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A New Second Order Finite Difference Conservative Scheme

Un Nuevo Método Conservativo de Segundo Orden en Diferencias Finitas

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Abstract

A complete analytical and numerical study of a new second order finite difference discretization for derivatives and its associated scheme for the Laplace's operator is presented. It is based on a one side approximation for the gradient at boundary nodes in a non-uniform staggered (point distributed) grid. It is shown that the numerical scheme applied to the discretization of the Laplacian operator has a global quadratic convergence rate. In addition, it is also proved that this new discretization scheme is conservative,as its formulation is naturally motivated. That is, it is not necessary to introduce artifacts such as ghost points or extended grid concepts to formulate it. Illustrative numerical tests provide evidence that our new scheme is a better choice than standard finite difference and/or support operator schemes to find numerical solution of boundary-layer like problems formulated in terms of the diffusion equation.

Key words and phrases: mimetic discretizations; finite difference; partial differential equations; diffusion equation; Robin boundary conditions; boundary layer.

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Resumen

Se presenta un estudio analítico y numérico completo de un nuevo esquema de discretizaciones en diferencia finitas de segundo orden para discretizar derivadas y, como un ejemplo, el esquema de discretización correspondiente al operador de Laplace. El esquema se basa en la aproximación lateral para el gradiente en nodos en la frontera en una malla escalonada (de puntos distribuidos) no uniforme. Se muestra que al aplicar el esquema numérico a la discretización del operador de Laplace, el mismo tiene una tasa de convergencia global de segundo orden. Además, también se prueba que este nuevo esquema de discretización es conservativo, siendo muy natural su formulación. Es decir, en su formulación no es necesario recurrir a ideas artificiales como el de puntos fantasmas o de mallas extendidas. Pruebas numéricas ilustrando el método dan evidencias que nuestro nuevo esquema es mejor elección que los esquemas de diferencia finita normal y/o el de operador de soporte para encontrar soluciones numéricas de problemas del tipo de capa límite formulado en términos de la ecuación de difusión.

Palabras y frases clave:discretizaciones miméticas; diferencias finitas; ecuaciones diferenciales parciales; ecuación de difusión; condiciones de borde tipo Robin; capa límite (boundary layer).

1 Introduction

By its simplicity and efficiency the finite difference methods are widely used to solve partial differential equations. In particular, second order conservative schemes can be easily obtained on uniform grids by using the so-called ghost point extension. However, such approach assumes that the discretization of both the partial differential equation and the boundary conditions are simultaneously valid at the extended boundary nodes. To avoid this problem one side finite difference approximations has been proposed on the boundary but most of them produce no conservative schemes having low order truncation error. If non-uniform grids are used then convergence rates of these schemes deteriorates. This is particularly true for the Laplace operator. All these deficiencies of finite difference schemes are well documented in textbooks [1, 2, 3, 4]. Overall it may be said that a correct discretization of the boundary conditions is one of the main difficulties to be addressed in order to improve or develop new discretization methods close to standard finite difference schemes.

In the last ten years a new generation of numerical methods named by the generic label of mimetic methods has been developed. A partial review of them, up to 2002, can be found in [5]. A key distinguishing feature of the mimetic discretizations approach is to produce discretizations of the operators (i.e gradient, divergence, and/or curl) in terms of which the differential equation of the physical problem of interest is written, preserving symmetry and conservation properties that are true in the continuum and satisfying discrete version of the Green-Stokes theorem. This last condition ensures that the discretization of the boundary conditions and of the differential equation are compatible. It has been known for some time that numerical methods based on such discretization produce better physically faithful results than standard finite differences. Mimetic discretizations also has the great advantage that their formulation is not more complex than standard finite differences.

In recent publications, Castillo and coworkers [6, 7] developed a systematic procedure to obtain high order mimetic discretizations for the divergence and gradient operators, attaining the same order of approximation at boundary and inner points. Mimetic schemes for the steady state diffusion equation based on Castillo-Grone approach have been reported in [8, 9, 10, 11, 12]. Each one of those articles provides evidence of quadratic convergence rates for such schemes but a rigorous proof of it has not been published so far.

