatorname{Ext}_R^i(X, Y) = 0$ for all $i \geq 1, X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. $(\mathcal{X}, \mathcal{Y})$ is called *perfect* provided that $\mathcal{X}$ is a covering class and $\mathcal{Y}$ is an enveloping class. We know that a cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is a complete cotorsion pair if it is cogenerated by a set [7, Theorem 7.4.1].

(4) An R-module M is called *Gorenstein flat* [7, 9, 20] if there exists an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat R-modules such that $M = \ker(F^0 \to F^1)$ and that remains exact whenever $E \otimes_R -$ is applied for any injective right R-module E. The class of Gorenstein flat modules is denoted by $\mathcal{GF}$. An R-module N is called *Gorenstein cotorsion* [9] if $\operatorname{Ext}_R^1(M, N) = 0$ for any Gorenstein flat R-module M. The class of Gorenstein cotorsion modules is denoted by $\mathcal{GC}$. Over right coherent rings, $(\mathcal{GF}, \mathcal{GC})$ is a hereditary and perfect cotorsion pair [9, Theorem 3.1.9]. So we can define the Gorenstein cotorsion dimension $\operatorname{Gcd}(M)$ of an R-module M as the least nonnegative integer n such that there is an exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to 0$ with $C^i \in \mathcal{GC}$ for $0 \leq i \leq n$.

In Section 2, we introduce the concept of FP-Gorenstein cotorsion modules. We show that the class of FP-Gorenstein cotorsion modules is closed under extensions, pure submodules, pure quotients, direct products and direct limits (and so direct sums) over coherent rings. Some basic properties of FP-Gorenstein cotorsion modules are given.

In Section 3, we prove that over coherent rings, every R-module M has a surjective FP-Gorenstein cotorsion cover and an injective FP-Gorenstein cotorsion preenvelope.

In Section 4, we introduce and investigate the FP-Goresntein cotorsion dimension of modules and rings. We characterize some rings through FP-Gorenstein cotorsion dimensions.

2. Some properties of FP-Gorenstein cotorsion modules

We begin with the following definition.

Definition 2.1. An R-module N is called *FP-Gorenstein cotorsion* if $\text{Ext}_R^1(F, N) = 0$ for all finitely presented Gorenstein flat R-modules F.

Proposition 2.2. *The following hold:*

(1) *Injective modules, FP-injective modules and Gorenstein cotorsion modules are FP-Goresntein cotorsion.*
(2) *Every direct product of FP-Goresntein cotorsion modules is FP-Gorenstein cotorsion.*
(3) *Every finite direct sum of FP-Gorenstein cotorsion modules is FP-Goresntein cotorsion.*
(4) *Suppose $N = N_1 \oplus N_2$, then N is FP-Gorenstein cotorsion if and only if N_1 and N_2 are both FP-Gorenstein cotorsion.*

Proof. By Definition 2.1. □

Recall that a ring R is called *left coherent* (resp. *right coherent*) if every finitely generated left (resp. right) ideal is finitely presented. A ring R is coherent if it is both left and right coherent. A ring R is left coherent if and only if every finitely generated submodule of a finitely presented R-module is also finitely presented.

Proposition 2.3. *Suppose R is a coherent ring and N an FP-Gorenstein cotorsion R-module. Then $\text{Ext}_R^i(F, N) = 0$ for any finitely presented Gorenstein flat R-module F and for all $i \geq 1$.*

Proof. Let F be a finitely presented Gorenstein flat R-module. By Definition 2.1, we need only to prove that $\text{Ext}_R^i(F, M) = 0$ for $i \geq 2$. Since R is coherent, we have a finitely generated free resolution of F

$$\cdots \to F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} F \to 0.$$

Then every $\ker f_i$ (for $i \geq 0$) is also finitely presented and Gorenstein flat by [9, Corollary 2.1.8]. Hence $\text{Ext}_R^{i+1}(F, N) \cong \text{Ext}_R^1(\ker f_{i-1}, M) = 0$ for all $i \geq 1$. □

Corollary 2.4. *Let R be a coherent ring and $0 \to N' \to N \to N'' \to 0$ a short exact sequence. If N' is FP-Goresntein cotorsion, then N is FP-Goresntein cotorsion if and only if N'' is FP-Gorenstein cotorsion.*

Proof. Let F be any finitely presented Gorenstein flat R-module, we get the following exact sequence

$$0 = \text{Ext}_R^1(F, N') \to \text{Ext}_R^1(F, N) \to \text{Ext}_R^1(F, N'') \to \text{Ext}_R^2(F, N').$$

By Proposition 2.3, $\text{Ext}_R^2(F, N') = 0$. Hence the result follows. □

Lemma 2.5. *Let R be a coherent ring. Then $\varinjlim N_i$ is FP-Gorenstein cotorsion, where $((N_i),(f_{ji}))$ is a direct system of FP-Gorenstein cotorsion R-mdoules. In particular, the class $\mathcal{FGC}$ of FP-Gorenstein cotorsion R-modules is closed under direct sums.*

Proof. Let F be a finitely presented Gorenstein flat R-module. By [18, Theorem 3.2], we get

$$\mathrm{Ext}_R^1(F, \varinjlim N_i) \cong \varinjlim \ \mathrm{Ext}_R^1(F, N_i) = 0.$$

Then the result follows. □

It is not hard to see that the condition "R is commutative" can be dropped in [2, Proposition 1.3]. Then we have the next lemma.

Lemma 2.6. *If R is coherent, then a finitely presented R-module is Gorenstein flat if and only if it is Gorenstein projective.*

Remark 2.7.

