Classes of Nonrigid Carnot Groups

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Abstract. The main purpose of this paper is to provide examples of nonrigid Carnot groups that do not appear in the literature. We use a condition in a previous article in order to construct such examples. It comes out that suitable semidirect products of any nilpotent stratified Lie group with some euclidean space are nonrigid Carnot groups. Moreover, we single out conditions using the language of root system in order to define a class of nonrigid Hessenberg manifolds.

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1. Introduction

Let \( n \) be a Lie algebra over \( \mathbb{R} \). A nilpotent Lie algebra \( n \) has a \( p \)-step stratification if it can be written as

\[
\begin{align*}
n = \sum_{i=1}^{p} g_i,
\end{align*}
\]

a direct sum of vector spaces such that

\[
\begin{align*}
[g_1, g_j] &= g_{j+1},
\end{align*}
\]

for every \( j \geq 1 \). We will use the usual symbol of sum when referred to vector spaces direct sum, whereas the symbol \( a \oplus b \) denote the direct sum of Lie algebras, meaning in particular \( [a, b] = 0 \). A Carnot group \( N \) is a connected, simply connected nilpotent Lie group, whose Lie algebra is stratified and equipped with an inner product such that \( g_i \perp g_j, \ i \neq j \). By left translation, \( n \) defines the tangent bundle \( TN \) to \( N \) and the subspace \( g_1 \) defines a subbundle \( HN \) of \( TN \) which is called horizontal bundle or contact bundle. Equation (1.1) implies that the horizontal bundle has the property that it generates at each point the whole tangent space to \( N \). A diffeomorphism

\[
\phi : \mathcal{U} \to \mathcal{V}
\]

between open sets of \( N \) is called a contact mapping if the differential \( \phi_* \) preserves the horizontal bundle. A contact vector field \( V \) on an open set of \( N \) is a smooth vector field

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which generates a local one parameter flow $\phi_t$ of contact mappings. If $\tilde{X}$ is a left invariant vector field corresponding to a vector $X \in \mathfrak{g}_1$, we have
\[
\frac{d}{dt} (\phi_t)_* (X) \bigg|_{t=0} = -\mathcal{L}_V (X) = [X, V],
\]
where $\mathcal{L}$ denotes the Lie derivative. Thus a smooth vector field $V$ is a contact vector field if and only if
\[
[V, \tilde{X}] \in \mathcal{H}N, \quad \text{for every } \tilde{X} \in \mathcal{H}N,
\]
that is, $\text{ad} V$ preserves the horizontal bundle.

If the Lie group of contact mappings, or equivalently the Lie algebra of contact vector fields, is finite dimensional, then $N$ is said to be rigid, whereas $N$ is nonrigid if the group of contact mappings (contact vector fields) is infinite dimensional.

The study of contact mappings has consequences in the theory of quasiconformal mappings and nonintegrable differential systems. For this reason, contact mappings have been studied in several examples of Carnot groups. A remarkable piece of work concerning this circle of ideas is [8], by Yamaguchi. The main result leads to a complete classification of the rigid nilpotent Lie groups which arise as nilpotent parts of the parabolic subgroups of semisimple Lie groups. In [6], Reimann showed that $H$-type groups are rigid provided that the dimension of the center is strictly greater than two. In a recent work, Cowling, De Mari, Koranyi and Reimann [1] have proved rigidity results for homogeneous spaces $G/P$ with $G$ simple and $P$ minimal parabolic. This is the situation considered by Yamaguchi, but their results are independent of classification and rely on entirely elementary techniques. In [4], the author applied some of the techniques used in [1] to treat the case of Hessenberg manifolds, that can be locally viewed as quotients of Iwasawa nilpotent Lie groups and that under certain circumstances turn out to be rigid. In [7], Warhurst shows that jet spaces are nonrigid Carnot groups. Jet spaces contain all nonrigid examples known so far.

In [5], the author gives a sufficient condition for nonrigidity. In this paper we discuss some nonrigid examples that arise from that condition thus extending the class of nonrigid Carnot groups. In particular, we define a class of nonrigid Hessenberg manifolds.

2. Nonrigid Carnot groups

The main result in [5] is the following.

**Theorem 2.1.** Let $N$ be a Carnot group with Lie algebra $\mathfrak{n} = \sum_{i=1}^{p} \mathfrak{g}_i$. If there exists $X \in \mathfrak{g}_1$ such that $\text{ad} X : \mathfrak{n} \to \mathfrak{n}$ has rank $\leq 1$, then $N$ is nonrigid.

The proof of the above theorem relies on the following Kirillov type lemma, that is proved in [5].

