Some Algebraic Aspects of Quadratic Forms over Fields of Characteristic Two

Ricardo Baeza[1]

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Abstract. This paper is intended to give a survey in the algebraic theory of quadratic forms over fields of characteristic two. The relationship between differential forms and quadratics and bilinear forms over such fields discovered by Kato is used to reduced some problems on quadratics forms to concrete questions about differential forms, which in general are easier to handle.

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1 Introduction.

In his historical account on the algebraic theory of quadratic forms (s [Sch]), Scharlau remarks that fields of characteristic two have remained the pariahs of the theory. Nevertheless, as he also mentions right before the above remark (s. loc. cit.), some aspects of the theory over these fields are more interesting and richer, because of the interplay of symmetric bilinear and quadratic forms, as well as both separable and purely inseparable quadratic extensions have to be considered. The purpose of this brief survey article is to show how these aspects work, and how some questions related to Milnor's conjecture for fields with $2 \neq 0$, can be answered in a more elementary way in the case of characteristic two.

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We will focus our attention on the $W(F)$-module structure of $W_q(F)$, where $W(F)$ is the Witt-ring of a field F with $2 = 0$ and $W_q(F)$ is the Witt-group of quadratic forms over F (s. [Mi]$_2$, [Sa] and section 2). If $I \subset W(F)$ is the maximal ideal of $W(F)$, then we have the graded Witt-ring

$$gr_I W(F) = \bigoplus_{n=0}^{\infty} I^n / I^{n+1}$$

and the graded $gr_I W(F)$- module

$$gr_I W_q(F) = \bigoplus_{n=0}^{\infty} I^n W_q(F) / I^{n+1} W_q(F).$$

The structure of this module is explained in sections 3 and 4. Section 3 deals with the relationship established by Kato between differential forms over F and symmetric bilinear and quadratic forms. If $k_*(F)$ denotes Milnor's graded k-ring of F, we introduce in section 4 a graded $k_*(F)$-module, defined by generators and relations, which describes the graded $gr_I W(F)$-module $gr_I W_q(F)$. In section 5 we examine the behaviour of this module under certain field extensions, particularly function field extensions of quadrics defined by Pfister-forms. As an application of these results we mention, how Knebusch's degree conjecture for fields with $2 = 0$ follows from them. The results of section 5, (c.f. (5.10), (5.11), (5.14), (5.16)), cited from [Ar-Ba]$_3$ and [Ar-Ba]$_4$ have not been published yet, but these manuscripts can be found at the server "Linear Algebraic Groups and Related Structures" http://www.mathematik.uni-bielefeld.de/LAG/.

2 Basic definitions.

Let F be a field of characteristic two. A symmetric bilinear form $b : V \times V \longrightarrow F$ defined on an n-dimensional F-vector space V is non-singular if $b(x, y) = 0$ for all $x \in V$ implies $y = 0$. (V, b) is anisotropic if $b(x, x) \neq 0$ for all $x \neq 0$, and in this case it is easy to see that (V, b) admits an orthogonal basis (s. [Mi]$_2$ for example). If $a \in F^* = F \setminus \{0\}$ we will denote by $< a >$ the one dimensional form axy, and by $< a_1, \cdots, a_n >$ $(a_i \in F^*)$ the orthogonal sum $< a_1 > \perp \cdots \perp < a_n >$ A non singular quadratic form on V is a map $q : V \longrightarrow F$ such that $q(\lambda x) = \lambda^2 q(x)$ and $b_q(x, y) = q(x + y) - q(x) - q(y)$ is a symmetric non singular bilinear form on V. Since $b_q(x, x) = 0$, n must be even. The most simple non singular quadratic forms over F are the forms $ax^2 + xy + by^2$ with $a, b \in F$ (i.e. $q : Fe \oplus Ff \longrightarrow F, \quad q(e) = a, \quad q(f) = b, \quad b_q(e, f) = b_q(f, e) = 1$), which we will denote by $[a, b]$. Any non singular quadratic form over F is of the form $[a_1, b_1] \perp \cdots \perp [a_m, b_m]$. Scaling a quadratic form q by $a \in F^*$ means $(aq)(x) = aq(x)$. This extends to an operation of bilinear forms on quadratic forms by $< a_1, \cdots a_n > \cdot q = a_1 q \perp \cdots \perp a_n q$. Besides the dimension, the most simple invariant of a symmetric bilinear form $b =< a_1, \cdots a_n >$ is its

