Presentations of Subshifts and Their Topological Conjugacy Invariants

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Abstract. We introduce the notions of symbolic matrix system and λ-graph system that are presentations of subshifts. They are generalized notions of symbolic matrix and λ-graph for sofic subshifts to general subshifts. We then formulate strong shift equivalence and shift equivalence between symbolic matrix systems and show that two subshifts are topologically conjugate if and only if the associated canonical symbolic matrix systems are strong shift equivalent. We construct several kinds of shift equivalence invariants for symbolic matrix systems. They are the dimension groups, the Bowen-Franks groups and the nonzero spectrum that are generalizations of the corresponding notions for nonnegative matrices. The K-groups for symbolic matrix systems are introduced. They are also shift equivalence invariants and stronger than the Bowen-Franks groups but weaker than the dimension triples. These kinds of shift equivalence invariants naturally induce topological conjugacy invariants for subshifts.

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Contents

1. Introduction

The classification of symbolic dynamical systems has been a very important and one of central problems in the theory of topological dynamical systems and the ergodic theory. The classification problem has been first examined for a class of symbolic dynamical systems called subshifts of finite type or topological Markov shifts. Each dynamical system of the class is determined by a single square matrix with entries in nonnegative integers. Hence the behavior of such a dynamical system depends on the underlying matrix. In [Wi], R. F. Williams introduced the notions of strong shift equivalence and shift equivalence between nonnegative matrices and showed that two topological Markov shifts are topologically conjugate if and only if the associated matrices are strong shift equivalent. He also showed that strong shift equivalence implies shift equivalence. Although the converse implication had been a long standing problem, Kim-Roush [KimR2] has recently solved negatively for even irreducible matrices. There is a class of subshifts called sofic subshifts that are generalized class of Markov shifts and that are determined by square matrices with entries in alphabet (see [Kit], [Kr3], [LM], [We], etc.). A square matrix with entries in alphabet is simply called a symbolic matrix. It is an equivalent object to a labeled graph called a λ-graph. M. Nasu in [N], [N2] generalized the notion of strong shift equivalence to symbolic matrices. He showed that two sofic subshifts are topologically conjugate if and only if their canonical symbolic matrices are strong shift equivalent ([N], [N2], see also [HN]). M. Boyle and W. Krieger in [BK] introduced the notion of shift equivalence for symbolic matrices and studied topologically conjugacy for sofic subshifts.

In this paper, we first introduce the notions of symbolic matrix system and λ-graph system. They are generalized notions of symbolic matrix and λ-graph for sofic subshifts. We will show that they are presentations of subshifts. A symbolic matrix system consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{N}$. The matrices $\mathcal{M}_{l,l+1}$ have entries in symbols and the matrices $I_{l,l+1}$ have entries in $\{0,1\}$. They satisfy the following commutation relations

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{N}.$$

A λ-graph system is an inductive sequence of Bratteli diagrams, that come from the theory of operator algebras, with labeled edges by symbols. We will know that the symbolic matrix systems and the λ-graph systems are the same objects and give rise to subshifts. There is a canonical method to construct a symbolic matrix system from an arbitrary subshift (Theorem 3.7). The obtained symbolic matrix system is said to be canonical for the subshift. If a subshift is sofic, the canonical symbolic matrix system corresponds to the symbolic matrix of its left Krieger cover graph.

As a generalization of the notion of strong shift equivalence for nonnegative matrices and symbolic matrices, we will introduce the notion of strong shift equivalence for our symbolic matrix systems. We will prove

Theorem A (Theorem 4.2 and Theorem 4.15). *Two subshifts are topologically conjugate if and only if their canonical symbolic matrix systems are strong shift equivalent.*

Hence classification problem for subshifts are completely reduced to the classification of symbolic matrix systems up to strong shift equivalence in our sense. In the proof of the only if part of Theorem A, we provide the notion of bipartite λ-graph system. We then essentially use Nasu's factorization theorem for topological conjugacy between subshifts into bipartite codes and symbolic conjugacies.

We will next define shift equivalence between two symbolic matrix systems. That is a generalization of the corresponding notion for symbolic matrices defined by Boyle-Krieger in [BK]. We will see that strong shift equivalence implies shift equivalence even in our setting (Theorem 6.2). Similarly to the case of topological Markov shifts, we can prove that shift equivalence between two canonical symbolic matrix systems gives rise to an eventual conjugacy for the associated subshifts, that is, a topological conjugacy for their corresponding higher power shifts (Proposition 6.3). This result was motivated by a question raised by W. Krieger at a workshop at Kyushu University, Japan, March 1998.

For nonnegative matrices, there are two crucial shift equivalence invariants consisting of abelian groups. One is the dimension groups defined by W. Krieger in [Kr], [Kr2] and the other one is the Bowen-Franks groups in [BF]. They induce topological conjugacy invariants for the associated topological Markov shifts. We will generalize the two shift equivalence invariants to our symbolic matrix systems. For a symbolic matrix system $(\mathcal{M}, I)$, let $M_{l,l+1}$ be the nonnegative rectangular matrix obtained from $\mathcal{M}_{l,l+1}$ by setting all the symbols equal to 1 for each $l \in \mathbb{N}$. Then the resulting pair (M, I) still satisfies the following relations.

$$I_{l,l+1}M_{l+1,l+2} = M_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{N}.$$

We call (M, I) the nonnegative matrix system for $(\mathcal{M}, I)$. We say (M, I) to be canonical when $(\mathcal{M}, I)$ is canonical. More generally, for a sequence $M_{l,l+1}, l \in \mathbb{N}$ of rectangular matrices with entries in nonnegative integers and a sequence $I_{l,l+1}, l \in \mathbb{N}$ of rectangular matrices with entries in $\{0, 1\}$, the pair (M, I) is called a nonnegative matrix system if they satisfy the relations above. For a single $n \times n$ nonnegative square matrix A, if we set $M_{l,l+1} = A$ and $I_{l,l+1} = I_n$: the $n \times n$ identity matrix for all $l \in \mathbb{N}$, the pair (M, I) is a nonnegative matrix system. We will similarly formulate strong shift equivalence and shift equivalence between nonnegative matrix systems. These equivalences are generalizations of the corresponding equivalences for single nonnegative square matrices.

We will define the following three kinds of objects for a nonnegative matrix system (M, I).

(i) The dimension triple: $(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)})$.
(ii) The K-groups: $K_0(M, I)$, $\quad K_1(M, I)$.
(iii) The Bowen-Franks groups: $BF^0(M, I)$, $\quad BF^1(M, I)$.

The dimension triple $(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)})$ consist of an ordered abelian group $\Delta_{(M,I)}$ with positive cone $\Delta^+_{(M,I)}$ and an ordered automorphism $\delta_{(M,I)}$ on it. The K-groups $K_i(M,I), i = 0,1$ consist of a pair of abelian groups. The Bowen-Franks groups $BF^i(M,I), i = 0,1$ also consist of a pair of abelian groups. Let $m(l)$ be the row size of the matrix $I_{l,l+1}$ for each $l \in \mathbb{N}$. Let $\mathbb{Z}_{I^t}$ be the abelian group defined by the inductive limit $\mathbb{Z}_{I^t} = \varinjlim_l \{I^t_{l,l+1} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}\}$. The sequence $M^t_{l,l+1}, l \in \mathbb{N}$ of the transposes of $M_{l,l+1}$ naturally yields an endomorphism on $\mathbb{Z}_{I^t}$ that is denoted by $\lambda_{(M,I)}$. The dimension group and the K-groups are defined as follows:

$$\Delta_{(M,I)} = \varinjlim \{\lambda_{(M,I)} : \mathbb{Z}_{I^t} \to \mathbb{Z}_{I^t}\}$$

and

$$K_0(M,I) = \mathbb{Z}_{I^t}/(id - \lambda_{(M,I)})\mathbb{Z}_{I^t}, \qquad K_1(M,I) = \mathrm{Ker}(id - \lambda_{(M,I)}) \text{ in } \mathbb{Z}_{I^t}.$$

The positive cone $\Delta^+_{(M,I)}$ of $\Delta_{(M,I)}$ is $\varinjlim\{\lambda_{(M,I)} : \mathbb{Z}^+_{I^t} \to \mathbb{Z}^+_{I^t}\}$ where $\mathbb{Z}^+_{I^t}$ is the natural positive cone of $\mathbb{Z}_{I^t}$ and the automorphism $\delta_{(M,I)}$ on $\Delta_{(M,I)}$ is induced one from $\lambda_{(M,I)}$. Set the projective limit of the abelian group as $\mathbb{Z}_I = \varprojlim_l \{I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}\}$. The sequence $M_{l,l+1}, l \in \mathbb{N}$ acts on $\mathbb{Z}_I$ as an endomorphism that we denote by M. The identity on $\mathbb{Z}_I$ is denoted by I. The Bowen-Franks groups for (M,I) are defined by

$$BF^0(M,I) = \mathbb{Z}_I/(I - M)\mathbb{Z}_I, \qquad BF^1(M,I) = \mathrm{Ker}(I - M) \text{ in } \mathbb{Z}_I.$$

The above notions of dimension triple and Bowen-Franks group of degree zero for a nonnegative matrix system are generalizations of the corresponding notions for a single nonnegative square matrix. We will prove that the following Universal Coefficient Theorem holds (Theorem 9.6). It says that there exists a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(M,I),\mathbb{Z}) \xrightarrow{\delta} BF^0(M,I) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(K_1(M,I),\mathbb{Z}) \longrightarrow 0$$

that splits unnaturally. We also see that

$$BF^1(M,I) \cong \mathrm{Hom}_{\mathbb{Z}}(K_0(M,I),\mathbb{Z}).$$

The three kinds of objects above are proved to be invariant under shift equivalence in nonnegative matrix systems. Hence they are naturally induce topological conjugacy invariants for subshifts by taking their canonical nonnegative matrix systems.

