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Approximation of the Sobolev Trace Constant

Aproximación de la Constante Traza de Sobolev

Julio D. Rossi (jrossi@dm.uba.ar)
Departamento de Matemática, FCEyN
UBA (1428) Buenos Aires, Argentina.

Abstract

In this paper we study the Sobolev trace immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ with $1 < q < p^* = \frac{p(N-1)}{N-p}$ if $p > N$. We present an approximation procedure for the determination of the Sobolev trace constant and extremals, that is the best constant that verifies $S^{1/p}\|u\|_{L^q(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}$ and the functions where this constant is attained.
Key words and phrases: numerical approximations, p-Laplacian, nonlinear boundary conditions, Sobolev trace constant.

Resumen

En este artículo se estudia the inmersión traza de Sobolev $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ con $1 < q < p^* = \frac{p(N-1)}{N-p}$ si $p > N$. Se presenta un procedimiento de aproximación para la determinación de la constante traza de Sobolev y las extremales, esto es la mejor constante que verifica $S^{1/p}\|u\|_{L^q(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}$ y las funciones para las cuales se alcanza esta constante.
Palabras y frases clave: aproximación numérica, p-Laplaciano, condiciones de borde no lineales, constante traza de Sobolev.

1 Introduction

Let Ω be a bounded domain in R^N with smooth boundary. In this paper we deal with the Sobolev trace immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ with $1 < q <$

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$p^* = \frac{p(N-1)}{N-p}$ if $p < N$. This immersion is a continuous, compact operator and therefore there exists a constant S such that

$$S^{1/p}\|u\|_{L^q(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}.$$

This *Sobolev trace constant* S can be characterized as

$$S = \inf_{u\in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p + \int_\Omega |u|^p, \qquad \int_{\partial\Omega} |u|^q = 1 \right\}. \tag{1.1}$$

Using the compactness of the embedding it is easy to prove that there exists extremals, that is functions where the constant is attained. The extremals are weak solutions in $W^{1,p}(\Omega)$ of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. See [4] for a detailled analysis of the behaviour of extremals and best Sobolev constants in expanding domains for the linear case, $p = 2$.

In the case $p = q$ we have a nonlinear eigenvalue problem and the extremals are eigenfunctions of the first eigenvalue. In the linear case, that is for $p = 2$, this eigenvalue problem is known as the *Steklov* problem, [2]. In [5] it is proved that there exists a sequence of eigenvalues λ_n of (1.2) such that $\lambda_n \to +\infty$ as $n \to +\infty$. Also it is known that the first eigenvalue λ_1 is isolated and simple with a positive eigenfunction (see [8]). For the same type of results for the $p-$Laplacian with Dirichlet boundary conditions see [1], [6] and [7].

Our interest here is to approximate S. We remark that we are dealing with a nonlinear problem, (1.2), in the Banach space $W^{1,p}(\Omega)$. Let us describe a general approximation procedure. The idea is to replace the space $W^{1,p}(\Omega)$ with a subspace V_h in the minimization problem (1.1). To this end, let V_h be an increasing sequence of closed subspaces of $W^{1,p}(\Omega)$, such that

$$\begin{gathered} \left\{ v \in V_h : \int_{\partial\Omega} |v|^q = 1 \right\} \neq \emptyset \\ \text{and} \\ \lim_{h\to 0} \inf_{v\in V_h} \|u - v\|_{W^{1,p}(\Omega)} = 0, \quad \forall \|u\|_{W^{1,p}(\Omega)} = 1. \end{gathered} \tag{1.3}$$

With this sequence of subspaces V_h we define our approximation of S by

$$S_h = \inf_{u_h\in V_h} \left\{ \int_\Omega |\nabla u_h|^p + \int_\Omega |u_h|^p, \qquad \int_{\partial\Omega} |u_h|^q = 1 \right\}, \tag{1.4}$$

We prove that under hypothesis (1.3) S_h approximates S,

Theorem 1.1. *Let u be an extremal for* (1.1). *Then, there exists a constant C independent of h such that,*

$$|S - S_h| \leq C \inf_{v \in V_h} \|u - v\|_{W^{1,p}(\Omega)},$$

for every h small enough.