discriminant $d(b) = a_1 \cdots a_n \in F^*/F^{*^2}$ If $q = [a_1, b_1] \perp \cdots \perp [a_n, b_n]$ is a quadratic form the analogue of the discriminant is its Arf-invariant $A(q) = a_1b_1 + \cdots + a_nb_n \in F/\wp F$, where $\wp F = \{a^2 - a \backslash a \in F\}$.
One can write $[a, b] =< a > [1, ab]$ if $a \neq 0$, so that in general one usually writes a quadratic form q as $q =< a_1 > [1, b_1] \perp \cdots \perp < a_n > [1, b_n]$, and hence its Arf-invariant is $A(q) = b_1 + \cdots + b_n \in F/\wp F$ (s. [A], [Ba]$_1$, [Sa]). For quadratic forms (V, q) we have also the Clifford - algebra $C(q)$, which defines an element $w(q) \in Br(F) =$ Brauer group of F. If $q = \overset{m}{\underset{1}{\perp}} < a_i > [1, b_i]$, then $w(q) = \overset{m}{\underset{1}{\otimes}} (a_i, b_i] \in Br(F)$, where $(a, b]$ denotes the quaternion algebra $F \oplus Fe \oplus Ff \oplus Fef$ with $e^2 = a,\ f^2 + f = b,\ ef + fe = e$.
A symmetric bilinear form (V, b) is called metabolic if V contains a subspace $W \subseteq V$ with $W = W^\perp$ ($dim\ W = \frac{1}{2}\ dim\ V$). Two bilinear forms $b_1,\ b_2$ are Witt-equivalent if $b_1 \perp m_1 \cong b_2 \perp m_2$, where $m_1,\ m_2$ are metabolic. The set of classes $W(F)$ of symmetric non singular bilinear forms is a ring, additively generated by the classes $< a >,\ a \in F^*$ with relations $< a > + < b > = < a + b > + < ab(a + b) >$ if $a + b \neq 0$, $< a > + < a > = 0$ and $< a > \cdot < b > = < ab >$. We denote by $I_F \subset W(F)$ the maximal ideal of even dimensional forms (s. [Mi]$_2$, [Sa] for basic facts on $W(F)$). A quadratic form (V, q) is hyperbolic if V contains a totally isotropic subspace $W \subset V$ with $dim\ W = \frac{1}{2}\ dim\ V$. The form $[0, 0] = \mathbb{H}$ is the hyperbolic plane and every hyperbolic space is of the form $\mathbb{H} \perp \ldots \perp \mathbb{H}$. The forms $q_1,\ q_2$ are Witt-equivalent if $q_1 \perp r \times \mathbb{H} \cong q_2 \perp s \times \mathbb{H}$ $(r, s \geq 0)$ and we denote by $W_q(F)$ the Witt-group of such classes. The action defined above of bilinear forms on quadratic forms induces a $W(F)$-module structure on $W_q(F)$.
I_F is additively generated by the 1-fold Pfister forms $< 1, a >$, $a \in F^*$, so that for all $n \geq 1$, I_F^n is generated by the n-fold bilinear Pfister forms $\ll a_1, \cdots, a_n \gg = < 1, a_1 > \otimes \cdots \otimes < 1, a_n >$. These ideals define submodules $I_F^n \cdot W_q(F)$ of $W_q(F)$, which are additively generated by the n-fold quadratic Pfister forms $\ll a_1, \cdots a_n, a\, |] = \ll a_1, \cdots a_n \gg \otimes [1, a]$, $a_i \in F^*,\ a \in F$ (s. [Ba]$_1$, [Sa] for details on these forms).
Thus we have now two filtrations

$$W(F) \supseteq I_F \supset I_F^2 \supset \cdots \supset I_F^n \cdots$$

$$W_q(F) \supseteq IW_q(F) \supset I^2W_q(F) \supset \cdots \supset I^nW_q(F) \supset \cdots$$

and we will be mainly concerned with the quotients I_F^n/I_F^{n+1} and $I^nW_q(F)/I^{n+1}W_q(F)$ which we denote by $\overline{I}_F^n$ and $\overline{I^nW_q}(F)$ respectively. One easily checks that $dim : \overline{I}_F^0 \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$, $d : \overline{I}_F \xrightarrow{\sim} F^*/F^{*^2}$ and $A : \overline{I^0W_q}(F) \xrightarrow{\sim} F/\wp F$. The main result of [Sa] states that $w : \overline{IW_q}(F) \xrightarrow{\sim} Br(F)_2 =$ 2-torsion part of $Br(F)$. The surjectivity of w is a consequence of well-known results on p-algebras for $p = 2$ (s. [Al]), and the injectivity is shown in [Sa] by an elementary induction argument (notice

that the isomorphism $\overline{IW_q}(F) \xrightarrow{\sim} Br(F)_2$ is the analogue of Merkurjev's result $I_F^2/I_F^3 \xrightarrow{\sim} Br(F)_2$ for fields with $2 \neq 0$).
The higher groups $\overline{I}_F^n$ and $\overline{I^n W_q}(F)$ will be studied in the next section.

3 Differential forms and its relationship to quadratic and bilinear forms

The basic reference for what follows is Kato's fundamental paper [Ka]$_1$. Let Ω_F^1 be the F-vector space generated (over F) by the symbols da, $a \in F$, with the relations $d(ab) = bda + adb$. In particular $d(F^2) = 0$, and hence the map $d: F \longrightarrow \Omega_F^1$ is F^2-linear. Let $\Omega_F^n = \bigwedge^n \Omega_F^1$ be the F-space of n-differential forms over F. The map $d: F \longrightarrow \Omega_F^1$ extends to $d: \Omega_F^n \longrightarrow \Omega_F^{n+1}$ for all $n \geq 1$ by $d(xdx_1 \wedge \cdots \wedge dx_n) = dx \wedge dx_1 \wedge \cdots \wedge dx_n$. Recall that a 2-basis of F is a set $\{a_i\ ,\ i \in I\} \subset F$ such that the elements $\{a^\varepsilon = \prod_{i \in I} a_i^{\varepsilon_i},\quad \varepsilon = (\varepsilon_i,\ i \in I),\quad \varepsilon_i \in \{0,1\}$ and almost all $\varepsilon_i = 0\}$ form a F^2-basis of F. If $\{a_1, a_2, \dots\}$ is a 2-basis of F, then the forms $\dfrac{da_{i_1}}{a_{i_1}} \wedge \dots \wedge \dfrac{da_{i_n}}{a_{i_n}}$ $\ 1 \leq i_1 < \cdots < i_n$ form a F-basis of Ω_F^n. Fixing such a 2-basis, we define

$$[\Omega_F^n]^2 = \{ \sum_{i_1 < \cdots < i_n} c_{i_1 \cdots i_n}^2 \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_n}}{a_{i_n}}, \quad c_{i_1 \cdots i_n} \in F\}$$

which depends on the choice of the 2-basis. Then in [Ca] it is shown that the space $Z_F^n = ker(d : \Omega_F^n \longrightarrow \Omega_F^{n+1})$ has a direct-sum decomposition $Z_F^n = [\Omega_F^n]^2 \oplus d\Omega_F^{n-1}$.