We will describe relationships among the equivalences and the invariants for the matrix systems as in the following way :

Theorem B. *For two symbolic matrix systems $(\mathcal{M}, I), (\mathcal{M}', I')$ and their nonnegative matrix systems $(M, I), (M', I')$, consider the following situations:*

(S1) $(\mathcal{M}, I) \approx (\mathcal{M}', I)$ *: strong shift equivalence,*
(N1) $(M, I) \approx (M', I)$ *: strong shift equivalence,*
(S2) $(\mathcal{M}, I) \sim (\mathcal{M}', I)$ *: shift equivalence,*
(N2) $(M, I) \sim (M', I)$ *: shift equivalence,*
(3) $(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}) \cong (\Delta_{(M',I')}, \Delta^+_{(M',I')}, \delta_{(M',I')})$ *: isomorphic dimension triples,*
(4) $(\Delta_{(M,I)}, \delta_{(M,I)}) \cong (\Delta_{(M',I')}, \delta_{(M',I')})$ *: isomorphic dimension pairs,*
(5) $K_*(M, I) \cong K_*(M', I)$ *: isomorphic K-groups,*
(6) $BF^*(M, I) \cong BF^*(M', I)$ *: isomorphic Bowen-Franks groups.*

Then we have the following implications:

$$\begin{array}{lllllll}
(S1) \Longrightarrow (S2) & & & & & \\
\Downarrow \qquad\quad \Downarrow & & & & & \\
(N1) \Longrightarrow (N2) & \Longrightarrow (3) & \Longrightarrow (4) & \Longrightarrow (5) & \Longrightarrow (6). &
\end{array}$$

It is well-known that the set of all nonzero eigenvalues of a nonnegative matrix A is also a shift equivalence invariant. The set for A is called the nonzero spectrum of A and plays an important rôle for studying dynamical properties of the associated topological Markov shift (cf.[LM], [Kit]). We introduce eigenvalues and eigenvectors of a nonnegative matrix system and then generalize the notion of the nonzero spectrum of a single nonnegative matrix to a nonnegative matrix system (M, I). We denote by $Sp^\times(M, I)$ the set of all nonzero eigenvalues of (M, I). A nonnegative matrix system (M, I) in general is an infinite sequence of pairs of matrices $M_{l,l+1}, I_{l,l+1}, l \in \mathbb{N}$ for which sizes of matrices are increasing. Hence it seems to be natural to deal with eigenvalues having a certain boundedness condition on the corresponding eigenvectors. We denote by $Sp^\times_b(M, I)$ the set of all nonzero eigenvalues of (M, I) with the boundedness condition on the corresponding eigenvectors. We will prove, in Section 10, that the both of the nonzero spectrums $Sp^\times(M, I)$ and $Sp^\times_b(M, I)$ are not empty and invariant under shift equivalence of (M, I). In particular, if (M, I) is the canonical nonnegative matrix system for a subshift, the set $Sp^\times_b(M, I)$ is bounded by the topological entropy of the subshift. We then define the nonzero spectrum and the nonzero bounded spectrum for subshifts by the corresponding sets for the canonical nonnegative matrix systems (Theorem 10.14).
In the final section, we present an example of the canonical symbolic matrix system for a certain nonsofic subshift, called the context free shift in [LM;Example 1.2.9]. Its K-groups and Bowen-Franks groups are calculated. We see that the types of the invariants can not appear in those of sofic shifts. The maximum of the absolute values of the bounded spectrums of the canonical nonnegative matrix system for the subshift is $1 + \sqrt{1 + \sqrt{3}}$. The value is the maximum in the bounded spectrum and coincides with the topological entropy of the subshift.

The author has recently constructed the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ ([Ma]). The C^*-algebra $\mathcal{O}_\Lambda$ has a canonical action of the one dimensional torus group, called gauge action and written as α. The fixed point algebra $\mathcal{F}_\Lambda$ of $\mathcal{O}_\Lambda$ under α is an AF-algebra which is stably isomorphic to the crossed product $\mathcal{O}_\Lambda \times_\alpha \mathbb{T}$ ([Ma2]). Let (M, I) be the canonical nonnegative matrix system for the subshift Λ. The invariants mentioned above are described in terms of the K-theory for the C^*-algebras as in the following way:

$$(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}) = (K_0(\mathcal{F}_\Lambda), K_0(\mathcal{F}_\Lambda)_+, \hat{\alpha}_*),$$
$$K_i(M, I) = K_i(\mathcal{O}_\Lambda), \qquad i = 0, 1,$$
$$BF^i(M, I) = \mathrm{Ext}^{i+1}(\mathcal{O}_\Lambda), \qquad i = 0, 1$$

where $\hat{\alpha}$ denotes the dual action of α and $\mathrm{Ext}^1(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda), \mathrm{Ext}^0(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda \otimes C_0(\mathbb{R}))$. The normalized nonnegative eigenvectors of (M, I) exactly correspond to the KMS-states for α on the C^*-algebra $\mathcal{O}_\Lambda$. Hence the set of all bounded spectrums with nonnegative eigenvectors are the set of all inverse temperatures for the admitted KMS states.

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2. SYMBOLIC MATRIX SYSTEMS AND λ-GRAPH SYSTEMS

We fix a finite set Σ and call it the alphabet. Each element of Σ is called a symbol. We always write the empty symbol $\emptyset$ in Σ as 0. We denote by $\mathfrak{S}_\Sigma$ the set of all finite formal sums of elements of Σ. A square matrix with entries in $\mathfrak{S}_\Sigma$ is called a symbolic matrix over Σ.

DEFINITION. Let $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{N}$ be a pair of sequences of rectangular matrices such that the following four conditions for each $l \in \mathbb{N}$ are satisfied:

(1) $\mathcal{M}_{l,l+1}$ is an $m(l) \times m(l+1)$ rectangular matrix with entries in $\mathfrak{S}_\Sigma$.
(2) $I_{l,l+1}$ is an $m(l) \times m(l+1)$ rectangular matrix with entries in $\{0, 1\}$ satisfying the following two conditions:
(2-a) For i, there exists j such that $I_{l,l+1}(i, j) \neq 0$.
(2-b) For j, there uniquely exists i such that $I_{l,l+1}(i, j) \neq 0$.
(3) $m(l) \leq m(l + 1)$.
(4) $I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}$.

The pair $(\mathcal{M}, I)$ is called a *symbolic matrix system* over Σ. For $i = 1, \ldots, m(l), j = 1, \ldots, m(l+1)$, we denote by $\mathcal{M}_{l,l+1}(i, j)$, $I_{l,l+1}(i, j)$ the (i, j)-components of $\mathcal{M}_{l,l+1}$, $I_{l,l+1}$ respectively. A symbolic matrix system $(\mathcal{M}, I)$ is said to be *essential* if it satisfies the following further conditions:

(5-i) For i, there exists j such that $\mathcal{M}_{l,l+1}(i, j) \neq 0$.
(5-ii) For j, there exists i such that $\mathcal{M}_{l,l+1}(i, j) \neq 0$.

We henceforth study essential symbolic matrix systems and call them symbolic matrix systems for simplicity.
The following notion of specified equivalence between symbolic matrices has been introduced by M. Nasu in [N1], [N2].
For two symbolic matrices $\mathcal{A}$ over alphabet Σ and $\mathcal{A}'$ over alphabet Σ' and bijection ϕ from a subset of Σ onto a subset of Σ', we call $\mathcal{A}$ and $\mathcal{A}'$ are specified equivalence under specification ϕ if $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by replacing every symbol a appearing in $\mathcal{A}$ by $\phi(a)$. We write it as $\mathcal{A} \overset{\phi}{\simeq} \mathcal{A}'$. We call ϕ a specification from Σ to Σ'.
Two symbolic matrix systems $(\mathcal{M}, I)$ over Σ and $(\mathcal{M}', I')$ over Σ' are said to be isomorphic if there exists a specification ϕ from Σ to Σ' and an $m(l) \times m(l)$-square permutation matrix P_l for each $l \in \mathbb{N}$ such that

$$P_l \mathcal{M}_{l,l+1} \overset{\phi}{\simeq} \mathcal{M}'_{l,l+1} P_{l+1}, \qquad P_l I_{l,l+1} = I'_{l,l+1} P_{l+1} \qquad \text{for} \quad l \in \mathbb{N}.$$

The notion of symbolic matrix system is a generalized notion of symbolic matrix. We say a symbolic matrix system $(\mathcal{M}, I)$ to be *sofic* if there exists a number $L \in \mathbb{N}$ such that

$$\mathcal{M}_{l,l+1} = \mathcal{M}_{L,L+1}, \qquad I_{l,l+1} = I_{L,L+1}$$

for all $l \geq L$. Hence in this case, we see $m(L) = m(l)$ for all $l \geq L$.
A symbolic matrix corresponds to a labeled graph, called a λ-graph, that is a presentation of a sofic subshift. We will next consider a generalization of λ-graphs corresponding to symbolic matrix systems.
We first explain the notion of Bratteli diagram that appears in the theory of operator algebras (see [Bra], [Ef], [El]). A Bratteli diagram consists of a vertex set V and an edge set E satisfying the following conditions. We have a decomposition of V as a disjoint union $V = V_1 \cup V_2 \cup \cdots$ where each V_l is finite and nonempty. Similarly E decomposes as a disjoint union $E = E_{1,2} \cup E_{2,3} \cup \cdots$ where each $E_{l,l+1}$ is finite and nonempty. Moreover we have maps $s, r : E \to V$ such that $s(E_{l,l+1}) \subset V_l$, $r(E_{l,l+1}) \subset V_{l+1}$. They are called a source map and a range map respectively. A Bratteli diagram (V, E) is said to be essential if it satisfies the condition that $s^{-1}(v)$ is nonempty for all $v \in V$ and $r^{-1}(v)$ is nonempty for all $v \in V \setminus V_1$. For $u \in V_l, v \in V_{l+1}$, put

$$E_{l,l+1}(u, v) = \{e \in E_{l,l+1} | s(e) = u, r(e) = v\}.$$

We next introduce the notion of *labeled Bratteli diagram*. A labeled Bratteli diagram over alphabet Σ consists of a Bratteli diagram (V, E) and a map λ from E to Σ.
Definition. A λ-*graph system* over alphabet Σ consists of a labeled Bratteli diagram (V, E, λ) over Σ and a surjective map ι from $V \setminus V_1$ to V satisfying the following two conditions:

(1) $\iota(V_{l+1}) = V_l$ for $l \in \mathbb{N}$.

