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Equations Involving Arithmetic Functions of Factorials

Ecuaciones que Involucran Funciones Aritméticas de Factoriales

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Abstract

For any positive integer k let $\phi(k)$, $\sigma(k)$, and $\tau(k)$ be the Euler function of k, the divisor sum function of k, and the number of divisors of k, respectively. Let f be any of the functions ϕ, σ, or τ. In this note, we show that if a is any positive real number then the diophantine equation $f(n!) = am!$ has only finitely many solutions (m, n). We also find all solutions of the above equation when $a = 1$.
Key words and phrases: arithmetical function, factorial, diophantine equations.

Resumen

Para k entero positivo sean $\phi(k)$, $\sigma(k)$ y $\tau(k)$ la función de Euler de k, la función suma de divisores de k y el número de divisores de k, respectivamente. Sea f cualquiera de las funciones ϕ, σ o τ. En esta nota se muestra que si a is cualquier número real positivo entonces la ecuación diofántica $f(n!) = am!$ tiene sólo un número finito de soluciones (m, n). También se hallan todas las soluciones de la mencionada ecuación cuando $a = 1$.
Palabras y frases clave: función aritmética, factorial, ecuaciones diofánticas.

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1 Introduction

For any positive integer k let $\phi(k), \sigma(k)$ and $\tau(k)$ be the Euler's totient function, the divisor sum and the number of divisors of k, respectively. In this note, we prove the following theorem:

Theorem.

Let a be any positive rational number and let f be any of the arithmetical functions ϕ, σ or τ. Then, the equation

$$\frac{f(n!)}{m!} = a \tag{1}$$

has only finitely many solutions $(m,\ n)$.

We also find all the solutions of equation (1) when $a = 1$.

Corollary.

(i) The only solutions of the equation

$$\phi(n!) = m! \tag{2}$$

are obtained for $n = 0,\ 1,\ 2,\ 3$.

(ii) The only solutions of the equation

$$\sigma(n!) = m! \tag{3}$$

are obtained for $n = 0,\ 1$.

(iii) The only solutions of the equation

$$2\sigma(n!) = m! \tag{4}$$

are obtained for $n = 2,\ 3,\ 4,\ 5$.

(iv) The only solutions of the equation

$$\tau(n!) = m! \tag{5}$$

are obtained for $n = 0,\ 1,\ 2$.

The only reason that we have also treated equation (4) is because it has a rather interesting set of solutions given by

$$2\sigma(n!) = (n+1)! \qquad \text{for } n = 2,\ 3,\ 4,\ 5,$$

which is, in Richard Guy's terminology, just another manifestation of the "law of small numbers".

Related to equations (3) and (4) above Pomerance (see [4]) showed that the only positive integers n such that $n!$ is multiply perfect (that is, a divisor of $\sigma(n!)$) are $n = 1,\ 3,\ 5$.

Various other diophantine equations involving factorials have been previously treated in the literature. Erdős & Obláth (see [2]) have studied the equations $n! = x^p \pm y^p$ and $n! \pm m! = x^p$ and Erdős & Graham have studied the equation $y^2 = a_1!a_2!\dots a_r!$ (see [1]). The reader interested in results and open problems concerning diophantine equations involving factorials or arithmetic functions should consult Guy's excellent book [3].

2 The Proofs

In what follows p denotes a prime number. For a positive integer n, we denote by $\mu_p(n)$ the sum of the digits of n written in base p.

2.1 The Proof of the Theorem.

When $f \in \{\phi,\ \sigma\}$, we use the fact that

$$\frac{n}{2\log\log n} < \frac{n}{\phi(n)} < \frac{\sigma(n)}{n} \qquad \text{for all } n > 2\cdot 10^9, \tag{6}$$

(see, for example [6]) to conclude that equation (1) has only finitely many solutions $(m,\ n)$ with $m \neq n$. We then show that equation (1) has only finitely many solutions $(m,\ n)$ with $m = n$ as well.

Assume, for example, that $f = \phi$.

We first show that equation (1) has finitely many solutions with $n < m$. Indeed, if $n \leq m-1$, we get

$$am! = \phi(n!) < n! \leq (m-1)!,$$

which implies that $m \leq 1/a$.