In this article we supply a rigorous proof of quadratic convergence for a particular and unique finite difference discretizations of the Laplacian proposed in [8], which is asymptotically equivalent to a second order mimetic scheme based on the Castillo-Grone approach. In addition, an illustrative test problem in one dimension is developed, providing a solid evidence of the advantage for this second order scheme at numerically solving a boundary-layer like problem, formulated in terms of the diffusion equation.

The rest of article is divided in six small sections. In the first section, a short description of the continuous model used in the discretizations is presented. After that the second order scheme for the Laplace operator along with the gradient and divergence discretization are described. The proof of its quadratic convergence rate and conservative properties are provided in the following two sections. Next, the formulation and solution of an illustrative numerical test problem is given, and then the conclusions of the present work are summarized.

2 Continuous Model

In fact, being one of the most important and widely used equation of the mathematical-physics, the range of physical and engineering problems modeled by the diffusion equation (equation 1 below) includes heat transfer, flow through porous medium, and the pricing of some financial instruments. Accordingly, this wide range of applications of the diffusion equation some how justify the effort and time devoted in finding ways of obtaining high quality numerical solution of it on different contexts. Correspondingly, will be illustrating the robustness of the new scheme in solving one dimensional boundary-layer like problems formulated in terms of this equation, which has the form.

$$-\nabla\cdot\left(\overleftrightarrow{K}(\vec{x})\cdot\nabla f(\vec{x})\right) = F(\vec{x}) \tag{1}$$

where $\overleftrightarrow{K}(\vec{x})$ is a symmetric tensor, $f(\vec{x})$ is the target property we are looking for, and $F(\vec{x})$ is a source term. For instance, in a heat transfer problem, $\overleftrightarrow{K}(\vec{x})$, $f(\vec{x})$, and $F(\vec{x})$ are respectively the thermal conductivity, the temperature, and a source of heat influencing the domain of interest; in a porous media flow they are, respectively, the permeability tensor, the pressure driving the flow, and a source term (i.e. a producer or injector well in a oil field) affecting the fluid flow in the region of interest.

In the one dimensional case, equation (1) takes the form,

$$-\frac{d}{dx}\left(K(x)\frac{df(x)}{dx}\right) = F(x) \tag{2}$$

which in terms of the discretized operators via mimetic technique, is written in the form,

$$-\mathbf{D}(K(x)\mathbf{G}f) = F \tag{3}$$

where $\mathbf{D}$ and $\mathbf{G}$ represents the discretized version of the divergence ($\nabla\cdot$) and the gradient (∇) operators. That is, rather than discretizing a particular differential equation, the mimetic approach gives attention to the discretization of the operators itself. In this form, once we have the discrete version of the differential operators of interest, one could discretize any equation written in terms of them by means of matrix computations.

To have a boundary value problem posed by equation 2, we will be imposing boundary conditions of the Robin (mixed) type, which in its general form can be written in the form

$$\begin{aligned} \alpha_0 f(0) - \beta_0(K(x)\nabla f(x)|_{x=0} = \gamma_0 \\ \alpha_1 f(1) + \beta_1(K(x)\nabla f(x)|_{x=1} = \gamma_1 \end{aligned} \tag{4}$$

Let's mention that the one-dimensional boundary value problem given by equations (2) and (4) has a unique solution [15], unless there is a nontrivial solution of the associated homogeneous problem of equation (2), satisfying the boundary conditions (4).

In this article we are analyzing the case where $K(x)$ is the identity on the above equations. In this situation, it has been found [16] that the problem posed by (1) and Robin boundary conditions written in the form $\alpha\, f + \beta\, \frac{\partial\, f}{\partial\, n}$ on $\partial\, R$, the surface bounding the domain of interest, has unique solution when $\frac{\beta}{\alpha}$ has a single sign for all its possible values.

