# Multirelative $K$-Theory and Axioms for the $K$-Theory of Rings

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Abstract. $K$-groups are defined for a special type of $m$-tuples of ideals in a ring. It is shown that some of the properties of this multirelative $K$-theory characterize the $K$-theory of rings.

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## Introduction

Multirelative $K$-groups $K_n(R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)$ of an $m$-tuple $(\mathfrak{a}_1, \ldots, \mathfrak{a}_m)$ of ideals of a ring $R$ are recently used to derive properties of the absolute $K$-groups, e.g. by Levine [4] and by Bloch and Lichtenbaum [1]. Here it is shown how $K$-theory as defined in [3] can easily be extended to the multirelative case and that some of its properties can be taken as axioms for the $K$-theory of rings. Special types of $m$-tuples of ideals—the 'normal' $m$-tuples—play a crucial role. In fact we will only define multirelative $K$-groups for such $m$-tuples. The notion of normal $m$-tuple of ideals is introduced in Section 2. It already appeared in 1981 in a paper by Dayton and Weibel [2] on the $K$-theory of affine glued schemes under the name of 'condition (CRT)' (= Chinese Remainder Theorem).

In Section 4 we review briefly higher $K$-theory as defined in [3]. In Section 6 multirelative $K$-groups are defined, and in Section 7 it is shown that from some of their properties one can reconstruct the $K$-theory of rings.

## 1 Notations

In this paper 'ring' stands for a non-unital ring. Non-unital rings form a category which is denoted by $\mathcal{R}$.

Since the functors $GL$, $E$ and $K_1$ are product preserving functors from unital rings to groups, they can be extended to functors defined on $\mathcal{R}$ in the usual way: if $T$ is one of these functors, then put

$$T(R) := \mathrm{Ker}(T(R^+) \to T(\mathbb{Z})),$$

where $R^+ = R \times \mathbb{Z}$ with multiplication given by

$$(r,k)(s,l) = (rs + ks + lr, kl)$$

is a ring with $(0,1)$ as unity element.

Here 'ideal' will always stand for 'twosided ideal'.

By $\mathcal{A}$ we will denote the category of Abelian groups, by $\mathcal{G}$ the category of all groups, and by $\mathcal{S}$ the category of sets. The category of simplicial objects in a category $\mathcal{C}$ is denoted by $s\mathcal{C}$.

## 2 $m$-CUBES AND NORMAL $m$-TUPLES

In this section the notion of normality of an $m$-tuple of ideals is considered. Only the group structure is involved in its definition, and since we can use later a similar notion for groups instead of rings we give a more general definition. By $\underline{m}$ we will denote the set $\{1, \ldots, m\}$.

DEFINITION 1. An $m$-tuple $(B_1, \ldots, B_m)$ of normal subgroups of a group $A$—also denoted as $(A, B_1, \ldots, B_m)$—is called *normal* if for all subsets $I$ and $J$ of $\underline{m}$

$$\bigcap_{i \in I} B_i \cdot \prod_{j \in J} B_j = \bigcap_{i \in I} \Big( B_i \cdot \prod_{j \in J} B_j \Big).$$

The condition is trivially fulfilled when $I \cap J \neq \emptyset$. In the case of Abelian groups it reads in the additive notation as

$$\bigcap_{i \in I} B_i + \sum_{j \in J} B_j = \bigcap_{i \in I} \Big( B_i + \sum_{j \in J} B_j \Big).$$

Note that in the special case of an $m$-tuple of ideals in a commutative ring the condition is a local one since it involves only intersections and sums of ideals.

The subsets of $\underline{m}$ are ordered by inclusion. This ordered set determines in the usual way a category $\mathcal{C}_m$. For every pair $(I, J)$ of subsets with $I \subseteq J$ there is the unique morphism $\rho^I_J$ from $I$ to $J$ in $\mathcal{C}_m$.

DEFINITION 2. Let $\mathcal{D}$ be a category. An *$m$-cube* in $\mathcal{D}$ is a functor

$$D \colon \mathcal{C}_m \to \mathcal{D}, \quad I \mapsto D_I, \quad \rho^I_J \mapsto r^I_J.$$

The morphisms in $\mathcal{C}_m$ are generated by the $\rho^I_J$, where $\#J = \#I + 1$. An $m$-cube in a category $\mathcal{D}$ is a commutative diagram in $\mathcal{D}$ having the shape of an $m$-dimensional cube. The edges of the cube correspond to the images of these generating morphisms.

DEFINITION 3. Let $D \colon \mathcal{C}_m \to \mathcal{D}$ be an $m$-cube in $\mathcal{D}$. It is said to be a *split* $m$-cube if for every pair of subsets $(I, J)$ of $\underline{m}$ satisfying $I \subseteq J$ there is a morphism $s^J_I \colon D_J \to D_I$ in $\mathcal{D}$ such that

(S1) $s^J_I s^K_J = s^K_I$ for all $I \subseteq J \subseteq K$,

(S2) $r^I_J s^J_I = 1_{D_J}$ for all $I \subseteq J$, (S3) $r_J^{I\cap J} s_{I\cap J}^I = s_J^{I\cup J} r_{I\cup J}^I$ for all $I$ and $J$.

(Of course such a split $m$-cube can also be seen as a functor defined on a category which is obtained from $\mathcal{C}_m$ by adjoining extra morphisms $\sigma_I^J : J \to I$.)

In condition (S3) one only needs the case where $\#(I \setminus J) = \#(J \setminus I) = 1$. It then reads

(S3') $r_{I\cup\{k\}}^I s_I^{I\cup\{j\}} = s_{I\cup\{k\}}^{I\cup\{j,k\}} r_{I\cup\{j,k\}}^{I\cup\{j\}}$ for all $j, k \notin I$ with $j \neq k$.

