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Mean Residual Life of *k*th Records Under Double Monitoring

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Abstract. In the study of reliability of technical systems and shock models, the mean residual life function plays an important role. In this paper, we consider the mean residual life of *k*th records under double monitoring. We introduce the notion of the mean residual life of *k*th records under the condition that the *m*th and $(m+1)$st shocks arrived before and after t_1, respectively, and the $(n+1)$st $(1 \le m < n)$ shock arrived after t_2 $(0 < t_1 < t_2)$. We study their respective monotonicity and aging properties. Some stochastic ordering properties are also investigated.

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1. Introduction

Record values and associated statistics arise naturally in many practical problems, and there are several situations pertaining to meteorology, hydrology, largest insurance claims, and athletic events wherein only record values may be recorded. The model of record values can be also used in reliability theory. For example, consider a technical system or a piece of equipment which is subject to shocks, e.g. peaks of voltage (records). Let $\{X_i, i \ge 1\}$ be a sequence of independent identically distributed (iid) random variables (r.v.'s) with common continuous cumulative distribution function (cdf) F, probability density function (pdf) f and survival function $\overline{F} = 1 - F$. Record values are closely connected with the occurrence times of some corresponding non-homogeneous Poisson process (NHPP) (cf. [7]). Let $X_{i:n}$ be the ith order statistic from a sample of size n. For a fixed integer $k \ge 1$, we define the corresponding kth record times, $U(n), n \ge 0$, and the kth record values, as follows:

$$U(0) = k,$$
$$U(n) = \min\left\{ j : j > U(n-1), X_j > X_{U(n-1)-k+1:U(n-1)} \right\}, \ n \ge 1, k \ge 1.$$

The r.v.'s $X_{U(n)}, n \ge 0$ are called upper kth record values, [6]. Note that $U(n)$'s are the epochs at which the top kth sample value jumps. However, these records can be viewed as

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ordinary record values ($k = 1$) from the distribution function $G(x) = 1-(1-F(x))^k, k \geq 1$. Let $\{N(t), t \geq 0\}$ denote the number of kth record values less than or equal to t. Any event of the process occurs whenever a record value is observed. Here, $N(t)$ is an NHPP having hazard rate $k\, h_F(.)$ with $h_F(.) = f(.)/\overline{F}(.)$ and the kth record values are the epoch times of this NHPP. In the same context, the sequence of the waiting times between successive shocks (records) are of special interest and it can be considered as another possibility to fit a record model.

The join pdf of three kth records $X_{U(m)}$, $X_{U(n)}$ and $X_{U(p)} (0 \leq m < n < p)$ is given by

$$f_{m,n,p}(x,y,z) = k^{p+1} \frac{[H(x)]^m}{m!} \frac{[H(y)-H(x)]^{n-m-1}}{(n-m-1)!} \frac{[H(z)-H(y)]^{p-n-1}}{(p-n-1)!}$$
$$\times e^{-k\,H(z)}\, h(x)h(y)h(z), \quad -\infty < x < y < z < \infty \tag{1.1}$$

where $H(x) = -\log \overline{F}(x)$, is the cumulative hazard function and $h(x) = f(x)/\overline{F}(x)$, is the hazard rate of the distribution function F. In reliability theory, X is increasing failure rate (IFR) if $h(t)$ is increasing in t. Also, X is decreasing failure rate (DFR) if $h(t)$ in decreasing in t. The mean residual lifetime (MRL, for short) function $m(t)$ of a component with life distribution F pertaining to a life length X, is defined by the conditional expectation of $X-t$ given $X > t$:

$$m(t) = \mathbb{E}(X-t|X > t) = \frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)},$$

