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Hopf bifurcation for the equation $\ddot{x}(t) + f(x(t))\dot{x}(t) + g(x(t-r)) = 0.$

Bifurcación de Hopf para la ecuación $\ddot{x}(t) + f(x(t))\dot{x}(t) + g(x(t-r)) = 0.$

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Abstract

In this paper, by using the Hopf's bifurcation theorem we will discuss the existence of small amplitude periodic solutions of the equation $\ddot{x}(t)+f(x(t))\dot{x}(t) + g(x(t-r)) = 0$, taking as bifurcation parameter c either d or r. We assume that $r > 0$, $f \in C^1$, $f(0) = c > 0$, $g(0) = 0$ and $\dot{g}(0) = d > 0$.
Key words and phrases: Hopf's bifurcation, delay equation.

Resumen

En este artículo estudiamos la existencia de soluciones periódicas de amplitud pequeña de la ecuación diferencial con retardo $\ddot{x}(t) + f(x(t))\dot{x}(t) + g(x(t-r)) = 0$, vía bifurcación de Hopf. Suponemos que g es una función de clase C^1, $f(0) = c > 0$, $g(0) = 0$ y $\dot{g}(0) = d > 0$.
Palabras y frases clave: bifurcación de Hopf, ecuación con retardo.

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1 Introduction

In the analysis of the existence of nonconstant periodic solutions of the equation

$$\ddot{x}(t) + f(x(t))\dot{x}(t) + g(x(t-r)) = 0, \tag{1.1}$$

it is necessary to have a detailed information about the behavior of roots of the characteristic equation for the linear part of equation (1.1); namely the equation

$$\lambda^2 + c\lambda + de^{-\lambda r} = 0. \tag{1.2}$$

Hereafter, we will assume that $r > 0$, f is continuous, g is continuous together with its first derivative, $f(0) = c > 0$, $g(0) = 0$ and $\dot{g}(0) = d > 0$.

The main goal of this paper is to give necessary and sufficient conditions for all roots of the equation (1.2) to have negative real parts. By using the Hopf's bifurcation theorem and the above mentioned result, we will discuss the existence of small amplitude periodic solutions of equation (1.1), taking as bifurcation parameter c either d or r.

Equation (1.1) has been studied by many authors under the assumption that $d = 1$, for details see for instance [2, pp. 348–355]. To the author's knowledge this equation has not been studied just requiring $d > 0$, which can not be transformed to an equivalent one with $d = 1$. Thus why, along this work we have to perform again the study of the location of roots of equation (1.2) and we can not use the known results in the literature about equation (1.2) for $d = 1$.

Finally, we point out that equation (1.1) arises in many applications, a special case is $f(x) = k(x^2 - 1)$, $k > 0$; which is the famous van der Pol equation with a retardation, see [2, p. 355].

2 Stability of the equation $\lambda^2 + c\lambda + de^{-\lambda r} = 0$

The main goal in this section is to discuss the location of roots of the transcendental equation (1.2). More precisely, we will obtain a necessary and sufficient condition in order that all roots of equation (1.2) lie to the left of the imaginary axis. We are not going to use Pontriaguin's techniques outlined in Hale-Lunel [2, Appendix A]. Instead of that we will give a direct proof, using some ideas contained in Baptistini-Táboas [3].

Let us denote by

$$z = \lambda r \quad , \quad \alpha = \frac{1}{dr^2} \quad , \quad \beta = \frac{c}{dr} \; . \tag{2.1}$$

In terms of α, β and z the equation (1.2) can be rewritten as follows

$$(\alpha z^2 + \beta z)e^z + 1 = 0 \quad . \tag{2.2}$$

Let us denote by $z = a + ib$, $a, b \in R$. A straightforward computation shows that equation (2.2) is equivalent to the system

$$e^a[(\alpha(a^2 - b^2) + \beta a)\cos b - (2\alpha a + \beta)b \sin b] + 1 = 0 \tag{2.3}$$

$$(\alpha(a^2 - b^2) + \beta a)\sin b + (2\alpha a + \beta)b \cos b = 0 \tag{2.4}$$

