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NEW THEOREMS FOR SIGNLESS LAPLACIAN EIGENVALUES[1]

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(Dedicated to Ivan Gutman on the occasion of his 1000-th published scientific paper)

A b s t r a c t. *We extend our previous surveys of properties of spectra of signless Laplacians of graphs. Some new theorems for the signless Laplacian eigenvalues are given. In particular, a theorem on spectral characterization of graphs with Q-index not exceeding 4 is proved. The signless Laplacian eigenvalues of connected graphs on six vertices are given in Appendix.*

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1. *Introduction*

Let G be a simple graph with n vertices. The characteristic polynomial $\det(xI - A)$ of a (0,1)-adjacency matrix A of G is called the *characteristic*

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polynomial of G and denoted by $P_G(x)$. The eigenvalues of A (i.e. the zeros of $\det(xI - A)$) and the spectrum of A (which consists of the n eigenvalues) are also called the *eigenvalues of* G and the *spectrum of* G, respectively. The eigenvalues of G are real because A is symmetric, and the largest eigenvalue is called the *index* of G.

Together with the spectrum of an adjacency matrix of a graph we shall consider the spectrum of another matrix associated with the graph.

Let n, m, R be the number of vertices, the number of edges and the vertex-edge incidence matrix of a graph G. The following relations are well-known

$$RR^T = D + A, \quad R^T R = A(L(G)) + 2I, \tag{1}$$

where D is the diagonal matrix of vertex degrees and $A(L(G))$ is the adjacency matrix of the line graph $L(G)$ of G.

Since the non-zero eigenvalues of RR^T and $R^T R$ are the same, we deduce from the relations (1) that

$$P_{L(G)}(x) = (x+2)^{m-n} Q_G(x+2), \tag{2}$$

where $Q_G(x)$ is the characteristic polynomial of the matrix $Q = D + A$.

The polynomial $Q_G(x)$ will be called the *Q-polynomial* of the graph G. The eigenvalues and the spectrum of Q will be called the *Q-eigenvalues* and the *Q-spectrum*, respectively.

To avoid possible confusion, the eigenvalues and the spectrum of the adjacency matrix will be sometimes called *adjacency* eigenvalues and the spectrum.

Let A denote the adjacency matrix of a graph G and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ the spectrum of A, where the eigenvalues are such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Let $(q_1, q_2, \ldots, q_n)$ be the spectrum of Q, where the eigenvalues are such that $q_1 \geq q_2 \geq \ldots \geq q_n$. The largest eigenvalue q_1 is called the Q-index of G.

The matrix $L = D - A$, known as the *Laplacian* of G, features prominently in the literature (see, for example, [3]). The matrix $D + A$ is called the *signless Laplacian* and it appears very rarely in published papers before 2003 [8]. Only recently has the signless Laplacian attracted the attention of researchers [2, 8, 9, 1, 11, 10, 14, 19].

The present paper extends the surveys [8, 9] by providing further results, comments and numerical data. The new results include some equalities for the eigenvalues of $D + A$, while [9] mainly contains inequalities.

As usual, K_n, C_n and P_n denote respectively the *complete graph*, the *cycle* and the *path* on n vertices. We write $K_{m,n}$ for the *complete bipartite* graph with parts of size m and n. The graph $K_{n-1,1}$ is called a *star* and is denoted by S_n.

A unicyclic graph containing an even (odd) cycle is called *even-unicyclic* (*odd-unicyclic*). The *union* of disjoint graphs G and H is denoted by $G \cup H$, while mG denotes the union of m disjoint copies of G. The subdivision graph $S(G)$ of a graph G is obtained from G when each edge of G is subdivided by a new vertex.

The paper [9] contains 30 conjectures related to the Q-eigenvalues of a graph have been formulated after some computer experiments. Almost all the conjectures are in the form of inequalities which provide upper or lower bounds for spectrally based graph invariants.

For some conjectures we have indicated that they are, at least partially, resolved (previously in the literature or in the paper, explicitly or implicitly). The conjectures left unresolved appear to include some difficult research problems.