provided that $\overline{F}(t) > 0$. The MRL function of X, $m(t)$ can be considered as the conditional tail measure given that the object did not fall in $(0,t)$. The MRL function is quite useful in actuarial analysis, survivorship analysis and reliability. Along with the Poisson process applications, we define the MRL of records as the conditional tail measure of the $(n+1)$st shock time given that the mth and $(m+1)$st shocks arrived before and after t_1, respectively, and the $(n+1)$st shock did not fall in $(0,t_2)$, $t_1 < t_2$. Specifically, the MRL of records can be defined as

$$K_{m,n}(t_1,t_2) = \mathbb{E}(S_{m,n}(t_1,t_2)), \tag{1.2}$$

where

$$S_{m,n}(t_1,t_2) = (X_{U(n)} - t_2 | X_{U(m-1)} < t_1 < X_{U(m)}, X_{U(n)} > t_2). \tag{1.3}$$

This is called the MRL of kth records under double monitoring condition. Function (1.2) estimates and evaluates the characteristics of the future epoch time (say T_n) based on knowing the lower limit value of this event and lower and upper limits for the previous epoch time T_m $(1 \leq m < n)$ of NHPP. Raqab and Asadi [14] studied the MRL of records under the condition that all the record values exceed a time $t > 0$. Asadi and Raqab [4] discussed the MRL of records under the condition that the mth record value $(0 \leq m < n)$ exceeds $t > 0$. Zhao and Balakrishnan [17, 18] carried out stochastic comparisons of inactive record values and generalized order statistics, respectively. Other related works on a k-out-of-$n(1 \leq k \leq n)$ system can be found in [2, 3, 5, 10, 11, 19]. Recently, Poursaeed and Nematollahi [13] have studied the MRL function of a parallel system under the double monitoring condition.

The most important and common ordering measures considered in this paper are the hazard rate ordering, likelihood ratio ordering and the stochastic ordering. Many detailed notions of the stochastic ordering is given in [16]. We give a brief review of these here. Throughout this paper, increasing means nondecreasing and decreasing means nonincreasing. Let X and Y be two random variables with distribution functions F and G and survivals $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively. Let $l_x(l_y)$ and $u_x(u_y)$ be the left and the right extremity of support of $X(Y)$. Then X is said to be smaller than Y in the hazard rate ordering (denoted by $X \leq_{hr} Y$) iff $\overline{G}(t)/\overline{F}(t)$ is increasing in $t \geq 0$. In case the hazard rates exist, then $X \leq_{hr} Y$, if and only if, $h_G(x) \leq h_F(x)$, $\forall x$. X is said to be smaller than Y in the MRL order (denoted by $X \leq_{mrl} Y$) iff $E(X)$ and $E(Y)$ exist and the ratio $\int_t^\infty \overline{F}(u)du / \int_t^\infty \overline{G}(u)du$ is decreasing in $t \geq 0$. X is said to be stochastically smaller than Y (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$, $\forall x$. X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in $x \in (-\infty, max(u_X, u_Y))$. The likelihood ratio ordering implies the hazard rate ordering which implies the stochastic ordering. For the stochastic ordering of residual records, one may refer to recent works of Khaledi and Shojaei [8] and Khaledi *et al.* [9].

The main aim of this paper is to examine the average of the MRL of *k*th records under double monitoring condition from sequences of iid r.v.'s, explore some of its aging properties and present their respective stochastic ordering results.

In the following section we derive a formula for $K_{m,n}(t_1,t_2)$ in terms of cdf F, then we give some monotonicity and aging properties for $K_{m,n}(t_1,t_2)$.