Proposition 1. *The system* (2.3) − (2.4) *is equivalent to the following system*

$$(2\alpha a + \beta)b = e^{-a}\sin b \tag{2.5}$$

$$\alpha(a^2 - b^2) + \beta a = -e^{-a}\cos b \tag{2.6}$$

Proof. Let us denote by

$$u_1 = (\cos b, \sin b)\ ,\ u_2 = (-\sin b, \cos b)\ ,\ v = ((2\alpha a + \beta)b, \alpha(a^2 - b^2) + \beta a).$$

Thus, (2.3) − (2.4) are equivalent to the following system

$$v \cdot u_2 = -e^{-a} \quad , \quad v \cdot u_1 = 0 \tag{2.7}$$

where "$\cdot$" denotes the inner product in $\mathbb{R}^2$. Taking into account that $u_1 \cdot u_2 = 0$ and $v \cdot u_1 = 0$, we obtain that $v = \delta u_2$, for some δ in $\mathbb{R}$. From (2.7) we get that $\delta = -e^{-a}$. So, $v = -e^{-a}u_2$ is equivalent to

$$((2\alpha a + \beta)b, \alpha(a^2 - b^2) + \beta a) = -e^{-a}(-\sin b, \cos b)$$

which in turn implies (2.5) − (2.6). □

The following result is inspired in Theorem 2.1 in [3].

Lemma 2. *Let $v(a, b)$ and $w(b)$ be vectors define by*

$$v(a, b) = e^a((2\alpha a + \beta)b, \alpha(a^2 - b^2) + \beta a) \quad , \quad w(b) = (\sin b, -\cos b) \quad .$$

Then, for a giving $a \geq 0$ and a nonnegative integer n, there exist unique numbers $b_n(a) \in (2n\pi, (2n+1)\pi)$ and $\lambda_n(a) > 0$ such that

$$v(a, b_n(a)) = \lambda_n(a)w(b_n(a)) \quad . \tag{2.8}$$

Moreover, $b_n(a)$ and $\lambda_n(a)$ depend continuously on a.

Proof. For each $a \geq 0$, equations $\gamma = e^a(2\alpha a + \beta)b \;\;,\;\; \eta = e^a(\alpha(a^2 - b^2) + \beta a)$, with $b \in R$, describe a parabola in the $(\gamma, \eta)-$plane. Therefore, when $b \geq 0$ increases the vector $v(a, b)$ describes clockwise an unbounded arc of parabola, meanwhile $w(b)$ describes counterclockwise the unit circle.The way in which those curves are oriented implies that in each interval of the form $(2n\pi, (2n+1)\pi)$, $n = 0, 1, ...,$ there exists a unique number $b_n(a)$, that depends continuously on a, such that $v(a, b_n(a))$ is a positive multiple of $w(b_n(a))$. This proves (2.8). □

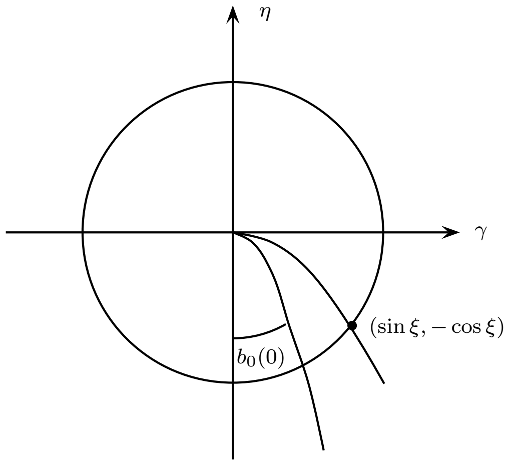

Figure 1:

Theorem 3. *All roots of the system (2.5)-(2.6) have negative real part, if and only if* $\beta > \frac{\sin\xi}{\xi}$*, where* ξ *is the only root on the interval* $(0, \frac{\pi}{2})$ *of the equation* $\alpha\xi^2 = \cos\xi$*.*

Proof. Let us assume first that all roots of the system (2.5)-(2.6) have negative real part, i.e. $a < 0$. However, $\beta \leq \frac{\sin\xi}{\xi}$, where ξ is the only root on the interval $(0, \frac{\pi}{2})$ of the equation $\alpha\xi^2 = \cos\xi$. If $\beta = \frac{\sin\xi}{\xi}$, then the pair $(a, b) = (0, \xi)$ is a solution of (2.5)-(2.6), and this contradicts the fact that $a < 0$. Now, let us suppose that $\beta < \frac{\sin\xi}{\xi}$. Applying Lemma 2 with $a = 0$, $n = 0$ see fig. 1, we obtain that $v(0, b_0(0)) = \lambda_0(0)w(b_0(0))$, with $0 < b_0(0) < \xi$. Moreover,

$$\lambda_0^2(0) = \beta^2 b_0^2(0) + \alpha^2 b_0^4(0) = b_0^2(0)(\beta^2 + \frac{\cos^2\xi}{\xi^4} b_0^2(0)) < \xi^2(\frac{\sin^2\xi}{\xi^2} + \frac{\cos^2\xi}{\xi^2}) = 1.$$

Thus, $\lambda_0(0) < 1$. On the other hand, by the continuity of $\lambda_0(a)$ and the fact that $\lim_{a\to\infty} \lambda_0(a) = \infty$, it follows that there exists a positive number a^*, such that $\lambda_0(a^*) = 1$, and the pair $(a^*, b_0(a^*))$ is a solution of (2.5)-(2.6), which is a contradiction. This completes the proof of the necessity.

Let us prove now the sufficiency. In order to accomplish our goal let us assume that $\beta > \frac{\sin\xi}{\xi}$, where ξ is the unique real number on the interval $(0, \frac{\pi}{2})$ such that $\alpha\xi^2 = \cos\xi$. Let us begin remarking that no matter constants α and β be, the pair $(a, 0)$, with $a \geq 0$, is not a solution of system (2.5)-(2.6). Let us suppose that there exists a pair (a, b) , with $a \geq 0$ and $b > 0$, which is a solution of the system (2.5)-(2.6). We will establish that under the hypothesis on β, it can not occur. The discussion is splitted in two cases $a = 0$, $b > 0$ and $a > 0$, $b > 0$.

If $a = 0$ and $b > 0$ then, from (2.5)-(2.6) we obtain that $\alpha = \frac{\cos b}{b^2}$ and $\beta = \frac{\sin b}{b}$. If $b \in (0, \frac{\pi}{2})$ then it must be equal to ξ which is the only root on the interval $(0, \frac{\pi}{2})$ of the equation $\alpha\xi^2 = \cos\xi$. Henceforth $\beta > \frac{\sin\xi}{\xi} = \frac{\sin b}{b} = \beta$, which is a contradiction. Let us assume that $b \geq \frac{\pi}{2}$. Since α and β are positive, we get that $\cos b > 0$ and $\sin b > 0$ and those inequalities imply that $b > 2\pi$. Now, $\frac{1}{b} < \frac{1}{2\pi}$ and then $\frac{\sin b}{b} < \frac{1}{2\pi} < \frac{2}{\pi}$. Combining this with the fact that $\frac{2}{\pi} < \frac{\sin\xi}{\xi}$, due to $\xi \in (0, \frac{\pi}{2})$ and on this interval the function $g(x) = \frac{\sin x}{x}$ is decreasing, we obtain that $\beta > \frac{\sin\xi}{\xi} > \frac{\sin b}{b} = \beta$, which is a contradiction as well.