(2) For $u \in V_l, w \in V_{l+2}$, there exists a bijective correspondence between the edge sets

$$E_{l,l+1}(u, \iota(w)) \quad \text{and} \quad \bigcup_{v \in V_{l+1}, \iota(v)=u} E_{l+1,l+2}(v, w)$$

that is compatible with the labeling λ.

We denote by (V, E, λ, ι) the λ-graph system.
The following two conditions are implied from the above condition (2).

(2-i) For $e \in E_{l+1,l+2}$, there exists $e' \in E_{l,l+1}$ such that

$$\iota(s(e)) = s(e'), \quad \iota(r(e)) = r(e') \quad \text{and} \quad \lambda(e) = \lambda(e').$$

(2-ii) For $f \in E_{l,l+1}$, $v \in V_{l+2}$ with $\iota(v) = r(f)$, there exists $e \in E_{l+1,l+2}$ such that

$$\iota(s(e)) = s(f), \quad r(e) = v \quad \text{and} \quad \lambda(e) = \lambda(f).$$

A λ-graph system (V, E, λ, ι) is said to be *essential* if the Bratteli diagram (V, E) is essential. We always treat an essential λ-graph system and call it a λ-graph system for simplicity. We remark that by the condition (1) in Definition of λ-graph system the cardinality of the set V_{l+1} is greater than or equal to that of the set V_l.

Two λ-graph systems (V, E, λ, ι) over alphabet Σ and $(V', E', \lambda', \iota')$ over alphabet Σ' are said to be isomorphic if there exist bijections $\Phi_V : V \to V'$, $\Phi_E : E \to E'$ and a specification $\phi : \Sigma \to \Sigma'$ such that

(1) $\Phi_V(V_l) = V'_l \quad$ and $\quad \Phi_E(E_{l,l+1}) = E'_{l,l+1} \quad$ for $l \in \mathbb{N}$,
(2) $\Phi_V(s(e)) = s(\Phi_E(e)) \quad$ and $\quad \Phi_V(r(e)) = r(\Phi_E(e)) \quad$ for $e \in E$,
(3) $\iota'(\Phi_V(v)) = \Phi_V(\iota(v)) \quad$ for $v \in V$,
(4) $\lambda'(\Phi_E(e)) = \phi(\lambda(e)) \quad$ for $e \in E$.

Proposition 2.1. *There exists a bijective correspondence between the set of all isomorphism classes of symbolic matrix systems and the set of all isomorphism classes of λ-graph systems.*

Proof. 1. From symbolic matrix systems to λ-graph systems: Let $(\mathcal{M}, I)$ be a symbolic matrix system over Σ. We are always assuming that it is essential. For each $l \in \mathbb{N}$, let $V_l = \{1, \dots, m(l)\}$ be the set of all rows of the matrix $\mathcal{M}_{l,l+1}$ and $E_{l,l+1}$ the disjoint union of elements appearing in the components of $\mathcal{M}_{l,l+1}$. For each $e \in E_{l,l+1}$ we put $s(e) = i$ and $r(e) = j$ if e appears in $\mathcal{M}_{l,l+1}(i, j)$. The map $\iota : V \setminus V_1 \to V$ is defined as $\iota(j) = i$ for $j \in V_{l+1}$ if $I_{l,l+1}(i, j) = 1$. The map $\lambda : E \to \Sigma$ is defined by $\lambda(e) = e$. Then it is straightforward to see that (V, E, λ, ι) is a λ-graph system.

2. From λ-graph systems to symbolic matrix systems : Let (V, E, ι, λ) be a λ-graph system over Σ. We denote by $m(l)$ the cardinality of the vertex set V_l. We identify V_l with the set $\{1, \dots, m(l)\}$. We define $m(l) \times m(l+1)$ matrices as follows: For $i \in V_l, j \in V_{l+1}$, set $I_{l,l+1}(i, j) = 1$ if $\iota(j) = i$ otherwise $I_{l,l+1}(i, j) = 0$. For $e_k \in E_{l,l+1}, k = 1, \dots, n$ with $s(e_k) = i, r(e_k) = j$, we put $\mathcal{M}_{l,l+1}(i, j) = \lambda(e_1) + \cdots + \lambda(e_n)$. It is straightforward to see that the relations $I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}$ for $l \in \mathbb{N}$ hold.

3. Presentations of subshifts

As in the preceding section, symbolic matrix systems may be identified with λ-graph systems. We will in this section construct subshifts, a class of topological dynamical systems, from λ-graph systems. We will further show that any subshift comes from a λ-graph system. This is a generalized observation of the correspondences between the sofic subshits and the symbolic matrices. Hence studies of subshifts are completely reduced to the studies of λ-graph systems and hence symbolic matrix systems.

We will review on subshifts. Let Σ be an alphabet. Let $\Sigma^{\mathbb{Z}}$, $\Sigma^{\mathbb{N}}$ be the infinite product spaces $\prod_{i=-\infty}^{\infty}\Sigma_i$, $\prod_{i=1}^{\infty}\Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology respectively. The transformation σ on $\Sigma^{\mathbb{Z}}, \Sigma^{\mathbb{N}}$ given by $(\sigma(x_i)) = (x_{i+1}), i \in \mathbb{Z}, \mathbb{N}$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift. We denote $\sigma|_\Lambda$ by σ and write the subshift as Λ for short. We denote by $X_\Lambda(\subset \prod_{i=1}^{\infty}\Sigma_i)$ the set of all right-infinite sequences that appear in Λ. The dynamical system (X_Λ, σ) is called the right one-sided subshift for Λ. We will give examples of subshifts as follows (cf.[LM], [Kit]):
Let A be an $n \times n$ matrix with entries in nonnegative integers. Put $V_A = \{1, \dots, n\}$: the vertex set. Write $A(i,j)$ edges from $i \in V_A$ to $j \in V_A$. Hence we have a directed graph from A. We denote it by G_A. Let E_A be the set of all edges of the graph G_A. Let s_A, r_A be the map from E_A to V_A that assigns the source and the range of the edge. Let Λ_A be the set of all biinfinite sequences $(e_i)_{i\in\mathbb{Z}}$ of $e_i \in E_A$ with $r_A(e_i) = s_A(e_{i+1}), i \in \mathbb{Z}$. Then Λ_A becomes a subshift, called the topological Markov shift defined by A.
Let $\mathcal{A}$ be an $n \times n$ symbolic matrix over Σ. Each entry $\mathcal{A}(i,j), i,j = 1, \dots, n$ consists of elements of $\mathfrak{S}_\Sigma$. Similarly to the construction above, we have a directed graph $G_\mathcal{A}$ from the matrix $\mathcal{A}$ with labeled edges by the symbols in Σ. We denote by $\lambda(e) = \alpha \in \Sigma$ the label α of edge e. Let $\Lambda_\mathcal{A}$ be the set of all biinfinite sequences $\lambda(e_i)_{i\in\mathbb{Z}}$ of labels of the sequence of edges $e_i \in E_\mathcal{A}$ with $r_\mathcal{A}(e_i) = s_\mathcal{A}(e_{i+1}), i \in \mathbb{Z}$. Then $\Lambda_\mathcal{A}$ becomes a subshift, called the sofic subshift defined by $\mathcal{A}$. The labeled graph $G_\mathcal{A}$ is called a λ-graph for $\mathcal{A}$.
There are many nonsofic subshifts as seen in [LM]. We will see an example of nonsofic subshift in the final section.
A finite sequence $\mu = (\mu_1, ..., \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word. We denote by $|\mu|$ the length k of μ. A block $\mu = (\mu_1, ..., \mu_k)$ is said to occur or appear in $x = (x_i) \in \Sigma^{\mathbb{Z}}$ if $x_m = \mu_1, ..., x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$.

We will first construct subshifts from symbolic matrix systems.
Let $(\mathcal{M}, I)$ be a symbolic matrix system over Σ and (V, E, λ, ι) its corresponding λ-graph system. For $k < l$, let $P_{k,l}$ be the set of all paths from V_k to V_l, that is,

$$P_{k,l} = \{(e_k, e_{k+1}, \dots, e_{l-1}) | e_i \in E_{i,i+1}, r(e_i) = s(e_{i+1}) \text{ for } i = k, k+1, \dots, l-2\}.$$

We define the maps $s : P_{k,l} \to V_k$ and $r : P_{k,l} \to V_l$ by

$$s(e_k, e_{k+1}, \dots, e_{l-1}) = s(e_k), \quad r(e_k, e_{k+1}, \dots, e_{l-1}) = r(e_{l-1}).$$

The labeling $\lambda : P_{k,l} \to \Sigma^{l-k} = \underbrace{\Sigma \times \cdots \times \Sigma}_{l-k \text{ times}}$ is defined by

$$\lambda(e_k, e_{k+1}, \dots, e_{l-1}) = \lambda(e_k)\lambda(e_{k+1}) \cdots \lambda(e_{l-1}).$$

Set

$$L_{k,l} = \{\lambda(w) \in \Sigma^{l-k} | w \in P_{k,l}\}.$$

Put $L_l = L_{1,l+1}$ and endow it with discrete topology. The map $\pi_l : L_{l+1} \to L_l$ is defined by

$$\pi_l(\alpha_1, \dots, \alpha_{l+1}) = (\alpha_1, \dots, \alpha_l).$$

We set

$$X_{(\mathcal{M},I)} = \varprojlim\{\pi_l : L_{l+1} \to L_l\}$$

the projective limit in the category of compact Hausdorff spaces. That is

$$X_{(\mathcal{M},I)} = \{(\lambda(e_1), \lambda(e_2), \dots) \in \Sigma^{\mathbb{N}} | e_i \in E_{i,i+1}, r(e_i) = s(e_{i+1}) \text{ for } i \in \mathbb{N}\}$$

the set of all right infinite sequences consisting of labels along infinite paths. The topology on $X_{(\mathcal{M},I)}$ is defined from open sets of the form

$$U_{(\mu_1,\dots,\mu_k)} = \{(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)} | \alpha_i = \mu_i \text{ for } i = 1, \dots, k\}$$

for $(\mu_1, \dots, \mu_k) \in L_k$.

