A Hybrid Extragradient Method for Pseudomonotone Equilibrium Problems and Fixed Point Problems

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Abstract. In this paper, we introduce a new hybrid extragradient iteration method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for pseudomonotone and Lipschitz-type continuous bifunctions. The iterative process is based on two well-known methods: Hybrid and extragradient. We show that the iterative sequences generated by this algorithm converge strongly to the common element in a real Hilbert space.

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1. Introduction

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $f$ be a bifunction from $C \times C$ to $\mathbb{R}$. We consider the equilibrium problems given as:

\[ \text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \quad \forall y \in C. \quad EP(f, C) \]

The set of solutions of $EP(f, C)$ is denoted by $\text{Sol}(f, C)$.

If $f(x, y) = \langle F(x), y - x \rangle$ for every $x, y \in C$, where $F$ is a mapping from $C$ to $H$, then Problem $EP(f, C)$ becomes the following variational inequalities:

\[ \text{Find } x^* \in C \text{ such that } \langle F(x^*), y - x^* \rangle \geq 0 \quad \forall y \in C. \quad VI(F, C) \]

We denote $\text{Sol}(F, C)$ which is the set of solutions of $VI(F, C)$.

In recent years, equilibrium problems become an attractive field for many researchers both theory and applications [1, 2, 4, 11, 17]. There are myriad of literature related to equilibrium problems and their applications in electricity market, transportation, economics and network [3, 5].

For solving $VI(F, C)$ in the Euclidean space $\mathbb{R}^n$ under the assumption that a subset $C \subseteq \mathbb{R}^n$ is nonempty closed convex, $F$ is monotone, $L$-Lipschitz continuous and $\text{Sol}(F, C) \neq \emptyset$, communicating by Ahmad Izani Md. Ismail.

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Korpelevich in [7] introduced the following extragradient method:

$$
\begin{aligned}
  x^0 &\in C, \\
  y^n &= \text{Pr}_C (x^n - \lambda F(x^n)), \\
  x^{n+1} &= \text{Pr}_C (x^n - \lambda F(y^n)),
\end{aligned}
$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{L})$. The author showed that the sequences $\{x^n\}$ and $\{y^n\}$ converge to the same point $z \in \text{Sol}(F,C)$.

For each $x, y \in C$, $f_\varphi(x, y) := f(x, y) + \varphi(y) - \varphi(x)$, motivated by the results of Peng in [11] introduced a new iterative scheme for finding a common element of the sets $\text{Sol}(f_\varphi, C)$, $\text{Sol}(F,C)$ and $\text{Fix}(T)$ in a real Hilbert space. Let sequences $\{x^n\}, \{y^n\}, \{t^n\}$ and $\{z^n\}$ be defined by

$$
\begin{aligned}
  x^0 &\in H, \\
  f_\varphi(u^n, y) + \frac{1}{r_n} \langle y - u^n, u^n - x^n \rangle &\geq 0 \quad \forall y \in C, \\
  y^n &= \text{Pr}_C (u^n - \lambda_n F(u^n)), \\
  t^n &= \text{Pr}_C (u^n - \lambda_n F(y^n)), \\
  z^n &= \alpha_n t^n + (1 - \alpha_n) T(t^n), \\
  C_n &= \{z \in C : ||z^n - z||^2 \leq ||x^n - z||^2 - (1 - \alpha_n)(\alpha_n - \varepsilon) ||t^n - T(t^n)||\}, \\
  Q_n &= \{z \in H : \langle x^n - z, x - x^n \rangle \geq 0\}, \\
  x^{n+1} &= \text{Pr}_{C_n \cap Q_n} (x^0).
\end{aligned}
$$

Then, the author showed that under certain appropriate conditions imposed on $\{\alpha_n\}, \{\lambda_n\}$ and $\varepsilon$, the sequences $\{x^n\}, \{u^n\}, \{t^n\}, \{y^n\}$ and $\{z^n\}$ converge strongly to $\text{Pr}_\Omega(x^0)$, where $\Omega := \text{Sol}(f_\varphi, C) \cap \text{Sol}(F,C) \cap \text{Fix}(T)$.