3 Description of the Numerical Scheme

In order to describe the new method a staggered grid configuration is needed. For simplicity, it is represented in figure 1 defining the interval $[x_0 = 0, x_{N+1} = 1]$. This grid has $N+1$ blocks of size h, each one of which has a central node denoted by $x_{i+\frac{1}{2}}$ for $i = 0, N$. Notice that the first and last block contains an additional node at the boundaries denoted by x_0 and x_{N+1} respectively. This is not a standard grid and it receives several names in the literature. In mimetic articles, the authors refers to it as a *staggered uniform grid* while in the finite difference technical literature it is called a *non-uniform point distributed grid* [13]. The spacing h between edges (x_i) is obtained from $h = \frac{x_{N+1}-x_0}{N+1}$, and it follows $x_i = x_0 + i\, h$. In our notation $f_i = f(x_i)$.

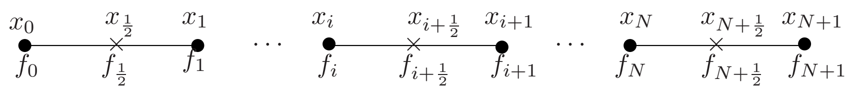

Figure 1: Staggered (non-uniform point distributed) grid.

Following the notation of figure 1, the new one-sided finite difference approximation for the gradient at the boundary points x_0 and x_{N+1} has the form.

$$(\mathbf{G}f)_0 = \frac{-\frac{8}{3}f_0 + 3f_{\frac{1}{2}} - \frac{1}{3}f_{\frac{3}{2}}}{h} \tag{5a}$$

$$(\mathbf{G}f)_{N+1} = \frac{\frac{8}{3}f_{N+1} - 3f_{N+\frac{1}{2}} + \frac{1}{3}f_{N-\frac{1}{2}}}{h} \tag{5b}$$

This approximation has the advantage that they may be obtained by straightforward application of Taylor expansions or applying the systematic approach developed in [6]. They have second order truncation error and thus they produce a better approximation of Robin's boundary conditions

$$\begin{array}{l} f(0) + \beta_0 f'(0) = \gamma_0 \\ f(1) + \beta_1 f'(1) = \gamma_1 \end{array} \tag{6}$$

than standard one side first order approximation for the derivatives. The coefficients β_0 and β_1 are, without lost of generality, positive constants while γ_0 and γ_1 have arbitrary values. At inner points (cell or edges), crosses in figure 1, the gradient and divergence approximations coincide with standard central difference schemes.

$$(\mathbf{G}f)_i = \frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{h} \quad ; \quad i = 1, \cdots, N \tag{7a}$$

$$(\mathbf{D}f)_{i+\frac{1}{2}} = \frac{f_{i+1} - f_i}{h} \quad ; \quad i = 0, \cdots, N \tag{7b}$$

It should be note that the discretized divergence operator (7b) is only defined at the inner nodes.

Under these conditions the discretization of the Laplacian at inner nodes is represented by the following expressions:

$$(\mathbf{DG}f)_{\frac{1}{2}} = \frac{1}{h^2}\left(\frac{8}{3}f_0 - 4f_{\frac{1}{2}} + \frac{4}{3}f_{\frac{3}{2}}\right) = F_{\frac{1}{2}} \tag{8a}$$

$$(\mathbf{DG}f)_{i+\frac{1}{2}} = \frac{1}{h^2}\left(f_{i+\frac{3}{2}} - 2f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}\right) = F_{i+\frac{1}{2}} \quad ; \quad i = 2, \cdots, N-1 \tag{8b}$$

$$(\mathbf{DG}f)_{N+\frac{1}{2}} = \frac{1}{h^2}\left(\frac{8}{3}f_{N+1} - 4f_{N+\frac{1}{2}} + \frac{4}{3}f_{N-\frac{1}{2}}\right) = F_{N+\frac{1}{2}}. \tag{8c}$$

These expressions are the same as standard finite difference, but at nodes $x_{\frac{1}{2}}$ and $x_{N+\frac{1}{2}}$ where one-sided approximations (5a) and (5b) are being used when computing the discretized Laplacian **DG**.