A few of the conjectures are related to the least Q-eigenvalue, and among them there is one according to which the minimal value of the least Q-eigenvalue among connected non-bipartite graphs of prescribed order is attained for the odd-unicyclic graph obtained from a triangle by appending a path. By the Interlacing Theorem such an extremal graph is an odd-unicyclic graph, and so we discuss the least eigenvalue in odd-unicyclic graphs. Investigations in [9] provide supporting evidence for this conjecture. The conjectire has been proved in [1].

The rest of this paper is organized as follows. Section 2 elaborates results from the survey paper [8] which will be required later. Section 3 contains several new results including those on trees, unicyclic graphs and subdsivision graphs. In Section 4 a theorem on spectral characterization of graphs with Q-index not exceeding 4 is given. Q-eigenvalues of connected graphs on six vertices are given in Appendix.

2. *Preliminaries*

In virtue of (1), the signless Laplacian is a positive semi-definite matrix, i.e. all its eigenvalues are non-negative. Concerning the least eigenvalue we have the following proposition (see [8, Proposition 2.1] or [13, Proposition

2.1]).

Proposition 2.1. *The least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.*

Corollary 2.2. *For any graph, the multiplicity of* 0 *as an eigenvalue of the signless Laplacian is equal to the number of bipartite components.*

The following proposition can be found in many places in the literature (see, for example, [15]), usually without a proof.

Proposition 2.3. *The Q-polynomial of a graph is equal to the characteristic polynomial of the Laplacian if and only if the graph is bipartite.*

P r o o f. Suppose that the graph G is bipartite, with parts U and V. Consider the determinant defining $Q_G(x)$. Multiply by -1 all rows corresponding to vertices in U and then do the same with the corresponding columns. The transformed determinant now defines the characteristic polynomial of the Laplacian of G.

The multiplicity of the eigenvalue 0 in the Laplacian spectrum is equal to the number of components, while for the signless Laplacian, the multiplicity of 0 is equal to the number of bipartite components. Therefore in non-bipartite graphs the two polynomials cannot coincide. $\square$

Let G be a connected graph with n vertices, and let

$$Q_G(x) = \sum_{j=0}^{n} p_j(G)x^{n-j} = p_0(G)x^n + p_1(G)x^{n-1} + \cdots + p_n(G).$$

A spanning subgraph of G whose components are trees or odd-unicyclic graphs is called a *TU-subgraph* of G. Suppose that a TU-subgraph H of G contain c unicyclic graphs and trees $T_1, T_2, \ldots, T_s$. Then the weight $W(H)$ of H is defined by $W(H) = 4^c \prod_{i=1}^{s}(1 + |E(T_i)|)$. Note that isolated vertices in H do not contribute to $W(H)$ and may be ignored.

We shall express coefficients of $Q_G(x)$ in terms of the weights of TU-subgraphs of G (cf. [12], [8]).

Theorem 2.4. *We have* $p_0(G) = 1$ *and*

$$p_j(G) = \sum_{H_j} (-1)^j W(H_j), \quad j = 1, 2, \ldots, n,$$

where the summation runs over all TU-subgraphs of G with j edges.

Definition 2.5. *A semi-edge walk (of length k) in an (undirected) graph G is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices $v_1, v_2, \ldots, v_{k+1}$ and edges $e_1, e_2, \ldots, e_k$ such that for any $i = 1, 2, \ldots, k$ the vertices v_i and v_{i+1} are end-vertices (not necessarily distinct) of the edge e_i.*

We shall say that the walk *starts* at the vertex v_1 and *terminates* at the vertex v_{k+1}.

The well known theorem concerning the powers of the adjacency matrix [3, p.44] has the following counterpart for the signless Laplacian [8].

Theorem 2.6. *Let Q be the signless Laplacian of a graph G. The (i, j)-entry of the matrix Q^k is equal to the number of semi-edge walks of length k starting at vertex i and terminating at vertex j.*

Let $T_k = \sum_{i=1}^{n} q_i^k$ $(k = 0, 1, 2, \ldots)$ be the k-th spectral moment for the Q-spectrum. Since $T_k = \text{tr } Q^k$, we have the following corollaries [8].

