Divulgaciones Matemáticas Vol. 14 No. 1(2006), pp. 39–46

Around a Quotient Space of Bennett-Lutzer's Space

Acerca de un Espacio Cociente del Espacio de Bennett-Lutzer

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Abstract

Let X be the Bennett-Lutzer's space and Y be the space obtained from X by shrinking the set of all rational numbers to a point. In his book, G.Gao claimed that the space Y is compact. In this paper, we prove that Y is neither countably compact nor Lindelöf, which shows that G.Gao's claim is not true. Moreover, we prove that Y is strongly paracompact. As an application of this result, we obtain that all covering properties which are between strong paracompactness and countable θ-refinability are not inversely preserved under closed Lindelöf mappings even if domain is Hausdorff. We also give an example to show that a closed Lindelöf inverse image of a compact space even need not be countably θ-refinable without requiring the regularity of domain involved.
Key words and phrases: Closed Lindelöf mapping, compact, Lindelöf, strongly paracompact, θ-refinable.

Resumen

Sea X el espacio de Bennett-Lutzer e Y el espacio obtenido a partir de X identificando el conjunto de los números racionales a un punto. En su libro, G. Gao afirma que el espacio Y es compacto. En este artículo se prueba que Y no es ni contablemente compacto ni Lindelöf, mostrando que la afirmacin de G. Gao no es verdadera. Más aún, se prueba que Y is fuertemente paracompacto. Como aplicación de este resultado, se obtiene que todas las propiedades de cubrimiento que están entre la paracompacidad fuerte y la θ-refinabilidad contable no son inversamente preservadas por aplicaciones de Lindelöf cerradas, aún si el dominio es

This project was supported by NSFC(No. 10571151)
Received 2005/05/26. Revised 2006/05/18. Accepted 2006/05/25.
MSC (2000): Primary 54B15, 54C10, 54D20, 54D30.

de Hausdorff. También se da un ejemplo para mostrar que la imagen inversa Lindelöf cerrada de un espacio compacto puede incluso no ser contablemente θ-refinable, sin requerir la regularidad del dominio.
Palabras y frases clave: Aplicacin cerrada de Lindelöf, compacto, Lindelöf, fuertemente paracompacto, θ-refinable.

1 Introduction

A continuous function $f : X \longrightarrow Y$ is called a Lindelöf mapping([2]) if $f^{-1}(y)$ is a Lindelöf subspace of X for each $y \in Y$. In [2], D.K.Burke gave the following result.

Theorem 1.1. *Suppose $f : X \longrightarrow Y$ is a continuous, onto, closed Lindelöf mapping, where X is regular. If Y is strongly paracompact (resp. paracompact, metacompact, θ-refinable), then X is strongly paracompact (resp. paracompact, metacompact, θ-refinable).*

By viewing the above result, we have the following question.

Question 1.2. Can the regularity of X in Theorem 1.1 be omitted or relaxed to Hausdorff?

Taking this question into account, G.Gao([5]) constructed a closed Lindelöf mapping f from a non-θ-refinable space X onto a space Y, where the space X was given by H.R.Bennett and D.J.Lutzer in [1] and Y is the space obtained from X by shrinking the set of all rational numbers to a point. Furthermore, G.Gao gave the following claim without proof.

Claim 1.3. The space Y is compact([5, Page 216]).

By Claim 1.3, the following conclusion was obtained.

Conclusion 1.4. All covering properties which are between compactness and θ-refinability are not inversely preserved under closed *Lindelöf* mappings without requiring the regularity of domains involved([5, Page 216]).

In this paper, we investigate Claim 1.3 and Conclusion 1.4. We prove that Y is neither countably compact nor Lindelöf, which shows that Claim 1.3 is not true. Moreover, we prove that Y is strongly paracompact. As an application of this result, we obtain that all covering properties which are between strong paracompactness and countable θ-refinability are not inversely preserved under closed Lindelöf mappings even if domain is Hausdorff. We also give an example to show that Conclusion 1.4 is true, although Claim 1.3 is not true.

Throughout this paper, all spaces are assumed to be T_1 at least, and all mappings are continuous and onto. The letter $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{U}$ be a family of subsets of a space X and let $x \in X$. $\bigcup\mathcal{U}$ denotes the union $\bigcup\{U : U \in \mathcal{U}\}$, and $ord(x,\mathcal{U})$ denotes the cardinal of the family $\{U \in \mathcal{U} : x \in U\}$. Let $\mathcal{U}$ and $\mathcal{V}$ be two families of (open) subsets of a space X. We say that $\mathcal{V}$ is a (open) partial refinement of $\mathcal{U}$, if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$; moreover, we say that $\mathcal{V}$ is a (open) refinement of $\mathcal{U}$, if in addition $\bigcup\mathcal{V} = \bigcup\mathcal{U}$ is also satisfied. For terms which are not defined here, refer to [4].