In this work only nontrivial Robin boundary conditions will be considered for the Poisson equation. Numerical experiments have shown that this new scheme does not have any advantage for Dirichlet conditions. The boundary conditions (6) are discretized using the one-sided approximations (5a) and (5b), resulting in the following equations

$$\left(1 - \frac{8\,\beta_0}{3\,h}\right) f_0 + \frac{3\,\beta_0}{h} f_{\frac{1}{2}} - \frac{\beta_0}{3\,h} f_{\frac{3}{2}} = \gamma_0 \tag{9a}$$

$$\left(1 + \frac{8\,\beta_1}{3\,h}\right) f_{N+1} - \frac{3\,\beta_1}{h} f_{N+\frac{1}{2}} + \frac{\beta_1}{3\,h} f_{N-\frac{1}{2}} = \gamma_1 \tag{9b}$$

Equations (8) through (9) represents the new finite difference scheme for the Laplace or Poisson equation developed in [8]. This system of equations is asymptotically equivalent to the mimetic finite difference approximation based on the Castillo-Grone approach. In references [8, 9, 10] this scheme is studied but the authors do not give any rigorous proof of its quadratic convergence rate. We will be filling that gap in this article. That is, in the next section we will be providing a rigorous proof of the quadratic convergence rate of the new discretization scheme just presented.

4 Convergence

The convergence proof of the new finite difference scheme presented in this section is achieved by making use of the following discrete version of the maximum principle for elliptic equations.

Theorem 1. *Suppose that region J is partitioned in two disjoint regions J_1, J_2 and a non-negative auxiliary mesh function $\phi(x)$ is defined on $J = J_1 \bigcup J_2$. If a discrete approximation L_h of an elliptic equation on a mesh satisfies $L_h\phi \geq C_1$ on J_1, $L_h\phi \geq C_2$ on J_2, and its truncation error T_i satisfies $|T_1| \leq Tr_1$ on J_1, $|T_2| \leq Tr_2$ on J_2, then the error between the approximated and exact solutions, e_i, is bounded in the maximum norm by* $([\max_{x\epsilon J_1}] \max\left\{\frac{Tr_1}{C_1}, \frac{Tr_2}{C_2}\right\})$.

This theorem and its proof can be found in page 177 of [3]. Our approach to prove convergence follows the same arguments given in [3], essentially by fulfilling the hypothesis of Theorem 1. In our work the auxiliary mesh function is

$$\phi(x) = (x-p)^2 \tag{10}$$

where p is a constant to be determined later.

As a first step substitute boundary conditions equations (9) in equations (8a) and (8c). When this is done, the following discretizations are obtained

$$-4\left(\frac{(2\,\beta_0 - h)}{h^2\,(8\,\beta_0 - 3\,h)}\right) f_{\frac{3}{2}} - 4\left(\frac{(2\,\beta_0 - 3\,h)}{h^2\,(8\,\beta_0 - 3\,h)}\right) f_{\frac{1}{2}} - \frac{8\,\gamma_0}{h\,(8\,\beta_0 - 3\,h)} = F_{\frac{1}{2}} \tag{11a}$$

$$-4\left(\frac{(2\,\beta_1 + h)}{h^2\,(8\,\beta_1 + 3\,h)}\right) f_{N-\frac{1}{2}} - 4\left(\frac{(2\,\beta_1 + 3\,h)}{h^2\,(8\,\beta_1 + 3\,h)}\right) f_{N+\frac{1}{2}} + \frac{8\,\gamma_1}{h\,(8\,\beta_1 + 3\,h)} = F_{N+\frac{1}{2}} \tag{11b}$$

