Linear operators that preserve term ranks of matrices over semirings*

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Abstract

The term rank of a matrix $A$ is the least number of lines (rows or columns) needed to include all the nonzero entries in $A$, and is a well-known upper bound for many standard and non-standard matrix ranks, and is one of the most important combinatorially. In this paper, we obtain a characterization of linear operators that preserve term ranks of matrices over antinegative semirings. That is, we show that a linear operator $T$ on a matrix space over antinegative semirings preserves term rank if and only if $T$ preserves any two term ranks $k$ and $l$ if and only if $T$ strongly preserves any one term rank $k$.

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1 Introduction

A semiring [2] is a set $S$ equipped with two binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative monoid with identity element 0 and $(S, \cdot)$ is a monoid with identity element 1. In addition, the operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates $S$.

A semiring $S$ is called antinegative if 0 is the only element to have an additive inverse. The following are some examples of antinegative semirings which occur in combinatorics. Let $B = \{0, 1\}$. Then $(B, +, \cdot)$ is an antinegative semiring (the binary

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**Boolean semiring** if arithmetic in $\mathbb{B}$ follows the usual rules except that $1 + 1 = 1$. If $\mathbb{F}$ is the real interval $[0, 1]$, then $(\mathbb{F}, +, \cdot) = (\mathbb{F}, \max, \min)$ is an antinegative semiring (the fuzzy semiring). Any nonnegative subring of the real numbers, such as the nonnegative integers, is an antinegative semiring.

Hereafter, $\mathbb{S}$ will denote an arbitrary commutative and antinegative semiring. Let $\mathcal{M}_{m,n}(\mathbb{S})$ be the set of all $m \times n$ matrices with entries in a semiring $\mathbb{S}$. Algebraic operations on $\mathcal{M}_{m,n}(\mathbb{S})$ are defined as if the underlying scalars were in a field.

The *term rank*, $\tau(A)$, of a matrix $A$ is the minimal number $k$ such that all the nonzero entries of $A$ are contained in $h$ rows and $k - h$ columns. Term rank plays a central role in combinatorial matrix theory and has many applications in network and graph theory (see [4]).

From now on we will assume that $2 \leq m \leq n$ unless specified otherwise. It follows that $1 \leq \tau(A) \leq m$ for all nonzero $A \in \mathcal{M}_{m,n}(\mathbb{S})$.

Let $T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S})$ be a linear operator. If $f$ is a function defined on $\mathcal{M}_{m,n}(\mathbb{S})$, then $T$ preserves the function $f$ if $f(T(A)) = f(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{S})$. If $\mathcal{X}$ is a subset of $\mathcal{M}_{m,n}(\mathbb{S})$, then $T$ preserves the set $\mathcal{X}$ if $A \in \mathcal{X}$ implies $T(A) \in \mathcal{X}$. Further, $T$ strongly preserves the set $\mathcal{X}$ if $A \in \mathcal{X}$ if and only if $T(A) \in \mathcal{X}$. There are many papers on linear operators that preserve matrix functions over $\mathbb{S}$ ([1]-[3], [5]-[6]). But there are few papers on term-rank preservers of matrices over semirings. Beasley and Pullman ([2]) have characterized linear operators on $\mathcal{M}_{m,n}(\mathbb{S})$ that preserve term rank, and the following are main results of their work: for a linear operator $T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S})$,

\begin{align*}
T \text{ preserves term rank if and only if } T \text{ preserves term ranks } 1 \text{ and } 2; \quad & (1.1) \\
T \text{ preserves term rank if and only if } T \text{ strongly preserves term rank } 1 \text{ or } m. \quad & (1.2)
\end{align*}

Kang and Song ([5]) also have characterized linear operators on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserve term rank, and the following are main results of their work: for a linear operator $T$ on $\mathcal{M}_{m,n}(\mathbb{B})$,

\begin{align*}
T \text{ preserves term rank if and only if } T \text{ preserves term ranks } 1(\text{or } 2) \text{ and } h, \quad & (1.3) \\
\text{for some } 3 \leq h \leq m.
\end{align*}

In this paper, we extend their works to any two term rank preservers.