Regarding the extremals we have,

Theorem 1.2. *Let u_h be a function in V_h where the infimum* (1.4) *is achived. Then from any sequence $h \to 0$ we can extract a subsequence $h_j \to 0$ such that u_{h_j} converges strongly to an extremal in $W^{1,p}(\Omega)$. That is, there exists an extremal of* (1.1)*, w, with*

$$\lim_{h_j \to 0} \|u_{h_j} - w\|_{W^{1,p}(\Omega)} = 0.$$

We observe that the only requirement on the subspaces V_h is (1.3). This allows us, for example, to choose V_h as the usual finite elements spaces.

2 Proofs of the Theorems

Along this section we write C for a constant that does not depend on h and may vary from one line to another.
Proof of Theorem 1.1: As $V_h \subset W^{1,p}(\Omega)$ we have that

$$S \leq S_h. \tag{2.1}$$

Let us choose $v \in V_h$ such that $\|u - v\|_{W^{1,p}(\Omega)} \leq \inf_{V_h} \|u - w\|_{W^{1,p}(\Omega)} + \varepsilon$. We have that

$$S_h^{1/p} = \|u_h\|_{W^{1,p}(\Omega)} \leq \frac{\|v\|_{W^{1,p}(\Omega)}}{\|v\|_{L^q(\partial\Omega)}} \leq \frac{\|v - u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)}}{\|v\|_{L^q(\partial\Omega)}}$$

$$= \left(\frac{\|v - u\|_{W^{1,p}(\Omega)} + S^{1/p}}{\|v\|_{L^q(\partial\Omega)}} \right).$$

Now we use that

$$|\|v\|_{L^q(\partial\Omega)} - 1| \leq |\|v\|_{L^q(\partial\Omega)} - \|u\|_{L^q(\partial\Omega)}| \leq \|v - u\|_{L^q(\partial\Omega)} \leq C\|v - u\|_{W^{1,p}(\Omega)}$$

and hypothesis (1.3) to obtain that for every h small enough,

$$S_h \leq \left(\frac{\|v-u\|_{W^{1,p}(\Omega)} + S^{1/p}}{1 - C\|v-u\|_{W^{1,p}(\Omega)}} \right)^p \leq S + C\|v-u_1\|_{W^{1,p}(\Omega)}. \tag{2.2}$$

From (2.1) and (2.2) the result follows.

Proof of Theorem 1.2: Theorem 1.1 and hypothesis (1.3) gives that

$$\lim_{h\to 0} \|u_h\|^p_{W^{1,p}(\Omega)} = \lim_{h\to 0} S_h = S.$$

Hence there exists a constant C such that for every h small enough,

$$\|u_h\|_{W^{1,p}(\Omega)} \leq C.$$

Therefore we can extract a subsequence, that we denote by u_{h_j}, such that

$$\begin{aligned} u_{h_j} &\rightharpoonup w && \text{weakly in } W^{1,p}(\Omega), \\ u_{h_j} &\to w && \text{strongly in } L^p(\Omega), \\ u_{h_j} &\to w && \text{strongly in } L^q(\partial\Omega). \end{aligned} \tag{2.3}$$

Hence, from the $L^q(\partial\Omega)$ convergence we have,

$$1 = \lim_{h_j\to 0} \int_{\partial\Omega} |u_{h_j}|^q = \int_{\partial\Omega} |w|^q.$$

Therefore w is an admissible function in the minimization problem (1.1). Now we observe that,

$$\begin{aligned} \|u\|^p_{W^{1,p}(\Omega)} &\leq \|w\|^p_{W^{1,p}(\Omega)} \leq \liminf_{h_j\to 0} \|u_{h_j}\|^p_{W^{1,p}(\Omega)} \\ &\leq \lim_{h_j\to 0} \|u_{h_j}\|^p_{W^{1,p}(\Omega)} = \lim_{h_j\to 0} S_h = S = \|u\|^p_{W^{1,p}(\Omega)}, \end{aligned}$$

and therefore,

$$\lim_{h_j\to 0} \|u_{h_j}\|_{W^{1,p}(\Omega)} = \|w\|_{W^{1,p}(\Omega)} = S^{1/p}. \tag{2.4}$$

The space $W^{1,p}(\Omega)$ being uniformly convex, the weak convergence, (2.3), and the convergence of the norms, (2.4), imply the convergence in norm. Therefore $u_{h_j} \to w$ in $W^{1,p}(\Omega)$. This limit w verifies $\|w\|^p_{W^{1,p}(\Omega)} = S$ and $\|w\|_{L^q(\partial\Omega)} = 1$. Hence it is an extremal and we have that

$$\lim_{h_j\to 0} \|u_{1,h} - w\|_{W^{1,p}(\Omega)} = 0,$$

as we wanted to prove.

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