One now defines a homomorphism
(3.1) $$C : Z_F^n \longrightarrow \Omega_F^n$$

by

$$C(\sum_{i_1 < \cdots < i_n} c_{i_1 \cdots i_n}^2 \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_n}}{a_{i_n}} + d\eta) = \sum_{i_1 < \cdots < i_n} c_{i_1 \cdots i_n} \frac{da_{i_1}}{a_{i_1}} \wedge \cdots \wedge \frac{da_{i_n}}{a_{i_n}}$$

C obviously does not depend on the choice of the 2-basis and induces an isomorphism $\overline{C} : Z_F^n / d\Omega_F^{n-1} \xrightarrow{\sim} \Omega_F^n$ of abelian groups.
We will call C the Cartier-operator. Let us define now the homomorphism $\wp = \overline{C}^{\,-1} - 1 : \Omega_F^n \longrightarrow \Omega_F^n / d\Omega_F^{n-1}$, which is given on generators by $\wp(x \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}) = (x^2 - x) \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \quad mod\ d\Omega_F^{n-1}$.
One can define a 2-basis dependent homomorphism $\wp : \Omega_F^n \to \Omega_F^n$ as follows. Fix a 2-basis $\mathcal{B} = \{a_1, a_2, \cdots\}$ of F. Then we set

$$\wp\left(\sum_{i_1<\cdots<i_n} c_{i_1\cdots i_n}\frac{da_{i_1}}{a_{i_1}}\wedge\cdots\wedge\frac{da_{i_n}}{a_{i_n}}\right) = \sum_{i_1<\cdots<i_n}(c^2_{i_1\cdots i_n}-c_{i_1\cdots i_n})\frac{da_{i_1}}{a_{i_1}}\wedge\cdots\wedge\frac{da_{i_n}}{a_{i_n}}.$$

If for $\omega = \sum_{i_1<\cdots<i_n} c_{i_1\cdots i_n}\frac{da_{i_1}}{a_{i_1}}\wedge\cdots\wedge\frac{da_{i_n}}{a_{i_n}}$ we set

$$\omega^{[2]} = \sum_{i_1<\cdots<i_n} c^2_{i_1\cdots i_n}\frac{da_{i_1}}{a_{i_1}}\wedge\cdots\wedge\frac{da_{i_n}}{a_{i_n}},$$

then $\wp\omega = \omega^{[2]}-\omega$.
Obviously if we change the 2-basis, the image of $\omega \in \Omega^n_F$ under the new $\wp$-operator differs from $\wp\omega$ by an exact form. We will use this type of operator in section 5.
Let $\nu_F(n) = Ker(\wp)$ and $H^{n+1}(F) = Coker(\wp)$, so that $0 \to \nu_F(n) \to \Omega^n_F \stackrel{\wp}{\to} \Omega^n_F/d\Omega^{n-1}_F \to H^{n+1}(F) \to 0$ is exact. An obvious characterization of $\nu_F(n)$ is the following

(3.2) Lemma. $\nu_F(n) = \{\omega \in \Omega^n_F \backslash d\omega = 0, C(\omega) = \omega\}$

In $[\mathrm{Ka}]_1$ it is shown that $\nu_F(n)$ is additively generated by the pure logarithmic differentials $\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}$, which is a direct consequence of lemma 2 in $[\mathrm{Ka}]_2$. Since we will refer frequently to this lemma, we will state it explicitly. Let $\mathcal{B} = \{a_i,\ i \in I\}$ be a 2-basis of F and endow I with a totally ordering. For any $j \in I$ set F_i resp. $F_{\leq j}$ for the subfield of F generated over F^2 by the elements a_i with $i < j$ resp. $i \leq j$. Endow with the lexicographic ordering the set $\sum_n$ of functions $\alpha : \{1,\cdots n\} \to I$ with $\alpha(i) < \alpha(j)$ whenever $i < j$. Then $\{da_{\alpha(1)}\wedge\cdots\wedge da_{\alpha(n)}, \alpha \in \sum_n\}$ is a F-basis of Ω^n_F and for any $\alpha \in \sum_n$ set $\Omega^n_{F,\alpha}$ resp. $\Omega^n_{F,<\alpha}$ for the subspace of Ω^n_F generated by the elements $da_{\beta(1)}\wedge\cdots\wedge da_{\beta(n)}$ with $\beta \leq \alpha$ resp. $\beta < \alpha$. Then Kato's lemma 2 in $[\mathrm{Ka}]_2$ asserts

(3.3) Lemma. *Let $y \in F$, $\alpha \in \sum_n$ and $\omega_\alpha = \frac{da_{\alpha(1)}}{a_{\alpha(1)}}\wedge\cdots\wedge\frac{da_{\alpha(n)}}{a_{\alpha(n)}} \in \Omega^n_F$, be such that*

$$(y^2-y)\omega_\alpha \in \Omega^n_{F,<\alpha} + d\Omega^{n-1}_F.$$

Then there exist $v \in \Omega^n_{F,<\alpha}$ and $a_i \in F^*_{\alpha(i)}$, $1 \leq i \leq n$, with

$$y\omega_\alpha = v + \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}.$$

It is clear that the last remark above follows immediately from this result, which we will quote as Kato's lemma in what follows.

One of the main results of $[\mathrm{Ka}]_1$ is the fact that there exist two natural isomorphisms

$$\alpha_F : \nu_F(n) \longrightarrow \overline{I}_F^{\,n} \tag{3.4}$$

$$\beta_F : H^{n+1}(F) \longrightarrow \overline{I^n W_q}(F) \tag{3.5}$$

given on generators by

$$\alpha_F\left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}\right) = \overline{\ll x_1, \cdots x_n \gg}$$

$$\beta_F\left(\overline{x\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}}\right) = \overline{\ll x_1, \cdots x_n, x\,]]}$$

Thus α and β translate many questions on bilinear and quadratic forms to corresponding problems in differential forms, which some times are easier to handle, in particular if one is able to choose a suitable 2-basis of the field F. Nevertheless the use of the isomorphism α can be some times difficult, since in order to compute $\alpha(\omega)$ one must first write $\omega \in \nu_F(n)$ as a sum of pure logarithmic differential forms.

