Double Entire Difference Sequence Spaces of Fuzzy Numbers

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Abstract. In this paper we introduce some double entire difference sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions \( M = (M_k, l) \). We also make an effort to study some topological properties and some inclusion relations between these spaces.

Keywords and phrases: Fuzzy numbers, Orlicz function, de La Vallee Poussin means, statistical convergence.

1. Introduction and preliminaries

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool.

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [37] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [17] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. For more details about sequence spaces of fuzzy numbers see [2,7,16,21,22,33] and references therein.

The initial works on double sequences is found in Bromwich [5]. Later on, it was studied by Hardy [11], Moricz [18], Moricz and Rhoades [19], Tripathy [34,35], Başarır and Sonalcan [3] and many others. Hardy [11] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [38] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [23] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between

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statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [20] and Mursaleen and Edely [24] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the \( M \)-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences \( x = (x_{m,n}) \) into one whose core is a subset of the \( M \)-core of \( x \). More recently, Altay and Başar [1] have defined the spaces \( BS, BS(t), CS, CS_p, CS_b, CS_r \) and \( BV \) of double sequences consisting of all double series whose sequence of partial sums are in the spaces \( M_a, M_a(t), CS_p, CS_b, CS_r \) and \( L_a \), respectively and also examined some properties of these sequence spaces and determined the \( \alpha \)-duals of the spaces \( BS, BS(t), CS_b \) and the \( \beta(v) \)-duals of the spaces \( CS_p \) and \( CS_r \) of double series. Now, recently Başar and Sever [4] have introduced the Banach space \( L_q \) of double sequences corresponding to the well known space \( \ell_q \) of single sequences and examined some properties of the space \( L_q \).

By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence \( x = (x_{k,l}) \) has Pringsheim limit \( L \) (denoted by \( P\text{-}lim x = L \)) provided that given \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) such that \( |x_{k,l} - L| < \epsilon \) whenever \( k, l > n \) see [27]. We shall write more briefly as \( P \)-convergent. The double sequence \( x = (x_{k,l}) \) is bounded if there exists a positive number \( M \) such that \( |x_{k,l}| < M \) for all \( k \) and \( l \).

An Orlicz function \( M : [0, \infty) \to [0, \infty) \) is a continuous, non-decreasing and convex function such that \( M(0) = 0, M(x) > 0 \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \).

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define the following sequence space,

\[
\ell_M = \left\{ x \in \mathbb{R} : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) < \infty \right\}
\]

which is called as an Orlicz sequence space. Also \( \ell_M \) is a Banach space with the norm

\[
||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}.
\]

Also, it was shown in [13] that every Orlicz sequence space \( \ell_M \) contains a subspace isomorphic to \( \ell_p(p \geq 1) \). The \( \Delta_2 \)- condition is equivalent to \( M(Lx) \leq LM(x) \), for all \( L \) with \( 0 < L < 1 \). An Orlicz function \( M \) can always be represented in the following integral form

\[
M(x) = \int_{0}^{x} \eta(t)dt
\]

where \( \eta \) is known as the kernel of \( M \), is right differentiable for \( t \geq 0, \eta(0) = 0, \eta(t) > 0, \eta \) is non-decreasing and \( \eta(t) \to \infty \) as \( t \to \infty \).

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping \( X : \mathbb{R}^n \to [0,1] \) which satisfies the following four conditions:

1. \( X \) is normal, i.e., there exist an \( x_0 \in \mathbb{R}^n \) such that \( X(x_0) = 1 \);
2. \( X \) is fuzzy convex, i.e., for \( x, y \in \mathbb{R}^n \) and \( 0 \leq \lambda \leq 1, X(\lambda x + (1 - \lambda)y) \geq \min[X(x), X(y)] \);
3. \( X \) is upper semi-continuous;
4. the closure of \( \{x \in \mathbb{R}^n : X(x) > 0\} \), denoted by \( [X]^0 \), is compact.

Let \( C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \) is compact and convex \} \). The spaces \( C(\mathbb{R}^n) \) has a linear structure induced by the operations

\[
A + B = \{a + b, a \in A, b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a : a \in A\}
\]
for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between $A$ and $B$ of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (non separable) metric space.

For $0 < \alpha \leq 1$, the $\alpha$-level set

$$X^\alpha = \{ x \in \mathbb{R}^n : X(x) \geq \alpha \}$$

is a nonempty compact convex, subset of $\mathbb{R}^n$, as is the support $X^0$. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces addition $X + Y$ and scalar multiplication $\lambda X$, $\lambda \in \mathbb{R}$, in terms of $\alpha$-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha \quad \text{and} \quad [\lambda X]^\alpha = \lambda [X]^\alpha$$

for each $0 \leq \alpha \leq 1$. Define for each $1 \leq q < \infty$

$$d_q(X, Y) = \left\{ \int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q \, d\alpha \right\}^{1/q}$$

and $d_\infty(X, Y) = \sup_{0 < \alpha \leq 1} \delta_\infty(X^\alpha, Y^\alpha)$. Clearly $d_\infty(X, Y) = \lim_{q \to \infty} d_q(X, Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover $(L(\mathbb{R}^n), d_\infty)$ is a complete, separable and locally compact metric space. Let $w$ denote the set of all fuzzy complex sequences $x = (x_k)_{k=1}^\infty$, and $M$ be an Orlicz function, or a modulus function. Consider

$$\Gamma_M = \left\{ x \in w : \lim_{k \to \infty} \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) = 0 \text{ for some } \rho > 0 \right\}$$

and

$$\Lambda_M = \left\{ x \in w : \sup_k \left( M \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) < \infty \text{ for some } \rho > 0 \right\}.$$ 

The spaces $\Gamma_M$ and $\Lambda_M$ are metric spaces with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right) \right) \leq 1 \right\}$$

for all $x = (x_k)$ and $y = (y_k)$ in $\Gamma_M$.