2 Around Bennett-Lutzer's Space

Definition 2.1. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) $\mathcal{P}$ is called star-finite if the family $\{Q \in \mathcal{P} : Q \cap P \neq \phi\}$ is finite for each $P \in \mathcal{P}$.

(2) $\mathcal{P}$ is called locally finite if for each $x \in X$ there exists a neighborhood W of x such that W has nonempty intersection with at most finitely many members of $\mathcal{P}$.

(3) $\mathcal{P}$ is called point finite if each point $x \in X$ is an element of at most finitely many members of $\mathcal{P}$.

Definition 2.2. Let X be a space.

(1) X is called strongly paracompact([7]) if each open cover of X has a star-finite open refinement.

(2) X is called paracompact([3]) if each open cover of X has a locally finite open refinement.

(3) X is called metacompact([4]) if each open cover of X has a point finite open refinement.

Definition 2.3. Let X be a space.

(1) X is called (countably) θ-refinable([9]) if each (countable) open cover $\mathcal{U}$ of X has a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open refinements of $\mathcal{U}$ such that the following condition (a) holds.

(2) X is called weak $\overline{\theta}$-refinable([8]) if each open cover $\mathcal{U}$ of X has a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open partial refinements of $\mathcal{U}$ such that the following conditions (a) and (b) hold.

(a) For each $x \in X$, there exists $n \in \mathbb{N}$ such that $0 < ord(x,\mathcal{U}_n) < \infty$.

(b) $\{\bigcup\mathcal{U}_n : n \in \mathbb{N}\}$ is point finite.

Remark 2.4. It is known that strongly paracompact $\Longrightarrow$ paracompact $\Longrightarrow$ metacompact $\Longrightarrow$ θ-refinable $\Longrightarrow$ weak $\overline{\theta}$-refinable.

Example 2.5. The Bennett-Lutzer's space([1, Example 1]).

Let X, Q and I be the set of all real numbers, the set of all rational numbers and the set of all irrational numbers respectively. Define a base $\mathcal{B}$ of X as follows.

$\mathcal{B} = \{\{x\} : x \in I\} \bigcup \{G(x,n) : x \in Q, n \in \mathbb{N}\}$, where $G(x,n) = \{x\} \bigcup \{y \in I : -1/n < y - x < 1/n\}$.

Then

(1) X is Hausdorff, and it is neither regular nor θ-refinable.

(2) X is not countably θ-refinable.

(3) X is weak $\overline{\theta}$-refinable.

Proof. (1) It is obtained from [1].

(2) Assume X is countably θ-refinable. Let $\mathcal{U}$ be an open cover of X. Then there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ which covers Q. Put $W = \bigcup \mathcal{V}$. Then W is clopen in X and $\mathcal{V}$ is a countable open cover of W. Notice that countable θ-refinability is hereditary to closed subspaces. W is countably θ-refinable, so there exists a sequence of open refinements $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of $\mathcal{V}$ such that for each $x \in W$, there exists $n \in \mathbb{N}$ such that $0 < ord(x, \mathcal{V}_n) < \infty$. Put $\mathcal{U}_n = \mathcal{V}_n \bigcup \{\{x\} : x \in X - W\}$ for each $n \in \mathbb{N}$. Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open refinements of $\mathcal{U}$. For each $x \in X$, if $x \in W$, then there exists $n \in \mathbb{N}$ such that $0 < ord(x, \mathcal{V}_n) < \infty$, hence $0 < ord(x, \mathcal{U}_n) = ord(x, \mathcal{V}_n) < \infty$; if $x \in X - W$, then $0 < ord(x, \mathcal{U}_n) = 1 < \infty$ for each $n \in \mathbb{N}$. Thus X is θ-refinable. This contradicts the above (1).

(3) Let $\mathcal{U}$ be an open cover of X. For each $x \in Q$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Let $G(x, n_x) \in \mathcal{B}$ such that $G(x, n_x) \subset U_x$. Put $\mathcal{U}_1 = \{G(x, n_x) : x \in Q\}$, $\mathcal{U}_2 = \{\{x\} : x \in I\}$, $\mathcal{U}_n = \emptyset$ for $n > 2$. Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open partial refinements of $\mathcal{U}$. For each $x \in X$, if $x \in Q$ then $ord(x, \mathcal{U}_1) = 1 < \infty$; if $x \in I$ then $ord(x, \mathcal{U}_2) = 1 < \infty$. It is clear that $\{\bigcup \mathcal{U}_n : n \in \mathbb{N}\}$ is point finite. So X is weak $\overline{\theta}$-refinable. □