(1) Let R be a coherent ring. Then each R-module with finite projective dimension is FP-Gorenstein cotorsion since finitely presented Gorenstein projective R-modules coincide with finitely presented Gorenstein flat R-modules by Lemma 2.6. Hence any R-module with finite injective dimension is also FP-Gorenstein cotorsion by [4, Lemma 2.1].

(2) Let $R = \mathbb{Z}$. Then $\mathrm{D}(R) = 1$, so every Goresntein flat R-module is flat. Since finitely presented flat R-modules are finitely generated projective, every R-module is FP-Gorenstein cotorsion by Definition 2.1. Note that the quotient field $\mathbb{Q}$ of R is a flat R-module, but it is not a projective R-module. So there is an R-module L such that $\mathrm{Ext}_R^1(\mathbb{Q}, L) \neq 0$, i.e., L is neither cotorsion nor Gorenstein cotorsion. This example shows that FP-Gorenstein cotorsion modules need not to be cotorsion or Gorenstein cotorsion. Then we get the following implications:

injective modules ⇒ Gorenstein cotorsion modules ⇒ cotorsion modules,
injective modules ⇒ FP-injective modules ⇒ FP-Gorenstein cotorsion modules.

Proposition 2.8. *Let R be a coherent ring.*

(1) *If an R-module N has finite FP-injective dimension, then N is FP-Gorenstein cotorsion.*

(2) *If a right R-module N has finite FP-injective dimension, then N^+ is FP-Gorenstein cotorsion.*

(3) *If an R-module M has finite flat dimension, then M is FP-Gorenstein cotorsion.*

Proof. (1). Suppose that $FP\text{-id}(N) = n < \infty$. Let F be a finitely presented Gorenstein flat R-module. Then there exists an exact sequence

$$0 \to F \to P^0 \to P^1 \to \cdots \to P^{n-1} \to L \to 0$$

such that P^i is finitely generated projective for $0 \le i \le n-1$ and L is a finitely presented Gorenstein flat R-module. Thus $\mathrm{Ext}^1_R(F, N) \cong \mathrm{Ext}^{n+1}_R(L, N) = 0$ and hence N is FP-Gorenstein cotorsion.

(2). Let F be a finitely presented Gorenstein flat R-module and E an injective right R-module. Then $\mathrm{Tor}^R_1(E, F) = 0$ and [7, Theorem 3.2.1] shows

$$\mathrm{Ext}^1_R(F, E^+) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_1(E, F), \mathbb{Q}/\mathbb{Z}) = 0,$$

which implies that E^+ is FP-Gorenstein cotorsion for every injective right R-module E.

Next, we assume that FP-id$(N) = n < \infty$. Then there exists an exact sequence

$$0 \to N \to E^0 \to E^1 \cdots \to E^{n-1} \to L \to 0$$

such that each E^i is injective for $0 \le i \le n-1$ and L is FP-injective by [18, Lemma 3.1]. This exact sequence induces the following exact sequence

$$0 \to L^+ \to (E^{n-1})^+ \to \cdots \to (E^1)^+ \to (E^0)^+ \to N^+ \to 0.$$

By Corollary 2.4, it is sufficient to prove that L^+ is FP-Gorenstein cotorsion. Since L is FP-injective, L is a pure submodule of any right R-module which contains L. Then we get a pure exact sequence

$$0 \to L \to E \to K \to 0$$

with E injective. Note that

$$0 \to K^+ \to E^+ \to L^+ \to 0$$

splits, so L^+ is FP-Gorenstein cotorsion since E^+ is FP-Gorenstein cotorsion by the proof above. This completes the proof.

(3). Let F be a finitely presented Gorenstein flat R-module and F' a flat R-module. Then $F' = \varinjlim P_i$ for some direct system $((P_i), (f_{ji}))$, where each P_i is projective. By [10, Lemma 3.1.6], we have

$$\begin{aligned} \mathrm{Ext}^1_R(F, F') &\cong \mathrm{Ext}^1_R(F, \varinjlim P_i) \\ &\cong \varinjlim \mathrm{Ext}^1_R(F, P_i) \\ &= 0. \end{aligned}$$

Hence any flat R-module is FP-Gorenstein cotorsion. Assume that fd$(M) = n$, then we have the exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where F_i is flat for $0 \leq i \leq n$. By the proof above, each F_i is FP-Gorenstein cotorsion and hence M is also FP-Gorenstein cotorsion by Corollary 2.4. □

Recall that a submodule T of an R-module N is said to be a *pure submodule* of N if $0 \to A \otimes_R T \to A \otimes_R N$ is exact for all right R-modules A, or equivalently, if $\mathrm{Hom}_R(A, N) \to \mathrm{Hom}_R(A, N/T) \to 0$ is exact for all finitely presented R-modules A. An exact sequence $0 \to T \xrightarrow{\lambda} N$ is said to be *pure exact* if $\lambda(T)$ is a pure submodule of N.

Proposition 2.9. *Let R be a ring and N an FP-Gorenstein cotorsion R-module. If the exact sequence $0 \to N' \to N \xrightarrow{\pi} N'' \to 0$ is pure, then N' is FP-Gorenstein cotorsion. In addition, if R is coherent, then N'' is also FP-Gorenstein cotorsion.*

Proof. Let F be a finitely presented Gorenstein flat R-module. Then we have an exact sequence

$$\begin{aligned}&\mathrm{Hom}_R(F, N) \xrightarrow{\pi_*} \mathrm{Hom}_R(F, N'') \to \mathrm{Ext}^1_R(F, N') \to \mathrm{Ext}^1_R(F, N)\ (= 0)\\ &\to \mathrm{Ext}^1_R(F, N'') \to \mathrm{Ext}^2_R(F, N').\end{aligned}$$

Since F is finitely presented and $0 \to N' \to N \xrightarrow{\pi} N'' \to 0$ is pure exact, π_* is epimorphic. So $\mathrm{Ext}^1_R(F, N') = 0$ and hence N' is FP-Gorenstein cotorsion. If R is coherent, then $\mathrm{Ext}^2_R(F, N') = 0$ by Proposition 2.3. So $\mathrm{Ext}^1_R(F, N'') = 0$ and then N'' is also FP-Gorenstein cotorsion. □

Corollary 2.10. *Suppose R is coherent. Then M is FP-Gorenstein cotorsion if and only if M^{++} is FP-Gorenstein cotorsion.*

Proof. Note that $0 \to M \to M^{++}$ is a pure exact sequence, then M is FP-Gorenstein cotorsion whenever M^{++} is by Proposition 2.9.

