On the Distinguishing Features of the Dobrakov Integral

T.V.Panchapagesan [1]
Departamento de Matemáticas, Facultad de Ciencias
Universidad de los Andes, Mérida, Venezuela.
e-mail: panchapa@@ciens.ula.ve

Abstract

The object of the present article is to describe some of the most important results in the theory of the Dobrakov integral, emphasizing particularly those which are not shared by other classical Lebesgue-type generalizations of the abstract Lebesgue integral.

1 Introduction

Among the various Lebesgue-type integration theories, the important ones are the following:

(a) Integration of scalar functions with respect to a σ-additive scalar measure-usual abstract Lebesgue integral.

(b) Integration of vector functions with respect to a σ-additive scalar measure-the Bochner and the Pettis integrals.

(c) Integration of scalar functions with respect to a σ-additive vector measure-spectral integrals,the Bartle-Dunford-Schwartz integral.

(d) Integration of vector functions with respect to a σ-additive vector measure-the Bartle bilinear and (*) integrals.

(e) Integration of vector functions with respect to a strongly σ-additive operator valued measure of finite variation on a δ-ring-the Dinculeanu integral.

(f) Integration of vector functions with respect to a strongly σ-additive operator valued measure of finite semivariation on a δ-ring-the Dobrakov integral.

To consider the abstract Lebesgue integral, let $(T, \mathcal{S})$ be a measurable space and let $\mu : \mathcal{S} \to [0, \infty]$ or $\mathbb{C}$ be σ-additive with $\mu(\emptyset) = 0$. Let $f : T \to \mathbb{C}$ be an $\mathcal{S}$-measurable function. Then f is μ-integrable if and only if $\int_T |f| dv(\mu) < \infty$ and hence if and only if $|f|$ is $v(\mu)$-integrable. We shall describe this as the property of absolute integrability of the abstract Lebesgue integral. In this terminology, the Bochner and the Dinculeanu integrals generalize the abstract Lebesgue integral so as to maintain the property of absolute integrability. See Section 8 for details.

[1]The research was partially supported by the C.D.C.H.T. project C-586 of the Universidad de los Andes, Mérida, and by the project of international cooperation between CONICIT-Venezuela and CNR-Italy.
1991 AMS subject classification:28-02,28B05,46G10

Again, for a μ-integrable scalar function f, the set function $\nu(.) = \int_{(\cdot)} f d\mu$ is σ-additive, so that $\sum_1^\infty \nu(E_i)$ is unconditionally convergent whenever $(E_i)_1^\infty$ is a disjoint sequence in $\mathcal{S}$. Let us refer to this as the property of unconditional convergence of the integral. Then the Bartle-Dunford-Schwartz integral, the Bartle bilinear and (*) integrals and the Dobrakov integral have only the property of unconditional convergence.

Dobrakov, adapting suitably the procedure followed in [2,27], developed a theory of integration exhaustively over a long period of 18 years since 1970, and published a series of papers [9-14,21,22,24] on the theme. The object of the present article is to describe some of the most important results in this theory, emphasizing how some of them are not shared by other Lebesgue-type generalizations. Here we also include some of his unpublished results such as Example 1, Theorem 16,etc. The present work is elaborated in our lecture notes [26].

Like the other integrals, the Dobrakov integral too coincides with the abstract Lebesgue integral when the functions and the measure are scalar valued. But, for the vector or operator case, the Bochner,the Dinculeanu and the Bartle (*) integrals are only special cases of the Dobrakov integral. In fact, the reader can observe in Section 8 that the Dobrakov integral gives a complete generalization of the abstract Lebesgue integral,whereas the Bochner and the Dinculeanu integrals give only a partial generalization. Moreover, the Dobrakov integral is related to the topological structure or dimension of the range space of the operators $m(E)$ of the measure m. (See Section 7.)

2 Preliminaries

In this section we fix notation and terminology and give some definitions and results from the theory of vector measures.

T denotes a non void set. $\mathcal{P}$(resp. $\mathcal{S}$) is a δ-ring (resp. a σ-ring) of subsets of T. $\sigma(\mathcal{P})$ denotes the σ-ring generated by $\mathcal{P}$. $I\!\!K$ denotes the scalar field $I\!\!R$ or $\mathbb{C}$. X, Y, Z are Banach spaces over $I\!\!K$ with norm denoted by $|\cdot|$. When X and Y are over the same scalar field $I\!\!K$, $L(X,Y)$ denotes the Banach space of all bounded linear transformations $T : X \to Y$, with $|T| = \sup\{|Tx| : |x| \leq 1\}$. The dual $X^\star$ of X is the Banach space $L(X, I\!\!K)$, $I\!\!K$ being the scalar field of X.

c_0 is the Banach space of all scalar sequences (λ_n) converging to zero, with $|(\lambda_n)| = \sup\limits_n |\lambda_n|$. The Banach space X is said to contain a copy of c_0 if there is a topological isomorphism Φ of c_0 onto a subspace of X, and in that case, we write $c_0 \subset X$. Otherwise, we say that X contains no copy of c_0 and write $c_0 \not\subset X$.

The following theorem of Bessaga-Pełczyński [5] characterizes the Banach spaces X which contain no copy of c_0.

Theorem 1 *The Banach space X contains no copy of c_0 if and only if every formal series $\sum_1^\infty x_n$ of vectors in X satisfying $\sum_1^\infty |x^\star(x_n)| < \infty$ for each $x^\star \in X^\star$ is unconditionally convergent in norm.*

Definition 1 *A set funtion $\gamma : \mathcal{P} \to X$ is called a vector measure if it is additive; i.e., if $\gamma(A \cup B) = \gamma(A) + \gamma(B)$ for $A, B \in \mathcal{P}$ with $A \cap B = \emptyset$. The vector measure $\gamma : \mathcal{P} \to X$ is said to be σ-additive if $|\gamma(\bigcup_1^\infty A_i) - \sum_1^n \gamma(A_i)| \to 0$ as $n \to \infty$, whenever $(A_i)_1^\infty$ is a disjoint sequence in $\mathcal{P}$ with $\bigcup_1^\infty A_i \in \mathcal{P}$. Then $\gamma(\bigcup_1^\infty A_i) = \sum_1^\infty \gamma(A_i)$.*

Definition 2 *A family $(\gamma_i)_{i \in I}$ of X-valued σ-additive vector measures defined on the σ-ring $\mathcal{S}$ is said to be uniformly σ-additive if, given $\epsilon > 0$ and a sequence $A_n \searrow \emptyset$ of members of $\mathcal{S}$, there exists n_0 such that $\sup_{i \in I} |\gamma_i(A_n)| < \epsilon$ for $n \geq n_0$.*

The following theorem plays a crucial role in the definition of the Dobrakov integral.

Theorem 2 (Vitali-Hahn-Saks-Nikodým(VHSN)) *Let $\gamma_n : \mathcal{S} \to X, n = 1, 2, \ldots,$ are σ-additive and let $\lim_n \gamma_n(E) = \gamma(E)$ exist in X for each $E \in \mathcal{S}$. Then γ_n, $n = 1, 2, \ldots,$ are uniformly σ-additive and consequently, γ is a σ-additive vector measure on $\mathcal{S}$.*

The above theorem is proved for a σ-algebra $\mathcal{S}$ in Chapter 1 of [7]. However, the result is easily extended to a σ-ring $\mathcal{S}$.

Definition 3 *A set function $\eta : \mathcal{S} \to [0, \infty]$ is called a submeasure if $\eta(\emptyset) = 0$, η is monotone (i.e., $\eta(A) \leq \eta(B)$ for $A, B \in \mathcal{S}$ with $A \subset B$) and subadditive (i.e., $\eta(A \cup B) \leq \eta(A) + \eta(B)$ for $A, B \in \mathcal{S}$). A submeasure η on $\mathcal{S}$ is said to be continuous (resp. σ-subadditive) if $\eta(A_n) \searrow 0$ whenever the sequence $A_n \searrow \emptyset$ in $\mathcal{S}$ (resp. if $\eta(\bigcup_1^\infty A_n) \leq \sum_1^\infty \eta(A_n)$ for any sequence $(A_n)_1^\infty$ in $\mathcal{S}$).*

Definition 4 *Let $\gamma : \mathcal{P} \to X$ be a vector measure. Then the* semivariation $\|\gamma\| : \sigma(\mathcal{P}) \to [0, \infty]$ *of γ is defined by*

$$\|\gamma\|(A) = \sup\left\{\left|\sum_1^r \lambda_i \gamma(A \cap A_i)\right| : (A_i)_1^r \subset \mathcal{P},\ disjoint,\ \lambda_i \in I\!\!K,\ |\lambda_i| \leq 1,\ r \in I\!\!N\right\}$$

for $A \in \sigma(\mathcal{P})$. We define $\|\gamma\|(T) = \sup\{\|\gamma\|(A) : A \in \sigma(\mathcal{P})\}$. The supremation $\overline{\gamma}$ *of γ is defined by*

$$\overline{\gamma}(A) = \sup\{|\gamma(B)| : B \subset A,\ B \in \mathcal{P}\}$$

for $A \in \sigma(\mathcal{P})$ and we define $\overline{\gamma}(T) = \sup\{\overline{\gamma}(A) : A \in \sigma(\mathcal{P})\}$.

Theorem 3 *Let $\gamma : \sigma(\mathcal{P}) \to X$ be a σ-additive vector measure. Then:*

(i) $\|\gamma\|, \overline{\gamma} : \sigma(\mathcal{P}) \to [0, \infty)$ are continuous σ-subadditive submeasures.