This can easily be seen as follows. Put $K = I \cap J$, $I \setminus K = \{i_1, \ldots, i_p\}$ and $J \setminus K = \{j_1, \ldots, j_q\}$. Then the result follows from the diagram

$$\begin{array}{ccccccc}
D_K & \longrightarrow & D_{K\cup\{i_1\}} & \longrightarrow & \cdots & \longrightarrow & D_I \\
\uparrow & & \uparrow & & & & \uparrow \\
D_{K\cup\{j_1\}} & \longrightarrow & D_{K\cup\{i_1,j_1\}} & \longrightarrow & \cdots & \longrightarrow & D_{I\cup\{j_1\}} \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
D_J & \longrightarrow & D_{J\cup\{i_1\}} & \longrightarrow & \cdots & \longrightarrow & D_{I\cup J}
\end{array}$$

where the horizontal maps are $r$-maps and the vertical maps are $s$-maps.

DEFINITION 4. An $m$-tuple $T = (A, B_1, \ldots, B_m)$ of normal subgroups determines an $m$-cube in $\mathcal{G}$:

$$I \mapsto T_I = A \Big/ \prod_{i\in I} B_i.$$

When $I \subseteq J$, then $\prod_{i\in I} B_i \subseteq J$ and $1_A$ induces a grouphomomorphism $r_J^I : T_I \to T_J$. This $m$-cube is said to be *induced* by the $m$-tuple $T$. Similarly for an $m$-tuple of ideals in a ring.

PROPOSITION 2.1. *Let $D : \mathcal{C}_m \to \mathcal{D}$ be an $m$-cube in $\mathcal{G}$, which is split as an $m$-cube in $\mathcal{S}$. Then $D$ is induced by a normal $m$-tuple of normal subgroups of $D_\emptyset$.*

*Proof.* For $i \in \underline{m}$ put

$$B_i = \operatorname{Ker}\left(r_{\{i\}}^\emptyset : D_\emptyset \to D_{\{i\}}\right).$$

We will first show that the cube is induced by the $m$-tuple $(D_\emptyset, B_1, \ldots, B_m)$. Since the cube splits in $\mathcal{S}$, the homomorphisms $D_\emptyset \to D_I$ are surjective. To show that for each $I \subseteq \underline{m}$

$$\operatorname{Ker}(D_\emptyset \to D_I) = \prod_{i\in I} B_i.$$

This can be done by induction on $\#(I)$. For $\#(I) = 0$ it is trivial. Let $\#(I) > 0$. Choose $k \in I$. By induction hypothesis

$$\operatorname{Ker}\left(D_\emptyset \to D_{I\setminus\{k\}}\right) = \prod_{i\in I\setminus\{k\}} B_i.$$

Since the cube splits in $\mathcal{S}$ we have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & B_k \cap \prod_{i\in I\setminus\{k\}} B_i & \longrightarrow & B_k & \longrightarrow & \mathrm{Ker}\left(r_I^{I\setminus\{k\}}\right) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \prod_{i\in I\setminus\{k\}} B_i & \longrightarrow & D_\emptyset & \longrightarrow & D_{I\setminus\{k\}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Ker}\left(r_I^{\{k\}}\right) & \longrightarrow & D_{\{k\}} & \longrightarrow & D_I & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Hence

$$\mathrm{Ker}(r_I^\emptyset)/B_k \cong \prod_{i\in I\setminus\{k\}} B_i \Big/ \left(B_k \cap \prod_{i\in I\setminus\{k\}} B_i\right) \cong \prod_{i\in I}(B_i/B_k),$$

and therefore,

$$\mathrm{Ker}(r_I^\emptyset) = \prod_{i\in I} B_i.$$

For the normality of the $m$-tuple let $I, J \subseteq \underline{m}$ and consider the commutative square

$$\begin{array}{ccc}
D_\emptyset & \xrightarrow{(r^\emptyset_{\{i\}})} & \bigtimes_{i\in I} D_\emptyset/B_i \\
\Big\downarrow r_J^\emptyset & & \Big\downarrow (r^{\{i\}}_{J\cup\{i\}}) \\
D_\emptyset \Big/ \prod_{j\in J} B_j & \xrightarrow{(r^J_{J\cup\{i\}})} & \bigtimes_{i\in I} D_\emptyset \Big/ \prod_{j\in J\cup\{i\}} B_j.
\end{array}$$

Since the $m$-cube is split in $\mathcal{S}$ the vertical homomorphisms have compatible sections in $\mathcal{S}$. So $r_J^\emptyset$ induces a surjective homomorphism on the kernels of the horizontal homomorphisms. This holds for all $I, J \subseteq \underline{m}$. Therefore, the $m$-tuple $(D_\emptyset, B_1, \ldots, B_m)$ is normal. $\square$

For the Abelian case we also prove the converse.

PROPOSITION 2.2. *Let $T = (A, B_1, \ldots, B_m)$ be a normal $m$-tuple of subgroups of an Abelian group $A$. Then the induced $m$-cube is split in the category $\mathcal{S}$.*

*Proof.* By taking kernels of the surjective homomorphisms in the induced $m$-cube it can be extended to a diagram of $3^m$ Abelian groups. We will give a detailed description of this diagram and show how a splitting of the cube can be obtained from it.

For each pair $(I, J)$ of disjoint subsets of $\underline{m}$ define

$$C_J^I = \bigcap_{i\in I} B_i + \sum_{j\in J} B_j \Big/ \sum_{j\in J} B_j.$$

Then for each such pair $(I,J)$ and each $k \notin I \cup J$ we have a surjective homomorphism $C^I_J \to C^I_{J\cup\{k\}}$, induced by $r^J_{J\cup\{k\}}\colon A_J \to A_{J\cup\{k\}}$, where we use the notation

$$A_J = A \Big/ \sum_{j\in J} B_j.$$

Thus $A_J = C^\emptyset_J$. The kernel of the surjective homomorphism $C^I_J \to C^I_{J\cup\{k\}}$ is

$$\Big(\bigcap_{i\in I} B_i + \sum_{j\in J} B_j\Big) \cap \Big(B_k + \sum_{j\in J} B_j\Big) \Big/ B_k + \sum_{j\in J} B_j.$$

