BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(2) (2012), 459–467

On the Sylow Normalizers of Some Simple Classical Groups

N. AHANJIDEH AND A. IRANMANESH
Department of Mathematics, Tarbiat Modares University, P. O. Box: 14115-137, Tehran, Iran
n_ahanjideh@yahoo.com, iranmana@modares.ac.ir

Abstract. Let G be a finite group and $\pi(G)$ be the set of prime divisors of the order of G. For $t \in \pi(G)$ denote by $n_t(G)$ the order of a normalizer of t-Sylow subgroup of G and put $n(G) = \{n_t(G) : t \in \pi(G)\}$. In this paper, we give an answer to the following problem, for the groups of Lie type B_n, C_n and D_n: "Let L be a finite non-abelian simple group and G be a finite group with $n(L) = n(G)$. Is it true that $L \cong G$?" In this paper, we find the first examples of non-abelian finite simple groups which are not isomorphic and they have the same set of orders of Sylow normalizers and hence, we show that the question above is not correct always. Let $\mathscr{A}$ be the set of prime numbers of order $2n$, $2(n-1)$ and $2(n-2)$ mod q. The latter condition is necessary if $n \geq 5$. Also, we show that $D_{n+1}(q)$ is determined uniquely by its order and $\{n_t(D_{n+1}(q)) : t \in \mathscr{A} \cup \{2\}\}$ and if $n = 2$ or $q \not\equiv \pm 1 \pmod 8$, then $B_n(q)$ and $C_n(q)$ are characterizable by their orders and orders of t-Sylow normalizers, where $t \in \mathscr{A} \cup \{2\}$. If $n \geq 3$ and $q \equiv \pm 1 \pmod 8$, then $B_n(q)$ and $C_n(q)$ are 2-characterizable by their orders and the orders of t-Sylow normalizers, where $t \in \mathscr{A} \cup \{2\}$.

2010 Mathematics Subject Classification: 20D06, 20D20, 20G40, 20C33

Keywords and phrases: Sylow subgroup, classical groups, simple group of Lie type, characterization.

1. Introduction

For a finite group K, let $\pi(K)$ and $n_t(K)$ denote the set of prime divisors of $|K|$ and the order of the normalizer of a t-Sylow subgroup of K, respectively. Also, let $n(K) = \{n_t(K) : t \in \pi(K)\}$. Characterization of finite groups is one of the central themes of research in group theory. There are various characterizations of finite groups by given properties, such as the set of orders of maximal abelian subgroup or order components, etc (see [1, 13]). A finite group G is said to be characterizable by the orders of its Sylow normalizers, if G is uniquely (up to isomorphism) determined by orders of its Sylow normalizers. A group G is said to be 2-characterizable by the orders of Sylow normalizers, if there is exactly one group H (up to isomorphism) such that H is not isomorphic to G and $n(H) = n(G)$. Characterization by the orders of their Sylow normalizers were first given by Bi [2]. Some finite non-abelian simple groups are characterizable by orders of Sylow normalizers (see [3, 4, 5, 6, 14, 15, 16]). Assume that $n \geq 3$ is a natural number, q is a power of p ($q = p^k$) and $\mathscr{A}$ is a set of prime

Communicated by Shum Kar Ping.
Received: May 21, 2010; *Revised:* September 30, 2010.

numbers of order $2n$, $2(n-1)$ and $2(n-2)$ mod q. The latter condition is necessary if $n \geq 5$. The main theorem of this paper is the following:

Theorem 1.1. *Let $S_{n,q} \in \{B_n(q), C_n(q), D_{n+1}(q)\}$ and, suppose that $|G| = |S_{n,q}|$ and*

$$\{n_t(S_{n,q}) : t \in \mathscr{A} \cup \{2\}\} = \{n_t(G) : t \in \mathscr{A} \cup \{2\}\}.$$

Then, if $S_{n,q} = D_{n+1}(q)$ or $n = 2$ or $q \not\equiv \pm 1$ $(mod\ 8)$, $G \cong S_{n,q}$. Further, if $q \equiv \pm 1$ $(mod\ 8)$ and $n \geq 3$, then either $G \cong S_{n,q}$ or $\{G, S_{n,q}\} = \{B_n(q), C_n(q)\}$.

Let $M_n(q)$ be the group of all $(n \times n)$ matrices with coefficients in $GF(q)$. If m is a natural number and r is prime, then $|m|_r$ denotes the r-part of m, in fact $|m|_r = r^t$ if $r^t \| m$. Also, $\mathbb{S}_t$ is the Symmetric group of degree t and (m,n) stands for the greatest common divisor of m and n. All further unexplained notations are standard and can be found in [8], [12] and [18].

2. On the orders of Sylow normalizers of some simple classical groups

In this section, we calculate the orders of Sylow normalizers of $B_n(q)$ and $C_n(q)$. First, let $s \in \pi(B_n(q)) = \pi(C_n(q))$. When we say $\exp_s(q) = m$, we mean s divides $q^m - 1$ and s doesn't divide $q^h - 1$ for all $0 < h < m$. Since $|B_n(q)| = |C_n(q)| = q^{n^2}(q^2-1)\cdots(q^{2n}-1)/(2,q-1)$, we have $s \in \{p\} \cup \{r : \exp_r(q) = i \text{ or } 2i \text{ such that } 1 \leq i \leq n\}$.