4 MILNOR'S K-THEORY.

For any field F Milnor defined in $[\mathrm{Mi}]_1$ its K-groups $K_n(F)$ in a purely algebraic manner as follows (s. also Pfister's survey [Pf] for more details). Let $K_1(F)$ be the multiplicative group of F written additively, i.e. $l : F^* \xrightarrow{\sim} K_1(F)$, $l(ab) = l(a) + l(b)$ for $a, b \in F^*$. Set $K_0(F) = \mathbb{Z}$ and $K_n(F) = K_1(F)^{\otimes n}/\mathfrak{I}_n \quad (n \geq 2)$, where $\mathfrak{I}_n$ is the subgroup of $K_1(F)^{\otimes n}$ generated by elements of the form $l(a_1) \otimes \cdots \otimes l(a_n)$ with $a_i + a_j = 1$ for some $i \neq j$. Denote by $l(x_1) \cdots l(x_n)$ the image of $l(x_1) \otimes \cdots \otimes l(x_n)$. Thus the main defining relation of these groups is $l(a)l(1-a) = 0$ in $K_2(F)$ for $a \neq 0, 1$.

Let $k_n(F) = K_n(F)/2K_n(F)$ and form the commutative ring $k_*(F) = k_0(F) \oplus k_1(F) \oplus \cdots$ with $k_0(F) = \mathbb{Z}/2\mathbb{Z}$, $k_1(F) \xrightarrow{\sim} F^*/F^{*^2}$. Milnor defines epimorphisms $s_n : k_n(F) \to \overline{I}_F^{\,n}$ by

$$s_n(l(a_1)\cdots l(a_n)) = \overline{\ll a_1, \cdots a_n \gg}$$

and conjectures that they are isomorphisms for all n. If 2=0 in F, then there are also natural homomorphisms (s. $[\mathrm{Ka}]_1$)

$$d\log : k_n(F) \longrightarrow \nu_F(n)$$

given by $\quad d\log(l(a_1)\cdots l(a_n)) = \dfrac{da_1}{a_1} \wedge \cdots \wedge \dfrac{da_n}{a_n}.$

A consequence of Kato's lemma is that $d\log$ is an epimorphism. In $[\mathrm{Ka}]_1$ it is shown that $d\log$ is an isomorphism, which combined with the isomorphism (3.3) gives us the following main result of $[\mathrm{Ka}]_1$

(4.1) Theorem (Kato) *For any field F with 2=0 there is a commutative diagram of isomorphisms*

$$\begin{tikzcd} k_n(F) \arrow[rr, "d\log"] \arrow[dr, "s_n"'] & & \nu_F(n) \arrow[dl, "\alpha_F"] \\ & \overline{I}^n_F & \end{tikzcd}$$

The defining relation $l(a)l(a-1) = 0 \quad (a \neq 0,\ 1)$ of the groups $k_n(F)$ corresponds in the case $2 \neq 0$ to the basic fact that the quaternion algebra $(a, 1-a)$ splits. Here (x, y) denotes the quaternion algebra $F \oplus Fe \oplus Ff \oplus Fef,\ e^2 = x,\ f^2 = y,\ ef = -fe.$

But if 2=0 we do not have such interpretation and the groups $k_n(F)$ are suitable only to describe symmetric bilinear forms and for quadratic forms, we need another universal object, which we introduce now. Thus in order to obtain groups which are appropriate to describe the quotients $\overline{I^n W_q}(F)$ by generators and relations one is led to alter Milnor's definition of k_n taking into account the basic relations of quaternion algebras over a field with 2=0. This has been done in $[\text{Ar-Ba}]_1$. Let $a \in F^*,\ b \in F$. The quaternion algebra $(a, b]$ is the algebra $F \oplus Fe \oplus Ff \oplus Fef$ with $e^2 = a,\ f^2 + f = b$ and $ef + fe = e$. It holds $(ax^2, b + y + y^2] \cong (a, b]$, and $(a, b]$ splits if and only if
$a \in D_F([1, b]) = \{x^2 + xy + by^2 /\ x, y \in F\}$, and $a \neq 0$. Thus the bilinear map

$$\phi : F^*/F^{*^2} \times F/\wp F \longrightarrow Br(F)_2, \qquad \phi(\bar{a}, \bar{b}) = (a, b]$$

satisfies $\phi(\bar{a}, \bar{b}) = 0$ iff $a \in D_F([1, b])$. The universal symbol for ϕ can be constructed as follows. Let $k_1(F) = F^*/F^{*^2}, \quad h_1(F) = F/\wp F$ and set

$$h_2(F) = \frac{k_1(F) \otimes h_1(F)}{< l(a) \otimes t(b) \quad a \in D_F[1,b],\ a \neq 0 >}$$

(here $t(b)$ is the image of b in $h_1(F) = F/\wp F$).
Thus one obtains a natural homomorphism

$$\phi_F :\ h_2(F) \longrightarrow Br(F)_2$$

which is in fact an isomorphism (s. [Ar-Ba]$_1$,[Sa]). On the other hand we also have a bilinear map

$$k_1(F) \times h_1(F) \longrightarrow H^2(F)$$

given by $\quad (l(a), t(b)) \longrightarrow b\frac{da}{a}$, which induce a natural homomorphism

$$d\log :\ h_2(F) \longrightarrow H^2(F).$$

This homomorphism is also an isomorphism (s. loc. cit), so that the group $h_2(F),\ H^2(F),\ Br(F)_2,\ \overline{IW}_q(F)$ are all isomorphic and we have a commutative diagram of isomorphisms

(4.2)
$$\begin{array}{ccc} h_2(F) & \xrightarrow{\phi_F} & Br(F)_2 \\ {\scriptstyle d\log}\downarrow & & \uparrow{\scriptstyle \omega} \\ H^2(F) & \xrightarrow[\beta_F]{} & \overline{IW}_q(F) \end{array}$$

