# Lower semicontinuous functions with values in a continuous lattice

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*Abstract.* It is proved that for every continuous lattice there is a unique semiuniform structure generating both the order and the Lawson topology. The way below relation can be characterized with this uniform structure. These results are used to extend many of the analytical properties of real-valued l.s.c. functions to l.s.c. functions with values in a continuous lattice. The results of this paper have some applications in potential theory.

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## 1. Introduction.

Real-valued lower and upper semicontinuous functions play an important role in various fields of mathematics, in particular in potential theory. If one wants to generalize potential theory to non-real-valued functions, for instance to describe solutions of systems of differential equations, then the first step should be to generalize the notion of semicontinuity to such functions. There have been some attempts to generalize semicontinuity in this direction (see [2], [6], [9] and [12]). However, analytical results of the type needed in potential theory (like Dini's theorem, in-between theorems, Choquet lemma) are usually not obtained. In this text, we focus on this kind of properties.

An application of the work in this text can be found in [8]. There a non-real and non-linear axiomatic potential theory is developed with a metrizable, arcwise connected, linked bicontinuous lattice as a value space. This axiomatic theory can be used to describe the solutions of systems of differential equations like the biharmonic equations: $\Delta u = v$, $\Delta v = 0$.

It is evident that, the more restrictions we impose on the value space R, the more results we can obtain. But also, the more restrictions needed on R, the less interesting the generalization. A very good compromise, at least for our purposes, seems to be the notion of a continuous lattice (see [7] for an introduction to continuous lattices).

After the preliminaries in Section 2, we will show in Section 3 that for every continuous lattice R, there is a unique semiuniform structure (as defined by Nachbin in [11]) generating both the order and the topology on R. Since in many analytical results, the uniform (or metric) structure of the real numbers is very important, this result will be crucial for the rest of this text. Then in Section 4, we will define lower semicontinuous functions on R and prove some basic results like lattice properties and some equivalent definitions. In the last section, we will prove analytical results like: every lower semicontinuous function on a completely regular space is the supremum of continuous functions; the in-between theorem on normal spaces; Dini's theorem and the Choquet lemma. For some of these results, R has to satisfy extra conditions.

**2. Preliminaries.**

For any topological space X and any $x \in \mathrm{X}$, $\mathcal{T}_{\mathrm{X}}(x)$ will be the collection of all open neighbourhoods of $x$.

A relation between elements of X and Y is a subset of $\mathrm{X} \times \mathrm{Y}$. A function of X to Y, when identified with its graph, is a special example of a relation. We will use the following extensions of notations commonly used for functions ($A$ is a relation between X and Y and $B$ is a relation between Y and Z):

- $A(x) = \{y \in \mathrm{Y} : (x,y) \in A\}$ (the image of $x \in \mathrm{X}$);
- $A(V) = \cup_{x \in V} A(x)$ (the image of $V \subset \mathrm{X}$);
- $A^{-1} = \{(y,x) \in \mathrm{Y} \times \mathrm{X} : (x,y) \in A\}$ (the dual or inverse relation);
- $B \circ A = \{(x,z) \in \mathrm{X} \times \mathrm{Z} : \exists y \in \mathrm{Y} : (x,y) \in A, (y,z) \in B\}$ (the composite relation).

A relation $\trianglelefteq$ on a set R (i.e. between elements of R and R) is called an order relation if it satisfies:

- $\trianglelefteq^{-1} \cap \trianglelefteq = \{(x,x) : x \in \mathrm{R}\}$ (anti-symmetry);
- $\{(x,x) : x \in \mathrm{R}\} \subset \trianglelefteq$ (reflexivity);
- $\trianglelefteq \circ \trianglelefteq \subset \trianglelefteq$ (transitivity).

A set, together with an order relation, is called an ordered set. If $\trianglelefteq$ is an order relation, then we will write $x \leq y$ if $(x,y) \in \trianglelefteq$. The notation $\leq$ will be extended to subsets of R in the following way: $F \leq G$ if for all $f \in F$ and $g \in G$ we have $f \leq g$. If $F = \{f\}$, then we will also say $f \leq G$ and then $f$ is called a lower bound of $G$. If $G = \{g\}$, then we will also say $F \leq g$ and then $g$ is called an upper bound of $F$. Evidently the dual relation $\trianglelefteq^{-1}$ is also an order relation and we will write $x \geq y$ if $(x,y) \in \trianglelefteq^{-1}$.

Let $F$ be a non-empty subset of R and let $G$ be the collection of all upper bounds of $F$. If there is an $f \in G$ with $f \leq G$, then $f$ is called the supremum of $F$ and is denoted by $\sup F$. If $F = \{a,b\}$, the $\sup F$ is also denoted by $a \vee b$. In a dual way the infimum of $F$ (notation $\inf F$ or $a \wedge b$) is defined. An ordered set is called a lattice if every non-empty finite subset has a supremum and an infimum; it is called a complete lattice if every non-empty subset has a supremum and an infimum.

Let $F$ be a subset of R. $F$ is called upper (lower) directed if it is non-empty and if every pair of elements of $F$ has an upper (lower) bound in $F$. $F$ is called increasing (decreasing) if it is non-empty and if $a \in F$ and $b \geq a$ ($b \leq a$) implies $b \in F$.

Let $f$ be a map between ordered sets R and T. We say that $f$ is isotone if $a \geq b$ implies $f(a) \geq f(b)$. We say that $f$ preserves directed sups if for all upper directed $F \subset \mathrm{R}$ such that $\sup F$ exists, we have that $\sup f(F)$ exists and $f(\sup F) = \sup f(F)$. We say that $f$ preserves directed infs if for all lower directed $F \subset \mathrm{R}$ such that $\inf F$ exists, we have that $\inf f(F)$ exists and $f(\inf F) = \inf f(F)$. Note that if $f$ preserves directed sups or infs, then $f$ is isotone.

## 3. Uniform ordered spaces and continuous lattices.

The main aim of this section is to prove a characterization of continuous lattices, as defined in [7], in terms of uniform ordered spaces, as defined in [11]. This characterization will prove to be very useful in later sections, where the uniform structure of continuous lattices will be used to prove many results about lower semicontinuous functions.

**Uniform ordered spaces.**

In [11], a uniform ordered space is defined to be a set R together with a filter $\mathcal{U}$ on $\mathrm{R}^2$ such that for all $A \in \mathcal{U}$ we have that both $\{(x,x) : x \in \mathrm{R}\} \subset A$ and there is a $B \in \mathcal{U}$ such that $B \circ B \subset A$. On a uniform ordered space R a uniformity is defined by the filter $\{A \cap A^{-1} : A \in \mathcal{U}\}$. Note that if this uniformity is separated, then $\trianglelefteq = \bigcap\{A : A \in \mathcal{U}\}$ defines an order relation on R, and hence R is both a uniform space and an ordered space. The filter $\mathcal{U}$ is called the semiuniformity on R.

Now let R be a topological space with an order relation $\trianglelefteq$. If $\trianglelefteq$ is closed in $\mathrm{R}^2$, then R is said to have a closed order. In [11, Proposition II-1.6] it is shown that every uniform ordered set has a closed order. Compact topological spaces with a closed order have very nice properties. One of the most important properties is similar to the well-known result that every compact Hausdorff space is generated by a unique uniform structure.

**Proposition 3.1.** *Let* R *be a compact topological space with a closed order. Then the filter of neighbourhoods of* $\trianglelefteq$ *in* $\mathrm{R}^2$ *is the unique semiuniformity generating both the topology and the order on* R.

Proof: By [11, Proposition II-2.13] there is a semiuniformity $\mathcal{U}$ generating both the topology and the order on R.

Take $A \in \mathcal{U}$, then evidently $A$ is a neighbourhood of $\trianglelefteq$ in $\mathrm{R}^2$. Now let $V$ be open in $\mathrm{R}^2$ such that there is no $A \in \mathcal{U}$ with $A \subset V$. Then $\{A \setminus V : A \in \mathcal{U}\}$ is a filter base in $\mathrm{R}^2$ and since $\mathrm{R}^2$ is compact it must have a cluster point $(x,y)$. For this cluster point we have $(x,y) \in \bigcap\{\overline{A \setminus V} : A \in \mathcal{U}\} = \trianglelefteq \setminus V$. So $V$ cannot be a neighbourhood of $\trianglelefteq$ in $\mathrm{R}^2$. Hence $\mathcal{U}$ must be the filter of neighbourhoods of $\trianglelefteq$ in $\mathrm{R}^2$ and consequently $\mathcal{U}$ is unique. □

Note that if R is not compact, then the semiuniformity, if it exists, need not be unique. As an example consider the following two semiuniformities on the Euclidean plane $E$:

- $\mathcal{U}_1$ is generated by $A_\epsilon = \{(x,y) \in E \times E : (x_1 - y_1) \leq \epsilon, (x_2 - y_2) \leq \epsilon\}$ where $\epsilon$ runs through all strictly positive real numbers;
- $\mathcal{U}_2$ is generated by $A_\epsilon = \{(x,y) \in E \times E : (x_1 - y_1) + \epsilon(x_2 - y_2) \leq \epsilon, \epsilon(x_1 - y_1) + (x_2 - y_2) \leq \epsilon\}$ where $\epsilon$ runs through all strictly positive real numbers.

These two semiuniformities both generate the usual topology and order on the plane, but they are essentially different, seen as semiuniform spaces.

A semimetric on R is a real-valued function $m$ on $\mathrm{R}^2$ such that:

- for all $x, y \in \mathrm{R}$ we have $m(x,y) \geq 0$;

- for all $x \in \mathrm{R}$ we have $m(x,x) = 0$;
- for all $x, y, z \in \mathrm{R}$ we have $m(x,z) \leq m(x,y) + m(y,z)$.