Recently, iterative algorithms for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a nonexpansive mapping in a real Hilbert space have further developed by some authors (see [11, 12, 14, 17, 18]). At each iteration $n$ in all of these algorithms, it requires solving approximation auxiliary equilibrium problems.

In this paper, we introduce a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a pseudomonotone, Lipschitz-type continuous bifunction. This method can be considered as an improvement of the iterative method in [11] via an improvement set of extragradient methods in [1, 2]. At each iteration $n$, we only solve strongly convex problems on $C$. We obtain a strong convergence theorem for four sequences generated by this process.

2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We list some well known definitions and the projection which will be required in our following analysis.

**Definition 2.1.** Let $C$ be a closed convex subset in $H$, we denote the projection on $C$ by $\text{Pr}_C(\cdot)$, i.e.,

$$
\text{Pr}_C(x) = \text{argmin}\{\|y - x\| : y \in C\} \quad \forall x \in H.
$$

The bifunction $f : C \times C \to \mathbb{R}$ is said to be
Thus Lipschitz-type continuous with constants $c$ function of two sets $\text{Sol}$ Choose Algorithm 2.1. Initialization.

In this paper, for finding a point of the set $\text{Sol}$ we have

$$f(x, y) + f(y, x) \leq -\gamma \|x - y\|^2;$$

(2) monotone on $C$ if for each $x, y \in C$, we have

$$f(x, y) + f(y, x) \leq 0;$$

(3) pseudomonotone on $C$ if for each $x, y \in C$, we have

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0;$$

(4) Lipschitz-type continuous on $C$ with constants $c_1 > 0$ and $c_2 > 0$, if for each $x, y \in C$, we have

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \|x - y\|^2 - c_2 \|y - z\|^2.$$

The mapping $F : C \rightarrow H$ is said to be

(5) monotone on $C$ if for each $x, y \in C$, we have

$$\langle F(x) - F(y), x - y \rangle \geq 0;$$

(6) pseudomonotone on $C$ if for each $x, y \in C$, we have

$$\langle F(y), x - y \rangle \geq 0 \Rightarrow \langle F(x), x - y \rangle \geq 0;$$

(7) $L$-Lipschitz continuous on $C$ if for each $x, y \in C$, we have

$$\|F(x) - F(y)\| \leq L \|x - y\|.$$

If $L = 1$, then $F$ is nonexpansive on $C$.

Note that if $F$ is $L$-Lipschitz on $C$, then for each $x, y \in C$, $f(x, y) = \langle F(x), y - x \rangle$ is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{L}{2}$ on $C$. Indeed,

$$2f(x, y) + f(y, z) - f(x, z)$$

$$= \langle F(x), y - x \rangle + \langle F(y), z - y \rangle - \langle F(x), z - x \rangle$$

$$= -\langle F(y) - F(x), y - z \rangle \geq -\|F(x) - F(y)\| \|y - z\| \geq -L \|x - y\| \|y - z\|$$

$$\geq -\frac{L}{2} \|x - y\|^2 - \frac{L}{2} \|y - z\|^2 = -c_1 \|x - y\|^2 - c_2 \|y - z\|^2.$$

Thus $f$ is Lipschitz-type continuous on $C$.

In this paper, for finding a point of the set $\text{Sol}(f, C) \cap \text{Fix}(T)$, we assume that the bifunction $f$ satisfies the following conditions:

(i) $f$ is pseudomonotone on $C$;

(ii) $f$ is Lipschitz-type continuous on $C$;

(iii) for each $x \in C$, $y \mapsto f(x, y)$ is convex and subdifferentiable on $C$;

(iv) $\text{Sol}(f, C) \cap \text{Fix}(T) \neq \emptyset$.

Now we are in a position to describe the extragradient algorithm for finding a common of two sets $\text{Sol}(f, C)$ and $\text{Fix}(T)$.