(ii) $\overline{\gamma}(A) \leq \|\gamma\|(A) \leq 4\overline{\gamma}(A)$ for $A \in \sigma(\mathcal{P})$ and moreover, $\|\gamma\|(T) < \infty$.

3 Semivariation and Scalar Semivariation of Operator Valued Measures

Since the Dobrakov integral of a vector valued function is given with respect to an operator valued measure, we devote this section to define an operator valued measure m on $\mathcal{P}$ with values in $L(X,Y)$ and to introduce two extended real valued set functions $\widehat{m}$ and $\|m\|$ associated with m.

Definition 5 *A set function $m : \mathcal{P} \to L(X,Y)$ is called an operator valued measure if $m(\cdot)x : \mathcal{P} \to Y$ is a σ-additive vector measure for each $x \in X$; in other words, if $m(\cdot)$ is σ-additive in the strong operator topology of $L(X,Y)$.*

Unless otherwise specified, m will denote an operator valued measure on $\mathcal{P}$ with values in $L(X,Y)$.

Note 1 A σ-additive scalar measure μ on $\mathcal{P}$ can be considered as an operator valued measure $\mu : \mathcal{P} \to L(X,Y)$ with $X = Y = I\!\!K$, if we define $\mu(E)x = \mu(E).x$ for $E \in \mathcal{P}$ and $x \in X$. A σ-additive vector measure $\gamma : \mathcal{P} \to Y$ can also be considered as an operator valued measure $\gamma : \mathcal{P} \to L(X,Y)$ with $X = I\!\!K$, the scalar field of Y, if we define $\gamma(E)x = x.\gamma(E)$ for $E \in \mathcal{P}$ and $x \in X$. Thus the notion of an operator valued measure subsumes those of σ-additive scalar and vector measures. We shall return to this observation in Note 2 below and later, in Section 8.

Notation 1 We write $(A_i)_1^r$ is (D) in $\mathcal{P}$ to mean that $(A_i)_1^r$ is a finite disjoint sequence of members of $\mathcal{P}$.

The concept given in Definition 4 is suitably modified to define the semivariation of an operator valued measure as below.

Definition 6 *Let $m : \mathcal{P} \to L(X,Y)$ be an operator valued measure. Then we define the semivariation $\widehat{m}(A)$, scalar semivariation $\|m\|(A)$ and variation $v(m,A)$ in $A \in \sigma(\mathcal{P}) \cup \{T\}$ by*

$$\widehat{m}(A) = \sup\left\{\left|\sum_1^r m(A \cap A_i)x_i\right| : (A_i)_1^r \textit{ is } (D) \textit{ in } \mathcal{P},\, x_i \in X,\, |x_i| \le 1,\, r \in I\!\!N\right\},$$

$$\|m\|(A) = \sup\left\{\left|\sum_1^r \lambda_i m(A \cap A_i)\right| : (A_i)_1^r \, is\, (D)\, in\, \mathcal{P},\, \lambda_i \in I\!\!K,\, |\lambda_i| \le 1,\, r \in I\!\!N\right\}$$

and

$$v(m,A) = \sup\left\{\sum_i^r |m(A \cap A_i)| : (A_i)_1^r \, is\, (D)\, in\, \mathcal{P}\right\}.$$

Note that the scalar semivariation $\|m\|$ is the same as that given in Definition 4, if we treat m as an $L(X,Y)$-valued vector measure. Also observe that $\|m\|(T) = \sup\{\|m\|(A) : A \in \sigma(\mathcal{P})\}$, $\widehat{m}(T) = \sup\{\widehat{m}(A) : A \in \sigma(\mathcal{P})\}$ and $v(m,T) = \sup\{v(m,A) : A \in \sigma(\mathcal{P})\}$.

Note 2 When μ(resp. γ) is a σ-additive scalar (resp. vector) measure, by Note 1 μ (resp. γ) can be considered as an operator valued measure, and in that case, $v(\mu,\cdot) = \|\mu\| = \widehat{\mu}$ (resp. $\|\gamma\| = \widehat{\gamma}$).

Note 3 For an operator valued measure m on $\mathcal{P}$, $\|m\| \leq \widehat{m} \leq v(m,.)$. Moreover, $\|m\|(A) = 0 \Leftrightarrow \widehat{m}(A) = 0,\ A \in \sigma(\mathcal{P})$.

4 X-valued $\mathcal{P}$-measurable Functions

Since the integral is defined on a subclass of measurable functions, we give the notion of X-valued $\mathcal{P}$-measurable functions in a very restricted sense, involving only $\sigma(\mathcal{P})$ and not the operator valued measure m on $\mathcal{P}$. This definition is a natural extension of that of measurability for scalar functions (see Halmos [29]). **The success of the integration theory of Dobrakov lies in adopting such a definition (See Definition 8) for $\mathcal{P}$-measurability of X-valued functions, in stead of adapting the classical measurability definition used in the theory of the Bochner integral**.

Definition 7 *An X-valued $\mathcal{P}$-simple function s on T is a function $s : T \to X$ with range a finite set of vectors $x_1, x_2, \ldots, x_k$ such that $f^{-1}(\{x_i\}) \in \mathcal{P}$ whenever $x_i \neq 0,\ i = 1, 2, \ldots, k$. Then an X-valued $\mathcal{P}$-simple function s is of the form*

$$s = \sum_1^r x_i \chi_{A_i},\quad (A_i)_1^r \text{ is } (D) \text{ in } \mathcal{P},\ x_i \neq 0,\ i = 1, 2, \ldots, r. \qquad (\star)$$

Convention 1 Whenever an X-valued $\mathcal{P}$-simple function s is written in the form $s = \sum_1^r x_i \chi_{A_i}$, it is tacitly assumed that the A_i and x_i satisfy the conditions given in $(\star)$.

Notation 2 $S(\mathcal{P}, X) = \{s : \mathcal{P} \to X :\ s\ \mathcal{P}-simple\}$ is a normed space under the operations of pointwise addition and scalar multiplication with norm $\|\cdot\|_T$ given by $\|s\|_T = \max_{t\in T}|s(t)|$. Let $\|f\|_T = \sup\{|f(t)| :\ t \in T\}$ for a function $f : T \to X$. Then $\overline{S}(\mathcal{P}, X)$ denotes the closure of $S(\mathcal{P}, X)$ in the space of all X-valued bounded functions on T with respect to norm $\|.\|_T$.

Definition 8 *An X-valued function f on T is said to be $\mathcal{P}$-measurable if there exists a sequence $(s_n)_1^\infty$ in $S(\mathcal{P}, X)$ such that $s_n(t) \to s(t)$ for each $t \in T$. The set of all X-valued $\mathcal{P}$-measurable functions is denoted by $\mathcal{M}(\mathcal{P}, X)$.*

Notation 3 For a function $f : T \to X$, $N(f)$ denotes the set $\{t \in T :\ f(t) \neq 0\}$.

Clearly, $\mathcal{M}(\mathcal{P}, X)$ is a vector space with respect to the operations of pointwise addition and scalar multiplication. The fact that $\mathcal{M}(\mathcal{P}, X)$ is also closed under the formation of pointwise sequential limits is an immediate consequence of the equivalence of (i) and (ii) of the following strengthened version of the classical Pettis measurability criterion (see Theorem III.6.11 of [27]). To prove the following theorem one can use the notions of X-valued σ-simple and $\mathcal{P}$-elementary functions and modify the arguments given in §1 of [32].

Theorem 4 *For an X-valued function f on T the following are equivalent:*

(i) f *is* $\mathcal{P}$*-measurable.*

(ii) f *has separable range and is weakly* $\mathcal{P}$*-measurable in the sense that* $x^\star f$ *is* $\mathcal{P}$*-measurable for each* $x^\star \in X^\star$*.*

(iii) f *has separable range and* $f^{-1}(E) \cap N(f) \in \sigma(\mathcal{P})$ *for each Borel set* E *in* X*.*

Consequently, if $f_n(t) \to f(t) \in X$ *for each* $t \in T$ *and if* $(f_n)_1^\infty \subset \mathcal{M}(\mathcal{P}, X)$*, then* $f \in \mathcal{M}(\mathcal{P}, X)$*.*

Definition 9 *A sequence* (f_n) *of* X*-valued functions on* T *is said to converge* m*-a.e. on* T *to an* X*-valued function* f*, if there exists a set* $N \in \sigma(\mathcal{P})$ *with* $\|m\|(N) = 0\,(\Leftrightarrow \widehat{m}(N) = 0)$ *such that* $f_n(t) \to f(t)$ *for each* $t \in T \setminus N$*. If* $\eta : \mathcal{P} \to [0, \infty]$ *is a submeasure, similarly we define* η*-a.e. convergence on* T*.*

The following theorem plays a vital role in the development of the theory. For example, see the proof of Theorem 6 below.

Theorem 5 (Egoroff-Lusin) *Let* η *be a continuous submeasure on* $\sigma(\mathcal{P})$ *and let* $(f_n)_1^\infty \subset \mathcal{M}(\mathcal{P}, X)$*. Suppose there is a function* $f_0 \in \mathcal{M}(\mathcal{P}, X)$ *such that* $f_n(t) \to f_0(t)$ η*-a.e. on* T*. If* $F = \bigcup_{n=0}^{\infty} N(f_n)$*, then there exists a set* $N \in \sigma(\mathcal{P})$ *with* $\eta(N) = 0$ *and a sequence* $F_k \nearrow F \setminus N$ *with* $(F_k)_1^\infty \subset \mathcal{P}$ *such that* $f_n \to f_0$ *uniformly on each* F_k*,* $k = 1, 2, \ldots$*.*

5 Dobrakov Integral of $\mathcal{P}$-measurable Functions

As is customary in such theories, we first define the integral for $s \in S(\mathcal{P}, X)$ and then extend the integral to a wider class of $\mathcal{P}$-measurable functions. **The reader should note that the wider class, called the class of the Dobrakov integrable functions, is not obtained as the completion of $S(\mathcal{P}, X)$ with respect to a suitable pseudonorm**. The extension procedure given here is an adaptation of that in [2] and its importance is highlighted in Note 6 below.