Now, let us analize the case when $a > 0$ and $b > 0$. If $a > 0$, then from (2.5) we obtain that $\sin b > 0$ and $\beta < \frac{\sin b}{b}$. Therefore, from our assumption on β, we obtain

$$\frac{\sin\xi}{\xi} < \frac{\sin b}{b} \quad . \tag{2.9}$$

Let us show that there not exist a $b > 0$ such that (2.9) and the system (2.5)-(2.6) are satisfied simultaneously. Since $\sin b > 0$, we have that $b \in \bigcup_{n=0}^{\infty}(2n\pi, (2n+1)\pi)$. If $b \in (0, \frac{\pi}{2})$ and $b \geq \xi$, and having in mind that the function $g(x) = \frac{\sin x}{x}$ is decreasing on $(0, \frac{\pi}{2})$, we obtain that $\frac{\sin\xi}{\xi} \geq \frac{\sin b}{b}$ which contradicts (2.9). If $b \in (0, \frac{\pi}{2})$ and $b < \xi$, then, using that $\cos b > \cos\xi$, we obtain from (2.6) the estimation $\alpha(a^2 - b^2) + \beta a < -e^{-a}\cos\xi$. This estimation together with the assumption on β and the fact that $\alpha = \frac{\cos\xi}{\xi^2}$ imply $\frac{\cos\xi}{\xi^2}a^2 - \frac{\cos\xi}{\xi^2}b^2 + \frac{\sin\xi}{\xi}a < -e^{-a}\cos\xi$, and this implies, multiplying both sides by $\frac{1}{\cos\xi}$ and using that $\frac{b}{\xi} < 1$, that

$$\frac{1}{\xi^2}a^2 + \frac{\tan\xi}{\xi}a - 1 < -e^{-a} \quad . \tag{2.10}$$

Now, for any $\xi \in (0, \frac{\pi}{2})$ and $x > 0$ the graph of the function $g_1(x) = \frac{1}{\xi^2}x^2 + \frac{\tan\xi}{\xi}x - 1$ is above the graph of $g_2(x) = -e^{-x}$. Therefore, under the assumption that $a > 0$, the inequality (2.10) has no solution and this gives us a contradiction which comes from the fact that equation (2.6) is satisfied.

Let us discuss now the case $b \in [\frac{\pi}{2}, \pi)$. If $b \in [\frac{\pi}{2}, \pi)$, then $\frac{\sin b}{b} < \frac{2}{\pi}$ and this together with the fact $\frac{\sin\xi}{\xi} > \frac{2}{\pi}$, imply $\frac{\sin\xi}{\xi} > \frac{\sin b}{b}$ which contradicts (2.9).

Finally, if $b \in \bigcup_{n=1}^{\infty}(2n\pi, (2n+1)\pi)$, then $b > 2n\pi$. Now, $\frac{1}{b} < \frac{1}{2n\pi}$ and then $\frac{\sin b}{b} < \frac{1}{2n\pi} < \frac{2}{\pi}$. Combining this with the fact that $\frac{2}{\pi} < \frac{\sin\xi}{\xi}$, which contradicts (2.9). This completes the proof of our claim. □

Taking into account Theorem 3 and going back to the original variables, we can state the main result of this section.

Theorem 4. *All roots of equation (1.2) lie to the left of the imaginary axis, if and only if* $\frac{c}{d} > \frac{\sin(r\xi)}{\xi}$, *where* ξ *is the only root on the interval* $(0, \frac{\pi}{2r})$ *of the equation* $\frac{\xi^2}{d} = \cos(r\xi)$.

The following result will play a fundamental role in applying the Hopf bifurcation theorem.

Proposition 5. *All roots of the equation* $\lambda^2 + c\lambda + de^{-\lambda r} = 0$ *with nonnegative real part are simple. Moreover, if* λ_0 *is a root with real part equal to zero, then all other roots* $\lambda_j \neq \lambda_0, \bar{\lambda}_0$ *satisfy* $\lambda_j \neq m\lambda_0$ *for any integer* m.

