On the Number of Square Classes of a Field of Finite Level

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Abstract. The *level question* is, whether there exists a field F with finite *square class number* $q(F) := |F^\times/F^{\times^2}|$ and finite level $s(F)$ greater than four. While an answer to this question is still not known, one may ask for lower bounds for $q(F)$ when the level is given.

For a nonreal field F of level $s(F) = 2^n$, we consider the filtration of the groups $D_F(2^i)$, $0 \leq i \leq n$, consisting of all the nonzero sums of 2^i squares in F. Developing further ideas of A. Pfister, P. L. Chang and D. Z. Djoković and by the use of combinatorics, we obtain lower bounds for the invariants $\overline{q}_i := |D_F(2^i)/D_F(2^{i-1})|$, for $1 \leq i \leq n$, in terms of $s(F)$. As a consequence, a field with finite level ≥ 8 will have at least 512 square classes. Further we give lower bounds on the cardinalities of the Witt ring and of the 2-torsion part of the Brauer group of such a field.

1 Introduction

Let F be a field. The *level of* F, denoted by $s(F)$, is defined as the least positive integer m such that -1 is a sum of m squares in F whenever such an integer exists and ∞ otherwise. For fields of positive characteristic this invariant can take only the values 1 and 2, depending just on whether -1 is a square in F or not. Fields of level ∞, i.e. in which -1 is not a sum of squares, are called *real fields* and an equivalent condition to $s(F) = \infty$ is the existence of an ordering on F. Fields of finite level are also called *nonreal fields.*

For a long time it has been an open question which values exactly occur as the level of some field. The complete solution to this problem was given by A. Pfister in [10] and it inspired a big part of later advances in the theory of quadratic forms, e.g. the development of the theory of *Pfister forms* and the investigation of isotropy behaviors of quadratic forms under function field extensions.

Pfister proved that the level of a nonreal field is always a power of 2 [10, Satz 4] and further that, if F is any real field (e.g. $\mathbb{Q}$ or $\mathbb{R}$) and $n \geq 0$, then the function field of the projective quadric $X_0^2 + \cdots + X_{2^n}^2 = 0$ over F has level 2^n [10, Satz 5]. These were the first examples of nonreal fields of level greater than 4 and, actually, still no examples of an essentially different kind are known.
In general it remains a difficult problem to determine the level of a given field of characteristic zero. For an overview on what is known about levels of common types of fields we refer to [8, Chap. XI, Section 2]. In the same book T. Y. Lam also mentions the following question [8, p. 333]:

1.1. Level Question. *Does there exist a field F such that $4 < s(F) < \infty$ and such that $F^\times/F^{\times 2}$ is finite?*

Here and in the sequel we denote by $F^\times$ the multiplicative group of F and by $F^{\times 2}$ the subgroup of nonzero squares in F. The quotient $F^\times/F^{\times 2}$ is called the *square class group of F*. We call $q(F) := |F^\times/F^{\times 2}|$ the *square class number of F*. Another subgroup of $F^\times$ of importance is the group of nonzero sums of squares in F, denoted as $\sum F^{\times 2}$.
Further, for any $m \in \mathbb{N}$ we denote by $D_F(m)$ the set of elements of $F^\times$ which can be written as a sum of m squares over F. Pfister has shown that $D_F(m)$ is a group whenever m is a power of 2 [10, Satz 9]. We thus have the following group filtration for $\sum F^{\times 2}$:

$$F^{\times 2} \subsetneq D_F(2) \subsetneq D_F(4) \subsetneq \cdots \subsetneq D_F(2^{i-1}) \subsetneq D_F(2^i) \subsetneq \cdots \subset \textstyle\sum F^{\times 2}. \quad (1.2)$$

If F is nonreal of level 2^n then we actually have $D_F(2^n+1) = \sum F^{\times 2} = F^\times$.
For $i \geq 1$ we define $\bar{q}_i(F) := |D_F(2^i)/D_F(2^{i-1})|$. Note that the quotients $F^\times/F^{\times 2}$ and $D_F(2^i)/D_F(2^{i-1})$ are 2-elementary abelian groups. So $q(F)$ and $\bar{q}_i(F)$ are each either a power of 2 or ∞.
From (1.2) we see that the inequality

$$q(F) \geq \bar{q}_1(F) \cdots \bar{q}_n(F) \quad (1.3)$$

holds for any $n \geq 1$. We will use this in particular when $s(F) = 2^n$.

While an answer to the level question is still not known, one may look for lower bounds on $|F^\times/F^{\times 2}|$ in terms of $s(F)$.
One approach is to search for lower bounds on the invariants $\bar{q}_i(F)$ and to use then (1.3) to obtain a bound for $q(F)$. Following this idea, A. Pfister obtained in [11, Satz 18.d] the following estimate for a field F of level 2^n:

$$q(F) \geq 2^{\frac{n(n+1)}{2}}. \quad (1.4)$$

His proof (see also [8, p. 325]) actually shows for $1 \leq i \leq n$ that

$$\bar{q}_i(F) \geq 2^{n+1-i}. \quad (1.5)$$

Our standard examples of fields of level $1, 2$ and 4, respectively, are the field of complex numbers $\mathbb{C}$, the finite field $\mathbb{F}_3$ and $\mathbb{Q}_2$, the field of dyadic numbers. These examples show that (1.4) is best possible for $n \leq 2$. For higher n, however, P. L. Chang has improved the bound using combinatorics. In [1] he shows that $q(F) \geq 128$ for a field F of level eight and further that $q(F) \geq 16 \cdot \frac{2^s}{s^2}$ for any nonreal field F of level $s \geq 16$. His approach has been refined by D. Ž. Djoković in [2], leading to the following estimate:

$$q(F) \;\geq\; 2 \cdot \sum_{i=1}^{s/2} \frac{1}{s+2-i} \binom{s+1}{i} > \frac{2^s}{s} \;. \tag{1.6}$$

Their method does not provide any information about the invariants $\bar{q}_i(F)$.

The aim of the present work is to extend this method and to get lower bounds for the invariants $\bar{q}_i(F)$ with respect to $s(F)$ which improve (1.5). The combinatorial aspect is postponed to the two appendices where a certain coloring problem for (hyper-)graphs is considered.

We use common notations and results from quadratic form theory; the standard references are [8] and [12]. (Note that the uncomfortable case of characteristic 2 is implicitly excluded whenever we deal with a field of level greater than 1.) For isometry of quadratic forms we use the symbol $\cong$. For a quadratic form φ over F we denote by $D_F(\varphi)$ the set of nonzero elements of F represented by φ. We sometimes say just "*form*" or "*quadratic form*" to mean "*non-degenerate quadratic form*".

A diagonalized quadratic form over F with coefficients $a_1, \ldots, a_m \in F^\times$ is denoted by $\langle a_1, \ldots, a_m \rangle$. An *m-fold Pfister form* is a quadratic form of the shape $\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_m \rangle$ and shortly written as $\langle\langle a_1, \ldots, a_m \rangle\rangle$; its dimension is 2^m. A *neighbor* of an m-fold Pfister form π is a quadratic form φ which is similar to a subform of π and of dimension greater than 2^{n-1}. We know that in this situation φ is isotropic if and only of π is hyperbolic.

By $W(F)$ we denote the Witt ring of F, further by $\mathrm{Br}(F)$ the Brauer group and by $\mathrm{Br}_2(F)$ its 2-torsion part. In (3.1), (5.4) and (5.5) we shall use Milnor K-theory. For definitions and properties of the Milnor ring k_*F and its homgenous components k_mF ($m \geq 0$) we refer to [9] and [3]. However, we use the notation $\{a_1, \ldots, a_m\}$ instead of $\ell(a_1) \cdots \ell(a_m)$ for a symbol in k_mF. We recall that this symbol is zero in k_mF if and only if the corresponding m-fold Pfister form $\langle\langle -a_1, \ldots, -a_m \rangle\rangle$ over F is hyperbolic (see [3, Main Theorem 3.2]). In particular, $s(F) = 2^n$ is equivalent to $\{-1\}^n \neq 0$ and $\{-1\}^{n+1} = 0$ in k_*F. Everywhere else in the text, $\{x_1, \ldots, x_n\}$ stands simply for the set of elements $x_1, \ldots, x_1$.

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2 Sums of squares in fields

Let F be a field. For an element $x \in F$ we define its *length (over F)* to be the least positive integer m such that x can be written as a sum of m nonzero squares over F if such an integer exists and ∞ otherwise (i.e. if x is not a nontrivial sum of nonzero squares over F). We denote this value in $\mathbb{N} \cup \{\infty\}$ by $\ell_F(x)$, or just by $\ell(x)$ whenever the context makes clear over which field F we are working. Obviously $\ell_F(x)$ depends on x only up to multiplication by a nonzero square in F; in other words, $\ell_F(x)$ is an invariant of the square class $xF^{\times 2}$ whenever $x \neq 0$.
For $m \geq 1$, $D_F(m)$ is by definition the set $\{x \in F^\times \mid \ell(x) \leq m\}$. Our investigation into lengths of field elements is based on the following famous result [10, Satz 2]:

2.1. Theorem (Pfister). *For any $i \geq 0$, $D_F(2^i)$ is a subgroup of $F^\times$.*

A simple proof within the theory of Pfister forms can be found in [12, 4.4.1. Lemma]. As a consequence of this theorem one gets an inequality linking the lengths of two elements to the length of their product. We include a proof of this result, which is [10, Satz 3].