We have the inclusions

$$\bigcap_{i\in I\cup\{k\}} B_i + \sum_{j\in J} B_j \subseteq \Big(\bigcap_{i\in I} B_i + \sum_{j\in J} B_j\Big) \cap \Big(B_k + \sum_{j\in J} B_j\Big) \subseteq \bigcap_{i\in I\cup\{k\}} \Big(B_i + \sum_{j\in J} B_j\Big).$$

By normality these groups are equal, so we have a short exact sequence

$$0 \to C^{I\cup\{k\}}_J \to C^I_J \to C^I_{J\cup\{k\}} \to 0.$$

For each pair $(I,J)$ of disjoint subsets of $\underline{m}$ satisfying $I\cup J = \underline{m}$ choose a section

$$t^I_J\colon C^I_J \to C^I_\emptyset (\subseteq C^\emptyset_\emptyset = A)$$

of the map $C^I_\emptyset \to C^I_J$ induced by $r^\emptyset_J\colon A \to A_J$ and satisfying $t^I_J(0) = 0$. Next define maps $t^I_J\colon C^I_J \to C^I_\emptyset$ for every disjoint pair $(I,J)$ using induction to the number of elements of the complement of $I\cup J$. So, let $(I,J)$ be a disjoint pair of subsets of $\underline{m}$ with $\#(I\cup J) = n < m$ and assume that sections $t^K_L\colon C^K_L \to C^K_\emptyset$ have already been defined for pairs $(K,L)$ with $K\cup L$ having more than $n$ elements.

Choose $k \in \underline{m} \setminus (I\cup J)$. Let $x \in C^I_J$, then for $y = r^\emptyset_J t^I_{J\cup\{k\}} r^J_{J\cup\{k\}}(x)$ we have

$$r^J_{J\cup\{k\}}(y) = r^\emptyset_{J\cup\{k\}} t^I_{J\cup\{k\}} r^J_{J\cup\{k\}}(x) = r^J_{J\cup\{k\}}(x),$$

so, $x - y \in C^{I\cup\{k\}}_J$. Now define $t^I_J$ by

$$t^I_J(x) = t^{I\cup\{k\}}_J(x-y) + t^I_{J\cup\{k\}} r^J_{J\cup\{k\}}(x).$$

It easily verified that this map is a section of $r\colon C^I_\emptyset \to C^I_J$. Furthermore it is independent of the choice of $k$: if also $l \notin I\cup J$, then in both cases the image of an $x \in C^I_J$ under $t^I_J$ is determined in the same way by the images of the same elements in the following four groups

$$C^{I\cup\{l,k\}}_J,\ C^{I\cup\{l\}}_{J\cup\{k\}},\ C^{I\cup\{k\}}_{J\cup\{l\}},\ \text{and } C^I_{J\cup\{k,l\}}:$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_J^{I\cup\{l,k\}} \arrow[r] \arrow[d] & C_J^{I\cup\{l\}} \arrow[r] \arrow[d] & C_{J\cup\{k\}}^{I\cup\{l\}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_J^{I\cup\{k\}} \arrow[r] \arrow[d] & C_J^{I} \arrow[r] \arrow[d] & C_{J\cup\{k\}}^{I} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{J\cup\{l\}}^{I\cup\{k\}} \arrow[r] \arrow[d] & C_{J\cup\{l\}}^{I} \arrow[r] \arrow[d] & C_{J\cup\{k,l\}}^{I} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Thus we obtain a splitting of the cube, where the sections $s_I^J$ of the homomorphisms $r_J^I$, where $I \subseteq J$, are the maps $r_I^{\emptyset} t_J^{\emptyset}$. In particular, condition (S3') follows from the above diagram for $I = \emptyset$. $\square$

## 3 Operations on normal $m$-tuples of ideals

By $\mathcal{R}_m$ we will denote the category of all normal $m$-tuples of ideals. Such an $m$-tuple is denoted as $(R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)$, where $R$ is a ring and $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ are ideals of $R$. A morphism $\phi\colon (R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m) \to (S, \mathfrak{b}_1, \ldots, \mathfrak{b}_m)$ is just a ringhomomorphism $\phi\colon R \to S$ satisfying $\phi(\mathfrak{a}_i) \subseteq \mathfrak{b}_i$ for all $i \in \underline{m}$.

The following notations will simplify notations for long exact sequences of multirelative $K$-theory. Another advantage will be that they are useful to indicate funtoriality properties.

For each $m \geq 1$ the functor $D\colon \mathcal{R}_m \to \mathcal{R}_{m-1}$ is the functor that deletes the last ideal:

$$D(R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m) = (R, \mathfrak{a}_1, \ldots, \mathfrak{a}_{m-1})$$

and which has no effect on morphisms.

For each $m \geq 1$ the functor $M\colon \mathcal{R}_m \to \mathcal{R}_{m-1}$ is the functor that deletes the last ideal and that takes the ring and the other ideals modulo this ideal:

$$M(R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m) = (R/\mathfrak{a}_m, \overline{\mathfrak{a}}_1, \ldots, \overline{\mathfrak{a}}_{m-1}),$$

where $\overline{\mathfrak{a}}_j = \mathfrak{a}_j + \mathfrak{a}_i/\mathfrak{a}_i$, and which maps a morphism to the induced morphism.

A functor morphism $\phi\colon D \to M$ of the functors $D, M\colon \mathcal{R}_m \to \mathcal{R}_{m-1}$ is defined as follows: let $A = (R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)$, then $\phi_A\colon D(A) \to M(A)$ is the canonical ringhomomorphism $R \to R/\mathfrak{a}_m$.

Every $A \in \mathcal{R}_m$ has an *underlying ideal* $I(A)$, which is defined as the intersection of the $m$ ideals in $A$: when $A = (R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)$, then

$$I(A) = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_m.$$

Thus defined, $I(A)$ is functorial in $A$.

