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A THEORY OF LINEAR DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVES

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A b s t r a c t. *We give the existence of solutions to the linear differential equation with fractional derivatives which are real numbers and two different types of initial conditions. Relations between these initial conditions is considered.*

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1. *Introduction*

In this paper we consider an equation of the form

$$\sum_{i=0}^{m} A_{i0} D_t^{\alpha_i} y(t) = f(t),\ 0 < t \le b < \infty, \tag{1.1}$$

where $\alpha_i = [\alpha_i] + \gamma_i,\ i = 0, ..., m,\ [\alpha_i] \in \mathbb{N}_0,\ \gamma_i \in [0, 1);\ 0 \le \alpha_0 < \alpha_1 < ... < \alpha_m,\ A_i \ne 0,\ i = 0, ..., m$. We find suitable conditions on a function f such that ${}_0D_t^\alpha f,\ \alpha = n-1+\gamma,\ n \in \mathbb{N}$ and $\gamma \in (0,1)$, exists and the relation between Cauchy's initial conditions given by $f^{(k)}(0^+),\ k = 0,...,n-1$, and initial conditions given by $\left({}_0I_t^{1-\gamma} f\right)^{(k)}(0^+)$. Here ${}_0I_t^{1-\gamma} f$ is the fractional integral of f of the order $1-\gamma$. With this we consider solvability of (1.1) with various assumptions described below.

Usually for the existence of ${}_0D_t^\alpha f$ the assumption that $f \in AC^n([0,b])$ is used in a slightly changed version. Here $AC^n([0,b])$ denotes the space of $n-$th order absolutely continuous functions (cf. [2], [6], [13] and [5], for example). Further, we shall determine the precise form of relations between $\left({}_0I_t^{1-\gamma} f\right)^{(k)}(0^+),\ k = 0,1,...,n-1$ and $f^{(k)}(0^+)$ (cf. Lemma 2.4). This permit us to use the these conditions.

The following notation for the left Riemann-Liouville fractional integral and derivative are used:

Let $\alpha = n-1+\gamma, n \in \mathbb{N}, \gamma \in [0,1)$. Then,

$$\begin{aligned}
{}_0I_t^\alpha y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau)\, d\tau, \\
\gamma &\in (0,1),\ t>0, \\
{}_0D_t^\alpha y(t) &= \frac{1}{\Gamma(1-\gamma)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(\tau)}{(t-\tau)^\gamma} = \left(\frac{d}{dt}\right)^n_0 I_t^{1-\gamma} y(t), \\
\gamma &\in [0,1),\ \ t>0, \\
{}_0D_t^\alpha y(t) &= D^n y(t),\ \ \alpha = n, \quad t>0,
\end{aligned}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function.

Equation (1.1) has been treated in the mathematical literature using different methods: Integral transforms, Laplace or Mellin [7], numerical methods [3], or other methods suitable for some special forms of (1.1). For example, sequential linear equations of fractional order are treated by special methods in [6] 7.1 (see also [11]). In [3], Section 3, authors analyzed equation (1.1) with Caputo fractional derivatives. They show that the solutions to (1.1) can be approximated by a system of linear fractional differential equations of rational orders. This allows good approximation for unknown solution to (1.1). All results in [3] on (1.1) are given with the limitations: $\alpha_i - \alpha_{i-1} \leq 1,\ i = 1,...,m$.

Papers [8] and [4] are of particular interest because their authors considered explicit forms of solutions to (1.1) with α_i real numbers. In [4] a modification of Mikusiński's operational calculus (cf. [9]) was used in order to obtain explicit form of solutions. The fractional derivatives are of

Riemann-Liouville's or Caputo's form. The basic spaces used in [4] is defined as follows:

A function $f(t),\ t > 0$ is said to belong to the space $C_\alpha,\ \alpha \in \mathbb{R}$ if there exists a real number $p,\ p > \alpha$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C([0,\infty))$.

A function $f(t),\ t > 0$ is said to belong to the space $\Omega_\alpha^\mu,\ \mu \geq 0$ if for any $\nu,\ 0 \leq \nu \leq \mu,\ {}_0D_t^\nu f \in C_\alpha$.

Moreover, the operational calculus is defined by the field M extending the ring $(C_{-1}, \circ, +)$, where "$\circ$" is the sign of convolution.

In Section of [4], using a version of the operational calculus, authors considered "general solution" of equation (1.1) of real orders. As they said: "for the sake of simplicity we restrict ourselves to the case when the highest order of the fractional derivatives is not grater than one". The initial conditions are:

$$(I_{0^+}^{1-\mu_m} y)(0) = d,\ (I_{0^+}^{1-\mu_i} y)(0) = 0,\ 1 \leq i \leq m-1. \tag{1.2}$$

However the topology is not defined, neither in the field M nor in the ring $(C_{-1}, \circ, +)$ so the convergence and operations with them used in [4] should be explained.

In this paper we give another approach to (1.1). The known results imply that the solutions to equation (1.1) in $(0,b]$ can not be always extended so that they are also solutions in $[0,b]$ (cf. [10] Section V, 4.). However such solutions could be of general interest. Consequently we supplement the well-known solving methods by a fine analysis at the point 0. Instead of the Laplace transform defined on $[0,\infty)$, we use the Laplace transform for functions defined on $[0,b],\ b < \infty$, (cf. Part 4.1 and [15]), in order to avoid the limitation on the growth of f in (1.1). Formal application of the Laplace transform to (1.1) gives

$$\hat{y}(s) = \frac{1}{\sum\limits_{i=0}^{m} A_i s^{\alpha_i}} \left(\hat{f}(s) + \sum_{i=0}^{m} \sum_{j=0}^{n_i} s^{n_i - j - 1} \left({}_0I_t^{1-\gamma_i} y \right)^{(j)} (0^+) \right). \tag{1.3}$$

Because of that we consider initial condition given by $\left({}_0I_t^{1-\gamma_i} y \right)^{(j)} (0^+)$ $i = 0, ..., m;\ j = 0, ..., n_m - 1$ and show how this initial conditions are related to the Cauchy initial conditions $y^{(j)}(0^+),\ j = 0, ..., n_m - 1$ (cf. Lemma 3.1). Note that these initial conditions have a meaning even though Cauchy initial conditions $(y^{(j)}(0^+))$ do not exist. Let $n_m \geq 2, {}_0I_t^{1-\gamma_m} y \in AC^{n_m}([0,b])$ and

$y^{(j)}(0^+) = 0,\ j = 0, ..., n_m - 2$. Then Proposition 4.2 gives that one can have a solution with an additional condition on $\left({}_0I_t^{1-\gamma_m} y\right)^{(n_m-1)}$, although the analytical form of the solution is the same. The added condition implies only some properties of the solution not changing its form. The solutions obtained by Proposition 4.2 or by their derivatives enable us to construct solutions of non homogeneous equations (1.1) with non-zero initial conditions.

On the basis of ideas presented above, we give a ful description of solutions to (1.1) in cases wwhen the leading derivative is non-integer, Theorem 4.1, and when it isan integer, Theorem 4.2. Moreover we discuss non-homogeneous conditions in Theorems 4.3 and 4.4.

2. *On the domain of the operator* ${}_0D_t^{\alpha}$

2.1 Preliminaries

We introduce the space $\mathcal{J}_n([0,b]) \subset AC^n([0,b])$ with the simple additional condition that $f^{(n)} \in L^1([0,b])$ is locally bounded on $(0,b]$. $\mathcal{J}_0([0,b])$ is just the space of $L^1\,([0,b])$ locally bounded on $(0,b]$.

If $F, G \in \mathcal{J}_0([0,b])$, then $F * G \in \mathcal{J}_0([0,b])$, where $*$ denotes convolution (cf. [1], p.113).

Lemma 2.1 *Let* $h \in \mathcal{J}_0([0,b])$, *and* $\beta < 1$. *If*

$$\lim_{t\to 0^+} \frac{h(t)}{t^{-\beta}} = A_\beta, \quad i.e.,\ h(t) \sim A_\beta t^{-\beta},\ t \to 0^+, \tag{2.1}$$

then

$$\left({}_0I_t^{1-\gamma} h\right)(t) \sim \frac{A_\beta \Gamma(1-\beta)}{\Gamma(2-\beta-\gamma)} t^{1-\gamma-\beta},\ t \to 0^+. \tag{2.2}$$

Proof. We start with

$$\begin{aligned} \left({}_0I_t^{1-\gamma} h\right)(t) &= \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{h(\tau)d\tau}{(t-\tau)^\gamma} = \frac{1}{\Gamma(1-\gamma)} \int_0^1 \frac{h(tu)t\,du}{(t-tu)^\gamma} \\ &= \frac{t^{1-\gamma-\beta}}{\Gamma(1-\gamma)} \int_0^1 \frac{h(tu)}{(tu)^{-\beta}} u^{-\beta}(1-u)^{-\gamma} du \\ &\sim \frac{A_\beta \Gamma(1-\beta)}{\Gamma(2-\beta-\gamma)} t^{1-\gamma-\beta},\ t \to 0^+. \end{aligned}$$

(cf. [14], p. 60). This proves Lemma 2.1. □

Remark 2.1. Lemma 2.1 shows that a function h can have the property that $h(0^+)$ does not exist, but $\left({}_0I_t^{1-\gamma}h\right)(0^+)$ is defined. That is the reason why we use ${}_0I_t^{1-\gamma}h$ instead of h.