Obviously the sets $\{(x,y) \in \mathrm{R}^2 : m(x,y) \leq \epsilon\}$ for $\epsilon > 0$ generate a semiuniformity on R. If R is a uniform ordered space such that there is a semimetric that generates the semiuniformity of R, then R is called semimetrizable. Obviously, if R is semimetrizable, then R is metrizable. If R is a metrizable ordered space, then we can define a semimetric $m$ on R in the following way: $m(x,y)$ is the distance (in $\mathrm{R}^2$) of the point $(x,y)$ to the set $\trianglelefteq$. In general, it is not true that this semimetric generates the order and topology we started with. However, if R is compact and has a closed order, then the sets $\{(x,y) \in \mathrm{R}^2 : m(x,y) \leq \epsilon\}$ for $\epsilon > 0$ are a neighbourhoodbase of $\trianglelefteq$ in $\mathrm{R}^2$. Hence 3.1 implies that every compact uniform ordered space is metrizable iff it is semimetrizable. This implies that a compact uniform ordered space is semimetrizable iff $\mathcal{U}$ is generated by a countable filter base.

**Lemma 3.2.** *Let* R *be a compact topological space with a closed order.*

1. *For all upper directed $F \subset \mathrm{R}$ we have that $\sup F$ exists and is the limit of $F$ considered as a net on* R.
2. *Suppose $\wedge$ exists as a map from $\mathrm{R}^2$ to* R. *If $\wedge$ is continuous, then $\wedge$ preserves directed sups.*
3. *For all open and increasing $U \subset \mathrm{R}$ and all $a \in U$ there is an $A \in \mathcal{U}$ with $A(a) \subset U$.*

Proof: (1) Note that we can consider $F$ as a net on R. Since R is compact, $F$ must have a cluster point $x$. For any $f \in F$, we define $F_f = \{g \in F : g \geq f\}$. Since $F$ is upper directed, $x$ must be a cluster point of the upper directed set $F_f$. Since $F_f$ is contained in the closed set $\{g \in \mathrm{R} : g \geq f\}$ we must have $x \geq f$. Now since $f$ was arbitrary, $x \geq F$. Now take $y \geq F$. Then $F$ is contained in the closed set $\{g \in \mathrm{R} : g \leq y\}$ and hence $x \leq y$. So $x = \sup F$. Since $x$ was an arbitrary cluster point, $F$ is convergent with limit $x$.

(2) If $F, G \subset \mathrm{R}$ are upper directed, then $(\sup F) \wedge (\sup G) = (\lim F) \wedge (\lim G) = \lim(F \wedge G) = \sup(F \wedge G)$.

(3) Now let $U$ be open and increasing and $a \in U$. Then $A = \mathrm{R}^2 \setminus (\{a\} \times (\mathrm{R} \setminus U))$ is an open neighbourhood of $\trianglelefteq$. So $A \in \mathcal{U}$ and $A(a) = U$. $\square$

Let R be a uniform ordered space with semiuniformity $\mathcal{U}$. In many cases, like with the usual uniform spaces, it is not convenient to use the total semiuniformity $\mathcal{U}$. For this reason we define two very useful filter bases of $\mathcal{U}$. For any $A \in \mathcal{U}$ we define $A^\bullet = \bigcap\{B \circ A \circ B : B \in \mathcal{U}\}$. It is very easy to prove that:

- $A \subset A^\bullet \in \mathcal{U}$;
- $\trianglelefteq \circ A^\bullet \circ \trianglelefteq = A^\bullet$;
- $A^\bullet$ is closed with respect to the product topology on $\mathrm{R}^2$;
- $A^{\bullet\bullet} = A^\bullet$;
- $\mathcal{U}^\bullet \equiv \{A \in \mathcal{U} : A^\bullet = A\}$ is a filter base of $\mathcal{U}$.

For any $A \in \mathcal{U}$ we define $A^\circ = \bigcup\{B : B \in \mathcal{U}, \exists C \in \mathcal{U}, C \circ B \circ C \subset A\}$. Again it is very easy to prove that:

- $A \supset A^\circ \in \mathcal{U}$;
- $\trianglelefteq \circ A^\circ \circ \trianglelefteq = A^\circ$;
- $A^\circ$ is open with respect to the product topology on $\mathrm{R}^2$;
- $A^{\circ\circ} = A^\circ$;
- $\mathcal{U}^\circ \equiv \{A \in \mathcal{U} : A^\circ = A\}$ is a filter base of $\mathcal{U}$.

In the real numbers, the strict inequality $<$ has many important applications. In general uniform ordered spaces, the place of $<$ is taken by one of the following relations:

- $a \ll b$ if there is a $A \in \mathcal{U}$ with $a \leq A(b)$;
- $a \gg b$ if there is a $A \in \mathcal{U}$ with $a \geq A^{-1}(b)$;
- $a \lll b$ if there is a $A \in \mathcal{U}$ with $A^{-1}(a) \leq A(b)$.

Note that, in general, these relations are not reflexive and hence they are no order relations. Note also that, in general, $a \ll b$ is not the same as $b \gg a$. For instance, if R is the set of extended real numbers, then $a \ll b$ is the same as $a < b$ or $a = -\infty$, and $b \gg a$ is the same as $b > a$ or $b = \infty$. In that example, $a \lll b$ is the same as $a < b$. The following results are immediate:

- if $a \lll b$ then $a \ll b$ and $b \gg a$;
- if there is a $c$ with $c \gg a$ and $c \ll b$ then $a \lll b$;
- if $a \ll b$ or $b \gg a$ then $a \leq b$;
- if $a \leq b$ and $b \ll c$ and $c \leq d$ then $a \ll d$;
- if $a \geq b$ and $b \gg c$ and $c \geq d$ then $a \gg d$;
- if $a \leq b$ and $b \lll c$ and $c \leq d$ then $a \lll d$;
- if $a \gg b$ and $c \gg d$ then $a \wedge c \gg b \wedge d$;
- if $a \ll b$ and $c \ll d$ then $a \vee c \ll b \vee d$.

From analytical point of view, the elements $a \in \mathrm{R}$ with $a \ll a$ are not very desirable. The next result shows that in a lot of situations, they are not very common.

**Lemma 3.3.** *Let* R *be an arcwise connected uniform ordered space. Then* $a \ll a \Leftrightarrow a = \inf \mathrm{R}$.

PROOF: Suppose $a \ll a$, so there is an $A \in \mathcal{U}^\circ$ with $a \leq A(a)$. Take $b \in \mathrm{R}$ and let $f \in \mathrm{C}([0,1], \mathrm{R})$ with $f(0) = b$, $f(1) = a$. Define $V = \{x \in [0,1] : f(x) \in A(a)\}$. Since $f(V) \subset A(a) \geq a$ we have $V = \{x \in [0,1] : f(x) \geq a\}$. Now, since $f$ is continuous, $V$ is both open and closed in $[0,1]$. Hence $V = [0,1]$ and so $f(0) = b \geq a$. Since $b$ was arbitrary we get $a = \inf \mathrm{R}$. The other implication is evident. □

**Continuous lattices.**

In [7], and for an arbitrary ordered space R, another strengthening of $\leq$ is defined, the 'way below' relation. They define: $a$ is way below $b$ if for every upper directed $F \subset \mathrm{R}$ with $\sup F \geq b$ there is an $f \in F$ with $f \geq a$. Then they define a continuous lattice to be a complete lattice R such that every $a \in \mathrm{R}$ is the supremum of all $b \in \mathrm{R}$ that are way below $a$. Furthermore they define some topologies on an ordered set

R:

- Scott topology: A set $U$ is Scott-open iff it is increasing and for every upper directed $F$ with $\sup F \in U$ we have $F \cap U \neq \emptyset$;
- lower topology: The sets $\{x \in \mathrm{R} : x \geq a\}$ with $a \in \mathrm{R}$ are a subbase of closed sets;
- Lawson topology: The coarsest topology finer than both the Scott and lower topology.

Note (see [7, Proposition III-1.6(i) and Exercise III-3.20(iv)]) that on a complete lattice, an increasing set is Lawson-open iff it is Scott-open and a decreasing set is Lawson-open iff it is open in the lower topology. The following results will show that there is a strong relation between the notions defined in [7] and the notions defined in this section.

**Proposition 3.4.** *Let* R *be a uniform ordered complete lattice, then:*

1. *if every upper directed set is convergent, then $a \ll b$ implies $a$ way below $b$ and every increasing and open set is Scott-open;*
2. *if for all $a \in \mathrm{R}$ we have $a = \sup\{b : b \ll a\}$, then $a$ way below $b$ implies $a \ll b$ and every Scott-open set is open and increasing;*
3. *if both every upper directed set is convergent and for all $a \in \mathrm{R}$ we have $a = \sup\{b : b \ll a\}$, then for all open $U$ and all $a \in U$ there is an $A \in \mathcal{U}$ with $\inf A(a) \in U$ and hence $\trianglelefteq(U)$ is open.*

Proof: (1) If $a \ll b$, then there is an $A \in \mathcal{U}^\circ$ with $a \leq A(b)$. Now let $F$ be upper directed with $\sup F \geq b$, then $A(b)$ is an open neighbourhood of $\sup F$ and so there is an $f \in F$ with $f \in A(b)$. But then $a \leq f$ and hence $a$ way below $b$. The second statement is evident.

(2) Let $a$ be way below $b$. Then $F = \{\inf A(b) : A \in \mathcal{U}\}$ is upper directed with supremum $b$ and so there is an $A \in \mathcal{U}$ with $a \leq \inf A(b)$ and hence with $a \ll b$. Now let $U$ be Scott-open and $a \in U$. Then $F = \{\inf A(a) : A \in \mathcal{U}\}$ is upper directed with supremum $a \in U$. So there is an $A \in \mathcal{U}$ with $\inf A(a) \in U$. But then, since $U$ is increasing, $A(a) \subset U$ and hence $U$ is open.