In this paper we define Orlicz space of double entire sequence of fuzzy numbers by using regular matrix $A = (a_{mnk}) (m, n, k, l = 1, 2, 3, \ldots)$. By the regularity of $A$ we mean that the matrix which transform convergent sequence into a convergent sequence leaving the limit for detail see [12]; we prove that these spaces are complete paranormed spaces. If $E$ is a linear space over the complex field $\mathbb{C}$, then a paranorm on $E$ is a function $p : E \to \mathbb{R}$ which satisfies the following axioms, for $x, y \in E$,

1. $p(x) \geq 0$, for all $x \in X$;
2. $p(-x) = p(x)$, for all $x \in X$;
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$;
4. if $(\sigma_n)$ is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$. 

Define for each $1 \leq q < \infty$
A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [36, Theorem 10.4.2, p.183]). For more details about sequence spaces (see [14,15,25,28,29,30,33]) and references therein.

The notion of difference sequence spaces was introduced by Kizmaz [12], who studied the difference sequence spaces $l_ω(Δ)$, $c(Δ)$ and $c_0(Δ)$. The notion was further generalized by Et and Çolak [8] by introducing the spaces $l_ω(Δ^n)$, $c(Δ^n)$ and $c_0(Δ^n)$.

Let $r$ be non-negative integers. Then for $Z = l_ω$, $c$, $c_0$, we have sequence spaces

$Z(Δ^r) = \{ x = (x_k) \in w : (Δ^r x_k) \in Z \}$,

where $Δ^r x_k = (Δ^r_{-1} x_k - Δ^r_{-1} x_{k+1})$ and $Δ^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$Δ^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}$.

A sequence $X = (X_k)$ of fuzzy numbers is a function $X$ from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R}^n)$. The fuzzy number $X_n$ denotes the value of the function $n \in \mathbb{N}$ and is called $n^{th}$ term of the sequence.

Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the Orlicz space of entire sequences of fuzzy numbers convergent to zero, written as $(M(\langle |X_k|^{1/k} \rangle / ρ)) \rightarrow 0$ as $k \rightarrow \infty$, for some arbitrarily fixed $ρ > 0$ and is defined by

$$\tilde{d}\left(M\left(\frac{|X_k|^{1/k}}{\rho}\right)\rightarrow 0 \text{ as } k \rightarrow \infty\right)$$

is denoted by $Γ_M(F)$, with $M$ being a Orlicz function.

Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the space consisting of all those fuzzy sequence $X = (X_k)$ such that $\sup \{M(\langle |X_k|^{1/k} \rangle / ρ)\} < \infty$ for some arbitrary $ρ > 0$ is denoted by $Λ_M(F)$ and is known as Orlicz space of analytic sequences.

A fuzzy double sequence is a double infinite array of fuzzy numbers. We denote a fuzzy double sequence by $(X_{m,n})$, where $X_{m,n}$’s are fuzzy numbers for each $m, n \in \mathbb{N}$. By $s''(F)$ we denote the set of all double sequences of fuzzy numbers.

A double sequence $X = (X_{k,l})$ of fuzzy numbers is said to be convergent in the Pringsheim’s sense or P-convergent to a fuzzy number $X_0$, if for every $ε > 0$ there exists $N \in \mathbb{N}$ such that

$$\tilde{d}(X_{k,l}, X_0) < ε \text{ for } k, l > N,$$

where $\mathbb{N}$ is the set of natural numbers, and we denote by $P - \lim X = X_0$. The number $X_0$ is called the Pringsheim limit of $X_{k,l}$. More exactly we say that a double sequence $(X_{k,l})$ converges to a finite number $X_0$ if $X_{k,l}$ tend to $X_0$ as both $k$ and $l$ tend to $∞$ independently of one another.