Algorithm 2.1. Initialization. Choose $x^0 \in C$, positive sequences $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions

$$\left\{ \begin{array}{ll}
\{\lambda_n\} \subset [a, b] & \text{for some } a, b \in (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}), \\
\{\alpha_n\} \subset [0, c] & \text{for some } c \in (0, 1).
\end{array} \right.$$
Step 1. Solve the strongly convex problems:
\[
\begin{align*}
    y^n &= \text{argmin}\left\{ \frac{1}{2}\|y - x^n\|^2 + \lambda_n f(x^n, y) : y \in C \right\}, \\
    t^n &= \text{argmin}\left\{ \frac{1}{2}\|t - x^n\|^2 + \lambda_n f(y^n, t) : t \in C \right\}, \\
    z^n &:= \alpha_n x^n + (1 - \alpha_n) T(t^n).
\end{align*}
\]

Step 2. Set \( P_n = \{ z \in C : \|z^n - z\| \leq \|x^n - z\| \} \) and \( Q_n = \{ z \in C : \langle x^n - z, x^n - x^n \rangle \geq 0 \} \).
Compute \( x^{n+1} = \text{Pr}_{P_n \cap Q_n}(x^n) \). Increase \( k \) by 1 and go to Step 1.

In order to prove the main result in Section 3, we shall use the following lemma in the sequel.

Lemma 2.1. [5] Let \( C \) be a convex subset of a real Hilbert space \( H \) and \( g : C \to \mathbb{R} \) be convex and subdifferentiable on \( C \). Then, \( x^* \) is a solution to the following convex problem
\[
\min \{ g(x) : x \in C \}
\]
if and only if \( 0 \in \partial g(x^*) + N_C(x^*) \), where \( \partial g(\cdot) \) denotes the subdifferential of \( g \) and \( N_C(x^*) \) is the (outward) normal cone of \( C \) at \( x^* \).

3. Main results

In this section, we show a strong convergence theorem of sequences \( \{x^n\}, \{y^n\}, \{z^n\} \) and \( \{t^n\} \) defined by Algorithm 2.1 based on the extragradient method which solves the problem of finding a common element of two sets \( \text{Sol}(f, C) \) and \( \text{Fix}(T) \) for a monotone, Lipschitz-type continuous bifunction \( f \) in a real Hilbert space \( H \).

Lemma 3.1. Suppose that \( x^* \in \text{Sol}(f, C) \), \( f(x, \cdot) \) is convex and subdifferentiable on \( C \) for all \( x \in C \), and \( f \) is pseudomonotone on \( C \). Then, we have
\[
\|x^n - x^*\|^2 \leq \|x^n - x^*\|^2 - (1 - 2\lambda_n c_1)\|x^n - y^n\|^2 - (1 - 2\lambda_n c_2)\|t^n - y^n\|^2 \quad \forall n \geq 0.
\]

Proof. Since \( f(x, \cdot) \) is convex on \( C \) for each \( x \in C \) and Lemma 2.1, we obtain
\[
t^n = \text{argmin}\left\{ \frac{1}{2}\|t - x^n\|^2 + \lambda_n f(y^n, t) : t \in C \right\}
\]
if and only if
\[
0 \in \partial_2 \{ \lambda_n f(y^n, t) + \frac{1}{2}\|t - x^n\|^2 \}(t^n) + N_C(t^n).
\]

Since \( f(y^n, \cdot) \) is subdifferentiable on \( C \), by the well known Moreau-Rockafellar theorem [5], there exists \( w \in \partial_2 f(y^n, t^n) \) such that
\[
f(y^n, t) - f(y^n, t^n) \geq \langle w, t - t^n \rangle \quad \forall t \in C.
\]

With \( t = x^* \), this inequality becomes
\[
f(y^n, x^*) - f(y^n, t^n) \geq \langle w, x^* - t^n \rangle.
\]

It follows from (3.1) that
\[
0 = \lambda_n w + t^n - x^n + \tilde{w},
\]
where \( w \in \partial_2 f(y^n, t^n) \) and \( \tilde{w} \in N_C(t^n) \).

