Divulgaciones Matem'aticas v. 4, No. 1/2 (1996), 49–54

Purity and Direct Summands *

Pureza y Sumandos Directos

Ángel V. Oneto R.
Laboratorio de Álgebra Teórica y Computacional (LATyC)
Departamento de Matemática y Computación
Facultad Experimental de Ciencias. Universidad del Zulia.
Apartado Postal 526. Maracaibo 4001. Venezuela.
(E-mail: aoneto@@luz.ve)

Abstract

A criteria for a pure submodule to be a direct summand is given and some applications are derived.
Key words and phrases: Pure submodule, flat module, regular ring.

Resumen

Se da un criterio para que un submódulo puro sea sumando directo y se deducen algunas aplicaciones.
Palabras y frases clave: Submódulo puro, módulo plano, anillo regular.

1 Preliminaries

In what follows R will denote an associative ring with identity and R-module will mean unitary left R-module. Recall that a short exact sequence of R-modules:

$$0 \longrightarrow N \longrightarrow M \longrightarrow F \longrightarrow 0$$

*Este artículo fué financiado parcialmente por la División de Investigación de la Facultad Experimental de Ciencias de La Universidad del Zulia a través del proyecto de investigaci'on No. PNC–002–95

is *pure* if it remains exact after being tensored with any right R-module. If N is a submodule of a R-module M and the canonical short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

is pure, then we say that N is a pure submodule of M. It follows at once that:

Lemma 1 *Every direct summand is a pure submodule.*

For completeness we sketch a proof of the following well known result

Lemma 2 *Let*

$$0 \longrightarrow N \longrightarrow P \longrightarrow F \longrightarrow 0$$

be a short exact sequence of R-modules with P flat. The sequence is pure exact $\Longleftrightarrow F$ is flat.

Proof: Both implications can be obtained by diagram chasing. For example, assume F flat. We have to prove that

$$0 \longrightarrow M \bigotimes N \longrightarrow M \bigotimes P \longrightarrow M \bigotimes F \longrightarrow 0$$

is exact for any right R-module M. Choose a short exact sequence

$$0 \longrightarrow S \longrightarrow L \longrightarrow M \longrightarrow 0$$

with L free. The result follows by diagram chasing applied to the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & S \bigotimes N & \longrightarrow & L \bigotimes N & \longrightarrow & M \bigotimes N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & S \bigotimes P & \longrightarrow & L \bigotimes P & \longrightarrow & M \bigotimes P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S \bigotimes F & \longrightarrow & L \bigotimes F & \longrightarrow & M \bigotimes F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Recall the characterization of a (Von Neumann) *regular* ring as a ring R such that every R-module is flat. From this and the above definition of purity it follows (noted by Gentile [2]) that:

Lemma 3 *R is a regular ring if and only if any submodule (of any R-module) is pure.*

Recall also the following characterization of purity due to P. M. Cohn [1]: a submodule N of an R-module M is pure if and only if for any finite family $(x_i)_{i=1}^m$ of elements of N, any finite family $(y_j)_{j=1}^n$ of elements of M, and relations

$$x_i = \sum_j a_{ij} y_j \quad (a_{ij} \in R\ ,\ i = 1, \dots, m\ ,\ j = 1, \dots, n)$$

there exist $z_1, \dots, z_n \in N$ such that

$$x_i = \sum_j a_{ij} z_j$$

2 Some purity results

The next theorem is a partial converse of Lemma 1:

Theorem 4 *If P is a projective R-module and N a finitely generated pure submodule of P, then N is a direct summand of P.*

Proof: Suppose first that P is free with basis $(e_j)_{j \in J}$. Choose a finite set $(x_i)_{i=1}^m$ of generators of N. We have

$$x_i = \sum_{j \in J_0} a_{ij} e_j \quad (i = 1, \dots, m)$$

for some $a_{ij} \in R$ and finite $J_0 \subset J$. By purity there exist $z_j \in N (j \in J_0)$ such that

$$x_i = \sum_{j \in J_0} a_{ij} z_j$$

Define $\alpha \colon P \longrightarrow N$ by $\alpha(e_j) = z_j$ if $j \in J_0$ and $\alpha(e_j) = 0$ if $j \notin J_0$. If $\beta \colon N \longrightarrow P$ is the inclusion map, we have $\alpha\beta = 1_N$ and so N is a direct summand of P. For the general case, there exists a free R-module L such that P is a direct summand of L. By the particular case N is a direct summand of L

$$L = N \bigoplus N'$$

then $P = N \bigoplus (N' \cap P)$ and N is a direct summand of P.