2.2. Lemma. *For any $x, y \in F$ we have the inequalities $\ell(x+y) \leq \ell(x) + \ell(y)$ and $\ell(xy) \leq \ell(x) + \ell(y) - 1$.*

Proof: The first inequality is obvious from the definition of the length.
The second inequality is trivial if xy is zero or if x or y is not a sum of squares. So we may suppose that both x and y are nonzero sums of squares in F. Let then r be the least nonnegative integer such that $x, y \in D_F(2^r)$. We will prove $\ell(xy) < \ell(x) + \ell(y)$ by induction on r. If $r = 0$ then x, y and xy are squares in F and the inequality is clear. Suppose now that $r > 0$. Since $D_F(2^r)$ is a group we know that $\ell(xy) \leq 2^r$. So the inequality is clear if $2^r < \ell(x) + \ell(y)$. Otherwise, we may suppose that $\ell(y) \leq 2^{r-1}$. By the choice of r we then have $2^{r-1} < \ell(x) \leq 2^r$ and may therefore write $x = a + z$ with $a, z \in F^\times$ such that $\ell(a) = 2^{r-1}$ and $\ell(z) = \ell(x) - 2^{r-1} \leq 2^{r-1}$. By the induction hypothesis we have $\ell(zy) < \ell(y) + \ell(z)$. As $D_F(2^{r-1})$ is a group we have $\ell(ay) \leq 2^{r-1}$. Since $xy = ay + zy$, using the first inequality of the statement we obtain finally $\ell(xy) \leq \ell(ay) + \ell(zy) < 2^{r-1} + \ell(y) + \ell(z) = \ell(x) + \ell(y)$. □

According to the definition we gave in the introduction, the *level* of F is the length of -1 in F. We may also conclude that $\ell_F(0) = s(F) + 1$. Therefore, from any of the inequalities of the lemma we obtain immediately:

2.3. Corollary. *For any $x \in F$ we have $\ell(x) + \ell(-x) \geq s(F) + 1$.* □

2.4. Corollary. *Let $a_1, \ldots, a_m \in F^\times$. If the quadratic form $\langle a_1, \ldots, a_m \rangle$ over F represents the element $x \in F$ nontrivially then $\ell(a_1) + \cdots + \ell(a_m) \geq \ell(x)$.*

Proof: If the form $\langle a_1, \dots, a_m\rangle$ represents $x \in F$ nontrivially, this means that there are $x_1, \dots, x_m \in F$, not all zero, such that $a_1x_1^2 + \cdots + a_mx_m^2 = x$. We may suppose that x_i is nonzero for $1 \le i \le m'$ and zero for $m' < i \le m$. From the first inequality of the lemma we obtain $\ell(x) \le \ell(a_1x_1^2) + \cdots + \ell(a_{m'}x_{m'}^2) = \ell(a_1) + \cdots + \ell(a_{m'})$. $\square$

For $i \ge 0$, we say that the elements $a_1, \dots, a_m \in F^\times$ are *independent modulo* $D_F(2^i)$ if in $F^\times/D_F(2^i)$, considered as an $\mathbb{F}_2$-vectorspace, the classes represented by $a_1, \dots, a_m$ are $\mathbb{F}_2$-linear independent.

2.5. Proposition. *For $i \ge 2$, let $a, b \in D_F(3\cdot2^{i-2})\backslash D_F(2^{i-1})$ and $c \in D_F(2^i)$ such that $\ell(a + b + c) > 2^{i+1}$. Then the elements a, b and c of $D_F(2^i)$ are independent modulo $D_F(2^{i-1})$.*

Proof: We have to show that $a, b, c, ab, ac, bc, abc \notin D_F(2^{i-1})$. For a and b this is already given. We put $x := a + b + c$. Each of the quadratic forms $\langle a, b, c\rangle$, $\langle a, b, abc\rangle$, $\langle 1, ab, ac\rangle$ and $\langle ac, bc, 1\rangle$ over F represents one of the elements x, abx, ax and cx and neither of these elements lies in the group $D_F(2^{i+1})$. We obtain from (2.4) that each of the numbers $\ell(a) + \ell(b) + \ell(c)$, $\ell(a) + \ell(b) + \ell(abc)$, $1 + \ell(ab) + \ell(ac)$ and $\ell(ac) + \ell(bc) + 1$ is greater than 2^{i+1}. Since $\ell(a) + \ell(b) \le 3 \cdot 2^{i-1}$ and $ab, ac, bc \in D_F(2^i)$ we obtain $\ell(c), \ell(abc) \ge 2^{i-1}$ and further $\ell(ab) = \ell(ac) = \ell(bc) = 2^i$. $\square$

For the rest of this section we fix a sum of squares

$$x = x_1^2 + \cdots + x_l^2 \tag{2.6}$$

with $x_1, \dots, x_l \in F^\times$, $x \in F$ and $l = \ell_F(x)$. For a subset $I \subset \{1, \dots, l\}$ we denote $x_I := \sum_{i\in I} x_i^2$. If I is not empty then we have $\ell(x_I) = |I|$.
For a real number z we denote by $\lceil z\rceil$ the least integer $\ge z$.

2.7. Theorem. *Let I and J be nonempty proper subsets of $\{1, \dots, l\}$. Let r be a nonnegative integer such that $x_Ix_J \in D_F(2^r)$. Then the following hold:*

(*i*) $\left\lceil \frac{|I|}{2^r} \right\rceil = \left\lceil \frac{|J|}{2^r} \right\rceil$, *in particular* $|\,|I| - |J|\,| < 2^r$,

(*ii*) $|I \setminus J|\ ,\ |J \setminus I| \ \le\ 2\ \ell(x_Ix_J) - 1 \ <\ 2^{r+1}$,

(*iii*) $|I \cup J| - |I \cap J| \ \le\ 2^{r+1} + \ell(x_Ix_J) - 1 \ \le\ 3 \cdot 2^r - 1$.

Proof: The hypothesis implies that x_I and x_J are nonzero elements of F. We set $m := \ell(x_Ix_J)$ and $a := \frac{x_J}{x_I}$. Then $\ell(a) = m \le 2^r$.
If ν is an integer such that $|I| \le \nu 2^r$ then we can write x_I as a sum of $\le \nu$ elements of $D_F(2^r)$. As $D_F(2^r)$ is a group, $x_J = ax_I$ can also be written as a sum of $\le \nu$ elements of $D_F(2^r)$ which means that $|J| = \ell(x_J) \le \nu 2^r$. By symmetry we obtain for any $\nu \in \mathbb{N}$ that $|I| \le \nu 2^r$ if and only if $|J| \le \nu 2^r$. This shows (*i*).

We compute $x_{I\cup J} = x_{I\setminus J} + x_J = (1+a)x_{I\setminus J} + ax_{I\cap J}$ and then substitute $y := (1+a)x_{I\setminus J}$ and $z := ax_{I\cap J}$ to have $x_{I\cup J} = y + z$.
If $y \neq 0$ then we have $\ell(y) \leq m + |I \setminus J|$ by (2.2), but also $\ell(y) \leq 2^{r+1}$ since $D_F(2^{r+1})$ is a group. If $z \neq 0$ then (2.2) yields $\ell(z) \leq m + |I \cap J| - 1$. Therefore, if at least one of y and z is nonzero then we obtain the inequalities $\ell(y+z) \leq |I| + 2m - 1$ and $\ell(y+z) \leq 2^{r+1} + m + |I\cap J| - 1$. Both inequalities remain valid in the case $y = z = 0$, since then necessarily $a = -1$, whence $\ell(y+z) = \ell(0) = m+1$. As $|I \cup J| = \ell(y+z)$ we obtain (*ii*) by symmetry from the first and (*iii*) from the second inequality. □

For $m = 1$ this leads to an observation made in the proof of [1, Theorem 1]:

2.8. Corollary (Chang). *Let I and J be as in the theorem. If x_I and x_J lie in the same square class then both sets have the same cardinality and differ by at most one element.* □

2.9. Corollary. *Let I and J be as in the theorem with $|I| = |J| = 2^i$, $i \geq 2$. If x_I and x_J represent the same class modulo $D_F(2^{i-1})$ then $|I\cap J| \geq 2^{i-2}+1$.*

Proof: If x_I and x_J lie in the same class modulo $D_F(2^{i-1})$ then $\ell(x_Ix_J) \leq 2^{i-1}$. Applying part (*iii*) of the theorem for $r = i-1$ we obtain $|I \cup J| - |I \cap J| \leq 3\cdot 2^{i-1} - 1$. But our hypothesis here gives $|I\cup J| = 2\cdot 2^i - |I\cap J|$. This together implies $|I \cap J| > 2^{i-2}$. □

3 The invariants $\bar{q}_i$

For a nonreal field F of level 2^n we are going to study the invariants $\bar{q}_i(F) = |D_F(2^i)/D_F(2^{i-1})|$ for $1 \leq i \leq n$. In particular, we are interested to know whether Pfister's bounds (1.5) can be improved.

First we note that the bound $\bar{q}_n(F) \geq 2$, obtained from (1.5) for $i = n$, just takes into account that -1 represents a nontrivial class in the group $D_F(2^n)/D_F(2^{n-1})$. In spite of the simple argument, this bound is optimal for every $n \geq 1$. More precisely, for any $n \geq 1$ there is a field F of level 2^n such that $F^\times = D_F(2^{n-1}) \cup -D_F(2^{n-1})$. The construction of such an example will be included in a forthcoming paper of the author.
We now turn to consider $\bar{q}_{n-1}(F)$. For $i = n-1$, (1.5) gives $\bar{q}_{n-1}(F) \geq 4$. The example $F = \mathbb{Q}_2$ shows that this bound is optimal for $n = 2$.