2. MRL of *k*th records under double monitoring

In the following theorem we give a simplified form for the MRL of *k*th records under double monitoring condition $K_{m,n}(t_1,t_2)$. For convenience, we use the following notation:

$$T_r(s;t_1,t_2) = \sum_{i=0}^{r} \frac{k^i}{i!} \, [H(t_2+s) - H(t_1)]^i .$$

Theorem 2.1. *Let $X_1, X_2, \ldots,$ be a sequence of iid r.v.'s from absolutely continuous distribution function F. Given that $X_{U(m-1)} < t_1 < X_{U(m)}, X_{U(n)} > t_2$, $t_2 > t_1 > 0$, the MRL of kth records for $1 \leq m < n$ is given by*

$$K_{m,n}(t_1,t_2) = \frac{1}{T_{n-m}(0;t_1,t_2)} \int_0^\infty T_{n-m}(s;t_1,t_2)\, e^{-k[H(t_2+s)-H(t_2)]}\, ds. \tag{2.1}$$

Proof. From (1.1), the joint pdf of three *k*th records $X_{U(m-1)}$, $X_{U(m)}$ and $X_{U(n)}$ $(1 \leq m < n)$ is given by

$$f_{m-1,m,n}(x,y,z) = k^{n+1} \frac{[H(x)]^{m-1}}{(m-1)!} \frac{[H(z)-H(y)]^{n-m-1}}{(n-m-1)!}\, e^{-kH(z)}\, h(x)h(y)h(z). \tag{2.2}$$

By (2.2), we have, for $s > 0$,

$$\begin{aligned}
&\mathbb{P}(X_{U(n)} - t_2 > s, X_{U(m-1)} < t_1 < X_{U(m)}, X_{U(n)} > t_2) \\
&\quad = \mathbb{P}\left(X_{U(m-1)} < t_1 < t_2 + s < X_{U(m)} < X_{U(n)}\right) \\
&\quad + \mathbb{P}\left(X_{U(m-1)} < t_1 < X_{U(m)} < t_2 + s < X_{U(n)}\right) \\
&\quad = \int_{t_2+s}^\infty \left[\int_{t_2+s}^z \int_0^{t_1} f_{m-1,m,n}(x,y,z)dxdydz + \int_{t_1}^{t_2+s} \int_0^{t_1} f_{m-1,m,n}(x,y,z)dxdy\right] dz \\
&\quad = \frac{k^{n+1}H^m(t_1)}{m!(n-m-1)!} \int_{t_2+s}^\infty \int_{t_1}^z [H(z)-H(y)]^{n-m-1} e^{-kH(z)} h(y)h(z)dydz
\end{aligned}$$

$$= \frac{k^{n+1}H^m(t_1)}{m!(n-m)!}\int_{t_2+s}^{\infty}[H(z)-H(t_1)]^{n-m}e^{-kH(z)}h(z)dz. \tag{2.3}$$

After substitution arguments, some simplifications and using the well-known relationship between incomplete gamma and Poisson sum of probabilities ([15, p.212]), Eq. (2.3) becomes

$$\begin{aligned}&\mathbb{P}(X_{U(n)}-t_2>s, X_{U(m-1)}<t_1<X_{U(m)}, X_{U(n)}>t_2)\\ &\qquad = \frac{k^m H^m(t_1)}{m!}T_{n-m}(s;t_1,t_2)e^{-kH(t_2+s)}.\end{aligned} \tag{2.4}$$

Similarly, we get

$$\begin{aligned}&\mathbb{P}(X_{U(m-1)}<t_1<X_{U(m)}, X_{U(n)}>t_2)\\ &\qquad = \frac{k^m H^m(t_1)}{m!}T_{n-m}(0;t_1,t_2)e^{-kH(t_2)}.\end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we obtain the survival function of $S_{m,n}(t_1,t_2)$ as follows:

$$\begin{aligned}R_{m,n}(s;t_1,t_2) &= \mathbb{P}(X_{U(n)}-t_2>s|X_{U(m-1)}<t_1<X_{U(m)}, X_{U(n)}>t_2)\\ &= \frac{T_{n-m}(s;t_1,t_2)}{T_{n-m}(0;t_1,t_2)}\, e^{-k[H(t_2+s)-H(t_2)]}.\end{aligned}$$