A sectional summary is as follows: Some definitions and preliminaries are presented in section 2. Section 3 generalizes (1.1)~(1.3) by showing that $T$ preserves the term rank of Boolean matrices if and only if $T$ preserves term ranks $k$ and $l$, where $1 \leq k < l \leq m \leq n$. In section 4, we show that $T$ preserves the term rank of
matrices over antinegative semiring $\mathcal{S}$ if and only if $T$ preserves any two term ranks if and only if $T$ strongly preserves any one term rank.

2 Preliminaries

The matrix $I_n$ is the $n \times n$ identity matrix, $O_{m,n}$ is the $m \times n$ zero matrix, and $J_{m,n}$ is the $m \times n$ matrix all of whose entries are 1. We will suppress the subscripts on these matrices when the orders are evident from the context and we write $I$, $O$ and $J$, respectively. Let $E_{i,j}$ be the $m \times n$ matrix whose $(i,j)$th entry is 1 and whose other entries are all 0, and we call $E_{i,j}$ a cell.

The following is obvious by the definition of term rank of matrices over antinegative semirings.

**Lemma 2.1.** For matrices $A$ and $B$ in $\mathcal{M}_{m,n}(\mathcal{S})$, we have $\tau(A + B) \leq \tau(A) + \tau(B)$ and $\tau(A) \leq \tau(A + B)$.

If $A$ and $B$ are matrices in $\mathcal{M}_{m,n}(\mathcal{S})$, we say that $B$ dominates $A$ (written $A \sqsubseteq B$ or $B \sqsupseteq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all $i$ and $j$. This provides a reflexive and transitive relation on $\mathcal{M}_{m,n}(\mathcal{S})$.

The following is also obvious by the definition of term rank of matrices over antinegative semirings.

**Lemma 2.2.** For matrices $A$ and $B$ in $\mathcal{M}_{m,n}(\mathcal{S})$, $A \sqsubseteq B$ implies that $\tau(A) \leq \tau(B)$.

As usual, for any matrix $A$ and lists $L_1$ and $L_2$ of row and column indices respectively, $A(L_1 \mid L_2)$ denotes the submatrix formed by omitting the rows $L_1$ and columns $L_2$ from $A$ and $A[L_1 \mid L_2]$ denotes the submatrix formed by choosing the rows $L_1$ and columns $L_2$ from $A$.

For matrices $A$ and $B$ in $\mathcal{M}_{m,n}(\mathcal{S})$, the matrix $A \circ B$ denotes the Hadamard or Schur product. That is, the $(i,j)$th entry of $A \circ B$ is $a_{i,j}b_{i,j}$. A nonzero $s \in \mathcal{S}$ is a zero divisor if $s's = 0$ for some nonzero $s' \in \mathcal{S}$.

If $P$ and $Q$ are permutation matrices and $B$ is a matrix in $\mathcal{M}_{m,n}(\mathcal{S})$ none of whose entries is a zero divisor or zero, then an operator $T : \mathcal{M}_{m,n}(\mathcal{S}) \to \mathcal{M}_{m,n}(\mathcal{S})$ is called a $(P, Q, B)$-operator if $T(X) = P(X \circ B)Q$ for all $X$, or $m = n$ and $T(X) = P(X^t \circ B)Q$ for all $X$, where $X^t$ denotes the transpose of $X$. If $B = J$ we say that $T$ is a $(P, Q)$-operator.
3 A characterization of term rank preservers of Boolean matrices.

For a linear operator $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$, we say that $T$

1. preserves term rank $k$ if $\tau(T(X)) = k$ whenever $\tau(X) = k$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$;

2. strongly preserves term rank $k$ if $\tau(T(X)) = k$ if and only if $\tau(X) = k$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$;

3. preserves term rank if it preserves term rank $k$ for every $k(\leq m)$.

In this section we provide characterizations of linear operators $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ that preserve term ranks $k$ and $l$, where $1 \leq k < l \leq m \leq n$.

**Lemma 3.1.** Let $1 \leq k \leq m \leq n$. Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves term rank $k$. If $A \in \mathcal{M}_{m,n}(\mathbb{B})$ and $\tau(A) \leq k$ then $\tau(T(A)) \leq k$.