Conversely, suppose that M is FP-Gorenstein cotorsion. Let F be a finitely presented Gorenstein flat R-module and $\mathbf{P}$ a finitely generated projective resolution of F. Then we have

$$\begin{aligned}\mathrm{Ext}^1_R(F, M^{++}) &= H_{-1}(\mathrm{Hom}_R(\mathbf{P}, M^{++}))\\ &\cong H_{-1}(\mathrm{Hom}_{\mathbb{Z}}(M^{+} \otimes_R \mathbf{P}, \mathbb{Q}/\mathbb{Z}))\\ &\cong \mathrm{Hom}_{\mathbb{Z}}(H_1(M^{+} \otimes_R \mathbf{P}), \mathbb{Q}/\mathbb{Z})\\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(H_{-1}(\mathrm{Hom}_R(\mathbf{P}, M)), \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})\\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}^1_R(F, M), \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})\\ &= 0.\end{aligned}$$

The second step is Hom-tensor adjointness. The fourth step follows from the proof of [17, Theorem 9.51] and [17, Remark, p.257]. Hence M^{++} is FP-Gorenstein cotorsion. □

3. **Existences of FP-Gorenstein cotorsion covers and preenvelopes**

In the rest of this article, $\mathcal{GF}_{fp}$ always denotes the class of finitely presented Gorenstein flat R-modules.

Theorem 3.1. *Let R be a coherent ring.*

(1) *Every R-module M has a surjective FP-Gorenstein cotorsion cover $f : C \to M$.*
(2) *The pair $(^{\perp}\mathcal{FGC}, \mathcal{FGC})$ is a complete and hereditary cotorsion pair. In particular, every R-module M has a special $^{\perp}\mathcal{FGC}$-precover and a special FP-Gorenstein cotorsion preenvelope.*

Proof. (1). Since the class of FP-Gorenstein cotorsion modules is closed under pure quotient modules by Proposition 2.9 and closed under direct sums by Lemma 2.5, every R-module M has an FP-Gorenstein cotorsion cover $f : C \to M$ by [12, Theorem 2.5]. Note that each projective R-module is FP-Gorenstein cotorsion by Remark 2.7, then f is surjective.

(2). Firstly. It is easy to see that $(^{\perp}\mathcal{FGC}, \mathcal{FGC}) = (^{\perp}(\mathcal{GF}_{fp}{}^{\perp}), \mathcal{GF}_{fp}^{\perp})$ is a cotorsion pair.

Secondly. For any finitely presented Gorenstein flat R-module F, $\mathrm{Card}(F) \leq \aleph_0 \cdot \mathrm{Card}(R)$. Let Y be the set of all finitely presented Gorenstein flat R-modules F such that $\mathrm{Card}(F) \leq \aleph_0 \cdot \mathrm{Card}(R)$. Then C is in $\mathcal{FGC}$ if and only if $\mathrm{Ext}_R^1(F, C) = 0$ for all $F \in Y$. This just says that the cotorsion pair $(^{\perp}\mathcal{FGC}, \mathcal{FGC})$ is cogenerated by the set Y and hence $(^{\perp}\mathcal{FGC}, \mathcal{FGC})$ is a complete cotorsion pair by [10, Theorem 3.2.1]. In particular, every R-module M has a special $^{\perp}\mathcal{FGC}$-precover and a special $\mathcal{FGC}$-preenvelope.

Thirdly. $\mathcal{FGC}$ is coresolving by Proposition 2.2 and Corollary 2.4, so $(^{\perp}\mathcal{FGC}, \mathcal{FGC})$ is a hereditary cotorsion pair by [8, Theorem 2.1.4]. □

Remark 3.2.

(1) Note that $\mathcal{FGC}$ contains all injective modules, then every $\mathcal{FGC}$-preenvelope $g : M \to C$ of an R-module M is a monomorphism. Clearly, $^{\perp}\mathcal{FGC}$ contains all projective R-modules, so each $^{\perp}\mathcal{FGC}$-precover $f : G \to N$ of an R-module N is an epimorphism.
(2) $\mathcal{GF} \supseteq {}^{\perp}\mathcal{FGC}$ since $\mathcal{GC} \subseteq \mathcal{FGC}$. So every R-module $M \in {}^{\perp}\mathcal{FGC}$ is Gorenstein flat. In general, $^{\perp}\mathcal{FGC}$ isn't closed under direct limits. If $^{\perp}\mathcal{FGC}$ is closed under direct limits, then $^{\perp}\mathcal{FGC}$ contains all flat R-modules since every flat module is a direct limit of finitely generated free R-modules. Even over the ring $\mathbb{Z}$, $^{\perp}\mathcal{FGC}$ doesn't contain all flat modules (see Remark 2.7(2)).