3.1. Theorem. *Let F be a field of level 2^n with $n \geq 3$. Then $\bar{q}_{n-1}(F) \geq 16$.*

Proof: Since $\ell(0) = 2^n+1$ and $n \geq 3$, we may choose elements $a_1, a_2, a_3 \in F^\times$ such that $a_1 + a_2 + a_3 = 0$ and $2^{n-2}+1 \leq \ell(a_i) \leq 3\cdot 2^{n-3}$ for $i = 1,2,3$. Then by (2.5), a_1, a_2 and a_3 are independent modulo $D_F(2^{n-2})$. Let H be the subgroup of $D_F(2^{n-1})$ generated by $D_F(2^{n-2})$ and the elements a_1, a_2 and a_3. Since $|H/D_F(2^{n-2})| = 8$ it remains to show that $H \neq D_F(2^{n-1})$.

To this aim, we will calculate in the Milnor ring k_*F. For $i = 1,2,3$ we fix the symbols $\beta_i := \{a_1a_2a_3, a_i\}$ and $\gamma_i := \{-a_1a_2a_3, -a_i\}$ in k_2F. Let ε denote the element $\{-1\}$ in k_1F. Since $s(F) = 2^n$ we have $\varepsilon^n \neq 0$. As $a_1, a_2, a_3 \in D_F(2^{n-1})$ we observe that $\beta_1+\beta_2+\beta_3 = \{-1, a_1a_2a_3\}$ is annihilated by ε^{n-2} and that $\varepsilon^{n-2}(\beta_i+\gamma_i) = \varepsilon^{n-2}(\{a_1a_2a_3,-1\}+\{-1,a_i\}+\{-1,-1\}) = \varepsilon^n$ for $i = 1,2,3$.

If $\varepsilon^{n-2}\beta_i \neq 0$ in k_nF for some i then by the above relations we may suppose that $\varepsilon^{n-2}\beta_i \neq 0$ for $i = 1,2$ and $\varepsilon^{n-2}\beta_3 \neq \varepsilon^n$, i.e. $\varepsilon^{n-2}\gamma_3 \neq 0$. Using that $a_1 + a_2 + a_3 = 0$ we compute $\{-a_2, -a_3\} = \{a_1, -a_2a_3\} = \beta_1$ and equally $\{-a_1, -a_3\} = \beta_2$. Since none of β_1, β_2 and γ_3 is annihilated by ε^{n-2}, the symbols $\varepsilon^{n-2}\{-a_2, -a_3\}$, $\varepsilon^{n-2}\{-a_1, -a_3\}$ and $\varepsilon^{n-2}\{-a_1a_2, -a_3\}$ in k_nF are all nonzero. Therefore the Pfister forms $2^{n-2}\times\langle\langle a_2, a_3\rangle\rangle$, $2^{n-2}\times\langle\langle a_1, a_3\rangle\rangle$ and $2^{n-2}\times\langle\langle a_1a_2, a_3\rangle\rangle$ are anisotropic. Further, $2^{n-2}\times\langle\langle 1, a_3\rangle\rangle \cong 2^n\times\langle 1\rangle$ is anisotropic since $s(F) = 2^n$. This shows that $-1, -a_1, -a_2, -a_1a_2 \notin D_F(2^{n-2}\times\langle\langle a_3\rangle\rangle)$. As the group $D_F(2^{n-2}\times\langle\langle a_3\rangle\rangle)$ contains the subgroup $D_F(2^{n-2})$ and the element a_3 we conclude that $D_F(2^{n-2}\times\langle\langle a_3\rangle\rangle) \cap -H = \emptyset$. On the other hand, since $\ell(-a_3) \leq \ell(a_1)+\ell(a_2) \leq 3\cdot 2^{n-2}$ we can write $-a_3 = x+y$ with $x \in D_F(2^{n-1})$, $y \in D_F(2^{n-2})$ and obtain $-x = y + a_3 \in D_F(2^{n-2}\times\langle\langle a_3\rangle\rangle) \cap -D_F(2^{n-1})$.

Now we study the case where $\varepsilon^{n-2}\beta_i = 0$ for $i = 1,2,3$. As $\varepsilon^{n-2}\beta_i = \varepsilon^{n-2}\{-a_1a_2a_3, a_i\}$, this means that the Pfister form $2^{n-2}\times\langle\langle a_1a_2a_3, -a_i\rangle\rangle$ is hyperbolic for $i = 1,2,3$. We conclude that $H \subset D_F(2^{n-2}\times\langle\langle a_1a_2a_3\rangle\rangle)$. As the Pfister form $2^{n-1}\times\langle\langle a_1a_2a_3\rangle\rangle \cong 2^n\times\langle 1\rangle$ is anisotropic we have $-1 \notin D_F(2^{n-2}\times\langle\langle a_1a_2a_3\rangle\rangle)$ and therefore $D_F(2^{n-2}\times\langle\langle a_1a_2a_3\rangle\rangle) \cap -H = \emptyset$. Since $-a_1a_2a_3 = a_1^2a_2 + a_2^2a_1$ we have $\ell(-a_1a_2a_3) \leq \ell(a_2) + \ell(a_1) \leq 3\cdot 2^{n-2}$ and may therefore write $-a_1a_2a_3 = x+y$ with $x \in D_F(2^{n-1})$ and $y \in D_F(2^{n-2})$ to obtain this time $-x = y + a_1a_2a_3 \in D_F(2^{n-2}\times\langle\langle a_1a_2a_3\rangle\rangle) \cap -D_F(2^{n-1})$.

In both cases we have found an element $x \in D_F(2^{n-1}) \setminus H$. □

While the lower bound on $\bar{q}_{n-1}$ of the last theorem is based upon several algebraic arguments, the improvement (with respect to (1.5)) for the lower bounds on $\bar{q}_i(F)$ for $2 \leq i \leq n-2$ which we present now, is obtained by combinatorial reasoning, developed in appendix A.

For integers $0 \leq k \leq l$ we denote by $\mathcal{P}_k^l$ the set of subsets of $\{1,\dots,l\}$ with exactly k elements.

3.2. Theorem. *Let F be a field of level 2^n. Then*

$$\bar{q}_i(F) \;\geq\; \begin{cases} 2^7 & \text{for } \; i = n-2 \geq 3\,, \\ 2^{(n-i)(2^{n-i}+1)+1} & \text{for } \; \frac{n+1}{2} < i \leq n-3\,, \\ 2^{(n-i)(2^{i-2}+1)+1} & \text{for } \; 2 \leq i \leq \frac{n+1}{2}\,. \end{cases}$$

Proof: We fix elements $x_1,\dots,x_{2^n} \in F^\times$ such that $x_1^2+\cdots+x_{2^n}^2 = -1$. For a subset $J \subset \{1,\dots,2^n\}$ we denote $x_J := \sum_{j\in J} x_j^2$.

Let $2 \leq i \leq \frac{n+1}{2}$. We consider the map $f : \mathcal{P}_{2^i}^{2^n} \longrightarrow D_F(2^i)/D_F(2^{i-1})$ which sends a 2^i-subset $J \subset \{1,\dots,2^n\}$ to the class $x_JD_F(2^{i-1})$. By (2.9), if $J_1, J_2 \in$

$\mathcal{P}^{2^n}_{2^i}$ are such that $f(J_1) = f(J_2)$ then $|J_1 \cap J_2| \geq 2^{i-2}+1$. Therefore (A.8) in appendix A shows $|D_F(2^i)/D_F(2^{i-1})| \geq |Im(f)| > 2^r$ for $r := (n-i)(2^{i-2}+1)$. Since $D_F(2^i)/D_F(2^{i-1})$ is a 2-elementary abelian group it must then have at least 2^{r+1} elements. This establishes the third case in the statement.

In the remaining cases we cannot apply (A.8) directly for i and $m := n$. In the case $\frac{n+1}{2} < i \leq n-3$ we have $n \geq 8$ and $i \geq 5$ and define $n' := 2(n-i+1)$ and $i' := n-i+2 = \frac{n'}{2}+1$. In the case $i = n-2$ and $n \geq 5$ we set instead $n' := 5$ and $i' := 3 = \frac{n'+1}{2}$. Note that in both cases $n' - i' = n - i$.

For $1 \leq \nu \leq 2^{n'}$ let $J_\nu := \{(\nu-1)\cdot 2^{n-n'}+1, \ldots, \nu \cdot 2^{n-n'}\}$ and $y_\nu := x_{J_\nu}$. This yields $y_1 + \cdots + y_{2^{n'}} = -1$ and $\ell(y_\nu) = |J_\nu| = 2^{n-n'}$ for $1 \leq \nu \leq 2^{n'}$. Now we consider the map $f' : \mathcal{P}^{2^{n'}}_{2^{i'}} \longrightarrow D_F(2^i)/D_F(2^{i-1})$ which sends a $2^{i'}$-subset $N \subset \{1, \ldots, 2^{n'}\}$ to the class $(\sum_{\nu \in N} y_\nu) D_F(2^{i-1})$.

Suppose that $f'(N_1) = f'(N_2)$ for $N_1, N_2 \in \mathcal{P}^{2^{n'}}_{2^{i'}}$. For $k = 1, 2$ let $I_k := \bigcup_{\nu \in N_k} J_\nu \in \mathcal{P}^{2^n}_{2^i}$. Since by hypothesis $\sum_{\nu \in N_1} y_\nu = x_{I_1}$ and $\sum_{\nu \in N_2} y_\nu = x_{I_2}$ lie in the same class of $D_F(2^i)/D_F(2^{i-1})$, (2.9) shows that $|I_1 \cap I_2| \geq 2^{i-2}+1$ and it follows that $|N_1 \cap N_2| \geq 2^{i-2-(n-n')}+1 = 2^{i'-2}+1$.