Definition 10 *For an* X*-valued* $\mathcal{P}$*-simple function* $s = \sum_1^r x_i \chi_{A_i}$*, we define*

$$m(s, A) = \int_A s\, dm = \sum_{i=1}^{r} m(A \cap A_i) x_i \in Y \text{ for } A \in \sigma(\mathcal{P}) \cup \{T\}.$$

It is easy to show that $m(s, A)$ is well defined.

Proposition 1 *Let* $s \in S(\mathcal{P}, X)$ *and* $A \in \sigma(\mathcal{P}) \cup \{T\}$*. Then:*

(i) $m(s, A) = m(s, A \cap N(s))$*.*

(ii) $m(s, \cdot) : \sigma(\mathcal{P}) \to Y$ *is a* σ*-additive vector measure.*

(iii) $m(\cdot, A) : S(\mathcal{P}, X) \to Y$ *is linear.*

(iv) When A *is fixed,* $m(\cdot, A) : S(\mathcal{P}, X) \to Y$ *is a bounded linear mapping if and only if* $\widehat{m}(A)$ *is finite.*

Since the finiteness of $\widehat{m}$ on $\mathcal{P}$ is essential for the present extension procedure, and since $\widehat{m}(E)$ can be infinite for some $E \in \mathcal{P}$ even though $\mathcal{P}$ is a σ-algebra (see Example 5, p. 517 of [9]), we make the following assumption to hold in the sequel.

<u>BASIC ASSUMPTION 1</u> The operator valued measure m on $\mathcal{P}$ satisfies the hypothesis that $\widehat{m}(E) < \infty$ for each $E \in \mathcal{P}$.

We emphasize that $\widehat{m}(T)$ is not assumed to be finite. If $\widehat{m}(T) < \infty$, then the integral can be easily extended to all $f \in \overline{S}(\mathcal{P}, X)$ (see Notation 2).

Notation 4 With Basic Assumption 1 holding for m, each $s \in S(\mathcal{P}, X)$ is called a $\mathcal{P}$-simple m-integrable function and $S(\mathcal{P}, X)$ is now denoted by $\mathcal{I}_s(m)$, or simply by $\mathcal{I}_s$ when there is no ambiguity about m.

The whole integration theory rests on the following theorem. Therefore, we also include its proof from [26].

Theorem 6 *Let $f \in \mathcal{M}(\mathcal{P}, X)$. Suppose there is a sequence $(s_n)_1^\infty \subset \mathcal{I}_s$ such that $s_n(t) \to f(t)$ m-a.e. on T. Let $\gamma_n(\cdot) = \int_{(\cdot)} s_n \, dm : \sigma(\mathcal{P}) \to Y$, for $n = 1, 2, \ldots$. Then the following are equivalent :*

(i) $\lim_n \gamma_n(A) = \gamma(A)$ *exists in Y for each $A \in \sigma(\mathcal{P})$.*

(ii) γ_n, $n = 1, 2, \ldots$, *are uniformly σ-additive on $\sigma(\mathcal{P})$.*

(iii) $\lim_n \gamma_n(A)$ *exists in Y uniformly with respect to $A \in \sigma(\mathcal{P})$.*

If anyone of (i),(ii) or (iii) holds, then the remaining hold. Moreover, for each $A \in \sigma(\mathcal{P})$, the limit is independent of the sequence (s_n).

Proof. By VHSN, (i)⇒(ii) and obviously (iii)⇒(i). Now let (ii) hold.
Let

$$\eta(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\overline{\gamma_n}(A)}{1 + \overline{\gamma_n}(T)}, \quad A \in \sigma(\mathcal{P}).$$

Then, by Theorem 3(i), η is a continuous submeasure on $\sigma(\mathcal{P})$. Let $F = \bigcup_1^\infty N(s_n) \bigcup N(f)$. By the Egoroff-Lusin theorem there exists a set $N \in \sigma(\mathcal{P})$ with $\eta(N) = 0$ and a sequence $F_k \nearrow F \setminus N$ in $\mathcal{P}$ such that $s_n \to f$ uniformly on each F_k. As $(F \setminus N) \setminus F_k \searrow \emptyset$, given $\epsilon > 0$, by (ii) there exists k_0 such that $\|\gamma_n\|((F \setminus N) \setminus F_{k_0}) < \frac{\epsilon}{3}$ for all n. Since $s_n \to f$ uniformly on F_{k_0} and since $\widehat{m}(F_{k_0}) < \infty$ as $F_{k_0} \in \mathcal{P}$, (there exists n_0 such that $\|s_n - s_p\|_{F_{k_0}} \widehat{m}(F_{k_0}) < \frac{\epsilon}{3}$ for $n, p \geq n_0$. Then it follows that

$$\left| \int_A s_n \, dm - \int_A s_p \, dm \right| \leq \left| \int_{(A \setminus N) \setminus F_{k_0}} s_n \, dm \right| +$$

$$\left| \int_{(A \setminus N) \setminus F_{k_0}} s_p \, dm \right| + \left| \int_{(A \setminus N) \cap F_{k_0}} (s_n - s_p) \, dm \right|$$

$$\leq \|\gamma_n\|(F \setminus N \setminus F_{k_0}) + \|\gamma_p\|(F \setminus N \setminus F_{k_0}) + \|s_n - s_p\|_{F_{k_0}} \widehat{m}(F_{k_0}) < \epsilon$$

for all $n, p \geq n_0$ and for all $A \in \sigma(\mathcal{P})$. Now (iii) holds as Y is complete.

Let $(h_n)_1^\infty \subset \mathcal{I}_s$ with $h_n(t) \to f(t)$ m-a.e. on T. Let $\gamma_n'(\cdot) = \int_{(\cdot)} h_n \, dm, \quad n = 1, 2, \ldots,$ and let anyone of (i), (ii) or (iii) hold for $(\gamma_n')_{n=1}^\infty$. Then by the first part γ_n', $n = 1, 2, \ldots$, are uniformly σ-additive. If $w_{2n} = h_n$, $w_{2n-1} = s_n$, $n = 1, 2, \ldots$,then $w_n(t) \to f(t)$ m-a.e. on T and $\gamma_n''(\cdot) = \int_{(\cdot)} w_n \, dm : \sigma(\mathcal{P}) \to Y$, $n = 1, 2, \ldots$, are uniformly σ-additive. Consequently, by the first part $\lim_n \gamma_n''(A)$ exists in Y for each $A \in \sigma(\mathcal{P})$. Then $\lim_n \gamma_n(A) = \lim_n \gamma_{2n-1}''(A) = \lim_n \gamma_n''(A)$ and similarly, $\lim_n \gamma_n'(A) = \lim_n \gamma_n''(A)$ for $A \in \sigma(\mathcal{P})$. Hence the last part holds.

The above theorem suggests the following definition for integrable functions.

Definition 11 *A function $f \in \mathcal{M}(\mathcal{P}, X)$ is said to be m-integrable (in the sense of Dobrakov) if there exists a sequence $(s_n)_1^\infty$ in $\mathcal{I}_s$ such that $s_n \to f$ m-a.e. on T and such that anyone of conditions (i), (ii) or (iii) of Theorem 6 holds. In that case, we define*

$$\int_A f \, dm = \lim_n \int_A s_n \, dm \ , \quad A \in \sigma(\mathcal{P}) \cup \{T\}.$$

The class of all m-integrable functions is denoted by $\mathcal{I}(m)$, or simply by $\mathcal{I}$ if there is no ambiguity about m.

In the following theorem we list the basic properties of $\mathcal{I}(m)$ and the integral. (Cf. Proposition 1.)

Theorem 7 *I. Let $f \in \mathcal{I}$ and let $\gamma(\cdot) = \int_{(\cdot)} f \, dm : \sigma(\mathcal{P}) \to Y$. Then the following hold:*

- *(a) $\mathcal{I}_s \subset \mathcal{I}$ and for $s \in \mathcal{I}_s$, the integrals given in Definitions 10 and 11 coincide.*
- *(b) $\gamma(\cdot)$ is a Y-valued σ-additive vector measure and hence the Dobrakov integral has the property of unconditional convergence (see Introduction).*
- *(c) $\gamma << \|m\|$ (resp. $\gamma << \widehat{m}$) in the sense that, given $\epsilon > 0$, there exists $\delta > 0$ such that $\|m\|(E) < \delta$ (resp. $\widehat{m}(E) < \delta$) for $E \in \sigma(\mathcal{P})$ implies $|\gamma(E)| < \epsilon$.*
- *(d) $\mathcal{I}$ is a vector space and for a fixed $A \in \sigma(\mathcal{P})$, the mapping $f \to \int_A f \, dm$ is linear on $\mathcal{I}$.*
- *(e) If φ is a bounded $\mathcal{P}$-measurable scalar function on T and if $f \in \mathcal{I}$, then $\varphi.f \in \mathcal{I}$. Consequently, if $s \in \mathcal{I}_s$ and if φ is a scalar valued bounded, $\mathcal{P}$-measurable function which is not $\mathcal{P}$-simple, then $\varphi.s \in \mathcal{I}$ and $\varphi.s \notin \mathcal{I}_s$. Thus, in general $\mathcal{I}_s$ is a proper subset of $\mathcal{I}$.*
- *(f) If f is a bounded $\mathcal{P}$-measurable function on T and if $\widehat{m}$ is continuous on $\mathcal{P}$, then $f\chi_A \in \mathcal{I}$ for each $A \in \mathcal{P}$.*

II. Let $U \in L(Y, Z)$. If $m : \mathcal{P} \to L(X, Y)$ is σ-additive in the strong (resp. uniform) operator topology, then the following hold:

- *(a) $Um : \mathcal{P} \to L(X, Z)$ is σ-additive in the strong (resp. uniform) operator topology.*
- *(b) $\widehat{Um} \leq |U|\widehat{m}$ on $\sigma(\mathcal{P})$. Thus $\widehat{Um}$ is finite on $\mathcal{P}$.*

(c) $\mathcal{I}(m) \subset \mathcal{I}(Um)$ and for $f \in \mathcal{I}(m)$

$$U\left(\int_A f\, dm\right) = \int_A f\, d(Um)\,, \quad A \in \sigma(\mathcal{P}) \cup \{T\}.$$

III. Let $\overline{\mathcal{I}}_s$ denote the closure of $\mathcal{I}_s$ with respect to $\|\cdot\|_T$ in the space of X-valued bounded functions. Then an X-valued function f on T belongs to $\overline{\mathcal{I}}_s$ if and only if the following conditions are satisfied:

(a) f is $\mathcal{P}$-measurable.