Corollary 3.3. *Let R be a coherent ring and $f : M \to N$ a monomorphism.*

(1) *If* coker$(f) \in {}^{\perp}\mathcal{FGC}$*, then* $gf : M \to C$ *is also an* $\mathcal{FGC}$*-preenvelope of* M *whenever* $g : N \to C$ *is an* $\mathcal{FGC}$*-preenvelope of* N*.*

(2) *If* $g : N \to C$ *is a special* $\mathcal{FGC}$*-preenvelope of* N*, then* coker$(f) \in {}^{\perp}\mathcal{FGC}$ *if and only if* $gf : M \to C$ *is a special* $\mathcal{FGC}$*-preenvelope of* M*.*

Proof. This is similar to the proof of [15, Proposition 2.6]. □

Proposition 3.4. *The following conditions are equivalent for a coherent ring* R*:*

(1) *Every* R*-module is* FP*-Gorenstein cotorsion.*

(2) *Every* R*-module* $M \in {}^{\perp}\mathcal{FGC}$ *is* FP*-Gorenstein cotorsion.*

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (1). Let M be an R-module. By Theorem 3.1, we have a short exact sequence:

$$0 \to C \to F \xrightarrow{f} M \to 0$$

such that $f : F \to M$ is a special ${}^{\perp}\mathcal{FGC}$-precover. So C is FP-Gorenstein cotorsion and hence M is FP-Gorenstein cotorsion by Corollary 2.4. □

4. FP-Gorenstein cotorsion dimension of modules and rings

Definition 4.1. Let R be a ring. For an R-module M, the *FP-Gorenstein cotorsion dimension* FP-Gcd(M) of M is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for any finitely presented Gorenstein flat R-module F. If there is no such n, set FP-Gcd$(M) = \infty$. The (left) *global FP-Gorenstein cotorsion dimension* FP-G-cot.D(R) of R is defined as the supremum of the FP-Gorenstein cotorsion dimensions of R-modules.

Dually, we can define the *${}^{\perp}\mathcal{FGC}$ dimension* of M, denoted by Gfd$^*(M)$. Note that ${}^{\perp}\mathcal{FGC}$ contains all projective R-modules, then Gfd$(M) \leq$ Gfd$^*(M) \leq$ pd(M) for all R-modules M. The (left) *global ${}^{\perp}\mathcal{FGC}$ dimension* of R is defined by G-wD$^*(R) = \sup\{\mathrm{Gfd}^*(M) | M \in {}_R\mathfrak{M}\}$.

Proposition 4.2. *Let* R *be coherent and* N *an* R*-module.*

(1) *Consider the following two exact sequences*

$$0 \to N \to G^0 \to G^1 \to \cdots \to G^{n-1} \to X \to 0,$$
$$0 \to N \to \tilde{G}^0 \to \tilde{G}^1 \to \cdots \to \tilde{G}^{n-1} \to \tilde{X} \to 0,$$

where $G^0, G^1, \cdots, G^{n-1}$ *and* $\tilde{G}^0, \tilde{G}^1, \cdots, \tilde{G}^{n-1}$ *are* FP*-Gorenstein cotorsion* R*-modules. Then* X *is* FP*-Gorenstein cotorsion if and only if* $\tilde{X}$ *is* FP*-Gorenstein cotorsion.*

(2) *Dually, consider the following two exact sequences*

$$0 \to K \to F_{m-1} \to F_{m-2} \to \cdots \to F_0 \to N \to 0,$$
$$0 \to \tilde{K} \to \tilde{F}_{m-1} \to \tilde{F}_{m-2} \to \cdots \to \tilde{F}_0 \to N \to 0,$$

where $F_0, \cdots, F_{m-1}$ and $\tilde{F}_0, \cdots, \tilde{F}_{m-1}$ are all in ${}^{\perp}\mathcal{FGC}$. Then $K \in {}^{\perp}\mathcal{FGC}$ if and only if $\tilde{K} \in {}^{\perp}\mathcal{FGC}$.

Proof. (1). Clearly, we can construct the following diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & G^0 & \longrightarrow & G^1 & \longrightarrow \cdots \longrightarrow & G^{n-1} & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow \cdots \longrightarrow & E^{n-1} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & \tilde{G}^0 & \longrightarrow & \tilde{G}^1 & \longrightarrow \cdots \longrightarrow & \tilde{G}^{n-1} & \longrightarrow & \tilde{X} & \longrightarrow & 0
\end{array}$$

where E^i is injective for $0 \leq i \leq n-1$. By mapping cone, we get the following two exact sequences:

$$0 \to N \to N \oplus G^0 \to E^0 \oplus G^1 \to \cdots \to E^{n-2} \oplus G^{n-1} \to E^{n-1} \oplus X \to L \to 0,$$
$$0 \to N \to N \oplus \tilde{G}^0 \to E^0 \oplus \tilde{G}^1 \to \cdots \to E^{n-2} \oplus G^{n-1} \to E^{n-1} \oplus \tilde{X} \to L \to 0.$$

Then we get two exact sequences by [7, Remark 1.4.14]:

$$0 \to G^0 \to E^0 \oplus G^1 \to \cdots \to E^{n-2} \oplus G^{n-1} \to E^{n-1} \oplus X \to L \to 0,$$
$$0 \to \tilde{G}^0 \to E^0 \oplus \tilde{G}^1 \to \cdots \to E^{n-2} \oplus G^{n-1} \to E^{n-1} \oplus \tilde{X} \to L \to 0.$$

By Corollary 2.4, X is FP-Gorenstein cotorsion if and only if L is FP-Gorenstein cotorsion if and only if $\tilde{X}$ is FP-Gorenstein cotorsion.

(2). The proof is dual to that of (1). □

Over coherent rings, it is easily to see $\mathrm{Gfd}^*(M) = \mathrm{Gfd}(M)$ for every finitely presented R-module M.

Theorem 4.3. *Let R be a coherent ring.*

(1) FP-$\mathrm{Gcd}(M) = 0$ *or* ∞ *for an R-module M.*

(2) FP-G-$\mathrm{cot.D}(R) = 0$ *or* ∞.