Lemma 2.1. [9] *Let $n > 2$ be a natural number. If $p^n \neq 2^6$, then there is a prime factor r of $p^n - 1$ such that $\exp_r(p) = n$. And if p is not a Mersenne prime, then there is a prime factor r of $p^2 - 1$ such that $\exp_r(p) = 2$.*

For a given number n, we define

$$\tau(n) = \begin{cases} 2n, & \text{if } n \text{ is odd,} \\ n, & \text{if } n \text{ is even,} \end{cases}$$

and

$$\tau'(n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

Lemma 2.2. *If $\exp_s(q) = m$ such that $s \neq 2$, then*

$$\begin{aligned} &|n_s(B_n(q))| = |n_s(C_n(q))| \\ &= |Sp_{a_0}(q)|(\tau(m)(q^{\tau'(m)} + (-1)^m))^{a_1}(a_1!)\Pi_{i=1}^{t}(n_s(\mathbb{S}_{s^i})\tau(m)(q^{\tau'(m)} + (-1)^m). \\ &\quad (|q^{\tau'(m)} + (-1)^m|_s)^{s^i - 1})^{a_{i+1}}(a_{i+1}!)/(2,q-1), \end{aligned}$$

where $t \in \mathbb{N} \cup \{0\}$, $s^t\tau(m) \leq 2n < s^{t+1}\tau(m)$ and

$$2n = a_0 + a_1\tau(m) + a_2 s\tau(m) + \cdots + a_{t+1}s^t\tau(m).$$

Proof. Frattini's argument and [19] complete the proof. ∎

Corollary 2.1. *If $\exp_s(q) = 2n$, then*

$$|n_s(B_n(q))| = |n_s(C_n(q))| = 2n(q^n + 1)/(2,q-1),$$

and if $\exp_s(q) = 2(n-1)$, then

$$|n_s(B_n(q))| = |n_s(C_n(q))| = 2q(n-1)(q^2-1)(q^{n-1}+1)/(2,q-1).$$

Using the orders of Cartan subgroups of $B_n(q)$ and $C_n(q)$, we can prove the following lemma.

Lemma 2.3. *If $\bar{P} \in \mathrm{Syl}_p(C_n(q))$ and $P' \in \mathrm{Syl}_p(B_n(q))$, then $|N_{C_n(q)}(\bar{P})| = |N_{B_n(q)}(P')| = q^{n^2}(q-1)^n/(2,q-1)$.*

Corollary 2.2. *If $p \neq 2$, then $n_t(B_n(q)) = n_t(C_n(q))$ for every odd prime t.*

Again using Frattini's argument and [19], we can reach to the following results:

Lemma 2.4. *If $\exp_r(q) = 2n$, then $|N_{D_{n+1}(q)}(\bar{R})| = 2(q^n+1)(q+1)n/(q^{n+1}-1,4)$, where $\bar{R} \in \mathrm{Syl}_r(D_{n+1}(q))$.*

Lemma 2.5. *Let $\exp_{r_1}(q) = 2(n-1)$ and $\bar{R_1} \in \mathrm{Syl}_{r_1}(D_{n+1}(q))$. If $n \geq 4$, then $|N_{D_{n+1}(q)}(\bar{R_1})| = 2q^2(q^4-1)(q^{n-1}+1)(n-1)/(4,q^{n+1}-1)$. Also, $|N_{D_4(q)}(\bar{R_1})| = 8(q^2+1)^2/(4,q^4-1)$.*

Corollary 2.3. *If $n \geq 3$. If $r,\ r_1 \in \pi(S_{n,q})$ such that $\exp_r(q) = 2n$ and $\exp_{r_1}(q) = 2(n-1)$, then $r_1 \nmid n_r(S_{n,q})$ and $r \nmid n_{r_1}(S_{n,q})$.*

Proof. We claim that $r \nmid n_{r_1}(S_{n,q})$. If not, $r \mid n_{r_1}(S_{n,q})$. Since $\exp_r(q) = 2n$, we have $2n \mid r-1$. Moreover, by Lemma 2.2 and Lemma 2.5, we have $r \mid n-1$. Thus $2n < n$, a contradiction. Similar to the previous procedure, by Corollary 2.1 and Lemma 2.4, $r_1 \nmid n_r(S_{n,q})$, which completes the proof. ∎

Lemma 2.6. [17, Corollary after Theorem 3]. *Let $n \geq 3$ and $p \neq 2$. Then $n_2(B_n(q)) = |B_n(q)|_2$. If $q \equiv \pm 3 \pmod 8$, then $n_2(C_n(q)) = 3^t|C_n(q)|_2$, where the number t can be found from the 2-adic expansion $2n = 2^{s_1} + \cdots + 2^{s_t}$, $s_1 > \cdots > s_t$. Unless, $n_2(C_n(q)) = |C_n(q)|_2$.*

Corollary 2.4. *Let $n \geq 3$ and q be any prime power. Then $n(B_n(q)) = n(C_n(q))$, unless $q \equiv \pm 3 \pmod 8$.*

3. Main theorem

Lemma 3.1. *Let G be a finite non-abelian simple group and $|G|_p = p^e$. If $p^{7e/3} \leq |G| < p^{8e/3}$, then G is isomorphic to one of the following groups: $L_n(q)$ (for $2 \leq n \leq 5$ and $(n,q) \neq (5,11)$); $U_n(q)$ (for $3 \leq n \leq 5$, $(n,q) \neq (3,2)$, $(n,q) \neq (4,3)$ and $(n,q) \neq (5,4)$); $B_n(q)$ (for $n = 2$ and $(n,q) \neq (2,3)$); ${}^2B_2(q)$; ${}^2F_4'(2)$.*

Proof. [6, Lemma 2] and the orders of all finite simple groups of Lie type in characteristic p complete the proof. ∎

Remark 3.1. Let G be a finite non-abelian simple group and $|G|_p = p^e$. If $|G| < p^{7e/3}$, then G is isomorphic to a simple group of Lie type in characteristic p. Moreover G is not isomorphic to any of the groups stated in Lemma 3.1.

