

Computing the Value Function for an Optimal Control Problem

*Determinación de la Función de Valor
para un Problema de Control Óptimo*

Jesús A. Pascal (pascal@math.lsu.edu)

Cecilia Galindez

Departamento de Matemáticas. Facultad de Ciencias
Universidad del Zulia. Maracaibo, Venezuela

Abstract

A one-dimensional infinite horizon deterministic singular optimal control problem with controls taking values in a closed cone in \mathbb{R} leads to a dynamic programming equation of the form:

$$\max F^1(x, v, v'), F^2(x, v, v') = 0, \quad \forall x \in \mathbb{R},$$

which is called the Hamilton Jacobi Bellman (HJB) equation that the value function must satisfy. In this paper we find explicitly the value function for an infinite horizon deterministic optimal control problem.

Key words and phrases: Singular optimal control, viscosity solutions, dynamic programming.

Resumen

Un problema de control óptimo singular determinista con horizonte infinito en una dimensión con controles que toman valores en un cono cerrado de \mathbb{R} conduce a una ecuación de programación dinámica de la forma:

$$\text{máx } F^1(x, v, v'), F^2(x, v, v') = 0, \quad \forall x \in \mathbb{R},$$

la cual es llamada ecuación de Hamilton Jacobi Bellman (HJB) y debe ser satisfecha por la función de valor. En este trabajo encontramos explícitamente la función de valor para un problema de control óptimo determinista con horizonte infinito.

Palabras y frases clave: control óptimo singular, soluciones de viscosidad, programación dinámica.

1 Introduction

Let's consider the scalar control system

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + u(t) \\ x(0) &= x\end{aligned}\tag{1}$$

where the control $u(\cdot)$ is a function of time in the family

$$\mathcal{U} = L^\infty([0, +\infty), [0, +\infty)).$$

Let's define the Cost Functional

$$J(x, u(\cdot)) = \int_0^\infty e^{-t} [L(x(t)) + u(t)] dt,\tag{2}$$

Let's set the value function

$$v(x) = \inf_{u \in \mathcal{U}} J(x, u(\cdot))\tag{3}$$

Ferreyra and Hijab [1] studied the optimal control problem (1), (2) and (3). Their main assumption in [1] is linearity of the function f and convexity of the function L . This enables them to present a complete analysis of this control problem. They used the dynamic programming approach and proved that the free boundary is just a single point giving its location in terms of the parameters of the problem. They proved that the value function v is convex, smooth enough and a classical solution of the HJB equation

$$\max\{v(x) - f(x)v'(x) - L(x), -v'(x) - 1\} = 0 \quad -\infty < x < \infty.\tag{4}$$

Substituting the function L in (2) by a quadratic expression, we compute explicitly the candidate value function using the HJB equation (4) and through evaluations of the Cost Functional (2) along trajectories of the system (1). Finally, we use the corresponding verification theorem in [3] to prove that the candidate value function constructed according to the location of the free boundary is in fact the value function.

1.1 A One Dimensional Control Problem

We consider the scalar control system,

$$\begin{aligned}\dot{x}(t) &= bx(t) + u \\ x(0) &= x\end{aligned}\tag{5}$$

where $b < 0$, and the control $u(\cdot)$ is a function of time in the family

$$\mathcal{U} = L^\infty([0, +\infty), [0, +\infty)).$$

We define the Cost Functional

$$J(x, u(\cdot)) = \int_0^\infty e^{-t} [(x(t) - k)^2 + u(t)] dt, \quad (6)$$

with $k \in \mathbb{R}$, tal que $k = \frac{1-b}{2}$ We set the value function

$$v(x) = \inf_{u \in \mathcal{U}} J(x, u(\cdot)) \quad (7)$$

1.2 The Cost of Using the Control Zero

We consider the control

$$\theta(t) \equiv 0, \quad \forall t \in [0, +\infty),$$

the control system becomes

$$\begin{aligned} \dot{x}(t) &= bx(t) + \theta(t) \\ x(0) &= x, \end{aligned}$$

then we have,

$$\begin{aligned} \dot{x} &= bx \\ x(0) &= x, \end{aligned}$$

The solution of this initial value problem is

$$x(t) = xe^{bt}, \quad \forall t \in [0, +\infty),$$

which is the associated trajectory to the null control, $\theta(t) \equiv 0, \forall t \in [0, +\infty)$.