Corollary 2.7. *The spectral moment T_k is equal to the number of closed semi-edge walks of length k.*

Corollary 2.8. *Let G be a graph with n vertices, m edges, t triangles and vertex degrees $d_1, d_2, \ldots, d_n$. We have*

$$T_0 = n, \quad T_1 = \sum_{i=1}^{n} d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6t + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

3. *Some new results*

We shall prove a number of propositions on several, not quite related, topics.

* * *

1. *The sum of graphs.* There are very few formulas for Q-spectra of graphs obtained by some operations on other graphs. This happens in the case of the *sum* of graphs (for the definition and the corresponding result for the adjacency spectra see, for example, [3], pp. 65-72).

Let G_1, G_2 be graphs with adjacency matrices A_1, A_2, degree matrices D_1, D_2 and signless Laplacans Q_1, Q_2, respectively. We have $Q_1 = A_1 + D_1, Q_2 = A_2 + D_2$.

It is known that $A_1 \otimes I_2 + I_1 \otimes A_2$ is the adjacency matrix of the sum $G_1 + G_2$ of graphs G_1 and G_2. Here I_1, I_2 are identity matrices with the same order as G_1, G_2 respectively. If $\lambda_i^{(1)}, \lambda_j^{(2)}$ are eigenvalues of G_1, G_2, then the eigenvalues of $G_1 + G_2$ are all possible sums $\lambda_i^{(1)} + \lambda_j^{(2)}$.

In a quite analogous manner, $(A_1 + D_1) \otimes I_2 + I_1 \otimes (A_2 + D_2) = Q_1 \otimes I_2 + I_1 \otimes Q_2$ is the signless Laplacian of the sum $G_1 + G_2$ and if $q_i^{(1)}, q_j^{(2)}$ are Q-eigenvalues of G_1, G_2, then the Q-eigenvalues of $G_1 + G_2$ are all possible sums $q_i^{(1)} + q_j^{(2)}$.

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2. *The girth.* The following proposition is easily obtained from Theorem 2.4, as noted in [9]. Let t_i be the number of vertices of the tree obtained by deleting an edge i outside the cycle in a unicyclic graph.

Proposition 3.1. *For a graph G on n vertices, with girth g, we have:*

$$p_n(G) = 0, \quad (-1)^{n-1} p_{n-1}(G) = ng$$

if G is an even-unicyclic graph, and

$$(-1)^n p_n(G) = 4, \quad (-1)^{n-1} p_{n-1}(G) = ng + 4 \sum t_i$$

if G is an odd-unicyclic graph, where the summation goes over all edges i outside the cycle.

Hence the girth can be determined from the Q-eigenvalues in the case of even-unicyclic graphs but not in the case of odd-unicyclic graphs. For (adjacency) eigenvalues we have exactly the opposite situation (cf. [7]). However, Laplacian eigenvalues perform best: the girth of a unicyclic graph can be determined in all cases.

In fact, we can formulate the following proposition.

Proposition 3.2. *Given the Laplacian spectrum of a graph, we can establish whether or not the graph is unicyclic and, if the answer is positive, determine its girth.*

P r o o f. From the Laplacian spectrum of a graph we can determine the number of vertices, the number of edges and the number of connected components. Suppose we have established that the graph is unicyclic. Then the coefficient of the linear term in the characteristic polynomial is equal to

$-n$ times the number N of spanning trees, and for unicyclic graphs, N is equal to the girth. □

Note that results concerning coefficients $p_n(G)$ and $p_{n-1}(G)$ for some other classes of graphs, in particular for trees, have been obtained in [16]. Extremal results concerning the coefficients $p_i(T)$ for a tree T have been obtained in [19]. In particular, it is proved that for $i = 3, 4, \ldots, n-1$ the coefficient $(-1)^i p_i$ is minimal in paths and maximal in stars.

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3. *Subdivision graphs.* As pointed out in [9], the following formula appears implicitly in the literature (see e.g., [3, p. 63] and [18]):

$$P_{S(G)}(x) = x^{m-n} Q_G(x^2), \tag{3}$$

where G is a graph with n vertices and m edges, and $S(G)$ is the subdivision graph of G. Together with (2), this formula provides a link to the theory of the adjacency spectra. While formula (2) has been used to some effect in this context (cf. [8]), the connection with subdivision graphs remains to be exploited. Here we present some examples in this direction.