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} ≤ \sup_{k,l} p_{k,l} = H$ and let $D = \max \{1, 2H^{-1}\}$. Then for the factorable sequences $\{a_{k,l}\}$ and $\{b_{k,l}\}$ in the complex plane, we have

$$(1.1) \quad |a_{k,l} + b_{k,l}|^{p_{k,l}} ≤ D(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}).$$
Let \( \mathcal{M} = (M_{k,l}) \) be a sequence of Orlicz functions and \( A = (a_{mnkl}) \) be a regular matrix. Then we define the following classes of sequences in the present paper:

\[
\Gamma_{\mathcal{M}}^2[F,A,p,\Delta'] = \left\{ X = (X_{k,l}) \in s''(F) : \lim_{m,n \to \infty} \frac{1}{mn} \sum_{k,l=1}^{m,n} a_{mnkl} \left( \frac{\bar{d}(\Delta'X_{k,l},\bar{0})^{1/k+l}}{\rho} \right) \right\}^{p_{k,l}} = 0, 
\]

and

\[
\Lambda_{\mathcal{M}}^2[F,A,p,\Delta'] = \left\{ X = (X_{k,l}) \in s''(F) : \sup_{m,n} \frac{1}{mn} \sum_{k,l=1}^{m,n} a_{mnkl} \left( \frac{\bar{d}(\Delta'X_{k,l},\bar{0})^{1/k+l}}{\rho} \right) \right\}^{p_{k,l}} < \infty 
\]

and call them respectively the spaces of double sequences of fuzzy numbers which are strongly entire and strongly analytic.

If we take \( a_{mnkl} = 1 \) for all \( m,n,k,l \in \mathbb{N} \) then \( \Gamma_{\mathcal{M}}^2[F,A,p,\Delta'] \) and \( \Lambda_{\mathcal{M}}^2[F,A,p,\Delta'] \) are reduced to \( \Gamma_{\mathcal{M}}[F,p,\Delta'] \) and \( \Lambda_{\mathcal{M}}^2[F,p,\Delta'] \), respectively, defined as

\[
\Gamma_{\mathcal{M}}^2[F,p,\Delta'] = \left\{ X = (X_{k,l}) \in s''(F) : \lim_{m,n \to \infty} \frac{1}{mn} \sum_{k,l=1}^{m,n} M_{k,l} \left( \frac{\bar{d}(\Delta'X_{k,l},\bar{0})^{1/k+l}}{\rho} \right) \right\}^{p_{k,l}} = 0, 
\]

and

\[
\Lambda_{\mathcal{M}}^2[F,p,\Delta'] = \left\{ X = (X_{k,l}) \in s''(F) : \sup_{m,n} \frac{1}{mn} \sum_{k,l=1}^{m,n} M_{k,l} \left( \frac{\bar{d}(\Delta'X_{k,l},\bar{0})^{1/k+l}}{\rho} \right) \right\}^{p_{k,l}} < \infty. 
\]

A metric \( \bar{d} \) on \( L(\mathbb{R}^n) \) is said to be translation invariant if \( \bar{d}(X+Z,Y+Z) = \bar{d}(X,Y) \) for \( X,Y,Z \in L(\mathbb{R}^n) \).

The main purpose of this paper is to introduce some double entire difference sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions. We also make an effort to study some topological properties and inclusion relations between the above defined sequence spaces.

2. Main results

**Proposition 2.1.** If \( \bar{d} \) is a translation invariant metric on \( L(\mathbb{R}^n) \), then

(i) \( \bar{d}(\Delta'X + \Delta'Y,\bar{0}) \leq \bar{d}(\Delta'X,\bar{0}) + \bar{d}(\Delta'Y,\bar{0}) \),

(ii) \( \bar{d}(\lambda \Delta'X,\bar{0}) \leq |\lambda| \bar{d}(\Delta'X,\bar{0}), |\lambda| > 1. \)

**Proof.** It is easy to prove so we omit the details.

**Theorem 2.1.** \( \Gamma_{\mathcal{M}}^2[F,p,\Delta'] \) is a complete metric space with the metric is given by

\[
\rho(X,Y) = \sup_{m,n} \left( \frac{1}{mn} \sum_{k,l=1}^{m,n} M_{k,l} \left( \frac{\bar{d}(\Delta'(X_{mn} + Y_{mn}),\bar{0})^{1/m+n}}{\rho} \right) \right)^p, 
\]

where \( X = (\Delta'X_{mn}) \) and \( Y = (\Delta'Y_{mn}) \) are the double difference sequences of fuzzy numbers.
Proof. It is easy to show that this is a metric space. We will show that it is complete. Let \( \{X^{(i)}\} \) be a Cauchy sequence in \( \Gamma^2_{d}[F, p, \Delta'] \). Then
\[
\left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}, \tilde{0})^{1/m+n}}{\rho} \right) \right)
\]
will be a Cauchy sequence in \( \Gamma_{d}[F, p, \Delta'] \). Therefore for each \( m \) and \( n \), we have
\[
\left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}, \tilde{0})^{1/m+n}}{\rho} \right) \right)
\]
is a Cauchy sequence in \( L(\mathbb{R}^n) \). Since \( L(\mathbb{R}^n) \) is complete, thus
\[
\left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}, \tilde{0})^{1/m+n}}{\rho} \right) \right) \right) \to 0 \quad \text{as} \quad i \to \infty.
\]
Put \( \Delta X = (\Delta^{i}X_{mn}) \), since
\[
\left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}, \tilde{0})^{1/m+n}}{\rho} \right) \right)
\]
is a Cauchy sequence in \( \Gamma_{d}[F, p, \Delta'] \), there exists \( n_0 \in N \) such that for all \( i \).
\[
\left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}, \tilde{0})^{1/m+n}}{\rho} \right) \right) \right) \to 0 \quad \text{as} \quad m, n \to \infty.
\]
Let \( j \to \infty \), we get
\[
\left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}, \tilde{0})^{1/m+n}}{\rho} \right) \right) \right) \to 0 \quad \text{as} \quad m, n \to \infty.
\]
Therefore \( (M_{k,l}((\tilde{d}(\Delta^{i}X_{mn}, \tilde{0})^{1/m+n})/\rho)) \to 0 \) as \( i \to \infty \).