## 4 Higher $K$-theory of rings

In [3] the definition of higher $K$-groups is as follows. Let $R \in \mathcal{R}$. Choose a simplicial ring $\mathbf{R}$ with an augmentation $\varepsilon\colon \mathbf{R} \to R$ such that

- $\mathbf{R}$ is aspherical, i.e. $\pi_n(\mathbf{R}) = 0$ for all $n \geq 1$,
- $\mathbf{R}_m$ is free for all $m \geq 0$, say $\mathbf{R}_m$ is free on a set $X_m$ of generators,
- the sets $X_m$ of free generators are stable under degeneracies: $s_j(X_m) \subseteq X_{m+1}$ for all $m \geq 0$,
- the augmentation $\varepsilon$ induces an isomorphism $\pi_0(\mathbf{R}) \xrightarrow{\sim} R$.

Then for $n \geq 3$ the group $K_n(R)$ is defined as the $(n-2)$nd homotopy group of the simplicial group $GL(\mathbf{R})$, and the groups $K_1(R)$ and $K_2(R)$ are given by the exactness of

$$0 \to K_2(R) \to \pi_0(GL\mathbf{R}) \to GL(R) \to K_1(R) \to 0.$$

The groups $K_n(R)$ for $n \geq 3$ are Abelian because $GL(\mathbf{R})$ is a simplicial group. The group $K_1(R)$ is Abelian since it is the cokernel of $GL(\mathbf{R}_0) \to GL(R)$, and $K_2(R)$ is Abelian because it is the cokernel of $GL(\mathbf{R}_1) \to GL(Z_0)$, where $Z_0 = \{\,(x_0, x_1) \mid \epsilon(x_0) = \epsilon(x_1)\,\}$. In [3] it is shown using a comparison theorem that the higher $K$-groups are thus well-defined and that they are actually functors. For the purpose of this paper we will confine to a functorial resolution $\mathbf{Fr}(R)$ of a ring $R$, which we now describe. Let $F\colon \mathcal{S} \to \mathcal{R}$ the free ring functor and let $U\colon \mathcal{R} \to \mathcal{S}$ be the underlying set functor, then the functor $FU\colon \mathcal{R} \to \mathcal{R}$ together with the obvious functor morphisms $\nu\colon FU \to (FU)^2$ and $\eta\colon FU \to I$ is a cotriple. Put

$$\mathbf{Fr}_n = (FU)^{n+1}.$$

Face and degeneracy morphisms are given by

$$d_i = (FU)^i \eta (FU)^{n-1-i} \quad \text{and} \quad s_j = (FU)^i \nu (FU)^{n-1-i}.$$

The augmentation is then given by $\eta$.

A property of this functorial resolution is that, when applied to a surjective ringhomomorphism $R \to S$, it gives a dimensionwise surjective homomorphism $\mathbf{Fr}R \to \mathbf{Fr}S$ of simplicial rings, and since the ringhomomorphisms are dimensionwise split it also gives a surjective simplicial grouphomomorphism $GL(\mathbf{Fr}R) \to GL(\mathbf{Fr}S)$. This is often convenient when considering homotopy fibres, because surjective simplicial grouphomomorphisms are fibrations themselves. So instead of taking a homotopy fibre one just takes a fibre, i.e. the kernel of the simplicial group homomorphism.

## 5 Cubes in a simplicial group

Let $\mathbf{A}$ be a simplicial group with augmentation $d_0\colon \mathbf{A}_0 \to A$. It is a contravariant functor $\mathbf{A}\colon \Omega_+^{\mathrm{op}} \to \mathcal{G}$ from the category $\Omega_+$ of finite ordered sets

$$[n] = \{0, \dots, n\} \qquad (n \geq -1)$$

(where $[-1] = \emptyset$) and monotone (= order preserving) maps to the category of groups. (Here we use the notation $\mathbf{A}_{-1} = A$.) We will show that $\mathbf{A}$ determines an $m$-cube of groups for every nonnegative integer $m$. In stead of the ordered set of subsets of $\underline{m}$ for the description of an $m$-cube the ordered set of subsets of $[m-1]$ will be used for this purpose.

Let $\Omega(m)$ be the category of injective monotone maps

$$\alpha\colon [k] \to [m-1].$$

A morphism from $\alpha\colon [k] \to [m-1]$ to $\beta\colon [l] \to [m-1]$ is a monotone map $\gamma\colon [k] \to [l]$ such that $\beta\gamma = \alpha$. It exists if and only if $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(\beta)$, and it is unique if it exists.

For each $I \subseteq [m-1]$ there is a unique injective monotone map

$$\alpha_I\colon [k] \to [m-1],$$

where $k = m - 1 - \#(I)$ and $\mathrm{Im}(\alpha_I) = [m-1] \setminus I$. If $I \subseteq J \subseteq [m-1]$, then $\mathrm{Im}(\alpha_I) \supseteq \mathrm{Im}(\alpha_J)$ , so then there is a unique

$$\gamma_I^J : \alpha_J \to \alpha_I,$$

i.e. a monotone $\gamma_I^J : [m-1-\#(J)] \to [m-1-\#(I)]$ such that $\alpha_I \gamma_I^J = \alpha_J$.

