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A Parametrization of the Abstract Ramsey Theorem

Una Parametrización del Teorema de Ramsey Abstracto

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Abstract

We give a parametrization with perfect subsets of 2^{∞} of the abstract Ramsey theorem (see [13]). Our main tool is an adaptation, to a more general context of Ramsey spaces, of the techniques developed in [8] by J. G. Mijares in order to obtain the corresponding result within the context of topological Ramsey spaces. This tool is inspired by Todorcevic's abstract version of the combinatorial forcing introduced by Galvin and Prikry in [6], and also by the parametrized version of this combinatorial technique, developed in [12] by Pawlikowski. The main result obtained in this paper (theorem 5 below) turns out to be a generalization of the parametrized Ellentuck theorem of [8], and it yields as corollary that the family of perfectly Ramsey sets corresponding to a given Ramsey space is closed under the Souslin operation. This enabled us to prove a parametrized version of the infinite dimensional Hales-Jewett theorem (see [13]).

Key words and phrases: Ramsey theorem, Ramsey space, parametrization.

Resumen

Damos una parametrización con subconjuntos perfectos de 2^{∞} del teorema de Ramsey abstracto (vea [13]). Para ello adaptamos, a un

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contexto más general de espacios de Ramsey, las técnicas desarrolladas en [8] por J. G. Mijares para obtener el resultado análogo en el contexto de los espacios de Ramsey topológicos. Nuestras herramientas están inspiradas en la versión abstracta dada por Todocervic del *forcing* combinatorio definido por Galvin y Prikry en [6], y también por la versión parametrizada de esta técnica combinatoria, desarrollada en [12] por Pawlikowski. El principal resultado obtenido en el presente trabajo (teorema 5 más adelante) es de hecho una generalización del teorema de Ellentuck parametrizado obtenido en [8], y de él se obtiene como corolario que la familia de los subconjuntos perfectamente Ramsey que corresponden a un espacio de Ramsey dado es cerrada bajo la operación de Souslin. Esto nos permitó demostrar una versión parametrizada del teorema de Hales-Jewett infinito-dimensional.
Palabras y frases clave: Teorema de Ramsey, espacio de Ramsey, parametrización (vea [13]).

1 Introduction

In [13], S. Todorcevic presents an abstract characterization of those topological spaces in which an analog of Ellentuck's theorem [4] can be proven. These are called *topological Ramsey spaces* and the main result about them is referred to in [13] as *abstract Ellentuck theorem.* In [8] J. G. Mijares gives a parametrization with perfect subsets of 2^∞ of the abstract Ellentuck theorem, obtaining in this way new proofs of parametrized versions of the Galvin-Prikry theorem [6] (see [9]) and of Ellentuck's theorem (see [12]), as well as a parametrized version of Milliken's theorem [10].

But topological Ramsey spaces are a particular kind of a more general type of spaces, in which the Ramsey property can be characterized in terms of the abstract Baire property. These are called *Ramsey spaces.* One of such spaces, known as the *Hales-Jewett space*, is described below (for a more complete description of this – non topological– Ramsey space, see [13]). S. Todorcevic has given a characterization of Ramsey spaces which is summed up in a result known as the *abstract Ramsey theorem.* It tunrs out that the abstract Ellentuck theorem is a consequence of the abstract Ramsey theorem (see [13]). Definitions of all this concepts will be given below.

In this work we adapt in a natural way the methods used in [8] and inspired by [6], [11], and [13], namely, combinatorial forcing and its properties, to the context of Ramsey spaces in order to obtain a parametrized version of the abstract Ramsey theorem. In this way, we not only generalize the results obtained in [8] but we also obtain, in corollary 1 below, a parametrization of the infinite dimensional version of the Hales-Jewett theorem [7] (see [13]), which is the analog to Ellentuck's theorem corresponding to the Hales-Jewett space. It should be noted that our results differ from those included in [13] concerning parametrized theory in two ways: our parametrization of the abstract Ramsey theorem deals with a parametrized version of the Ramsey property relative to any Ramsey space, and those included in [13] has to do with a parametrized version of the Ramsey property relative to a particular Ramsey space, namely, Ellentuck's space $\mathbb{N}^{[\infty]}$, of all the infinite subsets of $\mathbb{N}$. And, we use perfect sequences of 0's and 1's to perform the parametrizations, instead of the products of finite subsets of $\mathbb{N}$ used in [13]. For a detailed presentation of the parametrization with products of finite subsets of $\mathbb{N}$ of the Ramsey property relative to Ellentuck's space, see [3].

In the next section we summarize the definitions and main results related to Ramsey spaces given by Todorcevic in [13]. In section 3 we introduce the combinatorial forcing adapted to the context of Ramsey spaces and present our main result (theorem 5 below). Finally, we conclude that the generalization of the *perfectly Ramsey property* (see [2] and [12]) to the context of Ramsey spaces is preserved by the Souslin operation (see corollary 4 below).

We'll use the following definitions and results concerning perfect sets and trees (see [12]). For $x = (x_n)_n \in 2^\infty$, $x_{|k} = (x_0, x_1, \ldots, x_{k-1})$. For $u \in 2^{<\infty}$, let $[u] = \{x \in 2^\infty \colon (\exists k)(u = x_{|k})\}$ and let $|u|$ denote the *length* of u. If $Q \subseteq 2^\infty$ is a perfect set, we denote T_Q its asociated perfect tree. For u, $v = (v_0, \ldots, v_{|v|-1}) \in 2^{<\infty}$, we write $u \sqsubseteq v$ to mean $(\exists k \leq |v|)(u = (v_0, v_1, \ldots, v_{k-1}))$. Given $u \in 2^{<\infty}$, let $Q(u) = Q \cap [u(Q)]$, where $u(Q)$ is defined as follows: $\emptyset(Q) = \emptyset$. If $u(Q)$ is already defined, find $\sigma \in T_Q$ such that σ is the $\sqsubseteq$-extension of $u(Q)$ where the first ramification occurs. Then, set $(u^\frown i)(Q) = \sigma^\frown i$, $i = 1, 0$. Where "$^\frown$" is concatenation. Thus, for each n, $Q = \bigcup\{Q(u) \colon u \in 2^n\}$. For $n \in \mathbb{N}$ and perfect sets S, Q, we write $S \subseteq_n Q$ to mean $S(u) \subseteq Q(u)$ for every $u \in 2^n$. Thus "$\subseteq_n$" is a partial order and, if we have chosen $S_u \subseteq Q(u)$ for every $u \in 2^n$, then $S = \bigcup_u S_u$ is perfect, $S(u) = S_u$ and $S \subseteq_n Q$. The *property of fusion* of this order is: if $Q_{n+1} \subseteq_{n+1} Q_n$ for $n \in \mathbb{N}$, then $Q = \cap_n Q_n$ is perfect and $Q \subseteq_n Q_n$ for each n.