Lemma 2.2 *Let $n \geq 2$. A necessary and sufficient condition that $h \in \mathcal{J}_n([0,b])$ is that h is of the form:*

$$h(t) = H_n(t) + \sum_{j=0}^{n-1} h_j \frac{t^j}{\Gamma(j+1)},\ t \in (0,b], \tag{2.3}$$

where every $h^{(j)}(t)$ $j = 0,...,n-1$, can be extended on $[0,b]$ to be continuous on $[0,b]$, $h_j = h^{(j)}(0^+)$,

$$H_n(t) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} h^{(n)}(\tau) d\tau,\ t \in [0,b]. \tag{2.4}$$

Proof.

Suppose that $h \in \mathcal{J}_n([0,b])$. Then there exist $\lim\limits_{t\to 0^+} h^{(j)}(t) \equiv h_j,\ j = 0,...,n-1$, and that $h^{(j)}(t)$ are continuous on $[0,b]$ (cf. [1], p.100).

Let the collection $\{H_j\},\ j = 0,...,n$, be defined by

$$H_j(t) = \int_0^t (t-\tau)^{j-1} h^{(n)}(\tau) d\tau,\ j = 1,...,n$$

This implies

$$H_n(t) = h(t) - \sum_{j=0}^{n-1} h_j \frac{t^j}{\Gamma(j+1)},\ t \in (0,b].$$

This gives (2.6) and the necessity of the condition is proved.

Conversely, suppose that h is of form (2.6) with h_j constants. Then applying the n-th derivative to (2.6) we obtain that $h^{(n)} \in \mathcal{J}_0([0,b])$, this follows from $H_n^{(n)}(t) \in \mathcal{J}_0([0,b]),\ 0 < t \leq b$, (cf. [1] Satz 5, p.92) and the fact that $H_n^{(k)}(t)$ are continuous functions on $[0,b]$, $k = 0,...,n-1$ and $H_n^{(k)}(0^+) = 0,\ k = 0,...,n-1$.

□

Remark 2.2 If in Definition 2.1 part 2) we suppose additionally on $F \in \mathcal{J}_n([0,b])$ that $F^{(i)}(t), i = 0, ..., n-1$ are continuous on $[0,b]$, then in Lemma 2.2 we can omit the assumption $n \geq 2$.

2.2 Fractional derivatives within $\mathcal{J}_0([0,b])$

Let $\alpha = n-1+\gamma,\ n \in \mathbb{N}$ and $\gamma \in (0,1)$. Since

$$ {}_0D_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n {}_0I_t^{1-\gamma} f(t),\ 0 < t < b, \qquad b < \infty, $$

the existence of ${}_0D_t^{\alpha} f$ is equivalent to the existence of $({}_0I_t^{1-\gamma} f)^{(n)}$ on $(0,b)$; so we have the following lemma.

Lemma 2.3 *A necessary and sufficient condition that* ${}_0D_t^{\alpha} f = \varphi \in \mathcal{J}_0([0,b])$ *is that* ${}_0I_t^{1-\gamma} f \in \mathcal{J}_n([0,b])$, *i.e.,* $\left({}_0I_t^{1-\gamma} f\right)^{(n)} = \varphi \in \mathcal{J}_0([0,b])$.

The next proposition gives a sufficient condition on f so that ${}_0D_t^{\alpha} f \in \mathcal{J}_0([0,b])$.

Proposition 2.1 *Suppose that the function f has the following properties:*

$$ t^{1-\gamma} f(\cdot) = w(\gamma, \cdot) \in \mathcal{J}_n([0,b]) \tag{2.5} $$

where $w^{(n)}(\gamma,t) \equiv \psi(\gamma,t) \sim A_\gamma t^{\beta-\gamma}, t \to 0^+, \beta - \gamma > -1,$ *and* $\psi(\gamma,\cdot) \in \mathcal{J}_0([0,b])$. *Then* ${}_0D_t^{\alpha} f$ *belongs to* $\mathcal{J}_0([0,b])$, $\alpha = n-1+\gamma,\ n \in N,\ \gamma \in (0,1)$. *The analytical form of f is given by*

$$ f(t) = t^{\gamma-1}\left[W_n(t) + \sum_{j=0}^{n-1} w_j(\gamma)\frac{t^j}{\Gamma(j+1)}\right], \tag{2.6} $$

where $W_n(t) = \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1}\psi(\gamma,\tau)d\tau, t \in (0,b]$ *and* $w_j(\gamma) = w^{(j)}(\gamma, 0^+)$, $j = 0, ..., n-1$.

Proof By Lemma 2.2

$$ w(\gamma,t) = W_n(t) + \sum_{j=0}^{n-1} w_j(\gamma)\frac{t^j}{\Gamma(j+1)},\ t \in (0,b], $$

$$ f(t) = t^{\gamma-1}\left[W_n(t) + \sum_{j=0}^{n-1} w_j(\gamma)\frac{t^j}{\Gamma(j+1)}\right]. \tag{2.7} $$

We substitute this in the next integral,

$$\begin{aligned}\frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{f(\tau)}{(t-\tau)^\gamma}d\tau &= \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\tau^{\gamma-1}W_n(\tau)}{(t-\tau)^\gamma}d\tau \\ &\quad +\frac{1}{\Gamma(1-\gamma)}\sum_{j=0}^{n-1} w_j(\gamma)\int_0^t \frac{\tau^{j+\gamma-1}}{\Gamma(j+1)}\frac{d\tau}{(t-\tau)^\gamma} \\ &= I_1(t)+I_2(t),\ t\in(0,b]. \qquad (2.8)\end{aligned}$$

Since $I_1(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^1 u^{\gamma-1}W_n(ut)(1-u)^{1-\gamma-1}du$, it follows

$$\begin{aligned}\left(\frac{d}{dt}\right)^n I_1(t) &= \frac{1}{\Gamma(1-\gamma)}\int_0^1 W_n^{(n)}(ut)u^{n+\gamma-1}(1-u)^{1-\gamma-1}du \\ &= \frac{1}{\Gamma(1-\gamma)}\int_0^1 \psi(\gamma,ut)u^{n+\gamma-1}(1-u)^{1-\gamma-1}du \\ &= \frac{t^{\beta-\gamma}}{\Gamma(1-\gamma)}\int_0^1 \frac{\psi(\gamma,ut)}{(ut)^{\beta-\gamma}}u^{n-1+\beta}(1-u)^{1-\gamma-1}du. \qquad (2.9)\end{aligned}$$

As in Lemma 2.1, we prove that $\left(\frac{d}{dt}\right)^n I_1$ exists and

$$\left(\frac{d}{dt}\right)^n I_1(t) \sim A_\gamma \frac{\Gamma(n+\beta)}{\Gamma(n+\beta-\gamma+1)}t^{\beta-\gamma},\ t\to 0^+.$$

As regards I_2, we have:

$$\begin{aligned}I_2(t) &= \frac{1}{\Gamma(1-\gamma)}\sum_{j=0}^{n-1} w_j(\gamma)\frac{1}{\Gamma(j+1)}\int_0^t t^{j+\gamma-1}(t-\tau)^{1-\gamma-1}d\tau \\ &= \sum_{j=0}^{n-1} w_j(\gamma)\frac{1}{\Gamma(j+1)}\frac{\Gamma(j+\gamma)}{\Gamma(j+1)}t^j,\ t\in(0,b]. \qquad (2.10)\end{aligned}$$

Consequently, $\left(\frac{d}{dt}\right)^n I_2(t) = 0$. By (2.12) it follows (cf. Lemma 2.1)

$$\begin{aligned}\left(\frac{d}{dt}\right)^n {}_0I_t^{1-\gamma} f &= \frac{t^{\beta-\gamma}}{\Gamma(1-\gamma)}\int_0^1 \frac{\psi(\gamma,ut)}{(ut)^{\beta-\gamma}}u^{n+\beta-1}(1-u)^{-\gamma}du \equiv \varphi(t) \\ &\sim A_\gamma \frac{\Gamma(n+\beta)}{\Gamma(n+\beta-\gamma+1)}t^{\beta-\gamma},\ t\to 0. \qquad (2.11)\end{aligned}$$

¿From (2.14) it follows that ${}_0I_t^{1-\gamma}f \in \mathcal{J}_n([0,b])$. Lemma 2.3 gives the existence of ${}_0D_t^{\alpha}f$ and its properties. This proves Proposition 2.1.

□

Remark 2.3
With the results of Proposition 2.1 we can use the Laplace transform to solve equations with fractional derivatives although all derivatives of the solutions are not bounded at zero (cf. (1.3)).