**Proof.** If $\tau(A) = k$, then $\tau(T(A)) = k$ since $T$ preserves term rank $k$. Suppose that $\tau(A) = h < k$, and $\tau(T(A)) > k$. Then there exist a matrix $B$ such that $\tau(A + B) = k$ and hence $\tau(T(A + B)) = k$, but by Lemma 2.1, $\tau(T(A + B)) = \tau(T(A) + T(B)) \geq \tau(T(A)) > k$, a contradiction. Thus $\tau(T(A)) \leq k$. ■

**Theorem 3.2.** [5, Lemma 2.5] Let $1 < k \leq m \leq n$. Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves term rank $1$ and term rank $k$. Then $T$ is a $(P, Q)$-operator.

**Lemma 3.3.** Let $1 \leq k < l \leq m \leq n$. If $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is a linear operator that preserves term rank $k$ and term rank $l$, then $T$ is nonsingular.

**Proof.** Suppose that $T(A) = O$ for some nonzero $A \in \mathcal{M}_{m,n}(\mathbb{B})$. Then, there is some cell $E_{i,j}$ such that $T(E_{i,j}) = O$. Without loss of generality, assume that $T(E_{1,1}) = O$. Since $\tau(E_{1,1} + E_{2,2} + \cdots + E_{l,l}) = l$ and $T$ preserves term rank $l$, we have $\tau(T(E_{2,2} + E_{3,3} + \cdots + E_{l,l})) = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{l,l})) = l$. Let $X = T(E_{2,2} + \cdots + E_{l,l})$ then we can choose a set of cells $Y = \{F_1, F_2, \cdots, F_l\}$ such that $X \supseteq F_i$ for all $i = 1, \cdots, l$, and $\tau(F_1 + F_2 + \cdots + F_l) = l$. Since $T(E_{2,2} + \cdots + E_{l,l}) = X$, there is some cell in $\{E_{2,2}, \cdots, E_{l,l}\}$ whose image under $T$ dominates two cells in $Y$. Without loss of generality, we may assume that $T(E_{2,2}) \supseteq F_1 + F_2$. Further, there is some cell in $\{E_{2,2}, \cdots, E_{l,l}\}$ whose image under $T$ dominates $F_3$. Without loss of generality, we may assume that $T(E_{2,2} + E_{3,3}) \supseteq F_1 + F_2 + F_3$. Continuing this process, there are $k$ cells in $\{E_{2,2}, \cdots, E_{l,l}\}$ the sum of whose images under
Let $T$ dominate $F_1 + F_2 + \cdots + F_{k+1}$. Without loss of generality, we may assume that $T(E_{2,2} + \cdots + E_{k+1,k+1}) \supseteq F_1 + F_2 + \cdots + F_{k+1}$, a contradiction since $\tau(E_{2,2} + \cdots + E_{k+1,k+1}) = k$ and $\tau(F_1 + F_2 + \cdots + F_{k+1}) = k + 1$. This contradiction establishes the lemma.

**Lemma 3.4.** Let $2 < k \leq m \leq n$ and $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves term rank $k$. If $T$ does not preserve term rank 1, then $\tau(T(J)) \leq k + 2$.

**Proof.** By Lemma 3.3, if $T$ does not preserve term rank 1, then there is some rank 1 matrix whose image has term rank 2 or more. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \supseteq E_{1,1} + E_{2,2}$.

Suppose that $\tau(T(J)) \geq k + 3$. Then, $\tau(T(J)[3, \cdots , m|3, \cdots , n]) \geq k - 1$. Without loss of generality, we may assume that $T(J)[3, \cdots , m|3, \cdots , n] \supseteq E_{3,3} + E_{4,4} + \cdots + E_{k+1,k+1}$. Thus, there are $k - 1$ cells, $F_3, F_4, \cdots, F_{k+1}$ such that $T(F_3 + F_4 + \cdots + F_{k+1}) \supseteq E_{3,3} + E_{4,4} + \cdots + E_{k+1,k+1}$. Then, $T(E_{1,1} + E_{1,2} + F_3 + F_4 + \cdots + F_{k+1}) \supseteq I_{k+1}$. But, $\tau(E_{1,1} + E_{1,2} + F_3 + F_4 + \cdots + F_{k+1}) \leq k$ while $\tau(T(E_{1,1} + E_{1,2} + F_3 + F_4 + \cdots + F_{k+1})) \geq k + 1$, a contradiction. Thus, $\tau(T(J)) \leq k + 2$.