2 Abstract Ramsey theory

The following definitions and results are due to Todorcevic (see [13]). Our objects will be structures of the form $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ where $\leq$ and $\leq^0$ are relations on $\mathcal{S}\times\mathcal{S}$ and $\mathcal{R}\times\mathcal{S}$ respectively; and r, s give finite approximations:

$$r\colon \mathcal{R}\times\omega\to\mathcal{AR} \qquad s\colon \mathcal{S}\times\omega\to\mathcal{AS}.$$

We denote $r_n(A)=r(A,n)$, $s_n(X)=s(X,n)$, for $A\in\mathcal{R}$, $X\in\mathcal{S}$, $n\in\mathbb{N}$. The following three axioms are assumed for every $(\mathcal{P},p)\in\{(\mathcal{R},r),(\mathcal{S},s)\}$.

(A.1) $p_0(P)=p_0(Q)$, for all P, $Q\in\mathcal{P}$.

(A.2) $P\neq Q\Rightarrow p_n(P)\neq p_n(Q)$ for some $n\in\mathbb{N}$.

(A.3) $p_n(P)=p_m(Q)\Rightarrow n=m$ and $p_k(P)=p_k(Q)$ if $k<n$.

In this way we can consider elements of $\mathcal{R}$ and $\mathcal{S}$ as infinite sequences $(r_n(A))_{n\in\mathbb{N}}$, $(s_n(X))_{n\in\mathbb{N}}$. Also, if $a\in\mathcal{AR}$ and $x\in\mathcal{AS}$ we can think of a and x as finite sequences $(r_k(A))_{k<n}$, $(s_k(X))_{k<m}$ respectively; with n, m the unique integers such that $r_n(A)=a$ and $s_m(X)=x$. Such n and m are called the *length* of a and the *length* of x, which we denote $|a|$ and $|x|$, respectively.

We say that $b\in\mathcal{AR}$ is an end-extension of $a\in\mathcal{AR}$ and write $a\sqsubseteq b$, if $\forall B\in\mathcal{R}\ [\exists n\ (b=r_n(B))\ \Rightarrow\ \exists m\leq n\ (a=r_m(B))]$. In an analogous way we define the relation $\sqsubseteq$ on $\mathcal{AS}$.

(A.4) **Finitization:** There are relations $\leq_{fin}$ and $\leq^0_{fin}$ on $\mathcal{AS}\times\mathcal{AS}$ and $\mathcal{AR}\times\mathcal{AS}$, respectively, such that:

(1) $\{a\colon a\leq^0_{fin} x\}$ and $\{y\colon y\leq_{fin} x\}$ are finite for all $x\in\mathcal{AS}$.

(2) $X\leq Y$ iff $\forall n\ \exists m\ s_n(X)\leq_{fin} s_m(Y)$.

(3) $A\leq^0 X$ iff $\forall n\ \exists m\ r_n(A)\leq^0_{fin} s_m(X)$.

(4) $\forall a\in\mathcal{AR}\ \forall x,y\in\mathcal{AS}\ [a\leq^0_{fin} x\leq_{fin} y\Rightarrow (a\leq^0_{fin} y)]$.

(5) $\forall a,b\in\mathcal{AR}\ \forall x\in\mathcal{AS}\ [a\sqsubseteq b$ and $b\leq^0_{fin} x\Rightarrow \exists y\sqsubseteq x\ (a\leq^0_{fin} y)]$.

We deal with the *basic sets*

$$[a,Y]=\{A\in\mathcal{R}\colon A\leq^0 Y \text{ and } \exists n\,(r_n(A)=a)\}$$

$$[x,Y]=\{X\in\mathcal{S}\colon X\leq Y \text{ and } \exists n\,(s_n(X)=x)\}$$

for $a\in\mathcal{AR}$, $x\in\mathcal{AS}$ and $Y\in\mathcal{S}$. Notation:

$$[n,Y]=[s_n(Y),Y]$$

Also, we define the *depth of* $a\in\mathcal{AR}$ in $Y\in\mathcal{S}$ by

$$\operatorname{depth}_Y(a)=\begin{cases}\min\{k\colon a\leq^0_{fin} s_k(Y)\}, & \text{if } \exists k\,(a\leq^0_{fin} s_k(Y))\\ -1, & \text{otherwise}\end{cases}$$

The next result is immediate.

Lemma 1. *If $a \sqsubseteq b$ then $depth_Y(a) \leq depth_Y(b)$.*

□

Now we state the last two axioms:

(A.5) **Amalgamation:** $\forall a \in \mathcal{AR}$, $\forall Y \in \mathcal{S}$, if $\text{depth}_Y(a) = d$, then:

(1) $d \geq 0 \Rightarrow \forall X \in [d, Y]$ $([a, X] \neq \emptyset)$.