Let now

$$h_n(F) = k_1(F)^{\otimes(n-1)} \otimes h_1(F)/\mathcal{R}_n$$

where $\mathcal{R}_n$ is the subgroup generated by the elements $l(a_1) \otimes \cdots \otimes l(a_{n-1}) \otimes t(b)$ such that either $a_i + a_{i+1} = 1$ for some i or $a_i \in D_F[1,b]$. We denote by $l(a_1) \cdots l(a_{n-1}) t(b)$ in $h_n(F)$ the image of $l(a_1) \otimes \cdots \otimes l(a_{n-1}) \otimes t(b)$.
The natural product $k_r(F) \times h_s(F) \to h_{r+s}(F)$ induces a $k_*(F)$-module structure on $h_*(F) = h_1(F) \oplus h_2(F) \oplus \cdots$. There are natural epimorphisms

$$s_n : h_n(F) \longrightarrow \overline{I^{n-1}W}_q(F)$$

$$d\log : h_n(F) \longrightarrow H^n(F)$$

given by

$$s_n(l(a_1)\cdots l(a_{n-1})t(b)) = \overline{\ll a_1, \cdots a_{n-1}, b \,|]}$$

$$d\log(l(a_1)\cdots l(a_{n-1})t(b)) = \overline{b\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_{n-1}}{a_{n-1}}}$$

In $[\text{Ar-Ba}]_1$ it is shown that $d\log$ is an isomorphism, and combining it with Kato's isomorphism β_F, we conclude also that s_n is an isomorphism. Thus we have (s. $[\text{Ar-Ba}]_1$ and $[\text{Ka}]_1$)

(4.3) Theorem. *For all n there is a commutative diagram of isomorphisms*

$$\begin{tikzcd} h_n(F) \arrow[rr, "s_n"] \arrow[dr, "d\log"'] & & \overline{I^{n-1}W_q F(n)} \arrow[dl, "\beta_F"] \\ & H^n(F) & \end{tikzcd}$$

Remark. The groups $k_n(F)$ and $h_n(F)$ are related through Galois cohomology. If F_s is a separable closure of F and $G_F = Gal(F_s/F)$ then $k_n(F_s)$ is a G_F-module and it holds (s. $[\text{Ar-Ba}]_1$)

$$H^0(G_F, k_n(F_s)) \;\cong\; k_n(F)$$

$$H^1(G_F, k_n(F_s)) \;\cong\; h_{n+1}(F)$$

(s. [Ar]).

5 Behaviour of quadratic and bilinear forms under field extensions.

A natural question is the behaviour of the groups Ω_F^n, $\nu_F(n)$, $H^{n+1}(F)$ resp $\overline{I}_F^n$, $\overline{I^nW_q}(F)$ under field extensions. Since the isomorphisms α_F, β_F (s. (3.4) and (3.5)) are functorial, we only need to study the behaviour of the groups $\nu_F(n)$, $H^{n+1}(F)$, to get information about $\overline{I}_F^n$ and $\overline{I^nW_q}(F)$ (but, as mentioned before, care must be taken with the use of α_F). If L/F is a field extension, we denote by $\Omega_{L/F}^n$ the kernel $Ker(\Omega_F^n \to \Omega_L^n)$, and similarly we define $\nu_{L/F}(n)$, $H^{n+1}(L/F)$, $\overline{I}_{L/F}^n$ and $\overline{I^nW_q}(L/F)$. By the remark above

$\alpha_F : \nu_{L/F}(n) \xrightarrow{\sim} \overline{I}^{\,n}_{L/F}$ and $\beta_F : H^{n+1}(L/F) \xrightarrow{\sim} \overline{I^nW_q}(L/F)$. The easiest group to handle is $\Omega^n_{L/F}$ because a suitable choice (if possible!) of a 2-basis of F and L gives quickly the answer. Since

$$\nu_{L/F}(n) = \nu_F(n) \cap \Omega^n_{L/F} \tag{5.1}$$

one also gets information about $\nu_{L/F}(n)$ knowing $\Omega^n_{L/F}$. Let us now review what we know about these kernels for some field extensions.

(i) Purely Transcendental extensions. If $L = F(X)$, X any set of variables over F, and $\mathcal{B}$ is a 2-basis of F, then $\mathcal{B} \cup \{X\}$ is a 2-basis of $F(X)$. In particular $\Omega^n_F \to \Omega^n_{F(X)}$ is injective and $\Omega^n_{F(X)/F} = 0$. Hence $\nu_{F(X)/F}(n) = 0$. Using Kato's lemma (3.3) one can also show $H^{n+1}(F(X)/F) = 0$ (s. [Ar-Ba]$_3$)

(ii) Quadratic extensions. Let $L = F(\sqrt{b})$, $b \in F \backslash F^2$ be a purely inseparable quadratic extension of F. Choose a 2-basis $\mathcal{B} = \{b_i,\ i \in I\}$ with $b = b_{i_0}$, some $i_0 \in I$. Then $\{b_i, i \in I - \{i_0\}, \sqrt{b}\ \}$ is a 2-basis of $F(\sqrt{b})$ and it is easy to check that

$$\Omega^n_{F(\sqrt{b})/F} = \Omega^{n-1}_F \wedge \frac{db}{b} \tag{5.2}$$

Hence $\nu_{F(\sqrt{b})/F}(n) = \{\omega \wedge \frac{db}{b} \ /\ \omega \in \Omega^{n-1}_F,\ \omega \wedge \frac{db}{b} \in \nu_F(n)\}$. It follows from (5.11) below that

$$\begin{aligned}\nu_{F(\sqrt{b})/F}(n) = \{\omega \wedge \frac{db}{b} \ /\ \omega \in \Omega^{n-1}_F \text{ and } \wp\omega \in a[\Omega^{n-1}_F]^2 + \\ d\Omega^{n-2}_F + \Omega^{n-2}_F \wedge da\}\end{aligned} \tag{5.3}$$

(s. section 3 for the definition of $\wp\omega$).