DEFINITION 5. Let $\mathbf{A}$ be an augmented simplicial group and let $m$ be a nonnegative integer. Then the *$m$-cube of* $\mathbf{A}$ is the $m$-cube $\mathbf{A}(m)\colon \mathcal{C}_m \to \mathcal{G}$ with

$$\begin{cases} \mathbf{A}(m)_I = \mathbf{A}_{[m-1-\#(I)]} & \text{for all } I \subseteq [m-1], \\ r_J^I = \mathbf{A}(\gamma_I^J)\colon \mathbf{A}(m)_I \to \mathbf{A}(m)_J & \text{for all } I \subseteq J \subseteq [m-1]. \end{cases}$$

LEMMA 5.1. *Let the augmentation $d_0\colon \mathbf{A}_0 \to \mathbf{A}_{-1}$ induce a surjective homomorphism $\pi_0(\mathbf{A}) \to \mathbf{A}_{-1}$. Then for all integers $i, j, m$ such that $0 \le j < i \le m$*

$$d_i^{(m)}\left(\mathrm{Ker}\left(d_j^{(m)}\right)\right) = \mathrm{Ker}\left(d_j^{(m-1)}\right).$$

*Proof.* Let $x \in \mathrm{Ker}\left(d_j^{(m)}\right)$. Then, since $i > j$, $d_j d_i(x) = d_{i-1} d_j(x) = 1$. So $d_i(\mathrm{Ker}(d_j)) \subseteq \mathrm{Ker}(d_j)$. Now, let $y \in \mathrm{Ker}\left(d_j^{(m-1)}\right)$. There is an $x \in \mathbf{A}_m$ such that $d_j(x) = 1$ and $d_i(x) = y$. For $m > 1$ this is the case because a simplicial group is a Kan-complex, while for $m = 1$ it follows from the condition on the augmentation. □

PROPOSITION 5.1. *Let $\mathbf{A}$ be a simplicial group with an augmentation $d_0\colon \mathbf{A} \to A$ that induces an isomorphism $\pi_0(\mathbf{A}) \to A$. Then for all $m \ge 1$ the $m$-cube $\mathbf{A}(m)$ is induced by the $m$-tuple*

$$(\mathbf{A}_{m-1}, \mathrm{Ker}(d_0), \ldots, \mathrm{Ker}(d_{m-1}))).$$

*Proof.* All face maps are surjective, so it remains to show that for all $J \subseteq [m-1]$

$$\mathrm{Ker}(r_J^{\emptyset}) = \prod_{j \in J} \mathrm{Ker}\left(d_j^{(m-1)}\right).$$

For $J = \emptyset$ this is trivially true. Let $J$ be nonempty and proceed by induction. Let $x \in \operatorname{Ker}(r_J^{\emptyset})$. Let $k \in J$ be maximal. Then $r_{\{k\}}^{\emptyset}(x) = d_k(x) \in \operatorname{Ker}\left(r_J^{\{k\}}\right)$. By induction this group is equal to $\prod_{j\in J'} \operatorname{Ker}\left(d_j^{(m-2)}\right)$, where $J' = J \setminus \{k\}$. (Here we used the maximality of $k$ in $J$ and the same result for the $(m-1)$-cube $\mathbf{A}(m-1)$.) By the lemma we have

$$d_k\left(\prod_{j\in J'} \operatorname{Ker}\left(d_j^{(m-1)}\right)\right) = \prod_{j\in J'} \operatorname{Ker}\left(d_j^{(m-2)}\right).$$

Choose $y \in \prod_{j\in J'} \operatorname{Ker}(d_j^{(m-1)}$ such that $d_k(y) = d_k(x)$. Then $xy^{-1} \in \operatorname{Ker}(d_k)$. It follows that

$$\operatorname{Ker}(r_J^{\emptyset}) \subseteq \prod_{j\in J} \operatorname{Ker}\left(d_j^{(m-1)}\right).$$

For the other inclusion note that $d_j = r_{\{j\}}^{\emptyset}$ and

$$r_J^{\{j\}} r_{\{j\}}^{\emptyset} = r_J^{\emptyset}.$$

□

PROPOSITION 5.2. *Let $\mathbf{A}$ be as in Proposition 5.1 and assume moreover that $\mathbf{A}$ is aspherical. Then the $m$-tuple*

$$(\mathbf{A}_{m-1}, \operatorname{Ker}(d_0), \ldots, \operatorname{Ker}(d_{m-1}))$$

*is normal.*

*Proof.* The edges of the $m$-cube are face maps of the simplicial group $(A)$. Normality means that these maps preserve intersections of (the images of) the normal subgroups $\operatorname{Ker}(d_0), \ldots, \operatorname{Ker}(d_{m-1})$. By induction it suffices to show this for the face maps $d_i^{(m-1)}$. Let $J \subseteq [m-1]$. Then to show that

$$d_i\left(\bigcap_{j\in J} \operatorname{Ker}(d_j)\right) = \bigcap_{j\in J} d_i(\operatorname{Ker}(d_j)).$$

for $i \notin J$. The inclusion of the left hand side in the right hand side is trivial. So let $x \in \bigcap_{j\in J} d_i(\operatorname{Ker}(d_j))$. Then for $j \in J$ there is an $y_j \in \operatorname{Ker}(d_j)$ such that $x = d_i(y_j)$. For $j < i$ it follows that $d_j(x) = d_j d_i(y_j) = d_{i-1} d_j(x_j) = 1$. Similarly for $j > i$ we have $d_{j-1}(x) = 1$. So, since a simplicial group is a Kan-complex and for $J = [m-1]$ since $\mathbf{A}$ is aspherical, there is a $y \in \mathbf{A}_{m-1}$ such that $d_j(y) = 1$ for all $j \in J$ and $d_i(y) = x$. This shows that $x \in d_i\left(\bigcap_{j\in J} \operatorname{Ker}(d_j)\right)$. □

## 6 MULTIRELATIVE $K$-THEORY

A normal $m$-tuple of ideals $A = (R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)$ induces an $m$-cube in $\mathcal{R}$

$$A \colon I \mapsto R \Big/ \sum_{i\in I} \mathfrak{a}_i,$$

which by Proposition 2.2 is split in $\mathcal{S}$. Application of **Fr** to this $m$-cube gives an $m$-cube of simplicial rings which is dimensionwise split in $\mathcal{R}$. Put

$$\mathbf{Fr}(R,\mathfrak{a}_i) := \operatorname{Ker}(\mathbf{Fr}(R) \to \mathbf{Fr}(R/\mathfrak{a}_i)).$$