**Lemma 3.5.** Let $1 \leq k, k + 3 \leq l \leq m \leq n$. Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves term rank $k$ and term rank $l$, then $T$ preserves term rank 1.

**Proof.** By Lemma 3.3, $T$ is nonsingular. Let $A$ be a term rank 1 matrix and suppose that $T(A)$ is not term rank 1. Then, $A$ is dominated by a row or column and the image of the sum of two cells in that line has term rank at least two. Without loss of generality, we may assume that $T(E_{1,1} + E_{1,2}) \supseteq E_{1,1} + E_{2,2}$. Now, by Lemma 3.4, if $B = T(C)$ is in the image of $T$, $\tau(B) \leq k + 2 < l$. But if we take $B = T(I_l)$, then $T(I_l)$ must have term rank $l$, a contradiction.

That is, $\tau(T(A)) \leq 1$. Since $T$ is nonsingular and $A$ was an arbitrary term rank 1 matrix, $T$ preserves term rank 1.

**Lemma 3.6.** Let $1 \leq k \leq m \leq n$. If $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is a linear operator that preserves term rank $k$ and term rank $k + 2$, then $T$ strongly preserves term rank $k + 1$.

**Proof.** Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$.

Case 1. Suppose that $\tau(A) = k + 1$ and $\tau(T(A)) \geq k + 2$. Let $A_1, A_2, \cdots, A_{k+1}$ be matrices of term rank 1 such that $A = A_1 + A_2 + \cdots + A_{k+1}$. Without loss of generality we may assume that $T(A) \supseteq E_{1,1} + E_{2,2} + \cdots + E_{k+2,k+2}$ and, since the image of some $A_i$ must have term rank at least 2, we may assume that $\tau(T(A_i) +
Let we may assume that \( T_2 \), By Lemma 3.3 suppose that \( \tau(A) = k + 1 \), \( \tau(T(A_1 + A_2 + \cdots + A_k)) \geq k + 1 \), a contradiction. Thus if \( \tau(A) = k + 1 \), \( \tau(T(A)) \leq k + 1 \).

Case 2. Suppose that \( \tau(A) = k + 1 \) and \( \tau(T(A)) = s \leq k \). Without loss of generality, we may assume that \( A = E_{1,1} + E_{2,2} + \cdots + E_{k+1,k+1} \) and \( T(A) \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{s,s} \). Then there are \( s \) members of \( \{ T(E_{1,1}), T(E_{2,2}), \ldots, T(E_{k+1,k+1}) \} \) whose sum dominates \( E_{1,1} + E_{2,2} + \cdots + E_{s,s} \). Say, without loss of generality, that \( T(E_{1,1}) + E_{2,2} + \cdots + E_{s,s} \) \( \sqsupseteq E_{1,1} + E_{2,2} + \cdots + E_{s,s} \). Now, \( \tau(A + E_{k+2,k+2}) = k + 2 \) so that \( \tau(T(A + E_{k+2,k+2})) = k + 2 \). But since \( \tau(T(A + E_{k+2,k+2})) = \tau((T(A) + T(E_{k+2,k+2}))) = \tau(T(E_{k+2,k+2})) \), it follows that \( \tau(T(E_{k+2,k+2})) \geq k + 2 - s \) and there are \( s \) members of \( \{ T(E_{1,1}), T(E_{2,2}), \ldots, T(E_{k+1,k+1}) \} \) whose sum together with \( T(E_{k+2,k+2}) \) has term rank \( k + 2 \), say \( \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2})) = k + 2 \). Since \( s \leq k \), \( \tau(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2}) \leq k + 1 \) and \( \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{s,s} + E_{k+2,k+2})) = k + 2 \). By Case 1, we again arrive at a contradiction.

Therefore \( T \) strongly preserves term rank \( k + 1 \).

**Lemma 3.7.** Let \( 1 \leq k < m \leq n \). If \( \tau(E_{1,1} + \cdots + E_{k,k} + A) \geq k + 1 \) and \( A[k + 1, \cdots, m][k + 1, \cdots, n] = O \), then there is some \( i, 1 \leq i \leq k \), such that \( \tau(E_{1,1} + \cdots + E_{i-1,i-1} + E_{i+1,i+1}, \cdots E_{k,k} + A) \geq k + 1 \).