(3) Take $B \in \mathcal{U}^\circ$ such that $B(a) \cap B^{-1}(a) \subset U$. Now since $\{b : b \ll a\}$ is upper directed with supremum $a$, there is a $b \in B(a)$ with $b \ll a$. But then there is an $A \in \mathcal{U}$ with $b \leq \inf A(a) \leq a$ and hence with $\inf A(a) \in B(a)$. But then also $\inf A(a) \in U$. □

**Theorem 3.5.** *Let* R *be a compact uniform ordered lattice such that $a = \sup\{b : b \ll a\}$ for all $a \in \mathrm{R}$. Then* R *is a continuous lattice, $a \ll b$ iff $a$ way below $b$ and $U \subset \mathrm{R}$ is open iff $U$ is Lawson-open.*

Proof: Using 3.2 we get that R is a complete lattice and that upper directed sets are convergent. Now, using 3.4, we get that $a \ll b$ iff $a$ way below $b$ and hence that R is a continuous lattice. Furthermore since $\trianglelefteq(x)$ is closed we have that the Lawson-topology is coarser than the given topology. But since the Lawson-topology is Hausdorff and the given topology is compact, they must coincide. □

**Theorem 3.6.** *If* R *is a continuous lattice, then there is a unique semiuniformity* $\mathcal{U}$*, both generating the Lawson-topology and the order on* R*. Furthermore,* $\wedge$ *is continuous and* $a$ *way below* $b$ *iff* $a \ll b$*.*

PROOF: Using [7, Theorem VI-3.4] we get that, in the Lawson-topology, R is a compact topological space with a closed order and $\wedge$ is continuous. So, by 3.1, the filter of neighbourhoods of $\trianglelefteq$ in $\mathrm{R}^2$ is the unique semiuniformity $\mathcal{U}$ generating both the Lawson-topology and the order on R.

If $a$ way below $b$, then $U = \{x : a \text{ way below } x\}$ is open and increasing, by [7, Proposition II-1.10], and $b \in U$. So there is an $A \in \mathcal{U}$ with $A(b) \subset U$ and hence with $a \leq A(b)$. So we get that $a$ way below $b$ implies $a \ll b$ and the other implication follows from 3.4. □

**Examples and an application.**

The basic example of a continuous lattice is the set of extended real numbers (or equivalently the unit interval). Furthermore, it is easy to check that the Cartesian product of an arbitrary number of continuous lattices again is a continuous lattice. In 5.11 we will prove that the lower semicontinuous functions on a locally compact Hausdorff space, with values in a continuous lattice, again is a continuous lattice. Another nice example comes from potential theory.

**Lemma 3.7.** *Let* X *be a* $\mathcal{P}$*-harmonic space as defined in* [4]*. Then the set of positive hyperharmonic functions on* X *is a continuous lattice.*

PROOF: Evidently the set of positive hyperharmonic functions is a complete lattice. Now take any positive hyperharmonic function $f$ and any continuous real potential $p$, harmonic outside a compact set $K$, such that $f > p$ on $K$. Now let $F$ be an increasing set of positive hyperharmonic functions with $\sup F \geq f$. Since $K$ is compact, there is a $g \in F$ with $g \geq p$ on $K$ and since $p$ is continuous this implies $g \geq p$ on X. Hence $p \ll f$. But it is a well known result that $f$ is the supremum of such potentials and hence we have a continuous lattice. □

More examples can be found in [7]. A nice application of continuous lattices in potential theory is a general definition of capacity. Let X be a locally compact Hausdorff space. Similar to [4, Section 5.2], we define a capacity $\gamma$ on X as a map from the subsets of X to a continuous lattice R such that:

- If $(A_n)_n \subset \mathrm{X}$ is an increasing sequence, then $\sup_n \gamma(A_n) = \gamma(\cup_n A_n)$;
- If $(K_n)_n \subset \mathrm{X}$ is a decreasing sequence of compact sets, then $\inf_n \gamma(K_n) = \gamma(\cap_n K_n)$.

A subset $A$ of X is called $\gamma$-capacitable if $\gamma(A) = \sup\{\gamma(K) : A \supset K \text{ compact}\}$. Note that, by 5.11, R may be the set of positive lower semicontinuous, numerical functions on a locally compact Hausdorff space as in [4, Section 5.2].

**Proposition 3.8.** *Let* Y *be a locally compact Hausdorff space,* $\phi$ *a continuous map from* Y *to* X *and* $\gamma$ *a capacity on* X*. Define* $\gamma'$ *by* $\gamma'(A) = \gamma(\phi(A))$*. Then* $\gamma'$ *is a capacity and if* $A \subset \mathrm{Y}$ *is* $\gamma'$*-capacitable, then* $\phi(A)$ *is* $\gamma$*-capacitable.*

PROOF: If $(A_n)_n \subset \mathrm{Y}$ is an increasing sequence, then $(\phi(A_n))_n$ is also an increasing sequence and $\phi(\cup_n A_n) = \cup_n \phi(A_n)$. So we have $\sup_n \gamma'(A_n) = \gamma'(\cup_n A_n)$.

If $(K_n)_n \subset \mathrm{Y}$ is a decreasing sequence of compact sets, then, since $\phi$ is continuous, $(\phi(K_n))_n$ is a decreasing sequence of compact sets and $\phi(\cap_n K_n) = \cap_n \phi(K_n)$. So we have $\inf_n \gamma'(K_n) = \gamma'(\cap_n K_n)$.

If $A \subset \mathrm{Y}$ is $\gamma'$-capacitable, then $\gamma(\phi(A)) = \sup\{\gamma(\phi(K)) : A \supset K \text{ compact}\} \leq \sup\{\gamma(L) : \phi(A) \supset L \text{ compact}\} \leq \gamma(\phi(A))$ and so $\phi(A)$ is $\gamma$-capacitable. □

**Proposition 3.9.** *Every $K_{\sigma\delta}$ set $A$ in a locally compact Hausdorff space* X *is capacitable with respect to any capacity $\gamma$.*

Proof: Let $a \ll \gamma(A)$. There is a family $(K_{m,n})_{m,n}$ of compact sets such that for all fixed $m$ the sequence $(K_{m,n})_n$ is increasing and $A = \cap_m \cup_n K_{m,n}$.

For convenience, set $n_0 = 0$ and $K_{0,0} = A$. We construct inductively a sequence $(n_m)_m$ such that for all $i \in \{0, 1, \cdots\}$ we have $a \ll \gamma(\cap_{0\leq m\leq i} K_{m,n_m})$ on $L$. Assume the construction is performed for all $m < i$. Now $(K_{i,j} \cap (\cap_{0\leq m<i} K_{m,n_m}))_j$ is an increasing sequence of subsets of X whose union is $\cap_{0\leq m<i} K_{m,n_m}$. So since $L$ is compact and all $\gamma$-values are lower semicontinuous on $L$ there is a $j$ such that $a \ll \gamma(K_{i,j} \cap (\cap_{0\leq m<i} K_{m,n_m}))$ on $L$. Define $n_i$ to be that $j$.

Since $\cap_{m\geq 1} K_{m,n_m} = \cap_{m\geq 0} K_{m,n_m}$ is a compact subset of $A$ we now have

$$\sup\{\gamma(K)(x) : A \supset K \text{ compact}\} \geq \gamma(\cap_{m\geq 1} K_{m,n_m}) = \inf_i \gamma(\cap_{1\leq m\leq i} K_{m,n_m}) \geq a.$$

So since $a$ was arbitrary, we get that $A$ is $\gamma$-capacitable. □

As in [4, Section 5.2] we define a subset $A$ of a locally compact space X to be K-analytic if there exists a locally compact space Y, a $K_{\sigma\delta}$-set $A'$ of Y and a continuous map $\phi$ from Y to X such that $A = \phi(A')$. Using 3.8 and 3.9 it is clear that every K-analytic set is capacitable for any capacity. Furthermore in [4, Corolary 5.2.2] it is proved that in a locally compact space with a countable base, every Borel set is K-analytic and hence capacitable for any capacity.

## 4. Semicontinuous functions.

In this section, R is a uniform ordered space with semiuniformity $\mathcal{U}$. For every filter base $\mathcal{F}$ in R we define the following two sets:

- LIMINF $(\mathcal{F}) = \{r \in \mathrm{R} : \forall A \in \mathcal{U}\ \exists V \in \mathcal{F}, V \subset A(r)\}$;
- LIMSUP $(\mathcal{F}) = \{r \in \mathrm{R} : \forall A \in \mathcal{U}\ \exists V \in \mathcal{F}, V \subset A^{-1}(r)\}$.

These sets will be used to define lower and upper semicontinuous functions with values in R.

**Lemma 4.1.** *Let $\mathcal{F}$ be a filter base in* R*, then:*

1. *if $x \in V$ for all $V \in \mathcal{F}$, then $x \geq$* LIMINF $(\mathcal{F})$*;*
2. *if $x \leq V$ for some $V \in \mathcal{F}$, then $x \in$* LIMINF $(\mathcal{F})$*;*
3. LIMINF $(\mathcal{F}) \leq$ LIMSUP $(\mathcal{F})$*;*
4. $(\mathcal{F} \to x) \Leftrightarrow (x \in$ LIMINF $(\mathcal{F}) \cap$ LIMSUP $(\mathcal{F}))$*;*
5. LIMINF $(\mathcal{F})$ *is closed;*
6. LIMINF $(\mathcal{F})$ *is decreasing.*

Proof: (3) Take $a \in$ LIMINF $(\mathcal{F})$, $b \in$ LIMSUP $(\mathcal{F})$, $A \in \mathcal{U}$ and $B \in \mathcal{U}$ with $B \circ B \subset A$. Then there is a $V \in \mathcal{F}$ such that both $V \subset B(a)$ and $V \subset B^{-1}(b)$. This means that $(a,b) \in B \circ B \subset A$. Since $A$ was arbitrary, we have $(a,b) \in \trianglelefteq$ and hence $a \leq b$.