Lemma 3.2. [3] *Let G be a finite group, $N \lhd G$ and $R \in Syl_r(G)$. If $r \mid |G/N|$ and $r \nmid |N|$ (r is prime and $r \neq p$), and if, in addition, $p^e \parallel |N|$ and $p^t \parallel |C_N(R)|$, then $r \mid p^{e-t}-1$.*

In Lemma 3.3 and Lemma 3.4, let G be a finite group such that $n_t(G) \mid n_t(S_{n,q})$, for every $t \in \pi(G)$. Also, assume that $r,\ r_1 \in \pi(S_{n,q})$ such that $\exp_r(p) = 2nk$ and $\exp_{r_1}(p) = 2(n-1)k$. For every prime number t and a natural number m, if $\exp_t(p) = mk$, then it is obvious that $\exp_t(q) = m$. Thus we can use the results of Section 2, when $\exp_t(p) = mk$.

Lemma 3.3. *Let $S_{n,q} \in \{B_n(q), C_n(q)\}$, $N \lhd G$, r, $r_1 \in \pi(G)$ and $|N|_p = p^t$. If r, $r_1 \notin \pi(N)$, $R \in \mathrm{Syl}_r(G)$ and $R_1 \in \mathrm{Syl}_{r_1}(G)$, then $|C_N(R)|_p = |C_N(R_1)|_p = p^t$ or $|C_N(R)|_p = p^u$ and $p^{(2a-1)k}.p^u | 2(n-1)$ such that $p^u \in \{1,2\}$, $a \in \mathbb{N}$ and $t = 2nka + u$.*

Proof. Let $|C_N(R)|_p = p^u$ and $|C_N(R_1)|_p = p^v$. If $t-u=0$, we claim that $t-v=0$. If not, then Lemma 3.2 allows us to assume that

$$2(n-1) \mid t-v, \tag{3.1}$$

because $\exp_{r_1}(q) = 2(n-1)$. Recall that $n_r(G) \mid 2n(q^n+1)$ and $n_{r_1}(G) \mid 2(n-1)q(q^2-1)(q^{n-1}+1)$, considering Corollary 2.1. Since $u=t$, $n_r(G) \mid 2n(q^n+1)$ and $|C_G(R)| \mid n_r(G)$, we have $p^t \mid 2n$. Hence $t \le \log_p^{2n} \le \log_2^{2n} < 2n/2 < n$. Thus (3.1) implies that $2(n-1) < n$, which is impossible. Similarly, we can see that if $t-v=0$, then $t-u=0$. In addition, $(t-u, t-v) = (0,0)$ or there are a, $b \in \mathbb{N}$ such that $t-u = 2nka$ and $t-v = 2(n-1)kb$. If $(t-u, t-v) \neq (0,0)$, then we can consider similar to the previous argument that $v > u$, $a = b$ and $v = 2ka + u$. So, Corollary 2.1 completes the proof. ∎

Lemma 3.4. *Let $S_{n,q} = D_{n+1}(q)$, $n \ge 3$, $N \lhd G$, r, $r_1 \in \pi(G)$ and $|N|_p = p^t$. If r, $r_1 \notin \pi(N)$, $R \in \mathrm{Syl}_r(G)$ and $R_1 \in \mathrm{Syl}_{r_1}(G)$, then $|C_N(R)|_p = |C_N(R_1)|_p = p^t$ or $|C_N(R)|_p = p^u \in \{1,2\}$ and $p^{(2a-2)k}.p^u | 2(n-1)$ with $a \in \mathbb{N}$ and $t = 2nka + u$ or $u = 2$, $t = 18$ and $S_{n,q} = D_5(2)$.*

Proof. By long and easy calculation, similar to the procedure of the proof of Lemma 3.3, this lemma can be proved. ∎

Proof of Theorem 1.1. We note that $B_2(q) \cong C_2(q)$ and $D_3(q) \cong L_4(q)$. Also, $L_2(q)$, $C_2(q)$ and $L_n(q)$ have been characterized in [2, 3, 4]. Thus we may and do, assume that $n \ge 3$. Let $1 = G_0 \le G_1 \le \cdots \le G_f = G$ be a chief series of G. Let r, r_1 and r_2 be prime numbers such that $\exp_r(q) = 2n$, $\exp_{r_1}(q) = 2(n-1)$ and $\exp_{r_2}(q) = 2(n-2)$. Let $R \in \mathrm{Syl}_r(G)$ and $R_1 \in \mathrm{Syl}_{r_1}(G)$. We have the following cases:

(I) Let $S_{n,q} \neq D_{n+1}(q)$ or $n \ge 5$. If $(n,q) \neq (3,2)$, $(4,2)$, $(5,2)$, then by Lemma 2.1, $\pi(G)$ contains prime numbers r, r_1 and r_2. Let $j_0 = Max\{1 \le i \le f : r \in \pi(G_i/G_{i-1})\}$ and, fix $H := G_{j_0-1}$ and $K := G_{j_0}$. By Frattini's argument, $|G/K| = n_r(G)/n_r(K)$. According to Corollary 2.3, $r_1 \nmid n_r(G)$ and hence, $r_1 \notin \pi(G/K)$. We can repeat the same argument to show that $r_1 \in \pi(K/H)$ and if $n \ge 4$, then $r_2 \notin \pi(G/K)$ and $r_2 \in \pi(K/H)$. Since $|G|_p = |S_{n,q}|_p = p^e$ and $|G| = |S_{n,q}|$, we have $|G| < p^{7e/3}$, considering Lemma 3.1. Thus, there is $1 \le j \le f$ such that $|G_j/G_{j-1}|_p = p^{e_j} > p$ and $|G_j/G_{j-1}| < p^{7e_j/3}$. We claim that there exists $1 \le h \le f$ such that r or r_1 (or r_2, where $n \ge 4$) is an element of $\pi(G_h/G_{h-1})$, $|G_h/G_{h-1}|_p = p^{e_h} > p$ and $|G_h/G_{h-1}| < p^{5e_h/2}$. If not, then $r \notin \pi(G_j/G_{j-1})$ and $r_1 \notin \pi(G_j/G_{j-1})$ (moreover $r_2 \notin \pi(G_j/G_{j-1})$, for $n \ge 4$). Also, by Frattini's argument, $|G/K| = n_r(G)/n_r(K)$ and $|G/K| = n_{r_1}(G)/n_{r_1}(K)$. This implies that $|G/K| \mid (n_r(G), n_{r_1}(G))$ and hence, $|G/K| \mid (2q^2, 2n)$. If there exist $j_1, \cdots, j_k > j_0$ such that $|G_{j_i}/G_{j_i-1}|_p = p^{e_{j_i}} > p$ and $|G_{j_i}/G_{j_i-1}| < e^{7e_{i_j}/3}$, then $p^{e_{j_1}+\cdots+e_{j_k}} \le 2q^2$ and $\prod_{i=1}^{k} |G_{j_i}/G_{j_i-1}| > p^{2(e_{j_1}+\cdots+e_{j_k})}$. Hence, $|G|/(\prod_{i=1}^{k} |G_{j_i}/G_{j_i-1}|) \le p^{7ek/3-2(e_{j_1}+\cdots+e_{j_k})} < p^{5(ek-(e_{j_1}+\cdots+e_{j_k}))/2}$, because $e_{j_1}+\cdots+e_{j_k} < 2k+1$. Thus we may assume that $j < j_0$ and we continue the proof in the following subcases:

(i) If $p \nmid n$ and $|C_{G_j/G_{j-1}}(RG_{j-1}/G_{j-1})|_p = |C_{G_j/G_{j-1}}(R_1G_{j-1}/G_{j-1})|_p = p^{e_j}$, then $p^{e_j} \mid n_r(G)$, because

$$\begin{aligned} C_{G_j/G_{j-1}}(XG_{j-1}/G_{j-1}) &\le N_{G_j/G_{j-1}}(XG_{j-1}/G_{j-1}) \\ &\le N_{G/G_{j-1}}(XG_{j-1}/G_{j-1}) \cong N_G(X)/N_{G_{i-1}}(X), \end{aligned} \tag{3.2}$$

for every $x \in \pi(G)$ and $X \in \mathrm{Syl}_x(G)$. Note that $n_r(G) \mid 2n(q^n+1)(q+1)$, using Corollary 2.1 and Lemma 2.4. So, $p^{e_j} \mid 2n$ and hence, $p^{e_j} \mid 2$ which is a contradiction, because $e_j > 1$.

(ii) If $p \nmid n$, $|C_{G_j/G_{j-1}}(RG_{j-1}/G_{j-1})|_p \neq |C_{G_j/G_{j-1}}(R_1G_{j-1}/G_{j-1})|_p$ and $S_{n,q} \neq D_{n+1}(q)$, then applying Lemma 3.3 to G/G_{i-1} and G_i/G_{i-1} implies that $e_j = 2nka+u$ with $a \in \mathbb{N}$, $|C_{G_j/G_{j-1}}(RG_{j-1}/G_{j-1})|_p = p^u \in \{1,2\}$ and $q^{(2a-1)}p^u|2(n-1)$. Since $p^{(2a-1)k}.p^u|2(n-1)$ and $p \nmid n$, we can consider that $p \mid n-1$. Let $(n,q) \neq (4,3)$. Also, $(n,q) \notin \{(3,2),\ (5,2)\}$. Therefore, $p \mid n-1$ shows that $n \geq 6$ or $n = 5$ and $k \geq 2$. On the other hand, Lemma 3.2 allows us to assume that $r_2 \mid p^{e_j-c}-1$, where $p^c \mid |C_{G_i/G_{i-1}}(RG_{i-1}/G_{i-1})|$. Repeating the argument used for (3.2), we obtain that $c = e_j$ or $p^c \mid |n_{r_2}(G)|_p = |2q^4|_p$ (using the results appearing in [19]). In addition, we conclude that $2(n-2)k \mid 4ka-c+u$, because $\exp_{r_2}(p) = 2(n-2)k$ and $e_j = 2nka+u$. Hence

$$(3.3) \qquad 2(n-2)k \leq 4ka-c+u \quad \text{or} \quad 4ka-c+u = 0.$$

But $p^{2ka+u} \mid 2p^k(n-1)$. Hence, if p is even, then $n-1$ is even and if p is odd, then $p^{2ka+u} \mid p^k(n-1)$. Thus