Let $\eta(G)$ be the multiplicity of the eigenvalue 0 in the spectrum of a graph G.

Proposition 3.3. *For any tree T we have $\eta(S(T)) = 1$.*

P r o o f. If T is a tree on n vertices formula (3) yields $P_{S(T)}(x) = x^{-1} Q_T(x^2)$. Since T is a bipartite graph $Q_T(x)$ has a simple root 0 by Proposition 2.1 and this completes the proof. □

Proposition 3.3 has been proved in [17] in another way.

The quantity $\eta(T)$ is an important parameter of a tree T since it determines the size of the maximal matching. By theorem 8.1 of [3], the size of the maximal matching of a tree T on n vertices is equal to $\frac{1}{2}(n - \eta(T))$.

Corollary 3.4. *The subdivision of a tree with m edges has a matching of size m.*

Two graphs are said to be *Q-cospectral* if they have the same polynomial $Q_G(\lambda)$. By analogy with the notions of PING and cospectral mate we introduce the notions of Q-PING and Q-cospectral mate with obvious meaning.

Two graphs are called *S-cospectral* (*L-cospectral*) if their subdivision (line) graphs are cospectral.

Proposition 3.5. *If two graphs are Q-cospectral, then they are S-cospectral and L-cospectral.*

P r o o f. Since Q-cospectral graphs have the same number of vertices and the same number of edges, their S-cospectrality and L-cospectrality follow from formulas (3) and (2), respectively. □

However, two L-cospectral graphs need not be Q-cospectral as noted in [8].

Example. The smallest Q-PING, consisting of the graphs $K_{1,3}$ and $C_3 \cup K_1$ on 4 vertices, yields by Proposition 3.5 the PING consisting of the graphs $S(K_{1,3})$ and $C_6 \cup K_1$. This PING provides the smallest example when a line graph is cospectral with a graph which is not a generalized line graph (cf. [5]) (while the Q-PING provides the smallest example when a line graph is Q-cospectral with a graph which is not a line graph). □

Note that the smallest PINGs (consisting of the graphs $K_{1,4}$ and $C_4 \cup K_1$ on 5 vertices and the well known PING of two connected graphs on 6 vertices [3, p. 157]) are not Q-PINGs.

Proposition 3.5 explains partially the fact that PINGs are more frequent than Q-PINGs. Namely, for any Q-PING Proposition 3.5 yields two PINGs whose graphs belong to restricted classes of graphs (subdivision and line graphs).

There are cospectral unicyclic graphs with different girths (which are necessarily even in this case) [7]. By Proposition 3.1 these graphs are not Q-cospectral.

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4. *The diameter.* Let $e(G)$ be the number of distinct Q-eigenvalues of a graph G. As pointed out in [9], the diameter of G is bounded above by $e(G) - 1$. This statement and its proof is analogous to an existing result related to the adjacency spectrum [3, Theorem 3.13].

Theorem 3.6. *Let G be a connected graph of diameter D and $e(G)$ distinct Q-eigenvalues. Then $D \leq e(G) - 1$.*

P r o o f. By Theorem 2.6 the (i, j)-entry $q_{i,j}^{(k)}$ of Q^k is the number of semi-edge walks of length k from i to j. By the definition of the diameter, for some vertices i and j there is no semiedge walk of length k connecting i and j for $k < D$, whereas there is at least one for $k = D$. Hence we have $q_{i,j}^{(k)} = 0$ for $k < D$ and $q_{i,j}^{(k)} > 0$ for $k = D$. The minimal polynomial of the matrix Q is of degree $e(G) = e$ and yields a recurence relation connecting e consecutive members of the sequence $q_{i,j}^{(k)}, \quad k = 0, 1, 2, \ldots$. The assumption $D > e-1$ would cause that all members of the sequence $q_{i,j}^{(k)}, \quad k = 0, 1, 2, \ldots$ are equal to 0 what is impossible. This contradiction proves the proposition. □

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5. *Eigenvectors.* We shall also consider the eigenvectors.