Using (1.2), the result in (2.1) follows. ∎

Remark 2.1. The MRL of kth records under double monitoring can be rewritten in the form

$$K_{m,n}(t_1,t_2) = \int_0^{\infty}\frac{\sum_{i=0}^{n-m}\mathbb{P}(Z_{t_1}^{t_2+s}=i)}{\sum_{i=0}^{n-m}\mathbb{P}(Z_{t_1}^{t_2}=i)}ds,$$

where $Z_{t_1}^{t}$ is Poisson r.v. with mean $Q(0;t_1,t_2)$, where $Q(s;t_1,t_2)=k[H(t_2+s)-H(t_1)]$. Therefore,

$$K_{m,n}(t_1,t_2) = \int_{t_2}^{\infty}\frac{\sum_{i=0}^{n-m}\mathbb{P}(Z_{t_1}^{u}=i)}{\sum_{i=0}^{n-m}\mathbb{P}(Z_{t_1}^{t_2}=i)}du.$$

Remark 2.2. The MRL of kth records under double monitoring condition $K_{m,n}(t_1,t_2)$ is still true when $n=m$. For $m=0$ with $X_{U(-1)}=0$, $K_{m,n}(t_1,t_2)$ reduces to $\mathbb{E}(X_{U(n)}-t_2|t_1< X_{U(0)}=X_{1:k}, X_{U(n)}>t_2)$. Further, consider two sequences of records from iid X-sequence and Y-sequence having the distribution functions F and G, with hazard rates h_F and h_G, respectively. If $X\le_{hr}Y$, then by Shaked and Shanthikumar [16], $\frac{\bar{G}(t_2+s)}{\bar{G}(t_1)}\ge\frac{\bar{F}(t_2+s)}{\bar{F}(t_1)}$, for all $s\ge 0$ and consequently, $K_{n,n}^F(t_1,t_2)\le K_{n,n}^G(t_1,t_2)$.

The following lemma is quite useful in the subsequent results.

Lemma 2.1. *For $t_2>t_1>0$ and $s\ge 0$, we have*

$$\begin{aligned}D(s;t_1,t_2) &= T_{n-m}(s;t_1,t_2)T_{n-m-1}(0;t_1,t_2)-T_{n-m}(0;t_1,t_2)T_{n-m-1}(s;t_1,t_2)\\ &\ge 0.\end{aligned}$$

Proof. It is sufficient to show

$$[H(t_2+s)-H(t_1)]^{n-m}\,T_{n-m-1}(0;t_1,t_2)\ge[H(t_2)-H(t_1)]^{n-m}\,T_{n-m-1}(s;t_1,t_2),$$

or equivalently

$$\sum_{i=0}^{n-m-1} \frac{k^{i-n+m}[H(t_2)-H(t_1)]^{i-n+m}}{i!} \geq \sum_{i=0}^{n-m-1} \frac{k^{i-n+m}[H(t_2+s)-H(t_1)]^{i-n+m}}{i!},$$

which readily follows from the fact that $Q(s;t_1,t_2) \geq Q(0;t_1,t_2)$.

In the following theorems we study the monotonicity properties of $K_{m,n,k}(t_1,t_2)$ via the monotonicity of $S_{m,n}(t_1,t_1)$ in the sense of the usual stochastic ordering.

Theorem 2.2. *Under the same conditions and notations of Theorem 2.1, we have*
(i) For fixed values of m, $S_{m,n}(t_1,t_2)$ is increasing in n in the sense of the usual stochastic ordering.
(ii) For fixed values of n, $S_{m,n}(t_1,t_2)$ is decreasing in m in the sense of the usual stochastic ordering.