By the definition of the normal cone \( N_C \) we have, from the latter equality, that
\[
\langle t^n - x^n, t - t^n \rangle \geq \lambda_n \langle w, t^n - t \rangle \quad \forall t \in C.
\]
With \( t = x^* \in C \) we obtain
\begin{equation}
\langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \langle w, t^n - x^* \rangle.
\end{equation}
It follows from (3.3) and (3.5) that
\begin{equation}
\langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \{ f(y^n, t^n) - f(y^n, x^*) \}.
\end{equation}
Since \( x^* \in \text{Sol}(f, C), f(x^*, y) \geq 0 \) for all \( y \in C \), and \( f \) is pseudomonotone on \( C \), we have \( f(y^n, x^*) \leq 0 \). Then, (3.6) implies that
\begin{equation}
\langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n f(y^n, t^n).
\end{equation}
Now applying Lipschitzian of \( f \) with \( x = x^n, y = y^n \) and \( z = t^n \), we get
\begin{equation}
f(y^n, t^n) \geq f(x^n, t^n) - f(x^n, y^n) - c_1 \| y^n - x^n \|^2 - c_2 \| t^n - y^n \|^2.
\end{equation}
Combining (3.7) and (3.8), we have
\begin{equation}
\langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \{ f(x^n, t^n) - f(x^n, y^n) - c_1 \| y^n - x^n \|^2 - c_2 \| t^n - y^n \|^2 \}.
\end{equation}
Similarly, since \( y^n \) is the unique solution to the strongly convex problem
\[
\min \left\{ \frac{1}{2} \| y - x^n \|^2 + \lambda_n f(x^n, y) : y \in C \right\},
\]
we have
\begin{equation}
\lambda_n \{ f(x^n, y) - f(x^n, y^n) \} \geq (y^n - x^n, y^n - y) \ \forall y \in C.
\end{equation}
As \( y = t^n \in C \), we have
\begin{equation}
\lambda_n \{ f(x^n, t^n) - f(x^n, y^n) \} \geq (y^n - x^n, y^n - t^n).
\end{equation}
From (3.9), (3.11) and
\[
2 \langle t^n - x^n, x^* - t^n \rangle = \| x^n - x^* \|^2 - \| t^n - x^n \|^2 - \| t^n - x^* \|^2,
\]
it implies that
\[
\| x^n - x^* \|^2 - \| t^n - x^n \|^2 - \| t^n - x^* \|^2 \geq 2 (y^n - x^n, y^n - t^n) - 2 \lambda_n c_1 \| x^n - y^n \|^2 - 2 \lambda_n c_2 \| t^n - y^n \|^2.
\]
Hence, we have
\[
\| t^n - x^n \|^2 \leq \| x^n - x^* \|^2 - \| t^n - y^n \|^2 - 2 (y^n - x^n, y^n - t^n) + 2 \lambda_n c_1 \| x^n - y^n \|^2 + 2 \lambda_n c_2 \| t^n - y^n \|^2 = \| x^n - x^* \|^2 - \| (t^n - y^n) + (y^n - x^n) \|^2 - 2 (y^n - x^n, y^n - t^n) + 2 \lambda_n c_1 \| x^n - y^n \|^2 + 2 \lambda_n c_2 \| t^n - y^n \|^2 \leq \| x^n - x^* \|^2 - \| t^n - y^n \|^2 - \| x^n - y^n \|^2 + 2 \lambda_n c_1 \| x^n - y^n \|^2 + 2 \lambda_n c_2 \| t^n - y^n \|^2 = \| x^n - x^* \|^2 - (1 - 2 \lambda_n c_1) \| x^n - y^n \|^2 - (1 - 2 \lambda_n c_2) \| y^n - t^n \|^2.
\]
The lemma thus is proved.