Proposition 2.6. *There exists an uncountable closed subset of the reals R consisting of irrational numbers.*

Proof. Let R and Q be the set of all real numbers and the set of all rational numbers respectively. Since Q is countable, put $Q = \{x_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, put A_n is the open interval $(x_n - \frac{1}{2^n}, x_n + \frac{1}{2^n})$, and put $A = \bigcup \{A_n : n \in \mathbb{N}\}$, then $A \supset Q$. Let $m(C)$ denote the measure of a subset C of X. It is clear that $m(A) \leq 2$. Put $B = X - A$, then $m(B) = +\infty$. So B is uncountable closed subset of reals R and B consists of irrational numbers. □

Example 2.7. G.Gao's space([5, Page 216]).

Let X be the Bennett-Lutzer's space in Example 2.5. Define an equivalence relation $\sim$ on X as follows.

$x \sim y$ if and only if either $x, y \in Q$ or $x = y$.

Let Y be the quotient space $X/\sim$. That is, Y is the space obtained from X by shrinking the set of all rational numbers to a point. Put $f : X \longrightarrow Y$ is the quotient mapping.

Then

(1) Y is Hausdorff, strongly paracompact. So Y is normal.

(2) Y is not countably compact.

(3) Y is not *Lindelöf*.

(4) f is a closed mapping.

Proof. (1) It is clear that Y is Hausdorff. We only need to prove Y is strongly paracompact. Let $\mathcal{U}$ be an open cover. Pick $x_0 \in Q$. Put $y_0 = f(x_0)$. Pick $U \in \mathcal{U}$ such that $y_0 \in U$. Then $\{U\} \bigcup \{\{y\} : y \in Y - U\}$ is a discrete (hence star-finite) open refinement of $\mathcal{U}$, so Y is strongly paracompact.

(2) Put $\mathcal{U} = \{Y - \{f(\sqrt{2} + n) : n \in \mathbb{N}\}\} \bigcup \{\{f(\sqrt{2} + n)\} : n \in \mathbb{N}\}$. Then $\mathcal{U}$ is a countably infinite open cover of Y. It is easy to see that $\mathcal{U}$ has not any proper subcover. So Y is not countably compact.

(3) Let B be the set in the proof of Proposition 2.6. Then B is an uncountable set consisting of irrational numbers. Put $\mathcal{U} = \{Y - f(B)\} \bigcup \{\{y\} : y \in f(B)\}$. Then $\mathcal{U}$ is a uncountable open cover of Y. It is easy to see that $\mathcal{U}$ has not any proper subcover. So Y is not Lindelöf.

(4) Since Q is a closed subset of X, f is a closed mapping. □

Remark 2.8. By Example 2.7(2) or (3), Claim 1.3 is not true. In addition, G.Gao claimed that Y is T_1. In fact, Y is also normal from Example 2.7(1).

Example 2.9. There exists a closed *Lindelöf* mapping $f : X \longrightarrow Y$, where X is Hausdorff, weak $\overline{\theta}$-refinable, not countably θ-refinable, and Y is strongly paracompact.

Proof. Let X and Y be the spaces in Example 2.5 and Example 2.7 respectively. Then X is Hausdorff, weak $\overline{\theta}$-refinable, not countably θ-refinable from Example 2.5, and Y is strongly paracompact from Example 2.7(1). Let $f : X \longrightarrow Y$ be the quotient mapping. Since Q is a closed countable subset of X, f is a closed Lindelöf mapping. □

Remark 2.10. It follows from Example 2.9 that all covering properties which are between strong paracompactness and countable θ-refinability are not inversely preserved under closed *Lindelöf* mappings even if domain is Hausdorff. So the answer of Question 1.2 is negative.

3 Another Example of Closed Lindelöf Mappings

In this section, we investigate conclusion 1.4.

Example 3.1. Let R, Q and I be the set of all real numbers, the set of all rational numbers and the set of all irrational numbers, respectively. Define a base $\mathcal{B}$ for a new topology τ on R as follows.

$\mathcal{B} = \{\{x\} : x \in I\} \bigcup \{\{x\} \bigcup A : x \in Q,\ A \subset I,\ I - A \ is\ finite\}$.

Then

(1) (R, τ) is not countably θ-refinable.

(2) (R, τ) is weak $\overline{\theta}$-refinable.

Proof. (1) Put $\mathcal{U} = \{\{x\} \bigcup I : x \in Q\}$, then $\mathcal{U}$ is a countable open cover of (R, τ). If (R, τ) is countably θ-refinable, then there exists a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open refinements of $\mathcal{U}$ such that condition (a) in Definition 2.3 holds. We construct a countable subset E_n of R for each $n \in \mathbb{N}$ as follows.