LEMMA 3.1. *If* $(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$, *we have* $(\alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)}$.

Proof. The assertion is direct from the condition (2-i) of Definition of λ-graph system.

LEMMA 3.2. *For* $l > k$, *if* $(\alpha_k, \dots, \alpha_{l-1}) \in L_{k,l}$, *we have* $(\alpha_k, \dots, \alpha_{l-1}) \in L_{k+1,l+1}$.

Proof. For $(\alpha_k, \dots, \alpha_{l-1}) \in L_{k,l}$, take $f_i \in E_{i,i+1}$ such as $\alpha_i = \lambda(f_i)$ for $i = k, k+1, \dots, l-1$ and $r(f_i) = s(f_{i+1})$ for $i = k, k+1, \dots, l-2$. We find $v_{l+1} \in V_{l+1}$ with $\iota(v_{l+1}) = r(f_{l-1})$. By the condition (2-ii) of Definition of λ-graph system, there exists $e_l \in E_{l,l+1}$ such that $\iota(s(e_l)) = s(f_{l-1})$, $r(e_l) = v_{l+1}$ and $\lambda(e_l) = \lambda(f_{l-1})$. Put $v_l = s(e_l) \in V_l$. We continue theses procedures so that we get $e_i \in E_{i,i+1}$ for $i = k+1, k+2, \dots, l$ satisfying $\iota(s(e_i)) = s(f_{i-1})$, $r(e_i) = s(e_{i+1})$ and $\lambda(e_i) = \lambda(f_{i-1})$ for $i = k+1, k+2, \dots, l$. Hence $\alpha_i = \lambda(e_{i+1})$ and $(\alpha_k, \dots, \alpha_{l-1}) \in L_{k+1,l+1}$.

As in [LM; Definition 1.3.1], a set $\mathfrak{L}$ of words of alphabet Σ is called a language if it satisfies the following conditions:

(a) Every subword of a word w in $\mathfrak{L}$ belongs to $\mathfrak{L}$.
(b) For a word w in $\mathfrak{L}$, there are nonempty words u, v in $\mathfrak{L}$ such that uwv belongs to $\mathfrak{L}$.

Let $\mathfrak{L}(\mathcal{M}, I)$ be the set of all words appearing in $X_{(\mathcal{M},I)}$. That is

$$\mathfrak{L}(\mathcal{M}, I) = \cup_{k \le l} L_{k,l}.$$

Then we have

Proposition 3.3. *$\mathfrak{L}(\mathcal{M}, I)$ is a language.*

Proof. $\mathfrak{L}(\mathcal{M}, I)$ clearly satisfies the condition (a) above. For a word $w \in L_{k,l}$, we know $w \in L_{k+1,l+1}$ by Lemma 3.2. We write $w = (\lambda(e_{k+1}), \lambda(e_{k+2}), \dots, \lambda(e_l))$ for $e_i \in E_{i,i+1}$ with $r(e_i) = s(e_{i+1})$, $i = k+1, \dots, l-1$. Since both the sets $r^{-1}(s(e_{k+1}))$ and $s^{-1}(s(e_l))$ are not empty, we may find words $u, v \in \mathfrak{L}(\mathcal{M}, I)$ such that $uwv \in \mathfrak{L}(\mathcal{M}, I)$. Thus $\mathfrak{L}(\mathcal{M}, I)$ satisfies the condition (b).

By [LM;Proposition 1.3.4], we see

Theorem 3.4. *There exists a subshift Λ over alphabet Σ whose language is $\mathfrak{L}(\mathcal{M}, I)$. Namely the set of all admissible words of the subshift Λ is $\mathfrak{L}(\mathcal{M}, I)$.*

We denote by $\Lambda_{(\mathcal{M},I)}$ the subshift Λ in the theorem above and call it *the subshift associated with symbolic matrix system* $(\mathcal{M}, I)$.

It is also possible to construct the subshift $\Lambda_{(\mathcal{M},I)}$ by using projective limit method as in the folloing way.

Lemma 3.5. *For $(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$, there exists a symbol $\alpha_0 \in \Sigma$ such that $(\alpha_0, \alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$.*

Proof. Put $w_k = (\alpha_1, \alpha_2, \dots, \alpha_{k-1}) \in L_{1,k}$. By Lemma 3.2 and Proposition 3.3, there exists a symbol $\beta_k \in \Sigma$ such that $\beta_k w_k \in L_{1,k+1}$. Hence we may find $y_k \in X_{(\mathcal{M},I)}$ such that $\beta_k w_k y_k \in X_{(\mathcal{M},I)}$. As the alphabet Σ is a finite set, there exists a symbol $\alpha_0 \in \Sigma$ and a subsequence of $(\beta_k)_{k\in\mathbb{N}}$ such that $\beta_{k_n} = \alpha_0$ for $n = 1, 2, \dots$ and $k_1 < k_2 < \cdots$. Put $x_{k_n} = \alpha_0 w_{k_n} y_{k_n}, n \in \mathbb{N}$. They converge to an element

$$x = (\alpha_0, \alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}.$$

By Lemma 3.1, the following map

$$S : (\alpha_1, \alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)} \to (\alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)}$$

is well-defined, continuous and surjective. We set

$$\Lambda = \varprojlim \{S : X_{(\mathcal{M},I)} \to X_{(\mathcal{M},I)}\}$$

the projective limit in the category of compact Hausdorff spaces. Thus Λ is identified with the set of all biinfinite sequences arising from the sequences in $X_{(\mathcal{M},I)}$. That is

$$\Lambda = \{(\dots, \alpha_2, \alpha_1, \alpha_0, \alpha_1, \alpha_2, \dots) | (\alpha_n, \alpha_{n+1}, \dots) \in X_{(\mathcal{M},I)} \text{ for all } n \in \mathbb{Z}\}.$$

The map S induces a homeomorphism on it. We denote it by σ that satisfies $\sigma((\alpha_i)_{i\in\mathbb{Z}}) = (\alpha_{i+1})_{i\in\mathbb{Z}}$. Therefore we have a subshift (Λ, σ) from symbolic matrix system $(\mathcal{M}, I)$. It is nothing but the subshift $(\Lambda_{(\mathcal{M},I)}, \sigma)$ defined in the preceding discussion.

We will next construct symbolic matrix systems from subshifts.
For a subshift (Λ, σ) over Σ and a number $k \in \mathbb{N}$, let Λ^k be the set of all words of length k in $\Sigma^{\mathbb{Z}}$ occurring in some $x \in \Lambda$. Put $\Lambda^* = \cup_{k=0}^{\infty} \Lambda^k$ where Λ^0 denotes the empty word $\emptyset$. Set

$$\Lambda^l(x) = \{\mu \in \Lambda^l | \mu x \in X_\Lambda\} \qquad \text{for} \quad x \in X_\Lambda, \quad l \in \mathbb{N}.$$

We define a nested sequence of equivalence relations in the space X_Λ. Two points $x, y \in X_\Lambda$ are said to be *l-past equivalent* if $\Lambda^l(x) = \Lambda^l(y)$. We write this equivalence as $x \sim_l y$. We denote by $\Omega_l = X_\Lambda / \sim_l$ the quotient space by l-past equivalence classes of X_Λ ([Ma3]).

LEMMA 3.6. *For $x, y \in X_\Lambda$ and $\mu \in \Lambda^k$,*

(i) *if $x \sim_l y$, we have $x \sim_m y$ for $m < l$.*
(ii) *if $x \sim_l y$ and $\mu x \in X_\Lambda$, we have $\mu y \in X_\Lambda$ and $\mu x \sim_{l-k} \mu y$ for $l > k$.*

Hence we have the following sequence of surjections in a natural way:

$$\Omega_1 \leftarrow \Omega_2 \leftarrow \cdots \leftarrow \Omega_l \leftarrow \Omega_{l+1} \leftarrow \cdots .$$