**Lemma 3.2.** Suppose that Assumptions (i)-(iv) hold and \( T \) is nonexpansive on \( C \). Then, we have
\begin{enumerate}
\item[(a)] \( \text{Sol}(f, C) \cap \text{Fix}(T) \subseteq P_n \cap Q_n \) for all \( n \geq 0 \).
\item[(b)] \( \lim_{n \to \infty} \| x^{n+1} - x^n \| = \lim_{n \to \infty} \| x^n - z^n \| = \lim_{n \to \infty} \| x^n - y^n \| = \lim_{n \to \infty} \| x^n - t^n \| = 0. \)
\end{enumerate}
Since Lemma 3.1 and $\varepsilon_n = \alpha_n x^n + (1 - \alpha_n) T(t^n)$, for each $x^* \in \text{Sol}(f, C) \cap \text{Fix}(T)$ we have

$$
\|\varepsilon^n - x^*\|^2 = \|\alpha_n x^n + (1 - \alpha_n) T(t^n) - x^*\|^2 \\
= \|\alpha_n(x^n - x^*) + (1 - \alpha_n)\{T(t^n) - T(x^*)\}\|^2 \\
\leq \alpha_n\|x^n - x^*\|^2 + (1 - \alpha_n)\|T(t^n) - T(x^*)\|^2 \\
\leq \alpha_n\|x^n - x^*\|^2 + (1 - \alpha_n)\|t^n - x^*\|^2 \\
\leq 2\|x^n - x^*\|^2.
$$

(3.12)

Hence $\|\varepsilon^n - x^*\| \leq \|x^n - x^*\|$ for every $n \geq 0$ and $x^* \in P_n$. So, we have

$$
\text{Sol}(f, C) \cap \text{Fix}(T) \subseteq P_n \quad \forall n \geq 0.
$$

Next, we show by mathematical induction that

$$
\text{Sol}(f, C) \cap \text{Fix}(T) \subseteq Q_n \quad \forall n \geq 0.
$$

For $n = 0$ we have $Q_0 = C$, hence we have $\text{Sol}(f, C) \cap \text{Fix}(T) \subseteq Q_0$. Now we suppose that $\text{Sol}(f, C) \cap \text{Fix}(T) \subseteq Q_k$ for some $k \geq 0$. From $x^{k+1} = \text{Pr}_{P_k \cap Q_k}(x^0)$, it follows that

$$
\langle x^{k+1} - x, x^0 - x^{k+1} \rangle \geq 0 \quad \forall x \in P_k \cap Q_k.
$$

Using this and $\text{Sol}(f, C) \cap \text{Fix}(T) \subseteq Q_k$, we have

$$
\langle x^{k+1} - x, x^0 - x^{k+1} \rangle \geq 0 \quad \forall x \in \text{Sol}(f, C) \cap \text{Fix}(T)
$$

and hence $\text{Sol}(f, C) \cap \text{Fix}(T) \subseteq Q_{k+1}$. This proves (a).

It follows from (a) and $x^{n+1} = \text{Pr}_{P_n \cap Q_n}(x^0)$ that

$$
\|x^{n+1} - x^0\| \leq \|\text{Pr}_{\text{Sol}(f, C) \cap \text{Fix}(T)}(x^0) - x^0\| \quad \forall n \geq 0.
$$

(3.13)

Hence, we get that $\{x^n\}$ is bounded. Otherwise, for each $x \in Q_n$, we have

$$
\langle x^n - x, x^0 - x^n \rangle \geq 0,
$$

and hence $x^n = \text{Pr}_{Q_n}(x^0)$. Using this and $x^{n+1} \in P_n \cap Q_n \subseteq Q_n$, we have

$$
\|x^n - x^0\| \leq \|x^{n+1} - x^0\| \quad \forall n \geq 0.
$$

Therefore, there exists

$$
A = \lim_{n \to \infty} \|x^n - x^0\|.
$$

(3.14)

Since $x^n = \text{Pr}_{Q_n}(x^0)$ and $x^{n+1} \in Q_n$, using

$$
\|\text{Pr}_{Q_n}(x) - x\|^2 \leq \|x - y\|^2 - \|\text{Pr}_{Q_n}(x) - y\|^2 \quad \forall x \in H, y \in Q_n,
$$

we have

$$
\|x^{n+1} - x^n\|^2 \leq \|x^{n+1} - x^0\|^2 - \|x^n - x^0\|^2 \quad \forall n \geq 0.
$$

Combining this and (3.14), we get

$$
\lim_{n \to \infty} \|x^{n+1} - x^n\| = 0.
$$

It proves the first part of (b).