$$2ka+u \leq \log_p^{2(n-1)q} < (n-1)/2+k+1.$$

So, we can consider that $2ka+u \leq (n-1)/2+k$, because $p^{2ka+u} \mid 2p^k(n-1)$. We claim that $4ka-c+u=0$. If not, then (3.3) implies that $2(n-2)k \leq 4ka-c+u$ and hence, $n \leq 3+2/(2k-1)$. Thus $n \leq 4$ or $n=5$ and $k=1$, which is a contradiction. This shows that $4ka-c+u=0$. But $p^c \mid 2q^4$ and hence $a=1$. Since $n \geq 5$, $e_j = 2nk+u$ and G_j/G_{j-1} is a direct product of some simple groups of Lie type (by Lemma 3.1), we observe that $|G_j/G_{j-1}| > p^{2e_j}$ and it is easy to see that there exists $1 \leq i \leq j_0$ such that $i \neq j$, $|G_i/G_{i-1}|_p = p^{e_i} > p$ and $|G_i/G_{i-1}| < p^{5e_i/2}$. Again, repeating the argument used for (3.2), we can see that that

$$(3.4) \qquad \prod_{k=1}^{f} |C_{G_k/G_{k-1}}(R_2G_{k-1}/G_{k-1})|_p \mid \prod_{k=1}^{f} |N_{G_k}(R_2)|/|N_{G_{k-1}}(R_2)| = n_{r_2}(G),$$

where $R_2 \in \mathrm{Syl}_{r_2}(G)$ and hence using Corollary 2.1, $\prod_{k=1}^{f} |C_{G_k/G_{k-1}}(R_2G_{k-1}/G_{k-1})|_p \mid |2q^4|_p$. So, $|C_{G_i/G_{i-1}}(R_2G_{i-1}/G_{i-1})|_p \in \{1,2\}$, because it had been considered that

$$|C_{G_j/G_{j-1}}(R_2G_{j-1}/G_{j-1})|_p = p^{4ka+u}.$$

The same argument as the case G_j/G_{j-1} guarantees that

$$|C_{G_i/G_{i-1}}(RG_{i-1}/G_{i-1})|_p = |C_{G_i/G_{i-1}}(R_1G_{i-1}/G_{i-1})|_p$$

which is impossible, considering (i). If $(n,q) = (4,3)$, then $7, 13, 41 \| \, |G|$. Moreover, $\pi(n_7(G)) = \{2,3,7\}$, $\pi(n_{13}(G)) = \{2,3,13\}$ and $\pi(n_{41}(G)) = \{2,41\}$. Hence, we can see that G_j/G_{j-1} is a k_3-simple group and hence, G_j/G_{j-1} is isomorphic to one of the following groups (see [11]):

$$A_5,\ A_6,\ L_2(7),\ L_2(8),\ L_2(17),\ L_3(3),\ U_3(3),\ U_4(2),$$

which contradicts Remark 3.1.

(iii) If $p \nmid n$, $|C_{G_j/G_{j-1}}(RG_{j-1}/G_{j-1})|_p \neq |C_{G_j/G_{j-1}}(R_1G_{j-1}/G_{j-1})|_p$ and $S_{n,q} = D_{n+1}(q)$, then by Lemma 3.4, $e_j = 2nka+u$ ($a \in \mathbb{N}$), $|C_{G_j/G_{j-1}}(RG_{j-1}/G_{j-1})|_p = p^u \in \{1,2\}$ and

$q^{(2a-2)}p^u | 2(n-1)$. But $n \geq 5$. Thus applying the results of [19] and the argument of previous case, we can see that $r_2 \mid p^{e_j - c} - 1$, where $p^c \mid \ |n_{r_2}(G)|_p$ and $|n_{r_2}(G)|_p \mid 2q^6(n-2)$. Since $e_j = 2nka + u$ and $\exp_{r_2}(p) = 2(n-2)k$, we have $2(n-2)k \mid 4ka + u - c$. So,

$$(3.5) \qquad 2(n-2)k \leq 4ka - c + u \quad \text{or} \quad 4ka - c + u = 0.$$

Also, $q^{2a}p^u \mid 2q^2(n-1)$ and hence, we observe that $2ka + u \leq (n-1)/2 + 2k$. If $2(n-2)k \leq 4ka - c + u$, we conclude that $2(n-2)k \leq (n-1) + 4k$. Hence $n - 1 \leq 3 + 3/(2k-1)$. Thus $n \leq 7$. We claim that $a = 1$. If not, then $q^2 \mid 2(n-1)$. So, $(n,q,a) = (7,2,2)$, because $(n,q) \neq (5,2)$ and $5 \leq n \leq 7$. Therefore, $2(n-2)k = 10 \mid 4ka + u - c = 8 + u - c$. It follows that $u \geq 2$ which is a contradiction, because $p^u \mid 2$. In addition, we conclude that

$$(3.6) \qquad 4ka - c + u = 0 \quad \text{or} \quad a = 1 \quad \text{and} \quad 2(n-2)k \leq 4k + u - c.$$

On the other hand, the results of [19] imply that $|n_{r_1}(G)|_p \mid 2q^2(n-1)$ and $|n_{r_2}(G)|_p \mid 2q^6(n-2)$, and hence, repeating the argument used for (3.4), we observe that

$$(3.7) \qquad \prod_{y=1}^{f} |C_{G_y/G_{y-1}}(R_1 G_{y-1}/G_{y-1})|_p \mid 2q^2(n-1), \text{ where } R_1 \in \mathrm{Syl}_{r_1}(G),$$

$$(3.8) \qquad \prod_{y=1}^{f} |C_{G_y/G_{y-1}}(R_2 G_{y-1}/G_{y-1})|_p \mid 2q^6(n-2), \text{ where } R_2 \in \mathrm{Syl}_{r_2}(G).$$