The corresponding result for $\overline{I}^{\,n}$ is now (s. (5.12) below for a more general statement)

$$\overline{I}^{\,n}_{F(\sqrt{b})/F} = \sum_{x \in F^2(b)^*} \overline{I}^{\,n-1}_F < 1, x > \tag{5.4}$$

Let us now examine the kernel $H^{n+1}(F(\sqrt{b})/F)$.
We have (s.[Ar-Ba]$_3$)

$$H^{n+1}(F(\sqrt{b})/F) = \overline{\Omega^{n-1}_F \wedge \frac{db}{b}} \tag{5.5}$$

The proof of this fact is again based on Kato's lemma and runs briefly as follows. Take $\mathcal{B} = \{b_1 = b,\ b_2, \cdots\}$ a 2-basis of F (one can assume w.l.o.g. that $\mathcal{B}$ is enumerable or even finite), so that $\mathcal{B}' = \{\sqrt{b_1},\ b_2, \cdots\}$ is a 2-basis of $F(\sqrt{b})$. $\overline{\omega} \in H^{n+1}(F(\sqrt{b})/F)$ means $\omega \in \Omega^n_F$ and $\omega = \wp u + dv$ with $u \in \Omega^n_{F(\sqrt{b})}$, $v \in \Omega^{n-1}_{F(\sqrt{b})}$. Order $\mathcal{B}'$ such that $\sqrt{b} > b_i, i = 2, 3\cdots$. Since $\overline{\Omega^{n-1}_F \wedge \frac{db}{b}} \subseteq H^{n+1}(F(\sqrt{b})/F)$ we may assume that db does not appear in the 2-basis expansion of ω and let $\alpha \in \sum_n$ be the leading index of ω (notice $\alpha(i) > 1$ for all $i = 1, \cdots n$), and let $\beta \in \sum_n$ be the leading index of u. Using Kato's lemma one may assume $\beta \leq \alpha$, and we obtain

$$(\wp u_\alpha + \omega_\alpha)\frac{db_\alpha}{b_\alpha} \equiv dv \qquad \mod \Omega^n_{F(\sqrt{b}),<\alpha}$$

(here $\frac{db_\alpha}{b_\alpha}$ means $\frac{db_{\alpha(1)}}{b_{\alpha(1)}} \wedge \cdots \wedge \frac{db_{\alpha(n)}}{b_{\alpha(n)}}$)
with $v \in \Omega^{n-1}_{F(\sqrt{b})}$. Since $b_{\alpha(i)} < \sqrt{b}$ for all i, we conclude comparing coefficients that the leading coefficient of dv is in F, so that u_α is defined over F. Thus v may be taken also in Ω^{n-1}_F. Since $\Omega^n_{F(\sqrt{b})/F} = \Omega^{n-1}_F \wedge \frac{db}{b}$, we conclude in Ω^n_F

$$\omega_\alpha \frac{db_\alpha}{b_\alpha} \equiv \wp(u_\alpha)\frac{db_\alpha}{b_\alpha} + dv \qquad \mod \Omega^n_{F,<\alpha} + \Omega^{n-1}_F \wedge \frac{db}{b}$$

Inserting this relation in ω, we can lower the highest index in ω. This concludes the proof of the claim.
The corresponding kernel for $I^n W_q$ is now

$$(5.6) \qquad \overline{I^n W_q}(F(\sqrt{b})/F) = \overline{\ll b \gg I^{n-1}W_q(F)}$$

For quadratic separable extensions of F the corresponding kernels are much easier to compute. Let $L = F(z)$, $z^2 + z = b$ $(b \notin \wp F)$ be a quadratic separable extension of F. Since we can alter b by elements of $\wp F$, we can assume $b \in F^2$. Thus $z \in L^2$ and we see that any 2-basis of F remains a 2-basis of L. In particular $\Omega^n_L = \Omega^n_F \oplus z \cdot \Omega^n_F$. Thus $\Omega^n_{L/F} = 0$ and also $\nu_{L/F} = 0$. The computation of $H^{n+1}(L/F)$ is in this case also very easy. We claim

$$(5.7) \qquad H^{n+1}(L/F) = \overline{b\nu_F(n)}$$

For the proof, take $\overline{\omega} \in H^{n+1}(F)$ with $\omega = \wp u + dv$, $u \in \Omega^n_L$, $v \in \Omega^{n-1}_L$ and set $u = u_1 + zu_2$, $v = v_1 + zv_2$ with $u_i \in \Omega^n_F$, $v_i \in \Omega^{n-1}_F$. Inserting in the above equation it follows $\wp u_2 = dv_2 \in d\Omega^{n-1}_F$, and this means $u_2 \in \nu_F(n)$. Moreover $\omega = bu_2^{[2]} + \wp u_1 + dv_1$ in Ω^n_F. But $u_2 \in \nu_F(n)$ implies $u_2^{[2]} \equiv u_2$ ($\mod d\Omega^{n-1}_F$) and since $b \in F^2$, it follows $\omega \equiv bu_2 \mod (\wp\Omega^n_F + d\Omega^{n-1}_F)$, ie $\overline{\omega} = \overline{bu_2}$. This proves (5.7). The corresponding result for quadratic forms is

$$(5.8) \qquad \overline{I^n W_q}(L/F) = \overline{I^n_F \cdot [1, b]}$$

(iii) Function fields of Pfister forms. Let us fix an anisotropic bilinear n-fold Pfister-form $\phi = \ll a_1, \cdots, a_n \gg$. This means that $\{a_1, \cdots, a_n\}$

are part of 2-basis of F. Let $L = F(\phi)$ be the function field of the quadric $\{\phi(x,x) = 0\}$. Thus $L = F(X)(\sqrt{T})$, where $X = \{X_\mu,\ \mu \in S_n\}$ and $T = \sum_\mu a^\mu X_\mu^2$, $a^\mu = \prod_{i=1}^n a_i^{\mu(i)}$, for all $\mu \in S_n$ where S_n denotes the set of maps $\mu : \{1, \cdots, n\} \to \{0,1\}$ whith some $\mu(i) = 1$
In [Ar-Ba]$_3$ it is shown that

$$\Omega^m_{L/F} = 0 \qquad \text{if } m < n \tag{5.9}$$

$$\Omega^m_{L/F} = \Omega_F^{m-n} \wedge \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \qquad \text{if } m \geq n \tag{5.10}$$