Theorem 3.7. *The eigenspace of the Q-eigenvalue 0 of a graph G determines sets of vertices and bipartitions in bipartite components of G.*

P r o o f. Let $\mathbf{x}^T = (x_1, x_2, \ldots, x_n)$. For a non-zero vector $\mathbf{x}$ we have $Q\mathbf{x} = \mathbf{0}$ if and only if $R^T\mathbf{x} = \mathbf{0}$. The later holds if and only if $x_i = -x_j$ for every edge. If the graph is connected (and then necessarily bipartite), $\mathbf{x}$ is determined up to a scalar multiple by the value of its coordinate corresponding to any fixed vertex i. If G is disconnected, at least one component is bipartite. If a vertex i belongs to a non-bipartite component, then $x_i = 0$. Using Corollary 2.2 we determine the number of bipartite components as the multiplicity of eigenvalue 0. For each bipartite component we have an eigenvector with non-zero coordinates exactly for vertices in this component. Now, vertex sets of bipartite components are determined by non-zero coordinates in vectors of a suitably chosen ortogonal basis of the eigencpace of 0. The sign of these coordinates determines colour classes within bipartite components. □

4. *Vertex degrees and graphs with Q-index not exceeding 4*

Expressions for the spectral moments from Corollary 2.8 can be used to determine vertex degrees if we know that vertex degrees can take only a limited number of values. In particular, suppose that a graph has n_i vertices of degree e_i for $i = 0, 1, 2, 3$ and no other vertices. If we specify e_0, e_1, e_2, e_3, the corresponding numbers of vertices n_0, n_1, n_2, n_3 can be determined from the system of equations (provided the spectral moments T_0, T_1, T_2, T_3 are known)

$$\sum_{i=0}^{3} n_i = T_0 = n, \quad \sum_{i=0}^{3} n_i e_i = T_1 = 2m,$$

$$\sum_{i=0}^{3} n_i e_i^2 = T_2 - 2m, \quad \sum_{i=0}^{3} n_i e_i^3 = T_3 - 6t - 3(T_2 - 2m).$$

Interesting conclusions could be made in the case $e_0 = 0, e_1 = 1, e_2 = 2, e_3 = 3$.

Such a situation occurs in graphs with vertex degrees at most 3. These graphs are of interest in chemical applications of graph theory. If such a graph is bipartite, we have $t = 0$ and vertex degrees are determined in terms of spectral moments by the above system. If the graph is connected, we have $n_0 = 0$ and we can treat non-bipartite case as well. (The first three equations suffice to determine vertex degrees and, in addition, the fourth equation yields the number of triangles t).

* * *

A similar situation occurs in graphs with the Q-index not exceeding 4. By Proposition 6.1 of [8] components of such graphs are paths (including isolated vertices), cycles and stars $K_{1,3}$. In fact we have a subset of the set of chemically interesting graphs but we shall try here to say a little more.

We shall assume that the whole Q-spectrum of such a graph is given. Could we hope that the graph is determined up to isomorphism ?

The smallest Q-PING, consisting of the graphs $K_{1,3}$ and $C_3 \cup K_1$ on 4 vertices yields a counter example.

We can then, according to the suggestion in [8], assume that together with Q-spectrum also the number of components is given. This really distinguishes between the graphs in this Q-PING: if the number of components is eqal to 1 we get $K_{1,3}$ and if the number of components is equal to 2 we get $C_3 \cup K_1$.

In order to treat the general case let us introduce the following notation:

k_q multiplicity of the Q-eigenvalue q,

c the number of components,

b the number of bipartite components,

s the number of components isomorphic to the star $K_{1,3}$,

e the number of even circuits,

u the number of of odd circuits of length greater or equal to 5,

p the number of (non-trivial) paths,

v the number of isolated vertices.

We have the following relations connecting these parameters with the spectum:

$$k_0 = b, \quad k_4 = e + t + u + s.$$

Next, we have some relations connecting these parameters with quantities n_0, n_1, n_2, n_3:

$$v = n_0, \quad p = \frac{n_1 - 3n_3}{2}, \quad s = n_3,$$

Note also that $b = v + p + e + s$ and $c = b + t + u$.