(3) $(\mathcal{FGC}, \mathcal{FGC}^{\perp})$ *is a perfect, hereditary cotorsion pair.*

Proof. (1). Suppose that FP-$\mathrm{Gcd}(M) = n < \infty$ for some nonnegative integer n. Let F be a finitely presented Gorenstein flat R-module. Then there exists an exact sequence

$$0 \to F \to P^0 \to P^1 \to P^2 \to \cdots \to P^{n-1} \to F' \to 0$$

such that each P^i is finitely generated projective for $0 \le i \le n-1$ and F' is a finitely presented Gorenstein flat. So we get $\mathrm{Ext}^1_R(F, M) \cong \mathrm{Ext}^{n+1}_R(F', M) = 0$. Hence M is FP-Gorenstein cotorsion.

(2) is clear by (1).

(3). We first prove that $(\mathcal{FGC}, \mathcal{FGC}^\perp)$ is a cotorsion pair. Note that $({}^\perp(\mathcal{FGC}^\perp), \mathcal{FGC}^\perp)$ is a cotorsion pair, then we must prove $\mathcal{FGC} = {}^\perp(\mathcal{FGC}^\perp)$. $\mathcal{FGC} \subseteq {}^\perp(\mathcal{FGC}^\perp)$ is clear, so we need to prove $\mathcal{FGC} \supseteq {}^\perp(\mathcal{FGC}^\perp)$. For any R-module $M \in {}^\perp(\mathcal{FGC}^\perp)$, there exists an exact sequence $0 \to K \to C \to M \to 0$, where $C \to M$ is the FP-Gorenstein cotorsion cover of M by Theorem 3.1. Then $K \in \mathcal{FGC}^\perp$ by [20, Lemma 2.1.1] and so $\mathrm{Ext}^1_R(M, K) = 0$. Hence $0 \to K \to C \to M \to 0$ splits and then $M \in \mathcal{FGC}$. So $\mathcal{FGC} \supseteq {}^\perp(\mathcal{FGC}^\perp)$.

Note that $\mathcal{FGC}$ is resolving by Remark 2.7 and Theorem 4.3, then $(\mathcal{FGC}, \mathcal{FGC}^\perp)$ is a complete, hereditary cotorsion pair by Theorem 3.1 and [7, Proposition 7.1.7].

Since $\mathcal{FGC}$ is closed under direct limits by Proposition 2.9, $(\mathcal{FGC}, \mathcal{FGC}^\perp)$ is a perfect cotorsion pair by [7, Theorem 7.2.6]. □

Proposition 4.4. *Let R be a coherent ring and M an R-module. Then the following are equivalent for a nonnegative integer n:*

1. $\mathrm{Gfd}^*(M) \le n$.
2. $\mathrm{Ext}^{n+1}_R(M, C) = 0$ *for all FP-Gorenstein cotorsion R-modules C.*
3. $\mathrm{Ext}^{i}_R(M, C) = 0$ *for all FP-Gorenstein cotorsion R-modules C and all $i \ge n+1$.*
4. *If the sequence $0 \to G^n \to G^{n-1} \to \cdots \to G^0 \to M \to 0$ is exact such that $G^0, G^1, \cdots, G^{n-1}$ are all in ${}^\perp\mathcal{FGC}$, then G^n is also in ${}^\perp\mathcal{FGC}$.*
5. *If $f : M \to C$ is a special $\mathcal{FGC}$-preenvelope, then* $\mathrm{Gfd}^*(C) \le n$.

Consequently, the ${}^\perp\mathcal{FGC}$ dimension of M is determined by the formula:

$$\mathrm{Gfd}^*(M) = \sup\{i \in \mathbb{N}_0 | \exists C \in \mathcal{FGC} : \mathrm{Ext}^i_R(M, C) \neq 0\}.$$

Proof. By Definition 4.1, Proposition 4.2 and Theorem 3.1. □

Corollary 4.5. *Let R be a coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. If two of* $\mathrm{Gfd}^*(A)$, $\mathrm{Gfd}^*(B)$ *and* $\mathrm{Gfd}^*(C)$ *are finite, so does the third. Moreover,*

1. $\mathrm{Gfd}^*(B) \le \max\{\mathrm{Gfd}^*(A), \mathrm{Gfd}^*(C)\}$.
2. $\mathrm{Gfd}^*(C) \le \max\{\mathrm{Gfd}^*(A) + 1, \mathrm{Gfd}^*(B)\}$.
3. $\mathrm{Gfd}^*(A) \le \max\{\mathrm{Gfd}^*(B), \mathrm{Gfd}^*(C) - 1\}$.

In particular, if B is in ${}^\perp\mathcal{FGC}$ and $\mathrm{Gfd}^*(C) > 0$, *then* $\mathrm{Gfd}^*(C) = \mathrm{Gfd}^*(A) + 1$.

Corollary 4.6. *Let R be a coherent ring with* $\mathrm{D}(R) < \infty$. *Then* $G\text{-wD}^*(R) = \mathrm{D}(R)$. *In particular, R is left hereditary if and only if* $G\text{-wD}^*(R) \le 1$.

Proposition 4.7. *Let R be a coherent ring with G-wD$^*(R) = n$ for some nonnegative integer n and M an R-module. Then*

(1) $\mathrm{id}(M) \leq n$ *if* $\mathrm{fd}(M) < \infty$.
(2) $\mathrm{id}(M) \leq n$ *if* $\mathrm{pd}(M) < \infty$.
(3) $\mathrm{id}(M) < \infty$ *if and only if* $\mathrm{id}(M) \leq n$ *if and only if* FP-$\mathrm{id}(M) \leq n$ *if and only if* FP-$\mathrm{id}(M) < \infty$.