Proof. Let us set $F(\lambda) = \lambda^2 + c\lambda + de^{-\lambda r}$ and let us assume that there exists a solution $\lambda = a + ib$, with $a \geq 0$, of equations $F(\lambda) = 0$, which is not simple; i.e. $F(\lambda) = F'(\lambda) = 0$. Taking into account this fact a straightforward computation gives us

$$(\frac{2a+c}{d})b = e^{-ar}\sin(rb) \tag{2.11}$$

$$\frac{1}{d}(a^2 - b^2 + ca) = -e^{-ar}\cos(rb) \quad . \tag{2.12}$$

and

$$2a + c - dre^{-ar}\cos(br) = 0 \tag{2.13}$$

$$2b + dre^{-ar}\sin(br) = 0 \quad . \tag{2.14}$$

Combining (2.11) and (2.14) we obtain that $2a + c = -2/r$, which is a contradiction, due to $a \geq 0, c, r > 0$.

In order to establish the last part of the proposition, let us assume that there exists a λ_m such that $F(\lambda_m) = 0$ and $\lambda_m = m\lambda_0$, for some $m \neq -1, 0, 1$, where $\lambda_0 = ib$.

By using (2.11) and (2.12), we obtain that $\frac{cb}{d} = \sin(rb)$, $-\frac{b^2}{d} = -\cos(rb)$, $\frac{cmb}{d} = \sin(rmb)$, $-\frac{m^2b^2}{d} = -\cos(rmb)$, which in turn imply that

$$(\frac{cb}{d})^2 + (\frac{b^2}{d})^2 = (\frac{cmb}{d})^2 + (\frac{m^2b^2}{d})^2,$$

or

$$b^2m^4 + c^2m^2 - c^2 - b^2 = 0. \tag{2.15}$$

The roots of equation (2.15) are $m = \pm 1$, $m = \pm\sqrt{1 + (\frac{c}{b})^2}$. Henceforth, equation (2.15) have no integer solutions except $m = \pm 1$. This completes the proof. □

3 Hopf Bifurcation

In this section, by using the Hopf bifurcation theorem, we discuss the existence of nonconstant periodic solutions of small amplitude of equation (1.1).

Let us denote by $F(p, \lambda) = \lambda^2 + c\lambda + de^{-\lambda r}$, where p represents either c, d or r. Following Hale-Lunel [2, Chapter 11] and taking p as a bifurcation parameter, it follows that equation (1.1) has a nonconstant periodic solution of small amplitude if the following conditions are satisfied:

(H1) The characteristic equation $F(p, \lambda) = 0$ has a simple purely imaginary root $\lambda_0(p_0) = ib_0(p_0) \neq 0$ and all the other roots $\lambda_j(p_0) \neq \lambda_0(p_0), \overline{\lambda_0(p_0)}$ satisfy $\lambda_j(p_0) \neq m\lambda_0(p_0)$ for any integer m, for some $p_0 > 0$.

(H2) There exists an open interval containing p_0 such that the roots of $F(p, \lambda) = 0$ can be expressed as a function $\lambda = \lambda(p)$, for p on that interval. Also $\lambda(p)$ is a C^1 function and

$$Re\lambda'(p_0) \neq 0 \quad . \tag{3.1}$$

We are going to carry on all computations in the case that the delay r is taking as a bifurcation parameter. We point out that the condition (H1) follows from Proposition 5, and the condition (H2) is derived from the following lemma.