3. *Two lemmas on* ${}_0I_t^{1-\gamma}f$

Lemma 3.4 *Suppose that* $\alpha = n-1+\gamma,\ n \in \mathbb{N},\ \gamma \in [0,1)$. *Let* $g(\gamma,\cdot) = {}_0I_t^{1-\gamma}f \in \mathcal{J}_n([0,b])$. *Then:*

1) $g(\gamma,0^+)^{(j)} = 0,\ j = 0,...,k,$ *is the necessary and sufficient condition that* $f^{(k)}(0^+) = 0, k = 0,...,n-2,$ *and is a sufficient condition that* $f^{(n-1)} \in \mathcal{J}_0([0,b])$.

2) *A necessary condition that* $f^{(j)}(0^+)$ *exists,* $j = 0,...,n-1,$ *is that* $g^{(k)}(\gamma,0^+) = 0,\ k = 0,...,n-1$.

Proof.

Case $n = 2$.

1) By assumptions: $g(\gamma,\cdot) \in \mathcal{J}_2([0,b])$ and $g^{(k)}(\gamma,\cdot),\ k = 0,1$ are continuous on $[0,b]$. With these properties of $g(\gamma,t)$, one has the solution of the Abel equation ${}_0I_t^{1-\gamma}f = g(\gamma,\cdot)$ (cf. [12], pp.16-18 and [13], p.31):

$$f(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}g(\gamma,0^+) + \left(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} * g^{(1)}(\gamma,\tau)\right)(t),\ 0 < t \leq b. \tag{3.1}$$

Since $g^{(1)}(\gamma,t)$ is continuous on $[0,b]$, the second addend in (2.15) is zero for $t = 0$. Hence a necessary and sufficient condition that $f(0^+) = 0$, is $g(\gamma,0^+) = 0$.

By (2.15) and by derivation of the convolution (cf. [1], p.120) we have:

$$\begin{aligned} f^{(1)}(t) &= \frac{1}{\Gamma(\gamma)}\frac{d}{dt}\int_0^t \frac{g^{(1)}(\gamma,\tau)d\tau}{(t-\tau)^{1-\gamma}} \\ &= \frac{1}{\Gamma(\gamma)}\frac{g^{(1)}(\gamma,0^+)}{t^{1-\gamma}} + \frac{1}{\Gamma(\gamma)}\int_0^t \frac{g^{(2)}(\gamma,\tau)d\tau}{(t-\tau)^{1-\gamma}},\ t \in (0,b]. \end{aligned} \tag{3.2}$$

Now we can use the property: If $g^{(2)}(\gamma,\cdot) \in \mathcal{J}_0([0,b])$, then ${}_0I_t^{\gamma} g^{(2)}(\gamma,\cdot) \in \mathcal{J}_0([0,b])$, as well (cf. [1], Satz 4, p.112). It follows that $f^{(1)} \in \mathcal{J}_0([0,b])$ because $g^{(2)}(\gamma,\cdot) \in \mathcal{J}([0,b])$ and $t^{1-\gamma} \in \mathcal{J}_0([0,b])$.

2) By (2.16) $g^{(i)}(\gamma,0) = 0, i = 0,1$ is a necessary condition that $f^{(1)}(0^+)$ exists. This proves the assertion of Lemma 2.4 for $n = 2$.

Case $n > 2$.

We can also start with (2.15) and by total induction we can prove the first part of the Lemma 3.1. For the second part we start with

$$\begin{aligned} f^{(n-1)}(t) &= \frac{d}{dt}\frac{1}{\Gamma(\gamma)}\int_0^t \frac{g^{(n-1)}(\gamma,\tau) f d\tau}{(t-\tau)1-\gamma} \\ &= \frac{1}{\Gamma(\gamma)}\left(\frac{g^{(n-1)}(\gamma,0^+) f}{t^{1-\gamma}} + \int_0^t \frac{g^{(n)}(\gamma,\tau) f d\tau}{(t-\tau)^{1-\gamma}}\right), \ 0 < t \le b. \end{aligned} \tag{3.3}$$

and the procedure of the proof is just the same as in case $n = 2$.

□

Remark 3.1 Lemma 2.4 gives the mutual dependence between the initial condition of $\left({}_0I_t^{1-\gamma} f\right)$ and of y for the solution to equation (1.1). It is particularly interesting that if $f^{(k)}(0^+)$ exists for $k = 0,...,n-1$, then $\left({}_0I_t^{1-\gamma_m} f\right)^{(j)}(0^+) = 0, \ j = 0,...,n-1$.

The next lemma gives the mutual dependence between ${}_0I_t^{1-\gamma_1} f$ and ${}_0I_t^{1-\gamma_2} f$ for different values of γ_1 and γ_2.

Lemma 3.5 *Let* $f \in \mathcal{J}_0([0,b]), \ n_2 \ge 2$ *and*
1) $\alpha_i = n_i - 1 + \gamma_i, \ i = 1,2, \ 1 \ge \gamma_2 > \gamma_1 > 0, \ n_1 = n_2 = n;$
or
2) $\alpha_i = n_i - 1 + \gamma_i, \ i = 1,2, \ n_2 \ge n_1 + 1.$

If $\left({}_0I_t^{1-\gamma_2} f\right) \in \mathcal{J}_{n_2}([0,b])$ *and* $\left({}_0I_t^{1-\gamma_2} f\right)^{(j)}(0^+) = 0, \ j = 0,...,n_2-2,$ *then:*
a) ${}_0I_t^{1-\gamma_1} f \in \mathcal{J}_{n_1}([0,b]);$
b) ${}_0D_t^{\alpha_i} f, \ i = 1,2,$ *exist and* ${}_0D_t^{\alpha_i} f, \ i = 1,2,$ *belong to* $\mathcal{J}_0([0,b]);$
c) $\left({}_0I_t^{1-\gamma_1} f\right)^{(j)}(0^+) = 0, \ k = 0,...,n_1-2;$
d) $\left({}_0I_t^{1-\gamma_1} f\right)^{(n_1-1)}(0^+) = 0.$

Proof. 1) Case $n_1 = n_2 = n,\ \gamma_2 > \gamma_1$. Since fractional integrals have the semigroup property in $L^1([0,b])$ (cf. [6], p.73), we have:

$$\begin{aligned} {}_0I_t^{1-\gamma_1} f(t) &= {}_0I_t^{1-\gamma_2+\gamma_2-\gamma_1} f(x) \\ &= {}_0I_t^{\gamma_2-\gamma_1}\ {}_0I_t^{1-\gamma_2} f(t),\ t \in (0,b]. \end{aligned} \tag{3.4}$$

Since ${}_0I_t^{1-\gamma_2} f$ is continuous ($n_2 \geq 2$), it follows $\left({}_0I_t^{1-\gamma_1} f\right)(0^+) = 0$.

The derivative of convolution (cf. [1], p.119) gives ($n = n_1 = n_2$):

$$\begin{aligned} \left({}_0I_t^{1-\gamma_1} f\right)^{(n)}(t) &= \left(\left({}_0I_t^{1-\gamma_2} f\right)^{(n)}(\tau) * \frac{\tau^{\gamma_2-\gamma_1-1}}{\Gamma(\gamma_2-\gamma_1)}\right)(t) \\ &+ \sum_{k=0}^{n_2-1} \left({}_0I_t^{1-\gamma_2} f\right)^{(k)}(0^+) \left(\frac{\tau^{\gamma_2-\gamma_1-1}}{\Gamma(\gamma_2-\gamma_1)}\right)^{(n-1-k)}(t), \\ t &\in (0,b]. \end{aligned} \tag{3.5}$$

To prove that ${}_0I_t^{1-\gamma_1} f \in \mathcal{J}_{n_1}([0,b])$ it remains to show that $\left({}_0I_t^{1-\gamma_1} f\right)^{(n_1)} \in \mathcal{J}_0([0,b])$. With the assumption that $\left({}_0I_t^{1-\gamma_2} f\right)^{(k)}(0^+) = 0,\ k = 0,...,n_2-2$, and with the existence of $\left({}_0I_t^{1-\gamma_2} f\right)^{(n_2-1)}(0^+)$ equation (2.19) has the form:

$$\begin{aligned} \left({}_0I_t^{1-\gamma_1} f\right)^{(n_1)}(t) &= \left(\left({}_0I_t^{1-\gamma_2} f\right)^{(n_2)}(\tau) * \frac{\tau^{\gamma_2-\gamma_1-1}}{\Gamma(\gamma_2-\gamma_1)}\right)(t) + \\ &+ \left({}_0I_t^{1-\gamma_2} f\right)^{(n_2-1)}(0^+) \frac{t^{\gamma_2-\gamma_1-1}}{\Gamma(\gamma_2-\gamma_1)},\ t \in (0,b]. \end{aligned}$$

Since $\left({}_0I_t^{1-\gamma_2} f\right)^{(n_2)}, \frac{t^{\gamma_2-\gamma_1-1}}{\Gamma(\gamma_2-\gamma_1)} \in \mathcal{J}_0([0,b])$, it follows that $\left({}_0I_t^{1-\gamma_1} f\right)^{(n_1)} \in \mathcal{J}_0([0,b])$, as well.