In particular $\nu_{L/F}(m) = 0$ if $m < n$. The case $m \geq n$ has been considered in [Ar-Ba]$_4$ and the result is:

$$\nu_{L/F}(m) = \{\omega \wedge \tfrac{da_1}{a_1} \wedge \cdots \wedge \tfrac{da_n}{a_n} /\ \omega \in \Omega_F^{m-n},\ \wp\omega \in \sum_{\varepsilon \neq 0} a^\varepsilon [\Omega_F^{m-n}]^2 + d\Omega_F^{m-n-1} + \sum_{i=1}^n \Omega_F^{m-n-1} \wedge da_i\} \tag{5.11}$$

If $m = n$, this result looks nicer, namely

$$\nu_{L/F}(n) = \{a\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} /\ a^2 - a \in F^2(a_1, \cdots a_n)'\ \}$$

where $F^2(a_1, \cdots, a_n)' \subset F^2(a_1, \cdots a_n)$ is the subgroup consisting in the elements $\sum_{\varepsilon \neq 0} c_\varepsilon^2 a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$, $\varepsilon = (\varepsilon_1, \cdots \varepsilon_n) \in \{0,1\}^n$.

The corresponding result for bilinear forms is

$$I^m_{L/F} = \left\langle \overline{\psi \ll x_1, \cdots x_n \gg} /\ \psi \in \overline{I}_F^{\,m-n},\ \ x_1, \cdots, x_n \in F^2(a_1, \cdots a_n)^* \right\rangle \tag{5.12}$$

The case $m = n$ is particularly interesting, because

$$\overline{I}^{\,n}_{L/F} = \{\overline{\ll x_1, \cdots x_n \gg} /\ x_i \in F^2(a_1, \cdots a_n)^*\}$$

implies the following corollary

(5.13) COROLLARY. *Given $x_1, \cdots x_n, y_1, \cdots y_n \in F^2(a_1, \cdots a_n)^*$, then there exist $z_1, \cdots z_n \in F^2(a_1, \cdots a_n)^*$ such that*

$$\ll x_1, \cdots, x_n \gg + \ll y_1, \cdots, y_n \gg \equiv \ll z_1, \cdots z_n \gg \quad \bmod\ I_F^{n+1}$$

This is a kind of relative n-linkage property of the subfields $F^2(a_1, \cdots, a_n)$ of F.

Let us now turn our attention to H^{n+1}. The main result of [Ar-Ba]$_3$ is

(5.14) Theorem. *If* $\phi = \ll a_1, \cdots a_n \gg$ *is anisotropic over* F, *then*

$$H^{n+1}(F(\phi)/F) = \overline{F\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}}$$

The proof of this fact, although elementary, is rather long. For $\overline{\omega} \in H^{n+1}(F(\phi)/F)$ we get an equation $\omega = \wp u + dv$ with $u \in \Omega^n_{F(\phi)}$ and $v \in \Omega^{n-1}_{F(\phi)}$. Writing $F(\phi) = L(y)$, $L = F(X_\mu,\ \mu \in S_n)$, $y^2 = T = \sum_{\mu \in S_n} a^\mu X_\mu^2$, $a^\mu = a_1^{\mu(1)} \cdots a_n^{\mu(n)}$, we choose a 2-basis $\mathcal{B} = \{a_i,\ i \in I\}$ of F containing $a_1, \cdots a_n$, so that $\mathcal{B} \cup \{X_\mu,\ \mu \in S_n\}$ is a 2-basis of L and then we fix a 2-basis $\mathcal{B}' = \mathcal{B} \setminus \{a_1\} \cup \{X_\mu,\ \mu \in S_n\} \cup \{y\}$ of $F(\phi)$. We order the elements of this basis such that all $X_\mu > \mathcal{B} \setminus \{a_1\}$ and $y > X_\mu$ for all μ (i.e. y is maximal). Using these choices, and Kato's lemma, one sees that u and v can be chosen free of differentials of the form dX_μ or dy, and moreover that the scalar coefficients of u and v do not contain y in the 2-basis expansion. Thus u and v are defined over $L = F(X_\mu)$. But since $H^{n+1}(F(\phi)/L) = \overline{\Omega_L^{n-1} \wedge dT}$ by (5.5),we have

$$\omega = \wp u + dv + \lambda \wedge dT \tag{5.15}$$

in Ω^n_L, with some $\lambda \in \Omega^{n-1}_L$. Expanding with respect to the 2-basis $\mathcal{B} \cup \{X_\mu,\ \mu \in S_n\}$ and comparing coefficients, one can show that $u,\ v,\ \lambda$ can be taken in $\Omega^n_F \otimes M$ and $\Omega^{n-1}_F \otimes M$ respectively, where $M = F(X_\mu^2,\ \mu \in S_n)$. This is the start for long descent argument which leads to an equation $\omega = \wp u_0 + dv_0 + b da_1 \wedge \cdots \wedge da_n$ whith $b \in F$ and u_0, v_0 defined over F
The corresponding result for quadratic forms is

(5.16) Theorem

$$\overline{I^n W_q}(F(\phi)/F) = \{\overline{\ll a_1, \cdots, a_n, a\,]]}\ /\ a \in F\}$$