Lemma 6. *Fixing $c, d > 0$ there exists a unique pair $(r_0, \xi(r_0))$, with $r_0 > 0$, where ξ is a function of r such that*

$$\frac{1}{d}\xi^2(r_0) = \cos(r_0\xi(r_0)) \quad \textit{with} \quad \xi(r_0) \in (0, \frac{\pi}{2r_0}) \tag{3.2}$$

and

$$\frac{c}{d}\xi(r_0) = \sin(r_0\xi(r_0)) \quad . \tag{3.3}$$

Moreover,

(i) If $0 < r < r_0$ *, then* $\frac{c}{d} > \frac{\sin(r\xi(r))}{\xi(r)}$ *and all roots of the equation* $\lambda^2 + c\lambda + de^{-\lambda r} = 0$ *have negative real part.*

(ii) If $r = r_0$ *, then the equation* $\lambda^2 + c\lambda + de^{-\lambda r} = 0$ *has two roots on the imaginary axis and all the other roots lie to the left of the imaginary axis.*

(iii) If $r > r_0$ *, then* $\frac{c}{d} < \frac{\sin(r\xi(r))}{\xi(r)}$ *and the equation* $\lambda^2 + c\lambda + de^{-\lambda r} = 0$ *has no roots on the imaginary axis and it has a finite number of roots with positive real part.*

Finally, there exists an open interval containing r_0 *such that* $\lambda = \lambda(r)$ *the roots of* $F(r, \lambda) = 0$ *are a* C^1 *functions such that*

$$Re\lambda'(r_0) > 0 \quad . \tag{3.4}$$

Proof. Giving $r > 0$, there exists a unique $\xi = \xi(r) \in (0, \frac{\pi}{2r})$ such that $\frac{1}{d}\xi^2(r) = \cos(r\xi(r))$. We have that $\xi(r)$ is a decreasing function and this implies that there exists a unique r_0 such that $(r_0, \xi(r_0))$ satisfies (3.2) and (3.3). Indeed, functions $\Psi_1(r, \xi(r)) = \frac{c}{d}\xi(r)$, $\Psi_2(r, \xi(r)) = \sin(r\xi(r))$ have just one intersection point on the set $\{(r, \xi(r)) : r > 0\}$, namely $(r_0, \xi(r_0))$.

From the previous discussion and Theorem 4 we obtain parts (i),(ii) and (iii) of our claim.

In order to get the last part of the lemma, let us consider the function $F(r, a, b) = (a^2 - b^2 + ca + de^{-ra}\cos(rb), 2ab + cb - de^{-ra}\sin(rb))$. A straightforward computations gives us that

$$\begin{aligned} &D_{(a,b)}F(r_0, 0, \xi(r_0)) \\ &\qquad = \begin{pmatrix} c - dr_0\cos(r_0\xi(r_0)) & -2\xi(r_0) - dr_0\sin(r_0\xi(r_0)) \\ 2\xi(r_0) + dr_0\sin(r_0\xi(r_0)) & c - dr_0\cos(r_0\xi(r_0)) \end{pmatrix}, \end{aligned}$$

and therefore $\det D_{(a,b)}F(r_0, 0, \xi(r_0)) > 0$, due to $(0, \xi(r_0))$ is a simple root. Thus, the implicit function theorem implies there is an open interval I, containing r_0, and a unique solution $\lambda = \lambda(r) = (a(r), b(r))$ with $r \in I$ such that $\lambda(r_0) = (0, \xi(r_0))$ and $F(r, a(r), b(r)) = (0, 0)$. Moreover, after some computations we obtain that

$$\begin{aligned} Re\lambda'(r_0) &= a'(r_0) \\ &= \frac{d\xi(r_0)(c\sin(r_0\xi(r_0)) + 2\xi(r_0)\cos(r_0\xi(r_0)))}{\det D_{(a,b)}F(r_0, 0, \xi(r_0))} > 0. \end{aligned}$$

□

Using Proposition 5 and Lemma 6, we state our main result of this paper.

Theorem 7. *Equation (1.1) has a Hopf bifurcation at* $r = r_0$*, where* r_0 *is defined in Lemma 6. Moreover, if* $r \in (0, r_0)$ *then the trivial solution of (1.1) is locally asymptotically stable.*

Finally, we point out that similar results to Theorem 7 can be obtained taking as bifurcation parameter either c or d. The proof is basically the same of Lemma 6, except obvious modifications.

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