These expressions may be written in the following form

$$\left(\frac{4}{3\,h^2} f_{\frac{3}{2}} - \frac{4}{h^2} f_{\frac{1}{2}} + \frac{8}{3\,h^2} f_0 - F_{\frac{1}{2}}\right) + \tag{12a}$$
$$\frac{8}{h\,(8\,\beta_0 - 3\,h)}\left(\left(1 - \frac{8\,\beta_0}{3\,h}\right) f_0 + \frac{3\,\beta_0}{h} f_{\frac{1}{2}} - \frac{\beta_0}{3\,h} f_{\frac{3}{2}} - \gamma_0\right) = 0$$
$$\left(\frac{4}{3\,h^2} f_{N-\frac{1}{2}} - \frac{4}{h^2} f_{N+\frac{1}{2}} + \frac{8}{3\,h^2} f_{N+1} - F_{N+\frac{1}{2}}\right) + \tag{12b}$$
$$\frac{8}{h\,(8\,\beta_1 + 3\,h)}\left(\left(1 + \frac{8\,\beta_1}{3\,h}\right) f_{N+1} + \frac{3\,\beta_1}{h} f_{N+\frac{1}{2}} - \frac{\beta_1}{3\,h} f_{N-\frac{1}{2}} - \gamma_1\right) = 0$$

which contains the differential equations and boundary conditions approximations in single equations. A Taylor's expansion calculation shows that truncations error for equations (12a) and (12b), which will be denoted by T_1 and T_n, are only first order.

$$|T_1| \leq O(h) \text{ and } |T_n| \leq O(h) \tag{13}$$

Truncation error for (8b) at inner points will be denoted by T_i, and a standard calculation shows that it is second order.

$$|T_i| \leq O(h^2) \tag{14}$$

Let's define the grid function as follows.

$$L_h(\phi(x_i)) = \begin{cases} -4\frac{(h-2\,\beta_0)}{h^2\,(8\,\beta_0-3\,h)}\phi(x_{\frac{3}{2}}) - 4\frac{(2\,\beta_0-3\,h)}{h^2\,(8\,\beta_0-3\,h)}\phi(x_{\frac{1}{2}}) & \text{at } x_{\frac{1}{2}} \\ \frac{1}{h^2}\left(\phi(x_{i+\frac{3}{2}}) - 2\,\phi(x_{i+\frac{1}{2}}) + \phi(x_{i-\frac{1}{2}})\right) & \text{at } x_{i+\frac{1}{2}} \\ 4\frac{(h+2\,\beta_1)}{h^2\,(8\,\beta_1+3\,h)}\phi(x_{N-\frac{1}{2}}) - 4\frac{(2\,\beta_1+3\,h)}{h^2\,(8\,\beta_1+3\,h)}\phi(x_{N+\frac{1}{2}}) & \text{at } x_{N+\frac{1}{2}} \end{cases} \tag{15}$$

A quick substitution of (10) into (15) and after a simplification we obtain.

$$L_h(\phi(x_i)) = \begin{cases} 2\left(1 + \frac{4\,p(p-2\beta_0)}{h(8\,\beta_0-3\,h)}\right) & \text{at } x_{\frac{1}{2}} \\ 2 & \text{at } x_{i+\frac{1}{2}} \\ 2\left(1 + \frac{8(1-p)(p-2\beta_1-1)}{h(8\,\beta_1+3\,h)}\right) & \text{at } x_{N+\frac{1}{2}} \end{cases} \tag{16}$$

It is always possible to pick and appropriated p in (10) and constant K_1, K_2 in such a way that the following inequality holds

$$L_h(\phi(x_i)) \geq \begin{cases} \frac{K_1}{h} & \text{at } x_{\frac{1}{2}} \\ 2 & \text{at } x_{i+\frac{1}{2}} \\ \frac{K_2}{h} & \text{at } x_{N+\frac{1}{2}} \end{cases} \tag{17}$$

Combining inequality (17), truncations errors (13, 14), and the modulus maximum principle for grid functions results in the following estimate

$$|f^{\text{ex}} - f^{\text{num}}| \leq (\max_{x\in[0,1]} \phi)\cdot \max\left(\frac{|T_1|}{K_1/h}, \frac{|T_n|}{K_2/h}, \frac{|T_i|}{2}\right) \leq O(h^2) \tag{18}$$

which complete the proof.