We have two following subcases:

(a) If $p \mid n-1$, then $p \nmid n-2$. We claim that $4ka + u - c = 0$. If not, then (3.6) implies that $2(n-2)k \leq 4k + u - c$ and so, $n \leq 4$, which is a contradiction, as required. Hence, $4ka + u - c = 0$. Under our assumption, $p \nmid n-2$ and by (3.8), $p^{4ka+u} = p^c = |C_{G_j/G_{j-1}}(RG_{j-1}/G_{j-1})|_p \mid 2q^6(n-2)$. These imply that $a = 1$. Set

$$(3.9) \qquad \Pi = \{1 \leq i \leq j_0 : |G_i/G_{i-1}|_p = p^{e_i} > p, \ |G_i/G_{i-1}| < p^{5e_i/2}\}.$$

With the same reasoning in the previous sub-case and using (3.8), we can see that $\Pi = \{j\}$. On the other hand,

$$|G|/|G_j/G_{j-1}| < p^{5[(n(n+1)-2n)k-u]/2}$$

and $|G|_p/|G_j/G_{j-1}|_p = |G|_p/p^{4nk+2u} = p^{[(n(n+1)-4n)k-2u]}$. These imply that there exists $x \in \{1,..,j_0\}$, where $x \neq j$, $|G_x/G_{x-1}|_p = p^{e_x} > p$ and $|G_x/G_{x-1}| < p^{5e_x/2}$. Therefore, similar to the argument given for I(i), we observe that $x \in \Pi - \{j\}$, which is a contradiction.

(b) Let $p \nmid n-1$. If $p \nmid n-2$, then Subcase (a) completes the proof. If $p \mid n-2$, then by the same argument as used for Subcase (a), we can see that there exists $x \in \Pi = \{1 \leq i \leq j_0 : |G_i/G_{i-1}|_p = p^{e_i} > p, \ |G_i/G_{i-1}| < p^{5e_i/2}\}$. According to (3.7),

$$|C_{G_j/G_{j-1}}(R_1 G_{j-1}/G_{j-1})|_p . |C_{G_x/G_{x-1}}(R_1 G_{x-1}/G_{x-1})|_p \mid 2q^2.$$

But Lemma 3.3 and previous argument imply that $|C_{G_j/G_{j-1}}(R_1 G_{j-1}/G_{j-1})|_p = p^{2ak+u}$ and hence, $|C_{G_x/G_{x-1}}(R_1 G_{x-1}/G_{x-1})|_p \leq p$. Thus Lemma 3.3 allows us to assume that $|C_{G_x/G_{x-1}}(R_1 G_{x-1}/G_{x-1})|_p = |C_{G_x/G_{x-1}}(RG_{x-1}/G_{x-1})|_p$, which is a contradiction using Subcase (i).

(iv) If $p \mid n$ and $S_{n,q} \neq D_{n+1}(q)$, then by Lemma 3.3,

$$(3.10) \qquad |C_{G_j/G_{j-1}}(RG_{j-1}/G_{j-1})|_p = |C_{G_j/G_{j-1}}(R_1 G_{j-1}/G_{j-1})|_p = p^{e_j}$$

and

$$p^u = 1,\ p^{(2a-1)k} = 2. \tag{3.11}$$

If $p,\ k,\ u$ satisfy (3.11), then $(p,k,u) = (2,1,0)$. Hence $|C_{G_j/G_{j-1}}(R_1G_{j-1}/G_{j-1})|_p = 4$. On the other hand, with the same reasoning as in Subcase (ii), $\prod_{y=1}^{f}|C_{G_y/G_{y-1}}(R_1G_{y-1}/G_{y-1})|_p \mid 4$. Thus Lemma 3.3 implies that $\Pi = \{1 \le i \le j_0 : |G_i/G_{i-1}|_p = p^{e_i} > p,\ |G_i/G_{i-1}| < p^{5e_i/2}\} = \{j\}$. Since $q = 2,\ p \mid n$ and $(n,q) \ne (4,2)$, we have $n \ge 6$. Also, $e_j = 2n$ and $|S_{n,q}| \le q^{n^2}(q^2-1)\cdots(q^{2n}-1) < q^{n^2}q^{n(n+1)}$. Thus $n(n+1) - 2n < 4(n^2-2n)/3$, because $n \ge 6$. It follows that $|G|/|G_j/G_{j-1}| < (|G|_p/|G_j/G_{j-1}|_p)^{7/3}$ and hence, there exists $x \in \Pi - \{j\}$, which is a contradiction. Therefore e_j satisfies (3.10). We conclude that $p^{e_j} \mid (n_r(G), n_{r_1}(G))$ and thus $p^{e_j} \mid 2p^k$. We can consider similar to the previous procedure that for every $i \in \Pi$, e_i satisfies (3.10) and $n(n+1)k < 4[n^2k - k - 1]/3$. Since $|C_G(R_1)|_p \le 2p^k$, by continuation of this procedure, we can find $x \in \Pi$ such that $|C_{G_x/G_{x-1}}(R_1G_{x-1}/G_{x-1})|_p = 1$, which is a contradiction.