∎

Proof. (i) The difference between the survival functions of $S_{m,n}(t_1,t_2)$ and $S_{m,n-1}(t_1,t_2)$ can be written as
(2.6)

$$R_{m,n}(s;t_1,t_2) - R_{m,n-1}(s;t_1,t_2) = \left[\frac{T_{n-m}(s;t_1,t_2)}{T_{n-m}(0;t_1,t_2)} - \frac{T_{n-m-1}(s;t_1,t_2)}{T_{n-m-1}(0;t_1,t_2)}\right] e^{-k[H(t_2+s)-H(t_2)]}.$$

The positivity of the right hand side of 2.6 follows directly from the result of Lemma 2.1.
(ii) Here, we have

$$\begin{aligned} R_{m,n}(s;t_1,t_2) - R_{m-1,n}(s;t_1,t_2) &= \left[\frac{T_{n-m}(s;t_1,t_2)}{T_{n-m}(0;t_1,t_2)} - \frac{T_{n-m+1}(s;t_1,t_2)}{T_{n-m+1}(0;t_1,t_2)}\right] e^{-k[H(t_2+s)-H(t_2)]} \\ &= R_{m,n}(s;t_1,t_2) - R_{m,n+1}(s;t_1,t_2) \\ &\leq 0. \end{aligned}$$

This is true since $S_{m,n}(t_1,t_2)$ is increasing in n in the sense of the stochastic ordering. ∎

Theorem 2.3. *Let $X_1, X_2, ...,$ be a sequence of iid r.v.'s from continuous distribution F. Then*
(i) $S_{m,n}(t_1,t_2)$ is increasing in t_1 in the sense of the usual stochastic ordering.
(ii) If F is IFR then $S_{m,n}(t_1,t_2)$ is decreasing in t_2 in the sense of the usual stochastic ordering.

Proof. (i) The result follows directly using the fact that

$$\frac{\partial}{\partial t_1}\left(\frac{T_{n-m}(s;t_1,t_2)}{T_{n-m}(0;t_1,t_2)}\right) = \frac{k\, h(t_1) D(s;t_1,t_2)}{T_{n-m}^2(0,t_1,t_2)},$$

and applying Lemma 2.1.

(ii) Now, consider

$$\frac{\partial}{\partial t_2} R_{m,n}(s;t_1,t_2) = \frac{B(s;t_1,t_2)}{\left[T_{n-m}(0;t_1,t_2)e^{-kH(t_2)}\right]^2},$$

where

$$B(s;t_1,t_2) = -ke^{-k[H(t_2+s)+H(t_2)]}\left\{[h(t_2+s)-h(t_2)]\, T_{n-m}(0,t_1,t_2) T_{n-m}(s,t_1,t_2) + D(s;t_1,t_2)\right\}.$$

From the assumption F is IFR and Lemma 2.1, the result (ii) follows. ∎

3. Stochastic ordering of the MRL of *k*th records

In this section we study some stochastic properties of the MRL of *k*th records under the double monitoring condition.

Theorem 3.1. *Let $X_{U(1)}, X_{U(2)}, \ldots,$ and $Y_{(1)}, Y_{U(2)}, \ldots,$ be two sequences of kth and k′th records from the corresponding iid r.v.'s with absolutely continuous distribution functions F and G, respectively. If $X \leq_{lr} Y$ then $S^F_{m,n}(t_1,t_2) \leq_{lr} S^G_{m,n}(t_1,t_2)$ for $1 \leq m < n$.*

Proof. The survival function of $S^F_{m,n}(t_1,t_2)$, can be written as

$$R^F_{m,n}(s;t_1,t_2) = \frac{\frac{1}{(n-m)!}\int_{t_2+s}^{\infty} Q_F^{n-m}(0;t_1,z)f(z)dz}{C^F},$$

where

$$C_F = \frac{m!\,\mathbb{P}(X_{U(m-1)} < t_1 < X_{U(m)}, X_{U(n)} > t_2)}{[H_F(t_1)]^m}.$$

Similarly, the survival function of $S^G_{m,n}(t_1,t_2)$, can be written as

$$R^G_{m,n}(s;t_1,t_2) = \frac{\frac{1}{(n-m)!}\int_{t_2+s}^{\infty} Q_G^{n-m}(0;t_1,z)f(z)dz}{C_G},$$

where

$$C_G = \frac{m!\,\mathbb{P}(Y_{U(m-1)} < t_1 < Y_{U(m)}, Y_{U(n} > t_2)}{[H_G(t_1)]^m}.$$