(5) Take $a \notin$ LIMINF $(\mathcal{F})$. Then there is an $A \in \mathcal{U}$ such that $\forall V \in \mathcal{F}\ \exists y \in V, (a,y) \notin A$. Now take $B \in \mathcal{U}$ such that $B \circ B \subset A$ and take $b \in B(a)$. Now for all $V \in \mathcal{F}$ there is a $y \in V$ such that both $(a,y) \notin B \circ B$ and $(a,b) \in B$. So for all $V \in \mathcal{F}$ there is a $y \in V$ such that $(b,y) \notin B$ and hence $b \notin$ LIMINF $(\mathcal{F})$. So the complement of LIMINF $(\mathcal{F})$ is open and hence LIMINF $(\mathcal{F})$ is closed. $\square$

If R is a complete lattice, then we define:

- $\liminf(\mathcal{F}) = \sup \text{LIMINF } (\mathcal{F})$;
- $\limsup(\mathcal{F}) = \inf \text{LIMSUP } (\mathcal{F})$.

The next result shows that if R is the set of extended real numbers, this definition of lim inf is just the usual one.

**Lemma 4.2.** *Let $\mathcal{F}$ be a filter base in a continuous lattice* R*, then:*

1. $\liminf(\mathcal{F}) = \sup\{\inf V : V \in \mathcal{F}\}$*;*
2. LIMINF $(\mathcal{F}) = \{r \in \mathrm{R} : r \leq \liminf(\mathcal{F})\}$.

Proof: (1) Let $r \in$ LIMINF $(\mathcal{F})$ and $A \in \mathcal{U}^\circ$. Take $B \in \mathcal{U}$ such that $\inf B(r) \in A(r)$ and $V \in \mathcal{F}$ with $V \subset B(r)$. Then $\inf V \in A(r)$ and hence $\sup\{\inf V : V \in \mathcal{F}\} \in A(r)$. Since $A$ and $r$ were arbitrary we get $\sup\{\inf V : V \in \mathcal{F}\} \geq \liminf(\mathcal{F})$. On the other hand we have for all $V \in \mathcal{F}$ that $\inf V \in$ LIMINF $(\mathcal{F})$ and hence $\sup\{\inf V : V \in \mathcal{F}\} \leq \liminf(\mathcal{F})$.

(2) Take $a, b \in$ LIMINF $(\mathcal{F})$ and $A \in \mathcal{U}^\circ$. Now $\{a' \vee b' : a' \ll a, b' \ll b\}$ is upper directed with supremum $a \vee b$. So there are $a' \ll a$ and $b' \ll b$ such that $a' \vee b' \in A(a \vee b)$. But then there is a $B \in \mathcal{U}$ such that for all $a' \in B(a)$ and all $b' \in B(b)$ we have $a' \vee b' \in A(a \vee b)$. Now choose $V \in \mathcal{F}$ such that $V \subset B(a) \cap B(b)$. Then we have also $V \subset A(a \vee b)$ and, since $A$ was arbitrary, $a \vee b \in$ LIMINF $(\mathcal{F})$. So LIMINF $(\mathcal{F})$ is upper directed and hence $\liminf(\mathcal{F})$ is the limit of LIMINF $(\mathcal{F})$. The fact that LIMINF $(\mathcal{F})$ is closed and decreasing completes the proof. $\square$

Let X be a topological space and let $f$ be a function from X to R. Remember that $\mathcal{T}_{\mathrm{X}}(x)$ is the filter base of all open neighbourhoods of a point $x \in \mathrm{X}$. We say that $f$ is lower semicontinuous in $x \in \mathrm{X}$ if $f(x) \in$ LIMINF $(f(\mathcal{T}_{\mathrm{X}}(x)))$ and that $f$ is upper semicontinuous in $x$ if $f(x) \in$ LIMSUP $(f(\mathcal{T}_{\mathrm{X}}(x)))$. Furthermore, $f$ is called lower (upper) semicontinuous on $V \subset \mathrm{X}$ if it is lower (upper) semicontinuous in every $x \in V$. The collection of all lower (upper) semicontinuous functions on X is denoted by LSC (X) (USC (X)). As usual, C(X) is the collection of all continuous functions on X and NUM (X) is the collection of all functions on X.

Note that if we had defined the LIMINF of a filter base $\mathcal{F}$ to be $\{r \in \mathrm{R} : \forall U \in \mathcal{T}_{\mathrm{R}}(r)\ \exists V \in \mathcal{F}, V \subset \trianglelefteq(U)\}$, then our definition of semicontinuity would have been equivalent to the one in [12]. In general, our definition is not equivalent to the one given there. If we consider continuous lattices however, then we have that for all $r \in \mathrm{R}$ and all $U \in \mathcal{T}_R(r)$, there is an $A \in \mathcal{U}$ such that $A(r) \subset \trianglelefteq(U)$. If this is the case, then the two definitions are equivalent. A disadvantage of our definition is that it is not necessary that LSC (X) is a lattice, even if $\wedge$ and $\vee$ are continuous. The main advantage of our definition is that we can use the uniform structure, for instance in Dini-like theorems. In general, the best definition of lower semicontinuous functions depends on the applications one has in mind. However, if the values are in a continuous lattice, then it is most likely that every reasonable definition is equivalent to the one given here.

**Lemma 4.3.** *Let* X *and* Y *be topological spaces, then:*

1. $\mathrm{C(X)} = \mathrm{LSC\ (X)} \cap \mathrm{USC\ (X)}$;
2. *if* $V \subset \mathrm{X}$ *open,* $f \in \mathrm{LSC}\ (\overline{V})$, $g \in \mathrm{LSC}\ (\mathrm{X} \setminus V)$ *and* $f \geq g$ *on* $\partial V$, *then* $h = \begin{cases} f \text{ on } V \\ g \text{ on } \mathrm{X} \setminus V \end{cases} \in \mathrm{LSC\ (X)}$;
3. *if* $f \in \mathrm{LSC\ (X)}$, $g \in \mathrm{USC\ (Y)}$ *and* $A \in \mathcal{U}^\bullet$, *then* $\{(x,y) \in \mathrm{X} \times \mathrm{Y} : (f(x), g(y)) \in A\}$ *is closed;*
4. *if* $f \in \mathrm{LSC\ (X)}$, $g \in \mathrm{USC\ (Y)}$ *and* $A \in \mathcal{U}^\circ$, *then* $\{(x,y) \in \mathrm{X} \times \mathrm{Y} : (g(y), f(x)) \in A\}$ *is open;*
5. *if* $f \in \mathrm{LSC\ (X)}$ *and* $g \in \mathrm{USC\ (Y)}$, *then* $\{(x,y) \in \mathrm{X} \times \mathrm{Y} : f(x) \leq g(y)\}$ *is closed;*
6. *if* $f \in \mathrm{LSC\ (X)}$ *and* $g \in \mathrm{USC\ (Y)}$, *then* $\{(x,y) \in \mathrm{X} \times \mathrm{Y} : g(y) \lll f(x)\}$ *is open;*
7. *if* $f \in \mathrm{LSC\ (X)}$ *and* $r \in \mathrm{R}$ *then* $\{x \in \mathrm{X} : r \ll f(x)\}$ *is open;*
8. *if* $F \subset \mathrm{LSC\ (X)}$ *is a lower directed, locally uniformly convergent set and* $f = \inf F$, *then* $f \in \mathrm{LSC\ (X)}$.

Proof: (2) Evidently we have $h(x) \in \mathrm{LIMINF}\ (h(\mathcal{T}_{\mathrm{X}}(x)))$ for all $x \in \mathrm{X} \setminus \partial V$. So let $x \in \partial V$. Then $h(x) = g(x) \leq f(x)$. Take $A \in \mathcal{U}^\circ$, then there is a $U \in \mathcal{T}_{\mathrm{X}}(x)$ with both $f(U \cap \overline{V}) \subset A(f(x)) \subset A(h(x))$ and $g(U \setminus V) \subset A(g(x)) = A(h(x))$. This implies $h(U) \subset A(h(x))$ and hence $h(x) \in \mathrm{LIMINF}\ (h(\mathcal{T}_{\mathrm{X}}(x)))$.

(3) Suppose $(f(x), g(y)) \notin A$. Then there is a $B \in \mathcal{U}$ with $(f(x), g(y)) \notin B \circ A \circ B$. Take $C \in \mathcal{U}$ with $C \circ C \subset B$, $V \in \mathcal{T}_{\mathrm{X}}(x)$ with $f(V) \subset C(f(x))$ and $W \in \mathcal{T}_{\mathrm{Y}}(y)$ with $g(W) \subset C^{-1}(g(y))$. Now take $x' \in V$ and $y' \in W$ and suppose $(f(x'), g(y')) \in A$. Then $(f(x'), g(y')) \in C \circ A \circ C$ and hence $(f(x), g(y)) \in C \circ C \circ A \circ C \circ C \subset B \circ A \circ B$ which gives a contradiction. So $\{(x,y) \in \mathrm{X} \times \mathrm{Y} : (f(x), g(y)) \notin A\}$ is open.

(4) Suppose $(g(y), f(x)) \in A$. Then there is a $B \in \mathcal{U}$ and a $C \in \mathcal{U}$ with $C \circ B \circ C \subset A$ and $(g(y), f(x)) \in B$. Now take $D \in \mathcal{U}$ with $D \circ D \subset C$, $V \in \mathcal{T}_{\mathrm{X}}(x)$ with $f(V) \subset D(f(x))$ and $W \in \mathcal{T}_{\mathrm{Y}}(y)$ with $g(W) \subset D^{-1}(g(y))$. Then for all $x' \in V$ and all $y' \in W$ we have $(g(y'), f(x')) \in D \circ D \subset A$.

(6) Let $g(y) \lll f(x)$. Then there is an $A \in \mathcal{U}$ with $A(f(x)) \geq A^{-1}(g(y))$. Now take $B \in \mathcal{U}$ with $B \circ B \subset A$, $V \in \mathcal{T}_{\mathrm{X}}(x)$ with $f(V) \subset B(f(x))$ and $W \in \mathcal{T}_{\mathrm{Y}}(y)$ with $g(W) \subset B^{-1}(g(y))$. Then for all $x' \in V$ and all $y' \in W$ we have $B(f(x')) \subset A(f(x))$ and $B^{-1}(g(y')) \subset A^{-1}(g(y))$ and hence $B(f(x')) \geq B^{-1}(g(y'))$.