5 Conservative Properties

A very important property that must satisfy any good numerical scheme is to be conservative. In our context, a numerical scheme is conservative if it satisfies a discrete version of the fundamental calculus theorem.

$$\int_0^1 f''(x)\,dx = f'(1) - f'(0) \tag{19}$$

If the integral of the above expression is approximated by a simple quadrature rule on the staggered grid, then the following summations are obtained

$$\sum_{i=1}^{n} f''(x_i)\cdot h \tag{20}$$

The second derivatives can be approximated using equations (8a-8b) and we obtain the relation

$$\sum_{i=1}^{n} f''(x_i)\cdot h = \frac{1}{h^2}\left(\frac{8}{3}f_0 - 4f_{\frac{1}{2}} + \frac{4}{3}f_{\frac{3}{2}}\right)h \tag{21}$$

$$+\sum_{i=2}^{n-1}\frac{1}{h^2}\left(f_{i+\frac{3}{2}} - 2f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}\right)h + \frac{1}{h^2}\left(\frac{8}{3}f_{N+1} - 4f_{N+\frac{1}{2}} + \frac{4}{3}f_{N-\frac{1}{2}}\right)h$$

Most terms in the above summation are telescopic and we obtain this simplified expression

$$\begin{aligned} &\frac{8f_0 - 9f_{\frac{1}{2}} + f_{\frac{3}{2}} + 8f_{N+1} - 9f_{N+\frac{1}{2}} + f_{N-\frac{1}{2}}}{3h} \\ = &\frac{1}{h}\left(\frac{8}{3}f_{N+1} - 3f_{N+\frac{1}{2}} + \frac{1}{3}f_{N-\frac{1}{2}}\right) - \frac{1}{h}\left(-\frac{8}{3}f_0 + 3f_{\frac{1}{2}} - \frac{1}{3}f_{\frac{3}{2}}\right) \end{aligned} \tag{22}$$

which represents the discrete version of the right hand side of (19). Consequently the conservative property of our scheme is guarantee.

6 Numerical Test

This section presents a comparative study among the new method, standard finite difference scheme and support operator method by means of applying the schemes to solve a one dimensional boundary value problem. It is important to note that those methods are all conservative and they provide three different alternatives to discretize boundary conditions on a staggered grid. Maximum norm is used to quantify all errors in the numerical tests.

The implementation of the standard finite differences scheme is based on the ghost point formulation, which uses second order central difference scheme for Robin boundary conditions [2, 3], while the formulation of the support operator schemes is presented in [14]. It uses a one-sided or lateral first order finite difference scheme to approximate derivatives at boundary points. Our new scheme uses one-sided or lateral second order finite difference scheme (5) to obtain second order approximation (9) at the boundary.

These three methods cover all the possible alternatives for the discretization of boundary conditions using second order schemes on a one-dimensional staggered grid.

The one dimensional boundary value problem in this test is formulated in terms of the ordinary differential equation

$$\frac{d^2f}{dx^2} = \frac{\lambda \exp(\lambda\, x)}{\exp(\lambda) - 1} \tag{23}$$

defined on the interval (0,1), and it's solution must satisfy Robin boundary conditions of the form

$$\begin{aligned} \alpha f(0) - \beta f'(0) &= -1 \\ \alpha f(1) + \beta f'(1) &= 0 \end{aligned} \tag{24}$$

at the borders. Equations (23) and (24) form together a well posed problem for $\alpha = -\exp(\lambda)$, $\beta = (\exp(\lambda) - 1)/\lambda$, being λ an arbitrary non-null real number. This problem has a unique analytical solution given by $f(x) = (\exp(\lambda x) - 1)/(\exp(\lambda) - 1)$, and it represents a boundary layer for large values of λ. Correspondingly, it is an excellent test problem to evaluate numerical schemes with different discretization alternatives for boundary conditions.