(2) Given $X \in \mathcal{S}$,

$$(X \leq Y \text{ and } [a, X] \neq \emptyset) \Rightarrow \exists Y' \in [d, Y]\,([a, Y'] \subseteq [a, X])$$

(A.6) **Pigeon hole principle:** Suppose $a \in \mathcal{AR}$ has length l and $\mathcal{O} \subseteq \mathcal{AR}_{l+1} = r_{l+1}(\mathcal{R})$. Then for every $Y \in \mathcal{S}$ with $[a, Y] \neq \emptyset$, there exists $X \in [\text{depth}_Y(a), Y]$ such that $r_{l+1}([a, X]) \subseteq \mathcal{O}$ or $r_{l+1}([a, X]) \subseteq \mathcal{O}^c$.

Definition 1. We say that $\mathcal{X} \subseteq \mathcal{R}$ is $\mathcal{S}$-Ramsey if for every $[a, Y]$ there exists $X \in [\text{depth}_Y(a), Y]$ such that $[a, X] \subseteq \mathcal{X}$ or $[a, X] \subseteq \mathcal{X}^c$. If for every $[a, Y] \neq \emptyset$ there exists $X \in [\text{depth}_Y(a), Y]$ such that $[a, X] \subseteq \mathcal{X}^c$, we say that $\mathcal{X}$ is $\mathcal{S}$-Ramsey null.

Definition 2. We say that $\mathcal{X} \subseteq \mathcal{R}$ is $\mathcal{S}$-Baire if for every $[a, Y] \neq \emptyset$ there exists a nonempty $[b, X] \subseteq [a, Y]$ such that $[b, X] \subseteq \mathcal{X}$ or $[b, X] \subseteq \mathcal{X}^c$. If for every $[a, Y] \neq \emptyset$ there exists a nonempty $[b, X] \subseteq [a, Y]$ such that $[b, X] \subseteq \mathcal{X}^c$, we say that $\mathcal{X}$ is $\mathcal{S}$-meager.

It is clear that every $\mathcal{S}$-Ramsey set is $\mathcal{S}$-Baire and every $\mathcal{S}$-Ramsey null set is $\mathcal{S}$-meager.

Consider $\mathcal{AS}$ with the discrete topology and $\mathcal{AS}^{\mathbb{N}}$ with the completely metrizable product topology. We say that $\mathcal{S}$ is *closed* if it corrresponds to a closed subset of $\mathcal{AS}^{\mathbb{N}}$ via the identification $X \to (s_n(X))_{n \in \mathbb{N}}$.

Definition 3. We say that $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ is a *Ramsey space* if every $\mathcal{S}$-Baire subset of $\mathcal{R}$ is $\mathcal{S}$-Ramsey and every $\mathcal{S}$-meager subset of $\mathcal{R}$ is $\mathcal{S}$-Ramsey null.

Theorem 1 (Abstract Ramsey theorem)**.** *Suppose $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ satisfies (A.1)... (A.6) and $\mathcal{S}$ is closed. Then $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ is a Ramsey space.*

□

Example: The Hales-Jewett space
Fix a countable alphabet $L = \cup_{n \in \mathbb{N}} L_n$ with $L_n \subseteq L_{n+1}$ and L_n finite for all n; fix $v \notin L$ a *variable* and denote W_L and W_{Lv} the semigroups of words over L

and of variable words over L, respectively. Given $X = (x_n)_{n\in\mathbb{N}} \subseteq W_L \cup W_{Lv}$, we say that X is *rapidly increasing* if

$$|x_n| > \sum_{i=0}^{n-1} |x_i|$$

for all $n \in \mathbb{N}$. Put

$$W_L^{[\infty]} = \{X = (x_n)_{n\in\mathbb{N}} \subseteq W_L \colon X \text{ is rapidly increasing }\}$$

$$W_{Lv}^{[\infty]} = \{X = (x_n)_{n\in\mathbb{N}} \subseteq W_{Lv} \colon X \text{ is rapidly increasing }\}$$

By restricting to finite sequences with

$$r_n \colon W_L^{[\infty]} \to W_L^{[n]} \qquad s_n \colon W_{Lv}^{[\infty]} \to W_{Lv}^{[n]}$$

being the natural restriction maps, we have rapidly increasing finite sequences of words or variable words. The *combinatorial subspaces* are defined for every $X \in W_{Lv}^{[\infty]}$ by

$$[X]_L = \{x_n[\lambda_0]^\frown \cdots {}^\frown x_{n_k}[\lambda_k] \in W_L \colon n_o < \cdots < n_k,\ \lambda_i \in L_{n_i}\}$$

$$[X]_{Lv} = \{x_n[\lambda_0]^\frown \cdots {}^\frown x_{n_k}[\lambda_k] \in W_{Lv} \colon n_o < \cdots < n_k,\ \lambda_i \in L_{n_i} \cup \{v\}\}$$

where "$^\frown$" denotes concatenation of words and $x[\lambda]$ is the result of substituting every occurance of v in the variable word x with the letter λ.
For $w \in [X]_L \cup [X]_{Lv}$ we call *support of* w *in* X the unique set $\mathrm{supp}_X(w) = \{n_0 < n_1 < \cdots < n_k\}$ such that $w = x_n[\lambda_0]^\frown \cdots {}^\frown x_{n_k}[\lambda_k]$ as in the definition of the combinatorial subspaces $[X]_L$ and $[X]_{Lv}$. We say that $Y = (y_n)_{n\in\mathbb{N}} \in W_{Lv}^{[\infty]}$ is a *block subsequence* of $X = (x_n)_{n\in\mathbb{N}} \in W_{Lv}^{[\infty]}$ if $\forall n\ y_n \in [X]_{Lv}$ and

$$\max(\mathrm{supp}_X(y_n)) < \min(\mathrm{supp}_X(y_m))$$

whenever $n < m$, and write $Y \leq X$. We define the relation $\leq^0$ on $W_L^{[\infty]} \times W_{Lv}^{[\infty]}$ in the natural way. Then, if $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s) = (W_L^{[\infty]}, W_{Lv}^{[\infty]}, \leq, \leq^0, r, s)$ is as before, where r, s are the restrictions

$$r_n(X) = (x_0, x_1, \ldots, x_{n-1}) \qquad s_n(Y) = (y_0, y_1, \ldots, y_{n-1})$$

then (A.1)…(A.6) hold; particularly, (A.6) is the following well known result:

Theorem 2. *For every finite coloring of* $W_L \cup W_{Lv}$ *and every* $Y \in W_{Lv}^{[\infty]}$ *there exists* $X \leq Y$ *in* $W_{Lv}^{[\infty]}$ *such that* $[X]_L$ *and* $[X]_{Lv}$ *are monochromatic.* □

And as a particular case of theorem 1, we have (see [7])

Theorem 3 (Hales–Jewett). *The field of $W_{Lv}^{[\infty]}$-Ramsey subsets of $W_L^{[\infty]}$ is closed under the Souslin operation and it coincides with the field of $W_{Lv}^{[\infty]}$-Baire subsets of $W_L^{[\infty]}$. Moreover, the ideals of $W_{Lv}^{[\infty]}$-Ramsey null subsets of $W_L^{[\infty]}$ and $W_{Lv}^{[\infty]}$-meager subsets of $W_L^{[\infty]}$ are σ-ideals and they also coincide.* $\square$

3 The parametrization

Let $\mathbb{P}$ be the family of perfect subsets of 2^∞ and let us use the following notation:

$$M \in \mathbb{P} \upharpoonright Q \Leftrightarrow (M \in \mathbb{P}) \wedge (M \subseteq Q).$$

From now on we assume that $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ satisfies (A.1)...(A.6) and $\mathcal{S}$ is closed; that is, it is an Ramsey space. The following are the abstract versions of the *perfectly-Ramsey* and the $\mathbb{P} \times Exp(\mathcal{R})$-*Baire* properties, as defined in [8].

Definition 4. $\Lambda \subseteq 2^\infty \times \mathcal{R}$ is *perfectly $\mathcal{S}$-Ramsey* if for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \upharpoonright Q$ and $X \in [\mathrm{depth}_Y(a), Y]$ with $[a, X] \neq \emptyset$ such that $M \times [a, X] \subseteq \Lambda$ or $M \times [a, X] \subseteq \Lambda^c$. If for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \upharpoonright Q$ and $X \in [\mathrm{depth}_Y(a), Y]$ with $[a, X] \neq \emptyset$ such that $M \times [a, X] \subseteq \Lambda^c$, then we say that Λ is *perfectly $\mathcal{S}$-Ramsey null.*

Definition 5. $\Lambda \subseteq 2^\infty \times \mathcal{R}$ is *perfectly $\mathcal{S}$-Baire* if for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \upharpoonright Q$ and $[b, X] \subseteq [a, Y]$ such that $M \times [b, X] \subseteq \Lambda$ or $M \times [b, X] \subseteq \Lambda^c$. If for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \upharpoonright Q$ and $[b, X] \subseteq [a, Y]$ such that $M \times [b, X] \subseteq \Lambda^c$, then we say that Λ is *perfectly $\mathcal{S}$-meager.*

Now, the natural extension of combinatorial forcing will be given. From now on fix $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{AR}$ and $\Lambda \subseteq 2^\infty \times \mathcal{R}$. For every $X \in \mathcal{S}$ let

$$\mathcal{AR}[X] = \{b \in \mathcal{AR} \colon [b, X] \neq \emptyset\}.$$

Combinatorial forcing 1 Given $Q \in \mathbb{P}$, $Y \in \mathcal{S}$ and $(u, a) \in 2^{<\infty} \times \mathcal{AR}[Y]$; we say that (Q, Y) *accepts* (u, a) if for every $x \in Q(u)$ and for every $B \in [a, Y]$ there exist integers k and m such that $(x_{|k}, r_m(B)) \in \mathcal{F}$.

Combinatorial forcing 2 Given $Q \in \mathbb{P}$, $Y \in \mathcal{S}$ and $(u, a) \in 2^{<\infty} \times \mathcal{AR}[Y]$; we say that (Q, Y) *accepts* (u, a) if $Q(u) \times [a, Y] \subseteq \Lambda$.

For both combinatorial forcings we say that (Q,Y) *rejects* (u,a) if for every $M \in \mathbb{P} \upharpoonright Q(u)$ and for every $X \leq Y$ compatible with a; (M,X) does not accept (u,a). Also, we say that (Q,Y) *decides* (u,a) if it accepts or rejects it.

The following lemmas hold for both combinatorial forcings.

Lemma 2. *a) If (Q,Y) accepts (rejects) (u,a) then (M,X) also accepts (rejects) (u,a) for every $M \in \mathbb{P} \upharpoonright Q(u)$ and for every $X \leq Y$ compatible with a.*

b) If (Q,Y) accepts (rejects) (u,a) then (Q,X) also accepts (rejects) (u,a) for every $X \leq Y$ compatible with a.

c) For all (u,a) and (Q,Y) with $[a,Y] \neq \emptyset$, there exist $M \in \mathbb{P} \upharpoonright Q$ and $X \leq Y$ compatible with a, such that (M,X) decides (u,a).

d) If (Q,Y) accepts (u,a) then, for every $b \in r_{|a|+1}([a,Y])$, (Q,Y) accepts (u,b).

e) If (Q,Y) rejects (u,a) then there exists $X \in [depth_Y(a),Y]$ such that (Q,Y) does not accept (u,b) for every $b \in r_{|a|+1}([a,X])$.

f) (Q,Y) accepts (rejects) (u,a) iff (Q,Y) accepts (rejects) (v,a) for every $v \in 2^{<\infty}$ such that $u \sqsubseteq v$.

Proof. (a) and (b) follow from the inclusion: $M(u) \times [a,X] \subseteq Q(u) \times [a,Y]$, if $X \leq Y$ and $M \subseteq Q(u)$.

(c) Suppose that we have (Q,Y) such that for every $M \in \mathbb{P} \upharpoonright Q$ and every $X \leq Y$ compatible with a, (M,X) does not decide (u,a). Then (M,X) does not accept (u,a), if $M \in \mathbb{P} \upharpoonright Q(u)$; i.e. (Q,Y) rejects (u,a).