**Lemma 2.1.** Let $\mathfrak{n} = \sum_{i=1}^{p} \mathfrak{g}_i$ be a stratified nilpotent Lie algebra. Suppose that there exists $X \in \mathfrak{g}_1$ such that $\text{rank}(\text{ad} X) = 1$. Then $\mathfrak{n}$ can be written as
\[
\mathfrak{n} = \mathbb{R} Z_1 + \cdots + \mathbb{R} Z_m + \mathbb{R} X + \mathbb{R} Y + \mathfrak{w},
\]
a vector space direct sum, where
\[
[X, Y] = Z_1,
\]
\[
[Z_i, Y] = Z_{i+1}, \quad i = 1, \ldots, m - 1,
\]
\[
Z_m \in \mathfrak{z}(\mathfrak{g}).
\]
Moreover,
\[ g_0 = \mathbb{R}Z_1 + \cdots + \mathbb{R}Z_m + \mathbb{R}X + \mathfrak{w} \]
is an ideal and
\[ \mathcal{I} = \mathbb{R}Z_1 + \cdots + \mathbb{R}Z_m + \mathbb{R}X \]
is abelian. Finally, \( g_1 \) has a basis \( \{X, Y, U_1, \ldots, U_s\} \) so that
\[ [X, U_i] = 0, \quad \forall i = 1, \ldots, s \]
\[ [Z_j, U_i] = 0, \quad \forall j = 1, \ldots, m, \forall i = 1, \ldots, s. \]

The condition in Theorem 2.1 leads to the construction of many examples of nonrigid Carnot groups, that include but go beyond the examples in the literature, that are the Heisenberg group, the Engel group and in general the jet spaces [7]. An important remark here is that jet spaces have the property that the center coincides with the highest layer of the stratification. It is easy to see that Theorem 2.1 provides examples where the center is distributed along different strata. The next proposition reveals the structure of the Lie algebras that fall in the class of those singled out by Theorem 2.1.

**Proposition 2.1.** Let \( a \) be a stratified nilpotent Lie algebra. Then, for every \( q \geq 1 \) there exists a Lie algebra homomorphism \( \tau \) from \( a \) to \( \text{End}(\mathbb{R}^q) \) with \( \tau(a)\mathbb{R}^q = \mathbb{R}^{q-1} \) and nontrivial for \( q \geq 2 \) such that the semidirect sum \( n = a \oplus_{\tau} \mathbb{R}^q \) is a stratified nilpotent Lie algebra for which the rank(\( \text{ad}X \)) \( \leq 1 \) condition holds. Thus \( N = \text{expn} \) is nonrigid. On the other hand, any stratified nilpotent Lie algebra satisfying the hypothesis of Theorem 2.1 arises in this way.

**Proof.** Fix bases \( \{X_1, \ldots, X_s\} \) of \( a_1 \), the first stratum of \( a \) and \( \{e_1, \ldots, e_q\} \) of \( \mathbb{R}^q \). If \( q = 1 \), then \( \tau = 0 \) and \( n \) is a Lie algebra direct sum of \( a \) and \( \mathbb{R} \), which is trivially nonrigid. For \( q \geq 2 \), define \( \tau(X_1)e_i = -e_{i+1} \), for every \( i = 1, \ldots, q-1 \) and zero otherwise. Then it is easy to verify that \( \tau \) is a Lie algebra homomorphism from \( a \) to \( \text{End}(\mathbb{R}^q) \). Therefore, \( n = a \oplus_{\tau} \mathbb{R}^q \) is a stratified nilpotent Lie algebra for which \( g_1 = a_1 + \mathbb{R}e_1 \). Moreover, rank(\( \text{ad}e_1 \)) = 1. This implies by Theorem 2.1 that \( \text{expn} \) is nonrigid.

Suppose that rank(\( \text{ad}X \)) = 0 for some \( X \in g_1 \). Then we can choose a basis \( \mathcal{B} \) of \( n \) containing \( X \) and so that \( X \) commutes with all vectors in \( \mathcal{B} \). Therefore \( n = a \oplus \mathbb{R}X \), where \( a = \text{span}(\mathcal{B} \setminus X) \).

If rank(\( \text{ad}X \)) = 1 for some \( X \in g_1 \), then the hypothesis of Lemma 2.1 is satisfied. Using the notations as in (2.1) and (2.2), we can then write \( n = (\mathbb{R}Y + \mathfrak{w}) \oplus_{\tau} \mathcal{I} \), where the assignments \( \tau(Y)(X) = -Z_1 \) and \( \tau(Y)(Z_i) = -Z_{i+1} \) for \( i = 1, \ldots, m-1 \) are the only nonzero images of \( \tau \) and by linear expansion define the action on \( \mathcal{I} = \mathbb{R}^{m+1} \).