(7) Let $r \ll f(x)$. Then there is an $A \in \mathcal{U}$ with $r \leq A(f(x))$. Now take $B \in \mathcal{U}$ with $B \circ B \subset A$ and $V \in \mathcal{T}_{\mathrm{X}}(x)$ with $f(V) \subset B(f(x))$. Then for any $x' \in V$ we

have $B(f(x')) \subset A(f(x))$ and hence $r \leq B(f(x'))$.

(8) Take $A \in \mathcal{U}$ and $x \in \mathrm{X}$. Now take $B \in \mathcal{U}^\circ$ such that $B \circ B \subset A$. Then there is a $g \in F$ and a $U \in \mathcal{T}_{\mathrm{X}}(x)$ with $(g(y), f(y)) \in B$ for all $y \in U$. Furthermore there is a $V \in \mathcal{T}_U(x)$ with $g(V) \subset B(g(x))$. Since $f \leq g$ we get $f(V) \subset B^2(f(x)) \subset A(f(x))$ and since $A$ and $x$ were arbitrary this implies $f \in \mathrm{LSC}\ (\mathrm{X})$. □

**Theorem 4.4.** *Let* R *be a continuous lattice and* X *a topological space, then:*

1. *if* $F \subset \mathrm{LSC}\ (\mathrm{X})$, *then* $\sup F \in \mathrm{LSC}\ (\mathrm{X})$;
2. *if* $f, g \in \mathrm{LSC}\ (\mathrm{X})$, *then* $f \wedge g \in \mathrm{LSC}\ (\mathrm{X})$;
3. $f \in \mathrm{LSC}\ (\mathrm{X})$ *iff* $\{(x, r) \in \mathrm{X} \times \mathrm{R} : f(x) \leq r\}$ *is closed;*
4. $f \in \mathrm{LSC}\ (\mathrm{X})$ *iff* $\forall r \in \mathrm{R} : \{x \in \mathrm{X} : r \ll f(x)\}$ *is open.*

Proof: (1) First take $f, g \in \mathrm{LSC}\ (\mathrm{X})$, $A \in \mathcal{U}^\circ$ and $x \in \mathrm{X}$. Since $\{a \vee b : a \ll f(x), b \ll g(x)\}$ is upper directed with supremum $f(x) \vee g(x)$, there are $a \ll f(x)$ and $b \ll g(x)$ such that $a \vee b \in A(f(x) \vee g(x))$. But then there is a $B \in \mathcal{U}$ such that for all $a \in B(f(x))$ and all $b \in B(g(x))$ we have $a \vee b \in A(f(x) \vee g(x))$. Now choose $V \in \mathcal{T}_{\mathrm{X}}(x)$ such that $f(V) \subset B(f(x))$ and $g(V) \subset B(g(x))$. Then for all $y \in V$ we have that $f(y) \vee g(y) \in A(f(x) \vee g(x))$. So $f \vee g \in \mathrm{LSC}\ (\mathrm{X})$.

Now take $F \subset \mathrm{LSC}\ (\mathrm{X})$ and set $f = \sup F$ and $G = \{f_1 \vee \cdots \vee f_n : f_i \in F\}$. Then $\sup G = f$, $G \subset \mathrm{LSC}\ (\mathrm{X})$ and $G$ is upper directed (and hence pointwise convergent to $f$). Now take $x \in \mathrm{X}$, $A \in \mathcal{U}$ and $B \in \mathcal{U}^\circ$ such that $B \circ B \subset A$. Then we can find an $h \in G$ with $(f(x), h(x)) \in B$ and a $V \in \mathcal{T}_{\mathrm{X}}(x)$ with $h(V) \subset B(h(x))$. But then $h(V) \subset (B \circ B)(f(x))$ and hence, since $f \geq h$, also $f(V) \subset (B \circ B)(f(x)) \subset A(f(x))$. So $f(x) \in \mathrm{LIMINF}\ (f(\mathcal{T}_{\mathrm{X}}(x)))$ and hence $f \in \mathrm{LSC}\ (\mathrm{X})$.

(2) Take $A \in \mathcal{U}^\circ$ and $x \in \mathrm{X}$. Since $\wedge$ is continuous we have that there is a $V \in \mathcal{T}_{\mathrm{R}}(f(x))$ and a $W \in \mathcal{T}_{\mathrm{R}}(g(x))$ such that $V \wedge W \subset A(f(x) \wedge g(x))$. But then, using 3.2 and 3.4, there is a $B \in \mathcal{U}$ such that $B(f(x)) \wedge B(g(x)) \subset \trianglelefteq(V) \wedge \trianglelefteq(W)$. Now since $f$ and $g$ are lower semicontinuous, there is a $U \in \mathcal{T}_{\mathrm{X}}(x)$ with both $f(U) \subset B(f(x))$ and $g(U) \subset B(g(x))$. Combining these inclusions, we get that for all $y \in U$ we have $f(y) \wedge g(y) \in A(f(x) \wedge g(x))$. Hence $f \wedge g \in \mathrm{LSC}\ (\mathrm{X})$.

(3) See [12, Proposition 1.3].

(4) Take $x \in \mathrm{X}$ and $A \in \mathcal{U}^\circ$. Now take $r \in \mathrm{R}$ with $r \ll f(x)$ and $(f(x), r) \in A$. Now $V = \{y \in \mathrm{X} : r \ll f(y)\} \in \mathcal{T}_{\mathrm{X}}(x)$ and $f(V) \subset A(f(x))$. So $f(x) \in \mathrm{LIMINF}\ (f(\mathcal{T}_{\mathrm{X}}(x)))$ for all $x \in \mathrm{X}$ and hence $f \in \mathrm{LSC}\ (\mathrm{X})$. □

**Proposition 4.5.** *Let* R *be a continuous lattice,* X *a topological space,* $V \subset \mathrm{X}$ *and* $f \in \mathrm{NUM}\ (V)$. *Define* $\hat{f} = \sup\{h \in \mathrm{LSC}\ (\mathrm{X}) : f \geq h$ *on* $V\}$. *Then* $\hat{f} \in \mathrm{LSC}\ (\mathrm{X})$ *and for all* $x \in \overline{V}$ *we have* $\liminf(f(\mathcal{T}_V(x))) = \liminf(\hat{f}(\mathcal{T}_{\mathrm{X}}(x)))$.

Proof: The fact that $f \in \mathrm{LSC}\ (\mathrm{X})$ follows directly from 4.4. Now take $x \in \overline{V}$. Since $f \geq \hat{f}$ we have $\mathrm{LIMINF}\ (\hat{f}(\mathcal{T}_{\mathrm{X}}(x))) \subset \mathrm{LIMINF}\ (f(\mathcal{T}_V(x)))$. Now take $a \in \mathrm{LIMINF}\ (f(\mathcal{T}_V(x)))$, $A \in \mathcal{U}^\circ$ and $B \in \mathcal{U}$ such that $\inf B(a) \in A(a)$. Then there is a $U \in \mathcal{T}_{\mathrm{X}}(x)$ with $f(U \cap V) \subset B(a)$ and hence with $\inf f(U \cap V) \in A(a)$. Define $h : \mathrm{X} \to \mathrm{R}$ by $h = \inf f(U \cap V)$ on $U$ and $h = \inf f(V)$ on $\mathrm{X} \setminus U$. Then $h \in \mathrm{LSC}\ (\mathrm{X})$ and $h \leq f$ and so $\hat{f} \geq h$. But then $\hat{f}(U) \subset A(a)$ and so we have $a \in \mathrm{LIMINF}\ (\hat{f}(\mathcal{T}_{\mathrm{X}}(x)))$. □

**Lemma 4.6.** *Let* R *be a continuous lattice,* X *a topological space,* $f \in \mathrm{LSC}\ (\mathrm{X})$, $A \in \mathcal{U}^\circ$ *and define* $g(x) = \sup A^{-1}(f(x))$ *for all* $x \in \mathrm{X}$. *Then* $g \in \mathrm{LSC}\ (\mathrm{X})$.

PROOF: Let $x \in \mathrm{X}$, $y \in A^{-1}(f(x))$. Then there are $B, C \in \mathcal{U}$ with $B \circ C \subset A$ and $y \in B^{-1}(f(x))$. Now there is a $V \in \mathcal{T}_\mathrm{X}(x)$ with $(f(x), f(V)) \subset C$. Hence $(y, f(V)) \subset B \circ C \subset A$ and so $y \leq g(V)$. This implies $y \in \mathrm{LIMINF}\ (g(\mathcal{T}_\mathrm{X}(x)))$ and hence $g(x) = \sup A^{-1}(f(x)) \in \mathrm{LIMINF}\ (g(\mathcal{T}_\mathrm{X}(x)))$. □

## 5. Semicontinuous functions in special spaces.

In this section we generalize some well known results on real-valued semicontinuous functions. These results are used quite often in analysis, in particular in potential theory and in the theory of Radon measures, and they may be used as a starting point for non-real-valued potential theory (e.g. see [8]). For simplicity, we will assume that R is a continuous lattice, although several results are also true in a more general case. Note that our definitions of 'complete regular spaces', 'normal spaces', 'compact spaces', etc. do not contain the Hausdorff property.

Since NUM (X) is just a Cartesian product of copies of R, $\ll$ is defined on this set according to the product semiuniformity. However, since we are looking at NUM (X) as a space of functions, not as a product space, this is not a useful definition for our purposes. We will use the following two generalizations of $\ll$ to NUM (X):

- $f \ll g$ pointwise: $\forall x \in \mathrm{X} : f(x) \ll g(x)$;
- $f \ll g$ uniform: $\exists A \in \mathcal{U}\ \forall x \in \mathrm{X} : f(x) \leq A(g(x))$.

For $\gg$ and $\lll$ similar generalizations will be used.

### Completely regular spaces.

The following two results generalize [3, Proposition 1.5]. The fact that lower semicontinuous functions can be approximated from below by continuous functions is used extensively in the theory of Radon measures and in potential theory.