By Lemma 2.3, this proves the assertions a) and b). For c) and d) we have only to start with (2.19) taking the j-th derivative, $j = 1,...,n_1-2$, of ${}_0I_t^{1-\gamma_1} f$:

$$\begin{aligned} \left({}_0I_t^{1-\gamma_1} f\right)^{(j)}(t) &= \left(\left({}_0I_t^{1-\gamma_2} f\right)^{(j)}(\tau) * \frac{\tau^{\gamma_2-\gamma_1-1}}{\Gamma(\gamma_2-\gamma_1)}\right)(t) \\ &+ \sum_{k=0}^{j-1} \left({}_0I_t^{1-\gamma_2} f\right)^{(k)}(0^+) \left(\frac{\tau^{\gamma_2-\gamma_1-1}}{\Gamma(\gamma_2-\gamma_1)}\right)^{(j-1-k)},\ 0 < t \leq b. \end{aligned} \tag{3.6}$$

Now, with supposition that $\left({}_0I_t^{1-\gamma_2}f\right)^{(j)}(0^+) = 0,\ j = 1, ..., n_1 - 2$, c) is evident. For d) we have

$$\left({}_0I_t^{1-\gamma_1}f\right)^{(n_1-1)}(t) = \left(\left({}_0I_t^{1-\gamma_2}f\right)^{(n_2-1)}(\tau) * \frac{\tau^{\gamma_2-\gamma_1-1}}{\Gamma(\gamma_2-\gamma_1)}\right)(t),\ 0 < t \le b.$$

Since $\left({}_0I_t^{1-\gamma_2}f\right)^{(n_2-1)}(t)$ is continuous on $[0, b]$, it follows that

$$\left({}_0I_t^{1-\gamma_2}f\right)^{(n_2-1)}(0^+) = 0.$$

This proves Lemma 2.4 with supposition 1).

2) Case $n_2 \ge n_1 + 1$. As in 1) we have for $n_1 = n_2 - p,\ p \ge 1,\ t \in (0, b]$,

$$\begin{aligned}
\left({}_0I_t^{1-\gamma_1}f\right)^{(n_1)}(t) &= \left(\frac{d}{dt}\right)^{n_1} {}_0I_t^{1-\gamma_1}f(t) = \left(\frac{d}{dt}\right)^{n_2-p} {}_0I_t^{1-\gamma_1}f(t) \\
&= \left(\frac{d}{dt}\right)^{n_2} {}_0I_t^{p+1-\gamma_1}f(t) = \left(\frac{d}{dt}\right)^{n_2} {}_0I_t^{p+1+\gamma_2-\gamma_2-\gamma_1}f(t) \\
&= \left(\frac{d}{dt}\right)^{n_2} {}_0I_t^{p+\gamma_2-\gamma_1}\,{}_0I_t^{1-\gamma_2}f(t) \\
&= \left(\frac{d}{dt}\right)^{n_2} \frac{1}{\Gamma(\gamma_2-\gamma_1+p)} \int_0^t \frac{\left({}_0I_t^{1-\gamma_2}f\right)(\tau)\,d\tau}{(t-\tau)^{\gamma_1-\gamma_2+1-p}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\gamma_2-\gamma_1+p)} \int_0^t \frac{\left({}_0I_t^{1-\gamma_2}f\right)^{(n_2)}(\tau)\,d\tau}{(t-\tau)^{\gamma_1-\gamma_2+1-p}} \\
&+ \sum_{j=0}^{n_2-1} \left({}_0I_t^{1-\gamma_2}f\right)^{(j)}(0^+) \left(\frac{t^{\gamma_2-\gamma_1-1+p}}{\Gamma(\gamma_2-\gamma_1+p)}\right)^{(n_2-1-j)}.
\end{aligned} \tag{3.7}$$

¿From (2.21) it follows the assertion in a) and b) with supposition in 2).

The assertions c) and d) can be obtained as in case 1). This proves Lemma 2.5.

□

Remark 3.2 Lemma 2.5 implies: Let ${}_0D_t^{\alpha_i}f,\ i = 0, ..., m,\ 0 \le \alpha_0 < \alpha_1 < ... < \alpha_m \le n_m, {}_0D_t^{\alpha_m}f \in \mathcal{J}_0([0, b])$, Then there exist ${}_0D_t^{\alpha_i}f$ and ${}_0D_t^{\alpha_i}f \in$

$\mathcal{J}_0([0,b]),\ i=0,...,m-1$. Also, if $\left({}_0I_t^{1-\gamma_m}f\right)^{(j)}(0^+)=0,\ j=0,...,n_m-2$, then $\left({}_0I_t^{1-\gamma_i}f\right)^{(j)}(0^+)=0,\ j=0,...,n_{m_i},\ i=0,...,m-1$.

These relations are useful in solving equations by the Laplace transform method (cf. Part 4) and in finding the number of linearly independent solutions.

4. *Equation (1.1) with* $\alpha_m-\alpha_0>1$

We consider homogeneous and the nonhomogeneous forms of equation (1.1) with suppositions on α_i, as it is given in the Introduction.

4.1 Laplace transform on an bounded interval

Let $L^{\exp}[0,\infty)$ denote the space of functions f such that $f\in\mathcal{J}_0([0,b])$ for every $b<\infty$ and $|f(t)|\le Ce^{\omega t},\ t_0<t$ for any $t_0,\ 0<t_0<\infty$, where C depends on t_0 and $\omega\in\mathbb{R}_0$.

It is easily seen that every function $f\in\mathcal{J}_0([0,b])$ can be extended on $(0,\infty)$ so that the extension $\overline{f}$ belongs to $L^{\exp}[0,\infty)$. Then the restriction of $\overline{f}$ on $[0,b)$ is f $(\overline{f}|_{[0,b)}=f)$.

In $L^{\exp}[0,\infty)$ we define the following equivalence relation: $f\sim g\Leftrightarrow f-g\in L^{\exp}[b,\infty)$ for some $b>0$. Since $L^{\exp}[b,\infty)$ is a vector space, the quotient space

$$L_b^{\exp}\equiv L^{\exp}[0,\infty)/L^{\exp}[b,\infty) \tag{4.1}$$

is correctly defined. An element of $L_b^{\exp}$ is the class defined by an $f\in L^{\exp}[0,\infty)$: the function $g\in cl(f)\Leftrightarrow f-g\in L^{\exp}[b,\infty)$ and we write $f\simeq g$. We quote without the proof:

Lemma 4.6 $\mathcal{J}_0([0,b])$ *is algebraically isomorphic to* $L_b^{\exp}$.

The Laplace transform of $f\in\mathcal{J}_0([0,b])$, denoted also by $\mathcal{L}f$, is defined by use of the quotient space (2.22). This leads to

$$\mathcal{L}L_b^{\exp}=\mathcal{L}L^{\exp}[0,\infty)/\mathcal{L}L^{\exp}[b,\infty). \tag{4.2}$$

Definition 4.1 *If* $f\in\mathcal{J}_0([0,b])$*, then we write* $\mathcal{L}f$ *for* $cl\mathcal{L}(\overline{f})$ *where* $\overline{f}$ *is extension of* f, $\mathcal{L}f\cong cl\mathcal{L}(\overline{f})$.

By Lemma 2.6, to find the Laplace transform of $f\in\mathcal{J}_0([0,b])$, we find the Laplace transform of any element $\overline{f_0}\in cl(\overline{f})$, i.e., $\mathcal{L}(\overline{f_0})$ and by (2.23) it follows that $cl(\mathcal{L}(\overline{f_0}))\in\mathcal{L}L_b^{\exp}$. We quote the main properties of the Laplace transform in $\mathcal{J}_0([0,b])$. If $f,g\in\mathcal{J}_0([0,b])$, one has

1) $$\mathcal{L}(\mathcal{C}_\infty\{+\mathcal{C}_\in\}) \sim \mathcal{C}_\infty\mathcal{L}\overline{\{}+\mathcal{C}_\in\mathcal{L}\overline{\}}, \quad \mathcal{L}(\{*\}) \simeq \mathcal{L}(\overline{\{})\mathcal{L}(\overline{\}}).$$

2) Let $f^{(n)} \in \mathcal{J}_0([0,b])$. Then (for Re $s > s_0$),

$$\left(\mathcal{L}f^{(n)}\right)(s) \simeq \left(\mathcal{L}(\overline{f}^{(n)})\right)(s) = s^n(\mathcal{L}\overline{f})(s) - \overline{f}(0)s^{n-1} - \ldots - \overline{f}^{(n-1)}(0).$$

Using the properties 1) and 2) we have:

$$(\mathcal{L}\,{}_0D_t^\alpha f)(s) = s^\alpha \mathcal{L}\overline{f}(s) - \left({}_0I_t^{1-\gamma_1}\overline{f}\right)(0^+)\,. \tag{4.3}$$

Application of the Laplace transform on $[0,b]$ reduces to the application of classical Laplace transform to the equation

$$\sum_{i=0}^{m} A_i\,{}_0D_t^{\alpha_i}\overline{y}\,(t) = \overline{f}\,(t)\,, \quad t \geq 0, \tag{4.4}$$

where $\overline{y}$ and $\overline{f}$ belong to the space $L^{\exp}[0,\infty)$ and are extensions on $[0,\infty)$ of $y \in \mathcal{J}_0([0,b])$ and f respectively.

