

# Piecewise $C^2$ Viscosity Solutions for Stochastic Singular Control

*Soluciones de Viscosidad  $C^2$  a Trozos para Control Singular Estocástico*

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## Abstract

We consider a class of dynamic programming equations arising from stochastic singular optimal control problems. We show that when a candidate solution of this nonlinear second order equation is  $C^1$  and piecewise  $C^2$  then it is **always** a viscosity solution of said equation. The result is applied to the dynamic programming equation of a singular problem that has a convex non- $C^2$  viscosity solution. **Key words and phrases:** Singular stochastic optimal control, viscosity solutions, dynamic programming

## Resumen

Consideramos una clase de ecuaciones de programación dinámica provenientes de problemas de control óptimo estocástico singular. Mostramos que cuando una solución candidata para esta ecuación no lineal de segundo orden es  $C^1$  y  $C^2$  a trozos, entonces es **siempre** una solución de viscosidad para dicha ecuación. Este resultado es aplicadu a la

ecuación de la programación dinámica. **Palabras y frases clave:** control óptimo estocástico singular, soluciones de viscosidad, programación dinámica.

## 1 Introduction

Let  $\mathcal{S}_n$  denote the set of all symmetric real-valued  $n \times n$ -matrices. Some infinite horizon stochastic singular optimal control problems with the control taking values in a closed cone  $U \subset \mathbb{R}^k$  lead to a dynamic programming equation of the form

$$\max[F^1(x, v(x), Dv(x), D^2v(x)), F^2(x, v(x), Dv(x))] = 0 \quad (1)$$

for  $x \in \mathcal{O}$ , where  $\mathcal{O}$ , the state space of the control problem, is some nonempty connected open subset of  $\mathbb{R}^n$ ,  $F^1$  is a continuous real-valued function on  $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n$  and  $F^2$  is a continuous real-valued function on  $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n$ . The *coercivity assumption* that the function  $r \rightarrow F^1(x, u, p, r)$  is nonincreasing on  $\mathcal{S}_n$  is universal in the theory of viscosity solutions, and this assumption is made here. In fact, it can be shown in great generality that the value function of such control problems satisfies an equation of the form of (1) in the viscosity sense (see [7, Chapter VIII]). Thus, (1) is a necessary condition for the value function of the optimal control problem. The PDE (1) is called a *free boundary problem*, since the crucial step in solving it is to locate the subset  $B$  of  $\mathcal{O}$ , called the *free boundary*, where there is a switch between the conditions

$$F^1(\cdot, v, Dv, D^2v) = 0, \quad F^2(\cdot, v, Dv) \leq 0, \quad (2)$$

and

$$F^1(\cdot, v, Dv, D^2v) \leq 0, \quad F^2(\cdot, v, Dv) = 0. \quad (3)$$

It is well known, and trivial to prove, that a  $C^2$  solution of (1) is a viscosity solution of (1). Also known is the fact that a weaker notion of solution of (1) where the solution is only assumed to have a locally Lipschitz derivative, and (1) is satisfied a.e., is in general not sufficient to provide a viscosity solution of (1). Our theorem is a result which sits in between these two facts. In fact, we show that naturally arising  $C^1$ , piecewise- $C^2$  solutions of (1) are in fact viscosity solutions of (1).

A route commonly followed to explicitly solve an optimal control problem involves the following three steps.

- (1.a) Piece together a candidate solution of the dynamic programming equation using differential equations methods. This gives a candidate value function.
- (1.b) Identify a candidate optimal feedback control strategy.
- (1.c) Prove a verification theorem asserting that the candidates in (1.a) and (1.b) are actually the optimal control and the value function, respectively, for the control problem.

An alternate route involves the following steps.

- (2.a) Prove that the dynamic programming equation has a unique solution in the viscosity sense.
- (2.b) Piece together a candidate solution of the dynamic programming equation using differential equations methods.
- (2.c) Prove that this candidate solution is indeed a viscosity solution of the dynamic programming equation.

The advantage of this second route is that it produces, by methods of analysis, the explicit solution of the dynamic programming equation and avoids, in the case of stochastic control problems, the difficult probabilistic steps ([1, pp. 53-54]) of having to identify the optimal feedback control and optimal trajectories, and having to evaluate the costs of these in order to prove a verification theorem. The drawback of this method is that without the probabilistic step, only a formal understanding of the optimal control policy and the optimal process is achieved.