Having established this intersection property of f', we obtain from (A.8) that $|D_F(2^i)/D_F(2^{i-1})| \geq |Im(f')| > 2^{r'}$ holds for $r' := (n'-i')(2^{i'-2}+1)$. As before, we conclude that $|D_F(2^i)/D_F(2^{i-1})| \geq 2^{r'+1}$. This finishes the proof since $r' = 6$ in case $i = n-2$ and $r' = (n-i)(2^{n-i}+1)$ otherwise. □

4 Nonreal fields with $\bar{q}_1$ equal to the level

From (1.5) we know that $\bar{q}_1(F) \geq s(F)$ holds for any nonreal field F. This bound is optimal for fields of level $1, 2$ and 4 as the standard examples show (see introduction). For nonreal fields of higher level, however, there is still no known example where $\bar{q}_1(F) < \infty$.

We show that $\bar{q}_1(F) = s(F) < \infty$ is a rather strong condition, with several consequences on the quadratic form structure of F. In particular, for $s(F) \geq 8$ it implies that $\bar{q}_2(F) \geq \frac{s(F)^2}{2}$ (4.9).

Let ξ be an element of length $l \geq 3$ of F. We fix a representation of ξ as a sum of l squares

$$\xi = x_1^2 + \cdots + x_l^2 \tag{4.1}$$

with $x_1, \ldots, x_l \in F^\times$. Let $f : \mathcal{P}^l_2 \to D_F(2)/F^{\times 2}$ be the function which sends a (nonordered) pair of distinct $i, j \leq l$ to the square class of $x_i^2 + x_j^2$. Considering the elements of $D_F(2)/F^{\times 2}$ as a set of colors, we can interprete f as an edge-coloring of a complete graph in l vertices $v_1, \ldots, v_l$. We denote this graph together with its edge-coloring f by $\mathcal{G}$. If in this graph two edges $[v_i, v_j]$ and $[v_{i'}, v_{j'}]$ are of the same color (with $\{i,j\}, \{i',j'\} \in \mathcal{P}^l_2$) this means that $x_i^2 + x_j^2$

and $x_{i'}^2 + x_{j'}^2$ lie in the same square class of F, which by (2.8) implies that the sets $\{i,j\}$ and $\{i',j'\}$ intersect. In other words, two edges of the same color in $\mathcal{G}$ need to have a vertex in common, i.e. $\mathcal{G}$ is a *CC-graph* in the terminology of appendix B.
We get from (B.1) that at least $l-2$ colors appear in $\mathcal{G}$. Furthermore, since $x_1^2 + \cdots + x_l^2$ is of length l, no sum $x_i^2 + x_j^2$ with $i \neq j$ can be a square. This gives a proof of [13, Theorem 1]:

4.2. Proposition (Tort). *In* (4.1), *the partial sums* $x_i^2 + x_j^2$ *with* $1 \leq i < j \leq l$ *represent at least* $l-2$ *different nontrivial classes of* $D_F(2)/F^{\times 2}$. □

Let now F be a nonreal field of level $s = 2^n$. We then can choose $\xi := 0$, which is of length $s+1$ over F, and write (4.1) as

$$0 = x_1^2 + \cdots + x_{s+1}^2 \,. \tag{4.3}$$

By the above proposition the partial sums $x_i^2 + x_j^2$ (with $1 \leq i < j \leq s+1$) represent at least $s-1$ nontrivial classes of $D_F(2)/F^{\times 2}$. This shows:

4.4. Corollary. *Let* F *be a nonreal field of level* s. *Then* $\bar{q}_1(F) \geq s$. *Moreover, if* $\bar{q}_1(F) = s$ *then, given any representation* (4.3) *of zero as a sum of* $s+1$ *nonzero squares over* F, *every nontrivial class of* $D_F(2)/F^{\times 2}$ *is represented by a partial sum* $x_i^2 + x_j^2$ *with* $1 \leq i < j \leq s+1$. □

Given a subgroup $G \subset F^\times/F^{\times 2}$ of finite order 2^m we may choose an irredundant set of representatives $a_1, \ldots, a_{2^m} \subset F^\times$ of the square classes in G and define the quadratic form $\pi_G := \langle a_1, \ldots, a_{2^m} \rangle$. Up to isometry, this form does only depend on G and not on the particular choice of the a_i. If we choose the a_i such that $a_1, \ldots a_m$ are independent modulo $F^{\times 2}$ then π_G is equal to $\langle\langle a_1, \ldots, a_m \rangle\rangle$, hence π_G is an m-fold Pfister form. If $\bar{q}_1(F)$ is finite we write $\pi_{D(2)}$ for π_G with $G := D_F(2)/F^{\times 2}$.

4.5. Proposition. *Let* F *be a nonreal field with* $s(F) > 1$ *and* $\bar{q}_1(F) < \infty$. *Then* $\pi_{D(2)}$ *is hyperbolic.*

Proof: Let $s := s(F)$. Given a representation (4.3) of zero as sum of $s+1$ squares over F we define $a_i := x_{2i-1}^2 + x_{2i}^2$ for $1 \leq i \leq s/2$. By (2.8) the a_i lie in distinct nontrivial square classes. Since $a_1 + \cdots + a_{s/2} + x_{s+1}^2 = 0$ the form $\langle 1, a_1, \ldots, a_{s/2} \rangle$ is isotropic. On the other hand, this is a subform of the Pfister form $\pi_{D(2)}$, which then must be hyperbolic. □

4.6. Lemma. *Let* H *be a subgroup of* $F^\times$ *containing* $F^{\times 2}$ *such that* $H/F^{\times 2}$ *is of order* 2^m *with* $m \geq 2$. *If* $a,b,c,d \in H$, *lie in distinct square classes then there are* $a_3, \ldots, a_m \in H$ *such that* $\pi_H = \langle a,b,c,d \rangle \otimes \langle\langle a_3, \ldots, a_m \rangle\rangle$.

Proof: It is easy to verify that, given four distinct elements t, u, v, w in a 2-elementary abelian group G there exists a subgroup K of index 4 in G such that t, u, v, w represent the four classes of G/K.
We apply this fact to the square classes $aF^{\times 2}, bF^{\times 2}, cF^{\times 2}$ and $dF^{\times 2}$ in $G := H/F^{\times 2}$. A subgroup K with the stated property must have order 2^{m-2}. We choose elements $a_3, \ldots, a_m \in F^\times$ such that their square classes form an $\mathbb{F}_2$-basis of K. The rest is clear. □

4.7. Proposition. *Let F be a field with $\bar{q}_1(F) = s(F) = 2^n$, $n \geq 2$, and let a, b, c, d be elements of $D_F(2)$ which lie in distinct square classes.*

(a) If $a \notin F^{\times 2}$ then $D_F(\langle 1,1\rangle) \cap D_F(\langle 1,a\rangle) = \{1,a\}F^{\times 2}$.

(b) If $x \in D_F(\langle 1,a\rangle) \cap D_F(\langle 1,b\rangle) \cap D_F(\langle 1,c\rangle)$ then $\ell(-x) = 2^n$.

(c) $D_F(\langle a,b\rangle) \cap D_F(\langle c,d\rangle) = \emptyset$.

(d) If $n \geq 3$ then $D_F(\langle a,b\rangle) \cap D_F(\langle a,c\rangle) \cap D_F(\langle b,c\rangle) = \emptyset$.

(e) If $x \in D_F(\langle 1,a\rangle) \cap D_F(\langle 1,b\rangle)$ then $\ell(cx) = 4$ or $\ell(-x) \geq 2^n - 1$.

Proof: (a) Given a and b lying in distinct nontrivial classes of $D_F(2)/F^{\times 2}$ we may choose $a_3, \ldots, a_{2^{n-1}} \in D_F(2)$ such that $\varphi := \langle 1, a, b, a_3 \ldots, a_{2^{n-1}}\rangle$ is a neighbor of the Pfister form $\pi_{D(2)}$ which is hyperbolic by the last proposition. So φ is isotropic. Now $b \in D_F(\langle 1,a\rangle)$ would imply that φ is isometric to $\langle 1, 1, ab, a_3 \ldots, a_{2^{n-1}}\rangle$ which is a subform of $2^n \times \langle 1\rangle$. This is impossible since the latter form is anisotropic by the hypothesis that $s(F) = 2^n$.
(b) Let $x \in D_F(\langle 1,a\rangle) \cap D_F(\langle 1,b\rangle) \cap D_F(\langle 1,c\rangle)$ where $a, b, c \in D_F(2)$ are distinct modulo squares. Then clearly $\ell(x) \leq 3$ and we have also $x \in D_F(\langle 1, abc\rangle)$ (with $-a, -b$ and $-c$ also $-abc$ lies in $D_F(\langle 1, -x\rangle)$). It follows from (a) that $\ell(x) \neq 2$. If x is a square then $\ell(-x) = \ell(-1) = 2^n$. Otherwise we must have $\ell(x) = 3$. Then none of a, b, c, abc can be a square. Further $\ell(-x) \geq 2^n - 2$ by (2.3). Thus (4.2) shows that, in a representation of $-x$ as sum of $\ell(-x)$ squares over F, the partial sums of length two lie in at least $2^n - 4$ distinct nontrivial square classes. As $|D_F(2)/F^{\times 2}| = 2^n$ by hypothesis, at least one of these square classes must also be represented by one of a, b, c or abc. Without loss of generality we may suppose that $-x = y + at^2$ with $\ell(y) = \ell(-x) - 2$. Writing $x = u^2 + av^2$ yields $0 = x - x = y + u^2 + a(t^2 + v^2)$. Thus $2^n + 1 \leq \ell(y) + 3$ and $2^n \leq \ell(y) + 2 = \ell(-x)$. Then $-x = (-1) \cdot x \in D_F(2^n)$ implies $\ell(-x) = 2^n$.
(c) By the above lemma there are $a_3, \ldots, a_n \in D_F(2)$ such that $\pi_{D(2)}$ is equal to $\langle a, b, c, d\rangle \otimes \langle\langle a_3, \ldots, a_n\rangle\rangle$.
Suppose now that there exists an $x \in D_F(\langle a,b\rangle) \cap D_F(\langle c,d\rangle)$. Then $\langle a,b,c,d\rangle \cong \langle x, abx, x, cdx\rangle$, which is similar to $\langle 1,1,1,abcd\rangle$. Hence $\pi_{D(2)}$ is similar to $\langle 1,1,1,abcd\rangle \otimes \langle\langle a_3, \ldots, a_n\rangle\rangle \cong 2^{n-1} \times \langle 1\rangle \perp \langle\langle abcd, a_3, \ldots, a_n\rangle\rangle$. It follows that the form $(2^{n-1}+1) \times \langle 1\rangle$ is a Pfister neighbor of $\pi_{D(2)}$, hence isotropic since $\pi_{D(2)}$ is hyperbolic. This is a contradiction to $s(F) = 2^n$.