We now show that equation (1) has only finitely many solutions with $m < n$. Indeed, assume that $n \geq (m+1)$ and $n! > 2\cdot 10^9$. Since $(m+1)! < (m+1)^{m+1}$, it follows, by inequality (6), that

$$am! = \phi(n!) > \frac{n!}{2\log\log(n!)} \geq \frac{(m+1)!}{2\log\log(m+1)!} > \frac{(m+1)!}{2\log\log\big((m+1)^{m+1}\big)},$$

or

$$m + 1 < 2a \log\big((m+1)\log(m+1)\big). \tag{7}$$

Inequality (7) implies that m is bounded by a constant depending on a.

Hence, equation (1) has only finitely many solutions $(m,\ n)$ with $m \neq n$. Assume now that $m = n$. In this case, we get

$$\frac{1}{a} = \frac{n!}{\phi(n!)} = \prod_{p \le n}\Big(1 + \frac{1}{p-1}\Big). \tag{8}$$

Since the product from the right side of formula (8) diverges to infinity when n tends to infinity, it follows that equation (8) has only finitely many solutions as well.

Hence, equation (1) has only finitely many solutions $(m,\ n)$ when $f = \phi$. The case $f = \sigma$ is entirely analogous.

Assume now that $f = \tau$. In this case, we use only divisibility arguments to conclude that equation (1) has only finitely many solutions.

For every real number x let $\pi(x)$ be the number of primes less than or equal to x and $\pi_1(x)$ be the number of primes in the interval $(x/2,\ x]$. Since we are interested in proving that equation (1) has only finitely many solutions, we may assume that both m and n are very large. We use the notation $n >> 1$ and $m >> 1$ to indicate that we assume that n (respectively m) is large enough.

Write

$$n! = \prod_{p \le n} p^{\alpha_p(n)}.$$

It is well-known that

$$\alpha_p(n) = \frac{n - \mu_p(n)}{p-1} < n.$$

Write equation (1) as

$$\prod_{p \le n} (\alpha_p(n) + 1) = am! \tag{9}$$

We first investigate the order at which the prime 2 divides both sides of equation (9). On the one hand, since $\alpha_p(n) = 1$ for all primes $p \in (n/2,\ n]$, it follows that the order at which 2 divides the left hand side of equation (9) is at least $\pi_1(n)$. On the other hand, the order at which 2 divides the right hand side of equation (9) is at most $\alpha_2(m) + c < m + c$, where c is a constant that depends only on a. Hence,

$$\pi_1(n) < m + c. \tag{10}$$

From the prime number theorem, it follows that

$$\frac{n}{3\log n} < \pi_1(n) \qquad \text{for } n >> 1.$$

Hence,

$$\frac{n}{3\log n} < m + c, \tag{11}$$

when $n >> 1$. From inequality (11), it follows that

$$n < 4m\log m \qquad \text{for } n >> 1. \tag{12}$$

We now investigate the large primes dividing both sides of equation (9). Assume that $m >> 1$ is such that $m/2$ is bigger than the denominator of a. In this case, all primes $q \in (m/2,\ m]$ divide the right hand side of equation (9). In particular, every such prime divides at least one of the factors from the right hand side of (9). Since

$$\alpha_p(n) + 1 < n + 1 < 4m\log m + 1 < (m/2)^2,$$

for $m >> 1$, it follows that every prime $q \in (m/2,\ m]$ divides exactly one of the factors from the left hand side of equation (9). In particular, there are at least $\pi_1(m)$ primes $p \le n$ such that $\alpha_p(n) + 1$ is at least $m/2$. Let p be one of such primes. Since

$$\frac{m}{2} < \alpha_p(n) + 1 = \frac{n - \mu_p(n)}{p - 1} + 1 < \frac{4m\log m}{p - 1} + 1,$$

it follows that

$$p < 1 + \frac{8m\log m}{m - 2} < 9\log m,$$

for $m >> 1$. But this last inequality shows that there are at most $\pi(9\log m) < 9\log m$ primes p for which $\alpha_p(n) + 1$ can be larger than $m/2$. Hence, we get

$$\pi_1(m) < 9\log m,$$

which, combined with the fact that

$$\pi_1(m) > \frac{m}{3\log m} \qquad \text{for } m >> 1,$$

shows that, in fact, m is bounded. Hence, equation (1) has only finitely many solutions when $f = \tau$ as well.