If the candidate solution obtained by differential equations methods as in step (2.b) is  $C^2$ , then it is automatically a viscosity solution of (1). Such is the case for the well-known monotone follower problem ([1]), which has a  $C^2$  value function. Our theorem shows that after completing step (2.b), the naturally arising piecewise- $C^2$  (but perhaps not  $C^2$ ) solutions of (1) are likewise automatically viscosity solutions. That is, our theorem renders step (2.c) automatic. A result for multidimensional deterministic singular control problems related to ours appears in [6]. That result is based on a characterization of viscosity solutions across surfaces of nonsmoothness discovered by Crandall and Lions ([4]) for deterministic problems.

In section 3 we present a one-dimensional example for which the dynamic programming equation has more than one viscosity solution, one of which is  $C^1$ , convex, piecewise- $C^2$ , but not  $C^2$ . The proof that such a solution

of the dynamic programming equation is a viscosity solution follows from our main theorem. The fact that this solution is the value function of the posed stochastic control problem follows easily from a verification theorem. As we have said, viscosity solutions of the dynamic programming equation of this example are not unique. Therefore, what is needed is the discovery of an appropriate class of viscosity solutions for which there is a uniqueness theorem. Unfortunately, due to the quadratic growth of the value function it is not clear to the authors if there is a uniqueness theorem in the viscosity sense (see step (2.a)) that applies to this problem. Recent work on uniqueness of viscosity solutions for quadratically growing value functions appear in [2] and [3] but do not apply to this example. Thus, step (2.a) remains open for this example.

## 2 Main result

Let's recall the definition of viscosity solution (see [7, p. 66]). A continuous function  $v$  on an open set  $\mathcal{O} \subset \mathbb{R}$  is a *viscosity subsolution* of (1) if for each  $x_0 \in \mathcal{O}$

$$\max[F^1(x_0, v(x_0), D\phi(x_0), D^2\phi(x_0)), F^2(x_0, v(x_0), D\phi(x_0))] \leq 0,$$

for all test functions  $\phi \in C^2(\mathcal{O})$  such that  $v(x_0) = \phi(x_0)$ , and  $v - \phi$  attains a local maximum at  $x_0$ .

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for all test functions  $\phi \in C^2(\mathcal{O})$  such that  $v(x_0) = \phi(x_0)$ , and  $v - \phi$  attains a local minimum at  $x_0$ .

Finally,  $v$  is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

**Theorem 1.** *Assume that  $v_1$  and  $v_2$  are  $C^2$  on  $\mathcal{O}$ , and that*

$$F^1(\cdot, v_1, Dv_1, D^2v_1) = 0, \quad F^2(\cdot, v_2, Dv_2) = 0$$

*on  $\mathcal{O}$ . Further assume that  $\mathcal{O}$  has a decomposition  $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup B$ , where  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are nonempty disjoint open sets with  $B \subset \bar{\mathcal{O}}_1 \cap \bar{\mathcal{O}}_2$ , and that*

$$F^1(\cdot, v_2, Dv_2, D^2v_2) \leq 0 \quad \text{on } \mathcal{O}_2, \quad F^2(\cdot, v_1, Dv_1) \leq 0 \quad \text{on } \mathcal{O}_1.$$

Further assume that for all  $\phi \in C^2(\mathcal{O})$  such that  $v - \phi$  attains a local maximum at  $a$ , with  $v(a) = \phi(a)$ , there is a sequence  $\{x_n\} \subset \mathcal{O}_1$  such that  $x_n \rightarrow a$  as  $n \rightarrow \infty$ , and  $D^2v_1(x_n) \leq D^2\phi(x_n)$ .

If  $v_1 = v_2$  and  $Dv_1 = Dv_2$  on  $B$ , then the  $C^1$  function  $v$  that is equal to  $v_1$  on  $\mathcal{O}_1 \cup B$  and  $v_2$  on  $\mathcal{O}_2 \cup B$  is a viscosity solution of (1) on  $\mathcal{O}$ .