**Theorem 5.1.** *The following statements are equivalent:*

1. $\{r \in \mathrm{R} : r \ll \sup \mathrm{R}\}$ *is arcwise connected;*
2. *for all completely regular* X *we have* $\mathrm{LSC}\ (\mathrm{X}) = \{f \in \mathrm{NUM}\ (\mathrm{X}) : f = \sup\{g \in \mathrm{C}(\mathrm{X}) : g \leq f\}\}$;
3. *for all completely regular* X *we have* $\mathrm{LSC}\ (\mathrm{X}) = \{f \in \mathrm{NUM}\ (\mathrm{X}) : f = \sup\{g \in \mathrm{C}(\mathrm{X}) : g \ll f \text{ uniform}\}\}$.

PROOF: ($1 \Rightarrow 3$) Take $f \in \mathrm{LSC}\ (\mathrm{X})$ and $x \in \mathrm{X}$ and $a \ll f(x)$. Then there is a $U \in \mathcal{T}_\mathrm{X}(x)$ such that $a \ll f$ uniform on $U$. Now take $V \in \mathcal{T}_\mathrm{X}(x)$ with $\overline{V} \subset U$. Since X is completely regular and there is a path from $\inf \mathrm{R}$ to $a$, we can choose $h \in \mathrm{C}(\mathrm{X})$ so that $h(x) = a$, $h(\mathrm{X} \setminus V) = \inf \mathrm{R}$ and $h \leq a$. Then $h \ll f$ uniform and the statement follows since $a$ and $x$ were arbitrary.

($3 \Rightarrow 2$) Evident.

($2 \Rightarrow 1$) Take $\mathrm{X} = [0, 1]$ and $r \ll \sup \mathrm{R}$. Now define an $h \in \mathrm{LSC}\ (\mathrm{X})$ by $h(0) = \inf \mathrm{R}$ and $h(x) = \sup \mathrm{R}$ if $x \neq 0$. Then there must be a $g \in \mathrm{C}(\mathrm{X})$ such that $g \leq h$ and $g(1) \geq r$. So $g \wedge r$ is a continuous path from $r$ to $\inf \mathrm{R}$. □

**Lemma 5.2.** *Suppose* LSC (X) $= \{f \in$ NUM (X) $: f = \sup\{g \in$ C(X) $: g \leq f\}\}$. *Then either* $\#$R $= 1$ *or* X *is completely regular.*

Proof: Take $a, b \in$ R with $a < b$. Then for arbitrary $x \in$ X and $V \in \mathcal{T}_{\mathrm{X}}(x)$ we can define an $f \in$ LSC (X) by $f = b$ on $V$ and $f = a$ on X $\setminus V$. So there is an $h \in$ C(X) with $h \leq f$ and $h(x) \not\leq a$. Since R is completely regular, we can find a continuous function $g :$ R $\to [0,1]$ with $g(h(x)) = 1$ and $g = 0$ on $\{r : r \leq a\}$. For the continuous function $g \circ h$ we then have $g \circ h(x) = 1$ and $g \circ h = 0$ on X $\setminus V$. So X is completely regular. □

**Normal spaces.**

**Lemma 5.3.** *Suppose:*

- T *is a complete lattice;*
- $\wedge$ *preserves directed sups;*
- $\vee$ *preserves directed infs;*
- $F \subset$ T *is a sublattice;*
- $(f_n) \subset F$ *and* $f = \sup(f_n)$*;*
- $(g_n) \subset F$ *and* $g = \inf(g_n)$*;*
- $f \geq g$.

*Then there are* $(f^n) \subset F$ *and* $(g^n) \subset F$ *such that* $f \geq \sup(f^n) = \inf(g^n) \geq g$.

Proof: We may suppose that $(f_n)$ is an increasing sequence and that $(g_n)$ is a decreasing sequence. Define $f^n \in F$ and $g^n \in F$ by $g^1 = g_1$, $f^n = g^n \wedge f_n$ for $n \geq 1$ and $g^n = f^{n-1} \vee g_n$ for $n \geq 2$. Define $f' = \sup(f^n) \leq \sup(f_n) = f$ and $g' = \inf(g^n) \geq \inf(g_n) = g$. Evidently $g^n \geq f^n$. Since $g^n \geq f^{n-1}$ and $f_n \geq f_{n-1} \geq f^{n-1}$ we get $f^n \geq f^{n-1}$. Dually also $g^n \leq g^{n-1}$. Since $g^{n-1} \geq g^n \geq f^n \geq f^{n-1}$ we have $g' \geq f'$. So $f' \leq f \wedge g'$ and $g' \geq g \vee f'$.

Since $g^n = f^{n-1} \vee g_n \leq f' \vee g_n$ we have $g' = \inf(g^n) \leq \inf(f' \vee g_n) = f' \vee (\inf(g_n)) = f' \vee g$ and dually $f' \geq f \wedge g'$. So we have both $g' = g \vee f'$ and $f' = f \wedge g'$.

Now since $f \geq g$, we get that $g \leq f \wedge g' = f'$ and hence that $g \leq g' = g \vee f' = f' \leq f$. □

The following three results generalize [13, Theorem 2] and [10, Theorem 1]. Together with 5.1, the next result is very important in the theory of non-linear integration (see [8, Section 1]). As in [7], we define a linked bicontinuous lattice to be a uniform ordered space that is a continuous lattice in both the normal and the dual order.

**Theorem 5.4.** *Suppose* X *is normal and* R *is a metrizable, arcwise connected, linked bicontinuous lattice. Then for every* $f \in$ LSC (X), $g \in$ USC (X) *with* $f \geq g$, *there is a* $k \in$ C(X) *such that* $f \geq k \geq g$.

Proof: Take $A \in \mathcal{U}$. Then for all $a \in$ R there is a $b_a \in$ R with $b_a \ll a$ and $b_a \in A(a)$. Hence for all $a \in$ R, there is a $B_a \in \mathcal{U}^\bullet$ with $b_a \ll B_a(a)$.

Take $a \in$ R and define $F = \overline{\{x : f(x) \not\geq b_a\}}$, $G = \{x : g(x) \in B_a(a)\}$ and $U = \{x : b_a \ll f(x)\}$. Notice that $F \cap U = \emptyset$ and $G \subset U$. So $F$ and $G$ are disjoint closed sets and we can find an $h_a \in \mathrm{C(X)}$ such that $h_a = \inf \mathrm{R}$ on $F$, $h_a = b_a$ on $G$ and $h_a \leq b_a$ on X. Notice that $h_a \leq f$.

Since R is compact and $\mathrm{R} = \bigcup_{a\in \mathrm{R}} B_a(a) \cap A^{-1}(a)$, there is a finite set $I \subset \mathrm{R}$ such that $\mathrm{R} = \bigcup_{a\in I} B_a(a) \cap A^{-1}(a)$. Now for all $x \in \mathrm{X}$, there is an $a \in I$ with $g(x) \in B_a(a) \cap A^{-1}(a)$ and hence with $h_a(x) = b_a \in A(a) \subset A^2(g(x))$. So if we define $h = \sup\{h_a : a \in I\}$ then $h \in \mathrm{C(X)}$, $h \leq f$ and $h(x) \in A^2(g(x))$ for all $x \in \mathrm{X}$.

Now let $(A_n)$ be a countable base of $\mathcal{U}$, using the construction above, and its dual, we can find sequences $(f_n)$ and $(g_n)$ of continuous functions such that for all $n$ we have $f_n \leq f$, $(g, f_n) \in A_n^2$, $g_n \geq g$ and $(f, g_n) \in A_n^2$. So for $f' = \sup f_n$ and $g' = \inf g_n$ we have $f \geq f' \geq g' \geq g$. Using 5.3, we can now find a $k \in \mathrm{C(X)}$ with $f' \geq k \geq g'$. □

**Lemma 5.5.** *Suppose for every normal* X *and all* $f \in \mathrm{LSC\ (X)}$, $g \in \mathrm{USC\ (X)}$ *with* $f \geq g$, *there is a* $k \in \mathrm{C(X)}$ *such that* $f \geq k \geq g$. *Then* R *is arcwise connected.*

Proof: Take $\mathrm{X} = [0, 1]$ and $a \in \mathrm{R}$. Now define an $f \in \mathrm{LSC\ (X)}$ by $f(0) = \inf \mathrm{R}$ and $f(x) = a$ if $x \neq 0$ and define a $g \in \mathrm{USC\ (X)}$ by $g(1) = a$ and $g(x) = \inf \mathrm{R}$ if $x \neq 1$. Then there is a $k \in \mathrm{C(X)}$ with $f \geq k \geq g$, hence $\inf \mathrm{R}$ and $a$ are connected by a continuous path. □

**Lemma 5.6.** *Suppose for every* $f \in \mathrm{LSC\ (X)}$, $g \in \mathrm{USC\ (X)}$ *with* $f \geq g$, *there is a* $k \in \mathrm{C(X)}$ *such that* $f \geq k \geq g$. *Then either* $\#\mathrm{R} = 1$ *or* X *is normal.*

Proof: Take $a, b \in \mathrm{R}$ with $a < b$. Then for arbitrary $F, G \subset \mathrm{X}$ closed and disjunct we can define an $f \in \mathrm{LSC\ (X)}$ by $f = a$ on $F$ and $f = b$ on $\mathrm{X} \setminus F$ and a $g \in \mathrm{USC\ (X)}$ by $g = b$ on $G$ and $g = a$ on $\mathrm{X} \setminus G$. So there is a $k \in \mathrm{C(X)}$ with $f \geq k \geq g$. Since R is normal, we can find a continuous function $g : \mathrm{R} \to [0, 1]$ with $g = 0$ on $\{r : r \geq b\}$ and $g = 1$ on $\{r : r \leq a\}$. For the continuous function $g \circ h$ we then have $g \circ h = 1$ on $F$ and $g \circ h = 0$ on $G$. So X is normal. □

**Compact spaces.**

The next result is a generalization of Dini's theorem. From this result it immediately follows that an upper directed set of continuous functions on a compact set, pointwise convergent to a continuous function, is uniformly convergent.