(b) $f(T)$ is relatively compact in X.

(c) for each $\epsilon > 0$, there is a set $A \in \mathcal{P}$ such that $\|f\|_{T\setminus A} < \epsilon$.

Consequently, for $A \in \sigma(\mathcal{P})$ with $\widehat{m}(A) < \infty$, and for f in the $\|\cdot\|_T$-closure of bounded integrable functions(=$\overline{B\mathcal{I}}$), $f\chi_A \in \mathcal{I}$. Particularly, if $\widehat{m}(T) < \infty$, then $\overline{\mathcal{I}}_s \subset B\mathcal{I}$ and $\overline{B\mathcal{I}} = B\mathcal{I}$.

Note 4 *The hypothesis that $\widehat{m}$ is continuous on $\mathcal{P}$ is indispensable in (f) of part I of the above theorem. See [22,39].*

Theorem 8 *For $f \in \mathcal{I}$, there exists a sequence (s_n) in $\mathcal{I}_s$ such that $s_n(t) \to f(t)$ and $|s_n(t)| \nearrow |f(t)|$ for all $t \in T$ and such that*

$$\lim_n \int_A s_n\, dm = \int_A f\, dm\,, \quad A \in \sigma(\mathcal{P}) \cup \{T\}. \tag{8.1}$$

Consequently, for each $A \in \sigma(\mathcal{P})$

$$\widehat{m}(A) = \sup\left\{\left|\int_A f\, dm\right| : f \in \mathcal{I}(m),\ \|f\|_A \leq 1\right\}$$

so that

$$\left|\int_A f\, dm\right| \leq \widehat{m}(A).\|f\|_A \quad for\ f \in \mathcal{I}\ and\ for\ A \in \sigma(\mathcal{P}) \cup \{T\}.$$

Note 5 Unlike the abstract Lebesgue integral and the Bochner integral, there is no guarantee that (8.1) holds for any sequence (s_n) in $\mathcal{I}_s$ with $s_n(t) \to f(t)$ and $|s_n(t)| \nearrow |f(t)|$ for all $t \in T$. Cf. Corollary 1 of theorem 15 below.

Theorem 8 is needed to prove the following closure theorem, which is one of the important results that distinguish the Dobrakov integral from the other theories of Lebesge-type integration. See Note 6 below and Section 8.

Theorem 9 (Theorem of closure or interchange of limit and integral) *Let $(f_n)_1^\infty \subset \mathcal{I}$, $f \in \mathcal{M}(\mathcal{P}, X)$ and $f_n \to f$ m-a.e. on T. Let $\gamma_n(\cdot) = \int_{(\cdot)} f_n\, dm : \sigma(\mathcal{P}) \to Y$ for $n = 1, 2, \ldots$. Then the following are equivalent:*

(i) $\lim \gamma_n(A) = \gamma(A)$ *exists in Y for each $A \in \sigma(\mathcal{P})$.*

(ii) γ_n, $n = 1, 2, \ldots$, *are uniformly* σ*-additive.*

(iii) $\lim_n \gamma_n(A) = \gamma(A)$ *exists in* Y *uniformly with respect to* $A \in \sigma(\mathcal{P})$.

If anyone of (i), (ii) or (iii) holds, then the remaining hold, f *is also* m*-integrable and*

$$\int_A f\,dm = \int_A (\lim_n f_n)\,dm = \lim_n \int_A f_n\,dm\ , \quad A \in \sigma(\mathcal{P}). \tag{9.1}$$

Note 6 **(i) The above theorem is called closure theorem since the extension process stops with $\mathcal{I}(m)$. In other words, if the procedure in Theorem 6 is repeated starting with sequences of m-integrable functions instead of sequences in $\mathcal{I}_s(m)$, we only get back the class $\mathcal{I}(m)$ and no new function from $\mathcal{M}(\mathcal{P}, X)$ is included.**

(ii) Equation (9.1) shows that Theorem 9 gives necessary and sufficient conditions for the validity of the interchange of limit and integral. In the classical abstract Lebesgue integral, the bounded and the dominated convergence theorems give only sufficient conditions for its validity. Again, only these theorems are generalized to vector case in the distinct Lebesge-type theories of integration referred to in the introduction. Cf. Theorems 15 and 17 below.

(iii) We also note that $\mathcal{I}(m)$ is the smallest class for which Theorem 9 holds. More precisely, let $J(m)$ be another class of X-valued $\mathcal{P}$-measurable functions which are m-integrable in a different sense, and let the integral of $f \in J(m)$ be denoted by $(J)\int_{(\cdot)} f\,dm$. If $\int_A s\,dm = (J)\int_A s\,dm$ for $s \in \mathcal{I}_s(m)$ and for $A \in \sigma(\mathcal{P})$, and if Theorem 9 holds also for $J(m)$, then $\mathcal{I}(m) \subset J(m)$. This observation will be used later in Section 8 while studying the relation between the Dobrakov and the Bochner (resp. the Dinculeanu) integrals.

We now pass on to the discussion of weakly m-integrable functions.

Definition 12 *A function* $f \in \mathcal{M}(\mathcal{P}, X)$ *is said to be weakly* m*-integrable if* $f \in \mathcal{I}(y^\star m)$ *for each* $y^\star \in Y^\star$.

Theorem 10 *Let* $f \in \mathcal{M}(\mathcal{P}, X)$*. Then:*

(i) If $f \in \mathcal{I}(m)$*, then* f *is weakly* m*-integrable and*

$$y^\star\Big(\int_A f dm\Big) = \int_A f d(y^\star m), A \in \sigma(\mathcal{P}), y^\star \in Y^\star.$$

(ii) Suppose $c_0 \not\subset Y$*. Then* f *is* m*-integrable if and only if it is weakly* m*-integrable.*

(iii) $f \in \mathcal{I}(m)$ *if and only if it is weakly* m*-integrable and for each* $A \in \sigma(\mathcal{P})$ *there exists a vector* $y_A \in Y$ *such that*

$$y^\star(y_A) = \int_A f d(y^\star m)$$

for each $y^\star \in Y^\star$*. In that case,* $y_A = \int_A f dm, A \in \sigma(\mathcal{P})$.

Note 7 *If* $c_0 \subset Y$*, then we can give examples of functions* $f \in \mathcal{M}(\mathcal{P}, X)$ *which are weakly* m*-integrable, but not* m*-integrable. See Example on p.533 of [9].*

6 The L_1-Spaces Associated with m

In the classical Lebesgue-type integration theories, integrable functions are obtained as those measurable functions which belong to the completion of the class of all integrable simple functions with respect to a suitable pseudonorm. But, as the reader would have observed in the previous section, the class $I(m)$ is defined without any reference to a pseudonorm on $I_s(m)$-a distinguished feature of the Dobrakov integral. The proceedure adopted by Dobrakov is a modification of that of Bartle-Dunford-Schwartz [2,22] given in connection with integration of scalar functions with respect to a σ-additive vector measure. See Section 8 below.

Interpreting the semivariation $\widehat{m}(A)$ as $\widehat{m}(\chi_A)$, Dobrakov modified Definition 6 suitably in [8] to define $\widehat{m}(.,T) : \mathcal{M}(\mathcal{P}, X) \to [0, \infty]$ and showed that $\widehat{m}(f,T)$ is a pseudonorm whenever it is finite. Using $\widehat{m}(f,T)$ for $f \in \mathcal{M}(\mathcal{P}, X)$, four distinct complete pseudonormed spaces are defined, which we denote by $\mathcal{L}_1\mathcal{M}(m)$, $\mathcal{L}_1\mathcal{I}(m)$,$\mathcal{L}_1\mathcal{I}_s(m)$ and $\mathcal{L}_1(m)$. The corresponding quotient spaces, with respect to the equivalence relation "$f \sim g$ if and only if $f = g$ m-a.e.", are called the L_1-spaces associated with m. While the classical Lebesgue-type integration theories induce only one L_1-space, Dobrakov's theory, being most general, gives rise to four such spaces, and when the Banach space $c_0 \not\subset Y$ it turns out that all these spaces coincide.