If $\overline{y}$ is a solution to (1.1) and $\overline{y} \in L^{\exp}[0,\infty)$ then $y = \overline{y}|_{[0,b]}$ is a solution to (1.1). Here we use $\overline{y}|_{[0,b]}$ to denote the restriction of $\overline{y}$ on $[0,b]$.

4.2 Properties of $\mathcal{L}^{-1}(Q_\alpha(\cdot))$

Formally, applying the Laplace transform to (2.25) we have (cf [6], p.84):

$$\hat{\overline{y}}(s) = \frac{1}{\sum\limits_{i=0}^{m} A_i s^{\alpha_i}}\left(\hat{\overline{f}} + \sum_{i=0}^{m}\sum_{j=0}^{n_i-1} s^{n_i-j-1}\left({}_0I_t^{1-\gamma_i}y\right)^{(j)}(0^+)\right). \tag{4.5}$$

We see that the functions

$$Q_\alpha(s) \equiv \frac{1}{\sum\limits_{i=0}^{m} A_i s^{\alpha_i}},\ \mathrm{Re}\ s > x_1 \quad \text{and} \quad \eta(t) \equiv \mathcal{L}^{-1}\left(Q_\alpha\left(s\right)\right)(t),\ t > 0, \tag{4.6}$$

have the basic role in finding the analytical form for the solution $\bar{y}$, for (2.25). The number x_1 will be fixed later in the text.

Note that there exists $x_1 > 0$ such that the function $\sum\limits_{i=0}^{m-1}\frac{A_i}{A_m}s^{-(\alpha_m-\alpha_i)}$ is analytic for Re $(s) > x_1$ and that

$$\left|\sum_{i=0}^{m-1}\frac{A_i}{A_m}s^{-(\alpha_m-\alpha_i)}\right| < 1,\ \ \mathrm{Re}\ (s) > x_1.$$

By (2.27) we have

$$Q_\alpha(s) = \frac{1}{A_m s^{\alpha_m}}\left(1 + \hat{\phi}(s)\right), \quad \mathrm{R}e\ (s) > x_1, \tag{4.7}$$

where $\hat{\phi}(s) = \sum\limits_{r=1}^{\infty}(-1)^r\left(\sum\limits_{i=0}^{m-1}\frac{A_i}{A_m}s^{-(\alpha_m-\alpha_i)}\right)^r$, $\mathrm{R}e\ (s) > x_1$. We need the next proposition

Proposition 4.2 *Let ϕ_r and $\hat{\phi}_r$, $r = 1, \ldots$ denote the functions*

$$\begin{aligned} \phi_r(t) &= \left(\sum_{i=0}^{m-1}\frac{A_i}{A_m}\frac{1}{\Gamma(\alpha_m-\alpha_i)}t^{\alpha_m-\alpha_i-1}\right)^{*r}, \quad t > 0, \\ \hat{\phi}_r(s) &= \left(\sum_{i=0}^{m-1}\frac{A_i}{A_m}s^{-(\alpha_m-\alpha_i)}\right)^{r}, \quad Re\ (s) > s_1 \end{aligned} \tag{4.8}$$

where $(.)^{*r}$ means r-times convolution, $(.) * \ldots * (.)$ if $r \geq 2$ and $(\cdot)^{*1} \equiv (\cdot)$.

1) If $q \equiv \alpha_m - \alpha_{m-1} \geq 1$, then the series $\phi(t) = \sum\limits_{r=1}^{\infty}(-1)^r\phi_r(t)$ absolutely converges for $t \geq 0$ and there exists $x_1 > 0$ such that

$$\left(\mathcal{L}\phi\right)(s) = \sum_{r=1}^{\infty}(-1)^r\hat{\phi}_r(s) \equiv \hat{\phi}(s), \ Re\ (s) \geq x_1. \tag{4.9}$$

2) Let $0 < q < 1$ and $r_0 = \min r \in \mathbb{N}$ such that $r_0 q - 1 \geq 0$. Then the series $\sum\limits_{r=r_0}^{\infty}(-1)^r\phi_r(t)$ absolutely converges for $t \geq 0$ and for $\phi(t) = \sum\limits_{r=r_0}^{\infty}(-1)^r\phi_r(t)$, $t > 0$, we have

$$\mathcal{L}\left(\sum_{r=1}^{\infty}(-1)^r\phi_r\right)(s) = \sum_{i=1}^{\infty}(-1)^r\hat{\phi}_r(s), \ Re\ (s) > x_1. \tag{4.10}$$

Proof. For the proof we use the following Theorem (cf. [1], Satz 2. p.305) which gives the inverse Laplace transform of a function given by a series. We begin with the supposition 1), $q = \alpha_m - \alpha_{m-1} \geq 1$ and $p = \alpha_m - \alpha_0 > 1$,

$$\begin{aligned} |\phi_1(t)| &= \leq \sum_{i=0}^{m-1}\left|\frac{A_i}{A_m}\right|\frac{1}{\Gamma(\alpha_m-\alpha_i)}t^{\alpha_m-\alpha_i-1} \\ &\leq \frac{t^{\alpha_m-\alpha_0-1}}{\Gamma(\alpha_m-\alpha_0)}\sum_{i=0}^{m-1}\left|\frac{A_i}{A_m}\right|\frac{\Gamma(\alpha_m-\alpha_0)}{\Gamma(\alpha_m-\alpha_i)}t^{-(\alpha_i-\alpha_0)}, \ t \geq 1. \end{aligned}$$

Let $M_1 = \max\limits_{0 \le i \le m-1} \left|\frac{A_i}{A_m}\right| \frac{\Gamma(\alpha_m - \alpha_0)}{\Gamma(\alpha_m - \alpha_i)}$, then $|\phi_1(t)| \le M_1 m \frac{t^{\alpha_m - \alpha_0 - 1}}{\Gamma(\alpha_m - \alpha_0)}$, $t \ge 1$. Consequently,

$$|\phi_r(t)| \le (M_1 m)^r \frac{t^{rp-1}}{\Gamma(rp)}, \; r = 1, 2, ..., \; t \ge 1. \tag{4.11}$$

But if $0 \le t \le 1$, then

$$|\phi_1(t)| \le \frac{t^{\alpha_m - \alpha_{m-1} - 1}}{\Gamma(\alpha_m - \alpha_{m-1})} \sum_{i=0}^{m-1} \left|\frac{A_i}{A_m}\right| \Gamma(\alpha_m - \alpha_{m-1}) \frac{t^{\alpha_{m-1} - \alpha_i - 1}}{\Gamma(\alpha_m - \alpha_i)} \le K_1, \; 0 < t \le 1,$$

where K_1 is a positive constant. Hence,

$$|\phi_r(t)| \le K_1^r \frac{t^{r-1}}{\Gamma(r)}, \; 0 < t \le 1, \; r = 1, 2, ... \tag{4.12}$$

Finally if $q \equiv \alpha_m - \alpha_{m-1} < 1$, then

$$|\phi_r(t)| \le (m M_2)^q \frac{t^{qr-1}}{\Gamma(qr)}, \; 0 < t \le 1, \; r = 1, 2, ..., \tag{4.13}$$

where

$$M_2 = \max_{0 \le i \le m-1} \left|\frac{A_i}{A_m}\right| \Gamma(\alpha_m - \alpha_{m-1}).$$

Let $r_0 \le \min r \in \mathbb{N}$ such that $rq - 1 \le 1$. Then

$$\phi(t) = \phi_1(t) + ... + \phi_{r_0}(t) + \sum_{r=r_0+1}^{\infty} \phi_r(t), \; 0 < t \le 1$$

and

$$|\phi_r(t)| \le K_2^{r_0+i} \frac{t^{r_0+i-1}}{\Gamma(r_0+i)}, \; r = r_0 + i, \; i = 1, 2, ..., \; 0 < t \le 1. \tag{4.14}$$

Thus, with (2.32), (2.33), (2.34) and (2.35) we have:

1. If $p \equiv \alpha_m - \alpha_0 > 1, \; q \equiv \alpha_m - \alpha_{m-1} \ge 1$, then

$$|\phi_r(t)| \le K_1^r \frac{t^{r-1}}{\Gamma(r)} + (M_1 m)^r \frac{t^{rp-1}}{\Gamma(rp)}, \; r \in \mathbb{N}; \tag{4.15}$$

2. If $p \equiv \alpha_m - \alpha_0 > 1,\ q \equiv \alpha_m - \alpha_{m-1} < 1$, then

$$|\phi_r(t)| \leq K_1^r \frac{t^{r-1}}{\Gamma(r)} + K_2^{r_0+i} \frac{t^{r_0+i-1}}{\Gamma(r+i)},\ r = r_0 + i,\ i \in \mathbb{N}. \tag{4.16}$$

By (2.36) and (2.37) we can prove that the suppositions a) in Doetsch's theorem are satisfied in both cases $q \equiv \alpha_m - \alpha_{m-1} \geq 1$ and $q \equiv \alpha_m - \alpha_{m-1} < 1$. Consequently, the assertions of the proposition follow.