From all these relations it is easy to derive the following equation

$$2n_0 + n_1 - 3n_3 = 2c - 2k_4.$$

Previous equations for n_0, n_1, n_2, n_3 read now

$$n_0 + n_1 + n_2 + n_3 = T_0 = n, \quad n_1 + 2n_2 + 3n_3 = T_1 = 2m,$$

$$n_1 + 4n_2 + 9n_3 = T_2 - 2m, \quad n_1 + 8n_2 + 27n_3 = T_3 - 6t - 3(T_2 - 2m).$$

Using the above equations we can determine vertex degrees and, in particular, numbers of components of each type, provided the Q-spectrum and the number of components c are known. In fact, the first four out of these five equations are independent and yield unique values for n_0, n_1, n_2, n_3 and the fifth equation yields t. Then gradually all other parameters can be calculated.

Hence, we have proved the following theorem.

Theorem 3.8. *Let the Q-spectrum and the number c of components of a graph be given. If the Q-index does not exceed 4, then the numbers v, p, e, s, t, u, defined above, are uniquely determined.*

However, all this is not sufficient to determine the graph up to isomorphism.

Example. Graphs $C_4 \cup 2P_3$ and $C_6 \cup 2K_2$ are Q-cospectral. This is the smallest of the following family of Q-PINGs: $C_{2k} \cup 2P_l$ and $C_{2l} \cup 2P_k$ for $k, l \geq 2,\ k \neq l$, what can be verified since the Q-spectra of circuits and paths are known [8]. $\square$

This example shows that although the numbers of components of each type are determined, the distribution of vertices between components (in these cases between paths and even circuits) is not unique.

Remark. The paper [4] studies spectra of the adjacency matrix of graphs in which the largest eigenvalue does not exceed 2. This problem is analogous to the problem covered by Theorem 3.8, i.e., the problem of graphs with the Q-index not exceeding 4. There are more graphs in the first case and cospectral graphs appear more frequently. Once more we come across facts supporting the idea that Q-eigenvalues contain more information on graphs than the eigenvalues of the adjacency matrix.

Appendix

We have computed Q-spectra of graphs on 6 vertices. The graphs are ordered and labeled in the same way as in the paper [6] and the reader is referred to this paper for drawings of the graphs. (In fact, the graphs are ordered lexicographically by spectral moments of the adjacency matrix.) The number of edges m is given. The following 5 pairs of graphs form Q-PINGs: 5 6, 14 16, 53 56, 65 71, 82 88.

Q-spectra of graphs on up to 5 vertices are given in the appendix of [8].

Q-SPECTRA OF CONNECTED GRAPHS WITH 6 VERTICES

m=15	001.	10.0000	4.0000	4.0000	4.0000	4.0000	4.0000
m=14	002.	9.4641	4.0000	4.0000	4.0000	4.0000	2.5359
m=13	003.	9.0000	4.0000	4.0000	4.0000	3.0000	2.0000
	004.	8.8284	4.0000	4.0000	4.0000	3.1716	2.0000
m=12	005.	8.6056	4.0000	4.0000	3.0000	3.0000	1.3944
	006.	8.6056	4.0000	4.0000	3.0000	3.0000	1.3944
	007.	8.4495	4.0000	4.0000	3.5505	2.0000	2.0000
	008.	8.2588	4.0000	4.0000	3.2518	3.0000	1.4894
	009.	8.0000	4.0000	4.0000	4.0000	2.0000	2.0000
m=11	010.	8.2749	4.0000	3.0000	3.0000	3.0000	0.7251
	011.	8.1355	4.0000	3.6532	3.0000	2.0000	1.2113
	012.	7.9651	4.0000	3.7180	3.0000	2.0000	1.3169
	013.	8.0000	4.0000	4.0000	2.0000	2.0000	2.0000
	014.	7.7588	4.0000	3.3054	3.0000	3.0000	0.9358
	015.	7.7913	4.0000	3.6180	3.2087	2.0000	1.3820
	016.	7.7588	4.0000	3.3054	3.0000	3.0000	0.9358
	017.	7.5616	4.0000	3.4384	3.0000	3.0000	1.0000
	018.	7.5047	4.0000	4.0000	3.1354	2.0000	1.3600
m=10	019.	7.7264	3.8577	3.0000	3.0000	1.7093	0.7066