We easily see that (Λ, σ) is a sofic subshift if and only if $\Omega_l = \Omega_{l+1}$ for some $l \in \mathbb{N}$.
For a fixed $l \in \mathbb{N}$, let $F_i^l, i = 1, 2, \dots, m(l)$ be the set of all l-past equivalence classes of X_Λ. Hence X_Λ is a disjoint union of the subsets $F_i^l, i = 1, 2, \dots, m(l)$. We define two rectangular $m(l) \times m(l+1)$ matrices $I_{l,l+1}^\Lambda, \mathcal{M}_{l,l+1}^\Lambda$ with entries in $\{0, 1\}$ and entries in $\mathfrak{S}_\Sigma$ respectively as in the following way. For $i = 1, 2, \dots, m(l), j = 1, 2, \dots, m(l+1)$, the (i, j)-component $I_{l,l+1}^\Lambda(i, j)$ of $I_{l,l+1}^\Lambda$ is one if F_i^l contains F_j^{l+1} otherwise zero. Let $a_1, \dots, a_n$ be the set of all symbols in Σ for which $a_k x \in F_i^l$ for some $x \in F_j^{l+1}$. We then define the (i, j)-component of the matrix $\mathcal{M}_{l,l+1}^\Lambda(i, j)$ as $\mathcal{M}_{l,l+1}^\Lambda(i, j) = a_1 + \cdots + a_n$: the formal sum of $a_1, \dots, a_n$. We call $I_{l,l+1}^\Lambda$ the *inclusion matrices for* Λ and $\mathcal{M}_{l,l+1}^\Lambda$ the *symbolic representation matrices for* Λ respectively.
We next construct a labeled graph from subshift Λ for each $l \in \mathbb{N}$. The vertices of the graph consist of the sets $F_i^l, i = 1, \dots, m(l)$ and $F_j^{l+1}, j = 1, \dots, m(l+1)$ which we denote by V_l and V_{l+1} respectively. We write an arrow with label a, denoted by $\lambda^\Lambda(a)$, from the vertex F_i^l to F_j^{l+1} if $ax \in F_i^l$ for some $x \in F_j^{l+1}$. We denote by $E_{l,l+1}$ the set of all arrows from V_l to V_{l+1}. Since for each $j = 1, \dots, m(l+1)$ there uniquely exists $i = 1, \dots, m(l)$ such that $I_{l,l+1}(i, j) = 1$, we have a natural map ι_l^Λ from V_{l+1} to V_l. Set $V^\Lambda = \cup_{l=1}^{\infty} V_l$ and $E^\Lambda = \cup_{l=1}^{\infty} E_{l,l+1}$. We then see

Theorem 3.7. *For a subshift (Λ, σ), the pair $(\mathcal{M}^\Lambda, I^\Lambda)$ is a symbolic matrix system for which its λ-graph is $(V^\Lambda, E^\Lambda, \lambda^\Lambda, \iota^\Lambda)$. Moreover the subshift $\Lambda_{(\mathcal{M}^\Lambda, I^\Lambda)}$ associated with $(\mathcal{M}^\Lambda, I^\Lambda)$ coincides with the original subshift Λ.*

Proof. For each $l \in \mathbb{N}$, it is straightforward to check that the relation

$$I^\Lambda_{l,l+1}\mathcal{M}^\Lambda_{l+1,l+2} = \mathcal{M}^\Lambda_{l,l+1}I^\Lambda_{l+1,l+2}$$

holds. It then follows that the pair $(\mathcal{M}^\Lambda, I^\Lambda)$ is a symbolic matrix system whose associated λ-graph system is $(V^\Lambda, E^\Lambda, \lambda^\Lambda, \iota^\Lambda)$. It is also easy to see that the subshift associated with $(\mathcal{M}^\Lambda, I^\Lambda)$ coincides with the original subshift Λ because their forbidden words coincide.

Therefore we have a symbolic matrix system $(\mathcal{M}^\Lambda, I^\Lambda)$ and a λ-graph system $(V^\Lambda, E^\Lambda, \lambda^\Lambda, \iota^\Lambda)$ from subshift (Λ, σ). We call them the *canonical* symbolic matrix system for Λ and the *canonical* λ-graph system for Λ respectively.
It is now clear that sofic symbolic matrix systems exactly correspond to sofic subshifts.
For a symbolic matrix system $(\mathcal{M}, I)$, let $\Lambda_{(\mathcal{M},I)}$ be the associated subshift constructed from $(\mathcal{M}, I)$. Then its canonical symbolic matrix system $(\mathcal{M}^\Lambda, I^\Lambda)$ does not necessarily coincide with the original symbolic matrix system $(\mathcal{M}, I)$. We indeed see the following proposition. Its proof is direct.

Proposition 3.8. *For a subshift Λ, we have*
- (i) *the representation matrices $\mathcal{M}^\Lambda_{l,l+1}$ are left resolving, i.e. the incoming edges to each vertex carry different labels.*
- (ii) *the labeled Bratteli diagram $(V^\Lambda, E^\Lambda, \lambda^\Lambda)$ is predecessor-separated, i.e. distinct vertices at each level have distinct predecessor sets of labels.*

For example set, for each $l \in \mathbb{N}$, $\mathcal{M}_{l,l+1} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}$ and $I_{l,l+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The symbolic matrix system gives rise to the even shift that is denoted by Y. Its canonical symbolic matrix system is given by the following matrices:

$$\mathcal{M}^Y_{1,2} = \begin{bmatrix} a & a+b & b \\ b & 0 & 0 \end{bmatrix}, \qquad I^Y_{1,2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{M}^Y_{l,l+1} = \begin{bmatrix} a & a & b \\ 0 & b & 0 \\ b & 0 & 0 \end{bmatrix}, \qquad I^Y_{l,l+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{for} \quad l \geq 2.$$

We indeed have

Proposition 3.9. *If Λ is a sofic subshift, its canonical λ-graph system is eventually realized as the left Krieger cover graph for Λ. Hence the canonical symbolic matrix system for Λ is eventually realized as the symbolic representation matrix for the left Krieger cover graph.*

4. Strong shift equivalence.

In this section, we will define two kinds of strong shift equivalences between two symbolic matrix systems. One is called the properly strong shift equivalence that exactly reflects a bipartite decomposition of the associated λ-graph systems. The other one is called the strong shift equivalence that is weaker than the former strong shift equivalence. They coincide at least between canonical symbolic matrix systems and between sofic symbolic matrix systems. The latter is easier defined and treated than the former. We will see, in the next section, that the latter strong shift equivalence directly leads to the shift equivalence between symbolic matrix systems. The main result in this section is that topological conjugacy between two subshifts are completely characterized by strong shift equivalence between their canonical symbolic matrix systems. We first define properly strong shift equivalence in 1-step between two symbolic matrix systems as a generalization of strong shift equivalence in 1-step between two nonnegative matrices defined by R. Williams in [Wi] and between two symbolic matrices defined by M. Nasu in [N](see also [BK]).

For alphabets C, D, put $C \cdot D = \{cd | c \in C, d \in D\}$. For $x = \sum_j c_j \in \mathfrak{S}_C$ and $y = \sum_k d_k \in \mathfrak{S}_D$, define $xy = \sum_{j,k} c_j d_k \in \mathfrak{S}_{C \cdot D}$.

Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be symbolic matrix systems over alphabets Σ, Σ' respectively, where $\mathcal{M}_{l,l+1}, I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices and $\mathcal{M}'_{l,l+1}, I'_{l,l+1}$ are $m'(l) \times m'(l+1)$ matrices.

Definition. Two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly strong shift equivalent in 1-step* if there exist alphabets C, D and specifications

$$\varphi : \Sigma \to C \cdot D, \qquad \phi : \Sigma' \to D \cdot C$$

and increasing sequences $n(l), n'(l)$ on $l \in \mathbb{N}$ such that for each $l \in \mathbb{N}$, there exist an $n(l) \times n'(l+1)$ matrix $\mathcal{P}_l$ over C, an $n'(l) \times n(l+1)$ matrix $\mathcal{Q}_l$ over D, an $n(l) \times n(l+1)$ matrix X_l over $\{0, 1\}$ and an $n'(l) \times n'(l+1)$ matrix X'_l over $\{0, 1\}$ satisfying the following equations:

$$\mathcal{M}_{l,l+1} \overset{\varphi}{\simeq} \mathcal{P}_{2l} \mathcal{Q}_{2l+1}, \qquad \mathcal{M}'_{l,l+1} \overset{\phi}{\simeq} \mathcal{Q}_{2l} \mathcal{P}_{2l+1}, \tag{4.1}$$

$$I_{l,l+1} = X_{2l} X_{2l+1}, \qquad I'_{l,l+1} = X'_{2l} X'_{2l+1} \tag{4.2}$$

and

$$X_l \mathcal{P}_{l+1} = \mathcal{P}_l X'_{l+1}, \qquad X'_l \mathcal{Q}_{l+1} = \mathcal{Q}_l X_{l+1}. \tag{4.3}$$

We write this situation as

$$(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}', I').$$

It follows by (4.1) that $n(2l) = m(l)$ and $n'(2l) = m(l)$ for $l \in \mathbb{N}$.

Two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly strong shift equivalent in N-step* if there exists a sequence of symbolic matrix systems $(\mathcal{M}^{(i)}, I^{(i)}), i = 1, 2, \dots, N-1$ such that

$$
\begin{aligned}
(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) & \underset{1-pr}{\approx} (\mathcal{M}^{(2)}, I^{(2)}) \\
& \underset{1-pr}{\approx} \cdots \underset{1-pr}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-pr}{\approx} (\mathcal{M}', I')
\end{aligned}
$$

We denote this situation by

$$
(\mathcal{M}, I) \underset{N-pr}{\approx} (\mathcal{M}', I')
$$

and simply call it a *properly strong shift equivalence.*

Proposition 4.1. *Properly strong shift equivalence is an equivalence relation on symbolic matrix systems.*

Proof. It is clear that properly strong shift equivalence is symmetric and transitive. It suffices to show that $(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}, I)$. Put $C = \Sigma, D = \{0, 1\}$. Define $\varphi : a \in \Sigma \to a \cdot 1 \in C \cdot D$ and $\phi : a \in \Sigma \to 1 \cdot a \in D \cdot C$. Let E_k be the $k \times k$ identity matrix. Set

$$
\begin{aligned}
\mathcal{P}_{2l} = \mathcal{P}_{2l+1} = \mathcal{M}_{l,l+1}, \quad \mathcal{Q}_{2l} = E_{m(l)}, \quad \mathcal{Q}_{2l+1} = E_{m(l+1)}, \\
X_{2l} = E_{m(l)}, \quad X_{2l+1} = I_{l,l+1}, \quad X'_{2l} = I_{l,l+1}, \quad X'_{2l+1} = E_{m(l+1)}.
\end{aligned}
$$

It is straightforward to see that they give a properly strong shift equivalence in 1-step between $(\mathcal{M}, I)$ and $(\mathcal{M}, I)$.