**Proof.** Suppose that \( B = E_{1,1} + \cdots + E_{k,k} + A \) and \( \tau(B) \geq k + 1 \). Then there are \( k + 1 \) cells \( F_1, F_2, \cdots, F_{k+1} \) such that \( B \supseteq F_1 + F_2 + \cdots + F_{k+1} \) and \( \tau(F_1 + F_2 + \cdots + F_{k+1}) = k + 1 \). If \( F_1 + F_2 + \cdots + F_{k+1} \supseteq I_k \oplus O \) then one cell \( F_i \) must be a cell \( E_{p,q} \) where \( p, q \geq k + 1 \), which contradicts the assumption \( A[k + 1, \cdots, m][k + 1, \cdots, n] = O \). Thus \( F_1 + F_2 + \cdots + F_{k+1} \) does not dominate \( I_k \oplus O \). That is, there is some \( i, 1 \leq i \leq k \), such that \( \tau(E_{1,1} + \cdots + E_{i-1,i-1} + E_{i+1,i+1}, \cdots E_{k,k} + A) \geq k + 1 \).

**Lemma 3.8.** Let \( 2 \leq k + 1 \leq m \leq n \). If \( T : M_{m,n}(\mathbb{B}) \rightarrow M_{m,n}(\mathbb{B}) \) is a linear operator that preserves term rank \( k \) and term rank \( k + 1 \), then \( T \) preserves term rank \( 1 \).

**Proof.** By Lemma 3.3 \( T \) is nonsingular.

If \( k + 1 = m \), then \( T \) strongly preserves term rank \( m \) and hence, by [2, Theorem 2], \( T \) preserves term rank, and hence, term rank \( 1 \). Thus we assume that \( k + 1 < m \).

Suppose that \( T \) does not preserve term rank \( 1 \). Then there is some matrix of term rank \( 1 \) whose image has term rank at least \( 2 \). Without loss of generality, we may assume that \( T(E_{1,1} + E_{1,2}) \supseteq E_{1,1} + E_{2,2} \). By Lemma 3.4 we have that \( \tau(T(J)) \leq k + 2 \). Since \( T \) preserves term rank \( k + 1 \), \( \tau(T(J)) \geq k + 1 \).

Thus, \( \tau(T(J)) = k + i \) for either \( i = 1 \) or \( i = 2 \). Now, we may assume that for some \( r, s \) with \( r + s = k + i \), \( T(J)[r + 1, \cdots, p][s + 1, \cdots, q] = O \). Further, we may
assume, without loss of generality, that there are \( k+i \) cells \( F_1, F_2, \ldots, F_{k+i} \) such that \( T(F_1) \supseteq E_{l,k+i-1+l} \) for \( l = 1, \ldots, k+i \). Suppose the image of one of the cells in \( F_1, F_2, \ldots, F_{k+i} \) dominates more than one cell in \( \{E_{1,k+i}, E_{2,k+i-1}, \ldots, E_{k+1,i}\} \). Say, without loss of generality, that \( T(F_1) \supseteq E_{1,k+i} + E_{2,k+i-1} \), then, \( T(F_1 + F_3 + \cdots + F_{k+1}) \supseteq E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k+1,i} \), a contradiction since \( \tau(F_1 + F_3 + \cdots + F_{k+1}) \leq k \), and hence \( \tau(T(F_1 + F_3 + \cdots + F_{k+1})) \leq k \), and \( \tau(E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k+1,i}) = k+1 \). It follows that for each \( j = 1, \ldots, k+1 \), if \( T(F_j) \supseteq E_{j,k+i-j+1} \) then \( j = j \) since \( T(F_j) \supseteq E_{j,k+i-j+1} \) is unique. Further, by permuting we may assume that \( F_1 + F_2 + \cdots + F_k \equiv \begin{bmatrix} J_k & O_{k,n-k} \\ O_{m-k,k} & O_{m-k,n-k} \end{bmatrix} \).