Definition 13 *Let $g \in \mathcal{M}(\mathcal{P}, X)$ and $A \in \sigma(\mathcal{P})$. The L_1-gauge $\widehat{m}(g, A)$ of g on the set A is defined by*

$$\widehat{m}(g, A) = sup\{\left|\int_A f dm\right| : f \in \mathcal{I}_s(m), |f(t)| \leq |g(t)| \text{ for } t \in A\}$$

and the L_1-gauge $\widehat{m}(g,T) = sup\{\widehat{m}(g, A) : A \in \sigma(\mathcal{P})\}$.

The following proposition lists some of the basic properties of $\widehat{m}(\cdot, \cdot)$.

Proposition 2 *Let $f, g \in \mathcal{M}(\mathcal{P}, X)$ and let $A \in \sigma(\mathcal{P})$. Then:*

(i) $\widehat{m}(f, \cdot)$ *is a σ-subadditive submeasure on $\sigma(\mathcal{P})$.*

(ii) $\widehat{m}(f, A) \leq \widehat{m}(g, A)$ *if $|f(t)| \leq |g(t)|$ m-a.e. in A.*

(iii) $\widehat{m}(f, A) = sup\{|\int_A hdm| : h \in \mathcal{I}(m), |h(t)| \leq |f(t)| \text{ for } t \in A\}$ *and consequently,*

$$|\int_A fdm| \leq \widehat{m}(f, A) \text{ for } f \in \mathcal{I}(m).$$

(iv) $\widehat{m}(f + g, A) \leq \widehat{m}(f, A) + \widehat{m}(g, A)$ *for each $A \in \sigma(\mathcal{P})$ and consequently,*

$$\widehat{m}(f + g, T) \leq \widehat{m}(f, T) + \widehat{m}(g, T).$$

In the light of Proposition 2(iv),$\{f \in \mathcal{M}(\mathcal{P}, X) : \widehat{m}(f, T) < \infty\}$ is a pseudonormed space and so we are justified in calling $\widehat{m}(f, T)$ as L_1-psedonorm of f.

Definition 14 *A sequence* $(g_n)_1^\infty$ *of functions in* $\mathcal{M}(\mathcal{P},X)$ *is said to converge in* L_1*-mean (or in* L_1*-pseudonorm) to a function* $g \in \mathcal{M}(\mathcal{P},X)$ *if* $\widehat{m}(g_n - g, T) \to 0$ *as* $n \to \infty$*; the sequence* $(g_n)_1^\infty$ *is said to be Cauchy in* L_1*-mean if* $\widehat{m}(g_n - g_p, T) \to 0$ *as* $n, p \to \infty$*.*

We observe that for $g \in \mathcal{M}(\mathcal{P},X), \widehat{m}(g,T) = 0$ if and only if $g = 0$ m-a.e. on T.

Definition 15 *Two functions* f *and* g *in* $\mathcal{M}(\mathcal{P},X)$ *are said to be* m*-equivalent if* $f = g$ m*-a.e. on* T*. In that case, we write* $f \sim g$ $[m]$*, or simply* $f \sim g$ *when there is no ambiguity about* m*.*

Obviously, $\sim$ is an equivalence relation on $\mathcal{M}(\mathcal{P},X)$ and for $f, g \in \mathcal{M}(\mathcal{P},X)$, $f \sim g$ if and only if $\widehat{m}(f-g,T) = 0$. Also it is easy to verify that L_1-mean convergence determines the limit uniquely in the equivalence classes of $\mathcal{M}(\mathcal{P},X)$.

Theorem 11 *Let* $(f_n)_1^\infty \subset \mathcal{M}(\mathcal{P},X)$ *be Cauchy in* L_1*-mean. Then:*

(i) There exists $f \in \mathcal{M}(\mathcal{P},X)$ *such that* $f_n \to f$ *in* L_1*-mean.*

(ii) If each f_n *is* m*-integrable, then the same is true for* f*.*

(iii) If the submeasure $\widehat{m}(f_n,\cdot)$ *is continuous on* $\sigma(\mathcal{P})$ *for each* n*, then the submeasure* $\widehat{m}(f,\cdot)$ *is also continuous on* $\sigma(\mathcal{P})$*.(See Definition 3.)*

Now we give the definition of the $\mathcal{L}_1$- and L_1-spaces associated with the operator valued measure m.

Definition 16 *Let* $\mathcal{L}_1\mathcal{M}(m)$ *(resp.*$\mathcal{L}_1\mathcal{I}(m)$*) be the set* $\{f \in \mathcal{M}(\mathcal{P},X) : \widehat{m}(f,T) < \infty\}$ *(resp. the set* $\{f \in \mathcal{I}(m) : \widehat{m}(f,T) < \infty\}$*. The closure of* $\mathcal{I}_s(m)$ *in* $\mathcal{L}_1\mathcal{M}(m)$ *in* L_1*-mean is denoted by* $\mathcal{L}_1\mathcal{I}_s(m)$*. The set* $\{f \in \mathcal{M}(\mathcal{P},X) : \widehat{m}(f,\cdot)$ *continuous on* $\sigma(\mathcal{P})\}$ *is denoted by* $\mathcal{L}_1(m)$*.*

By Proposition 2(iv), $\mathcal{L}_1\mathcal{M}(m)$, $\mathcal{L}_1\mathcal{I}(m)$, $\mathcal{L}_1\mathcal{I}_s(m)$ and $\mathcal{L}_1(m)$ are pseudonormed spaces with respect to the pseudonorm $\widehat{m}(\cdot,T)$ and consequently, the corresponding quotient spaces with respect to $\sim$ are normed spaces and are denoted by $L_1\mathcal{M}(m), L_1\mathcal{I}(m), L_1\mathcal{I}_s(m)$ and $L_1(m)$, respectively. These spaces will be referred to as the $\mathcal{L}_1$- and L_1-spaces associated with m. Results (i) and (ii) of the following theorem are immediate from Theorem 11.

Theorem 12 *(i) The spaces* $\mathcal{L}_1\mathcal{M}(m)$*,* $\mathcal{L}_1\mathcal{I}(m)$*,* $\mathcal{L}_1\mathcal{I}_s(m)$ *and* $\mathcal{L}_1(m)$ *are complete pseudonormed spaces. Consequently,* $L_1\mathcal{M}(m)$*,* $L_1\mathcal{I}(m)$*,* $L_1\mathcal{I}_s(m)$ *and* $L_1(m)$ *are Banach spaces.*

(ii) $\mathcal{L}_1\mathcal{M}(m) \supset \mathcal{L}_1\mathcal{I}(m) \supset \mathcal{L}_1\mathcal{I}_s(m) \supset \mathcal{L}_1(m)$*.*

(iii) If the Banach space $c_0 \not\subset Y$*, then*

$$\mathcal{L}_1\mathcal{M}(m) = \mathcal{L}_1\mathcal{I}(m) = \mathcal{L}_1\mathcal{I}_s(m) = \mathcal{L}_1(m).$$

(iv) $\mathcal{L}_1\mathcal{I}_s(m) = \mathcal{L}_1(m)$ *if and only if the semivariation* $\widehat{m}(\cdot)$ *is continuous on* $\mathcal{P}$*.*

By using Theorem 1 it can be shown that $\widehat{m}$ is continuous on $\mathcal{P}$ and $\widehat{m}(g,\cdot)$ is continuous on $\sigma(\mathcal{P})$ for $g \in \mathcal{L}_1\mathcal{M}(m)$, whenever the Banach space $c_0 \not\subset Y$. This fact gets reflected as result (iii) of the above theorem.

Note 8 *When $c_0 \subset Y$, it can happen that $\mathcal{L}_1\mathcal{M}(m) \supsetneqq \mathcal{L}_1\mathcal{I}(m) \supsetneqq \mathcal{L}_1\mathcal{I}_s(m) \supsetneqq \mathcal{L}_1(m)$, as is illustrated in the following example.*

Example 1 *Let $T = \mathbb{N}, \mathcal{P} = \mathcal{P}(\mathbb{N}), X$ be the real space l_1 and Y the real space c_0. For $x = (x_1, x_2, ...) \in l_1$, let us define*

$$\left\{ \begin{array}{lcl} m(\{1\})x & = & (x_1, 0, 0, ...) \end{array} \right.$$

$$\left\{ \begin{array}{lcl} m(\{2\})x & = & (0, \frac{1}{2}x_3, 0, 0, ...) \\ m(\{3\})x & = & (0, \frac{1}{2}x_5, 0, 0, ...) \end{array} \right.$$

$$\left\{ \begin{array}{lcl} m(\{4\})x & = & (0, 0, \frac{1}{3}x_7, 0, 0, ...) \\ m(\{5\})x & = & (0, 0, \frac{1}{3}x_9, 0, 0, ...) \\ m(\{6\})x & = & (0, 0, \frac{1}{3}x_{11}, 0, 0, ...) \end{array} \right.$$

and so on. For $E \subset \mathbb{N}$,let $m(E)x = \sum_{n \in E} m(\{n\})x$ if $E \neq \emptyset$ and $m(E) = 0$ if $E = \emptyset$. Then it can be shown that $m : \mathcal{P} \to L(l_1, c_0)$ is σ-additive in the uniform operator topology.Clearly $\widehat{m}(T) = 1$.

