On a problem of Gulevich on nonexpansive maps in uniformly convex Banach spaces

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Abstract. Let $X$ be a uniformly convex Banach space, $D \subset X$, $f : D \to X$ a nonexpansive map, and $K$ a closed bounded subset such that $\overline{co}K \subset D$. If (1) $f|_K$ is weakly inward and $K$ is star-shaped or (2) $f|_K$ satisfies the Leray-Schauder boundary condition, then $f$ has a fixed point in $\overline{co}K$. This is closely related to a problem of Gulevich [Gu]. Some of our main results are generalizations of theorems due to Kirk and Ray [KR] and others.

Keywords: uniformly convex, Banach space, Hilbert space, contraction, nonexpansive map, weakly inward map, demi-closed, Rothe condition, Leray-Schauder condition, (KR)-bounded, Opial’s condition

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The well-known theorem of Browder-Göhde-Kirk assures existence of a fixed point for nonexpansive maps $f : K \to K$ where $K$ is a bounded closed convex subset of a uniformly convex Banach space $X$. In [KR], Kirk and Ray showed that $f$ can be replaced by a weakly inward nonexpansive map $f : K \to X$ while the boundedness of $K$ can be replaced by that of the geometric estimator

$$G(x, fx) = \{z \in K : \|z - x\| \geq \|z - fx\|\}$$

for some $x \in K$ or more general sets. Note that any fixed point of $f$ is contained in $G(x, fx)$.

On the other hand, Gulevich [Gu] considered the situation as follows: $H$ is a Hilbert space, $K$ is a nonempty bounded closed (not necessarily convex) subset of $H$, and $f : D \subset H \to H$ is a nonexpansive map, where $\overline{co}K \subset D$. Gulevich’s basic theorem [Gu, Theorem 1] states that $f$ has a fixed point in $\overline{co}K$ if $f$ satisfies the Rothe condition $f(\text{Bd} K) \subset K$. He also raised as a problem whether $H$ can be replaced by a uniformly convex Banach space.

In the present paper, we obtain some fixed point theorems on nonexpansive maps defined on closed (not necessarily bounded or convex) subsets of a Banach space. Our results are closely related to Gulevich’s theorem and extend some known results of Kirk and Ray [KR], Goebel and Kuczumow [Go], and Browder [B1], [B2]. Moreover, we adopt more general boundary conditions on those nonexpansive maps. In fact, the weakly inwardness or the so-called Leray-Schauder condition is used in our results instead of the Rothe condition used in [Gu].
Recall that \( f : K \to X \) is a \textit{contraction} if there exists a \( k \in [0, 1) \) such that
\[
\|fx - fy\| \leq k\|x - y\| \quad \text{for all } x, y \in K;
\]
and a \textit{nonexpansive map} if
\[
\|fx - fy\| \leq \|x - y\| \quad \text{for all } x, y \in K.
\]
We say that \( f \) is \textit{weakly inward} if \( fx \in \text{Int } K(x) \) for any \( x \in \text{Bd } K \) (equivalently, for any \( x \in K \)), where \( \text{Bd} \), \( \text{Int} \) denote the closure, boundary, and interior, respectively, and
\[
I_K(x) = \{ x + c(y - x) : y \in K, \ c \geq 1 \}.
\]
Note that any map satisfying the Rothe condition is weakly inward.

We begin with the following:

**Theorem 0.** Let \( K \) be a closed subset of a Banach space \( X \) and \( f : K \to X \) a contraction satisfying one of the following:

(i) \( f(\text{Bd } K) \subset K \).

(ii) \( f \) is weakly inward.

(iii) \( 0 \in \text{Int } K \) and \( f x \neq mx \) for all \( x \in \text{Bd } K \) and \( m > 1 \).

Then \( f \) has a unique fixed point.

Note that Theorem 0(i) is a particular case of Assad and Kirk [AK, Theorem 1], Theorem 0(ii) is due to Martinez-Yanez [M, Theorem] or, in a more general form, to Zhang [Z, Theorem 3.3], and Theorem 0(iii) to Gatica and Kirk [GK, Theorem 2.1]. There are more general results than Theorem 0. However, Theorem 0 is sufficient for our purpose. Note also that (iii) can be replaced by the following:

(iii) there exists a \( w \in \text{Int } K \) such that
\[
f x - w \neq m(x - w) \quad \text{for all } x \in \text{Bd } K \quad \text{and} \quad m > 1.
\]

Moreover, (i) \( \Rightarrow \) (ii) and, whenever \( K \) is convex and \( 0 \in \text{Int } K \), we have (ii) \( \Rightarrow \) (iii).

A subset \( K \) of a vector space is said to be \textit{star-shaped} if there exists a given point \( x_0 \in K \) such that \( tx_0 + (1 - t)x \in K \) for any \( t \in (0, 1) \) and \( x \in K \), where \( x_0 \) is called a \textit{center} of \( K \).