□

We can apply Proposition 2.2 and (2.27), (2.28) to give the analytical form of $\eta(t) = \left(\mathcal{L}^{-1} Q_\alpha(s)\right)(t),\ t > 0$:

$$\eta(t) = \frac{1}{A_m} \frac{t^{\alpha_m - 1}}{\Gamma(\alpha_m)} + \left(\frac{1}{A_m} \frac{\tau^{\alpha_m - 1}}{\Gamma(\alpha_m)} * \phi(\tau) \right)(t),\ t > 0, \tag{4.17}$$

where ϕ is given by $\phi = \sum\limits_{r=1}^{\infty} (-1)^r \phi_r$. In case 1), $q \geq 1$, η can be extended on $[0, b]$.

4.3 Homogeneous Equation (1.1) with non-integer α_m

Theorem 4.1 *Suppose that in equation (1.1) we have:*

$$a)\ n_m - 1 < \alpha_m < n_m; \quad b)\ n_m \geq 2;$$

$$c)\ \alpha_m - \alpha_0 > 1; \quad d)\ A_m = 1; \quad e)\ f(t) = 0,\ 0 < t \leq b.$$

If $\alpha_m - \alpha_{m-1} > 1$ then equation (1.1) has the unique solution y_1, ${}_0D_t^{\alpha_m} y_1 \in C([a,b])$ which has the property that $y_1^{(i)}(0^+) = 0,\ i = 0, ..., n_m - 2$, and $\left({}_0I_t^{1-\gamma_m} y_1\right)^{(n_m - 1)}(0^+) = 1$.

If $\alpha_m - \alpha_{m-1} \leq 1$ but $2(\alpha_m - \alpha_{m-1}) > 1$ and $\alpha_m - \alpha_{m-2} > 1$, then we have also a unique solution y_1 with the same initial condition but ${}_0D_t^{\alpha_m} y_1 \in \mathcal{J}_0([0,b])$, i.e., ${}_0I_t^{1-\gamma_m} y_1 \in \mathcal{J}_{n_m}([0,b])$. The solution y_1 is given by $y_1(t) = \eta(t), t \geq 0$, where η is given by (2.38).

Proof. We apply the Laplace transform, defined on $(0, \infty)$, to (2.25)

$$\sum_{i=0}^{m} A_i\ {}_0D_t^{\alpha_i} \bar{y}(t) = 0,\ t \geq 0,\ A_m = 1,$$

which gives (cf. (2.26) with $\hat{\bar{f}} = 0$)

$$\hat{\bar{y}}(s) = \frac{1}{\sum\limits_{i=0}^{m} A_i s^{\alpha_i}} \sum_{i=1}^{m} \sum_{j=0}^{n_i-1} s^{n_i-j-1} \left(\frac{d}{dt}^{j} {}_0I_t^{1-\gamma_i} \bar{y} \right)(0^+)\,, \quad \mathrm{Re}\; s > x_1$$

If ${}_0I_t^{1-\gamma_m}\bar{y} \in \mathcal{J}_{\backslash\Updownarrow}([\prime, \lfloor]),\ \backslash\Updownarrow \geq \in$, then $\left({}_0I_t^{1-\gamma_m}\bar{y}\right)^{(j)}(0^+)$ defines $\left({}_0I_t^{1-\gamma_i}\bar{y}\right)^{(j)}(0^+)$, $i = 1, ..., m-1$, for $j = 0, ..., n_i-1$ (cf. Remark 3.2). In addition by Lemma 2.4 it follows that the initial condition $y^{(i)}(0^+) = 0,\ i = 0, ..., n_m-2,\ \left({}_0I_t^{1-\gamma_m}\bar{y}\right)^{(n_m-1)}(0^+) = 1$ can be exchanged by

$$\begin{aligned} \left({}_0I_t^{1-\gamma_m}\bar{y}\right)^{(i)}(0^+) &= 0,\ i = 0, ..., n_m - 2, \\ \left({}_0I_t^{1-\gamma_m}\bar{y}\right)^{(n_m-1)}(0^+) &= 1. \end{aligned} \tag{4.18}$$

With all these remarks, (2.26) becomes $\hat{\bar{y}}(s) = 1/P_m(s),\ P_m(s) = \sum\limits_{i=0}^{m} A_i s^{\alpha_i}$, or $\hat{\bar{y}}(s) = Q_\alpha(s),\ \mathrm{Re}\; s > x_1$. Let $y_0(t) = \left(\mathcal{L}^{-1}\frac{1}{P_m(s)}\right)(t),\ t > 0$ and

$$y_1(t) = \left(\mathcal{L}^{-1}\frac{1}{P_m(s)}\right)(t),\ 0 < t \leq b. \tag{4.19}$$

The function y_1 given by (2.40) is the solution to (1.1) if ${}_0D_t^{\alpha_i} y_1 \in \mathcal{J}_0([0, b]),\ i = 1, ..., m$. By Lemma 2.3 and Lemma 2.5 this is satisfied if and only if ${}_0I_t^{1-\gamma_m} y_1 \in \mathcal{J}_{n_m}([0, b])$. Since the function ${}_0I_t^{1-\gamma_m} y_0(t),\ t \geq 0$ is an extension on $[0, \infty)$, of ${}_0I_t^{1-\gamma_m} y_0$ by a simple procedure we prove that ${}_0I_t^{1-\gamma_m} y_0 \in \mathcal{J}_{n_m}([0, b])$.

Using the properties of ${}_0I_t^{1-\gamma_m} y_0$ given by (2.39) we have

$$\begin{aligned} \psi(s) \;&\equiv\; \mathcal{L}\left(\left({}_0I_t^{1-\gamma_m} y_0\right)^{(n_m)}\right)(s) \;=\; \frac{1}{s^{1-\gamma_m}} \frac{s^{n_m}}{P_m(s)} - 1 \\ &= \frac{s^{\alpha_m}}{P_m(s)} - 1 = (-1)\frac{P_{m-1}(s)}{P_m(s)} \\ &= \frac{(-1)}{s^{\alpha_m - \alpha_{m-1}}} \frac{s^{-\alpha_{m-1}} P_{m-1}(s)}{s^{-\alpha_m} P_m(s)} \\ &= \frac{(-1)}{s^{\alpha_m - \alpha_{m-1}}} F(s),\ \ \mathrm{Re}\; s > x_1. \end{aligned} \tag{4.20}$$

Note that

$$s^{-\alpha_k}P_k(s)=\sum_{i=0}^{k}A_i s^{-(\alpha_k-\alpha_i)},\ \mathrm{R}e\ s>x_1,\quad k=0,...,m$$

is a bounded function $\mathrm{R}e\ s\geq x_0>0$. The function $F(s)=\frac{s^{-\alpha_{m-1}}P_{m-1}(s)}{s^{-\alpha_m}P_m(s)}$ has the properties

1) There exists $x_1>0$ such that $F(s)$ is analytical for $\mathrm{Re\,s}>\mathrm{x}_1$. Consequently, $\psi(s)$ has the same property.

2) $F(s)$ is bounded for $\mathrm{Re\,s}\geq \mathrm{x}_2>\mathrm{x}_1$.

Suppose that $\alpha_m-\alpha_{m-1}>1$. By Theorem 3 and Theorem 4, p.263 in [1] $\left({}_0I_t^{1-\gamma_m}y_0\right)^{(n_m)}$ is continuous function on $[0,b]$ and

$$\left({}_0I_t^{1-\gamma_m}y_0\right)^{(n_m)}(t)=\frac{1}{2\pi r}\int_{x-i\infty}^{x+i\infty}e^{ts}\psi(s)ds,\ x>x_1>0,\ t>0. \tag{4.21}$$

Consequently $\left({}_0I_t^{1-\gamma_m}y_0\right)\in\mathcal{J}_{n_m}([0,b])$.