This is a simplicial ideal. The $m$-cube is then induced by the $m$-tuple

$$(\mathbf{Fr}(R), \mathbf{Fr}(R,\mathfrak{a}_1), \ldots, \mathbf{Fr}(R,\mathfrak{a}_m)),$$

of simplicial ideals, an object of the category $s\mathcal{R}_m$ of normal $m$-tuples of simplicial ideals. We also define the simplicial ideal

$$\mathbf{Fr}(R,\mathfrak{a}_1,\ldots,\mathfrak{a}_m) := \bigcap_{i=1}^{m} \mathbf{Fr}(R,\mathfrak{a}_i).$$

Application of $GL$ gives an $m$-cube of simplicial groups, which is dimensionwise split in $\mathcal{G}$. This $m$-cube is induced by the $m$-tuple

$$(GL\mathbf{Fr}(R), GL\mathbf{Fr}(R,\mathfrak{a}_1), \ldots, GL\mathbf{Fr}(R,\mathfrak{a}_m))$$

of simplicial normal subgroups. For $n \geq 3$ we define multirelative $K_n$ by

$$K_n(R,\mathfrak{a}_1,\ldots\mathfrak{a}_m) := \pi_{n-2}(GL\mathbf{Fr}(R,\mathfrak{a}_1,\ldots,\mathfrak{a}_m)).$$

Multirelative $K_2$ and $K_1$ are then given by the exactness of

$$\begin{aligned} 0 \to K_2(R,\mathfrak{a}_1,\ldots\mathfrak{a}_m) \to \pi_0(GL\mathbf{Fr}(R,\mathfrak{a}_1,\ldots,\mathfrak{a}_m)) \to& \\ GL(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_m) \to K_1(R,\mathfrak{a}_1,\ldots\mathfrak{a}_m) \to 0.& \end{aligned}$$

These multirelative $K_1$ and $K_2$ are Abelian groups for the same reason as in the absolute case.

Now let $A \in \mathcal{R}_m$ with $m \geq 1$. Then $\phi_*\colon GL\mathbf{Fr}(DA) \to GL\mathbf{Fr}(MA)$ is a fibration with fibre $GL\mathbf{Fr}(A)$. The long exact sequence of homotopy groups is a long exact sequence of multirelative $K$-groups which can easily be extended to include multirelative $K_2$ and $K_1$.

PROPOSITION 6.1. *Let $A \in \mathcal{R}_m$ with $m \geq 1$. Then we have a functorial exact sequence*

$$\cdots \to K_n(A) \to K_n(DA) \to K_n(MA) \to K_{n-1}(A) \to \cdots \to K_1(MA).$$

□

The connecting map $K_n(MA) \to K_{n-1}(A)$ will be denoted by $\delta$ and the map $K_n(A) \to K_n(DA)$ by $\iota$. To put it in an even more functorial way, we have an exact sequence of functors and functor morphisms

$$\cdots \to K_n \xrightarrow{\iota} K_nD \xrightarrow{K_n(\phi)} K_nM \xrightarrow{\delta} K_{n-1} \to \cdots \to K_1M.$$

In the remaining part of this section multirelative $K_0$ is defined and the long exact sequence for multirelative $K$-theory is extended with multirelative $K_0$-groups.

DEFINITION 6. For a normal $m$-tuple $A$ of ideals we define

$$K_0(A) = K_0(IA).$$

Thus defined, $K_0$ is a functor from $\mathcal{R}_m$ to $\mathcal{A}$.

For $m = 1$ we take the long exact sequence to be the long exact sequence of an ideal in a ring. Now assume that $m \geq 1$ and that we have an extended long exact sequence

$$\cdots \to K_1D \to K_1M \to K_0 \to K_0D \to K_0M$$

of functors $\mathcal{R}_m \to \mathcal{A}$. We will show that there is also such a sequence of functors $\mathcal{R}_{m+1} \to \mathcal{A}$.

Let $A = (R, \mathfrak{a}_1, \dots, \mathfrak{a}_{m+1}) \in \mathcal{R}_{m+1}$. Put $\mathfrak{b} = IA = \bigcap_{i=1}^{m+1} \mathfrak{a}_i$ . We have exact sequences for the following $m$-tuples of ideals

$$B = DA = (R, \mathfrak{a}_1, \dots, \mathfrak{a}_m),$$
$$\overline{B} = (R/\mathfrak{b}, \mathfrak{a}_1/\mathfrak{b}, \dots, \mathfrak{a}_m/\mathfrak{b})$$

and

$$(R, \mathfrak{a}_1, \dots, \mathfrak{a}_{m-1}, \mathfrak{b}).$$

These $m$-tuples are normal and their $K$-groups fit into a commutative diagram

$$\begin{tikzcd}
 & & K_1(DB) \arrow[dr] \arrow[rr, bend left] & & K_1(MB) \arrow[dr] & & \\
 & K_1(B) \arrow[ur] \arrow[dr] & & K_1(D\overline{B}) \arrow[ur] \arrow[dr] & & K_0(B) \arrow[dr] & \\
K_2(MB) \arrow[ur] \arrow[rr, bend right] & & K_1(\overline{B}) \arrow[ur] \arrow[rr, dashed, bend right] & & K_0(\mathfrak{b}) \arrow[ur] \arrow[rr, bend right] & & K_0(DA).
\end{tikzcd}$$

Let the dashed arrow be the composition $K_1(\overline{B}) \to K_1(D\overline{B}) \to K_0(\mathfrak{b})$. By an easy diagram chase we see that the sequence with the dashed arrow is exact as well. The identity on $R$ is a morphism

$$(R, \mathfrak{a}_1, \dots, \mathfrak{a}_m, \mathfrak{b}) \to A$$

in $\mathcal{R}_{m+1}$. So we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
K_1(R,\mathfrak{a}_1,\ldots,\mathfrak{a}_m,\mathfrak{b}) & \longrightarrow & K_1(B) & \longrightarrow & K_1(\overline{B}) & \longrightarrow & K_0(\mathfrak{b}) \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \alpha} & & \\
K_1(A) & \longrightarrow & K_1(DA) & \longrightarrow & K_1(MA). & &
\end{array}$$