The Theorem is therefore proved. □

For the proof of the Corollary, we employ *ad hoc* divisibility arguments to deal with the equations involving ϕ and σ. For equation (5), we simply follow the procedure indicated in the proof of the Theorem.

2.2 The Proof of the Corollary.

The proof of (i)**.** The statement is true for $n \leq 4$. Now suppose that $n \geq 5$. Write $n! = 2^s \cdot t$ where t is odd. Then $\phi(n!) = 2^{s-1}\phi(t)$ where $\phi(t)$ is divisible by $\prod_{p\leq n}(p-1)$. In particular, $\phi(t)$ is divisible by $(3-1)(5-1) = 8$. It now follows that the exponent of 2 in the prime factor decomposition of $\phi(n!)$ is at least $s-1+3 > s$. On the other hand, since $m! = \phi(n!) < n!$, it follows that $m < n$. Thus, the exponent of 2 in the prime factor decomposition of $m!$ cannot exceed s. This gives the desired contradiction. □

The proof of (ii) **and** (iii)**.** One can check that the asserted solutions are the only ones for which $n \leq 8$. Assume now that $n \geq 9$. We have:

$$\frac{\sigma(n!)}{n!} < \frac{n!}{\phi(n!)} = \prod_{p\leq n}\frac{p}{p-1} \leq \prod_{\substack{2\leq k\leq n \\ k\neq 4,6,8,9}}\frac{k}{k-1} = n\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{8}{9} < \frac{n}{2}.$$

Hence, $n! < m! \leq 2\sigma(n!) < 2\cdot(n/2)\cdot n! < (n+1)!$, which is a contradiction. □

The proof of (iv)**.** We proceed in two steps.

Step I. *Suppose that* $(n,\ m)$ *is a solution of equation* (5)*. Then the following hold:*

1) *if* $n > 41$*, then*

$$m > \frac{3n}{10\log(n/2)};$$

2) *if* $m \geq 340$*, then*

$$n > \frac{m^2}{12}.$$

1) Suppose that $(n,\ m)$ is a solution of (5) with $n > 41$. Since $\tau(s) \leq s$ for all $s \geq 1$, it follows that $n \geq m$. Let

$$n! = p_1^{\alpha_1(n)}p_2^{\alpha_2(n)}\cdots p_{\pi(n)}^{\alpha_{\pi(n)}(n)},$$

where $2 = p_1 < 3 = p_2 < \cdots < p_{\pi(n)}$ are all the prime numbers less than or equal to n. Since

$$\alpha_i(n) = \left[\frac{n}{p_i}\right] + \left[\frac{n}{p_i^2}\right] + \cdots$$

for all $1 \leq i \leq \pi(n)$, it follows that $\alpha_i(n) \geq \alpha_j(n)$ whenever $i \leq j$. In particular

$$\alpha_1(n) = \max\bigl\{\alpha_i(n) \mid 1 \leq i \leq \pi(n)\bigr\}. \tag{13}$$

Equation (5) can now be rewritten as

$$(\alpha_1(n)+1)(\alpha_2(n)+1)\dots(\alpha_{p_{\pi(n)}}(n)+1)=m!. \tag{14}$$

We now use the inequality

$$\pi(2x)-\pi(x)>\frac{3x}{5\log x} \qquad \text{for } x>20.5 \tag{15}$$

(see [6]) with $x=n/2$, to conclude that at least $\dfrac{3n}{10\log(n/2)}$ of the $\alpha_i(n)$'s are equal to 1. Hence,

$$m\geq m-\mu_2(m)=\mathrm{ord}_2(m!)=\mathrm{ord}_2\Big(\prod_{i=1}^{\pi(n)}(\alpha_{p_i}(n)+1)\Big)>\frac{3n}{10\log(n/2)}, \tag{16}$$

which proves 1).