**Proposition 5.7.** *Let* X *be compact and* $F \subset \mathrm{LSC\ (X)}$ *upper directed. Then:*

1. *if* $g \in \mathrm{USC\ (X)}$ *and* $A \in \mathcal{U}^\circ$ *such that for all* $x \in \mathrm{X} : (g(x), \sup F(x)) \in A$, *then there is an* $f \in F$ *such that for all* $x \in \mathrm{X} : (g(x), f(x)) \in A$;
2. *if* $g \in \mathrm{USC\ (X)}$ *with* $g \lll \sup F$ *pointwise, then there is an* $f \in F$ *with* $g \lll f$ *pointwise;*
3. *if* $r \in \mathrm{R}$ *with* $r \ll \sup F$ *pointwise, then there is an* $f \in F$ *with* $r \ll f$ *pointwise.*

Proof: (1) Define $K_f = \{x \in \mathrm{X} : (g(x), f(x)) \notin A\}$.

(2) Define $K_f = \{x \in \mathrm{X} : g(x) \not\lll f(x)\}$.

(3) Define $K_f = \{x \in \mathrm{X} : r \not\ll f(x)\}$.

Then $K_f$ is closed for all $f \in F$, $\{K_f : f \in F\}$ is lower directed and $\bigcap_{f\in F} K_f = \emptyset$. Since X is compact there must be an $f \in F$ with $K_f = \emptyset$. This $f$ will do. □

The next result is a generalization of the theorem that on a compact space, every lower finite and lower semicontinuous function $f$ is lower bounded. It is well known that, if R is not totally ordered, in general there is no $x \in \mathrm{X}$ such that $f(x) = \inf f(\mathrm{X})$.

**Proposition 5.8.** *Suppose* X *is compact,* $a \in \mathrm{R}$ *and* $f \in \mathrm{LSC}\ (\mathrm{X})$. *Then the following statements are equivalent:*

1. $a \lll f$ *pointwise;*
2. $a \lll f$ *uniform;*
3. *There is a* $b \gg a$ *with* $b \ll f$ *uniform.*

*Furthermore, they imply* $\inf f(\mathrm{X}) \gg a$.

Proof: ($1 \Rightarrow 2$) Take $x \in \mathrm{X}$ and $A_x \in \mathcal{U}$ with $A_x^2(f(x)) \geq A_x^{-2}(a)$. Then there is a $V_x \in \mathcal{T}_\mathrm{X}(x)$ such that $f(V_x) \subset A_x(f(x))$. Since X is compact, only a finite set of the $V_x$ already covers X. Now choose $A \in \mathcal{U}$ with $A \subset A_x$ for that finite set of $x$. Then $A(f(x)) \geq A^{-1}(a)$ for all $x \in \mathrm{X}$.

($2 \Rightarrow 3$) Take $A \in \mathcal{U}$ with $A(f(x)) \geq A^{-1}(a)$ for all $x \in \mathrm{X}$ and set $b = \sup A^{-1}(a)$.

The other implications are evident. □

**Lemma 5.9.** *Suppose* X *is compact,* $g \in \mathrm{LSC}\ (\mathrm{X})$ *and* $\mathcal{F}$ *is a filter base of closed sets. Then* $\liminf(g(\mathcal{F})) = \sup\{\inf g(K) : K \in \mathcal{F}\} = \inf g(\cap\mathcal{F})$.

Proof: Note first that $\cap\mathcal{F} \neq \emptyset$. For all $K \in \mathcal{F}$ we have $\cap\mathcal{F} \subset K$ and hence $\inf(g(\cap\mathcal{F})) \geq \inf(g(K))$, so we also have $\inf g(\cap\mathcal{F}) \geq \sup\{\inf g(K) : K \in \mathcal{F}\}$.

Now take $A \in \mathcal{U}^\circ$ and for all $x \in \cap\mathcal{F}$ an open $V_x \in \mathcal{T}_\mathrm{X}(x)$ such that $g(V_x) \subset A(g(x)) \subset A(\inf g(\cap\mathcal{F}))$ and define $V = \cup_x V_x$. Then $V$ is an open neighbourhood of $\cap\mathcal{F}$ and hence there is a $K \in \mathcal{F}$ such that $K \subset V$. For that $K$ we have $g(K) \subset g(V) \subset A(\inf g(\cap\mathcal{F}))$ and hence $\inf g(\cap\mathcal{F}) \in \mathrm{LIMINF}\ (g(\mathcal{F}))$. But then $\inf g(\cap\mathcal{F}) \leq \liminf(f(\mathcal{F})) = \sup\{\inf g(K) : K \in \mathcal{F}\}$. □

**Locally compact spaces.**

The next result is a generalization of the lemma of [4, Proposition 1.1.2]. It is used there to show that it is equivalent to define the axiom of convergence of a harmonic space with increasing sequences or with upper directed sets. It is surprising that this is true when neither X nor R is metrizable.

**Theorem 5.10.** *Suppose* X *is locally compact and Hausdorff. Then for any lower directed* $F \subset \mathrm{LSC}\ (\mathrm{X})$ *we have that* $\inf F$ *is lower semicontinuous if the infimum of any decreasing sequence in* $F$ *is lower semicontinuous.*

Proof: Define $f = \inf F$. Choose $x \in \mathrm{X}$ and $A \in \mathcal{U}^\circ$. Now we shall construct by induction a decreasing sequence $(f_n)$ in $F$ and a decreasing sequence $(K_n)$ of compact neighbourhoods of $x$ such that:

$$\inf f_n(K_n) \in A^{-1}(\liminf(f(\mathcal{T}_\mathrm{X}(x)))) \quad \text{and} \quad \inf f_{n-1}(K_n) \in A(f(x)).$$

Start with an arbitrary compact neighbourhood $K_0$ of $x$. Then we have $\inf f(K_0) \leq \liminf(f(\mathcal{T}_\mathrm{X}(x)))$ and hence there is an $f_0 \in F$ with $\inf f_0(K_0) \in$

$A^{-1}(\liminf(f(\mathcal{T}_{\mathrm{X}}(x))))$. Now suppose the sequences have been constructed up to $n-1$. Choose $B \in \mathcal{U}$ such that $\inf B(f_{n-1}(x)) \in A(f_{n-1}(x)) \subset A(f(x))$. Since $f_{n-1}$ is lower semicontinuous, there is a closed neighbourhood $K_n \subset K_{n-1}$ of $x$ such that $f_{n-1}(K_n) \subset B(f_{n-1}(x))$ and hence with $\inf f_{n-1}(K_n) \in A(f(x))$. Again $\inf f(K_n) \leq \liminf(f(\mathcal{T}_{\mathrm{X}}(x)))$ and hence there is an $f_n \in F$ with $f_n \leq f_{n-1}$ such that $\inf f_n(K_n) \in A^{-1}(\liminf(f(\mathcal{T}_{\mathrm{X}}(x))))$.

Now let $g$ be the infimum of $(f_n)$ and put $K = \cap_n K_n$. Then, applying 5.9 on $\mathcal{F} = (K_n)_n$, we get $\inf(g(K)) = \sup\{\inf g(K_n) : n\}$. Now we have for all $n$ that $\inf g(K_n) \in A^{-1}(\liminf(f(\mathcal{T}_{\mathrm{X}}(x))))$ and hence that $\inf g(K) \in$ $A^{-2}(\liminf(f(\mathcal{T}_{\mathrm{X}}(x))))$. Furthermore, for all $n$ we have $\inf f_n(K) \in A(f(x))$ and hence that $\inf g(K) \in A^2(f(x))$. Combining these two results we get $(f(x), \liminf(f(\mathcal{T}_{\mathrm{X}}(x)))) \in A^4$. Since $A$ and $x$ were arbitrary we get $f \in \mathrm{LSC}\ (\mathrm{X})$. □

**Proposition 5.11.** *If* X *is locally compact and Hausdorff, then* LSC (X) (*with the usual order*) *is a continuous lattice.*

Proof: From 4.4 we have that LSC (X) is a complete lattice. Take $f \in \mathrm{LSC}\ (\mathrm{X})$, $x \in \mathrm{X}$ and $r \ll f(x)$. Then there is a compact neighbourhood $K$ of $x$ such that $r \ll f$ pointwise on $K$. Define $g = r$ on the interior of $K$ and $g = \inf \mathrm{R}$ elsewhere. Then evidently $g \leq f$ and $g \in \mathrm{LSC}\ (\mathrm{X})$. Now let $F \subset \mathrm{LSC}\ (\mathrm{X})$ be upper directed with $\sup F \geq f$. Then $r \ll \sup F$ pointwise on $K$ and hence there is an $h \in F$ with $r \ll h$ pointwise on $K$. But then $g \leq h$ on X and hence $g$ is way below $f$ in the lattice LSC (X). Since evidently $f$ is the supremum of such $g$ we get that LSC (X) is a continuous lattice. □

Now let $\mathcal{A} \subset \mathrm{USC}$ be a presheaf on a locally compact Hausdorff space X. The fine topology is defined to be the coarsest topology on X, finer than the original topology, making all $f \in \mathcal{A}$ continuous.