Now we have to show that \( X \in \Gamma_{d}[F, p, \Delta'] \). Since \( \Delta^{i}X_{mn}^{(i)} \in \Gamma_{d}[F, p, \Delta'] \), there exists \( \Delta^{i}X_{0}^{(i)} \in L(\mathbb{R}^n) \) such that
\[
\left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}^{(i)} - \Delta^{i}X_{0}^{(i)}, \tilde{0})^{1/m+n}}{\rho} \right) \right) \right) \to 0 \quad \text{as} \quad m, n \to \infty.
\]
Hence
\[
\left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}^{(i)} - \Delta^{i}X_{0}^{(i)}, \tilde{0})^{1/m+n}}{\rho} \right) \right) \right) ^{p} \leq \left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}^{(i)} - \Delta^{i}X_{0}^{(i)}, \tilde{0})^{1/m+n}}{\rho} \right) \right) \right) ^{p} + \left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}^{(i)} - \Delta^{i}X_{0}^{(i)}, \tilde{0})^{1/m+n}}{\rho} \right) \right) \right) ^{p} + \left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta^{i}X_{mn}^{(i)} - \Delta^{i}X_{0}^{(i)}, \tilde{0})^{1/m+n}}{\rho} \right) \right) \right) ^{p}.
\]
\[ \rightarrow 0 \text{ as } m, n \rightarrow \infty. \] Thus \( (M_{k,l}\left(\frac{\tilde{d}(\Delta'X_{mn}^{(i)} - \Delta'X_{0}, \bar{0})^{1/m+n}}{\rho}\right)) \) is a Cauchy sequence in \( L(\mathbb{R}^n) \).

Since \( L(\mathbb{R}^n) \) is complete, there exists \( \Delta'X_0 \in L(\mathbb{R}^n) \) such that
\[
\left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{mn}^{(i)} - \Delta'X_{0}, \bar{0})^{1/m+n}}{\rho} \right) \right) \right)^{p} \rightarrow 0 \text{ as } m, n \rightarrow \infty.
\]
Therefore
\[
\begin{align*}
&\leq \left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{mn}^{(i)} - \Delta'X_{mn}^{(j)}, \bar{0})^{1/m+n}}{\rho} \right) \right) \right)^{p} \\
&\quad + \left( \frac{1}{mn} \left( M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{mn}^{(i)} - \Delta'X_{0}, \bar{0})^{1/m+n}}{\rho} \right) \right) \right)^{p}
\end{align*}
\]
\[ \rightarrow 0 \text{ as } m, n \rightarrow \infty. \] This implies that \( X = (X_{k,l}) \in \Gamma_2^d[F, p, \Delta'] \). This completes the proof of the theorem.

**Theorem 2.2.** \( \Gamma_2^d[F, A, p, \Delta'] \) and \( \Lambda_2^d[F, A, p, \Delta'](\inf p_k > 0) \) are complete with respect to the topology generated by the paranorm \( h \) is defined by
\[
h(X) = \sup_{m,n} \left( \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}^{(i)} - \Delta'X_{k,l}^{(j)}, \bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \right), \quad \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \rightarrow 0 \text{ as } k, l \rightarrow \infty,
\]
where \( \tilde{d} \) is a translation invariant and \( X = (X_{k,l}) \) be a double sequence of fuzzy numbers.

**Proof.** Let \( (X^{(s)}) \) be a Cauchy sequence in \( \Gamma_2^d[F, A, p, \Delta'] \). Then
\[
\left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'(X^{(s)} - X^{(t)}), \bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \rightarrow 0 \text{ as } s, t \rightarrow \infty,
\]
that is
\[
\lim_{s,t \rightarrow \infty} \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'(X^{(s)}_{k,l} - X^{(t)}_{k,l}), \bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} = 0 \text{ for all } m,n.
\]
Hence
\[
\left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'(X^{(s)}_{k,l} - X^{(t)}_{k,l}), \bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \rightarrow 0 \text{ as } s, t \rightarrow \infty \text{ for all } k, l,
\]
which implies that \( \left[ M_{k,l} \left( (\tilde{d}(\Delta'X^{(s)}_{k,l}, \bar{0})^{1/k+l}) / \rho \right) \right]^{p_{k,l}} \) is a Cauchy sequence in \( \mathbb{R} \) for each \( k, l \) and so there exists \( Y = (\Delta'Y_{k,l}) \) such that \( \left[ M_{k,l} \left( (\tilde{d}(\Delta'X^{(s)}_{k,l}, \bar{0})^{1/k+l}) / \rho \right) \right]^{p_{k,l}} \rightarrow (\Delta'Y_{k,l}) \) as
\( s \to \infty \) for each \( k, l \). Now, from (2.2) we have, for \( \varepsilon > 0 \), there exists natural number \( N \) such that

\[
\left( \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y_{k,l}^{(t)}), 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \right) < \varepsilon
\]

for \( s, t > N \) and for all \( m, n \). Hence, for any natural number \( g \), we have from (2.3)

\[
\left( \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y_{k,l}^{(t)}), 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \right) < \varepsilon \quad \text{for } s, t > N \text{ for all } n.
\]