In this test all the numerical methods were implemented on the staggered grid described in figure 1. The value of the parameter λ was set equal to 20, although similar results and conclusions are obtained for any positive value of it. Numerical results are presented in tables 1 and 2 along with figures 2 and 3.

Grid Size	Error Finite Difference	Error Support Operator	Error New Method
16	0.3958	0.1861	0.0794
64	0.2206	0.0154	0.0045
256	0.0717	0.0010	0.0002

Table 1: Numerical Errors

Table 1 shows the numerical errors computed in the maximum norm. They indicate that on refined grids the new method achieved at least three exact digits in its approximation, while support operator and standard finite difference methods obtained only two and one exact digits respectively. Such results indicate a clear advantage of our new scheme.

Method	Rate
Standard Finite Differences	0.9104
Operator Support	2.0434
New Method	1.9796

Table 2: Numerical Convergence Rates

In Table 2 numerical convergence rates for each method are presented. A quadratic convergence rate was obtained for both the new and support operator methods. This is the optimum possible rate for these two schemes and it ratifies our theoretical result for the new scheme. Standard finite differences schemes get a first order numerical convergence rate, which is a direct effect of having a first order discretization for the Laplacian at nodes $x_{\frac{1}{2}}$ and $x_{N+\frac{1}{2}}$. In the extended ghost point grid, those two nodes become internal nodes away from the ghost boundary. Consequently, modulus maximum principle implies that first order truncation error in the Laplacian will be transferred completely to the convergence rate and it cannot be canceled or balanced with second order discretizations at boundary nodes. This low convergence rate for the standard finite difference method can be fixed if an uniform block centered grid is used in its implementation.

Figure 2 exhibits the error curves for each method based on a sequence of 256 runs. Their slope represents the numerical rates in table 2. This graph gives clear evidence that the error for the new scheme is one order of magnitude less than the support operator error. For standard finite differences case the error is at least four times larger than the other two schemes on the finest grid and this gap will increase under further grid refinements. If a uniform block centered grid is used for the implementation of the standard finite difference, then its error is comparable to the ones generated by the support operator method. The new method, however, still shows a better precision than the other two.

The approximated solutions computed by the three methods on a twenty blocks grid are presented in figure 3. This graph deserves several comments. The numerical solution computed with the new method lies over the analytical curve at all grid points. For standard finite difference and support operator solutions, theirs points agrees with the analytical curve only at the left boundary. At the right boundary standard finite difference does not reach analytical curve and this behavior push its points up, above the real solution. On the other hand, most of the points in support operator solution are below the analytical curve and its quality is comparable to standard finite difference at this level of discretization. The great accuracy exhibited by the new method in this test problem gives a strong evidence of its numerical advantages over well known numerical schemes.

There is an important property related to our new scheme, which cannot be matched by standard finite difference approximation. It essentially is the rigorous treatment given in the new method to both the boundary conditions and the differential equation. This advantage can easily be observed if the non-homogeneous term in the differential equation has a singularity at the boundary. In such case, the new method produces a robust code whose numerical results are of high accuracy. On the contrary, standard finite difference codes developed on any grid based on ghost point will break down because it requires the regularity of the non-homogeneous term up to the boundary. This last condition is artificial and it is one of the main deficiencies of standard finite difference schemes. Such deficiencies are eliminated in the new scheme.