	020.	7.5446	3.8329	3.0000	3.0000	2.0000	0.6224
	021.	7.7588	4.0000	3.3054	2.0000	2.0000	0.9358
	022.	7.5742	3.7337	3.6180	2.5076	1.3820	1.1845
	023.	7.3723	4.0000	3.0000	3.0000	1.6277	1.0000
	024.	7.4279	4.0000	3.3757	2.0000	2.0000	1.1965
	025.	7.3919	3.7904	3.2106	3.0000	1.6815	0.9256
	026.	7.1859	3.7200	3.3007	3.0000	2.0000	0.7933
	027.	7.1190	4.0000	3.6180	2.5684	1.3820	1.3126
	028.	7.2361	3.6180	3.6180	2.7639	1.3820	1.3820
	029.	7.0839	4.0000	3.2132	3.0000	2.0000	0.7029
	030.	6.8951	4.0000	3.3973	3.0000	1.7076	1.0000
	031.	6.8284	4.0000	4.0000	2.0000	2.0000	1.1716
	032.	6.9095	3.6093	3.0000	3.0000	3.0000	0.4812
m=9	033.	7.2724	3.7245	3.0000	2.0000	1.3437	0.6594
	034.	7.0604	3.6395	3.0000	2.4522	1.2270	0.6208
	035.	6.9095	3.6093	3.0000	2.0000	2.0000	0.4812
	036.	7.0000	4.0000	2.0000	2.0000	2.0000	1.0000
	037.	7.4641	4.0000	2.0000	2.0000	2.0000	0.5359
	038.	7.0839	3.2132	3.0000	3.0000	1.0000	0.7029
	039.	6.7982	3.7904	3.0000	2.5025	1.3626	0.5463
	040.	7.1156	3.6701	3.0971	2.0000	1.2393	0.8780
	041.	6.7321	3.4142	3.2679	2.0000	2.0000	0.5858
	042.	6.8284	3.6180	3.6180	1.3820	1.3820	1.1716
	043.	6.9576	3.6180	3.1215	2.0000	1.3820	0.9209
	044.	6.6458	4.0000	3.0000	2.0000	1.3542	1.0000
	045.	6.6648	3.3011	3.0000	3.0000	1.5713	0.4628
	046.	6.6058	3.7197	3.1897	2.4767	1.3225	0.6856
	047.	6.4081	3.6180	3.2934	2.5573	1.3820	0.7411
	048.	6.3234	4.0000	3.3579	2.0000	1.3187	1.0000
	049.	6.4940	4.0000	3.1099	2.0000	2.0000	0.3961
	050.	6.3419	3.5959	3.0000	3.0000	1.6324	0.4298
	051.	6.0000	4.0000	3.0000	3.0000	1.0000	1.0000
	052.	6.0000	3.0000	3.0000	3.0000	3.0000	0.0000
m=8	053.	6.9095	3.6093	2.0000	2.0000	1.0000	0.4812
	054.	6.6728	3.4142	2.6481	2.0000	0.6791	0.5858
	055.	6.3923	3.3254	2.0000	2.0000	2.0000	0.2823
	056.	6.9095	3.6093	2.0000	2.0000	1.0000	0.4812
	057.	6.4940	3.1099	3.0000	2.0000	1.0000	0.3961
	058.	6.7494	3.1469	3.0000	1.4577	1.0000	0.6460
	059.	6.4317	3.6180	2.7995	1.3820	1.2245	0.5443
	060.	6.2422	3.5496	2.6524	2.0000	1.0855	0.4703
	061.	6.6262	3.5151	2.0000	2.0000	1.0000	0.8587
	062.	6.0000	4.0000	2.0000	2.0000	1.0000	1.0000
	063.	6.1779	3.1905	3.0000	2.4204	0.7828	0.4284
	064.	6.1159	3.7195	2.7379	2.0000	1.0648	0.3619