Let $f(n) = e_{2n}, n \in \mathbb{N}$, where $e_n = (\underbrace{0, 0, ..., 0, 1}_{n}, 0, ...) \in l_1$. Let $g(n) = e_{2n-1}, n \in \mathbb{N}$. Then $f, g \in \mathcal{M}(\mathcal{P}, X)$. Clearly, f is m-integrable and obviously, $\int_A f dm = 0$ for each $A \in \sigma(\mathcal{P})$. By Proposition 2(iii) and Theorem 8, $\widehat{m}(f, T) \leq ||f||_T \widehat{m}(T) \leq 1$, and hence $f \in \mathcal{L}_1\mathcal{I}(m)$. Since $\widehat{m}(\cdot)$ is not continuous on $\sigma(\mathcal{P}) = \mathcal{P}$, it can be shown that f is not approximable by a sequence $(s_n)_1^\infty \subset \mathcal{I}_s(m)$ in L_1-mean. Thus $f \notin \mathcal{L}_1\mathcal{I}_s(m)$. This shows that $\mathcal{L}_1\mathcal{I}(m) \supsetneqq \mathcal{L}_1\mathcal{I}_s(m)$.
For the function g defined above we have

$$\int_{\{1\}} gdm = e_1 \quad ; \quad \int_{\{2\}} gdm = \int_{\{3\}} gdm = \frac{1}{2}e_2$$

$$\int_{\{4\}} gdm = \int_{\{5\}} gdm = \int_{\{6\}} gdm = \frac{1}{3}e_3$$

and so on. This shows that

$$\sum_1^\infty \int_{\{n\}} gdm = \sum_1^\infty e_k \notin c_0$$

and hence g is not m-integrable. However, $\widehat{m}(g, T) \leq ||g||_T \widehat{m}(T) = 1$. Thus $g \in \mathcal{L}_1\mathcal{M}(m)$, but $g \notin \mathcal{L}_1\mathcal{I}(m)$. This shows that $\mathcal{L}_1\mathcal{M}(m) \supsetneqq \mathcal{L}_1\mathcal{I}(m)$.

Since $\widehat{m}(\cdot)$ is not continuous on $\mathcal{P}$, by Theorem 12(iv) we have $\mathcal{L}_1\mathcal{I}_s(m) \supsetneqq \mathcal{L}_1(m)$.

Thus, for the present choice of $\mathcal{P}, X, Y$ and m we have shown that

$$\mathcal{L}_1\mathcal{M}(m) \supsetneqq \mathcal{L}_1\mathcal{I}(m) \supsetneqq \mathcal{L}_1\mathcal{I}_s(m) \supsetneqq \mathcal{L}_1(m).$$

From the above results we observe that the Dobrakov integral is related to the topological structure of the underlying Banach space Y such as $c_0 \not\subset Y$ or $c_0 \subset Y$. A similar involvement of the space Y is absent in other Lebesgue-type integration theories. See Section 8 below.

Now we take up the study of the separability of the L_1-spaces.

Definition 17 *Let $\mathcal{P}_1 = \{E \in \sigma(\mathcal{P}) : \widehat{m}(E) < \infty\}$. We define $\rho(E,F) = \widehat{m}(E\Delta F)$ for $E, F \in \mathcal{P}_1$.*

Clearly, ρ is a pseudometric on $\mathcal{P}_1$. It is routine to verify that $(\mathcal{P}_1, \rho)$ is complete. In terms of ρ we have the following sufficient condition for the continuity of $\widehat{m}(\cdot)$ on $\mathcal{P}$.

Theorem 13 *If $(\mathcal{P}, \rho)$ is separable, then the semivariation $\widehat{m}(\cdot)$ is continuous on $\mathcal{P}$. Consequently, $\mathcal{L}_1\mathcal{I}_s(m) = \mathcal{L}_1(m)$ (by Theorem 12(iv)). More generally, if Ω is anyone of the spaces $\mathcal{L}_1\mathcal{M}(m)$, $\mathcal{L}_1\mathcal{I}(m)$, $\mathcal{L}_1\mathcal{I}_s(m)$ or $\mathcal{L}_1(m)$ and if Ω is separable, then $\Omega = \mathcal{L}_1(m)$.*

Since any separable $\mathcal{L}_1$-space coincides with $\mathcal{L}_1(m)$, it follows that only the space $\mathcal{L}_1(m)$ can be separable. Now we shall give a characterization of separable $\mathcal{L}_1(m)$.

Theorem 14 *Let $\mathcal{L}_1(m)$ be non trivial. Then it is separable if and only if the space $(\mathcal{P}_0, \rho)$ and X are separable, where $\mathcal{P}_0 = \{A \in \mathcal{P} : \widehat{m}(A \cap E_n) \searrow 0$ for each sequence$E_n \searrow \emptyset$ in $\sigma(\mathcal{P})\}$. Consequently, if $\mathcal{P}_0$ is the δ-ring generated by a countable family of sets and if X is separable, then $\mathcal{L}_1(m)$ is separable.*

Note that the last part of the above theorem generalizes its corresponding classical analogue.

7 Generalizations of Classical Convergence Theorems to $\mathcal{L}_1(m)$

The Lebesgue dominated convergence theorem (shortly, LDCT),the Lebesgue bounded convergence theorem (shortly,LBCT) and the monotone convergence theorem (shortly, MCT) are suitably generalized to the space $\mathcal{L}_1(m)$. The space $\mathcal{L}_1(m)$ is characterized as the biggest class of m-integrable functions for which LDCT holds. Also Theorem 8 is strengthened for functions in $\mathcal{L}_1(m)$ as shown in Corollary 1 of Theorem 15. Finally, the complete analogue of the classical Vitali convergence theorem also holds for this class.

Theorem 15 *(LDCT)Suppose $(f_n)_1^\infty \subset \mathcal{M}(\mathcal{P}, X)$ and $f \in \mathcal{M}(\mathcal{P}, X)$ and suppose $f_n \to f$ m-a.e. on T. If there is a function $g \in \mathcal{L}_1(m)$ such that $|f_n(t)| \le |g(t)|$ m-a.e. on T for $n = 1, 2, ...,$ then $f, f_n \in \mathcal{L}_1(m)$ for $n = 1, 2, ...,$ and $\widehat{m}(f_n - f, T) \to 0$. Consequently,$f, f_n \in \mathcal{I}(m)$ for $n = 1, 2, ...,$ and*

$$\lim_n \int_A f_n dm = \int_A f dm$$

uniformly with respect to $A \in \sigma(\mathcal{P})$.

The following corollary gives a strengthened version of Theorem 8 for functions in $\mathcal{L}_1(m)$.

Corollary 1 *Let $f \in \mathcal{L}_1(m)$. Then for each sequence $(s_n)_1^\infty$ in $\mathcal{I}_s(m)$ with $s_n \to f$ and $|s_n| \nearrow |f|$ m-a.e. on T,*

$$\lim_n \int_A s_n dm = \int_A f dm$$

uniformly with respect to $A \in \sigma(\mathcal{P})$.

Now we give a characterization of the space $\mathcal{L}_1(m)$ in terms of LDCT.

Theorem 16 *(A CHARACTERIZATION OF $\mathcal{L}_1(m)$) A function $g \in \mathcal{M}(\mathcal{P}, X)$ belongs to $\mathcal{L}_1(m)$ if and only if every $f \in \mathcal{M}(\mathcal{P}, X)$ with $|f| \leq |g|$ m-a.e. on T is m-integrable. (In that case, $f \in \mathcal{L}_1(m)$.) Consequently, $\mathcal{L}_1(m)$ is the largest class of m-integrable functions for which LDCT holds in the sense that, if the hypotheses that $f, f_n, n = 1, 2, ...,$ are in $\mathcal{M}(\mathcal{P}, X), f_n \to f$ m-a.e. on T and there exists $g \in \mathcal{M}(\mathcal{P}, X)$ such that $|f_n| \leq |g|$ m-a.e. imply that $f, f_n \in \mathcal{I}(m)$ for $n = 1, 2, ...,$ then $g \in \mathcal{L}_1(m)$.*

Now we state the generalized Lebesgue bounded convergence theorem.

Theorem 17 *(LBCT) Suppose $\widehat{m}(\cdot)$ is continuous on $\sigma(\mathcal{P})$,or equivalently, suppose every bounded $f \in \mathcal{M}(\mathcal{P}, X)$ is m-integrable. Let $f, f_n, n = 1, 2, ...,$ be in $\mathcal{M}(\mathcal{P}, X)$ such that $f_n \to f$ m-a.e. on T. If there is a finite constant C such that $|f_n(t)| \leq C$ m-a.e. on T for $n = 1, 2, ...,$ then $f, f_n \in \mathcal{L}_1(m)$ for all n, $\widehat{m}(f_n - f, T) \to 0$ as $n \to \infty$ and*

$$\lim_n \int_A f_n dm = \int_A f dm$$

uniformly with respect to $A \in \sigma(\mathcal{P})$.

The reader is referred to [8] for the generalization of the Vitali convergence theorem to $\mathcal{L}_1\mathcal{I}_s(m)$, and to [10] for the generalizations of the MCT and the Vitali convergence theorem to $\mathcal{L}_1(m)$. Another theorem, called diagonal convergence theorem, is given in [9] with many interesting applications. Because of lack of space, we omit their discussion here.

8 Comparison with Classical Lebesgue-type Integration Theories

As mentioned in the introduction, the Dobrakov integral is now compared with the abstract Lebesgue integral, the Bochner and the Pettis integrals, the Bartle-Dunford-Schwartz integral, the Bartle bilinear integral and the Dinculeanu integral. As observed in Note 1, the reader can consider a σ-additive scalar or vector measure as a particular case of an operator valued measure by taking $\mathbb{K} = X$, or $\mathbb{K} = X$ and $X = Y$, respectively. Thus the comparison is possible.