Proof. (1). Since G-wD$^*(R) = n < \infty$, there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to N \to 0$$

for any R-module N such that $F_i \in {}^{\perp}\mathcal{FGC}$ for $0 \leq i \leq n$. Note that $M \in \mathcal{FGC}$ if $\mathrm{fd}(M) < \infty$ by Proposition 2.8, then we have $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for any R-module N. Hence $\mathrm{id}(M) \leq n$.

(2) is a consequence of (1).

(3). $\mathrm{id}(M) < \infty \Rightarrow \mathrm{id}(M) \leq n$ and FP-$\mathrm{id}(M) < \infty \Rightarrow \mathrm{id}(M) \leq n$ are similar to (1).

$\mathrm{id}(M) \leq n \Rightarrow FP$-$\mathrm{id}(M) \leq n \Rightarrow FP$-$\mathrm{id}(M) < \infty$ are trivial. □

Theorem 4.8. *Let R be a Noetherian ring. Then the following are equivalent:*

(1) *R is quasi-Frobenius (i.e., 0-Gorenstein).*
(2) *Every FP-Gorenstein cotorsion R-module is injective.*
(3) *Every Gorenstein cotorsion R-module is injective.*
(4) $\mathrm{Gfd}^*(M) = 0$ *for any R-module M.*

Proof. (1) $\Rightarrow$ (2). Since R is quasi-Frobenius, R/I is finitely presented Gorenstein flat for any left ideal I of R. Then for any FP-Gorenstein cotorsion R-module N, we have $\mathrm{Ext}_R^1(R/I, N) = 0$. So N is injective by Bear criterion.

(2) $\Rightarrow$ (3) and (2) $\Leftrightarrow$ (4) are trivial.

(3) $\Rightarrow$ (1). Since $(\mathcal{GF}, \mathcal{GC})$ is a cotorsion pair, every R-module is Gorenstein flat by (3). Then R is quasi-Frobenius by [7, Theorem 12.3.1]. □

Remark 4.9. In general, G-wD$(R) \leq G$-wD$^*(R) \leq$ D(R). Theorem 4.8 shows that the the second inequality may be strict. In fact, the first inequality may be also strict. For example, consider Small's triangular ring

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Since wD$(R) = 1$ and D$(R) = 2$ by [13, Example (5.62b)], we have G-wD$(R) =$ wD$(R) = 1 < G$-wD$^*(R) =$ D$(R) = 2$.

Following [5], a ring R is called an *n-FC ring* if R is left and right coherent with FP-$\mathrm{id}({}_RR) \leq n$ and FP-$\mathrm{id}(R_R) \leq n$ for an integer $n \geq 0$. An R-module M is said to be *torsionless* (or *semi-reflxive*) [13] if the natural map $i : M \to M^{**}$ is a monomorphism and an R-module M is called *reflexive* if $i : M \to M^{**}$ is an isomorphism, where $M^* = \operatorname{Hom}_R(M, R)$.

Theorem 4.10. *Let R be a coherent ring. Then the following are equivalent:*

(1) *R is an FC ring (i.e.,* 0*-FC ring).*
(2) *Every FP-Gorenstein cotorsion R-module is FP-injective.*

Proof. (1) ⇒ (2). Since R is FC, every R-module is Gorenstein flat by [14, Proposition 5.5]. For any FP-Gorenstein cotorsion R-module N, we have $\operatorname{Ext}^1_R(F, N) = 0$ for any finitely presented R-module F. Hence N is FP-injective.

(2) ⇒ (1). Let M be a finitely presented R-module. Since every FP-Gorenstein cotorsion R-module is FP-injective by (2), every finitely presented R-module M is Gorenstein flat and hence Gorenstein projective. Then M can be embedded in a free R-module and is torsionless by [13, Remarks 4.65]. By [18, Lemma 4.6], we have an exact sequence

$$0 \to M \to M^{**} \to \operatorname{Ext}^1_R(L, R) \to 0$$

for some finitely presented R-module L. Note that L is finitely presented Gorenstein projective and hence $\operatorname{Ext}^1_R(L, R) = 0$ since R is FP-Gorenstein cotorsion by Remark 2.7. Then M is reflexive and R is an FC ring by [18, Theorem 4.9]. □

Example 4.11. By Theorems 4.3, 4.8 and 4.10, we get

(1) If R is quasi-Frobenius (i.e., 0-Gorenstein), then the cotorsion pair $(\mathcal{FGC}, \mathcal{FGC}^{\perp})$ is exactly $(\mathcal{P}roj, {}_R\mathfrak{M})$, where $\mathcal{P}roj$ is the class of projective R-modules. In fact, by Theorem 4.8, FP-Gorenstein cotorsion R-modules coincide with injective R-modules. Note that R is quasi-Frobenius, so projective modules coincide with injective modules. Then the result holds. Similarly, we have
(2) If R is an FC ring, then the cotorsion pair $(\mathcal{FGC}, \mathcal{FGC}^{\perp})$ is exactly $(\mathcal{F}lat, Cot)$, where $\mathcal{F}lat$ (Cot) is the class of flat (cotorsion) R-modules.

Proposition 4.12. *Let R be a coherent ring. Then the following are equivalent:*

(1) *R is n-FC.*
(2) FP-id$(M) \leq n$ *for any FP-Goresntein cotorsion (left and right) R-module M.*

Proof. (1) ⇒ (2). Let N be a finitely presented R-module. Since R is n-FC, we get $\operatorname{Gfd}(N) \leq n$ by [5, Theorem 7]. Then $\operatorname{Ext}^{n+1}_R(N, M) = 0$ for any FP-Gorenstein cotorsion R-module M. So FP-id$(M) \leq n$ by [18, Theorem 3.1].