We will prove the following theorem.

Theorem 4.2. *Two subshifts Λ and Λ' are topologically conjugate if and only if their canonical symbolic matrix systems $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ and $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ are properly strong shift equivalent.*

We will first show the only if part of the theorem above. In our proof, we will use Nasu's factorization theorem for topological conjugacy between subshifts into bipartite codes ([N]).

We now introduce the notion of bipartite symbolic matrix system.

Definition. A symbolic matrix system $(\mathcal{M}, I)$ over alphabet Σ is said to be *bipartite* if there exist disjoint subsets $C, D \subset \Sigma$ and increasing sequences $n(l), n'(l)$ on $l \in \mathbb{N}$ with $m(l) = n(l) + n'(l)$ such that for each $l \in \mathbb{N}$, there exist an $n(l) \times n'(l+1)$ matrix $\mathcal{P}_{l,l+1}$ over C, an $n'(l) \times n(l+1)$ matrix $\mathcal{Q}_{l,l+1}$ over D, an $n(l) \times n(l+1)$ matrix $X_{l,l+1}$ over $\{0, 1\}$ and an $n'(l) \times n'(l+1)$ matrix $X'_{l,l+1}$ over $\{0, 1\}$ satisfying the following equations:

$$
\mathcal{M}_{l,l+1} = \begin{bmatrix} 0 & \mathcal{P}_{l,l+1} \\ \mathcal{Q}_{l,l+1} & 0 \end{bmatrix}, \qquad I_{l,l+1} = \begin{bmatrix} X_{l,l+1} & 0 \\ 0 & X'_{l,l+1} \end{bmatrix}.
$$

We thus see

LEMMA 4.3. *For a bipartite symbolic matrix system* $(\mathcal{M}, I)$ *as above, set*

$$\mathcal{P}_l = \mathcal{P}_{l,l+1}, \quad \mathcal{Q}_l = \mathcal{Q}_{l,l+1}, \quad X_l = X_{l,l+1}, \quad X'_l = X'_{l,l+1}$$

and

$$\mathcal{M}^{CD}_{l,l+1} = \mathcal{P}_{2l}\mathcal{Q}_{2l+1}, \qquad \mathcal{M}^{DC}_{l,l+1} = \mathcal{Q}_{2l}\mathcal{P}_{2l+1},$$
$$I^{CD}_{l,l+1} = X_{2l}X_{2l+1}, \qquad I^{DC}_{l,l+1} = X'_{2l}X'_{2l+1}.$$

Then the both pairs $(\mathcal{M}^{CD}, I^{CD})$ *and* $(\mathcal{M}^{DC}, I^{DC})$ *are symbolic matrix systems over alphabets* $C \cdot D$ *and* $D \cdot C$ *respectively and they are properly strong shift equivalent in 1-step.*

Proof. The relations $I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}$ and

$$I_{2l,2l+1}I_{2l+1,2l+2}\mathcal{M}_{2l+2,2l+3}\mathcal{M}_{2l+3,2l+4} = \mathcal{M}_{2l,2l+1}\mathcal{M}_{2l+1,2l+2}I_{2l+2,2l+3}I_{2l+3,2l+4}$$

shows that the both pairs $(\mathcal{M}^{CD}, I^{CD})$ and $(\mathcal{M}^{DC}, I^{DC})$ are symbolic matrix systems and they are properly strong shift equivalent in 1-step because we see

$$X_{l-1,l}\mathcal{P}_{l,l+1} = \mathcal{P}_{l-1,l}X'_{l,l+1}, \qquad X'_{l-1,l}\mathcal{Q}_{l,l+1} = \mathcal{Q}_{l-1,l}X_{l,l+1}.$$

DEFINITION. A λ-graph system (V, E, λ, ι) over alphabet Σ is said to be *bipartite* if there exist disjoint subsets $C, D \subset \Sigma$ such that $\Sigma = C \cup D$ and disjoint subsets $V_l^C, V_l^D \subset V_l$ for each $l \in \mathbb{N}$ such that $V_l^C \cup V_l^D = V_l$ and

(1) for each $e \in E_{l,l+1}$

$$s(e) \in V_l^D, \quad r(e) \in V_{l+1}^C \quad \text{if and only if} \quad \lambda(e) \in C,$$
$$s(e) \in V_l^C, \quad r(e) \in V_{l+1}^D \quad \text{if and only if} \quad \lambda(e) \in D.$$

(2)
$$\iota(V_{l+1}^D) = V_l^D, \qquad \iota(V_{l+1}^C) = V_l^C.$$

PROPOSITION 4.4. *A symbolic matrix system is bipartite if and only if its corresponding* λ*-graph system is bipartite.*

Proof. It is clear that a bipartite symbolic matrix system gives rise to a bipartite λ-graph system. Conversely, suppose that a λ-graph system (V, E, λ, ι) is bipartite. Let $n(l)$ and $n'(l)$ be the cardinalities of the sets V_l^D and V_l^C respectively. We may identify V_l^D and V_l^C with the sets $\{1, \dots, n(l)\}$ and $\{1, \dots, n'(l)\}$ respectively. For $i \in V_l^D, j \in V_{l+1}^C$, put $\mathcal{P}_{l,l+1}(i,j) = \lambda(e_1) + \cdots + \lambda(e_p)$ where $e_k \in E_{l,l+1}, k = 1, \dots, p$ are the set of all edges in $E_{l,l+1}$ satisfying $s(e_k) = i, r(e_k) = j$. Similarly we define for $i \in V_l^C, j \in V_{l+1}^D$, put $\mathcal{Q}_{l,l+1}(i,j) = \lambda(f_1) + \cdots + \lambda(f_q)$ where $f_k \in E_{l,l+1}, k = 1, \dots, q$ are the set of all edges in $E_{l,l+1}$ satisfying $s(f_k) = i, r(f_k) = j$. For $i \in V_l^D, j \in V_{l+1}^D$, put $X_{l,l+1}(i,j) = 1$ if $\iota(j) = i$ and $X_{l,l+1}(i,j) = 0$ otherwise. Similarly for $i \in V_l^C, j \in V_{l+1}^C$, put $X'_{l,l+1}(i,j) = 1$ if $\iota(j) = i$ and $X'_{l,l+1}(i,j) = 0$ otherwise. Then by these matrices, we know that the corresponding symbolic matrix system $(\mathcal{M}, I)$ for (V, E, λ, ι) is bipartite.

M. Nasu introduced the notion of bipartite subshift in [N] and [N2]. A subshift Λ over alphabet Σ is said to be bipartite if there exist disjoint subsets $C, D \subset \Sigma$ such that any $(x_i)_{i\in\mathbb{Z}} \in \Lambda$ is either

$$x_i \in C \text{ and } x_{i+1} \in D \text{ for all } i \in \mathbb{Z} \quad \text{or} \quad x_i \in D \text{ and } x_{i+1} \in C \text{ for all } i \in \mathbb{Z}.$$

Let $\Lambda^{(2)}$ be the 2-higher power shift for Λ. Put

$$\Lambda_{CD} = \{(c_i d_i)_{i\in\mathbb{Z}} \in \Lambda^{(2)} | c_i \in C, d_i \in D\},$$
$$\Lambda_{DC} = \{(d_i c_i)_{i\in\mathbb{Z}} \in \Lambda^{(2)} | c_i \in C, d_i \in D\}.$$

They are subshifts over alphabets $C \cdot D$ and $D \cdot C$ respectively. Hence $\Lambda^{(2)}$ is partitioned into the two subshifts Λ_{CD} and Λ_{DC}.

Proposition 4.5. *A subshift Λ is bipartite if and only if its canonical symbolic matrix system $(\mathcal{M}^\Lambda, I^\Lambda)$ is bipartite.*

Proof. It is clear that a bipartite canonical symbolic matrix system gives rise to a bipartite subshift from the preceding proposition. Suppose that Λ is bipartite with respect to alphabets C, D. It suffices to show that its canonical λ-graph system (V, E, λ, ι) is bipartite. As in the construction of the canonical λ-graph system, the vertex set V_l is the set of all l-past equivalence classes $\{F_i^l\}_{i=1,\dots,m(l)}$. Put

$$V_l^C = \{F_i^l | x_1 \in D \text{ for all } (x_1, x_2, \dots,) \in F_i^l\},$$
$$V_l^D = \{F_i^l | x_1 \in C \text{ for all } (x_1, x_2, \dots,) \in F_i^l\}$$

so that we have a disjoint union $V_l^C \cup V_l^D = V_l$. It is easy to see that this decomposition of $V_l, l \in \mathbb{N}$ yields a bipartite decomposition of the λ-graph system (V, E, λ, ι).

Let Λ be a bipartite subshift over Σ with respect to alphabets C, D. As in Lemma 4.3, we have two symbolic matrix systems $(\mathcal{M}^{CD}, I^{CD})$ and $(\mathcal{M}^{DC}, I^{DC})$ over alphabets $C \cdot D$ and $D \cdot C$ from the bipartite canonical symbolic matrix system $(\mathcal{M}^\Lambda, I^\Lambda)$ for Λ respectively. They are naturally identified with the canonical symbolic matrix systems for the subshifts Λ_{CD} and Λ_{DC} respectively.
We thus see by Lemma 4.3.

Corollary 4.6. *For a bipartite subshift Λ with respect to alphabets C, D, we have*

$$(\mathcal{M}^{CD}, I^{CD}) \underset{1-pr}{\approx} (\mathcal{M}^{DC}, I^{DC})$$

a properly strong shift equivalence in 1-step.