The Cost associated to this control is

$$\begin{aligned} J(x, \theta(\cdot)) &= \int_0^\infty e^{-t} [(x(t) - k)^2 + \theta(t)] dt \\ &= \int_0^\infty e^{-t} (xe^{bt} - k)^2 dt \\ &= \int_0^\infty e^{-t} [x^2 e^{2bt} - 2xke^{bt} + k^2] dt \end{aligned}$$

So,

$$J(x, \theta(\cdot)) = \frac{1}{1-2b} x^2 - x + k^2. \quad (8)$$

A Candidate Value Function

Remark 1. By Ferreyra and Hijab's work, see [1], we know that the value function is an affine function on the left hand side of zero (0), and a solution of the equation

$$v - fv' - L = 0,$$

on the right hand side of zero(0). Then we approach the value function v through a function W which must satisfy the conditions mentioned above.

Then we define the function W by

$$W(x) = \begin{cases} -x + k^2 & \text{if } x \leq 0 \\ \frac{1}{1-2b}x^2 - x + k^2 & \text{if } x > 0 \end{cases}$$

1.3 Conditions of the Verification Theorem

Observe that since (8) $W(x) = J(x, \theta(\cdot)), \forall x > 0$. Then

$$v(x) \leq W(x), \forall x > 0$$

According to Ferreyra and Hijab's work, see [2], the dynamic programming equation of the optimal control problem (5), (6) y (7) is

$$\max\{v(x) - bxv'(x) - (x - k)^2, -v'(x) - 1\} = 0 \quad -\infty < x < \infty \quad (9)$$

Lemma 1. The function $W(x)$ is a viscosity solution of the dynamic programming equation (9)

W is C^1 obviously. **Case of $x \leq 0$.**

$$\begin{aligned} W(x) &= -x + k^2; \quad W'(x) = -1 \\ W(x) - bxW'(x) - (x - k)^2 &= -x + k^2 + bx - (x^2 - 2xk + k^2) \\ &= -x + k^2 + bx - x^2 + 2xk - k^2 \\ &= -x^2 - (1 - b - 2k)x \\ &= -x - 2\left(\frac{1-b}{2} - k\right)x \\ &= -x^2 \leq 0 \end{aligned}$$

On the other hand,

$$-W'(x) - 1 = -(-1) - 1 = 1 - 1 = 0,$$

thus,

$$\max \left(W(x) - bxW'(x) - (x - k)^2, -W'(x) - 1 \right) = 0, \quad x \leq 0.$$

Case of $x > 0$.

$$\begin{aligned} W(x) &= \frac{1}{1-2b}x^2 - x + k^2; \quad W'(x) = \frac{2}{1-2b}x - 1 \\ W(x) - bxW'(x) - (x - k)^2 &= \frac{1}{1-2b}x^2 - x + k^2 - \frac{2b}{1-2b}x^2 + bx - x^2 + 2kx - k^2 \\ &= \left(\frac{1}{1-2b} - \frac{2b}{1-2b} - 1 \right) x^2 - (1 - b - 2k)x \\ &= \left(\frac{1-2b}{1-2b} - 1 \right) x^2 - 2 \left(\frac{1-b}{2} - k \right) x \\ &= 0 \end{aligned}$$

$$-W'(x) - 1 = -\frac{2}{1-2b}x + 1 - 1 = -\frac{2}{1-2b}x < 0, \quad \text{since } x > 0.$$

Then according to Ferreyra, Dorroh and Pascal's work, see [4], the function W is a viscosity solution of the equation

$$\max \left(v(x) - bxv'(x) - (x - k)^2, -v'(x) - 1 \right) = 0, \quad x > 0. \quad \square$$

Since W is C^1 , and since Shreve and Soner's work, see [6], W is a classical solution of this equation

Lemma 2. The function W satisfies

$$W(x) \leq C(1 + |x|)^2, \quad \text{for some } C > 0.$$

This is obvious since W is an affine function on the left hand side of zero and a quadratic function on the right hand side of zero. \square

Lemma 3. For each $M > 0$ the set

$$\{(bx + u)x : x \in \mathbb{R}, u(\cdot) \in \mathcal{U}_M\}$$

is bounded above.

where $\mathcal{U}_M = \{u(\cdot) \in \mathcal{U}; -\frac{1}{M} < u(\cdot) < \frac{1}{M}\}$.