Here it is observed that the Dobrakov integral is the same as the abstract Lebesgue integral when the functions and the measure are scalar valued (Theorem 18). Moreover, it is pointed out that the Dobrakov integral is the complete all pervading generalization of the abstract Lebesgue integral, while the other integrals such as the Bochner,the Bartle and the Dinculeanu integrals generalize only partially. See Theorem 19 and the comments following Example 2, and Theorems 21 and 22 along with the comments following Example 3. In the case of the Pettis integral, for X-valued $\mathcal{P}$-measurable functions, the concepts of integrability and integral coincide in both the theories.(See Theorem 20(i)).

(a) The abstract Lebesgue integral

Let $\mu : S \to [0, \infty]$ or $\mathbb{C}$ be σ-additive and let $\mathcal{P} = \{E \in S : v(\mu, E) < \infty\}$. Since each μ-integrable function f has $N(f)$ σ-finite, it follows that f is $\mathcal{P}$-measurable in the sense of Definition 8. By Note 1, μ is an operator valued measure with $\mu(E) \in L(\mathbb{K}, \mathbb{K})$ for $E \in \mathcal{P}$.

Theorem 18 *Let S, μ and $\mathcal{P}$ be as above. A scalar function f on T is μ-integrable in the usual sense if and only if it is Dobrakov μ-integrable and moreover, both integrals coincide on each $A \in S$. Thus $\mathcal{I}(\mu)$ coincides with the class of all μ-integrable (in the usual sense) scalar functions. Further,*

$$\widehat{\mu}(f, A) = \int_A |f| dv(\mu, \cdot), \quad A \in S$$

and $\mathcal{I}(\mu) = \mathcal{L}_1\mathcal{M}(\mu) = \mathcal{L}_1\mathcal{I}(\mu) = \mathcal{L}_1\mathcal{I}_s(\mu) = \mathcal{L}_1(\mu)$.

(b) The Bochner integral [22,24]

Let $S, \mu, \mathcal{P}$, be as in (a). If f is an X-valued Bochner μ-integrable function, then $N(f)$ is σ-finite and consequently, f is $\mathcal{P}$-measurable in the sense of Definition 8. Take $Y = X$ and consider $\mu(E)$ as the operator $\mu(E)I$, where I is the identity operator on X.

Theorem 19 *Let $S, \mu, \mathcal{P}$ be as in the above. If f is an X-valued Bochner μ-integrable function, then f is Dobrakov μ-integrable and both integrals coincide on each $A \in S$. Consequently, if θ is a complex Radon measure in the sense of Bourbaki [4] on a locally compact Hausdorff space T, and if μ_θ is the complex measure induced by θ in the sense of [29,31], then each function $f : T \to X$ which is θ-integrable in the sense of Bourbaki [4] is μ_θ-integrable in the sense of Dobrakov and both integrals coincide on each Borel subset of T. (See also [30]).*

It is well known that an X-valued $\mathcal{P}$-measurable function f is Bochner μ-integrable if and only if $\int_T |f| dv(\mu, \cdot) < \infty$. As the following example illustrates, when X is infinite dimensional there exist X-valued functions on T which are Dobrakov μ-integrable,but not Bochner μ-inegrable for a suitably chosen σ-additive scalar measure.

Example 2 *Let dim $X = \infty$ and choose by the Dvoretzky-Roger theorem in [5] a sequence $(x_n)_1^\infty$ in X such that $\sum x_n$ converges unconditionally in norm, with $\sum |x_n| = \infty$. Let $S = \mathcal{P}(\mathbb{N})$ and $\mu(E) = \sharp E$ if E is finite and $\mu(E) = \infty$ otherwise. Let $\mathcal{P} = \{E \subset \mathbb{N} : E$ finite$\}$. If $f(n) = x_n, n \in \mathbb{N}$, then f is $\mathcal{P}$-measurable and by the unconditional convergence of $\sum x_n$ it follows that f is Dobrakov μ-integrable. But f is not Bochner μ-integrable, since $\int_{\mathbb{N}} |f| d\mu = \sum_1^\infty |x_n| = \infty$.*

Since dim $X = \infty$ is the only hypothesis that was used in the above example, we can state the following:

When dim $X = \infty$, one can always define $\mathcal{P}$ and a σ-additive scalar measure μ on $\mathcal{P}$ such that the class of all Bochner μ-integrable X-valued functions is a proper subset of $\mathcal{I}(\mu)$. In that case, by Note 5(iii) the theorem on interchange of limit and integral is not valid for the class of the Bochner μ-integrable functions.

Recall that an X-valued $\mathcal{P}$-measurable function is Bochner μ-integrable if and only if $|f|$ is $v(\mu, .)$-integrable and hence, in terms of the terminology given in the introduction, **the Bochner integral generalizes the abstract Lebesgue integral in such a way as to maintain the property of absolute integrability. On the other hand, the Dobrakov integral maintains only the property of unconditional convergence, and not that of absolute integrability**.

Finally, for an X-valued $\mathcal{P}$-measurable function f it can be easily verified that $\widehat{\mu}(f, A) = \int_A |f| dv(\mu, \cdot)$ for $A \in \sigma(\mathcal{P})$ and hence f is Bochner μ-integrable if and only if $\widehat{\mu}(f, T) < \infty$. In that case, $\widehat{\mu}(f.\cdot) : \mathcal{S} \to [0, \infty)$ is a σ-additive finite measure and hence is continuous on $\mathcal{S}$. Thus **the class of all Bochner μ-integrable functions coincides with $\mathcal{L}_1\mathcal{M}(\mu) = \mathcal{L}_1\mathcal{I}(\mu) = \mathcal{L}_1\mathcal{I}_s(\mu) = \mathcal{L}_1(\mu)$ of Dobrakov.**

The above observation motivates the following

Definition 18 *For an operator valued measure m, the associted space $\mathcal{L}_1(m)$ (or $L_1(m)$) is called the Bochner class of m.*

(c) The Pettis integral [24]

Let $\mathcal{S}, \mu, \mathcal{P}$ be as in (a). Recall that an X-valued weakly $\mathcal{P}$-measurable function f is said to be Pettis integrable if $x^\star f$ is μ-integrable for each $x^\star \in X^\star$ and if for each $A \in \sigma(\mathcal{P})$ there exists a vector $x_A \in X$ such that

$$x^\star(x_A) = \int_A x^\star f d\mu.$$

In that case, the Pettis integral of f over A is defined by

$$(P) \int_A f d\mu = x_A, \quad A \in \sigma(\mathcal{P}).$$

Considering $\mu(E)$ as $\mu(E)I \in L(X, X)$, one can compare the Pettis integral with the Dobrakov integral. In fact, the following theorem describes their relationship.

Theorem 20 *Let $\mathcal{S}, \mu, \mathcal{P}$ be as in the above. Let f be an X-valued function on T. Then the following hold:*

(i) If $f \in \mathcal{M}(\mathcal{P}, X)$, then it is Pettis μ-integrable if and only if it is Dobrakov μ-integrable and both integrals coincide.

(ii) *There exist weakly $\mathcal{P}$-measurable functions which are not $\mathcal{P}$-measurable. Hence there exist functions which are Pettis integrable, but not Dobrakov integrable.*

(d) The Bartle-Dunford-Schwartz integral [2,22]

When $\mathcal{P}$ is a σ-algebra, the Dobrakov integral of a $\mathcal{P}$-measurable scalar function with respect to a σ-additive vector measure γ on $\mathcal{P}$ (see Note 1) is the same as the Bartle-Dunford-Schwartz integral. This is not surprising, since Dobrakov adapted the proceedure followed in [2] to define the integral of vector valued functions with respect to an operator valued measure and since $||\gamma|| = \widehat{\gamma}$ (see Note 2). Note that this integral maintains only the property of unconditional convergence and not that of absolute integrability.

(e) The Bartle integral [1]

Bartle has developed a theory of integral in [1] for X-valued functions with respect to a set function $\mu : \mathcal{S} \to L(X, Y)$, where μ is σ-additive in the uniform operator topology and $\mathcal{S}$ is a σ-algebra of sets. We shall refer to this integral as the Bartle integral. Further, when μ satisfies the condition $(\star)$ on p.346 of [1],we shall call it $(\star)$-integral of Bartle.From the discussion on p.535 of [7], we have the following results.

Theorem 21 *Let $m : \mathcal{S} \to L(X, Y)$ be σ-additive in the uniform operator topology,where $\mathcal{S}$ is a σ-algebra. Then the following hold:*

(i) *The Bartle integral maintains the property of unconditional convergence.*

(ii) *Each X-valued Bartle m-integrable function is Dobrakov m-integrable and both integrals coincide.*

(iii) *$\widehat{m}(\cdot)$ is continuous on $\mathcal{S}$ if and only if m satisfies the $(\star)$-condition of Bartle. Thus,***if $\widehat{m}(\cdot)$ is continuous on $\mathcal{S}$ (for example, if $c_0 \not\subset Y$),then the theory of $(\star)$-integral of Bartle is the same as the theory of the Dobrakov integral.**

(iv) *The function f in Example 1 above is not even measurable in the sense of Bartle (and hence not Bartle m-integrable). Thus* **the theorem on interchange of limit and integral is in general not valid for the class of all Bartle m-integrable functions (see Note 5(iii)).**

In [1] the Bartle $(\star)$-integral is compared with other Lebesgue-type integrals that are not discussed here. The reader is referred to [1].

(f) The Dinculeanu integral [6]

In [6] Dinculeanu has developed a theory of integral for vector valued functions with respect to an operator valued measure m of finite variation on a δ-ring $\mathcal{P}$. Since $\widehat{m}(\cdot) \leq v(m, \cdot)$ on $\sigma(\mathcal{P})$ and $N(f)$ is σ-finite with respect to $v(m, \cdot)$, the following result holds.