The following notion of bipartite conjugacy has been introduced by Nasu in [N], [N2]. The conjugacy from Λ_{CD} onto Λ_{DC} that maps $(c_i d_i)_{i\in\mathbb{Z}}$ to $(d_i c_{i+1})_{i\in\mathbb{Z}}$ is called the forward bipartite conjugacy. The conjugacy from Λ_{CD} onto Λ_{DC} that maps $(c_i d_i)_{i\in\mathbb{Z}}$ to $(d_{i-1} c_i)_{i\in\mathbb{Z}}$ is called the backward bipartite conjugacy. A topological conjugacy between subshifts is called a symbolic conjugacy if it is a 1-block map given by a bijection between the underlying alphabets of the subshifts. M. Nasu proved the following factorization theorem.

LEMMA 4.7(M.NASU [N]). *Any topological conjugacy ψ between subshifts is factorized into a composition of the form*

$$\psi = \kappa_n \zeta_n \kappa_{n-1} \zeta_{n-1} \cdots \kappa_1 \zeta_1 \kappa_0$$

where $\kappa_0, \dots, \kappa_n$ are symbolic conjugacies and $\zeta_1, \dots, \zeta_n$ are either forward or backward bipartite conjugacies.

Thanks to the Nasu's result above, we reach the following theorem

THEOREM 4.8. *For two subshifts Λ, Λ', let $(\mathcal{M}, I), (\mathcal{M}', I')$ be their canonical symbolic matrix systems for Λ, Λ' respectively. If Λ and Λ' are topologically conjugate, the symbolic matrix systems $(\mathcal{M}, I), (\mathcal{M}', I')$ are properly strong shift equivalent.*

We will prove the converse implication of the theorem above. We will indeed prove the following proposition.

PROPOSITION 4.9. *If two symbolic matrix systems are properly strong shift equivalent in 1-step, their associated subshifts are topologically conjugate.*

To prove the proposition, we provide a notation and a lemma.

Set the $m(l) \times m(l+k)$ matrices:

$$I_{l,l+k} = I_{l,l+1} \cdot I_{l+1,l+2} \cdots I_{l+k-1,l+k},$$
$$\mathcal{M}_{l,l+k} = \mathcal{M}_{l,l+1} \cdot \mathcal{M}_{l+1,l+2} \cdots \mathcal{M}_{l+k-1,l+k}$$

for each $l, k \in \mathbb{N}$.

LEMMA 4.10. *Assume that two symbolic matrix systems $(\mathcal{M}, I)$ over Σ and $(\mathcal{M}', I')$ over Σ' are properly strong shift equivalent in 1-step. Let $\varphi : \Sigma \to C\cdot D$ and $\phi : \Sigma' \to D\cdot C$ be specifications that give a properly strong shift equivalence in 1-step between them. For any word $x_1 x_2 \in (\Lambda_{(\mathcal{M},I)})^2$ of length two in the associated subshift $\Lambda_{(\mathcal{M},I)}$, put $\varphi(x_i) = c_i d_i, i = 1, 2$ where $c_i \in C, d_i \in D$. Then there uniquely exists a symbol $y_0 \in \Sigma'$ such that $\phi(y_0) = d_1 c_2$.*

Proof. Note that by definition the specification ϕ is not necessarily defined on all the elements of Σ'. It suffices to show the existence of y_0. Since $x_1 x_2 \in (\Lambda_{(\mathcal{M},I)})^*$, for any fixed $l \geq 3$, we find $j = 1, 2, \dots, m(l+2)$ and $k =$

$1, 2, \ldots, m(l)$ such that x_1x_2 appears in $\mathcal{M}_{l,l+2}(k,j)$. Take $i = 1, 2, \ldots, m(l-2)$ with $I_{l-2,l}(i,k) = 1$. Hence x_1x_2 appears in $I_{l-2,l}\mathcal{M}_{l,l+2}(i,j)$. As we know the equality:

$$I_{l-2,l}\mathcal{M}_{l,l+2} \overset{\varphi}{\simeq} I_{l-2,l-1}X_{2l-1}\mathcal{P}_{2l-1}\mathcal{Q}_{2l}\mathcal{P}_{2l+1}\mathcal{Q}_{2l+2}X_{2l+3},$$

the word $\varphi(x_1x_2) = c_1d_1c_2d_2$ appears in a component of the right hand symbolic matrix above. Thus the word d_1c_2 appears in a component of $\mathcal{Q}_{2l}\mathcal{P}_{2l+1}$. By the equality $\mathcal{M}'_{l,l+1} \overset{\phi}{\simeq} \mathcal{Q}_{2l}\mathcal{P}_{2l+1}$, we can find a symbol y_0 in the corresponding component of the matrix $\mathcal{M}'_{l,l+1}$ so that $\phi(y_0) = d_1c_2$.

Proof of Proposition 4.9. Suppose that $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are properly strong shift equivalent in 1-step. We use the same notation as in Definition of properly strong shift equivalence. Set $\Lambda = \Lambda_{(\mathcal{M},I)}$ and $\Lambda' = \Lambda_{(\mathcal{M}',I')}$. By the preceding lemma, we have a 2-block map Φ from Λ^2 to Σ' defined by $\Phi(x_1x_2) = y_0$ where $\phi(y_0) = d_1c_2$ and $\varphi(x_i) = c_id_i, i = 1, 2$. Let Φ_∞ be the sliding block code induced by Φ so that Φ_∞ is a map from Λ to ${\Sigma'}^{\mathbb{Z}}$. We also write as Φ the map from Λ^* to the set of all words of Σ' defined by

$$\Phi(x_1x_2\cdots x_n) = \Phi(x_1x_2)\Phi(x_2x_3)\cdots\Phi(x_{n-1}x_n).$$

We will prove that $\Phi_\infty(\Lambda) \subset \Lambda'$. To prove this, it suffices to show that for any word w in Λ, $\Phi(w)$ is an admissible word in Λ'. For $w = w_1w_2\cdots w_n \in \Lambda^n$ and any fixed $l \geq n+1$, we find $j = 1, 2, \ldots, m(l+n)$ and $k = 1, 2, \ldots, m(l)$ such that w appears in $\mathcal{M}_{l,l+n}(k,j)$. Take $i = 1, 2, \ldots, m(l-n)$ with $I_{l-n,l}(i,k) = 1$. Hence w appears in $I_{l-n,l}\mathcal{M}_{l,l+n}(i,j)$. Put $\varphi(w_i) = c_id_i, i = 1, 2, \ldots, n$. By the equality

$$I_{l-1,l}\mathcal{M}_{l,l+n} \overset{\varphi}{\simeq} X_{2l-2}\mathcal{P}_{2l-1}\mathcal{Q}_{2l}\mathcal{P}_{2l+1}\mathcal{Q}_{2l+2}\cdots\mathcal{P}_{2l+2n-3}\mathcal{Q}_{2l+2n-2}X_{2l+2n-1},$$

the word $d_1c_2d_2c_3\cdots d_{n-1}c_n$ appears in a component of $\mathcal{Q}_{2l}\mathcal{P}_{2l+1}\mathcal{Q}_{2l+2}\cdots\mathcal{P}_{2l+2n-3}$. Hence the word $\phi^{-1}(d_1c_2)\phi^{-1}(d_2c_3)\cdots\phi^{-1}(d_{n-1}c_n)$ appears in a component of $\mathcal{M}'_{l,l+1}\cdot\mathcal{M}'_{l+1,l+2}\cdots\mathcal{M}'_{l+n-2,l+n-1}$. Thus we see that $\Phi(w)$ is an admissible word in Λ' and that the sliding block code Φ_∞ maps Λ to Λ'. Similarly, we can construct a sliding block code Ψ_∞ from Λ' to Λ that is an inverse of Φ_∞. Thus two subshifts Λ' and Λ are topologically conjugate.

Therefore we conclude the following theorem

Theorem 4.11. *If two symbolic matrix systems are properly strong shift equivalent, their associated subshifts are topologically conjugate.*

By Theorem 4.8 and Theorem 4.11, we conclude Theorem 4.2.

Remark. If there exist the matrices $\mathcal{P}_l, \mathcal{Q}_l$ for all sufficiently large number l in Definition of properly strong shift equivalence in 1-step, we may show that the associated subshifts are topologically conjugate because of the proof of Proposition 4.9.

Properly strong shift equivalence exactly corresponds to a finite sequence of bipartite decompositions of symbolic matrix systems and λ-graph systems. The definition of properly strong shift equivalence for symbolic matrix systems however needs rather complicated formulations than that of strong shift equivalence for nonnegative matrices. We will next introduce the notion of strong shift equivalence between two symbolic matrix systems that is simpler and weaker condition than properly strong shift equivalence. It is also a generalization of the notion of strong shift equivalence between nonnegative matrices defined by Williams in [Wi] and between symbolic matrices defined by Nasu in [N]. Let $(\mathcal{M}, I), (\mathcal{M}', I)$ be two symbolic matrix systems over alphabet Σ, Σ' respectively. Let $m(l), m'(l)$ be the sequences for which $\mathcal{M}_{l,l+1}, I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices and $\mathcal{M}'_{l,l+1}, I'_{l,l+1}$ are $m'(l) \times m'(l+1)$ matrices respectively.