For each $n \in \mathbb{N}$ and each $x \in Q$, there exists $U(n, x) \in \mathcal{U}_n$ with $x \in U(n, x)$. Because $\{x\} \bigcup I$ is the only member of $\mathcal{U}$ that contains x, we know that $U(n, x)$ has the form $U(n, x) = \{x\} \bigcup A(n, x)$, where $A(n, x) \subset I$ and $I - A(n, x)$ is finite. Put $A_n = \bigcap\{U(n, x) : x \in Q\} = \bigcap\{A(n, x) : x \in Q\}$. Then $R - A_n = Q \bigcup (\bigcup\{I - A(n, x) : x \in Q\}$ is countable. Put $E_n = \{y \in R : 0 < ord(y, \mathcal{U}_n) < +\infty\}$. For each $y \in A_n$, $y \in U(n, x)$ for each $x \in Q$, so $ord(y, \mathcal{U}_n) = +\infty$, i.e., $y \notin E_n$. Thus $E_n \subset R - A_n$, so E_n is countable. Thus E_n is constructed.

For each $y \in R$, there exists $n \in \mathbb{N}$ such that $0 < ord(y, \mathcal{U}_n) < \infty$, so $R = \bigcup\{E_n : n \in \mathbb{N}\}$. Thus R is countable. This is a contradiction. So (R, τ) is not countably θ-refinable.

(2) Let $\mathcal{U}$ be an open cover of (R, τ). For each $x \in Q$, pick $U_x \in \mathcal{U}$ such that $x \in U_x$, put $V_x = \{x\} \bigcup (U_x - Q)$. Put $\mathcal{U}_1 = \{V_x : x \in Q\}$, $\mathcal{U}_2 = \{\{x\} : x \in I\}$, $\mathcal{U}_n = \emptyset$ for $n > 2$. Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open partial refinements of $\mathcal{U}$. For each $x \in X$, if $x \in Q$ then $ord(x, \mathcal{U}_1) = 1 < \infty$; if $x \in I$ then $ord(x, \mathcal{U}_2) = 1 < \infty$. It is clear that $\{\bigcup \mathcal{U}_n : n \in \mathbb{N}\}$ is point finite. So (R, τ) is weak $\overline{\theta}$-refinable. □

Example 3.2. Let (R, τ) be the space in Example 3.1. Define an equivalence relation $\sim$ on R as follows.

$x \sim y$ if and only if either $x, y \in Q$ or $x = y$.

Let Z be the quotient space $(R, \tau)/\sim$. That is, Z is the space obtained from (R, τ) by shrinking the set of all rational numbers to a point. Then Z is Hausdorff and compact.

Proof. It is clear that Z is Hausdorff. We only need to prove Z is compact. Let $\mathcal{U}$ be an open cover. Pick $x_0 \in Q$. Put $y_0 = f(x_0)$, where $f : R \longrightarrow Z$ is the quotient mapping. Pick $U \in \mathcal{U}$ such that $y_0 \in U$. Then $Z - U$ is finite, thus $\mathcal{U}$ has a finite subcover. so Z is compact. □

Example 3.3. There exists a closed *Lindelöf* mapping $f : (R, \tau) \longrightarrow Z$, where (R, τ) is weak $\overline{\theta}$-refinable but not countably θ-refinable, and Z is compact.

Proof. Let (R, τ) and Z be the spaces in Example 3.1 and Example 3.2 respectively. Then (R, τ) is weak $\overline{\theta}$-refinable, not countably θ-refinable from Example 3.1, and Z is compact from Example 3.2. Let $f : (R, \tau) \longrightarrow Z$ be the quotient mapping. Since Q is a closed countable subset of (R, τ), f is a closed Lindelöf mapping. □

Remark 3.4. Although Claim 1.3 is not true, Conclusion 1.4 is still true from Example 3.3. Unfortunately, the domain in Example 3.3 is not Hausdorff.

We proved that closed Lindelöf mappings inversely preserve weak $\overline{\theta}$-refinability if domain is regular([6, Theorem 2]). Notice that both the space X in Example 2.9 and the space (R, τ) in Example 3.1 are weak $\overline{\theta}$-refinable. We do not know whether the regularity of domain in [6, Theorem 2] can be omitted. So the following question is still open.

Question 3.5. Do closed *Lindelöf* mappings inversely preserve weak $\overline{\theta}$-refinability if the domain is not assumed to be regular?

Acknowledgements

The author would like to thank the referee for his many valuable amendments and suggestions.

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