(d) Follows from: $a \sqsubseteq b$ and $[a,Y] \subseteq [b,Y]$, if $b \in r_{|a|+1}([a,Y])$.

(e) Suppose (Q,Y) rejects (u,a) and define $\phi\colon \mathcal{AR}_{|a|+1} \to 2$ by $\phi(b) = 1$ if (Q,Y) accepts (u,b). By (A.6) there exist $X \in [\mathrm{depth}_Y(a),Y]$ such that ϕ is constant in $r_{|a|+1}([a,X])$. If $\phi(r_{|a|+1}([a,X])) = 1$ then (Q,X) accepts (u,a), which contradicts (Q,Y) rejects (u,a) (by part (b)). The completes the proof of (e).

(f) $(\Leftarrow)$Obvious.

$(\Rightarrow)$ Follows from the inclusion: $Q(v) \subseteq Q(u)$, if $u \sqsubseteq v$. $\square$

We say that a sequence $([n_k, Y_k])_{k\in\mathbb{N}}$ is a *fusion sequence* if:

1. $(n_k)_{k\in\mathbb{N}}$ is nondecreasing and converges to ∞.

2. $X_{k+1} \in [n_k, X_k]$ for all k.

Note that since $\mathcal{S}$ is closed, for every fusion sequence $([n_k, Y_k])_k \in \mathbb{N}$ there exist a unique $Y \in \mathcal{S}$ such that $s_{n_k}(Y) = s_{n_k}(X_k)$ and $Y \in [n_k, X_k]$ for all k. Y is called the *fusion* of the sequence and is denoted $lim_k X_k$.

Lemma 3. *Given $P \in \mathbb{P}$, $Y \in \mathcal{S}$ and $N \geq 0$; there exist $Q \in \mathbb{P} \restriction P$ and $X \leq Y$ such that (Q, X) decides every $(u, a) \in 2^{<\infty} \times \mathcal{AR}[X]$ with $N \leq depth_X(a) \leq |u|$.*

Proof. We build sequences $(Q_k)_k$ and $(Y_k)_k$ such that:

1. $Q_0 = P$, $Y_0 = Y$.
2. $n_k = N + k$.
3. (Q_{k+1}, Y_{k+1}) decides every $(u, b) \in 2^{n_k} \times \mathcal{AR}[Y_k]$ with $\text{depth}_{Y_k}(b) = n_k$.

Suppose we have defined (Q_k, Y_k). List

$$\{b_0, \dots, b_r\} = \{b \in \mathcal{AR}[Y_k] \colon \text{depth}_{Y_k}(b) = n_k\}$$

and $\{u_0, \dots, u_{2^{n_k}-1}\} = 2^{n_k}$. By lemma 1(c) there exist $Q_k^{0,0} \in \mathbb{P} \restriction Q_k(u_0)$ and $Y_k^{0,0} \in [n_k, Y_k]$ compatible with b_0 such that $(Q_k^{0,0}, Y_k^{0,0})$ decides (u_0, b_0). In this way we can obtain $(Q_k^{i,j}, Y_k^{i,j})$ for every $(i, j) \in \{0, \dots, 2^{n_k} - 1\} \times \{0, \dots, r\}$, which decides (u_i, b_j) and such that $Q_k^{i,j+1} \in \mathbb{P} \restriction Q_k^{i,j}(u_i)$, $Y_k^{i,j+1} \leq Y_k^{i,j}$ is compatible with b_{j+1}, $Q_k^{i+1,0} \in \mathbb{P} \restriction Q_k(u_{i+1})$ and $Y_k^{i+1,0} \leq Y_k^{i,r}$.
Define

$$Q_{k+1} = \bigcup_{i=0}^{2^{n_k}-1} Q_k^{i,r} \qquad , \qquad Y_{k+1} = Y_k^{2^{n_k}-1,r}$$

Then, given $(u, b) \in 2^{n_k} \times \mathcal{AR}[Y_{k+1}]$ with $\text{depth}_{Y_{k+1}}(b) = n_k = \text{depth}_{Y_k}(b)$, there exist $(i, j) \in \{0, \dots, 2^{n_k} - 1\} \times \{0, \dots, r\}$ such that $u = u_i$ and $b = b_j$. So $(Q_k^{i,j}, Y_k^{i,j})$ decides (u, b) and, since

$$Q_{k+1}(u_i) = Q_k^{i,r} \subseteq Q_k^{i,j}(u_i) \subseteq Q_k^{i,j} \text{ and } Y_{k+1} \leq Y_k$$

we have (Q_{k+1}, Y_{k+1}) decides (u, b) (by lemma 1(a)) We claim that $Q = \cap_k Q_k$ and $X = lim_k Y_k$ are as required: given $(u, a) \in 2^{<\infty} \times \mathcal{AR}[X]$ with $N \leq \text{depth}_X(a) \leq |u|$, we have $\text{depth}_X(a) = n_k = \text{depth}_{Y_k}(a)$ for some k. Then, if $|u| = n_k$, (Q_{k+1}, Y_{k+1}) from the construction of X decides (u, a) and hence (Q, X) decides (u, a). If $|u| > n_k$ (Q, X) decides (u, a) by lemma 1(f). $\square$

Lemma 4. *Given $P \in \mathbb{P}$, $Y \in \mathcal{S}$, $(u,a) \in 2^{<\infty} \times \mathcal{AR}[Y]$ with depth$_Y(a) \leq |u|$ and (Q,X) as in lemma 2 with $N =$ depth$_Y(a)$; if (Q,X) rejects (u,a) then there exist $Z \leq X$ such that (Q,Z) rejects (v,b) if $u \sqsubseteq v$, $a \sqsubseteq b$ and depth$_Z(b) \leq |v|$.*