It now suffices to show that the morphism $\alpha$ in this diagram is an isomorphism. The $(m+1)$-tuple $(R/\mathfrak{b},\mathfrak{a}_1/\mathfrak{b},\ldots,\mathfrak{a}_{m+1}/\mathfrak{b})$ induces an exact sequence

$$K_1(R/\mathfrak{b},\mathfrak{a}_1/\mathfrak{b},\ldots,\mathfrak{a}_{m+1}/\mathfrak{b}) \to K_1(\overline{B}) \to K_1(MA).$$

The group $K_1(R/\mathfrak{b},\mathfrak{a}_1/\mathfrak{b},\ldots,\mathfrak{a}_{m+1}/\mathfrak{b})$ is a quotient of $GL((\mathfrak{a}_1/\mathfrak{b})\cap\cdots\cap(\mathfrak{a}_{m+1}/\mathfrak{b})) = \{1\}$, so $\alpha$ is injective. On the other hand, since the $(m+1)$-tuple $A$ of ideals is normal, the identity on $R$ induces an isomorphism $I(\overline{B}) \to I(MA)$ and hence also an isomorphism

$$GL(I(\overline{B})) \xrightarrow{\sim} GL(I(MA)).$$

Since the multirelative $K_1$ is a quotient of the general linear group of the underlying ideal, the map $\alpha$ is surjective. This proves:

THEOREM 1. *Let $A \in \mathcal{R}_m$ for $m \geq 1$. Then we have a functorial exact sequence*

$$\cdots \to K_n(A) \to K_n(DA) \to K_n(MA) \to K_{n-1}(A) \to \cdots \to K_0(MA).$$

□

## 7 AXIOMS FOR MULTIRELATIVE $K$-THEORY

It will be shown in this section that an axiomatic approach to multirelative $K$-theory is possible. We take some of the properties of multirelative $K$-groups as axioms and show that they determine all of multirelative $K$-theory.

### AXIOMS

MULTIRELATIVE $K$-THEORY consists of functors

$$K_n\colon \mathcal{R}_m \to \mathcal{A} \qquad \text{for } m \text{ and } n \text{ integers} \geq 0,$$

morphisms

$$\delta\colon K_{n+1}M \to K_n \qquad (\text{for } m \text{ and } n \text{ integers} \geq 0)$$

of functors $\mathcal{R}_{m+1} \to \mathcal{A}$ and morphisms

$$\iota\colon K_n \to K_nD \qquad (\text{for } m \text{ and } n \text{ integers} \geq 0)$$

of functors $\mathcal{R}_{m+1} \to \mathcal{A}$, such that

(MK1) the following sequence is an exact sequence of functors $\mathcal{R}_{m+1} \to \mathcal{A}$ for all non-negative integers $m$ and $n$

$$K_{n+1}D \xrightarrow{K_{n+1}\phi} K_{n+1}M \xrightarrow{\delta} K_n \xrightarrow{\iota} K_nD \xrightarrow{K_n\phi} K_nM.$$

(MK2) $K_n(R) = 0$ for all $n \geq 0$ and all free associative non-unital rings $R$,

(MK3) $K_0(A) = K_0(IA)$ for all $A \in \mathcal{R}_m$ for all $m$.

Loosely speaking, the multirelative $K$-groups are only defined for normal $m$-tuples of ideals and they fit into exact sequences the way one can expect, the (absolute) $K$-groups of free non-unital rings are trivial and the multirelative $K_0$ is just the Grothendieck group of the intersection of the ideals.

Let $(R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m)$ be a normal $m$-tuple of ideals. It induces an $m$-cube

$$I \mapsto R_I = R \Big/ \sum_{i \in I} \mathfrak{a}_i,$$

which is split in $\mathcal{S}$. Application of $\mathbf{Fr}$ gives an $m$-cube

$$I \mapsto \mathbf{Fr}(R_I)$$

of aspherical simplicial rings, which is dimensionwise split in $\mathcal{R}$.

PROPOSITION 7.1. *Let $m$ and $n$ be positive integers. Then the $(m+n)$-tuple*

$$\left(\mathbf{Fr}(R)_{n-1}, \mathbf{Fr}(R, \mathfrak{a}_1)_{n-1}, \ldots, \mathbf{Fr}(R, \mathfrak{a}_m)_{n-1}, \operatorname{Ker}\left(d_0^{(n-1)}\right), \ldots, \operatorname{Ker}\left(d_{n-1}^{(n-1)}\right)\right)$$

*is normal.*

*Proof.* First we show that the induced $(m+n)$-cube is

$$(I_1, I_2) \mapsto \mathbf{Fr}(R_{I_1})_{n-1-\#(I_2)},$$

where the cube is indexed by pairs of subsets of $\underline{m}$ and $[n-1]$. This set of pairs is ordered by componentwise inclusion:

$$(I_1, I_2) \leq (J_1, J_2) \iff I_1 \subseteq J_1 \quad \text{and} \quad I_2 \subseteq J_2.$$

The homomorphism

$$\mathbf{Fr}(R)_{n-1} \to \mathbf{Fr}(R_{I_1})_{n-1-\#(I_2)}$$

is the composition

$$\mathbf{Fr}(R)_{n-1} \to \mathbf{Fr}(R_{I_1})_{n-1} \to \mathbf{Fr}(R_{I_1})_{n-1-\#(I_2)},$$

the first map being induced by $\emptyset \subseteq I_1$ and the second by $[n-1] \setminus I_2 \subseteq [n-1]$. Both homomorphisms are surjective. The first one has kernel $\bigcap_{i \in I_1} \mathbf{Fr}(R, \mathfrak{a}_i)_{(n-1)}$ and the second one $\bigcap_{i \notin I_2} \operatorname{Ker}(d_i)$, where the $d_i$ are face maps of $\mathbf{Fr}(R_{I_1})$. Since $\mathbf{Fr}(R)$ and $\mathbf{Fr}(R_{I_1})$ are both aspherical, elements of the second kernel can be lifted to elements of $\bigcap_{i \notin I_2} \operatorname{Ker}(d_i)$, where the $d_i$ are face maps of $\mathbf{Fr}(R)$.