(v) If $p \mid n$ and $S_{n,q} = D_{n+1}(q)$, then by Lemma 3.4,

$$|C_{G_j/G_{j-1}}(RG_{j-1}/G_{j-1})|_p = |C_{G_j/G_{j-1}}(R_1G_{j-1}/G_{j-1})|_p = p^{e_j} \tag{3.12}$$

or

$$p^u \mid 2, a = 1. \tag{3.13}$$

Since $|C_{G_j/G_{j-1}}(R_1G_{j-1}/G_{j-1})|_p = q^2p^u$, $\prod_{k=1}^{f}|C_{G_k/G_{k-1}}(R_1G_{k-1}/G_{k-1})|_p \mid |C_G(R_1)|$ and $|C_G(R_1)|_p \mid 2q^2p^u$, similar to the previous procedure, we can get a contradiction.

Therefore, we conclude that there is $1 \le h \le f$ such that r or r_1 (or r_2, for $n \ge 5$) is an element of $\pi(G_h/G_{h-1})$, $|G_h/G_{h-1}|_p = p^{e_h} > p$ and $|G_h/G_{h-1}| < p^{5e_h/2}$. Hence G_h/G_{h-1} is a direct product of isomorphic simple groups of Lie type in characteristic p. But $r_i \in \pi(G_h/G_{h-1})$, for some $i \in \{0,1,2\}$ such that $r_0 = r$ (if $n \ge 5$, then i can be equal to 2). Hence G_h/G_{h-1} is a simple group of Lie type in characteristic p. We claim that $h = j_0$. If not, then we conclude by checking the orders of all finite simple groups of Lie type in characteristic p that $r \notin \pi(K/H)$ or $r_1 \notin \pi(K/H)$ (or $r_2 \notin \pi(K/H)$, where $n \ge 5$) which is a contradiction. Thus $h = j_0$. Therefore K/H is a simple group of Lie type in characteristic p. It follows that for all $1 \le i \le f$ $(i \ne j_0)$, $r,\ r_1 \notin \pi(G_i/G_{i-1})$ (moreover, $r_2 \notin \pi(G_i/G_{i-1})$, where $n \ge 5$). Hence, $r \nmid |H|$. Let $|H|_p = p^t$ and $|G|_p = p^e$. Lemma 3.2 allows us to assume that $r \mid p^{t-x} - 1$, where $|C_H(R)|_p = p^x$ and hence, we may consider that $2nk \mid t - x$. This implies that either $|H/K|_p < q^{e-2n}$ or $t = x$. But

$$|C_H(R)||K/H| \mid (|N_H(R)|n_r(K))/|N_H(R)|, \tag{3.14}$$

using Frattini's argument. According to [19], $|n_r(G)|_p \le |2n|_p < p^n$. Therefore, if $t = x$, then $|H/K|_p > p^{ek-n}$. In addition, we conclude that $|H/K|_p < q^{e-2n}$ or $|K/H|_p > p^{ek-n}$. Also, $|K/H| \mid |S_{n,q}|$, $\exp_r(p) \in Max\{\exp_s(p) : s \in \pi(S_{n,q})\}$ and $\exp_{r_1}(p) \in Max\{\exp_s(p) : s \in \pi(S_{n,q})$ and $\exp_s(p) \ne \exp_r(p)\}$ (moreover for $n \ge 4$, $\exp_{r_2}(p) \in Max\{\exp_s(p) : s \in \pi(S_{n,q})$ and $\exp_s(p) \ne \exp_r(p), \exp_{r_1}(p)\}$). It follows that by checking the orders of all finite simple groups of Lie type, if $S_{n,q} = D_{n+1}(q)$, then $K/H \cong D_{n+1}(q)$ and if $S_{n,q} \ne D_{n+1}(q)$, then $K/H \cong B_n(q)$ or $K/H \cong C_n(q)$. So, $|G| = |S_{n,q}| = |K/H|$. Therefore $G = K$ and $H = 1$. In addition, by Lemma 2.6, we conclude that $G \cong S_{n,q}$ if and only if $q \not\equiv \pm 1 \pmod 8$ or $n \le 3$ or $S_{n,q} = D_{n+1}(q)$. Unless, $S_{n,q} \cong B_n(q)$ or $S_{n,q} \cong C_n(q)$.

(II) If $(n,q) = (3,2)$, then similar to the above argument, there is $1 \leq j \leq f$ such that $|G_j/G_{j-1}|_p = p^{e_j} > p$ and $|G_j/G_{j-1}| < p^{7e_j/3}$. Since $|G_j/G_{j-1}| \mid |S_{n,q}|$ and by remark (3.1), we can consider that $G_j/G_{j-1} \cong S_{n,q}$. Thus $G \cong S_{n,q}$ and the proof is complete.

(III) If $(n,q) = (4,2)$, then $r = 17$. Let $j_0 = Max\{1 \leq i \leq f : 17 \in \pi(G_i/G_{i-1})\}$, $H := G_{j_0-1}$ and $K := G_{j_0}$. We claim that $j_0 \in \Pi$. If not, then there is $i \in \Pi$ such that $j_0 \neq i$. We can see that by the orders of Sylow normalizers of G, $i < j_0$ and $\pi(G_i/G_{i-1}) = \{2,3,5\}$. Hence G_i/G_{i-1} is isomorphic to one of the following groups (see [11]):

$$A_5,\ A_6,\ U_4(2)$$

which is a contradiction to remark 3.1. Hence $j_0 \in \Pi$ and we conclude that by remark 3.1, $K/H \cong S_{n,q}$. Therefore, $G \cong S_{n,q}$ and the proof is complete. The same argument applies if $(n,q) = (5,2)$ and $(n,q) = (3,4)$.