(d) After multiplying by a in the statement we may suppose that $a = 1$. Suppose that there exists $x \in D_F(\langle 1, b\rangle) \cap D_F(\langle 1, c\rangle) \cap D_F(\langle b, c\rangle)$. It follows $-b, -c \in D_F(\langle 1, -x\rangle)$, thus $bc \in D_F(\langle 1, -x\rangle) \cap D_F(\langle 1, 1\rangle) \subset D_F(\langle 1, x\rangle)$. Therefore we have $\langle 1, b, c, bc\rangle \cong \langle 1, x, bcx, bc\rangle \cong \langle bc, bcx, bcx, bc\rangle$, whence $\langle 1, b, c, bc\rangle \cong \langle 1, 1, x, x\rangle$. Next we choose $a_3, \ldots, a_n \in D_F(2)$ such that $\pi_{D(2)} \cong \langle 1, b, c, bc\rangle \otimes \langle\langle a_3, \ldots, a_n\rangle\rangle$ and obtain $\pi_{D(2)} \cong \langle\langle 1, x, a_3, \ldots, a_n\rangle\rangle \cong 2^{n-1} \times \langle\langle x\rangle\rangle \cong 2^n \times \langle 1\rangle$, since $a_3, \ldots, a_n \in D_F(2)$, $n \geq 3$ and $x \in D_F(4)$. This is contradictory since $\pi_{D(2)}$ is hyperbolic but $s(F) = 2^n$.
(e) Let $x \in D_F(\langle 1, a\rangle) \cap D_F(\langle 1, b\rangle)$. Then, certainly, x and cx belong to $D_F(4)$. If $\ell(cx) \leq 2$ then $\ell(x) \leq 2$ and (2.3) yields $\ell(-x) \geq 2^n - 1$. Suppose now $\ell(cx) = 3$ and write $cx = e + t^2$ with $t \in F^\times$ and $e \in D_F(2)$. We have $cx \in D_F(\langle c, ac\rangle) \cap D_F(\langle c, bc\rangle) \cap D_F(\langle 1, e\rangle)$. Since $1, c, ac$ and bc represent distinct square classes, we conclude with (c) that e and c lie in the same square class. Therefore $x \in D_F(\langle 1, a\rangle) \cap D_F(\langle 1, b\rangle) \cap D_F(\langle 1, c\rangle)$, which by (b) implies $\ell(-x) = 2^n$. □

4.8. Theorem. *Let F be a nonreal field of level s, equal to $\bar{q}_1(F)$. Any representation* (4.3) *of zero as a nontrivial sum of $s+1$ squares over F may be reordered in such way that the following holds: for $\{i, j\}, \{i', j'\} \in \mathcal{P}_2^{s+1}$ the partial sums $x_i^2 + x_j^2$ and $x_{i'}^2 + x_{j'}^2$ lie in the same square class if and only if* $\max\{i, j, 3\} = \max\{i', j', 3\}$.

Proof: Let $\mathcal{G}$ be a complete graph in $s+1$ vertices $v_1, \ldots, v_{s+1}$ and with the edge-coloring given by $f : \mathcal{P}_2^{s+1} \to D_F(2)/F^{\times 2}, \{i, j\} \mapsto (x_i^2 + x_j^2)F^{\times 2}$ (see at the beginning of this section). We know from (4.4) that exactly $s - 1$ colors appear in $\mathcal{G}$. Further, $\mathcal{G}$ does not contain any triangle with three different colors; indeed, such a triangle would correspond to a partial sum of three squares $x := x_i^2 + x_j^2 + x_k^2$ with $1 \leq i < j < k \leq s+1$ where $a := x_i^2 + x_j^2$, $b := x_i^2 + x_k^2$ and $c := x_j^2 + x_k^2$ lie in three distinct square classes which is impossible by part (b) of the last proposition since $\ell(-x) = s - 2$. Therefore by (B.3), $\mathcal{G}$ is a total CC-graph.
Since $\mathcal{G}$ has precisely $(s+1) - 2$ colors we obtain from the definition of a total CC-graph in appendix B and the subsequent remarks: the vertices in $\mathcal{G}$ (and at the same time the x_i) may be renumbered in such way that for $\{i, j\} \in \mathcal{P}_2^{s+1}$ the color of the edge between v_i and v_j (i.e. the square class of $x_i^2 + x_j^2$) depends precisely on $\max\{i, j, 3\}$. □

4.9. Corollary. *Let F be a nonreal field of level $s = \bar{q}_1(F) \geq 8$. Then* $\bar{q}_2(F) \geq \frac{s^2}{2}$.

Proof: Let $0 = x_1^2 + \cdots + x_{s+1}^2$ be a representation of zero as a nontrivial sum of $s+1$ squares over F. By the theorem we may, after reordering the indices, suppose that for $\{i, j\} \in \mathcal{P}_2^{s+1}$ the square class of $x_i^2 + x_j^2$ depends precisely on $\max\{i, j, 3\}$.

Defining $a_i := x_{i+1}^2 + x_{i+2}^2$ for $1 \le i \le s-1$, we get a system of representatives $a_1, \dots, a_{s-1}$ of the $s-1$ nontrivial classes of $D_F(2)/F^{\times 2}$. Further we set $c_{jk} := x_1^2 + x_{j+2}^2 + x_{k+2}^2$ for $1 \le j < k \le s-1$.
Suppose now that $b\, c_{jk} = c_{j'k'}$ for $b \in D_F(2)$ and $1 \le j' < k' \le s-1$. Then $c_{j'k'} \in D_F(\langle 1, a_{j'}\rangle) \cap D_F(\langle 1, a_{k'}\rangle) \cap D_F(\langle b, b\, a_j\rangle) \cap D_F(\langle b, b\, a_k\rangle)$. In view of (b), (c) and (d) of the proposition this is only possible if $b \in F^{\times 2}$, $j = j'$ and $k = k'$.
This shows that the elements c_{jk} for $1 \le j < k \le s-1$ represent distinct nontrivial classes of $D_F(4)/D_F(2)$. Therefore $\bar{q}_2(F) > \binom{s-1}{2}$. Since s is a power of 2, at least 8, and $\bar{q}_2(F)$ is a power of 2 or infinite we obtain $\bar{q}_2(F) \ge \frac{s^2}{2}$. □

5 Lower bounds for the square class number

We start this section with Djoković's proof of his bound (1.6), rephrased in the terminology of appendix A.

5.1. Theorem (Djoković). *If F is a nonreal field of level $s \ge 8$ then*

$$q(F) \;\ge\; 2 \cdot |D_F(s/2)/F^{\times 2}| \;\ge\; 2 \cdot \sum_{i=1}^{s/2} \frac{1}{s+2-i} \binom{s+1}{i} \;.$$

Proof: The first inequality is clear since $|F^\times/D_F(s/2)| \ge 2$.
Next we consider a representation $0 = x_1^2 + \cdots + x_{s+1}^2$ of zero as a sum of $s+1$ nonzero squares over F. We denote by $\mathcal{P}$ the set of nonempty subsets of $\{1, \dots, s+1\}$ of cardinality not greater than $s/2$. We define $f : \mathcal{P} \to D_F(s/2)/F^{\times 2}, J \mapsto (\sum_{j \in J} x_j^2) F^{\times 2}$. For $1 \le k \le s/2$ we write f_k for the restriction of f to $\mathcal{P}_k^{s+1}$. By (2.8), for $k \ne k'$ the images of f_k and $f_{k'}$ are disjoint. Also by (2.8), f_k is $(k-1)$-connected for any $k \le s/2$ and therefore $|Im(f_k)| \ge \frac{1}{(s+1)-k+1} \binom{s+1}{k}$ by (A.4, c). All together we obtain

$$|D_F(s/2)/F^{\times 2}| \;\ge\; \sum_{k=1}^{s/2} |Im(f_k)| \;\ge\; \sum_{k=1}^{s/2} \frac{1}{s-k+2} \binom{s+1}{k}$$

which shows the second inequality. □

5.2. Remark. For an integer $s \ge 8$, let $\sum(s)$ denote the term on the right hand side in the inequality of the above theorem. Djoković showed by an elementary counting argument that $\sum(s) > \frac{2^s}{s}$ [2]. As was pointed out by David B. Leep, the argument may be improved to obtain the bound $\sum(s) > \frac{2^{s+1}}{s}$ for every even $s \ge 8$. Under the hypothesis of the last theorem one has thus $q(F) > \frac{2^{s+1}}{s}$; further, since $s = s(F)$ is a power of 2 and $q(F)$ is also a power of 2 or infinite, it follows that $q(F) \ge \frac{2^{s+2}}{s}$.
Our calculations have shown that, at least for s a power of 2 in the range between 8 and 2^{13}, actually one has $\frac{2^{s+1}}{s} < \sum(s) \le \frac{2^{s+2}}{s}$.