Since $x^{n+1} = \text{Pr}_{P_n \cap Q_n}(x^0)$, we have $x^{n+1} \in P_n$, $\|z^n - x^{n+1}\| \leq \|x^n - x^{n+1}\|$ and hence

$$
\|x^n - z^n\| \leq \|x^n - x^{n+1}\| + \|x^{n+1} - z^n\| \leq 2\|x^n - x^{n+1}\| \quad \forall n \geq 0.
$$
From \( \lim_{n \to \infty} |x^{n+1} - x^n| = 0 \), we have
\[
\lim_{n \to \infty} \|x^n - z^n\| = 0.
\]
This proved the second apart of \((b)\).

From (3.12) and Lemma 3.1, it implies that
\[
\begin{align*}
\|z^n - x^*\|^2 & \leq \alpha_n \|x^n - x^*\|^2 + (1 - \alpha_n) \|y^n - x^*\|^2 \\
& \leq \alpha_n \|x^n - x^*\|^2 + (1 - \alpha_n) \{\|x^n - x^*\|^2 - (1 - 2\lambda_n c_1) \|x^n - y^n\|^2 \} \\
& \leq (1 - \alpha_n)(1 - 2\lambda_n c_1) \|x^n - y^n\|^2.
\end{align*}
\]
Therefore, we have
\[
\begin{align*}
\|x^n - y^n\|^2 & \leq \frac{1}{(1 - \alpha_n)(1 - 2\lambda_n c_1)} \{\|x^n - x^*\|^2 - \|z^n - x^*\|^2 \} \\
& = \frac{1}{(1 - \alpha_n)(1 - 2\lambda_n c_1)} (\|x^n - x^*\|^2 - \|z^n - x^*\|^2) (\|x^n - x^*\| + \|z^n - x^*\|) \\
& \leq \frac{1}{(1 - \alpha_n)(1 - 2\lambda_n c_1)} \|x^n - z^n\| (\|x^n - x^*\| + \|z^n - x^*\|) \\
& \leq (1 - \alpha_n)(1 - 2\lambda_n c_1) \|x^n - y^n\|^2.
\end{align*}
\]
Since \( \lim_{n \to \infty} \|x^n - z^n\| = 0 \) and the sequences \( \{x^n\}, \{z^n\} \) are bounded, we get
\[
\lim_{n \to \infty} \|x^n - y^n\| = 0.
\]
This proves the third apart of \((b)\).

By similar way, we also obtain that \( \lim_{n \to \infty} \|t^n - y^n\| = 0 \). Then we have
\[
\lim_{n \to \infty} \|x^n - t^n\| \leq \lim_{n \to \infty} (\|x^n - y^n\| + \|y^n - t^n\|) = 0,
\]
and hence \( \lim_{n \to \infty} \|x^n - t^n\| = 0 \). This proves the last part of \((b)\).

Using \((b)\) and \( z^n = \alpha_n x^n + (1 - \alpha_n) T(t^n) \), we have
\[
\begin{align*}
(1 - c) \|T(t^n) - t^n\| & \leq (1 - \alpha_n) \|T(t^n) - t^n\| \\
& = \|\alpha_n (t^n - x^n) + (z^n - t^n)\| \\
& \leq \alpha_n \|t^n - x^n\| + \|z^n - t^n\|, \\
& \leq (1 + \alpha_n) \|t^n - x^n\| + \|z^n - t^n\|,
\end{align*}
\]
and hence \( \lim_{n \to \infty} \|t^n - T(t^n)\| = 0 \).