3. Hessenberg manifolds

Let \( g \) be a simple Lie algebra with Killing form \( B \) and Cartan involution \( \theta \). Let \( \mathfrak{k} + \mathfrak{p} \) be the Cartan decomposition of \( g \). Fix a maximal abelian subspace \( a \) of \( \mathfrak{p} \), and denote by \( \Sigma \) the set of restricted roots, a subset of the dual \( \alpha' \) of \( a \). The Killing form \( B \) induces a scalar product \( \langle \cdot, \cdot \rangle \) and thus a norm \( \| \cdot \| \) on \( \alpha' \). We choose an ordering on \( \alpha' \), and consequently define the subsets \( \Sigma_+ \) and \( \Delta = \{ \delta_1, \ldots, \delta_r \} \) of positive and simple positive restricted roots. Since we shall always work with the restricted root spaces, we forget the adjective “restricted” when it is referred to roots. Every positive root \( \alpha \) can be written as \( \alpha = \sum_{i=1}^r n_i \delta_i \) for uniquely defined non-negative integers \( n_1, \ldots, n_r \), and the positive integer \( \text{ht}(\alpha) = \sum_{i=1}^r n_i \) is called
the *height* of $\alpha$. It is well-known that there is exactly one root $\omega$, called the highest root, that satisfies $\omega \succ \alpha$ (strictly) for every other root $\alpha$. The root space decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{m} + \alpha + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$, where $\mathfrak{m} = \{X \in \mathfrak{t} : [X, H] = 0, H \in \mathfrak{a}\}$ and $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \forall H \in \mathfrak{a}\}$. The sum of two roots is still a root if the corresponding root spaces do not commute. The nilpotent Iwasawa algebra $\mathfrak{n}$ is $\sum_{\gamma \in \Sigma_+} \mathfrak{g}_\gamma$ and we denote with $\pi$ its counterpart $\theta(\mathfrak{n})$. It is well-known that $\mathfrak{n}$ is a stratified Lie algebra in the usual sense, that is $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$, where $\mathfrak{g}_i = \sum_{\gamma, i = 1, \ldots, \text{ht}(\omega)} \mathfrak{g}_\gamma$.

Let $G$ be a Lie group whose Lie algebra is $\mathfrak{g}$. Let $P = \text{MAN}$ be a minimal parabolic subgroup of $G$. We may assume that the center of $G$ is trivial. Indeed, if $Z$ is the center of $G$, then $Z \subset P$, and so $G/P$ and $(G/Z)/(P/Z)$ may be identified. Moreover, the action of $G$ on $G/P$ factors to an action of $G/Z$. Among all groups with trivial centers and the same Lie algebra $\mathfrak{g}$, the largest is the group $\text{Aut}(\mathfrak{g})$ of all automorphisms of $\mathfrak{g}$, and the smallest is the group $\text{Int}(\mathfrak{g})$ of the inner automorphisms of $\mathfrak{g}$, the connected component of the identity of $\text{Aut}(\mathfrak{g})$. Any group $G_1$ such that $\text{Int}(\mathfrak{g}) \subseteq G_1 \subseteq \text{Aut}(\mathfrak{g})$, with corresponding minimal parabolic subgroup $P_1$, gives rise to the same space, meaning that $G_1/P_1$ may be identified with $\text{Aut}(\mathfrak{g})/P$ if $P$ is a minimal parabolic subgroup of $\text{Aut}(\mathfrak{g})$. For the purposes of this paper the correct assumption is that $G$ is connected and centerless, and hence we can assume $G = \text{Int}(\mathfrak{g})$ and that $P$ is a minimal parabolic subgroup of $G$.

Let $\mathcal{R}$ be some proper subset of the set of the positive roots $\Sigma_+$. We call it of *Hessenberg type* if it satisfies the following property:

If $\alpha \in \mathcal{R}$ and $\beta$ is any negative root such that $\alpha + \beta \in \Sigma_+$, then $\alpha + \beta \in \mathcal{R}$.

Write $b_\mathcal{R} = a + \pi + \sum_{\gamma \in \mathcal{R}} \mathfrak{g}_\gamma$ and fix a regular element $H$ in the Cartan subspace $\mathfrak{a}$. Then

$$\text{Hess}_\mathcal{R}(H) = \{ (g)_P \in G/P : \text{Ad}g^{-1}H \in b_\mathcal{R} \}. $$

Denote by $m_\alpha$ the multiplicity of the root $\alpha$, that is, the dimension of the root space $\mathfrak{g}_\alpha$.

**Proposition 3.1.** [2] $\text{Hess}_\mathcal{R}(H)$ is a smooth submanifold of $G/P$ of dimension $\sum_{\alpha \in \mathcal{R}} m_\alpha$.