However, for level 8 and 16 we get stronger bounds on $q(F)$.

5.3. Theorem. *Let F be a field. If $s(F) = 8$ then $q(F) \geq 512$. If $s(F) = 16$ then $q(F) \geq 2^{15}$.*

Proof: Under the hypothesis $s(F) = 8$ we have $\bar{q}_3(F) \geq 2$, $\bar{q}_2(F) \geq 16$ (3.1) and $\bar{q}_1(F) \geq 8$ (1.5). Moreover, by (4.9) one of the last two inequalities must be proper. From $|F^\times/{F^\times}^2| \geq \bar{q}_1(F)\cdot\bar{q}_2(F)\cdot\bar{q}_3(F)$ we get therefore $q(F) \geq 512$, since $F^\times/{F^\times}^2$ is an elementary abelian 2-group.
For $s(F) = 16$ we have by the previous sections $\bar{q}_4(F) \geq 2$, $\bar{q}_3(F) \geq 16$, $\bar{q}_2(F) \geq 32$ and $\bar{q}_1(F) \geq 16$ and one of the last two inequalities must be proper. As $|F^\times/{F^\times}^2| \geq \bar{q}_1(F)\cdots\bar{q}_4(F)$ this leads to $q(F) \geq 2^{15}$. □

For $s(F) = 2^n$ with $n \geq 5$ the analogous arguments are not sufficient to improve Djoković's result. For $s(F) = 32$, for example, we may get in this way $q(F) \geq 2^{25}$ while (5.1) yields $q(F) \geq 2^{29}$.

5.4. Theorem. *Let F be a field of level 2^n with $n \geq 3$. Then $|k_{n-1}F| \geq 128$. More precisely, the subgroup $\{-1\}^{n-2}k_1F$ of $k_{n-1}F$ is of index at least 4 and order at least 32.*

Proof: Again, we use the notation $\varepsilon := \{-1\} \in k_1F$. The homomorphism $F^\times \to \{-1\}^{n-2}k_1F$ which maps $x \in F^\times$ to the symbol $\varepsilon^{n-2}\cdot\{x\}$, has kernel $D_F(2^{n-2})$. Since $\bar{q}_n(F) \geq 2$ and $\bar{q}_{n-1}(F) \geq 16$ by (3.1), we have $|F^\times/D_F(2^{n-2})| \geq \bar{q}_n(F)\cdot\bar{q}_{n-1}(F) \geq 32$. Therefore $\{-1\}^{n-2}k_1F$ has at least 32 elements.
To show that the index of this group in $k_{n-1}F$ is at least 4 we just need to find $\alpha, \beta, \gamma \in k_{n-1}F \setminus \{-1\}^{n-2}k_1F$ such that $\alpha + \beta + \gamma \in \{-1\}^{n-2}k_1F$.
By the hypothesis there are $a, b, c \in D_F(3\cdot 2^{n-3}) \setminus D_F(2^{n-2})$ such that $a + b + c = 0$. In k_2F we compute $\{-a,-b\} + \{-a,-c\} + \{-b,-c\} = \{-a, bc\} + \{a, -bc\} = \{-1, abc\}$. Therefore we are finished if we show that none of the symbols $\varepsilon^{n-3}\{-a,-b\}, \varepsilon^{n-3}\{-a,-c\}$ and $\varepsilon^{n-3}\{-b,-c\}$ in $k_{n-1}F$ lies actually in $\{-1\}^{n-2}k_1F$.
If this is not true we may by case symmetry suppose that $\varepsilon^{n-3}\{-a,-b\} = \varepsilon^{n-2}\{-x\}$ for some $x \in F^\times$. Then the $(n-1)$-fold Pfister forms $2^{n-3}\times\langle\langle a,b\rangle\rangle$ and $2^{n-2}\times\langle\langle x\rangle\rangle$ over F are isometric, i.e. the quadratic form $\varphi := 2^{n-3}\times\langle 1, x, x, -a, -b, -ab\rangle$ over F is hyperbolic. It follows that any subform of φ of dimension greater than $\frac{1}{2}\dim(\varphi) = 3\cdot 2^{n-3}$ is isotropic. In particular, the form $2^{n-2}\times\langle -ax\rangle \perp 2^{n-3}\times\langle 1\rangle \perp \langle b\rangle$, similar to a subform of φ, must be isotropic. It follows that $ax \in D_F(2^{n-2})\cdot D_F(2^{n-3}\times\langle 1\rangle \perp \langle b\rangle) \subset D_F(2^{n-1})$ whence $x \in D_F(2^{n-1})$. On the other hand, $\varphi \cong 2^{n-3}\times\langle 1, x, x, c, abc, -ab\rangle$ shows that $2^{n-2}\times\langle x\rangle \perp 2^{n-3}\times\langle 1\rangle \perp \langle c\rangle$ is isotropic. This in turn implies that $-x \in D_F(2^{n-2})\cdot D_F(2^{n-3}\times\langle 1\rangle \perp \langle c\rangle) \subset D_F(2^{n-1})$. Together this leads to $-1 \in D_F(2^{n-1})$ which contradicts $s(F) = 2^n$. □

5.5. COROLLARY. *Let F be a nonreal field with $s(F) \geq 8$. Then $|\mathrm{Br}_2(F)| \geq 128$ and $|W(F)| \geq 2^{18}$.*

Proof: If $s(F) = 8$ then the theorem shows $|k_2F| \geq 128$. But this is also true if $s(F) = 2^n > 8$ since then already the subgroup $\{-1\}k_1F$, isomorphic to $F^\times/D_F(2)$, has order at least $\bar{q}_n(F) \cdot \bar{q}_{n-1}(F) \cdot \bar{q}_{n-2}(F)$ which is sufficiently large by the results of section 3. By Merkuriev's theorem, $\mathrm{Br}_2(F)$ is isomorphic to k_2F, so in particular we have $|\mathrm{Br}_2(F)| \geq 128$. (In fact, the arguments to estimate the size of k_2F work similarly for $\mathrm{Br}_2(F)$, so it is not necessary to invoke Merkuriev's theorem here.)
Let I denote the fundamental ideal of $W(F)$ and let $\bar{I}^i := I^i/I^{i+1}$ for $i \geq 0$. For $i = 0, 1, 2$ it follows from [9] that $\bar{I}^i \cong k_iF$. Thus $|\bar{I}^0| = 2$, $|\bar{I}^1| = q(F) \geq 512$ and $|\bar{I}^2| \geq 128$. Moreover, $s(F) \geq 8$ implies $|\bar{I}^3| \geq 2$. Therefore $|W(F)| \geq |\bar{I}^0| \cdot |\bar{I}^1| \cdot |\bar{I}^2| \cdot |\bar{I}^3| \geq 2^{18}$. □

A HYPERGRAPHS WITH CONNECTED COLORINGS

In this appendix t, k and n denote nonnegative integers with $t \leq k \leq n$. We briefly say *k-set* for a set of cardinality k. A *k-hypergraph* is a system $\mathcal{H} = (V, \mathcal{E})$ where V is a set whose elements are called *vertices* and $\mathcal{E}$ a collection of distinct k-subsets of V called *edges*. A graph in the usual sense is then just a 2-hypergraph.
Let $\mathcal{H} = (V, \mathcal{E})$ be a k-hypergraph. Its number of vertices $|V|$ is called the *order of $\mathcal{H}$*. We say that $\mathcal{H}$ is *complete* if each k-subset of V is actually an edge, i.e. if $\mathcal{E} = \{E \subset V \mid |E| = k\}$. By an *edge-coloring* of $\mathcal{H}$ we mean a function $f : \mathcal{E} \to C$. We consider the elements of C as *colors* and for $E \in \mathcal{E}$ we call $f(E)$ the *color of E*. For $t > 0$ we say that the edge-coloring f is *t-connected* if any two edges of the same color meet in at least t vertices, i.e. if for any $E, E' \in \mathcal{E}$ with $f(E) = f(E')$ we have $|E \cap E'| \geq t$.