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Suppose that Assumptions (i)-(iv) hold and \( T \) is nonexpansive on \( C \). Then, the sequences \( \{x^n\}, \{y^n\}, \{z^n\} \) and \( \{t^n\} \) generated by Algorithm 2.1 converge strongly to the same point \( x^* \), where
\[
x^* = \text{Pr}_{\text{Sol}(f,C) \cap \text{Fix}(T)}(x^0).
\]

**Proof.** Since \( \{x^n\} \) is bounded, there exists a subsequence \( \{x^{n_j}\} \) of \( \{x^n\} \) such that \( \{x^{n_j}\} \) converges weakly to some \( \bar{x} \) as \( j \to \infty \). Then, it follows from \((b)\) of Lemma 3.2 that \( \{t^{n_j}\} \) also converges weakly to some \( \bar{x} \) as \( j \to \infty \). We can obtain that \( \bar{x} \in \text{Sol}(f,C) \cap \text{Fix}(T) \). First,
we show \( \bar{x} \in \text{Fix}(T) \). Assume that \( \bar{x} \notin Fix(T) \). Since Opial’s condition in [6], i.e., for any sequence \( \{x_n\} \) with \( x_n \rightharpoonup \bar{x} \) the inequality
\[
\liminf_{n \to \infty} \|x_n - \bar{x}\| < \liminf_{n \to \infty} \|x_n - y\|
\]
holds for every \( y \in H \) with \( y \neq \bar{x} \), we have
\[
\liminf_{j \to \infty} \|t^{n_j} - \bar{x}\| < \liminf_{j \to \infty} \|t^{n_j} - T(\bar{x})\|
\leq \liminf_{j \to \infty} (\|t^{n_j} - T(t^{n_j})\| + \|T(t^{n_j}) - T(\bar{x})\|)
= \liminf_{j \to \infty} \|T(t^{n_j}) - T(\bar{x})\|
\leq \liminf_{j \to \infty} \|t^{n_j} - \bar{x}\|.
\]
This is contradiction. Thus, \( \bar{x} = T(\bar{x}) \).

From \((b)\) of Lemma 3.2 and \( x^{n_j} \to \bar{x} \) as \( j \to \infty \), it follows
\[
y^{n_j} \to \bar{x}, t^{n_j} \to \bar{x} \text{ as } j \to \infty.
\]
Then, using (3.10), \( \{\lambda_n\} \subset [a, b] \subset (0, 1) \) and assumptions of \( f \), we have
\[
\lambda_n \{f(x^{n_j}, y) - f(x^{n_j}, y^{n_j})\} \geq \langle y^{n_j} - x^{n_j}, y^{n_j} - y \rangle \quad \forall y \in C.
\]
As \( j \to \infty \), we get \( f(\bar{x}, y) \geq 0 \) for all \( y \in C \). It means that \( \bar{x} \in \text{Sol}(f, C) \). So, we have
\[
\bar{x} \in \text{Sol}(f, C) \cap \text{Fix}(T).
\]
Since \( x^* = \text{Pr}_{\text{Sol}(f, C) \cap \text{Fix}(T)}(x^0) \), \( \bar{x} \in \text{Sol}(f, C) \cap \text{Fix}(T) \) and (3.13), we have
\[
\|x^* - x^0\| \leq \|\bar{x} - x^0\| \leq \liminf_{j \to \infty} \|x^{n_j} - x^0\| \leq \limsup_{j \to \infty} \|x^{n_j} - x^0\| \leq \|x^* - x^0\|.
\]
So, we get
\[
\lim_{j \to \infty} \|x^{n_j} - x^0\| = \|\bar{x} - x^0\|.
\]
Since \( x^{n_j} - x^0 \) converges weakly to \( \bar{x} - x^0 \) as \( j \to \infty \), we have \( x^{n_j} - x^0 \) converges strongly to \( \bar{x} - x^0 \) as \( j \to \infty \). By \( x^n = \text{Pr}_{Q_n}(x^0) \) and \( x^* \in \text{Sol}(f, C) \cap \text{Fix}(T) \subset P_n \cap Q_n \subset Q_n \), we have
\[
\langle x^* - x^{n_j}, x^0 - x^0 \rangle \leq \langle x^* - x^{n_j}, x^0 - x^0 \rangle + \langle x^* - x^{n_j}, x^{n_j} - x^0 \rangle = -\|x^* - x^{n_j}\|^2.
\]
As \( j \to \infty \), we have
\[
\langle x^* - \bar{x}, x^0 - x^0 \rangle \leq -\|x^* - \bar{x}\|^2.
\]
Combining this, \( \bar{x} \in \text{Sol}(f, C) \cap \text{Fix}(T) \), \( \langle x^* - \bar{x}, x^0 - x^0 \rangle \geq 0 \) and \( x^* = \text{Pr}_{\text{Sol}(f, C) \cap \text{Fix}(T)}(x^0) \), we obtain \( \bar{x} = x^* \). This implies that \( \lim \|x^{n_j} - x^*\| = 0 \). From \((b)\) of Lemma 3.2, it follows
\[
\lim_{n \to \infty} \|y^n - x^*\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|t^n - x^*\| = 0.
\]
4. Applications