Now fix \( s > N \) and let \( t \to \infty \). Then from (2.3), we have

\[
\left( \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y_{k,l}^{(t)}), 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \right) < \varepsilon \quad \text{for } s > N \text{ for all } n.
\]

Since this is valid for any natural number \( g \), we have

\[
\left( \sum_{k,l=1}^{m,n} \frac{a_{mnkl}}{mn} \left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y_{k,l}^{(t)}), 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \right) < \varepsilon \quad \text{for } s > N \text{ for all } m, n,
\]

that is,

\[
\left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y_{k,l}^{(t)}), 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \to 0 \quad \text{as } s \to \infty,
\]

and thus \( \left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y, 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \to (\Delta)Y \) as \( s \to \infty \), and therefore

\[
\left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y, 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \in \Gamma_{\Delta}^2[F, A, p, \Delta'].
\]

Hence \( \Gamma_{\Delta}^2[F, A, p, \Delta'] \) is complete. Similarly we can prove the completeness of \( \Lambda_{\Delta}^2[F, A, p, \Delta'] \).

**Theorem 2.3.** Let \( X = (X_{k,l}) \) be a double sequence of fuzzy numbers and \( \bar{d} \) a translation invariant. Let \( A = (a_{mnkl}), (m, n, k, l = 1, 2, 3, \ldots) \) be an infinite matrix. Then \( \Gamma_{\Delta}^2[F, A, p] \subset \Gamma_{\Delta}^2[F, A, p, \Delta'] \) if and only if given \( \varepsilon > 0 \), there exists \( N = N(\varepsilon) > 0 \) such that

\[
|\Delta^{-1} a_{mnkl} - \Delta^{-1} a_{m+1,n+1,k,l}| < \varepsilon^{m,n} \Delta K_{l} \quad (m, n, k, l = 1, 2, 3, \ldots).
\]

**Proof.** Let \( X = (X_{k,l}) \in \Gamma_{\Delta}^2[F, A, p] \) and let

\[
Y_{m,n} = \left( \sum_{k,l=1}^{m,n} a_{mnkl} \left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y_{k,l}^{(t)}), 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \right) \quad (m, n = 1, 2, 3, \ldots),
\]

so that \( \Delta Y_{m,n} = \left( \sum_{k,l=1}^{m,n} \left( \Delta^{-1} a_{mnkl} - \Delta^{-1} a_{m+1,n+1,k,l} \right) \right) \left[ M_{k,l} \left( \frac{\bar{d}(\Delta r(X_{k,l}^{(s)} - Y_{k,l}^{(t)}), 0)^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \).

Then \( \Delta Y_{m,n} \in \Gamma_{\Delta}^2[F, A, p] \) if and only if given any \( \varepsilon > 0 \) there exists \( N = N(\varepsilon) > 0 \) such that \( |\Delta^{-1} a_{mnkl} - \Delta^{-1} a_{m+1,n+1,k,l}| < \varepsilon^{m,n} \Delta K_{l} \). Now \( \Delta Y_{m,n} \in \Gamma_{\Delta}^2[F, A, p] \) if and only if \( (Y_{m,n}) \in \Gamma_{\Delta}^2[F, A, p, \Delta'] \). Thus \( \Gamma_{\Delta}^2[F, A, p] \subset \Gamma_{\Delta}^2[F, A, p, \Delta'] \) if and only if the condition holds. This completes the proof.
**Theorem 2.4.** Let $X = (X_{k,l})$ be a double sequence of fuzzy numbers and $\tilde{d}$ be a translation invariant. If $A = (a_{mnlk})$ transform $\Gamma_{\Delta}^2 [F, A, p]$ into $\Gamma_{\Delta}^2 [F, A, p, \Delta']$. Then $\lim_{n \to \infty} (\Delta^{-1} a_{mnlk} - \Delta^{-1} a_{m+1,n+1,k,l}) q^{m,n} = 0$ for all integers $q > 0$ and each fixed $k, l = 1, 2, 3, \ldots$.

**Proof.** Let $Y_{m,n} = (\sum_{k,l=1}^{\infty} a_{mnlk}) [M_{k,l} ((\Delta^j X_{k,l}, 0)^{1/k+l}) / \rho] (m, n = 1, 2, 3, \ldots)$ formally. Let $(X_{k,l}) \in \Gamma_{\Delta}^2 [F, A, p]$ and $(Y_{m,n}) \in \Gamma_{\Delta}^2 [F, A, p, \Delta']$. But then $(\Delta^j Y_{m,n}) \in \Gamma_{\Delta}^2 [F, A, p]$,

$$\Delta^j Y_{m,n} = \left( \sum_{k,l=1}^{\infty} (\Delta^{-1} a_{m,n,k,l} - \Delta^{-1} a_{m+1,n+1,k,l}) M_{k,l} \left( \frac{(\Delta X_{k+1,l+1}, 0)^{1/k+l}}{\rho} \right) \right) (m, n = 1, 2, 3, \ldots).$$