If $\alpha_m-\alpha_{m-1}\leq 1$, we change the analytical form of F as $F(s)=\frac{s^{-\alpha_{m-1}}P_{m-1}(s)}{s^{-\alpha_m}P_m(s)}-A_{m-1}+A_{m-1}$, $\mathrm{R}e\ s>x_1$, so that (for $\mathrm{R}e\ s>x_1$)

$$\begin{aligned}\frac{-F(s)}{s^{\alpha_m-\alpha_{m-1}}} &= \frac{-1}{s^{\alpha_m-\alpha_{m-1}}}\frac{s^{-\alpha_{m-1}}P_{m-1}(s)}{s^{-\alpha_m}P_m(s)}=\\ &= \frac{-1}{s^{\alpha_m-\alpha_{m-1}}}\left(\frac{s^{-\alpha_{m-1}}P_{m-1}(s)}{s^{-\alpha_m}P_m(s)}-A_{m-1}+A_{m-1}\right)\\ &= \frac{-1}{s^{\alpha_m-\alpha_{m-1}}}\times\\ &\quad\left(\frac{A_{m-1}+s^{-\alpha_{m-1}}P_{m-2}(s)-A_{m-1}(1+s^{-\alpha_m}P_{m-1}(s))}{s^{-\alpha_m}P_m(s)}\right.\\ &\quad\left.+A_{m-1}\right)\\ &= \frac{-1}{s^{\alpha_m-\alpha_{m-1}}}\left(\frac{s^{-\alpha_{m-1}}P_{m-2}(s)-A_{m-1}s^{-\alpha_m}P_{m-1}(s)}{s^{-\alpha_m}P_m(s)}+A_{m-1}\right).\end{aligned}$$

The last equation may be written as

$$\frac{-F(s)}{s^{\alpha_m-\alpha_{m-1}}} = \frac{-1}{s^{\alpha_m-\alpha_{m-1}}}\left(\frac{1}{s^{\alpha_{m-1}-\alpha_{m-2}}}\frac{s^{\alpha_{m-2}}P_{m-2}(s)}{s^{-\alpha_m}P_m(s)}\right.$$

$$
\begin{aligned}
&- \quad A_{m-1}\frac{-1}{s^{\alpha_m-\alpha_{m-1}}}\frac{s^{-\alpha_{m-1}}P_{m-1}(s)}{s^{-\alpha_m}P_m(s)}+A_{m-1}\Bigg) \\
&= \quad \frac{-1}{s^{\alpha_m-\alpha_{m-2}}}\frac{s^{\alpha_{m-2}}P_{m-2}(s)}{s^{-\alpha_m}P_m(s)}-\frac{A_{m-1}}{s^{2(\alpha_m-\alpha_{m-1})}}\frac{s^{-\alpha_{m-1}}P_{m-1}(s)}{s^{-\alpha_m}P_m(s)} \\
&+ \quad \frac{A_{m-1}}{s^{\alpha_m-\alpha_{m-1}}}, \quad \mathrm{Re}\, s > x_1. \qquad (4.22)
\end{aligned}
$$

If $\alpha_m-\alpha_{m-2}>1$ and $2(\alpha_m-\alpha_{m-1})>1$, we can use once more Theorem 3 and Theorem 4 in [1], p. 263 for functions in right-hand side of (2.43). Then it follows from (2.43) that

$$
\mathcal{L}^{-1}\left(\frac{-F(s)}{s^{\alpha_m-\alpha_{m-1}}}\right)\Big|_{(0,b]} \in \mathcal{J}_0([0,b]),
$$

i.e., $\left({}_0I_t^{1-\gamma_m}y_0\right)^{(n_m)} \in \mathcal{J}_0([0,b])$, or, $\left({}_0I_t^{1-\gamma_m}y_0\right) \in \mathcal{J}_{n_m}([0,b])$. This proves Theorem 2.1.

□

Remark 4.1

1) Usually with equation (2.25) we have also the initial condition expressed by $\left({}_0I_t^{1-\gamma_i}y_0\right)^{(j)}(0^+)$, as (2.39). In that case we can use (2.26). But if the initial conditions are given by y and its derivatives, then by Lemma 2.4 we can find mutual dependence between initial condition given by y and by ${}_0I_t^{1-\gamma_m}y$, so that we can use (2.39).

2) In Theorem 2.1 we have mixed initial condition, because if

$$
\left({}_0I_t^{1-\gamma_m}y_0\right)^{(n_m-1)}(0^+)=1,
$$

then $y^{(n_m-1)}(0^+)$ does not exist (cf. Lemma 2.4).

3) In Theorem 2.1 we supposed that $n_m\geq 2$. If $n_m\leq 1$, then equation (1.1) reduces to the Volterra integral equation with the Kernel of the form $K(t-\tau)$ or it reduces to Abel integral equation.

4.4 Homogeneous equation with $\alpha_m=[\alpha_m]\in\mathbb{N}$

By Theorem 2.1 with $n_m-1<\alpha_m<n_m$, we know that there exists a solution, y_1 to (1.1) with $f=0$, which has the properties:

1) $\left({}_0I_t^{1-\gamma_m}y_1\right)^{(j)}$, $j=0,...,n_m-1$ are continuous functions on $[0,b]$ and

$\left({}_0I_t^{1-\gamma_m} y_1\right)^{(n)} \in \mathcal{J}_0([0,b])$;

2) $\left({}_0I_t^{1-\gamma_m} y_1\right)(0^+) = 0,\ j = 0,...,n_m - 2$ and $\left({}_0I_t^{1-\gamma_m} y_1\right)^{(n_m-1)}(0^+) = 1$.

In this case, if $\alpha_m - \alpha_{m-1} > 1$, we can construct new linearly independent solutions using only derivatives. But in case $\alpha_m = [\alpha_m]$ this possibility can be better used. Recall that if $\alpha_m = [\alpha_m] \in \mathbb{N}$, then ${}_0D_t^{\alpha_m} = D^{[\alpha_m]}$ (cf. Introduction).

Theorem 4.2 *We suppose that assumptions b),...,e) from Theorem 2.1 are satisfied. Let $0 \leq \alpha_0 < ... < \alpha_{m-1} < \alpha_m = [\alpha_m] \in \mathbb{N}$. Then equation (1.1) has the unique solution y_1 in the space $\mathcal{J}_{[\alpha_m]}([0,b])$ which satisfies the initial condition $y_1^{(j)}(0^+) = 0,\ j = 0,...,[\alpha_m] - 2$ and $y_1^{([\alpha_m]-1)}(0^+) = 1$. Let additionally $\alpha_m - \alpha_{m-1} = k + \varepsilon,\ k \in \mathbb{N},\ \varepsilon \in (0,1)$ and let $y_1^{([\alpha_m]+k)} \in \mathcal{J}_0([0,b])$. Then $y_q = y_1^{(q-1)},\ q = 1,...,k+1$, represents a linearly independent system of $k+1$ solutions to (1.1) ($f = 0$) with the properties:*
1) for $q = 1,...,k$:

$$\begin{aligned} y_q &\in \mathcal{J}_{[\alpha_m]}([0,b]),\ y_k^{([\alpha_m])} \in C[0,b]; \\ y_q^{([\alpha_m]-q)} &= 1; \\ y_q^{(i)}(0^+) &= 0,\ i = 0,...,[\alpha_m],\ i \neq [\alpha_m] - q; \end{aligned}$$

2) for $q = k+1$

$$y_{k+1}^{(i)}(0^+) = 0,\ i = 0,...,[\alpha_m] - 1,\ i \neq [\alpha_m] - (k+1).$$

If the assumption that $y_1^{([\alpha_m]+k)} \in \mathcal{J}_0([0,b])$ is not satisfied, we can assert that only k linearly independent solutions exist on $[0,b]$; y_{k+1} is a solution on $(0,b]$.

Proof. Let $\alpha_m = [\alpha_m]$. Then (2.25) with initial condition $y^{(j)}(0^+) = 0,\ j = 0,...,[\alpha_m] - 2;\ y^{([\alpha_m]-1)}(0^+) = 1$ and with $f = 0$, becomes (cf. Lemma 2.4)

$$\left(\sum_{i=0}^{m} A_i s^{\alpha_i}\right) \bar{y}(t) = 1.$$

By the same method as in the proof of Theorem 2.1 we obtain

$$\left(\mathcal{L} y_1^{([\alpha_m])}\right)(s) = \frac{(-1)}{s^{\alpha_m - \alpha_{m-1}}} F(s), \quad \text{Re } s > x_1 \tag{4.23}$$

where

$$F(s)=\frac{\sum\limits_{i=0}^{m-1} A_i s^{\alpha_i-\alpha_{m-1}}}{\sum\limits_{i=0}^{m} A_i s^{\alpha_i-\alpha_m}} \equiv \frac{s^{-\alpha_{m-1}}P_{m-1}(s)}{s^{-\alpha_m}P_m(s)}, \quad \mathrm{Re}\ s > x_1. \tag{4.24}$$

Now we recall some known results to prove the existence of other solutions y_q to (1.1) with $f=0$:

a) Let $\mathrm{Re}\,\alpha \geq 0,\ k \in \mathbb{N}$ and $D^k = \left(\frac{d}{dx}\right)^k$. If $\left({}_0D_t^\alpha y\right)(x)$ and $\left({}_0D_x^{\alpha+k} y\right)(x)$ exist, then $D^k\left({}_0D_t^\alpha y\right)(x) = \left({}_0D_x^{\alpha+k} y\right)(x)$. (cf. [6], p.74).

b) Let y_1 have k continuous derivatives on $[0,b]$. Let $\alpha_i > 0$. If either $D^{\alpha_i}[D^k y_1]$ or $D^{k+\alpha_i} y_1$ exists on $(0,b]$, then

$$\begin{aligned} {}_0D_t^{k+\alpha_i} y_1(t) &= {}_0D_t^{\alpha_i}[D^k y_1(t)] + \\ &+ \sum_{j=1}^{k} \frac{t^{-\alpha_i-j}}{\Gamma(1-\alpha_i-j)} D^{k-j} y_1(0),\ 0 < t \leq b. \end{aligned} \tag{4.25}$$

(cf. [10], p.110). Hence, by (2.46) we have

$$D^k[{}_0D_t^{\alpha_i} y_1(t)] = {}_0D_t^{\alpha_i}[D^k y_1(t)],\ t \in (0,b],\ \alpha_i > 0. \tag{4.26}$$

We apply these results to homogeneous (1.1). Then with (2.47), (2.46) and initial condition $y_1^{(j)}(0^+) = 0,\ j = 0, ..., [\alpha_m] - 2$, we get

$$\sum_{i=0}^{m} A_i\ {}_0D_t^{\alpha_i} y_1^{(k)}(t) = D^k \sum_{i=0}^{m} A_i\ {}_0D_t^{\alpha_i} y_1 = 0,$$

because $1 \leq k \leq [\alpha_m] - 1$. In such a way with y_1 we have also $y_q = y_1^{(q-1)},\ q = 2, ..., k+1,\ k+1$ solutions to homogeneous form of equation (1.1), if we prove that a) and b) are satisfied.