**Proposition 5.12.** *Every point $x \in \mathrm{X}$ has a fine neighbourhood base of compact sets.*

Proof: Let $W$ be a fine neighbourhood of $x$. By Theorem 4.4, it is sufficient to consider sets of the form $W = \{y \in U : r \ll f(y)\}$, where $U$ is an open neighbourhood of $x$, $f \in \mathcal{A}(U)$ and $r \in \mathrm{R}$. Now there is an $A \in \mathcal{U}$ with $r \leq A(f(x))$ and a $B \in \mathcal{U}^\bullet$ with $B \circ B \subset A$. Let $V$ be a relatively compact neighbourhood of $x$ with $\overline{V} \subset U$ and define $K = \{y \in \overline{V} : f(y) \in B(f(x))\}$. Then $K$ is closed and hence compact. Furthermore, if $y \in K$, then $B(f(y)) \subset A(f(x))$ and hence $r \ll f(y)$. So $K \subset W$. Now take $s \in \mathrm{R}$ with $s \ll f(x)$ and $s \in B(f(x))$. Then $x \in \{y \in V : r \ll f(y)\} \subset K$ and hence, using Theorem 4.4, $K$ is a fine neighbourhood of $x$. □

**Theorem 5.13.** X *equipped with the fine topology is a Baire space.*

Proof: Let $(G_n)$ be a sequence fine open, fine dense subsets of X and $G$ fine open in X. We construct inductively a sequence $(K_n)$ of non-empty compact sets such that $K_n \subset G \cap G_n$ and $K_{n+1}$ is contained in the fine interior of $K_n$ for all $n$. Since $\emptyset \neq \cap K_n \subset G \cap (\cap G_n)$ we have that $\cap G_n$ is fine dense. □

**Spaces with a countable base.**

The next result is a generalization of [1, Lemma I-1.7].

**Proposition 5.14.** *Let* X *be a topological space with a countable base and let* R *be metrizable. Then for every family* $F \subset \mathrm{LSC}\,(\mathrm{X})$, *there is a countable family* $F_0 \subset F$ *with* $\sup F_0 = \sup F$.

PROOF: From [7, Corollary III-4.10] it follows there is a countable subset $Q$ of R such that for all $a \in \mathrm{R}$ we have $a = \sup\{b \in Q : b \ll a\}$. Let $G = \{f_1 \vee \cdots \vee f_n : f_i \in F\}$. Now for every $s \in Q$ there is a countable family $G_s \subset G$ such that $\bigcup_{f \in G_s}\{x : s(x) \ll f(x)\} = \bigcup_{f \in G}\{x : s(x) \ll f(x)\}$. Let $F_s$ be the countable family $F_s \subset F$ such that $G_s = \{f_1 \vee \cdots \vee f_n : f_i \in F_s\}$. Let $F_0 = \bigcup_{s \in Q} F_s$. Then $F_0$ is countable. Now take $x \in \mathrm{X}$ and $a \ll \sup F(x)$. Then there is an $s \in Q$ with $a \leq s \ll \sup F(x)$. But then there is an $f \in G$ such that $s \ll f(x)$ and hence there is also an $f \in G_s$ with $s \ll f(x)$. So now we have $a \leq s \ll \sup G_s(x) = \sup F_s(x) \leq \sup F_0(x)$. □

The next result is a generalization [1, Lemma I-1.8].

**Proposition 5.15.** *Let* X *be a topological space with a countable base and let* R *be metrizable, linked bicontinuous lattice. Then for every family* $F \subset \mathrm{NUM}\,(\mathrm{X})$, *there is a countable family* $F_0 \subset F$ *with* $\widehat{\inf F_0} = \widehat{\inf F}$.

PROOF: Let $V \subset \mathrm{X}$ be open. By the dual of 5.14 on a one-point-space, there is a countable family $F_V \subset F$ such that $\inf\{\inf f(V) : f \in F\} = \inf\{\inf f(V) : f \in F_V\}$. Let $F_0$ be the union of $F_V$ for all $V$ in the countable base of X. Take $x \in \mathrm{X}$ and $a \ll \widehat{\inf F_0}(x)$. Then there is a $V$ in the countable base such that $a \leq \inf F_0(V) = \inf\{\inf f(V) : f \in F_V\} = \inf\{\inf f(V) : f \in F\} = \inf F(V)$. Hence $a \leq \widehat{\inf F}(x)$. □

**Isotone maps.**

**Theorem 5.16.** *Suppose* X *is a topological space and* T *a uniform ordered space. Now let* $f$ *be a map from* X *to* R *and* $g$ *an isotone map from* R *to* T. *If* $f$ *is lower semicontinuous in* $x \in \mathrm{X}$ *and* $g$ *is lower semicontinuous in* $f(x)$, *then* $g \circ f$ *is lower semicontinuous in* $x$.

PROOF: Let $\mathcal{V}$ be the semiuniformity on T and take $A \in \mathcal{V}^\circ$ and $x \in \mathrm{X}$. Since $g$ is isotone and lower semicontinuous in $f(x)$, there is an increasing neighbourhood $U$ of $f(x)$ with $g(U) \subset A(g(f(x)))$. Since R is a continuous lattice, there is a $B \in \mathcal{U}$ with $B(f(x)) \subset U$ and hence with $g(B(f(x))) \subset A(g(f(x)))$. Now since $f$ is lower semicontinuous in $x$, there is an $f(V) \in \mathcal{T}_{\mathrm{X}}(x)$ with $f(V) \subset B(f(x))$ and hence with $g(f(V)) \subset A(g(f(x)))$. □

For a map $f$ between two ordered sets, the property that $f$ preserves directed sups is defined purely in terms of order. No topology has to be present. Still, the property resembles something like "one-sided continuity". Therefore, it is not surprising that there is a strong connection with the "one-sided continuity" that is the main subject of this text.

**Lemma 5.17.** *Suppose:*

- $\mathrm{R}_1$ *is a uniform ordered space with semiuniformity* $\mathcal{U}_1$ *such that for all* $f \in \mathrm{R}_1$ *we have* $f = \sup\{g \in \mathrm{R}_1 : g \ll f\}$ *and such that every upper directed subset of* $\mathrm{R}_1$ *with supremum is convergent (as a net);*
- $\mathrm{R}_2$ *is a uniform ordered space with semiuniformity* $\mathcal{U}_2$ *such that every upper directed subset of* $\mathrm{R}_2$ *with supremum is convergent (as a net);*
- $f$ *is an isotone map from* $\mathrm{R}_1$ *to* $\mathrm{R}_2$.

*Then the following statements are equivalent:*

1. $f$ *preserves directed sups;*
2. $f$ *is lower semicontinuous.*

Proof: $(1 \Rightarrow 2)$ Take $r \in \mathrm{R}_1$. Then $\{s \in \mathrm{R}_1 : g \ll r\}$ is upper directed with supremum $r$ and hence $f(\{s \in \mathrm{R}_1 : g \ll r\})$ is upper directed with supremum $f(r)$. Hence for all $A \in \mathcal{U}_2{}^\circ$ there is an $s \ll r$ such that $f(s) \in A(f(r))$. On the other hand, $s \ll r$ means that there is a $B \in \mathcal{U}_1$ such that $s \leq B(r)$. Now we have that $f(B(r)) \subset A(f(r))$.

$(2 \Rightarrow 1)$ Let $F \subset \mathrm{R}_1$ be upper directed with $\sup F \in \mathrm{R}_1$ and take $A \in \mathcal{U}_2{}^\circ$ and an upper bound $b$ of $f(F)$. Now there is a $B \in \mathcal{U}_1$ such that $f(B(\sup F)) \subset A(f(\sup F))$ and there is an $a \in F$ with $a \in B(\sup F)$. So $f(a) \in A(f(\sup F))$ and hence $b \in A(f(\sup F))$. Since $A$ was arbitrary we get $b \geq f(\sup F)$ and since $b$ was arbitrary this implies $\sup f(F)$ exists and $\sup f(F) = f(\sup F)$. □

**Proposition 5.18.** *Let* $\mathrm{R}$ *and* $\mathrm{T}$ *be two continuous lattices and* $f$ *an isotone map from* $\mathrm{R}$ *to* $\mathrm{T}$. *Then* $f$ *preserves directed sups iff* $f$ *is lower semicontinuous.*

Proof: Use 5.17. □

**Proposition 5.19.** *Suppose* $\mathrm{R}$ *is arcwise connected,* $\mathrm{X}$ *and* $\mathrm{Y}$ *are compact and* $f$ *is an isotone map from* $\mathrm{C(X)}$ *to* $\mathrm{C(Y)}$. *Then the following statements are equivalent:*

1. $f$ *preserves directed sups;*
2. $f$ *is lower semicontinuous with respect to the topology of uniform convergence on* $\mathrm{C(X)}$ *and the topology of pointwise convergence on* $\mathrm{C(Y)}$*;*
3. $f$ *is lower semicontinuous with respect to the topology of uniform convergence on* $\mathrm{C(X)}$ *and the topology of uniform convergence on* $\mathrm{C(Y)}$.

Proof: Use 5.17, 5.1 and Dini's theorem. □

## References


[1] Bliedtner J., Hansen W., *Potential Theory. An Analytic and Probabilistic Approach to Balayage*, Springer-Verlag, Berlin, 1986.

[2] Borwein J.M., Théra M., *Sandwich Theorems for Semicontinuous Operators*, preprint, 1990.

[3] Bourbaki N., *Topologie Générale, ch. IX*, Hermann & Cie, Paris, 1948.

[4] Constantinescu C., Cornea A., *Potential Theory on Harmonic Spaces*, Springer-Verlag, Berlin, 1972.

[5] Fletcher P., Lindgren W.F., *Quasi-uniform spaces*, Lecture Notes in Pure and Applied Mathematics 77, Marcel Dekker inc., New York, 1982.

[6] Gerritse G., *Lattice-valued Semicontinuous Functions*, Report 8532, Catholic University of Nijmegen, 1985.

[7] Gierz G., Hofmann K., Keimel K., Lawson J., Mislove M., Scott D., *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin, 1980.

[8] van Gool F.A., *Non-linear Potential Theory*, Preprint 606, University of Utrecht, 1990.

[9] Holwerda H., *Closed Hypographs, Semicontinuity and the Topological Closed-graph Theorem: A unifying Approach*, Report 8935, Catholic University of Nijmegen, 1989.

[10] Katětov M., *On real-valued functions in topological spaces*, Fundamenta Mathematicae **38** (1951), 85–91; Correction in Fund. Math. **40** (1953), 203–205.

[11] Nachbin L., *Topology and Order*, Van Nostrand, Princeton, 1965.

[12] Penot J.P., Théra M., *Semi-continuous mappings in general topology*, Arch. Math. **38** (1982), 158–166.

[13] Tong H., *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. **19** (1952), 289–292.

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