Take $(X_{k,l}) = (0, 0, 0, \ldots, 1, 0, 0, \ldots, 1)$ in the $k^{th}$ place and zero’s elsewhere. Then $(X_{k,l}) \in \Gamma_{\Delta}^2 [F, A, p]$. We have $\Delta^j Y_{m,n} = \Delta^j a_{m,n,k,l} - \Delta^j a_{m+1,n+1,k,l}$. But $(\Delta^j Y_{m,n}) \in \Gamma_{\Delta}^2 [F, A, p]$. Hence $\sum_{k,l=1}^{\infty} (\Delta^{-1} a_{m,n,k,l} - \Delta^{-1} a_{m+1,n+1,k,l}) q^{m,n} < \infty$ for all integers $q$ and each fixed $k, l = 1, 2, 3, \ldots$. This completes the proof. $\blacksquare$

**Theorem 2.5.** Let $X = (X_{k,l})$ be a double sequence of fuzzy numbers and $\tilde{d}$ be a translation invariant. If $A = (a_{mnlk})$ transform $\Gamma_{\Delta}^2 [F, A, p, \Delta']$ into $\Gamma_{\Delta}^2 [F, A, p]$. Then $\lim_{m,n \to \infty} a_{mnlk} q^{m,n} = 0$ for all integers $q$.

**Proof.** Let

$$t_{m,n} = \left( \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left( M_{k,l} \left( \frac{(\Delta X_{k+1,l+1}, 0)^{1/k+l}}{\rho} \right) \right) \right),$$

with $(X_{k,l}) \in \Gamma_{\Delta}^2 [F, A, p, \Delta']$, $(t_{m,n}) \in \Gamma_{\Delta}^2 [F, A, p]$ and

$$s_{m,n} = \left( \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left( M_{k,l} \left( \frac{(\Delta X_{k+1,l+1}, 0)^{1/k+l}}{\rho} \right) \right) \right),$$

$(s_{m,n}) \in \Gamma_{\Delta}^2 [F, A, p]$. Then

$$Y_{m,n} = (t_{m,n} - s_{m,n}) = \left( \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left( M_{k,l} \left( \frac{(\Delta X_{k+1,l+1}, 0)^{1/k+l}}{\rho} \right) \right) \right),$$

and $\Delta^j X_{k,l} \in \Gamma_{\Delta}^2 [F, A, p]$ and $(Y_{m,n}) \in \Gamma_{\Delta}^2 [F, A, p]$. Hence $(a_{m,n,k,l}) q^{m,n} \to 0$ as $n \to \infty$ for all $k, l$. This completes the proof of the theorem. $\blacksquare$

**Theorem 2.6.** Let $X = (X_{k,l})$ be a double sequence of fuzzy numbers and $\tilde{d}$ be a translation invariant. If $A = (a_{mnlk})$ transforms $\Gamma_{\Delta}^2 [F, A, p, \Delta']$ into $\Gamma_{\Delta}^2 [F, A, p, \Delta']$, then $(a_{mnlk}) q^{m,n} \to 0$ and $(a_{m+1,n+1,k,l}) q^{m,n} \to 0$ as $m, n \to \infty$.

**Proof.** From Theorems 2.3 and 2.4 we have $a_{mnlk} q^{m,n} \to 0$ and

$$(\Delta^{-1} a_{m,n,k,l} - \Delta^{-1} a_{m+1,n+1,k,l}) q^{m,n} \to 0$$

for all positive integers $q$ and for all $k, l$.

$\Rightarrow (a_{mnlk}) q^{m,n} \to 0$ and $(\Delta^{-1} a_{mnlk} - \Delta^{-1} a_{m+1,n+1,k,l}) q^{m,n} \to 0$

$\Rightarrow (a_{m+1,n+1,k,l}) q^{m,n} \to 0$ and $(a_{mnlk}) q^{m,n} \to 0$ as $m, n \to \infty$ for all $k, l$. 

This completes the proof.

3. $\Delta'$-statistical convergence

The idea of statistical convergence of a sequence was introduced by Fast [9]. Statistical convergence was generalized by Buck [6] and studied by many other authors, using a regular nonnegative summability matrix $A$ in place of Cesaro matrix. The existing literature on statistical convergence have been restricted to real or complex analysis, but at the first time Nuray and Savas [26] extended the idea to apply the sequences of fuzzy numbers. For more details on fuzzy sequence spaces and statistical convergence see [10,31,32] and references therein. The generalized de la Vallee-Pousin mean is defined by $t_n(X) = \frac{1}{\lambda_n} \sum_{k \in I_n} X_k$, where $\lambda = (\lambda_n)$ is non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$.