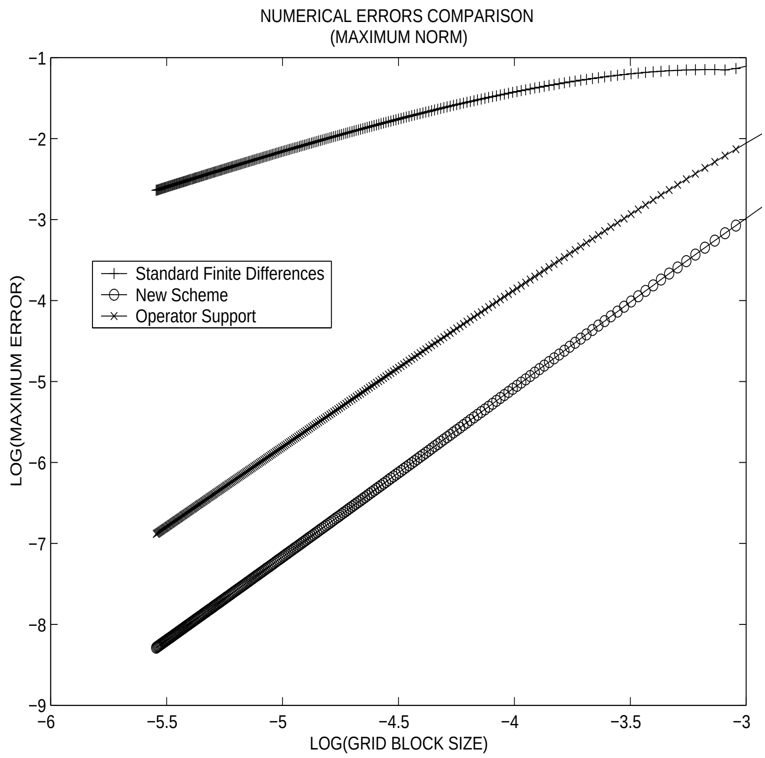

Figure 2: One dimensional convergence rates

7 Conclusions and Discussion

A complete analytical and numerical study for a new second order conservative finite difference scheme has been presented. Theoretical and numerical analysis of its quadratic convergence rate is a new contribution. This is not an obvious result in view of the first order truncations errors in its mathematical formulation.

The new scheme was applied to a selected test problem. The numerical results indicate its main advantages over most common second order conservative methods for problems with boundary layer.

The most important advantages of the new scheme are: it is conservative; its formulations at inner and boundary nodes is consistent; its numerical

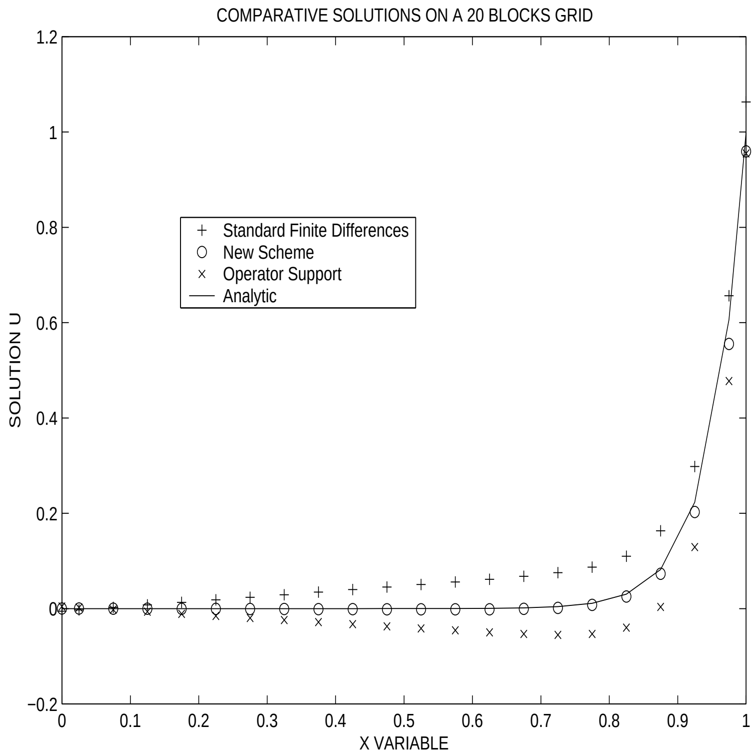

Figure 3: Comparing numerical solutions

implementation is more robust than most common second order finite differences schemes; it is not based on ghost point techniques; and it gives a rigorous discretization of both the boundary conditions and the differential equation.

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