For \( K \) and \( f \) in Theorem 0, we say that \( K \) is \( \text{(KR)-bounded} \) or bounded in the sense of Kirk-Ray [KR] if, for some bounded set \( A \subset K \), the set
\[
G(A) = \bigcap_{u \in A} G(u, f u)
\]
is either empty or bounded.

The following is a generalization of the almost fixed point property of bounded closed subsets of a Banach space for nonexpansive maps.
Theorem 1. Let $X$ be a Banach space, $K$ a closed subset of $X$, and $f : K \to X$ a nonexpansive map such that $K$ is $(KR)$-bounded and one of the following holds:

(i) $K$ is star-shaped and $f(\text{Bd} \ K) \subseteq K$.

(ii) $K$ is star-shaped and $f$ is weakly inward.

(iii) $0 \in \text{Int} \ K$ and $fx \neq mx$ for all $x \in \text{Bd} \ K$ and $m > 1$.

Then there exists a bounded sequence $\{x_n\}$ in $K$ such that $\|x_n - fx_n\| \to 0$ as $n \to \infty$.

Proof: For cases (i) and (ii) we may without loss of generality assume that 0 is the center. For $\alpha \in (0, 1)$, define $f_\alpha : K \to X$ by $f_\alpha x = \alpha fx$ for $x \in K$. Then clearly $f_\alpha$ is a contraction. We show that one of (i)–(iii) in Theorem 0 holds for $f_\alpha$:

(i) Since $K$ is star-shaped at center 0, we have $\alpha K \subseteq K$. Since $f(\text{Bd} \ K) \subseteq K$, for $x \in \text{Bd} \ K$, we have $f_\alpha x = \alpha fx \in \alpha K \subseteq K$. Therefore, $f_\alpha(\text{Bd} \ K) \subseteq K$.

(ii) From $fx \in \text{I}_K(x)$, we have $f_\alpha x = \alpha fx \in \alpha \text{I}_K(x) \subseteq \text{I}_K(x)$ since $\text{I}_K(x)$ is a star-shaped set with center 0. See Zhang [Z, Theorem 1.2]. Note that (i) $\Rightarrow$ (ii).

(iii) Suppose that $f_\alpha x = mx$ for some $x \in \text{Bd} \ K$ and $m > 1$, then $fx = \alpha^{-1} f_\alpha x = (\alpha^{-1} m)x$ and $\alpha^{-1} m > 1$, which contradicts our assumption.

Therefore, by Theorem 0, $f_\alpha$ has a fixed point $x_\alpha \in K$. Suppose that the set $\{x_\alpha : \alpha \in (0, 1)\}$ is not bounded. Then it is possible to choose $\alpha \in (0, 1)$ so that

$$\sup_{u \in A} \|fu\| \leq \inf_{u \in A} \|x_\alpha - u\|$$

and in addition, if $G(A) \neq \emptyset$, then $\alpha$ may also be chosen so that

$$\|x_\alpha\| > \sup\{\|x\| : x \in G(A)\}.$$ 

Therefore, for each $u \in A$,

$$\|x_\alpha - fu\| = \|\alpha fx_\alpha - fu\| \leq \alpha \|fx_\alpha - fu\| + (1 - \alpha)\|fu\| \leq \alpha \|x_\alpha - u\| + (1 - \alpha)\|x_\alpha - u\| = \|x_\alpha - u\|.$$ 

This implies $x_\alpha \in G(A)$, which is a contradiction. Thus $M = \sup\{\|x_\alpha\| : \alpha \in (0, 1)\} < \infty$ and we have

$$\|x_\alpha - fx_\alpha\| = (\alpha^{-1} - 1)\|x_\alpha\| \leq (\alpha^{-1} - 1)M,$$

yielding $\|x_\alpha - fx_\alpha\| \to 0$ as $\alpha \to 1$. This completes our proof. □

Note that Kirk and Ray [KR, Theorem 2.3] obtained Theorem 1(ii) for the case $K$ is convex. In the second half of the proof of Theorem 1, we followed that of [KR, Theorem 2.3]. Note that Theorem 1(i) generalizes Dotson [D, Theorem 1].
A Banach space $X$ is said to satisfy Opial’s condition if, whenever a sequence $\{x_n\}$ converges weakly to $x_0 \in X$, then
\[
\liminf_{n \to \infty} \|x_n - x\| > \liminf_{n \to \infty} \|x_n - x_0\|
\]
for all $x \in X$, $x \neq x_0$. Opial [O] showed that if $C$ is a weakly compact subset of a Banach space $X$ satisfying this condition and $f : C \to X$ is nonexpansive, then $I - f$ is demi-closed ([B2], [Gö]); that is, if $\{x_n\} \subset C$ satisfies $x_n \to x$ weakly while $(I - f)x_n \to y$ strongly, then $(I - f)x = y$, where $I$ is the identity map on $C$.

Examples of spaces satisfying Opial’s condition are Hilbert spaces, $l^p$ ($1 \leq p < \infty$), and uniformly convex Banach spaces with weakly continuous duality maps.