*Proof.* Since  $v$  is a classical solution off  $B$ ,  $v$  is a viscosity solution off  $B$ . Let  $a \in B$ . Let  $\phi \in C^2(\mathcal{O})$  be such that  $v - \phi$  attains a local maximum at  $a$ , with  $v(a) = \phi(a)$ . Then  $Dv(a) = D\phi(a)$  and

$$F^2(a, \phi(a), D\phi(a)) = F^2(a, v_2(a), Dv_2(a)) = 0.$$

Therefore, to show that  $v$  is a viscosity subsolution of (1) at  $x_0 = a$  we only need to show that  $F^1(a, \phi(a), D\phi(a), D^2\phi(a)) \leq 0$ . Since  $v - \phi$  has a local maximum at  $a$  and since  $v = v_1$  on  $\mathcal{O}_1$ , then there is a sequence  $\{x_n\} \subset \mathcal{O}_1$  such that  $x_n \rightarrow a$  as  $n \rightarrow \infty$ , and  $D^2v_1(x_n) \leq D^2\phi(x_n)$ . Therefore, by continuity and the coercivity assumption, we have

$$\begin{aligned} F^1(a, \phi(a), D\phi(a), D^2\phi(a)) &= F^1(a, v_1(a), Dv_1(a), D^2\phi(a)) \\ &= \lim_{n \rightarrow \infty} F^1(x_n, v_1(x_n), Dv_1(x_n), D^2\phi(x_n)) \\ &\leq \lim_{n \rightarrow \infty} F^1(x_n, v_1(x_n), Dv_1(x_n), D^2v_1(x_n)) = 0. \end{aligned}$$

The proof that  $v$  is a viscosity supersolution of (1) is trivial. In fact, let  $\phi \in C^2(\mathcal{O})$  be such that  $v(a) = \phi(a)$ , and  $v - \phi$  attains a local minimum at  $a$ . Then  $Dv(a) = D\phi(a)$ , so  $F^2(a, v(a), D\phi(a)) = F^2(a, v(a), Dv(a)) = 0$ . Therefore, (4) holds at  $x_0 = a$ .  $\square$

### 3 Example

Consider the dynamic programming equation

$$\max[F^1(x, v(x), v'(x), v''(x)), F^2(x, v(x), v'(x))] = 0 \tag{5}$$

for all  $x \in \mathbb{R}$ , where

$$\begin{aligned} F^1 &= v(x) - (x - K)^2 - bxv'(x) - \frac{1}{2}\epsilon^2x^2v''(x), \\ F^2 &= -v'(x) - 1, \end{aligned}$$

with given parameters  $b < 0$ ,  $K$  and a small enough  $\epsilon > 0$ . The equation (5) is satisfied in the viscosity sense (see [9]) by the value function  $v(x)$  which is the infimum over all progressively measurable control pairs  $(\xi(\cdot), u(\cdot))$  such that  $\xi(\cdot)$  is nondecreasing, real valued, left continuous with  $\xi(0) = 0$ ,  $E|\int_0^\infty u(s)d\xi(s)|^m < \infty$ , for  $m = 1, 2, \dots$ , and  $u$  is nonnegative of the costs

$$J^{\xi, u}(x) = E^x \int_0^\infty e^{-t} [(x(t) - K)^2 dt + u(t) d\xi(t)]. \quad (6)$$

Here  $E^x$  denotes the conditional expectation given that  $x(0) = x$ , and the one dimensional state  $x(t)$  satisfies the stochastic differential equation

$$dx(t) = bx(t) dt + u(t) d\xi(t) + \epsilon x(t) dW(t), \quad (7)$$

with the usual hypotheses on the Brownian process  $W(t)$ . This is a stochastic singular optimal control problem. More details on this problem appear in [9].