For the $(m+n)$-tuple to be normal it suffices that the intersections of the images of the $m+n$ ideals are preserved under the maps on the edges of the induced $(m+n)$-cube. These are the homomorphisms

$$\mathbf{Fr}(R_J)_l \to \mathbf{Fr}(R_{J \cup \{k\}})_l,$$

where $J \subseteq \underline{m}$, $k \in \underline{m} \setminus J$ and $l \in [n-1]$, and also the face maps

$$d_i \colon \mathbf{Fr}(R_J)_p \to \mathbf{Fr}(R_J)_{p-1},$$

where $p \in [n-1]$ and $0 \leq i \leq p$. Without loss of generality we may assume that $J = \underline{m}$, $l = n-1$ and $p = n-1$.

Because the $m$-cube $J \mapsto \mathbf{Fr}(R_J)$ is dimensionwise split we have short exact sequences

$$0 \to \bigcap_{i \in I \cup \{k\}} \mathbf{Fr}(R, \mathfrak{a}_i) \to \bigcap_{i \in I} \mathbf{Fr}(R, \mathfrak{a}_i) \to \bigcap_{i \in I} \mathbf{Fr}(R/\mathfrak{a}_k, \overline{\mathfrak{a}}_i) \to 0$$

of aspherical simplicial rings. It follows that for all $J \subseteq [n-1]$ we have

$$\bigcap_{i \in I} \mathbf{Fr}(R, \mathfrak{a}_i)_{n-1} \cap \bigcap_{j \in J} \operatorname{Ker}(d_j) = \bigcap_{j \in J} \operatorname{Ker}(d'_j),$$

where the $d'_j$ are the face maps of $\bigcap_{i \in I} \mathbf{Fr}(R, \mathfrak{a}_i)$. Under $\mathbf{Fr}(R) \to \mathbf{Fr}(R/\mathfrak{a}_k)$ this maps onto

$$\bigcap_{j \in J} \operatorname{Ker}(d''_j) = \bigcap_{i \in I} \mathbf{Fr}(R/\mathfrak{a}_k, \overline{\mathfrak{a}}_i)_{n-1} \cap \bigcap_{j \in J} \operatorname{Ker}(d'''_j),$$

where the $d''_j$ are the face maps of $\bigcap_{i \in I} \mathbf{Fr}(R/\mathfrak{a}_k, \overline{\mathfrak{a}}_i)$ and $d'''_j$ those of $\bigcap_{i \in I} \mathbf{Fr}(R/\mathfrak{a}_k)$.

Because the simplicial rings $\bigcap_{i \in I} \mathbf{Fr}(R, \mathfrak{a}_i)$ are aspherical also the face maps $d_i \colon \mathbf{Fr}(R)_{n-1} \to \mathbf{Fr}(R)_{n-2}$ preserve intersections

$$\bigcap_{i \in I} \mathbf{Fr}(R, \mathfrak{a}_i)_{n-1} \cap \bigcap_{j \in J} \operatorname{Ker}(d_j).$$

□

THEOREM 2. *Let $A = (R, \mathfrak{a}_1, \ldots, \mathfrak{a}_m) \in \mathcal{R}$. Then for all $n \geq 0$ it follows from the axioms (MK1) and (MK2) that $K_n(A)$ is naturally isomorphic to $K_0$ of the following object of $\mathcal{R}_{m+n}$:*

$$(\mathbf{Fr}(R)_{n-1}, \mathbf{Fr}(R, \mathfrak{a}_1)_{n-1}, \ldots, \mathbf{Fr}(R, \mathfrak{a}_m)_{n-1}, \operatorname{Ker}(d_0), \ldots, \operatorname{Ker}(d_{n-1})).$$

From axiom (MK3) it then follows that $K_n(A)$ is determined. So (MK1), (MK2) and (MK3) can be taken as axioms for the (multirelative) $K$-theory of rings.

*Proof.* The proof follows from the following three lemmas. □

LEMMA 7.1. *Let $m \geq -1$ and $q, n \geq 0$. Then*

$$K_q(\mathbf{Fr}(R)_n, \mathbf{Fr}(R, \mathfrak{a}_1)_n, \ldots, \mathbf{Fr}(R, \mathfrak{a}_m)_n) = 0.$$

*Proof.* Since for $m \geq 0$ the $(m-1)$-tuples $D(A)$ and $M(A)$ are of the same type, the proof reduces by (MK1) to the case $m = -1$. For $m = -1$ the lemma follows from (MK2). □

Put

$$A[n,p] = (\mathbf{Fr}(R)_n, \mathbf{Fr}(R,\mathfrak{a}_1)_n, \dots, \mathbf{Fr}(R,\mathfrak{a}_m)_n, \mathrm{Ker}(d_0), \dots, \mathrm{Ker}(d_p)),$$

where $-1 \leq p \leq n$. It is an object of $\mathcal{R}_{m+p+1}$.

LEMMA 7.2. *For all $p < n$ and all $q > 0$ we have*

$$K_q(A[n,p]) = 0.$$

*Proof.* For $p \geq 0$ we have

$$D(A[n,p]) = A[n,p-1] \quad \text{and} \quad M(A[n,p]) = A[n-1,p-1].$$

By (MK1) the problem reduces to the case $p = -1$, which is covered by the previous lemma. □

LEMMA 7.3. *For all $q, n \geq 0$ we have*

$$K_q(A[n,n]) \cong K_{q+1}(A[n-1,n-1]).$$

*Proof.* This follows from (MK1) and the previous lemma. □

From this lemma the theorem follows:

$$K_n(A) = K_n(A[-1,-1]) \cong K_{n-1}(A[0,0]) \cong \cdots \cong K_0(A[n-1,n-1]).$$

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