Denote by $\mathcal{C}$ the complement in $\Sigma_+$ of $\mathcal{R}$. It is rather easy to check that the direct sum

$$n_\mathcal{C} = \sum_{\alpha \in \mathcal{C}} \mathfrak{g}_\alpha$$

is an ideal in $\mathfrak{n}$. In [4] it is shown that the quotient group $N/N_\mathcal{C}$ is a local model of $\text{Hess}_\mathcal{R}(H)$: if $\mathcal{U} \subset \text{Hess}_\mathcal{R}(H)$ is open and dense, then it is diffeomorphic to the nilpotent stratified group $N/N_\mathcal{C}$. In this order of ideas, the study of contact mappings on $\text{Hess}_\mathcal{R}(H)$ is reduced to that of $N/N_\mathcal{C}$. The tangent space at the identity is $n/n_\mathcal{C}$, which is isomorphic to the vector space $n_\mathcal{R} = \sum_{\gamma \in \mathcal{R}} \mathfrak{g}_\gamma$ and the horizontal space is isomorphic to $g_1 = \sum_{\delta \in \mathcal{R}\setminus\mathcal{C}} \mathfrak{g}_\gamma$.

Call a positive root $\mu$ in $\mathcal{R}$ *maximal* if $\mu + \alpha \notin \mathcal{R}$ for any other root $\alpha \in \Sigma_+$. Since, by definition of $\mathcal{R}$ one has that $\mu + \alpha \notin \mathcal{R}$ if $\alpha \in \mathcal{C}$, it suffices to check maximality for all $\alpha \in \mathcal{R}$. Denote by $\mathcal{R}_M$ the set of maximal roots. For a fixed $\mu \in \mathcal{R}_M$, we call *shadow* of $\mu$ the set

$$S_\mu = \{ \alpha \in \mathcal{R} : \alpha \preceq \mu \}.$$

It is easy to show [4] that the union $\bigcup_{\mu \in \mathcal{R}_M} S_\mu$ covers $\mathcal{R}$. We assume for the following that $\mathfrak{g}$ is split, which implies in particular that each root space has dimension exactly one.

**Theorem 3.1.** Let $\mathcal{R}$ be a set of Hessenberg type containing a shadow $S_\mu$ such that

(i) $S_\mu$ contains 1 or 2 simple roots;
(ii) $ht(\mu) \leq 4$;
(iii) if $ht(\mu) \geq 2$, then $S_\mu$ has nontrivial intersection with at most one $S_\mu'$, $\mu' \neq \mu$, and the intersection coincides with the simple root of $S_\mu$ with smallest norm. Then $N/N_C$ is nonrigid.

Proof. We prove that in the hypotheses of the theorem, there exists $X \in \mathfrak{g}_1$ such that rank $(\text{ad}X) \leq 1$. Then Theorem 2.1 implies nonrigidity of $N/N_C$.

Suppose first that $\mu$ contains one simple root, then $\mu + \alpha$ is not in $\mathcal{R}$ for any root $\alpha \in \mathcal{R}$, by the maximality of $\mu$. We have that $\mathfrak{g}_\mu = \mathbb{R}X$ commutes with all of $n/n_C$ and so $\text{ad}X$ has rank zero.

Let now $\alpha$ and $\beta$ be the simple roots in $S_\mu$. Then $\alpha + \beta$ is a root in $S_\mu$ and it is well-known that $\alpha$ and $\beta$ can be summed to form a rank two root system, namely $A_2$, $B_2$ or $G_2$. For an insight on the classification of root systems, see e.g., [3]. By assumption (ii) only three cases arise for $S_\mu$: $S_\mu = \{\alpha, \beta, \alpha + \beta\}$, $S_\mu = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$, $S_\mu = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta\}$. If $S_\mu$ does not intersect any other shadow, then $n/n_C$ splits into the direct sum of two Lie algebras, one of those being generated by the root spaces lying in $S_\mu$.

In particular, by writing $\mathfrak{g}_\alpha = \mathbb{R}X$ we have that $X$ is a rank one element in the first stratum of $n/n_C$. On the other hand, if $S_\mu$ intersects other shadows, condition (iii) implies that the intersection is $\beta$. Indeed, in all cases $\|\alpha\| \geq \|\beta\|$ (refer to [3]). It then follows that within $\mathcal{R}$, the root $\alpha$ can be summed only to $\beta$, so that the corresponding one dimensional root space $\mathfrak{g}_\alpha$ does not commute only with the one dimensional space $\mathfrak{g}_\beta$. Hence, by writing $\mathfrak{g}_\alpha = \mathbb{R}X$, we have that $\text{ad}X$ is a rank one element in the first stratum of $n/n_C$.

References