(2) ⇒ (1). Suppose $n \geq 1$. Let N be a finitely presented R-module and M an FP-Gorenstein cotorsion R-module. We get a finitely generated projective resolution of N:

$$0 \to K \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to N \to 0.$$

Since FP-id$(M) \leq n$, $0 = \text{Ext}_R^{n+1}(N, M) \cong \text{Ext}_R^1(K, M)$. Then K is finitely presented Gorenstein flat and hence R is n-FC by [5, Theorem 7] again.

Suppose $n = 0$. By Theorem 4.10, we easily get that R is an FC ring. □

Corollary 4.13. *Let R be an n-FC ring. Then the following are equivalent:*

(1) *${}^{\perp}\mathcal{FGC}$ is closed under direct limits.*

(2) *$\mathcal{FGC} = \mathcal{GC}$.*

Proof. (1) ⇒ (2). Since R is an n-FC ring, every Gorenstein flat R-module M is isomorphic to $\varinjlim P_i$ for some inductive system $((P_i), (f_{ji}))$ by [5, Theorem 5], where each P_i is a finitely presented Gorenstein flat R-module. By (1), every Gorenstein flat R-module is in ${}^{\perp}\mathcal{FGC}$, so (2) follows.

(2) ⇒ (1). Since $({}^{\perp}\mathcal{FGC}, \mathcal{FGC})$ and $(\mathcal{GF}, \mathcal{GC})$ are both cotorsion pairs, we get ${}^{\perp}\mathcal{FGC} = \mathcal{GF}$ by (2). Hence ${}^{\perp}\mathcal{FGC}$ is closed under direct limits by [9, Corollary 2.1.9]. □

Theorem 4.14. *Let R be a coherent ring.*

(1) *If every FP-Gorenstein cotorsion R-module is Gorenstein cotorsion, then R is left perfect.*

(2) *If R is an n-FC ring and N is a pure-injective R-module, then N is FP-Gorenstein cotorsion if and only if N is Gorenstein cotorsion.*

(3) *If R is left perfect, then* $\text{Gfd}^*(F) = 0$ *or* ∞ *for any Gorenstein flat R-module F. Furthermore, if G-$\text{wD}^*(R) < \infty$, then an R-module M is Gorenstein cotorsion if and only if it is FP-Gorenstein cotorsion.*

Proof. (1). For any flat R-module F, we have a short exact sequence

$$0 \to K \to P \to F \to 0.$$

Note that K is flat and so it is FP-Gorenstein cotorsion by Proposition 2.8. Then we have $\text{Ext}_R^1(F, K) = 0$ and so the sequence splits. Thus F is projective and then R is left perfect.

(2). The sufficiency is trivial.

Necessity. Suppose $n \geq 1$. Let M be a Gorenstein flat R-module. Note that R is n-FC, $M \cong \varinjlim C_i$ for some inductive system $((C_i), (f_{ji}))$, where each C_i is a finitely presented Gorenstein projective R-module by [5, Theorem 5]. Note that N is pure-injective, then [10, Lemma 3.3.4] implies

$$\begin{aligned}\text{Ext}_R^1(M, N) &\cong \text{Ext}_R^1(\varinjlim C_i, N)\\ &\cong \varprojlim \text{Ext}_R^1(C_i, N)\\ &= 0.\end{aligned}$$

So N is Gorenstein cotorsion.

Suppose $n = 0$. Note that an R-module N is FP-Gorenstein cotorsion if and only if it is FP-injective by Theorem 4.10, the rest proof is similar to the case $n \geq 1$.

(3). Let F_0 be a Gorenstein flat R-module. Suppose $\mathrm{Gfd}^*(F_0) = n < \infty$ and let $f : G \to F_0$ be a special ${}^\perp\mathcal{FGC}$-precover. Then $K = \ker(f)$ is FP-Gorenstein cotorsion and Gorenstein flat. There exists an exact sequence

$$0 \to F_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to K \to 0$$

with each P_i projective and $F_n \in {}^\perp\mathcal{FGC}$. It is easy to see that F_n is FP-Gorenstein cotorsion. Note that there is an exact sequence

$$0 \to L \to P \to F_n \to 0$$

with P projective and $L \in \mathcal{FGC}$. The sequence splits and then F_n is projective. It is not hard to prove that every Gorenstein flat R-module is Gorenstein projective when R is coherent and left perfect. Hence we get that K is projective and so the short exact sequence $0 \to K \to G \to F_0 \to 0$ splits. Hence F_0 is a direct summand of G and so $F_0 \in {}^\perp\mathcal{FGC}$. Then $\mathrm{Gfd}^*(F_0) = 0$ or ∞.

Now, the last statement is obvious. □

Remark 4.15. The condition $G\text{-wD}^*(R) < \infty$ in Theorem 4.14 (3) can be replaced by $\mathrm{Gfd}^*(F) < \infty$ for all Gorenstein flat R-modules F.

Corollary 4.16. *Let R be a coherent ring. Then the following hold:*

(1) *every FP-Gorenstein cotrosion R-module is Gorenstein cotorsion if and only if R is left perfect and* $\mathrm{Gfd}^*(F) < \infty$ *for all Gorenstein flat R-modules F.*

(2) $\mathrm{Gfd}(M) \leq \mathrm{Gfd}^*(M) \leq \mathrm{pd}(M)$ *for any R-module M. Furthermore, if R is left perfect, then*

  (a) $\mathrm{Gfd}(M) = \mathrm{Gfd}^*(M)$ *if* $\mathrm{Gfd}^*(M) < \infty$.

  (b) $\mathrm{Gfd}(M) = \mathrm{Gfd}^*(M) = \mathrm{pd}(M)$ *if* $\mathrm{pd}(M) < \infty$.