Consider now (2.45), where $\alpha_m - \alpha_{m-1} = k + \varepsilon,\ k \in \mathbb{N},\ \varepsilon \in (0,1)$. Thus we can write (2.44) as:

$$\left(\mathcal{L}\bar{y}_1^{([\alpha_m])}\right)(s) = (-1)\frac{1}{s^{k-1}}\frac{1}{s^{1+\varepsilon}}F(s), \quad \mathrm{Re}\ s > x_1.$$

We have seen (cf. (2.33)) that $\left(\mathcal{L}^{-1}\frac{1}{s^{1+\varepsilon}}F(s)\right)(t), t > 0$ is continuous function on $[0,b]$ which tends to zero when $t \to 0^+$. Consequently, if $k = 1$, $\bar{y}_1^{([\alpha_m])}$

has the same property: $\bar{y}_1^{([\alpha_m])}$ is continuous on $[0,b]$ and $y_1^{([\alpha_m])}(0^+) = 0$. Also, by supposition, $y_1^{([\alpha_m]+1)}$ exists on $[0,b] \setminus \{0\}$. By total induction we can prove that for $j = 1,...,k$ we have

$$\left(\mathcal{L}\bar{y}_1^{([\alpha_m]+j-1)}\right)(s) = (-1)\frac{s^{j-1}}{s^{k-1}}\frac{F(s)}{s^{1+\varepsilon}} \quad \mathrm{Re}\, s > x_1. \tag{4.27}$$

Hence the properties of the function y_1 are: $y_1^{([\alpha_m]+j-1)}$, $j = 1,...,k$, are continuous functions and $y_1^{([\alpha_m]+j-1)}(0^+) = 0$. (cf. Theorem on the Laplace transform of derivation in [1], p.100). Also, by supposition, $y_1^{([\alpha_m]+k)}$ exists on $[0,b] \setminus \{0\}$.

Consider the functions:

$$y_q = y_1^{(q-1)} \quad \text{and} \quad y_q^{(i)} = y_1^{(i+q-1)},\ q = 1,...,k+1, \tag{4.28}$$

with their properties.

By (2.49), for a fixed $q = 2,...,k+1$, the properties of the function y_1 moves to the function y_q and we have the system $\{y_q\}$, $q = 1,...,k+1$ with the properties. For $q = 1,...,k$ we have

$$y_q^{(i)}(0^+) = \begin{cases} y_1^{(i+q-1)}(0^+) = 0, & 0 \le i \le [\alpha_m] - 1 - q \\ y_1^{([\alpha_m]-1)}(0^+) = 1, & i = [\alpha_m] - q \\ y^{(i+q-1)}(0^+) = 0, & [\alpha_m] - q + 1 \le i \le [\alpha_m]. \end{cases}$$

All y_q, $q = 1,...,k$, belong to $\mathcal{J}_{[\alpha_\Updownarrow]}([\prime, \lfloor])$ and $y_q^{([\alpha_m])} \in C([0,b])$. For y_{k+1} it follows that $y_{k+1}^{([\alpha_m]-(k+1))}(0^+) = 1$, $y_{k+1}^{(i)}(0^+) = 0$, $0 \le i \le [\alpha_m] - 1$, $i \ne [\alpha_m] - (k+1)$ and $y_{k+1}^{([\alpha_m])}$ exist on $[0,b] \setminus \{0\}$, by the supposition.

With these properties every y_q, $q = 2,...,k+1$, satisfies (2.38) and is a solution to (1.1) with $f = 0$.

To prove that the solutions $y_1,...,y_{k+1}$ are linearly independent, let us consider

$$\sum_{j=1}^{k+1} c_j y_j(t) = 0, \text{ i.e., } \sum_{j=1}^{k+1} c_j y_1^{(j-1)} = 0, \tag{4.29}$$

where c_j, $j = 1,...,k+1$ are constants. The solutions of (2.50) belong to $C^\infty(-\infty,\infty)$, but y_1 has not this property. Hence $c_j = 0$, $j = 1,...,k+1$ and y_j, $j = 1,...,k+1$ are linearly independent. This proves Theorem 2.2.

□

Remark 4.2 When we find the solution y_1 guaranteed by the first part of Theorem 2.2, the condition that $y_1^{([\alpha_m]+k)}(t)$ exists can be checked using theorems which give the inverse Laplace transform. Also, our procedure to find (2.41) and (2.43) can be useful.

4.5 Nonhomogeneous equation (1.1)

Theorem 4.3 *Suppose that* $\alpha_m \notin \mathbb{N},\ n \geq 2$ *and:*

1) f *is bounded function on* $[0,b]$*;*
2) *the initial conditions* $y^{(i)}\left(0^{+}\right), i=0,...,n_m-1$ *exist but are not specified;*
3) ${}_0I_t^{1-\gamma_m} y \in \mathcal{J}_{n_m}([0,b])$*;*

Then the solution to equation (1.1), is of the form

$$y\left(t\right)=\left(\eta\left(\tau\right)*f\left(\tau\right)\right)\left(t\right),\quad 0\leq t\leq b. \tag{4.30}$$

The function $\eta(t)$ *is given by (2.38).*

Proof. Assumption 2) implies that $\left({}_0I_t^{1-\gamma_i}y\right)^{(j)}\left(0^{+}\right)=0,\ i=0,...,m,\ j=0,...,n_m-1$ (cf. Lemma 3.1 and Lemma 3.2). Consequently (2.26) becomes $\hat{\bar{y}}_0(s)=Q_\alpha\left(s\right)\hat{\bar{f}},\quad \mathrm{Re}\ s>x_1$, and $y_0\left(t\right)=\left(\eta*\overline{f}\right)\left(t\right),\quad 0\leq t\leq b$. That y_0 has the supposed properties follows from the properties of the function η proved in Subsection 4.2. We have only to use derivative of the convolution (cf. [1], p.119-120).

□

Remark 4.3 The constants $y^{(i)}\left(0^{+}\right), i=0,...,n_m-1$, depend only on $\left({}_0I_t^{1-\gamma_m}y\right)^{(n_m)}$; the homogeneous part of equation (1.1) with 2) and 3) has only trivial solution $y\left(t\right)\equiv 0,\ 0\leq t\leq b$.

Theorem 4.4 *Suppose that,*

1) *conditions c) and d) of Theorem 4.1 are satisfied;*
2) $0\leq\alpha_0<\alpha_1<...<\alpha_m=[\alpha_m]\in\mathbb{N}$*;*
3) $\alpha_m-\alpha_{m-1}=k+\varepsilon,\ k\in\mathbb{N},\ \varepsilon\in(0,1)$*;*

4) $y_1^{([\alpha_m]+k)}$ *exist on* $[0,b]\backslash\{0\}$, *where* y_1 *is the unique solution to (1.1) with* $f=0$ *and* $y_1^{(j)}(0^+)=0,\ j=0,...,[\alpha_m]-2,\ y_1^{([\alpha_m]-1)}(0^+)=1$

5) $f\in\mathcal{J}_0([0,b])$.

Then the solution $Y(t)$ *to equation (1.1), with the initial conditions*

$$\begin{aligned} Y^{(i)}(0^+) &= c_i,\ \ i=[\alpha_m]-j,\ j=1,...,k+1, \\ Y^{(i)}(0^+) &= 0,\ \ i=0,...,[\alpha_m],\ \ i\neq[\alpha_m]-j,\ j=1,...,k+1, \end{aligned}$$

is given by

$$Y=(y_1*f)+\sum_{i=0}^{m}c_iy_{i+1},$$

where $y_1,...,y_{k+1}$ *are linearly independent solutions to (1.1) with* $f=0$ *and* c_i *are arbitrary constants. The solution* Y *and its derivatives* $Y^{([\alpha_m]-1)}$ *are continuous functions, but* $Y^{([\alpha_m])}$ *exist on* $[0,b]\backslash\{0\}$.

Proof. The proof is direct consequence of Theorem 4.2 and Theorem 4.3. □

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