The double sequence $\lambda_2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$
\lambda_{m+1,n} \leq \lambda_{m,n} + 1, \quad \lambda_{m,n+1} \leq \lambda_{m,n} + 1,
$$

$$
\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \quad \lambda_{1,1} = 1,
$$

and $I_{m,n} = \{(k,l) : m - \lambda_{m,n} + 1 \leq k \leq m, \ n - \lambda_{m,n} + 1 \leq l \leq n\}$. The generalized double de la Vallee-Poussin mean is defined by

$$
t_{m,n} = t_{m,n}(x_{k,l}) = \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} x_{k,l}.
$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be $\Delta'$-statistical convergent to fuzzy number zero if

$$
\lim_{m,n \to \infty} \frac{1}{mn} \left\{ k, l \leq m, n : \left| M_{k,l} \left( \frac{d(\Delta'X_k, \bar{\theta})^{1/k+l}}{\rho} \right) \right| \geq \epsilon \right\} = 0,
$$

i.e., $\left[ M_{k,l} \left( (d(\Delta'X_k, \bar{\theta})^{1/k+l})/\rho \right) \right] < \epsilon$. In this case we write

$$
St(\Delta') - \lim_{k,l \to \infty} M_{k,l} \left( \frac{(X_k, \bar{\theta})^{1/k+l}}{\rho} \right) = 0.
$$

Let $X = (X_{k,l})$ be a sequence of fuzzy numbers and $p = (p_{k,l})$ be a sequence of strictly positive real numbers. Then the sequence $X = (X_{k,l})$ is said to be strongly $\Delta'$-convergent if there is a fuzzy number zero such that

$$
\lim_{m,n \to \infty} \frac{1}{mn} \sum_{k \leq n} M_{k,l} \left( \frac{d(\Delta'X_k, \bar{\theta})^{1/k+l}}{\rho} \right)^{p_{k,l}} = 0.
$$

Let $X = (X_{k,l})$ be a sequence of fuzzy numbers. Then the space consisting of all those fuzzy sequences $X = (X_{k,l})$ such that $\sup(M_{k,l}(X_k, \bar{\theta})/\rho) < \infty$ for some arbitrary $\rho > 0$ is denoted by $\Lambda_M(\Delta')$ and is known as $\Delta'$-Orlicz space of analytic sequences of fuzzy numbers.

**Theorem 3.1.** If $(X_{k,l}), (Y_{k,l}) \in St(\Delta')$ and $c \in L(\mathbb{R}^n)$, then

(i) $St(\Delta') - \lim \left[ c M_{k,l} \left( \frac{(\Delta'X_{k,l}, \bar{\theta})^{1/k+l}}{\rho} \right) \right] = c St(\Delta') - \lim M_{k,l} \left( \frac{(\Delta'X_{k,l}, \bar{\theta})^{1/k+l}}{\rho} \right),$
By Minkowski’s inequality we get

\[ (ii) \ St(\Delta^r) - \lim \left[ M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right] \]

\[ = St(\Delta^r) - \lim \left[ M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right] + St(\Delta^r) - \lim \left[ M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right], \]

where \( \bar{d} \) is a translation invariant.

**Proof.** (i) Let \( St(\Delta^r) - \lim \left[ M_{k,l} \left( \left( \left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1} \right) / \rho \right) \right] \) \( c \in L(\mathbb{R}) \) and \( \varepsilon > 0 \) be given. Then the proof follows from the following inequality

\[ \frac{1}{mn} \left\{ k, l \leq m, n : \left| M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right| \geq \varepsilon \right\} \]

\[ \leq \frac{1}{mn} \left\{ k, l \leq m, n : \left| M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right| \geq \varepsilon \right\}. \]

(ii) Suppose that

\[ St(\Delta^r) - \lim \left[ M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right] = 0 \]

and

\[ St(\Delta^r) - \lim \left[ M_{k,l} \left( \frac{\left( \Delta'Y_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right] = 0. \]

By Minkowski’s inequality we get

\[ \left[ M_{k,l} \left( \frac{\left( \Delta'X_{k,l} + \Delta'Y_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right] \]

\[ = \left[ M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right] + \left[ M_{k,l} \left( \frac{\left( \Delta'Y_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right]. \]

Therefore given \( \varepsilon > 0 \) we have

\[ \frac{1}{mn} \left\{ k, l \leq m, n : \left| M_{k,l} \left( \frac{\left( \Delta'X_{k,l} + \Delta'Y_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right| \geq \varepsilon \right\} \]

\[ \leq \frac{1}{mn} \left\{ k, l \leq m, n : \left| M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right| \geq \varepsilon \right\} \]

\[ + \frac{1}{mn} \left\{ k, l \leq m, n : \left| M_{k,l} \left( \frac{\left( \Delta'Y_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right| \geq \varepsilon \right\}. \]

Hence \( St(\Delta^r) - \lim \left[ M_{k,l} \left( \left( \left( \Delta'X_{k,l} + \Delta'Y_{k,l} \right), \bar{0} \right)^{1/k+1} \right) / \rho \right] = 0 \). This completes the proof.

**Theorem 3.2.** If a sequence \( X = (X_{k,l}) \) is \( \Delta^r \)-statistically convergent to the fuzzy number zero and \( \liminf_{mn} (\lambda_{mn}/mn) > 0 \), then it is \( \Delta^r \)-statistically convergent to zero.