Now we give a criteria of purity:

Proposition 5 *Let N be a submodule of a R-module M. If N is projective and every map $f\colon N \longrightarrow R$ can be extended to a map $f'\colon M \longrightarrow R$, then N is a pure submodule of M.*

Proof: Consider the situation

$$x_i = \sum_j a_{ij} y_j$$

where $x_i \in N$, $y_j \in M$, $a_{ij} \in R$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. Being N projective there exist a set of generators $(e_h)_{h\in H}$ of N and a set $(f_h)_{h\in H}$ of linear functionals $f_h\colon N \longrightarrow R$ such that for each $x \in N$, $f_h(x) = 0$ for almost all h, and

$$x = \sum_h f_h(x) e_h$$

By hypothesis f_h extends to $f'_h\colon M \longrightarrow R$ and then

$$f_h(x_i) = \sum_j a_{ij} f'_h(y_j) \qquad \forall h \in H.$$

So

$$x_i = \sum_h f_h(x_i) e_h = \sum_j a_{ij} (\sum_h f'_h(y_j) e_h)$$

and N is a pure submodule of M.

3 Applications

In this section we show the ubiquity of Theorem 4, obtaining results that arises in several different contexts.
The first application is a classical theorem due to Villamayor:

Corollary 6 *A finitely presented flat module is projective.*

Proof: Let F be a finitely presented and flat R-module. We have an exact sequence

$$0 \longrightarrow N \longrightarrow L \longrightarrow F \longrightarrow 0$$

where L is free and N a finitely generated submodule of L. By Lemma 2, N is a pure submodule of L and, by Theorem 4, a direct summand of L (i.e. the sequence splits). Hence F is projective.
The next result is due to Kaplansky ([3], Th.1.11).

Corollary 7 *If P is a projective module over a regular ring then every finitely generated submodule of P is a direct summand.*

Proof: Since over a regular ring every submodule (of any module) is pure (Lemma 3), the result follows from Theorem 4.

As a final application we give a variation of a result due to Gentile ([2], Prop. 3.1). Recall that a *left semihereditary ring* is a ring such that every finitely generated submodule of a finitely generated projective module is also projective.

Corollary 8 *A ring R is regular if and only if it is left semihereditary and for any finitely generated projective R-module P, and any finitely generated submodule N of P, every map $f\colon N \longrightarrow R$ extends to $f\colon P \longrightarrow R$.*

Proof: Assume R regular and let N be a finitely generated submodule of a finitely generated projective R-module P. By Lemma 3 and Theorem 4, N is a direct summand of P. It follows that R is left semihereditary and the property of extension of maps holds. Conversely (recall the characterization of a regular ring as a ring such that any finitely generated left ideal is a direct summand) let I be a finitely generated left ideal of R. Being R semihereditary I is projective and then by Proposition 5 it is a pure submodule of R. Finally, by Theorem 4, I is a direct summand of R.

4 Final comment

Theorem 4, as one of the referees pointed out to the author, may also be obtained as a consequence of a result of O. Villamayor (see Lemma 2.2 in [4]).

References

[1] Cohn, P. M. *On the Free Product of Associative Rings 1*, Math. Zeit., **71**, (1959), 380–398.

[2] Gentile, E. R. *Purity and Algebraic Closure*, Rev. Union Mat. Arg. **24**(1), (1968) 37–47.

[3] Goodearl, K. R. *Von Neumann Regular Rings*, Pitman, London, l979.

[4] Ware, R. *Endomorphism Rings of Projective Modules*, Trans. Am. Math. Soc. **155**(1), (1971), 233–256.