In fact, for each $x \in \mathbb{R}$ and some $u(\cdot) \in \mathcal{U}_M$

$$\begin{aligned} (bx + u)x &= bx^2 + ux \leq b\left(-\frac{u}{2b}\right)^2 + u\left(-\frac{u}{2b}\right), \\ &= \frac{u^2}{4b} - \frac{u^2}{2b} = -\frac{u^2}{4b} \leq -\frac{M^2}{4b} < \infty. \quad \square \end{aligned}$$

Lemma 4. Given $x \in \mathbb{R}$ and given $M > 0$, for each $u(\cdot) \in \mathcal{U}_M$

$$e^{-t}W(x(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

where $x(t)$ is the solution of (5), with initial value $x(0) = x$, associated to the control $u(\cdot)$.

Lemma 2 and 3 allow us to apply a result of Fleming and Soner, see [5, Chapter I, page 27], then we have,

$$\lim_{t \rightarrow \infty} e^{-t}W(x(t)) = 0. \quad \square$$

Lemma 5. For each $x \in \mathbb{R}$, $W(x) \leq v(x)$.

Given $M > 0$ we define

$$v_M(x) = \inf_{u(\cdot) \in \mathcal{U}_M} J(x, u(\cdot)) \quad (10)$$

Let $x \in \mathbb{R}$, we shall show that for each $M > 0$

$$W(x) \leq v_M(x),$$

and hence,

$$W(x) \leq \inf_{M > 0} v_M(x) = v(x).$$

Let $M > 0$, we consider the optimal control problem (5), (6) and (10). By Lemma 4, the hypothesis of the Verification Theorem in Ferreyra and Pascal's work, see [3], holds, thus we can write,

$$W(x) \leq v_M(x). \quad \square$$

Lemma 6. Let $x \in (0, \infty)$. Then

$$v(x) = W(x)$$

Observe that for the null control

$$\begin{aligned}\theta(t) &\equiv 0, \forall t \geq 0 \\ J(x, \theta) &= \frac{1}{1-2b}x^2 - x + k = W(x) \text{ for } x > 0\end{aligned}$$

thus $v(x) \leq W(x)$ for $x > 0$, and since we proved that $W(x) \leq v(x)$, $\forall x \in \mathbb{R}$ then we have $v(x) = W(x) \forall x \in (0, \infty)$. \square

Lemma 7. Let $x \in (-\infty, 0)$. Then

$$v(x) = W(x)$$

Given $x < 0$, let's consider a sequence of controls $\{u_n(t)\}_{n=1}^{\infty}$, defined by, for each $n \in \mathbb{N}$

$$u_n(t) = \begin{cases} -nx & \text{si } 0 \leq t \leq \frac{1}{n} \\ 0 & \text{si } t > \frac{1}{n} \end{cases}$$

For each n , $1 \leq n < \infty$, the trajectory $x_n(t)$ associated to the control $u_n(t)$ is defined by

$$x_n(t) = \begin{cases} e^{bt} \left(-\int_0^t e^{-bs} nx \, ds \right) + xe^{bt} & \text{if } 0 \leq t \leq \frac{1}{n} \\ \alpha_n e^{b(t-\frac{1}{n})} & \text{if } t > \frac{1}{n} \end{cases}$$

with $\alpha_n = x_n(\frac{1}{n})$ Then

$$x_n(t) = \begin{cases} e^{bt} \left(x - \frac{nx}{b} \right) + \frac{nx}{b} & \text{if } 0 \leq t \leq \frac{1}{n} \\ \alpha_n e^{b(t-\frac{1}{n})} & \text{if } t > \frac{1}{n} \end{cases}$$