As it is shown in [Ar-Ba]$_2$, this result implies the following one. Let $p = \ll a_1, \cdots, a_n, a\,]]$ be now an anisotropic quadratic n-fold Pfister form and let $F(p)$ be the function field of the quadric $\{p(x) = 0\}$. Then

(5.17) Theorem

$$H^{n+1}(F(p)/F) = \{0,\ \bar{p}\}$$

Remark. One may expect that (5.14) generalizes to the following assertion

$$H^{m+1}(F(\phi)/F) = \overline{\Omega_F^{m-n} \wedge \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}} \qquad , m \geq n.$$

6 An application:

generic splitting of quadratic forms.
One can develop a generic splitting theory for non singular quadratic forms over a field with $2 = 0$ in the same way as it has been done for the case $2 \neq 0$ in $[\text{Kn}]_{1,2}$, because in the case $2 = 0$ one has:

(i) the analogue of Pfister's subform theorem (s. [Am], $[\text{Ba}]_3$ and [Le])

(ii) The analogue of Knebusch's norm theorem (s. $[\text{Ba}]_2$).

With these tools one defines a generic splitting tower of a non singular quadratic form q over F and obtains a leading form, which is similar to a Pfister form. The degree of this form is called the degree of q. Now define $\mathfrak{I}(n) = \{\bar{q} \in W_q(F) \ /deg\ q \geq n\}$. Then $\mathfrak{I}(n)$ is a $W(F)$-submodule of $W_q(F)$ and one easily sees that $I^n W_q(F) \subseteq \mathfrak{I}(n)$. In $[\text{Ar-Ba}]_3$ it is shown that the equality $\mathfrak{I}(n) = I^n W_q(F)$ for all n (over a field of any characteristic) is equivalent with the statement of theorem (5.17) above for any n. Thus we have

(6.1) Theorem *For any field F with $2 = 0$, it holds*

$$\mathfrak{I}(n) = I^n W_q(F)$$

Remark. The corresponding result for (5.17) over fields with $2 \neq 0$ has been announced by Orlov-Vishik-Voevodsky (s. [Pf]).

References

[Al] Albert, A. Structure of Algebras. Amer. Math. Soc. Publ. 24, Providence, AMS, (1939).

[A] Arf, C. Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. J. reine ang. Math. 183, 148-167 (1941).

[Am] Amer, M. Quadratische Formen über Funktionenkörpern. Unpublished dissertation, Johannes Gutenberg Universität, Mainz, (1976).

[Ar] Arason, J. Wittring und Galoiscohomologie bei Charakteristik 2 J. reine ang. Math. 307/308, 247-256, (1979).

$[\text{Ar-Ba}]_1$ Aravire, R., Baeza, R. Milnor's K-theory and quadratic forms over fields of characteristic two. Comm. Alg. 20(4) 1087-1107, (1992).

$[\text{Ar-Ba}]_2$ Aravire, R., Baeza, R. A note on generic splitting of quadratic forms. Comm. Alg., 27(7), 3473-3477, (1999).

$[\text{Ar-Ba}]_3$ Aravire, R., Baeza, R. The behaviour of quadratic and differential forms under function field extensions in characteristic two. Preprint 2000. (Submitted) http://www.mathematik.uni-bielefeld.de/LAG/

[Ar-Ba]$_4$ Aravire, R., Baeza, R. Linkage of fields in characteristic two. Preprint 2001. (Submitted) http://www.mathematik.uni-bielefeld.de/LAG/

[Ba]$_1$ Baeza, R. Quadratic Forms over semi-local Rings. LNM 655, Springer-Verlog, (1978).

[Ba]$_2$ Baeza, R. The norm theorem for quadratic forms over a field of characteristic two. Comm. Alg. 18(5), 1337-1348 (1990).

[Ba]$_3$ Baeza, R. Ein Teilformensatz für quadratische Formen in Charakteristik 2. Math. Zeit. 135, 175-184 (1974).

[Ca] Cartier, P. Questions de rationalité des diviseurs en géometrie algebrique. Bull. Soc. Math. France, 86, 177-251, 1958.

[Ka]$_1$ Kato, K. Symmetric bilinear forms, quadratic forms and Milnor's K-theory in characteristic two. Inv. Math. 66, 493 - 510, (1982).

[Ka]$_2$ Kato, K. Galois cohomology of complete discrete valuation fields. Algebraic K-theory, Proc. Oberwolfach, Part II (R. K. Dennis, Ed.) LNM vol. 967, Springer-Verlag, (1982).

[Kn]$_1$ Knebusch, M. Generic splitting of quadratic forms I. Proc. London Math. Soc. (3) 33, 65-93 (1976).

[Kn]$_2$ Knebusch, M. Generic splitting of quadratic forms II. Proc. London Math. Soc. (3) 34, 1-31 (1977).

[Le] Leep, D. The Amer-Brumer theorem over arbitrary fields. Preprint 2001.

[Mi]$_1$ Milnor, J. Algebraic K-theory and quadratic forms. Invent. Math. 9, 318-344 (1970).

[Mi]$_2$ Milnor, J. Symmetric inner products in characteristic 2. Prospects in Mathematics, Ann. of Math. Studies, Princeton Univ. Press. 59-75 (1971).

[Pf] Pfister, A. On the Milnor Conjectures History, Influence, Applications. Jber. d. Dt. Math. Verein. 102, 15-41 (2000).

[Sa] Sah, C-H. Symmetric bilinear forms and quadratic forms. J. of Algebra 20, 144-160 (1972).

[Sch] Scharlau, W. On the history of the algebraic theory of quadratic forms. Proc. of the Dublin Conference on Quadratic Forms and their Applications. Contemporary Mathematics 272 (E. Bayer-Fluckiger, D. Lewis, A. Ranicki, Ed.) A.M.S, 229-259, (2000).

Ricardo Baeza
Instituto de Matematica
Universidad de Talca
Casilla 721
Talca, Chile
rbaeza@inst-mat.utalca.cl