V) Let $q \neq 2$, $S_{n,q} = D_5(q)$ and $r,\ r_1,\ r_2 \in \pi(S_{n,q})$ such that $\exp_r(p) = 8k$, $\exp_{r_1}(p) = 6k$ and $\exp_{r_2}(p) = 5k$. We can consider that by Corollary 2.1,

$$\begin{aligned} n_r(G) &= 8(q+1)(q^4+1)/(q^5-1,4); \\ n_{r_1}(G) &= 6q^2(q^4-1)(q^3+1)/(q^5-1,4); \\ n_{r_2}(G) &= 5(q^5-1)/(q^5-1,4), \end{aligned}$$

which easily, we can complete the proof. Also, if $q \neq 2$ and $S_{n,q} = D_4(q)$, then we assume that $r,\ r_1 \in \pi(S_{n,q})$ such that $\exp_r(p) = 6k$ and $\exp_{r_1}(p) = 4k$. We can consider that by Corollary 2.1, $n_r(G) = 6(q+1)(q^3+1)/(q^4-1,4)$ and $n_{r_1}(G) = 8(q^2+1)^2/(q^4-1,4)$, which easily, we can complete the proof. ∎

Corollary 3.1. *Let $S_{n,q} \in \{B_n(q), C_n(q), D_{n+1}(q)\}$ and suppose that $n(S_{n,q}) = n(G)$. If $S_{n,q} = D_{n+1}(q)$ or $n = 2$ or $q \not\equiv \pm 1$ (mod 8), then $G \cong S_{n,q}$. Further, if $n \geq 3$ and $q \equiv \pm 1$ (mod 8), then either $G \cong S_{n,q}$ or $\{G, S_{n,q}\} = \{B_n(q), C_n(q)\}$.*

Using the results about the characterization of finite groups by the orders of Sylow normalizers, we can put forward the following conjecture:

Conjecture 3.1. *If S is a non-abelian finite simple group, then either S is characterizable by the orders of Sylow normalizers or $S \in \{B_n(q), C_n(q)\}$, where $n \geq 3$ and $q \equiv \pm 1$ (mod 8), and in the latter case, S is 2-characterizable.*

Acknowledgement. The authors would like to thank the referees for the valuable comments.

References

[1] N. Ahanjideh and A. Iranmanesh, A characterization of $B_n(q)$ by the set of orders of maximal abelian subgroups, *Internat. J. Algebra Comput.* **19** (2009), no. 2, 191–211.

[2] J. Bi, A characterization of $L_2(q)$, *J. Liaoning Univ (Natural Sciences Edition),* **19** (1992), no. 2, 1–4.

[3] J. Bi, A characterization of $L_n(q)$ by the normalizers' orders of their Sylow subgroups, *Acta Math. Sinica (N.S.)* **11** (1995), no. 3, 300–306.

[4] J. Bi, On the group with the same orders of Sylow normalizers as the finite simple group $S_4(q)$, *Algebras Groups Geom.* **18** (2001), no. 3, 349–355.

[5] J. Bi, Characterization of alternating groups by orders of normalizers of Sylow subgroups, *Algebra Colloq.* **8** (2001), no. 3, 249–256.

[6] J. Bi, On the group with the same orders of Sylow normalizers as the finite projective special unitary group, *Sci. China Ser. A* **47** (2004), no. 6, 801–811.

[7] R. Carter and P. Fong, The Sylow 2-subgroups of the finite classical groups, *J. Algebra* **1** (1964), 139–151.
[8] R. W. Carter, *Simple Groups of Lie Type*, John Wiley & Sons, London, 1972.
[9] W. Feit, On large Zsigmondy primes, *Proc. Amer. Math. Soc.* **102** (1988), no. 1, 29–36
[10] The GAP group, *GAP-Groups, Algorithms, and Programming*, Version 4.3, Aachen, St Andrews (2002) http://www.gap-system.org.
[11] M. Herzog, On finite simple groups of order divisible by three primes only, *J. Algebra* **10** (1968), 383–388.
[12] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134 Springer, Berlin, 1967.
[13] A. Iranmanesh, S. H. Alavi and B. Khosravi, A characterization of $\mathrm{PSL}(3,q)$ for $q=2^m$, *Acta Math. Sin. (Engl. Ser.)* **18** (2002), no. 3, 463–472.
[14] A. Iranmanesh and N. Ahanjideh, A characterization of ${}^2D_n(p^k)$ by order of normalizer of Sylow subgroups, *Int. J. Algebra* **2** (2008), no. 17-20, 853–865.
[15] B. Khosravi and B. Khosravi, Groups with the same orders of Sylow normalizers as the Janko groups, *J. Appl. Algebra Discrete Struct.* **3** (2005), no. 1, 23–31.
[16] B. Khosravi and B. Khosravi, Two new characterizations of the Mathieu simple groups, *Internat. J. Math. Math. Sci.*, **9** (2005), 1449–1453.
[17] A. S. Kondrat'ev, Mat. Zametki **78** (2005), no. 3, 368–376; translation in *Math. Notes* **78** (2005), no. 3–4, 338–346.
[18] W. R. Scott, *Group Theory*, second edition, Dover, New York, 1987.
[19] M. Stather, Constructive Sylow theorems for the classical groups, *J. Algebra* **316** (2007), no. 2, 536–559.