Theorem 22 *Let $m : \mathcal{P} \to L(X, Y)$ be an operator valued measure of finite variation. If $f : T \to X$ is m-integrable in the sense of Dinculeanu, then f is Dobrakov m-integrable, both integrals coincide on each $A \in \sigma(\mathcal{P})$ and f belongs to the Bochner class of m (see*

Definition 18).Moreover, a function $f \in \mathcal{M}(\mathcal{P}, X)$ is Dinculeanu m-integrable if and only if $\int_T |f| dv(m, \cdot) < \infty$ and hence the Dinculeanu integral maintains the property of absolute integrability.

Example 3 *Let $T = I\!N$ and $\mathcal{P} = \{E \subset I\!N, E\ finite\}$. Then $\sigma(\mathcal{P}) = \mathcal{P}(I\!N)$. Let dim $Y = \infty$ and let $X = I\!K$, the scalar field of Y. By the Dvoretzky-Roger theorem in [5], choose $(y_n)_1^\infty$ in Y such that $\sum y_n$ converges unconditionally with $\sum |y_n| = \infty$. Define $m(E) = \sum_{n \in E} y_n$ for $E \in \sigma(\mathcal{P})$. Then m is a well defined Y-valued σ-additive vector measure on $\sigma(\mathcal{P})$ and hence by Note 2, $\widehat{m}(T) = ||m||(T)$ is finite. Note that $v(m, E)$ is finite for each $E \in \mathcal{P}$ and hence the Dinculeanu integral can be defined with respect to m (see Note 2). However, observe that $v(m, T) = \sum |y_n| = \infty$.*

Clearly, the function χ_T is $\mathcal{P}$-measurable. Let $s_k = \chi_{\{1,2,\ldots,k\}}$ Then $s_k \to \chi_T$ pointwise and for $A \subset T$,

$$|\int_A s_k dm - \int_A s_{k+p} dm| \to 0$$

as $k \to \infty$. Thus χ_T is Dobrakov m-integrable. Moreover, $\chi_T \in \mathcal{L}_1\mathcal{I}(m)$, since $\widehat{m}(\chi_T, T) = \widehat{m}(T) < \infty$. On the other hand, $\int_T \chi_T dv(m, \cdot) = \sum_{n=1}^\infty |y_n| = \infty$ and hence χ_T is not Dinculeanu m-integrable. Further, if $c_0 \not\subset Y$, then by Theorem 12(iii), $\mathcal{L}_1\mathcal{M}(m) = \mathcal{L}_1\mathcal{I}(m) = \mathcal{L}_1\mathcal{I}_s(m) = \mathcal{L}_1(m)$ and hence $\chi_T \in \mathcal{L}_1(m)$.

The above example establishes the following:

If dim $Y = \infty$, then there exists a δ-ring $\mathcal{P}$ and an operator valued measure $m : \mathcal{P} \to L(X, Y)$ of finite variation such that the class $\mathcal{D}_i(m)$ of all Dinculeanu m-integrable functions is a proper subset of $\mathcal{I}(m)$, so that the theorem on interchange of limit and integral does not hold for $\mathcal{D}_i(m)$ by Note 5(iii). Moreover, one can also have $\mathcal{L}_1(m) \supsetneqq \mathcal{D}_i(m)$ (for example,when $c_0 \not\subset Y$). LDCT holds for $\mathcal{D}_i(m)$ and $\mathcal{L}_1(m)$ (cf. Theorem 16).

For X-valued vector functions the Dinculeanu integral with respect to a σ-additive scalar measure on $\mathcal{P}$ (see Note 1) is the same as the Bochner integral and hence the Dinculeanu integral is a direct generalization of the Bochner integral for operator valued measures of finite variation on a δ-ring. Note that the Bochner and the Dinculeanu integrals share the property of absolute integrability of the abstract Lebesgue integral.

Since the assumption that the operator valued measure m is of finite variation on $\mathcal{P}$ is very restrictive in comparison with the finiteness of the semivariation $\widehat{m}$ on $\mathcal{P}$ (see Example 3), the Dobrakov theory permits the m-integrability of vector functions when the operator valued measure m has only finite semivariation on $\mathcal{P}$. Note that such functions cannot be integrated in the sense of Dinculeanu, when m is not of finite variation on $\mathcal{P}$.

The advantage of the Dobrakov integral over that of Dinculeanu is that it permits an integral representation for weakly compact operators $U : C_0(T, X) \to Y$, where $C_0(T, X) = \{f :$

$T \to X, f$ continuous and vanishes at $\infty\}$ and T is a locally compact Hausdorff space.(See[20].)

9 Concluding Remarks

Maynard obtained a powerful Radon-Nikodým theorem for the Dobrakov integral in [26]. Dobrakov proved the Fubini type theorem for operator valued product measures under certain restrictions in [9], and later, in 1988, he proved it in the most general form in [13]. The validity of the Fubini type theorem also asserts that the Dobrakov integral is the apt generalization of the abstract Lebesgue integral. Because of lack of space, we omit the discussion of these interesting topics. The reader is referred to the bibliography.

The techniques and the ideas found in the Dobrakov theory are so powerful and profound as to permit him to formulate a theory of multilinear integration of vector functions with respect to an operator valued multimeasure.See [14-19,21]. This theory is more complete and exhaustive than its particular cases given in [27,28,32].

Finally, the author wishes to acknowledge sincerely the financial assistance received from Intercambios Científicos of the Universidad de los Andes, CONICIT and Fundación Polar of Venezuela, to visit the Mathematical Institute in Bratislava. Also he wishes to express his indebtedness to Professor Dobrakov for many fruitful and thought provoking discussions on the subject during his visit to Bratislava.

REFERENCES

1. R. G. Bartle, A general bilinear vector integral, Studia Math. 15(1956), 337–352.
2. R. G. Bartle, N. Dunford and J. T. Schwartz, Weak compactness and vector measures, Canadian J. Math. 7(1955),289–305.
3. C.Bessaga and A.Pelczyński, On bases and unconditional convergence of series in Banach spaces, Studia Math. 17(1958), 151–164.
4. N. Bourbaki, Éléments de mathématique, Livre VI, Intégration, (Chs.1–4), Herman et cie,1965.
5. J. Diestel and J. J. Uhl, Vector measures, Amer. Math. Soc. Surveys, No. 15, Providence, 1977.
6. N. Dinculeanu, Vector measures, Pergamon Press, New York, 1967.
7. I. Dobrakov, On integration in Banach spaces, I, Czech. Math. J. 20(1970), 511–536.
8. I. Dobrakov, On integration in Banach spaces, II, Czech. Math. J. 20(1970), 680–695.
9. I. Dobrakov, On integration in Banach spaces, III, Czech, Math. J. 29(1979), 478–491.
10. I. Dobrakov, On integration in Banach spaces, IV, Czech. Math. J. 30 (1980), 259–279.
11. I. Dobrakov, On integration in Banach spaces, V, Czech. Math. J. 30(1980), 610–628.
12. I. Dobrakov, On integration in Banach spaces, VI, Czech. Math. J. 35(1985), 173–187.
13. I. Dobrakov, On integration in Banach spaces, VII, Czech. Math. J. 38(1988), 434–449.

14. I. Dobrakov, On integration in Banach spaces, VIII, Czech. Math. J. 37(1987), 487–506.

15. I. Dobrakov, On integration in Banach spaces, IX, Czech. Math. J. 38(1988), 589–601.

16. I. Dobrakov, On integration in Banach spaces, X, Czech. Math. J. 38(1988), 713–725.

17. I. Dobrakov, On integration in Banach spaces, XI, Czech. Math. J. 40(1990), 8–24.

18. I. Dobrakov, On integration in Banach spaces, XII, Czech. Math. J. 40(1990), 424–440.

19. I. Dobrakov, On integration in Banach spaces, XIII, Czech. Math. J.40(1990),566-582.

20. I. Dobrakov, On representation of linear operators on $C_0(T, X)$, Czech. Math. J.39(1971),13-30.

21. I. Dobrakov, On representation of multilinear operators on $XC_0(T_i)$, Czech. Math. J.39(1989), 288–302.

22. N.Dunford and J.T.Schwartz, Linear operators,Part I: General theory, Interscience, New York, 1957.

23. P.R.Halmos, Measure theory, Van Nostrand, New York, 1950.

24. E.Hille and R. S. Phillips, Functional analysis and semigroups, Amer. Math. Soc. Colloq. Publ. 1957.

25. J. L. Kelley and T. P. Srinivasan, On the Bochner integral, in Vector and operator valued measures and applications, Academic Press, New York, 1973, 165–174.

26. H. B. Maynard, A general Radon Nikodým theorem, ibid., 233–246.

27. M.Morse and W.Transue, C-bimeasures Δ and their superior integral $\Delta^\star$, Rend. Circ. Mat. Palermo (2) 4 (1955), 270–300.

28. M.Morse and W.Transue, C-bimeasures and their integral extensions, Ann. of math. (2) 64 (1956), 480–504.

29. T. V. Panchapagesan, Medida e integración,Parte I, Teoría de la medida, Tomos 1 y 2, Coedition of the C.D.C.H.T., n.E.P, and Facultad de Ciencias, Universidad de los Andes, Mérida, Venezuela, 1991.

30. T. V. Panchapagesan, Integral de Radon en espacios localmente compactos y de Hausdorff, Published by IV Escuela Venezolana de Matemáticas, Universidad de los Andes, Mérida, Venezuela, 1991.

31. T. V. Panchapagesan, On complex Radon measures I, Czech. Math. J. 42(1992), 599–612.

32. K.Ylinen, On vector bimeasures, Annali Mat. Pura Appl. 117(1978), 115–138.