Proof. Let's build a fusion sequence $([n_k, Z_k])_k$, with $n_k = |u| + k$. Let $Z_0 = X$. Then (Q,Z_0) rejects (u,a) (and by lemma 1(f) it rejects (v,a) if $u \sqsubseteq v$). Suppose we have (Q,Z_k) which rejects every (v,b) with $v \in 2^{n_k}$ extending u, $a \sqsubseteq b$ and $\text{depth}_{Z_k}(b) \leq n_k$. List $\{b_0,\ldots,b_r\} = \{b \in \mathcal{AR}[Z_k] \colon a \sqsubseteq b \text{ and } \text{depth}_{Z_k}(b) \leq n_k\}$ and $\{u_0,\ldots,u_s\}$ the set of all $v \in 2^{n_k+1}$ extending u. By lemma 1(f) (Q,Z_k) rejects (u_i,b_j), for every $(i,j) \in \{0,\ldots,s\} \times \{0,\ldots,r\}$. Use lemma 1(e) to find $Z_k^{0,0} \in [n_k, Z_k]$ such that $(Q, Z_k^{0,0})$ rejects (u_0,b) if $b \in r_{|b_0|+1}([b_0, Z_k^{0,0}])$. In this way, for every $(i,j) \in \{0,\ldots,s\} \times \{0,\ldots,r\}$, we can find $Z_k^{i,j} \in [n_k, Z_k]$ such that $Z_k^{i,j+1} \in [n_k, Z_k^{i,j}]$, $Z_k^{i+1,0} \in [n_k, Z_k^{i,r}]$ and $(Q, Z_k^{i,j})$ rejects (u_i, b) if $b \in r_{|b_j|+1}([b_j, Z_k^{i,j}])$. Define $Z_{k+1} = Z_k^{s,r}$. Note that if $(v,b) \in 2^{<\infty} \times \mathcal{AR}[Z_{k+1}]$, $a \sqsubseteq b$, $u \sqsubseteq v$ and $\text{depth}_{Z_{k+1}}(b) = n_k + 1$ then $v = u_i$ for some $i \in \{0,\ldots,s\}$ and $b = r_{|b|}(A)$, $a = r_{|a|}(A)$ for some $A \leq^0 Z_{k+1}$; by (A.4)(5) there exist $m \leq n_k$ such that $b' = r_{|b|-1}(A) \leq^0_{fin} s_m(Z_{k+1})$, so $\text{depth}_{Z_{k+1}}(b') \leq n_k$, i.e. $b' = b_j$ for some $j \in \{0,\ldots,r\}$. Then $b \in r_{|b_j|+1}([b_j, Z_k^{i,j}])$. Hence, by lemma 1(f), (Q, Z_{k+1}) rejects (v,b). Then $Z = lim_k Z_k$ is as required: given (v,b) with $u \sqsubseteq v$, $a \sqsubseteq b$ and $\text{depth}_Z(b) \leq |v|$ then $\text{depth}_Z(b) = \text{depth}_Y(a) + k \leq n_k$ for some k and $b \in r_{|b_j|+1}([b_j, Z_k^{i,j}])$ for some $j \in \{0,\ldots,r\}$ from the construction of Z (again, by (A.4)(5)). So (Q,Z_k) (from the construction of Z) rejects (v,b) and, by lemma 1(a), (Q,Z) also does it. □

The following theorem is an extension of theorem 3 of [8] and its proof is analogous.

Theorem 4. *For every $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{AR}$, $P \in \mathbb{P}$, $Y \in \mathcal{S}$ and $(u,a) \in 2^{<\infty} \times \mathcal{AR}$ there exist $Q \in \mathbb{P} \upharpoonright P$ and $X \leq Y$ such that one of the following holds:*

1. *For every $x \in Q$ and $A \in [a,X]$ there exist integers k, $m > 0$ such that $(x_{|k}, r_m(A)) \in \mathcal{F}$.*
2. *$(T_Q \times \mathcal{AR}[X]) \cap \mathcal{F} = \emptyset$.*

Proof. Whitout loss of generality, we can assume $(u,a) = (\langle\rangle, \emptyset)$. Consider combinatorial forcing 1. Let (Q,X) as in lemma 3 ($N = 0$). If (Q,X) accepts $(\langle\rangle, \emptyset)$, part (1) holds. If (Q,X) rejects $(\langle\rangle, \emptyset)$, use lemma 4 to obtain $Z \leq X$ such that (Q,X) rejects $(u,a) \in 2^{<\infty} \times \mathcal{AR}[Z]$ if $\text{depth}_Z(b) \leq |u|$. If $(t,b) \in$

$(T_Q \times \mathcal{AR}[Z]) \cap \mathcal{F}$, find $u_t \in 2^{<\infty}$ such that $Q(u_t) \subseteq Q \cap [t]$. Thus, (Q, Z) accepts (u, b). In fact: for $x \in Q(u_t)$ and $B \in [b, Z]$ we have $(x_{|k}, r_m(A)) = (t, b) \in \mathcal{F}$ if $k = |t|$ and $m = |b|$. By lemma 2(f), (Q, Z) accepts (v, b) if $u_t \sqsubseteq v$ and $\text{depth}_Z(b) \leq |v|$. But this is a contradiction with the choice of Z. Hence, $(T_Q \times \mathcal{AR}[X]) \cap \mathcal{F} = \emptyset$. $\square$

The following theorem, which extends theorem 2 of [8], is our main result.

Theorem 5. *For $\Lambda \subseteq 2^\infty \times \mathcal{R}$ we have:*

1. *Λ is perfectly $\mathcal{S}$-Ramsey iff it is perfectly $\mathcal{S}$-Baire.*
2. *Λ is perfectly $\mathcal{S}$-Ramsey null iff it is perfectly $\mathcal{S}$-meager.*

Proof. (1) We only have to prove the implication from right to left. Suppose that $\Lambda \subseteq 2^\infty \times \mathcal{R}$ is perfectly $\mathcal{S}$-Baire and fix $Q \times [a, Y]$. Again, whitout a loss of generality, we can assume $a = \emptyset$. Using combinatorial forcing and lemma 3, we have the following:

Claim 1. *Given $\hat{\Lambda} \subseteq 2^\infty \times \mathcal{R}$, $P \in \mathbb{P}$ and $Y \in \mathcal{S}$, there exists $Q \in \mathbb{P} \upharpoonright P$ and $X \leq Y$ such that for each $(u, b) \in 2^{<\infty} \times \mathcal{AR}[X]$ with $depth_X(b) \leq |u|$ one of the following holds:*

i) $Q(u) \times [b, X] \subseteq \hat{\Lambda}$

ii) $R \times [b, Z] \not\subseteq \hat{\Lambda}$ for every $R \subseteq Q(u)$ and every $Z \leq X$ compatible with b.