Let  $K = (1 - b)/2 > 0$ , and  $A = (1 - \epsilon^2 - 2b)^{-1} > 0$ . Define

$$V_1(x) = Ax^2 - x + K^2, \quad (8)$$

$$V_2(x) = K^2 - x, \quad (9)$$

$$V_3(x) = \begin{cases} V_2(x), & x \leq 0, \\ V_1(x), & x \geq 0. \end{cases} \quad (10)$$

Next, define the feedback control which is zero for  $x \geq 0$  and impulsive, producing a jump to  $x = 0$  at  $t = 0^+$ , for  $x < 0$  as

$$u^*(t, x) = \begin{cases} -x, & t = 0, x < 0, \\ 0, & \text{otherwise,} \end{cases} \quad (11)$$

$$\xi^*(t, x) = \begin{cases} 0, & t = 0, x < 0, \\ 1, & t = 0^+, x < 0, \\ 1 + t, & \text{otherwise.} \end{cases} \quad (12)$$

**Proposition 1.** 1.  $V_2$  is  $C^\infty$  in  $\mathbb{R}$ , and it is a classical solution of (5).

2.  $V_3$  is not  $C^2$  at  $x = 0$ .  $V_3$  is nonnegative on  $\mathbb{R}$ , but  $V_2$  is not.

3.  $V_2$  and  $V_3$  are convex viscosity solutions of (5).

4.  $V_3$  is the value function and  $(\xi^*(\cdot, x), u^*(\cdot, x))$  is the optimal feedback control for (6)-(7).

*Proof.* It is trivial to verify the convexity and stated continuous differentiability properties of  $V_2$  and  $V_3$ . Also, the statements on the signs of  $V_2$  and  $V_3$  are trivial. It also trivial to verify that  $V_2$  satisfies (5). To conclude the proof of (3), apply Theorem 1 to  $V_3$  once it has been verified that  $V_1$  and  $V_2$  satisfy

$$F^1(\cdot, V_1, DV_1, D^2V_1) = 0, \quad F^2(\cdot, V_2, DV_2) = 0$$

on  $\mathbb{R}$ , and that

$$F^1(\cdot, V_2, DV_2, D^2V_2) \leq 0, \quad x < 0; \quad F^2(\cdot, V_1, DV_1) \leq 0, \quad x > 0.$$

The proof of (4) involves a verification theorem ([7]). If  $x \geq 0$  then the solution of (7) corresponding to  $(u^*, \xi^*)$  is

$$x^*(t) = x \exp[(b - \epsilon^2/2)t - \epsilon W(t)].$$

Then it is easy to check that the cost  $J^{\xi^*, u^*}(x)$  produces the function  $V_3$ .  $\square$

Remark: The above proposition implies that in order to obtain a uniqueness theorem for viscosity solutions of (5) one has to keep in mind that the desirable solution must be positive. This example also shows that the principle of smooth fit can fail even for convex problems, cf. [7, below Theorem 4.2 in Chapter VIII]. This unusual feature of the value function is due to the fact that the diffusion (7) is degenerate at  $x = 0$  and that is precisely where the free boundary lies.

Another example where our theorem applies is provided by the non- $C^2$  value function found by Lehoczky and Shreve in [8, Thm. 6.1.b].

## References

- [1] BENEŠ, V. E., SHEPP, L. A. AND WITSENHAUSEN, H. S., Some solvable stochastic control problems, *Stochastics* **4**(1980)39-83.
- [2] BARDI, M. AND DA LIO, F., On the Bellman equation for some unbounded control problems, *Nonlinear Differential Equations and Applications* **4**(1997)491-510.
- [3] DA LIO, F., On the Bellman equation for infinite horizon problems with unbounded cost functionals, preprint.
- [4] CRANDALL, M. G. AND LIONS, P. L., Viscosity solutions of Hamilton-Jacobi equations, *Trans. AMS* **277**(1983)1-42.

- [5] CRANDALL, M. G. ISHII, H. AND LIONS, P. L., User's guide to viscosity solutions of second order partial differential equations, *Bull. AMS* **27**(1992)1-67.
- [6] DORROH, J. R. AND FERREYRA, G. S., Piecewise  $C^1$  viscosity solutions of some Bellman equations, *Communications in Applied and Nonlinear Analysis* **4** (1997) 63-65.
- [7] FLEMING, W. H. AND SONER, H. M., *Controlled Markov Processes and Viscosity Solutions*, Appl. Math., vol 25, Springer-Verlag, New York, 1993.
- [8] LEHOCZKY, J. P. AND SHREVE, S. E., Absolutely continuous and singular stochastic control, *Stochastics* **17** (1997) 91-109.
- [9] PASCAL, J., *Ph. D. Thesis*.