Proof. (1). The sufficiency follows from Theorem 4.14 and Remark 4.15.

Necessity. Since $(\mathcal{GF}, \mathcal{GC})$ and $({}^\perp\mathcal{FGC}, \mathcal{FGC})$ are both cotorsion pairs, we easily get $\mathcal{GF} = {}^\perp\mathcal{FGC}$ by hypothesis and hence $\mathrm{Gfd}^*(F) = 0 < \infty$ for any Gorenstein flat R-module F.

(2). $\mathrm{Gfd}(M) \leq \mathrm{Gfd}^*(M) \leq \mathrm{pd}(M)$ are obvious. (a) holds by Theorem 4.14.

For (b), we claim that if an R-module is Gorenstein flat, then it is Gorenstein projective. Let F be a Gorenstein flat R-module. Note that R is left perfect, then we get an exact sequence of projective R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $F = \ker(P^0 \to P^1)$ such that $E \otimes_R -$ is exact for any injective right R-module E. For any projective R-module Q, Q^+ is right injective, then

$$\mathrm{Ext}_R^i(F, Q^{++}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_i^R(Q^+, F), \mathbb{Q}/\mathbb{Z}) = 0$$

for all $i \geq 1$ by [7, Theorem 3.2.1] and [11, Theorem 3.6]. Since

$$0 \to Q \to Q^{++} \to Q^{++}/Q \to 0$$

is a pure short exact sequence, Q^{++}/Q is flat by [13, Corollary 4.86] and hence projective. This sequence splits and so Q is a direct summand of Q^{++}. We get $\mathrm{Ext}_R^i(F, Q) = 0$ for all $i \geq 1$ and then F is Gorenstein projective by [11, Proposition 2.3]. Thus (b) follows. □

Proposition 4.17. *If R is an n-FC ring with $n \geq 0$, then the following are equivalent:*

(1) $\mathrm{wD}(R) < \infty$.
(2) *Every finitely presented Gorenstein flat R-module is projective.*
(3) *Every R-module is FP-Gorenstein cotorsion.*
(4) *Every quotient of an FP-Gorenstein cotorsion R-module is FP-Gorenstein cotorsion.*
(5) *Every submodule of an FP-Gorenstein cotorsion R-module is FP-Gorenstein cotorsion.*
(6) *The left/right symmetric of* (1) ∼ (5).

Proof. (1) ⇒ (2). Since $\mathrm{fd}(M) = 0$ or ∞ for any Gorenstein flat R-module M, M is flat by hypothesis. Hence every finitely presented Gorenstein flat R-module is projective.

(2) ⇒ (3) is trivial.

(3) ⇔ (4) ⇔ (5) hold by Theorems 3.1 and 4.3.

(3) ⇒ (1). Since ${}^{\perp}\mathcal{FGC} \subseteq \mathcal{GF}$, we easily get every finitely presented Gorenstein flat R-module is projective by hypothesis. For a Gorenstein flat R-module F, $F = \varinjlim G_i$ for some direct system $((G_i), (f_{ji}))$ by [5, Theorem 5], where each G_i is finitely presented Gorenstein flat. Note that each G_i is projective and hence $F = \varinjlim G_i$ is flat, then $\mathrm{wD}(R) < \infty$ by [5, Theorem 13].

(1) ⇔ (6). The proofs are similar to those of (1) ∼ (5). □

Proposition 4.18. *Let R be a commutative coherent ring and M an R-module. Then the following are equivalent:*

(1) $M \in \mathcal{FGC}$.
(2) $\mathrm{Hom}_R(P, M) \in \mathcal{FGC}$ *for any projective R-module P.*
(3) $G \otimes_R M \in \mathcal{FGC}$ *for any flat R-module G.*

Proof. (1) $\Rightarrow$ (2). Let P be a projective R-module and F a finitely presented Gorenstein flat R-module. Then there exists another projective R-module Q such that $P \oplus Q = R^{(X)}$ for some set X. So we have

$$\begin{aligned} \mathrm{Ext}^1_R(F,\ \mathrm{Hom}_R(P \oplus Q, M)) &\cong\ \mathrm{Ext}^1_R(F,\ \mathrm{Hom}_R(R^{(X)}, M)) \\ &\cong\ \mathrm{Ext}^1_R(F, (\ \mathrm{Hom}_R(R, M))^X) \\ &\cong (\ \mathrm{Ext}^1_R(F, M))^X \\ &= 0. \end{aligned}$$

Hence $\mathrm{Hom}_R(P, M) \in \mathcal{FGC}$ by Proposition 2.2.

(1) $\Rightarrow$ (3). Let G be a flat R-module. Then $G = \varinjlim F_i$ for some direct system $((F_i), (f_{ji}))$, where each F_i is a free R-module. For any finitely presented Gorenstein flat R-module F, we have

$$\begin{aligned} \mathrm{Ext}^1_R(F, G \otimes_R M) &\cong\ \mathrm{Ext}^1_R(F, \varinjlim F_i \otimes_R M) \\ &\cong\ \mathrm{Ext}^1_R(F, \varinjlim(F_i \otimes_R M)) \\ &\cong \varinjlim\ \mathrm{Ext}^1_R(F, F_i \otimes_R M) \\ &\cong \varinjlim\ \mathrm{Ext}^1_R(F, M^{(X)}) \\ &= 0. \end{aligned}$$

The second isomorphism holds since $- \otimes_R -$ commutes with $\varinjlim$, the third follows by [10, Lemma 3.1.6] and the fourth holds since $\mathcal{FGC}$ is closed under direct sums. Hence $G \otimes_R M$ is FP-Goresntein cotorsion.

(2) $\Rightarrow$ (1) holds by letting $P = R$ and (3) $\Rightarrow$ (1) holds by letting $G = R$. □

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