The density functions of $S^F_{m,n}(t_1,t_2)$ and $S^G_{m,n}(t_1,t_2)$ are respectively,

$$g^F_{m,n}(s) = \frac{Q_F^{n-m}(s;t_1,t_2)f(t_2+s)}{C_F},$$

and

$$g^G_{m,n}(s) = \frac{Q_G^{n-m}(s;t_1,t_2)g(t_2+s)}{C_G}.$$

To show the result, it is enough to show that $\triangle(s) = \frac{g^G_{m,n}(s)}{g^F_{m,n}(s)}$ is increasing function of s. Now,

$$\triangle(s) = \frac{k'}{k}\frac{C_F}{C_G}\left(\frac{log\,\bar{G}(t_1) - log\,\bar{G}(t_2+s)}{log\,\bar{F}(t_1) - log\,\bar{F}(t_2+s)}\right)^{n-m}\frac{g(t_2+s)}{f(t_2+s)}.$$

Now,

$$(3.1)\quad \frac{d}{ds}\left(\frac{log\,\bar{G}(t_1) - log\,\bar{G}(t_2+s)}{log\,\bar{F}(t_1) - log\,\bar{F}(t_2+s)}\right) = \frac{h_F(t_2+s)\,log\,\frac{\bar{G}(t_2+s)}{\bar{G}(t_1)} - h_G(t_2+s)\,log\,\frac{\bar{F}(t_2+s)}{\bar{F}(t_1)}}{\left(log\,\frac{\bar{F}(t_2+s)}{\bar{F}(t_1)}\right)^2}.$$

Under the assumption $X \leq_{lr} Y$, we have $X \leq_{hr} Y$. As a consequence of that, we have $h_F(x) \geq h_G(x)$ and then $\frac{\bar{G}(t_2+s)}{\bar{G}(t_1)} \geq \frac{\bar{F}(t_2+s)}{\bar{F}(t_1)}$ for all $s \geq 0$. This turns out that the right hand side of (3.1) is positive. By this and $g(x)/f(x)$ is increasing in x, $\triangle(s)$ is increasing function of s. This completes the proof. ∎

Remark 3.1. Under the assumptions of Theorem 3.1, we conclude that if $X \leq_{lr} Y$, then $S^F_{m,n}(t_1,t_2) \leq_{st} S^G_{m,n}(t_1,t_2)$. By Theorem 3.1, this ordering relation is still valid for $F = G$.

4. Exponential case

Let $Z_i, i \geq 1$ be a sequence of iid standard exponential r.v.'s with $h(t) = 1$, then it is known that

$$\left(X_{U(m)}, X_{U(n)}\right) \stackrel{d}{=} \left(F^{-1}\left(1 - e^{-Z_{U(m)}}\right), F^{-1}\left(1 - e^{-Z_{U(n)}}\right)\right), \tag{4.1}$$

where $Z_{U(i)}$ stands for the ith value of the kth records from standard exponential distribution (cf. [12]). Let $T_i, i \geq 1$ be the time of the ith event of the NHPP, then for $1 \leq m < n$,

$$(T_m, T_n) \stackrel{d}{=} \left(F^{-1}\left(1 - e^{-T_m^*}\right), F^{-1}\left(1 - e^{-T_n^*}\right)\right), \tag{4.2}$$

where $T_i^*, i \geq 1$ is the time of the ith event of Poisson process with $h(t) = 1$ (cf. [1]).