Now, let \( O \neq E \in \mathcal{M}_{m,n}(\mathbb{B}) \) have term rank 1, and suppose that \( A[1,2,\ldots,k][1,2,\ldots,n] = O \) and \( A[K = 1, \ldots, m][1, \ldots, k] = O \). So that \( A = \begin{bmatrix} O_k & O_{k,n-k} \\ O_{m-k,k} & A_{11} \end{bmatrix} \). If \( T(A)[k+1, \ldots, p][1, i] = O \), then, since \( \tau(F_1 + \cdots + F_k + A) = k+1, \tau(T(F_1 + \cdots + F_k + A)) = k+1 \). Applying Lemma 3.7, we have that there is some \( j \) such that \( \tau(T(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A)) = k+1 \). But \( \tau(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A) = k \) while \( \tau(T(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + A)) = k+1 \), a contradiction. So we must have that \( T(E_{k+1,1})[k+1, \ldots, p][1, i] \neq O \). If \( T(E_{k+1,1})[k+1, \ldots, p][1, i] \neq O \) then \( \tau(T(F_1 + \cdots + F_k + E_{k+1,1})) = k+1 \), a contradiction since \( \tau(F_1 + \cdots + F_k + E_{k+1,1}) = k \). Suppose that the \((k,i+1)\) entry of \( T(E_{k+1,1}) \) is nonzero, then, \( \tau(T(F_1 + \cdots + F_k + E_{k+1,1})) = k+1 \), a contradiction, since \( \tau(F_1 + \cdots + F_k + E_{k+1,1}) = k \).

Consider \( T(F_1 + \cdots + F_{k-1} + E_{k,k+1} + E_{k+1,k+2}) \). This must have term rank \( k+1 \) and dominates \( E_{1,k+i} + E_{2,k+i-1} + \cdots + E_{k-1,i+2} + E_{k+1,j} \) for some \( j \in \{1, i\} \). Thus, \( T(E_{k+1,1}) \) must have some cell \( F_j \) such that \( \tau(T(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + E_{k+1,1})) = k+1 \). But \( \tau(F_1 + \cdots + F_{j-1} + F_{j+1} + \cdots + F_k + E_{k+1,1}) = k \), a contradiction.

It follows that \( T \) must preserve term rank 1.

**Lemma 3.9.** Let \( 1 \leq k \leq m \leq n \). If \( T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B}) \) is a linear operator that strongly preserves term rank \( k \), then \( T \) preserves term rank \( 1 \).

**Proof.** Suppose that \( T \) does not preserve term rank \( k-1 \). The there is some matrix \( A \in \mathcal{M}_{m,n}(\mathbb{B}) \) such that \( \tau(A) = k-1 \) and \( \tau(T(A)) = s \leq k-2 \) since \( \tau(T(A)) < k \). Without loss of generality, we may assume that \( A = E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1} \). Then \( T(E_{k,k}) \) must have term rank at least \( k-s \) since using Lemma 2.1 \( k = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1} + E_{k,k})) = \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1}) + T(E_{k,k})) \leq \tau(T(E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1})) + \tau(T(E_{k,k})) = s + \tau(T(E_{k,k})) \) without loss of generality, we also assume that \( T(E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1} + E_{k,k}) \supseteq E_{1,1} + E_{2,2} + \cdots + E_{k-1,k-1} + E_{k,k} \) and \( T(E_{k,k}) \supseteq E_{s+1,s+1} + \cdots + E_{k,k} \). But then, there
are \( s \) cells in \( \{ E_{1,1}, E_{2,2}, \ldots, E_{k,k} \} \) the image of whose sum dominates \( E_{1,1} + E_{2,2} + \cdots + E_{s,s} \). Say \( T(E_{i_1,i_1} + E_{i_2,i_2} + \cdots + E_{i_s,i_s}) \supseteq E_{1,1} + E_{2,2} + \cdots + E_{s,s} \). Then,
\[
T(E_{i_1,i_1} + E_{i_2,i_2} + \cdots + E_{i_s,i_s} + E_{k,k}) \supseteq E_{1,1} + E_{2,2} + \cdots + E_{k,k},
\]
a contradiction since \( \tau(E_{i_1,i_1} + E_{i_2,i_2} + \cdots + E_{i_s,i_s} + E_{k,k}) < k \) while \( \tau(T(E_{i_1,i_1} + E_{i_2,i_2} + \cdots + E_{i_s,i_s} + E_{k,k})) = k \).
Thus \( T \) preserves term rank \( k - 1 \). By Lemma 3.8, \( T \) preserves term rank 1.