It is enough to show that

$$\lim_{n \rightarrow \infty} J(x, u_n(\cdot)) = W(x),$$

since

$$v(x) \leq J(x, u_n(\cdot)), \forall n,$$

then

$$v(x) \leq \lim_{n \rightarrow \infty} J(x, u_n(\cdot)) = W(x),$$

By the Verification Theorem $W(x) \leq V(x), \forall x$, and hence we obtain the equality. The functional cost associated to this control is,

$$\begin{aligned}
 J(x, u_n(\cdot)) &= \int_0^\infty e^{-t} \left[(x_n(t) - K)^2 + u_n(t) \right] dt \\
 &= \int_0^\infty e^{-t} (x_n(t) - K)^2 dt + \int_0^\infty e^{-t} u_n(t) dt \\
 &= \int_0^{\frac{1}{n}} e^{-t} (x_n(t) - K)^2 dt + \int_{\frac{1}{n}}^\infty e^{-t} (x_n(t) - K)^2 dt \\
 &\quad - nx \int_0^{\frac{1}{n}} e^{-t} dt + \int_{\frac{1}{n}}^\infty e^{-t} 0 dt \\
 &= \int_0^{\frac{1}{n}} e^{-t} (x_n(t) - K)^2 dt + \int_0^\infty e^{-t} (x_n(t) - K)^2 X_{[\frac{1}{n}, \infty)} dt \\
 &\quad - nx \int_0^{\frac{1}{n}} e^{-t} dt + 0
 \end{aligned}$$

Now we compute separately these three integrals, For the first integral, we designe by

$$F_n(t) = \int_0^{\frac{1}{n}} e^{-t} \left[e^{bt} \left(\int_0^t e^{-bs} \beta_n ds + x e^{bt} \right) - K \right]^2 dt$$

Let's show that $\lim_{n \rightarrow \infty} F_n(t) = 0$. Observe that $x'_n(t) = (b - n)x e^{bt} > 0$, thus $x_n(t)$ es nondecreasing. Then

$$x \leq x_n(t) \leq x_n\left(\frac{1}{n}\right), \quad \forall t, 0 \leq t \leq \frac{1}{n}$$

On the other hand,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} x_n\left(\frac{1}{n}\right) &= \frac{nx}{b} + \left(x - \frac{nx}{b} + x \right) e^{\frac{b}{n}} \\
 &= -\frac{nx}{b} \left(e^{\frac{b}{n}} - 1 \right) + x e^{\frac{b}{n}} \\
 &= -x \frac{e^{\frac{b}{n}} - 1}{\frac{b}{n}} + x e^{\frac{b}{n}} \\
 &= -x + x = 0,
 \end{aligned}$$

Thus

$$x \leq x_n(t) \leq 0, \quad \forall t, 0 \leq t \leq \frac{1}{n}$$

Then

$$F_n(t) \leq \int_{\frac{1}{n}}^{\infty} e^{-t} M dt$$

for some $M > 0$, therefore,

$$\lim_{n \rightarrow \infty} F_n(t) = 0$$

For the second integral, we designe

$$G_n(t) = \int_0^{\infty} e^{-t} (x_n(t) - K)^2 X_{[\frac{1}{n}, \infty)} dt,$$

By the Dominated Convergence Theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} G_n(t) &= \int_0^{\infty} e^{-t} (0 - k)^2 dt \\ &= \int_0^{\infty} e^{-t} k^2 dt \\ &= k^2, \end{aligned}$$

For the third integral, we designe

$$H_n = -nx \int_0^{\frac{1}{n}} e^{-t} dt.$$

Then,

$$\begin{aligned} \lim_{n \rightarrow \infty} H_n(t) &= \lim_{n \rightarrow \infty} -x \frac{e^{-\frac{1}{n}} - 1}{-\frac{1}{n}}, \\ &= -x \end{aligned}$$

The computations of these three integrals allow us to write

$$\lim_{n \rightarrow \infty} J(x, U_n(\cdot)) = 0 + K^2 - x = W(x). \quad \square$$

References

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