$\square$

By applying the claim to Λ, P and Y, we find $Q_1 \in \mathbb{P} \upharpoonright P$ and $X_1 \leq Y$ such that for each $(u, b) \in 2^{<\infty} \times \mathcal{AR}[X_1]$ with $\text{depth}_{X_1}(b) \leq |u|$ one of the following holds:

- $Q_1(u) \times [b, X_1] \subseteq \Lambda$ or
- $R \times [b, Z] \not\subseteq \Lambda$ for every $R \subseteq Q_1(u)$ and every $Z \leq X_1$ compatible with b.

For each $t \in T_{Q_1}$, choose $u_1^t \in 2^{<\infty}$ with $u_1^t(Q_1) \sqsubseteq t$. If we define the family

$$\mathcal{F}_1 = \{(t, b) \in T_{Q_1} \times \mathcal{AR}[X_1] \colon Q_1(u_1^t) \times [b, X_1] \subseteq \Lambda\}$$

then we find $S_1 \subseteq Q_1$and $Z_1 \leq X_1$ as in theorem 4. If (1) of theorem 4 holds, we are done. If part (2) holds, apply the claim to Λ^c, S_1 and Z_1 to find $Q_2 \in \mathbb{P} \upharpoonright P$ and $X_2 \leq Y$ such that for each $(u, b) \in 2^{<\infty} \times \mathcal{AR}[X_2]$ with $\text{depth}_{X_2}(b) \leq |u|$ one of the following holds:

- $Q_2(u) \times [b, X_2] \subseteq \Lambda^c$ or
- $R \times [b, Z] \nsubseteq \Lambda^c$ for every $R \subseteq Q_2(u)$ and every $Z \leq X_2$ compatible with b.

Again, for each $t \in T_{Q_2}$, choose $u_2^t \in 2^{<\infty}$ with $u_2^t(Q_2) \sqsubseteq t$; define the family

$$\mathcal{F}_2 = \{(t, b) \in T_{Q_2} \times \mathcal{AR}[X_2] \colon Q_2(u_2^t) \times [b, X_1] \subseteq \Lambda\}$$

and find $S_2 \subseteq Q_2$ and $Z_2 \leq X_2$ as in theorem 4. If (1) holds, we are done and part (2) is not possible since Λ is perfectly $\mathcal{S}$-Baire (see [8]). This proves (1). To see part (2), notice that, as before, we only have to prove the implication from right to left, which follows from part (1) and the fact that Λ is perfectly $\mathcal{S}$-meager. This completes the proof of theorem 5. □

Corollary 1 (Parametrized infinite dimensional Hales-Jewett theorem)**.** *For $\Lambda \subseteq 2^\infty \times W_L^{[\infty]}$ we have:*

1. *Λ is perfectly Ramsey iff it has the $\mathbb{P} \times W_{Lv}^{[\infty]}$-Baire property.*
2. *Λ is perfectly Ramsey null iff it is $\mathbb{P} \times W_{Lv}^{[\infty]}$-meager .*

□

Making $\mathcal{R} = \mathcal{S}$ in $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$, we obtain the following:

Corollary 2 (Mijares)**.** *If $(\mathcal{R}, \leq, (p_n)_{n\in\mathbb{N}})$ satisfies (A.1)...(A.6) and $\mathcal{R}$ is closed then:*

1. *$\Lambda \subseteq \mathcal{R}$ is perfectly Ramsey iff has the $\mathbb{P} \times Exp(\mathcal{R})$-Baire property.*
2. *$\Lambda \subseteq \mathcal{R}$ is perfectly Ramsey null iff is $\mathbb{P} \times Exp(\mathcal{R})$-meager.*

□

Corollary 3 (Pawlikowski)**.** *For $\Delta \subseteq 2^\infty \times \mathbb{N}^{[\infty]}$ we have:*

1. *Λ is perfectly Ramsey iff it has the $\mathbb{P} \times Exp(\mathbb{N}^{[\infty]})$-Baire property.*
2. *Λ is perfectly Ramsey null iff it is $\mathbb{P} \times Exp(\mathbb{N}^{[\infty]})$-meager .*

□

Now we will show that the family of perfectly $\mathcal{S}$-Ramsey and perfectly $\mathcal{S}$-Ramsey null subsets of $2^\infty \times \mathcal{R}$ are closed under the Souslin operation. Recall that the result of applying the Souslin operation to a given $(\Lambda_a)_{a\in\mathcal{AR}}$ is

$$\bigcup_{A\in\mathcal{R}} \bigcap_{n\in\mathbb{N}} \Lambda_{r_n(A)}$$

Proposition 1. The perfeclty $\mathcal{S}$-Ramsey null subsets of $2^\infty \times \mathcal{R}$ form a σ-ideal.