A.1. PROBLEM. *Let t, k, n be nonnegative integers with $t \leq k \leq n$. Let $\mathcal{H} = (V, \mathcal{E})$ be a complete k-hypergraph of order n. What is the least integer m such that there exists a t-connected edge-coloring $f : \mathcal{E} \to C$ on $\mathcal{H}$ with $|C| = m$?*

The integer m which meets the condition in the problem depends only on the values of t, k and n and will be denoted by $M(t, k, n)$. We recall our notation $\mathcal{P}_k^n$ for the set of all k-subsets of $\{1, \ldots, n\}$. A complete k-hypergraph of order n is then given by $\mathcal{K}_k^n := (\{1, \ldots, n\}, \mathcal{P}_k^n)$. So $M(t, k, n)$ is just the least integer m such that there exists a function $f : \mathcal{P}_k^n \to C$ where $|C| = m$ and such that $f(X) = f(X')$ implies $|X \cap X'| \geq t$ for any $X, X' \in \mathcal{P}_k^n$. To study $M(t, k, n)$ as a function in t, k and n we use the theory of *intersecting families* in combinatorics.
Let $\mathcal{F}$ be a family of sets. We write $\bigcup \mathcal{F}$ (resp. $\bigcap \mathcal{F}$) for the union (resp. the intersection) of all sets belonging to $\mathcal{F}$. If $|U \cap V| \geq t$ holds for every $U, V \in \mathcal{F}$ then we say that the family $\mathcal{F}$ is *t-intersecting* (just *intersecting* for $t = 1$). A

coloring $f : \mathcal{E} \to C$ of a k-hypergraph $\mathcal{H} = (V, \mathcal{E})$ is thus t-connected if and only if $f^{-1}(\{c\})$ is a t-intersecting family for every $c \in C$.
The crucial result on intersecting families is the Erdös-Ko-Rado theorem [4] which we state in the slightly stronger version of [14]:

A.2. Theorem (Erdös-Ko-Rado). *Let $n \geq (k-t+1)(t+1)$. If $\mathcal{F}$ is a t-intersecting family of k-subsets of an n-set then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.*

This theorem gives the optimal bound. Indeed, if N is an n-set and T a t-subset then $\mathcal{F} := \{U \subset N \mid |U| = k, T \subset U\}$ is a t-intersecting family with precisely $\binom{n-t}{k-t}$ elements. However, under the additional condition $|\bigcap \mathcal{F}| < t$, better bounds on $|\mathcal{F}|$ can be given. In the case $t = 1$ this is the following main result of [6]. (A short proof of this can be found in [5] where the case $t > 1$ is also treated.)

A.3. Theorem (Hilton-Milner). *Let $\mathcal{F}$ be a family of pairwise intersecting k-subsets of an n-set such that $\bigcap \mathcal{F} = \emptyset$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

Now we begin with the investigation $M(t,k,n)$ as a function in t, k and n with $0 < t \leq k \leq n$. We first treat the easy cases when t and k take extremal values. Part (c) is implicitly shown in [2].

A.4. Proposition. *(a) $M(t,k,n) = 1$ is equivalent to $n \leq 2k - t$.*

(b) $M(t,k,n) = \binom{n}{k}$ is equivalent to $k = t$.

(c) $M(k-1,k,n) = M(n-k-1, n-k, n) \geq \frac{1}{n-k+1}\binom{n}{k}$ for $1 \leq k \leq n/2$.

Proof: (a) $M(t,k,n)$ is equal to 1 if and only if $\mathcal{P}_k^n$ is t-intersecting; this is the case if and only if $n \leq 2k - t$.
(b) Each condition holds if and only if any nonempty t-intersecting family of k-subsets of $\{1, \ldots, n\}$ consists of just one k-set.
(c) It is quite obvious that a family $\mathcal{F} \subset \mathcal{P}_k^n$ is $(k-1)$-intersecting if and only if the family of complement sets $\{\{1, \ldots, n\} \setminus U \mid U \in \mathcal{F}\}$ is $(n-k-1)$-intersecting. So $f : \mathcal{P}_k^n \to C$ is $(k-1)$-connected if and only if $f' : \mathcal{P}_{n-k}^n \to C, V \mapsto f(\{1, \ldots, n\} \setminus V)$ is $(n-k-1)$-connected. This shows in particular $M(k-1,k,n) = M(n-k-1, n-k, n)$.
For a $(k-1)$-intersecting family $\mathcal{F} \subset \mathcal{P}_k^n$ it is easy to check that either $|\bigcap \mathcal{F}| \geq k-1$ or $|\bigcup \mathcal{F}| \leq k+1$. In the first case we conclude $|\mathcal{F}| \leq n-k+1$ and in the second case $|\mathcal{F}| \leq k+1 \leq n-k+1$. If now $f : \mathcal{P}_k^n \to C$ is $(k-1)$-connected then $\mathcal{P}_k^n$ is covered by the $(k-1)$-intersecting families $f^{-1}(\{c\})$ for $c \in C$, which implies that $\binom{n}{k} = |\mathcal{P}_k^n| \leq (n-k+1) \cdot |C|$. □

A.5. Examples. (1) The function $f : \mathcal{P}_k^n \to \mathcal{P}_t^{n-k+t}$ which associates to $X \in \mathcal{P}_k^n$ the set of the t smallest numbers in X is a t-connected edge-coloring of $\mathcal{K}_k^n$.

(2) If $n \geq 2k-1$ then a 1-connected edge-coloring of $\mathcal{K}_k^n$ is given by

$$f : \mathcal{P}_k^n \longrightarrow \{1,\dots,n-2k+2\}\,,\ X \longmapsto \max\,(X \cup \{2k-1\}) - 2k + 2\,.$$

(3) Let $t < k < n$. If $f : \mathcal{P}_k^n \to C$ be a t-connected edge-coloring of $\mathcal{K}_k^n$ and $g : \mathcal{P}_{k+1}^n \to C'$ is a $(t+1)$-connected edge-coloring of $\mathcal{K}_{k+1}^n$, where C and C' are disjoint sets, then a $(t+1)$-connected edge-coloring of $\mathcal{K}_{k+1}^{n+1}$ is defined by

$$h : \mathcal{P}_{k+1}^{n+1} \longrightarrow C \cup C'\,,\ X \longmapsto \begin{cases} f\,(X \backslash \{n+1\}) & \text{if } n+1 \in X, \\ g\,(X) & \text{otherwise.} \end{cases}$$

From these examples we conclude:

A.6. Proposition. *(a)* $M(t,k,n) \leq \binom{n-k+t}{t}$.

(b) If $n \geq 2k-1$ *then* $M(1,k,n) \leq n-2k+2$.

(c) If $t<k<n$ *then* $M(t+1,k+1,n+1) \leq M(t,k,n)+M(t+1,k+1,n)$. □

For lower bounds on $M(t,k,n)$ we first consider the case $t \geq 2$.

A.7. Theorem. *Let* $2 \leq t < k$. *Then for* $n \geq (k-t+1)(t+1)$ *we have*

$$M(t,k,n) \ \geq\ \prod_{i=0}^{t-1} \frac{n-i}{k-i} \ >\ \left(\frac{n}{k}\right)^t .$$

Proof: Let $f : \mathcal{P}_k^n \to C$ be a t-connected edge-coloring of $\mathcal{K}_k^n$ with $n \geq (k-t+1)(t+1)$. For each $c \in C$ we have then by the Erdös-Ko-Rado theorem $|f^{-1}(\{c\})| \leq \binom{n-t}{k-t}$. As $\mathcal{P}_k^n = \bigcup_{c\in C} f^{-1}(\{c\})$ we get $\binom{n}{k} \leq |C| \cdot \binom{n-t}{k-t}$. Therefore $|C| \geq \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-t+1}{k-t+1}$ and an easy computation shows the second inequality. □

For the purposes of section 3 we state the following particular case:

A.8. Corollary. *Let i and m be positive integers satisfying either* $2 \leq i \leq \frac{m}{2}$ *or* $3 \leq i = \frac{m+1}{2}$ *or* $5 \leq i = \frac{m}{2}+1$. *Then* $M(2^{i-2}+1, 2^i, 2^m) > 2^{(m-i)(2^{i-2}+1)}$. □

Now we come to the case $t = 1$.

A.9. Lemma. *For* $k > 1$ *we define the polynomial*

$$F_k(X) := \prod_{i=0}^{k-1}(X-i) - k\,(X-2k+1)\left(\prod_{i=1}^{k-1}(X-i) - \prod_{i=1}^{k-1}(X-k-i) + (k-1)!\right).$$

If $k \leq n$ *and* $f : \mathcal{P}_k^n \to C$ *is such that* $\bigcap f^{-1}(\{c\}) = \emptyset$ *for every* $c \in C$ *then either* $|C| \geq n-2k+2$ *or* $F_k(n) \leq 0$.

Proof: Suppose that f has the stated property. Then the Hilton-Milner theorem implies $\binom{n}{k} \leq |C| \cdot [\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1]$. On the other hand, $(k!)^{-1} \cdot F_k(n) = \binom{n}{k} - (n-2k+1) \cdot [\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1]$. Thus $F_k(n) > 0$ implies $|C| > (n-2k+1)$. $\square$

A.10. Remark. The polynomial F_k defined in the lemma is monic of degree k. In particular, we have $F_k(n) > 0$ for all n sufficiently large. Computation for small values of k yields: $F_2(X) = X^2 - 7X + 18$, $F_3(X) = X^3 - 21X^2 + 140X - 240$ and $F_4(X) = X^4 - 54X^3 + 731X^2 - 3534X + 5880$. Thus we have $F_2(n) > 0$ for any $n \in \mathbb{N}$, $F_3(n) > 0$ for $n \geq 3$ and $F_4(n) > 0$ for $n \geq 37$ whereas $F_4(36) < 0$.

A.11. Theorem. *For any $k \geq 1$ there is a constant $c_k \geq 2k-2$ such that for all $n \in \mathbb{N}$ sufficiently large we have*

$$M(1,k,n) = n - c_k.$$

For $k \leq 3$ we have, more precisely, $M(1,k,n) = n - 2k + 2$ for $n \geq 2k-1$.