**Theorem 3.10.** Let \( 1 \leq k < l \leq m \leq n \). If \( T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S}) \) is a linear operator that preserves term rank \( k \) and term rank \( l \), or if \( T \) strongly preserves term rank \( k \), then \( T \) is a \((P,Q)\)-operator.

**Proof.** Application of Lemmas 3.5, 3.6, 3.8, and 3.9, give that \( T \) preserves term rank 1. By Lemma 3.2, \( T \) is a \((P,Q)\)-operator.

## 4 Term rank preservers of matrices over antinegative semirings.

Throughout this section, \( \mathbb{S} \) will denote any commutative and antinegative semiring without zero divisors.

In this section we provide characterizations of linear operators \( T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S}) \) that preserve term rank. Let \( A \in \mathcal{M}_{m,n}(\mathbb{S}) \) and define \( \overline{A} \in \mathcal{M}_{m,n}(\mathbb{B}) \) to be the matrix \([\overline{a}_{i,j}]\) where \( \overline{a}_{i,j} = 1 \) if and only if \( a_{i,j} \neq 0 \). \( \overline{A} \) is called the support or pattern of \( A \). Clearly \( \tau(\overline{A}) = \tau(A) \). Let \( T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S}) \) be a linear operator. Define \( \overline{T} : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B}) \) by \( \overline{T}(E_{i,j}) = \overline{T(E_{i,j})} \), and extend linearly.

**Lemma 4.1.** Let \( T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S}) \) be a linear operator. Then \( T \) preserves term rank \( k \) if and only if \( \overline{T} \) preserves term rank \( k \), for any \( 1 \leq k \leq m \).

**Proof.** Since the term rank of a matrix depends only upon the entries being zero or nonzero, we have \( \tau(\overline{A}) = \tau(A) \). Further, \( \overline{T}(\overline{A}) = \overline{T(A)} \). Thus, \( \tau(T(A)) = \tau(\overline{T(A)}) \), and hence the results follows.

**Theorem 4.2.** Let \( T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S}) \) be a linear operator. Then the following are equivalent:

1. \( T \) preserves term rank;
2. \( T \) preserves term rank \( k \) and term rank \( l \), with \( 1 \leq k < l \leq m \leq n \);
3. \( T \) strongly preserves term rank \( h \), with \( 1 \leq h \leq m \leq n \);
4. $T$ is a $(P, Q, B)$-operator.

Proof. It is obvious that 1 implies 2 and 3, and 4 implies 1, 2 and 3. In order to show that 2 (or 3) implies 4, assume that $T$ preserves term rank $k$ and term rank $l$, with $1 \leq k < l \leq m \leq n$. By Lemma 4.1, $T$ preserves term rank $k$ and term rank $l$, with $1 \leq k < l \leq m \leq n$. Thus, by Theorem 3.10, $T$ is a $(P, Q)$-operator. It follows that for every cell $E_{i,j}$, there is some nonzero $b_{i,j} \in S$ such that for $B = [b_{i,j}]$ either

1. $T(E_{i,j}) = b_{i,j}(PE_{i,j}Q)$, and

$$T(X) = T \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j}E_{i,j} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j}T(E_{i,j})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j}b_{i,j}(PE_{i,j}Q) = P(X \circ B)Q$$

for every $X \in M_{m,n}(S)$; or

2. $m = n$ and $T(E_{i,j}) = b_{i,j}(PE_{i,j}^tQ)$, and

$$T(X) = T \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j}E_{i,j} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j}T(E_{i,j})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j}b_{i,j}(PE_{i,j}^tQ) = P(X^t \circ B)Q$$

for every $X \in M_{m,n}(S)$.

Thus, $T$ is a $(P, Q, B)$-operator.

In order to show that 3 implies 4, if we apply Lemma 4.1 and Theorem 3.10, the proof is parallel to the above.

In conclusion, we have obtained new characterizations of the linear operators that preserve term rank, which generalized the previous results in [2] and [5]. Further research should include application of these results to characterize linear transformations between the set of $m \times n$ matrices and $p \times q$ matrices.

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