2) Suppose that $(n,\ m)$ is a solution of equation (5) with $m\geq 340$. Applying inequality (16) for $x=m/2$, it follows that there are at least

$$k=\Big[\frac{3m}{10\log(m/2)}\Big]+1$$

primes q such that $m/2<q\leq m$. Since all these primes divide

$$\prod_{i=1}^{\pi(n)}(\alpha_i+1),$$

we conclude that one of the following situations must occur:

CASE 1. *There exist two primes p and q such that $m/2<p<q\leq m$ and $pq\mid\alpha_i(n)+1$ for some $i\geq 1$.*

In this case

$$n\geq n-\mu_2(n)+1\geq\alpha_1(n)+1\geq\alpha_i(n)+1\geq pq>\frac{m^2}{4}>\frac{m^2}{12}. \tag{17}$$

CASE 2. *For every $i\geq 1$ the number $\alpha_i(n)+1$ is divisible by at most one prime $p>m/2$.*

By the arguments employed at CASE 1, we may assume that none of the numbers $\alpha_i(n)+1$ is divisible by two distinct primes $p>m/2$. Since there are k such primes and each one of the numbers $\alpha_i(n)+1$ is divisisble by at most one of them, it follows that k of the numbers $\alpha_i(n)+1$ are larger than $m/2$. Since the sequence $(\alpha_i(n)+1)_{i\geq 1}$ is decreasing, it follows that $\alpha_k(n)+1>m/2$. Hence,

$$\frac{n}{p_k-1}+1>\alpha_k(n)+1>\frac{m}{2}$$

or

$$n>\frac{1}{2}(m-2)(p_k-1). \tag{18}$$

Since $p_s>s\log s$ for all $s\geq 1$ (see [5]), it follows that

$$n>\frac{1}{2}(m-2)(k\log k-1)>\frac{1}{2}(m-2)\Big(\Big(\frac{3m}{10\log(m/2)}\Big)\log\Big(\frac{3m}{10\log(m/2)}\Big)-1\Big). \tag{19}$$

From inequality (19), it follows that in order to prove that $n>m^2/12$ it suffices to show that

$$\frac{1}{2}(m-2)\Big(\Big(\frac{3m}{10\log(m/2)}\Big)\log\Big(\frac{3m}{10\log(m/2)}\Big)-1\Big)>\frac{m^2}{12}\qquad\text{for } m\geq 340$$

or, with $x=m/2$, that

$$f(x)=\Big(1-\frac{1}{x}\Big)\Big(\Big(\frac{3}{5\log(x)}\Big)\log\Big(\frac{3x}{5\log(x)}\Big)-\frac{2}{x}\Big)>\frac{1}{3}\qquad\text{for } x>170. \tag{20}$$

One can now check, using Mathematica for example, that $f(x)>1/3$ for $x>161.5$.

Step II. *The only solutions* $(n,\ m)$ *of equation* (5) *are the asserted ones.*

We first show that if $(n,\ m)$ is a solution, then $m<340$ and $n<9608$.

Suppose that $m\geq 340$. In this case, by 2) of Step I, it follows that

$$n>\frac{m^2}{12}\geq\frac{340^2}{12}>41.$$

By 1) of Step I it follows that

$$m>\frac{3n}{10\log(n/2)}.$$

Since the function $g(x)=\dfrac{3x}{10\log(x/2)}$ is increasing for $x>2e$ and since $n>m^2/12$, it follows that

$$m>\frac{3n}{10\log(n/2)}>\frac{m^2}{40\log(m^2/24)} \tag{21}$$

or

$$m < 40 \log\Big(\frac{m^2}{24}\Big). \tag{22}$$

Inequality (22) implies that $m < 338.95 < 340$.

We now show that $n < 9608$. Suppose that $n > 41$. By 1) of Step I, it follows that

$$\frac{3n}{10\log(n/2)} < m < 340.$$

Hence,

$$n < \frac{3400}{3} \log\Big(\frac{n}{2}\Big). \tag{23}$$

Inequality (23) implies that $n < 9607.5 < 9608$.

One can now use Mathematica to test that the asserted solutions are the only ones in the range $m < 340$ and $n < 9608$.

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References

[1] P. Erdös, R. Graham, *On products of factorials*, Bull. Inst. Math. Acad. Sinica, Taiwan, **4** (1976), 337–355.

[2] P. Erdös, R. Obláth, *Über diophantische Gleichungen der Form* $n! = x^p \pm y^p$ *und* $n! \pm m! = x^p$, Acta Szeged. **8** (1937), 241–255.

[3] R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, New York, 1994.

[4] C. Pomerance, *Problem 10331*, Amer. Math. Monthly **100** (1993), 796.

[5] B. Rosser, *The n-th prime is greater than* $n \log n$, Proc. Lond. Math. Soc. (2) **45** (1939), 21–44.

[6] B. Rosser, *Approximate formulas for some functions of prime numbers*, Illinois J. of Math. **6** (1962), 64–94.