Proof: For $k = 1$ there is nothing to show since $M(1,1,n) = n$. For $k \geq 2$ let $F_k(X)$ be defined as in the lemma. By the above remark we may choose the least integer $n_k \geq 2k-1$ such that $F_k(n) > 0$ for all $n \geq n_k - 1$. In particular we have $n_2 = 3$ and $n_3 = 5$. Let $c_k := n_k - M(1,k,n_k)$. Then (A.6, b) implies $c_k \geq 2k-2$ and we check that equality holds for $k = 2, 3$.
We want to prove by induction that $M(1,k,n) = n - c_k$ for $n \geq n_k$. For $n = n_k$ this is trivial statement. Suppose it is true for $n - 1 \geq n_k$. Let $f : \mathcal{P}_k^n \to C$ be a 1-connected edge-coloring of $\mathcal{K}_k^n$. If $\bigcap f^{-1}(\{c\}) = \emptyset$ for each $c \in C$ then by the lemma we have $|C| \geq n - 2k + 2 \geq n - c_k$. On the other hand, if there is $c \in C$ such that the intersection $\bigcap f^{-1}(\{c\})$ is not empty then we may suppose that it contains the element n. Then the restriction $f' : \mathcal{P}_k^{n-1} \to C \setminus \{c\}$ of f to $\mathcal{P}_k^{n-1}$ is a 1-connected edge-coloring of $\mathcal{K}_k^{n-1}$. By the induction hypothesis we have $|C \setminus \{c\}| \geq M(1,k,n-1) = (n-1) - c_k$ and thus $|C| \geq n - c_k$. This implies $M(1,k,n) \geq n - c_k$. But (A.6, c) shows $M(1,k,n) \leq M(1,k,n-1) + M(0,k-1,n-1) = n - c_k$ since $M(0,k-1,n-1) = 1$. Hence $M(1,k,n) \geq n - c_k$ which finishes the induction step. $\square$

A.12. Question. *Does $M(1,k,n) = n - 2k + 2$ hold for all $n \geq 2k-1$, even if $k > 3$?*

B CC-Graphs

In this appendix we study connected edge-colorings for usual complete graphs. Here we are not only interested in the minimal number of colors but also in the distribution of the colors in the graph.

Let $\mathcal{G}$ denote a complete graph of order n with vertices $v_1, \ldots, v_n$ and colored edges. The distribution of colors in $\mathcal{G}$ can be equivalently represented by an edge-coloring of $\mathcal{K}_2^n$ (see appendix A), i.e. by a function $f : \mathcal{P}_2^n \to C$, where C stands for the set of colors in $\mathcal{G}$ and f associates to $\{i, j\} \in \mathcal{P}_2^n$ the color of the edge between the vertices v_i and v_j.
A set of all the edges of a certain color shall be called a *color-component.* If such a color-component consists of $r \geq 3$ edges all together having a vertex x in common we call it an r*-star* and x its *center.* By a *triangle* in $\mathcal{G}$ we mean a complete subgraph of order 3 of $\mathcal{G}$. A *triangle* is said to be *monochrome* (resp. *three-colored*) if the three edges are of the same color (resp. of three different colors). A second complete colored graph $\mathcal{G}'$ of order n is said to be *equivalent to* $\mathcal{G}$ if there is a bijection between the sets of vertices of $\mathcal{G}$ and $\mathcal{G}'$ such that the induced bijection on the sets of edges preserves the color-components (in both directions).
We call $\mathcal{G}$ *color-connected* or a *CC-graph* if in $\mathcal{G}$ any two edges of the same color are adjacent. This is equivalent to the edge-coloring f being 1-connected. The only possible color-components in $\mathcal{G}$ are then single edges, pairs of edges with a vertex in common, stars and monochrome triangles.
Theorem (A.11) says that $M(1, 2, n) = n - 2$ for $n \geq 3$. This corresponds to a result of [13]. We rephrase it as follows and give a direct proof.

B.1. PROPOSITION (TORT). *A CC-graph of order $n \geq 3$ has at least $n - 2$ colors.*

Proof: For $n = 3$ the statement is trivial. If $n > 3$ and $\mathcal{G}$ has less than n colors then one of its color-components must be a star. Deleting the center of this star yields a CC-graph $\mathcal{G}'$ of order $n - 1$ with less colors. By induction hypothesis $\mathcal{G}'$ has at least $n - 3$ and therefore $\mathcal{G}$ at least $n - 2$ colors. □

For any $n \geq 3$ the complete graph $\mathcal{K}_2^n$, whose vertices are the integers $1, \ldots, n$, together with the 1-connected coloring $f_n : \mathcal{P}_2^n \to \{1, \ldots, n - 2\}$, $\{i, j\} \mapsto \max\{i, j, 3\} - 2$ defines a particular CC-graph $\mathcal{G}_n$ of order n with $n-2$ colors (compare with example (A.5, 2)). The color-components of $\mathcal{G}_n$ are one monochrome triangle and one i-star for each $3 \leq i \leq n-1$. For $3 \leq n \leq 5$, every CC-graph with $n-2$ colors is equivalent to $\mathcal{G}_n$. This is not true for $n = 6$, since there is a CC-graph of order 6 with color-components a triangle and three 4-stars.

B.2. PROPOSITION. *Let $\mathcal{G}$ be a CC-graph with $n \geq 3$ vertices and $n-2$ colors. Then $\mathcal{G}$ has as color-components one monochrome triangle and $n - 3$ stars. Moreover, each vertex of $\mathcal{G}$ lies either on the monochrome triangle or is the center of exactly one star.*

Proof: Let $\mathcal{G}'$ be the complete subgraph spanned by all vertices of $\mathcal{G}$ which are not the center of a star in $\mathcal{G}$. We want to show that $\mathcal{G}'$ is a monochrome triangle. Then the vertices of $\mathcal{G}$ outside of $\mathcal{G}'$ will be the centers of $n - 3$ stars and as $\mathcal{G}$ has just $n - 2$ colors the entire statement follows.

Let n' be the order of $\mathcal{G}'$. The $n-n'$ vertices of $\mathcal{G}$ outside of $\mathcal{G}'$ are all centers of stars whose colors do not appear in $\mathcal{G}'$. As a consequence, $\mathcal{G}'$ has at least $n-n'$ colors less than $\mathcal{G}$. Then by (B.1), $\mathcal{G}'$ has exactly $n'-2$ colors. Since $\mathcal{G}'$ is a graph without stars each color appears at most three times, counting the edges yields $3(n'-2) \geq \frac{n'(n'-1)}{2}$ whence $n' \leq 5$. As $\mathcal{G}'$ has $n'-2$ colors and contains no star, we have $n'=3$ and $\mathcal{G}'$ is a monochrome triangle. □

A CC-graph $\mathcal{G}$ will be called *total* if there is a permutation $\sigma \in \mathcal{S}_n$ such that for any $\{i,j\} \in \mathcal{P}_2^n$ the color of the edge between v_i and v_j depends only on $\max\{\sigma(i),\sigma(j)\}$. After renumbering the vertices $\mathcal{G}$ we may then suppose that the permutation σ is the identity on $\{1,\dots,n\}$.
Let $\mathcal{G}$ be a total CC-graph of order n with vertices $v_1,\dots,v_n$ enumerated in such a way that the color of any edge linking v_i and v_j depends only on $\max\{i,j\}$. Then $\mathcal{G}$ has at most $n-1$ different colors. From (B.1) it follows that the number of colors in $\mathcal{G}$ is either $n-2$ or $n-1$. Further, by (B.2) the number of colors is $n-2$ if and only if v_1, v_2 and v_3 form a monochrome triangle and then the color of the edge between v_i and v_j depends precisely on $\max\{i,j,3\}$. In both cases the enumeration of the vertices is unique up to changing the first three respectively the first two indices. Moreover, $\mathcal{G}$ contains exactly $n-3$ stars. More precisely, for each $4 \leq i \leq n$ there is exactly one $(i-1)$-star in $\mathcal{G}$ whose center is v_i. It is clear from the definition that a complete subgraph of a total CC-graph is also a total CC-graph.

B.3. Proposition. *A CC-graph $\mathcal{G}$ is total if and only if it contains no three-colored triangle.*

Proof: The necessity of the condition follows from the definition of a total CC-graph. Suppose now that $\mathcal{G}$ is a CC-graph with n vertices with no three-colored triangle. We show by induction on n that $\mathcal{G}$ is total. For $n \leq 3$ this is evident. If $n \geq 4$ then any complete subgraph with 4 vertices contains a star since otherwise it would contain a three-colored triangle. So we can choose an r-star in $\mathcal{G}$ where r is as large as possible. For the ease of imagination say, it is of red color. We may suppose that v_n is the center of this star. Let $\mathcal{G}'$ be the complete subgraph of $\mathcal{G}$ with all the vertices of $\mathcal{G}$ except v_n. Then $\mathcal{G}'$ is also a CC-graph with $n-1$ vertices and contains no three-colored triangle. So, by the induction hypothesis, $\mathcal{G}'$ is total, i.e. its vertices can be enumerated as $v_1,\dots,v_{n-1}$ in such a way that the color of an edge connecting vertices v_i and v_j depends just on $\max\{i,j\}$. This would still be true for the enumeration of the vertices $v_1,\dots,v_n$ of $\mathcal{G}$, if v_n is connected with each of the $v_1,\dots,v_{n-1}$ by an edge of red color. So we just have to show that $r=n-1$. Suppose that $r<n-1$. Then certainly $n>4$ since $r \geq 3$ by the definition of an r-star. But v_{n-1} is the center of an $n-2$-star in $\mathcal{G}'$, say of blue color. By the maximality of r we see that the edge between v_{n-1} and v_n cannot be blue and that $r=n-2$. So there must be exactly one vertex v_k with $1 \leq k \leq n-1$ which is connected with v_n with an edge of color different from red. It cannot be of blue color either so say that its color is green. Now we see that there is a triangle of colors red, blue and green contained in $\mathcal{G}$, formed by v_k, v_{n-1}, v_n if $k < n-1$ and by v_1, v_{n-1}, v_n if $k = n-1$, which gives the desired contradiction. □

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