Proof. This proof is also analogous to its corresponding version in [8] (lemma 4). So we just expose the main ideas. Given an increasing sequence of perfectly $\mathcal{S}$-Ramsey null subsets of $2^\infty \times \mathcal{R}$ and $P \times [\emptyset, Y]$, we proceed as in lemma 3 to build fusion sequences $(Q_n)_n$ and $[n+1, X_n]$ such that

$$Q_n \times [b, X_n] \cap \Lambda_n = \emptyset$$

for every $n \in \mathbb{N}$ and $b \in \mathcal{AR}[X_n]$ with $\text{depth}_{X_n}(b) \leq n$. Thus, if $Q = \cap_n Q_n$ and $X = lim_n X_n$, we have $Q \times [\emptyset, X] \cap \bigcup_n \Lambda_n = \emptyset$. □

Recall that given a set X, two subsets A, B of X are *compatible* with respect to a family $\mathcal{F}$ of subsets X if there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$. And $\mathcal{F}$ is *M-like* if for $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| < |\mathcal{F}|$, every member of $\mathcal{F}$ which is not compatible with any member of $\mathcal{G}$ is compatible with $X \setminus \bigcup \mathcal{G}$. A σ-algebra $\mathcal{A}$ of subsets of X together with a σ-ideal $\mathcal{A}_0 \subseteq \mathcal{A}$ is a *Marczewski pair* if for every $A \subseteq X$ there exists $\Phi(A) \in \mathcal{A}$ such that $A \subseteq \Phi(A)$ and for every $B \subseteq \Phi(A) \setminus A$, $B \in \mathcal{A} \to B \in \mathcal{A}_0$. The following is a well known fact:

Theorem 6 (Marczewski)**.** *Every σ-algebra of sets which together with a σ-ideal is a Marczeswki pair, is closed under the Souslin operation.*

□

Let $\mathcal{E}(\mathcal{S}) = \{[n, Y] \colon n \in \mathbb{N},\, Y \in \mathcal{S}\}$.

Proposition 2. If $|\mathcal{S}| = 2^{\aleph_0}$, then the family $\mathcal{E}(\mathcal{S})$ is M-like.

Proof. Consider $\mathcal{B} \subseteq \mathcal{E}(\mathcal{S})$ with $|\mathcal{B}| < |\mathcal{E}(\mathcal{S})| = 2^{\aleph_0}$ and suppose that $[n, Y]$ is not compatible with any member of $\mathcal{B}$, i. e. for every $[m, Y'] \in \mathcal{B}$, $[m, Y'] \cap [n, Y]$ does not contain any member of $\mathcal{E}(\mathcal{S})$. We claim that $[n, Y]$ is compatible with $\mathcal{R} \smallsetminus \bigcup \mathcal{B}$. In fact:
Since $|\mathcal{B}| < 2^{\aleph_0}$, $\bigcup \mathcal{B}$ is $\mathcal{S}$-Baire (it is $\mathcal{S}$-Ramsey). So, there exist $[b, X] \subseteq [n, Y]$ such that:

1. $[b, X] \subseteq \bigcup \mathcal{B}$ or
2. $[b, X] \subseteq \mathcal{R} \smallsetminus \bigcup \mathcal{B}$

(1) is not possible because $[n, Y]$ is not compatible with any member of $\mathcal{B}$. And (2) implies that $[n, Y]$ is compatible with $\mathcal{R} \smallsetminus \bigcup \mathcal{B}$. □

As consequences of the previous proposition and theorem 6, the following facts hold.

Corollary 4. *If $|\mathcal{S}| = 2^{\aleph_0}$, then the family of perfectly $\mathcal{S}$-Ramsey subsets of $2^\infty \times \mathcal{R}$ is closed under the Souslin operation.*

□

Corollary 5. *The field of perfectly $W_{Lv}^{[\infty]}$-Ramsey subsets of $2^\infty \times W_L^{[\infty]}$ is closed under the Souslin operation.*

□

Now we would like to point out an interesting consequence of these facts.

Given $u \in 2^{<\infty}$, remember that the set

$$[u] = \{x \in 2^\infty : (\exists k)(u = x_{|k})\}$$

is a basic open neighborhood of the metric topology of 2^∞.

Also notice that given $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$, we can consider $\mathcal{R}$ as a metric space. Actually, if we regard $\mathcal{AR}$ as a discrete space, then by identifying each $A \in \mathcal{R}$ with the sequence $(r_n(A))_n$ we can consider $\mathcal{R}$ as a metric subspace of $\mathcal{AR}^{\mathbb{N}}$, with the product topology. The basic open neighborhoods of the metric topology of $\mathcal{R}$ are of the form:

$$[a] = \{A \in \mathcal{R} : (\exists n)(a = r_n(A))\}, \text{for } a \in \mathcal{AR}.$$

It is easy to prove the following:

Lemma 5. *$\forall u \in 2^{<\infty}$, $\forall a \in \mathcal{AR}$ the set $[u] \times [a]$ is perfectly S-Baire.*

□

Thus, corollary 4 yields:

Corollary 6. *Consider $2^\infty \times \mathcal{R}$ with the product topology, regarding 2^∞ and $\mathcal{R}$ as metric spaces (as described above). If $|\mathcal{S}| = 2^{\aleph_0}$, then every analytic subset of $2^\infty \times \mathcal{R}$ is perfectly $\mathcal{S}$-Ramsey. In particular, every Borel subset of $2^\infty \times \mathcal{R}$ is perfectly $\mathcal{S}$-Ramsey.*

□

Corollary 7. *Consider $2^\infty \times W_L^{[\infty]}$ with the product topology, regarding 2^∞ and $W_L^{[\infty]}$ as metric spaces (as described above). Then, every analytic subset of $2^\infty \times W_L^{[\infty]}$ is perfectly $W_{Lv}^{[\infty]}$-Ramsey. In particular, every Borel subset of $2^\infty \times W_L^{[\infty]}$ is perfectly $W_{Lv}^{[\infty]}$-Ramsey.*

□

Finally, making $\mathcal{R} = \mathcal{S}$ in $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$, we obtain these known results:

Corollary 8 (Mijares)**.** *If $(\mathcal{R}, \leq, r)$ satisfies (A.1)... (A.6), $\mathcal{R}$ is closed, and $|\mathcal{R}| = 2^{\aleph_0}$ then the family of perfectly Ramsey subsets of $2^\infty \times \mathcal{R}$ is closed under the Souslin operation.*

□

Corollary 9 (Pawlikowski)**.** *The field of perfectly Ramsey subsets of $2^\infty \times \mathbb{N}^{[\infty]}$ is closed under the Souslin operation.*

□

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