From Theorem 1, we have the following:

**Theorem 2.** Let $X$ be a Banach space, $K$ a weakly compact subset of $X$, and $f : K \to X$ a nonexpansive map satisfying one of (i)–(iii) in Theorem 1.

(a) If $I - f$ is demi-closed on $K$, then $f$ has a fixed point.

(b) If $X$ satisfies Opial’s condition, then $f$ has a fixed point.

**Proof:** Since $K$ is closed and bounded, $f$ satisfies all the requirements of Theorem 1. Hence, there exists a sequence $\{x_n\}$ in $K$ such that $\|x_n - fx_n\| \to 0$ as $n \to \infty$. Since $K$ is weakly compact, we may assume that $x_n \to x$ weakly to some $x \in K$. Since $\|x_n - fx_n\| \to 0$ strongly, $x_n - fx_n = (I - f)x_n$, and $I - f$ is demi-closed, we conclude that $(I - f)x = 0$ and hence $x = fx$. This completes our proof.

Note that Zhang [Z, Theorem 3.8 and Corollaries 3.10, 3.11] obtained the multi-valued version of Theorem 2(ii), with different proof, and that Theorem 2(i) generalizes Dotson [D, Theorem 2]. Note also that if $K$ is compact, then $f$ has a fixed point in Theorem 2 without assuming the demi-closedness of $I - f$.

From Theorem 1, we also have the following:

**Theorem 3.** Let $X$ be a uniformly convex Banach space, $D$ a subset of $X$, and $f : D \to X$ a nonexpansive map. Let $K$ be a closed (KR)-bounded subset of $X$ such that $\overline{co}K \subset D$ and one of (i)–(iii) of Theorem 1 holds for $f|_K$. Then $f$ has a fixed point in $\overline{co}K$.

**Proof:** Since $f|_K$ satisfies all the requirements of Theorem 1, there exists a bounded sequence $\{x_n\}$ in $K$ such that $\|x_n - fx_n\| \to 0$. Since $\{x_n\}$ is contained in a bounded closed convex subset $L \subset \overline{co}K$ and $L$ is weakly compact, we may assume $x_n \to x_0$ weakly to some $x_0 \in L$. Since $I - f$ is demi-closed on $L$ ([B2], [Gö]) and $(I - f)x_n \to 0$ strongly, we conclude that $(I - f)x = 0$, and hence $x = fx$. This completes our proof.

If we can eliminate the star-shapedness in (i), then Theorem 3(i) will be the required affirmative answer to Gulevich’s problem. Moreover, Gulevich [Gu] noted that, for case (i) of Theorem 3 in a Hilbert space $H$, $f$ has a fixed point in $K$. 
In case $0 \in \text{Int } K$, Theorem 3(iii) generalizes [Gu, Theorem 1].

Note that the set $A \subset K$ for the (KR)-boundedness can be chosen so that $A \subset D$ and $f(A) \subset K$. See the proof of Theorem 1. Therefore, Theorem 3 generalizes Ray [R, Lemma 1].

For $D = K = \overline{\text{co}} K$, Theorem 3 reduces to the following:

**Theorem 4.** Let $X$ be a uniformly convex Banach space, $K$ a closed convex subset, and $f : K \to X$ a nonexpansive map such that $K$ is (KR)-bounded and one of (i)–(iii) in Theorem 0 holds. Then $f$ has a fixed point.

Note that Theorem 4(i) and (ii) are due to Kirk and Ray [KR, Theorem 2.3], which extends Goebel and Kuczumow [Go, Theorem 6]. Also note that Theorem 4(iii) extends Browder [B2, Theorem 1] for nonexpansive maps. For a Hilbert space $X$ and a closed ball in $X$, Theorem 4(i) is due to Browder [B1, Theorem 1], which was used to show existence of periodic solutions for nonlinear equations of evolution.

Recently Canetti, Marino, and Pietramala [CMP] obtained multi-valued versions of Theorem 4(ii) and, under the stronger assumption of convexity, some other results similar to Theorems 1–3 for case (ii).

Finally, we note that the so-called Rothe condition (i) was first adopted by Knaster, Kuratowski, and Mazurkiewicz [KKM]. Also, the origin of the so-called Leray-Schauder condition (iii) seems to be Schaefer [S], and the following are well-known examples of that condition:

(A) $\|fx - x\|^2 \ge \|fx\|^2 - \|x\|^2$ for $x \in \text{Bd } K$.

(K) $\text{Re } \langle fx, x \rangle = \|x\|^2$ for $x \in \text{Bd } K$, $x \neq 0$, in a Hilbert space $H$.

Or more generally,

(P) $\langle fx, Jx \rangle \le \langle x, Jx \rangle$ for $x \in \text{Bd } K$, $0 \in \text{Int } K$, where $J$ is any duality map of $X$ into $2^{X^*}$.

Condition (A) is due to Altman [A], (K) to Krasnosel’skii [K] and Shinbrot [Sh], and (P) to Petryshyn [P].

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**References**


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