In this section, we discuss about two applications of Theorem 3.1 to find a common point of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problems for a monotone, Lipschitz continuous mapping. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, for each pair $x, y \in C$,

$$f(x, y) := \langle F(x), y - x \rangle,$$

where $F : C \rightarrow H$.

In Algorithm 2.1, the subproblems needed to solve at Step 1 are of the form

$$\begin{aligned}
y^n &= \arg\min \left\{ \frac{1}{2} \| y - x^n \|^2 + \lambda_n \langle F(x^n), y - x^n \rangle : y \in C \right\}, \\
t^n &= \arg\min \left\{ \frac{1}{2} \| t - x^n \|^2 + \lambda_n \langle F(y^n), t - y^n \rangle : t \in C \right\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
y^n &= \arg\min \left\{ \frac{1}{2} \| y - x^n - \lambda_n F(x^n) \|^2 : y \in C \right\} = \Pr_C (x^n - \lambda_n F(x^n)), \\
t^n &= \arg\min \left\{ \frac{1}{2} \| t - x^n - \lambda_n F(y^n) \|^2 : t \in C \right\} = \Pr_C (x^n - \lambda_n F(y^n)).
\end{aligned}$$

Thus, in this case Algorithm 2.1 and its convergence become the following results:

**Theorem 4.1.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $F : C \rightarrow H$ be a monotone, $L$-Lipschitz continuous mapping and $T : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(T) \cap \text{Sol}(F, C) \neq \emptyset$. Let $\{x^n\}, \{y^n\}$ and $\{z^n\}$ be the sequences generated by

$$\begin{aligned}
x^0 &\in C, \\
y^n &= \Pr_C (x^n - \lambda_n F(x^n)), \\
t^n &= \Pr_C (x^n - \lambda_n F(y^n)), \\
z^n &= \alpha_n x^n + (1 - \alpha_n)T(t^n), \\
P_n &= \{ z \in C : \| x^n - z \| \leq \| y^n - z \| \}, \\
Q_n &= \{ z \in C : \langle x^n - z, x^n - y^n \rangle \geq 0 \}, \\
x^{n+1} &= \Pr_{P_n \cap Q_n}(x^n),
\end{aligned}$$

for every $n \geq 0$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{4})$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x^n\}, \{y^n\}$ and $\{z^n\}$ converge strongly to $\Pr_{\text{Sol}(F,C) \cap \text{Fix}(T)}(x^0)$.

Using Theorem 4.1, we prove the following theorem proposed by Nakajo and Takahashi:

**Theorem 4.2.** [9] Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Let $\{x^n\}$ and $\{y^n\}$ be the sequences generated by

$$\begin{aligned}
x^0 &\in C, \\
y^n &= \alpha_n x^n + (1 - \alpha_n)T(\Pr_C(x^n)), \\
P_n &= \{ z \in C : \| x^n - z \| \leq \| y^n - z \| \}, \\
Q_n &= \{ z \in C : \langle x^n - z, x^n - y^n \rangle \geq 0 \}, \\
x^{n+1} &= \Pr_{P_n \cap Q_n}(x^n),
\end{aligned}$$

for every $n \geq 0$, where $\{\alpha_n\} \subset [0,c]$ for some $c \in [0,1)$. Then, the sequences $\{x^n\}$ and $\{y^n\}$ converge strongly to $\Pr_{\text{Fix}(T)}(x^0)$.

Proof. For $f = 0$, by Theorem 4.1, we have the desired results. 

References