	065.	5.8781	3.5834	3.0000	2.0000	1.2296	0.3089
	066.	6.2491	3.4142	2.8536	2.0000	0.8972	0.5858
	067.	6.0280	3.2953	3.0000	2.0000	1.2849	0.3918
	068.	5.9452	3.6180	3.0856	1.3820	1.2963	0.6728
	069.	5.7093	3.4142	3.1939	2.0000	1.0968	0.5858
	070.	5.5616	4.0000	3.0000	1.4384	1.0000	1.0000
	071.	5.8781	3.5834	3.0000	2.0000	1.2296	0.3089
	072.	5.5887	3.5463	3.0000	2.4537	1.0000	0.4113
	073.	6.0000	4.0000	2.0000	2.0000	2.0000	0.0000
	074.	5.5616	3.0000	3.0000	3.0000	1.4384	0.0000
m=7	075.	6.4940	3.1099	2.0000	1.0000	1.0000	0.3961
	076.	6.1563	3.4142	2.0000	1.3691	0.5858	0.4746
	077.	5.9452	3.0856	2.6180	1.2963	0.6728	0.3820
	078.	5.8781	3.5834	2.0000	1.2296	1.0000	0.3089
	079.	6.3723	3.0000	2.0000	1.0000	1.0000	0.6277
	080.	5.6458	3.4142	2.0000	2.0000	0.5858	0.3542
	081.	5.8154	3.0607	2.0000	2.0000	0.8638	0.2602
	082.	5.4893	3.2892	2.0000	2.0000	1.0000	0.2215
	083.	5.7217	3.5127	2.0000	1.3098	1.0000	0.4558
	084.	5.0000	4.0000	2.0000	1.0000	1.0000	1.0000
	085.	5.6597	3.1461	2.7357	1.3736	0.7772	0.3077
	086.	5.3615	3.1674	2.6180	2.0000	0.4711	0.3820
	087.	5.2647	3.5378	2.6491	1.2987	1.0000	0.2497
	088.	5.4893	3.2892	2.0000	2.0000	1.0000	0.2215
	089.	5.0664	3.2222	3.0000	1.3478	1.0000	0.3636
	090.	5.5141	3.5720	2.0000	2.0000	0.9139	0.0000
	091.	5.2361	3.0000	3.0000	2.0000	0.7639	0.0000
	092.	5.0000	3.0000	3.0000	2.0000	1.0000	0.0000
	093.	4.9032	3.4142	2.8061	2.0000	0.5858	0.2907
m=6	094.	6.2015	2.5451	1.0000	1.0000	1.0000	0.2534
	095.	5.5344	3.0827	1.5929	1.0000	0.4889	0.3010
	096.	5.2361	2.6180	2.6180	0.7639	0.3820	0.3820
	097.	5.3839	2.7424	2.0000	1.0000	0.6721	0.2015
	098.	4.9809	3.0420	2.0000	1.2938	0.4629	0.2204
	099.	4.8422	3.5069	1.4931	1.0000	1.0000	0.1578
	100.	4.6554	3.2108	2.0000	1.0000	1.0000	0.1338
	101.	5.2361	3.0000	2.0000	1.0000	0.7639	0.0000
	102.	4.8136	3.0000	2.5293	1.0000	0.6571	0.0000
	103.	4.7321	3.4142	2.0000	1.2679	0.5858	0.0000
	104.	4.5616	3.0000	2.0000	2.0000	0.4384	0.0000
	105.	4.4383	3.1386	2.6180	1.1798	0.3820	0.2434
	106.	4.0000	3.0000	3.0000	1.0000	1.0000	0.0000
m=5	107.	6.0000	1.0000	1.0000	1.0000	1.0000	0.0000
	108.	5.0861	2.4280	1.0000	1.0000	0.4859	0.0000

109.	4.5616	3.0000	1.0000	1.0000	0.4384	0.0000
110.	4.3028	2.6180	2.0000	0.6972	0.3820	0.0000
111.	4.2143	3.0000	1.4608	1.0000	0.3249	0.0000
112.	3.7321	3.0000	2.0000	1.0000	0.2679	0.0000

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