Now, let $N(t)$ denote the number of kth records less than or equal to t. Using (4.1) and (4.2), it follows that $N(t)$ is a counting process of events where an event is said to occur at a time which is a kth record value. Here $N(t)$ is a NHPP with hazard rate $k\, h_F$ and the k^{th} records are epoch times of this NHPP. Suppose we know that $T_{m-1} < t_1 < T_m$ and $T_n > t_2$ $(t_1 < t_2)$, where $T_i, i \geq 1$ is the time of the ith event of the NHPP. Since the sequence $\{T_i, i \geq 1\}$ is stochastically the same as the sequence $\{X_{U(i)}, i \geq 1\}$, we can apply the results obtained in Section 3 in the context of minimal repair of an item which is an important topic in the reliability theory. In fact, the times that minimal repairs occurred are distributed according to the epoch times of NHPP.

Let $\{Z_i^\lambda, i \geq 1\}$ be a sequence of iid exponential r.v.'s each with $h(t) = \lambda$. Then

$$\mathbb{E}\left(Z^\lambda_{U(n)} - t | Z^\lambda_{U(m-1)} < t_1 < Z^\lambda_{U(m)}, Z^\lambda_{U(n)} > t_2\right) = \int_0^\infty e^{-\lambda k s} \frac{\sum_{i=0}^{n-m} \frac{(\lambda k)^i}{i!} (t_2 - t_1 + s)^i}{\sum_{i=0}^{n-m} \frac{(\lambda k)^i}{i!} (t_2 - t_1)^i} ds$$

$$= \frac{\sum_{i=0}^{n-m} \frac{(\lambda k)^i}{i!} I_i}{\sum_{i=0}^{n-m} \frac{(\lambda k)^i}{i!} (t_2 - t_1)^i}, \tag{4.3}$$

where

$$I_i = \int_0^\infty (t_2 - t_1 + s)^i e^{-\lambda k s}\, ds. \tag{4.4}$$

Using integration by parts, Eq. (4.4) may be simplified as

$$I_i = \frac{1}{\lambda k}\left[(t_2 - t_1)^i + i\, I_{i-1}\right]. \tag{4.5}$$

Repeating the same process for I_{i-1} in Eq. (4.5), I_i could be evaluated to be

$$I_i = \frac{1}{\lambda k} \sum_{j=0}^{i} \frac{i!}{(i-j)!(\lambda k)^j} (t_2 - t_1)^{i-j}. \tag{4.6}$$

Substituting (4.6) in (4.3), we get

(4.7)

$$\mathbb{E}\left(Z^\lambda_{U(n)} - t | Z^\lambda_{U(m-1)} < t_1 < Z^\lambda_{U(m)}, Z^\lambda_{U(n)} > t_2\right) = \frac{\sum_{i=0}^{n-m} \frac{(\lambda k)^{i-1}}{i!} \sum_{j=0}^{i} \frac{i!}{(i-j)!(\lambda k)^j} (t_2 - t_1)^{i-j}}{\sum_{i=0}^{n-m} \frac{(\lambda k)^i}{i!} (t_2 - t_1)^i}.$$

Many characteristics of $S_{m,n}(t_1,t_2)$ defined in (1.3) cannot be easily computed when the cdf F is either known or unknown.

Below we provide an upper bounds for the MRL of kth records under double monitoring. Let $\{X_i, i \geq 1\}$ be a sequence of iid r.v.'s each with $h(t)$ which is not completely known, but it is known that for all t, $h(t) \geq L$. This means that $X_1 \leq_{hr} Z_1^L$. (cf. [16]). Under the assumption of Theorem 3.1 and using (4.7), it follows that

$$K_{m,n}(t_1,t_2) \leq \frac{\sum_{i=0}^{n-m} \frac{(L\,k)^{i-1}}{i!} \sum_{j=0}^{i} \frac{(t_2-t_1)^{i-j} i!}{(i-j)!(L\,k)^j}}{\sum_{i=0}^{n-m} \frac{(L\,k)^i}{i!} (t_2-t_1)^i},$$

where $K_{m,n}(t_1,t_2)$ is the MRL of kth records under double monitoring given in (1.2).

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