**Proof.** Given \( \varepsilon > 0 \) we have

\[ \frac{1}{mn} \left\{ k, l \leq m, n : \left| M_{k,l} \left( \frac{\left( \Delta'X_{k,l}, \bar{0} \right)^{1/k+1}}{\rho} \right) \right| \geq \varepsilon \right\} \]
\[ \frac{1}{mn} \left\{ k, l \leq I_{m,n}: \left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \varepsilon \right\} \]

Therefore,
\[ \frac{1}{mn} \left\{ k, l \leq m, n: \left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \varepsilon \right\} \]
\[ \geq \frac{1}{mn} \left\{ k, l \leq I_{m,n}: \left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \varepsilon \right\} \]
\[ \geq \frac{\lambda_{mn}}{mn} \frac{1}{\lambda_{mn}} \left\{ k, l \leq I_{m,n}: \left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right] \geq \varepsilon \right\}. \]

Taking limit as \( m, n \to \infty \) and using \( \liminf_{mn}(\lambda_{mn}/mn) > 0 \), we get \( X \) is \( \Delta' \)-statistically convergent to zero. This completes the proof.

**Theorem 3.3.** Let \( 0 \leq p_{k,l} \leq q_{k,l} \) and let \( \{q_{k,l}/p_{k,l}\} \) be bounded. Then \( \Gamma^2_{\Delta}[F,q,\Delta'] \subset \Gamma^2_{\Delta}[F,p,\Delta']. \)

**Proof.** Let \( X \in \Gamma^2_{\Delta}[F,q,\Delta'] \) and given \( \varepsilon > 0 \), we have

\[ (3.1) \quad \frac{1}{mn} \left\{ k, l \leq m, n: \left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right]^{q_{k,l}} \geq \varepsilon \right\} \to 0 \text{ as } m, n \to \infty. \]

Let \( t_{k,l} = \left[ M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right]^{q_{k,l}} \) and \( \lambda_{k,l} = p_{k,l}/q_{k,l} \). Since \( p_{k,l} \leq q_{k,l} \), we have
\[ 0 \leq \lambda_{k,l} \leq 1. \]

Take \( 0 < \lambda < \lambda_{k,l} \). Define \( u_{k,l} = t_{k,l}(t_{k,l} \geq 1); u_{k,l} = 0(t_{k,l} < 1) \) and \( v_{k,l} = 0(t_{k,l} \geq 1); v_{k,l} = t_{k,l}(t_{k,l} < 1), t_{k,l} = u_{k,l} + v_{k,l} \), i.e., \( t_{k,l} = \lambda_{k,l} u_{k,l} + \lambda_{k,l} v_{k,l} \). Now it follows that

\[ (3.3) \quad u_{k,l} \leq u_{k,l} \leq t_{k,l} \quad \text{and} \quad v_{k,l} \leq v_{k,l}. \]

Since \( t_{k,l} = \lambda_{k,l} u_{k,l} + \lambda_{k,l} v_{k,l} \), then \( t_{k,l} \leq \lambda_{k,l} u_{k,l} + \lambda_{k,l} v_{k,l} \),

\[ \frac{1}{mn} \left\{ k, l \leq m, n: \left( M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right)^{q_{k,l}} \geq \varepsilon \right\} \leq \frac{1}{mn} \left\{ k, l \leq m, n: \left( M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right)^{q_{k,l}} \geq \varepsilon \right\} \]
\[ \Rightarrow \frac{1}{mn} \left\{ k, l \leq m, n: \left( M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right)^{q_{k,l}} \geq \varepsilon \right\} \]
\[ \leq \frac{1}{mn} \left\{ k, l \leq m, n: \left( M_{k,l} \left( \frac{\tilde{d}(\Delta'X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right)^{q_{k,l}} \geq \varepsilon \right\} \]

where
\[ \lambda_{k,l} = p_{k,l}/q_{k,l} \text{ and } \lambda_{k,l} \leq 1. \]
From (3.1) and (3.4) we get

\[ \Gamma \]

Hence

\[ \text{(3.4)} \]

Therefore

\[ \text{(3.4)} \]

\[ 1/(mn) \left\{ \{ k, l \leq m, n : \left[ M_{k,l} \left( \frac{d(\Delta' X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right]^{q_{k,l}} \geq \varepsilon \} \right\} \]

But

\[ 1/(mn) \left\{ \{ k, l \leq m, n : \left[ M_{k,l} \left( \frac{d(\Delta' X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \} \right\} \]

\[ 0 \text{ as } m, n \to \infty \text{ by (3.2).} \]

Therefore

\[ 1/(mn) \left\{ \{ k, l \leq m, n : \left[ M_{k,l} \left( \frac{d(\Delta' X_{k,l}, \bar{0})^{1/k+l}}{\rho} \right) \right]^{q_{k,l}} \geq \varepsilon \} \right\} \]

\[ 0 \text{ as } m, n \to \infty. \]

Hence

\[ \text{(3.4)} \]

\[ X \in \Gamma_{\gamma}^2 [F, p, \Delta']. \]

From (3.1) and (3.4) we get \( \Gamma_{\gamma}^2 [F, q, \Delta'] \subset \Gamma_{\gamma}^2 [F, p, \Delta'] \). This completes the proof of the theorem.

\[ \square \]

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**References**