DEFINITION. Two symbolic matrix systems $(\mathcal{M}, I), (\mathcal{M}', I)$ are said to be *strong shift equivalent in 1-step* if there exist alphabets C, D and specifications

$$\varphi : \Sigma \to C \cdot D, \qquad \phi : \Sigma' \to D \cdot C$$

such that for each $l \in \mathbb{N}$, there exist an $m(l-1) \times m'(l)$ matrix $\mathcal{H}_l$ over C and an $m'(l-1) \times m(l)$ matrix $\mathcal{K}_l$ over D satisfying the following equations:

$$I_{l-1,l}\mathcal{M}_{l,l+1} \overset{\varphi}{\simeq} \mathcal{H}_l\mathcal{K}_{l+1}, \qquad I'_{l-1,l}\mathcal{M}'_{l,l+1} \overset{\phi}{\simeq} \mathcal{K}_l\mathcal{H}_{l+1}$$

and

$$\mathcal{H}_l I'_{l,l+1} = I_{l-1,l}\mathcal{H}_{l+1}, \qquad \mathcal{K}_l I_{l,l+1} = I'_{l-1,l}\mathcal{K}_{l+1}.$$

We write this situation as

$$(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}', I').$$

Two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *strong shift equivalent in N-step* if there exist symbolic matrix systems $(\mathcal{M}^{(i)}, I^{(i)}), i = 1, 2, \ldots, N-1$ such that

$$\begin{aligned}(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) &\underset{1-st}{\approx} (\mathcal{M}^{(2)}, I^{(2)}) \\ &\underset{1-st}{\approx} \cdots \underset{1-st}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (\mathcal{M}', I').\end{aligned}$$

We denote this situation by

$$(\mathcal{M}, I) \underset{N-st}{\approx} (\mathcal{M}', I')$$

and simply call it a *strong shift equivalence*.

Similarly to the case of properly strong shift equivalence, we see that strong shift equivalence on symbolic matrix systems is an equivalence relation.

Proposition 4.12. *Properly strong shift equivalence in 1-step implies strong shift equivalence in 1-step.*

Proof. Let $\mathcal{P}_l, \mathcal{Q}_l, X_l$ and X'_l be the matrices in Definition of properly strong shift equivalence in 1-step between $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$. We set

$$\mathcal{H}_l = X_{2l-1}\mathcal{P}_{2l-1}, \qquad \mathcal{K}_l = X'_{2l-1}\mathcal{Q}_{2l-1}.$$

They give rise to a strong shift equivalence in 1-step between $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$.

Conversely we have

Proposition 4.13. *Suppose that both $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are canonical. If they are strong shift equivalent in 1-step, they are properly strong shift equivalent in 1-step. Hence strong shift equivalence on canonical symbolic matrix systems is completely the same as properly strong shift equivalence.*

Proof. Let Λ, Λ' be the associated subshifts for $(\mathcal{M}, I), (\mathcal{M}', I')$ respectively. Suppose that $(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}', I')$. We use the same notation as in Definition of strong shift equivalence. Set

$$\begin{aligned}\Lambda_\varphi = \{(\dots, c_{-1}, d_{-1}, \dot{c}_0, d_0, c_1, d_1, \dots) |\\ \text{there exists } (x_i)_{i\in\mathbb{Z}} \in \Lambda; \varphi(x_i) = c_i d_i \text{ for all } i \in \mathbb{Z}\},\\ \Lambda'_\phi = \{(\dots, d_{-1}, c_0, \dot{d}_0, c_1, d_1, c_2, \dots) |\\ \text{there exists } (y_i)_{i\in\mathbb{Z}} \in \Lambda'; \phi(y_i) = d_i c_i \text{ for all } i \in \mathbb{Z}\}\end{aligned}$$

where $\dot{c}_0, \dot{d}_0$ locate at the position of the 0-th coordinate in the sequences. Put

$$\Lambda_o = \Lambda_\varphi \cup \Lambda'_\phi$$

that becomes a subshift over $C \cup D$ because of strong shift equivalence between $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$. It is clear that Λ_o is a bipartite subshift with respect to the alphabets C, D. Hence the 2-higher power shift $\Lambda_o^{(2)}$ is decomposed as $\Lambda_o^{(2)} = \Lambda_\varphi^{(2)} \cup \Lambda_\phi^{(2)}$. As there exist symbolic conjugacies:

$$\Lambda \overset{\varphi}{\simeq} \Lambda_\varphi^{(2)}, \qquad \Lambda' \overset{\phi}{\simeq} \Lambda_\phi^{(2)},$$

the canonical symbolic matrix systems for the subshifts Λ and Λ' are properly strong shift equivalence in 1-step by the previous discussions.

By a similar argument to the proof of Proposition 4.9, we obtain

Proposition 4.14. *If two symbolic matrix systems (not necessarily canonical) are strong shift equivalent in 1-step, their associated subshifts are topologically conjugate.*

Thus we conclude

Theorem 4.15. *If two symbolic matrix systems (not necessarily canonical) are strong shift equivalent, their associated subshifts are topologically conjugate.*

5. Higher λ-graph systems

In studies of symbolic dynamics, the operation of taking higher block presentation plays important rôles (cf.[Kit], [LM]). In topological Markov shifts, the operation of taking 2-higher block presentation is a typical example of giving strong shift equivalence in 1-step. The N-higher block presentation of an edge shift corresponds to the edge shift of the N-higher edge graph. We in this section introduce higher λ-graph systems and correspondingly higher symbolic matrix systems. It follows that the subshift associated with the N-higher λ-graph system is the N-higher block presentation of the subshift associated with the original λ-graph system. We see that a symbolic matrix system is properly strong shift equivalent in N-step to its N-higher symbolic matrix system. We treat a left resolving λ-graph system, that is, the incoming edges to each vertex carry different labels. General case and also general state splitting procedure of λ-graph systems will be treated in a forthcoming paper.

For a left resolving λ-graph system (V, E, λ, ι) over alphabet Σ and a natural number $N \in \mathbb{N}$, we will define a λ-graph system $(V^{[N]}, E^{[N]}, \lambda^{[N]}, \iota^{[N]})$ over $\Sigma^{[N]} = \underbrace{\Sigma \cdots \Sigma}_{\text{N-times}}$ as follows:

$$
\begin{aligned}
V_l^{[N]} = \{(e_1, e_2, \dots, e_{N-1}) \in E_{l,l+1} \times E_{l+1,l+2} \times \cdots \times E_{l+N-2,l+N-1} | \\
r(e_i) = s(e_{i+1}) \text{ for } i = 1, 2, \dots, N-2\}, \\
E_{l,l+1}^{[N]} = \{((e_1, \dots, e_{N-1}),(f_1, \dots, f_{N-1})) \in V_l^{[N]} \times V_{l+1}^{[N]} | \\
e_{i+1} = f_i \text{ for } i = 1, 2, \dots, N-2\}.
\end{aligned}
$$

The maps

$$
s^{[N]} : E_{l,l+1}^{[N]} \to V_l^{[N]}, \qquad r^{[N]} : E_{l,l+1}^{[N]} \to V_{l+1}^{[N]}
$$

are defined by

$$
\begin{aligned}
s^{[N]}((e_1, \dots, e_{N-1}), (f_1, \dots, f_{N-1})) &= (e_1, \dots, e_{N-1}), \\
r^{[N]}((e_1, \dots, e_{N-1}), (f_1, \dots, f_{N-1})) &= (f_1, \dots, f_{N-1}).
\end{aligned}
$$

Set $V^{[N]} = \cup_{l \in \mathbb{N}} V_l^{[N]}$ and $E^{[N]} = \cup_{l \in \mathbb{N}} E_{l,l+1}^{[N]}$. Hence $(V^{[N]}, E^{[N]}, s^{[N]}, r^{[N]})$ is a Bratteli diagram. A labeling $\lambda^{[N]}$ on $(V^{[N]}, E^{[N]})$ is defined by

$$
\lambda^{[N]}((e_1, \dots, e_{N-1}), (f_1, \dots, f_{N-1})) = \lambda(e_1)\lambda(e_2) \dots \lambda(e_{N-1})\lambda(f_{N-1}) \in \Sigma^{[N]}
$$

for $((e_1, \dots, e_{N-1}), (f_1, \dots, f_{N-1})) \in E^{[N]}$. A sequence of surjections $\iota^{[N]} : V_{l+1}^{[N]} \to V_l^{[N]}, l \in \mathbb{N}$ is defined as follows. For $(e_1, \dots, e_{N-1}) \in V_{l+1}^{[N]}$, since the λ-graph system (V, E, λ, ι) is left resolving, there uniquely exist $e'_i \in E_{l+i-1,l+i}$ for $i = 1, 2, \dots, N-2$ such that

$$
\iota(s(e_i)) = s(e'_i), \quad \iota(r(e_i)) = r(e'_i), \quad \lambda(e_i) = \lambda(e'_i).
$$

As we know $(e'_1, \ldots, e'_{N-1}) \in V_l^{[N]}$, by setting $\iota^{[N]}(e_1, \ldots, e_{N-1}) = (e'_1, \ldots, e'_{N-1})$. We get a λ-graph system $(V^{[N]}, E^{[N]}, \lambda^{[N]}, \iota^{[N]})$ over $\Sigma^{[N]}$.

Definition. We call the λ-graph system $(V^{[N]}, E^{[N]}, \lambda^{[N]}, \iota^{[N]})$ the *N-higher λ-graph system for* (V, E, λ, ι). For a symbolic matrix system $(\mathcal{M}, I)$, the *N-higher symbolic matrix system* $(\mathcal{M}^{[N]}, I^{[N]})$ is defined to be the symbolic matrix system associated with the N-higher λ-graph system for the λ-graph system of $(\mathcal{M}, I)$.

It is routine to show the following proposition.

Proposition 5.1. $\Lambda_{(\mathcal{M}^{[N]}, I^{[N]})} = (\Lambda_{(\mathcal{M}, I)})^{[N]}$.

As seen in the case of nonnegative matrices, we see

Proposition 5.2. $(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}^{[2]}, I^{[2]})$: *a properly strong shift equivalence in 1-step.*

Proof. Let (V, E, λ, ι) and $(V^{[2]}, E^{[2]}, \lambda^{[2]}, \iota^{[2]})$ be the associated λ-graph systems for $(\mathcal{M}, I)$, $(\mathcal{M}^{[2]}, I^{[2]})$ over alphabets Σ and $\Sigma^{[2]}$ respectively. We will construct a bipartite λ-graph system $(\hat{V}, \hat{E}, \hat{\lambda}, \hat{\iota})$ that gives rise to a properly strong shift equivalence in 1-step between the λ-graph systems. We set for $l \in \mathbb{N}$

$$\hat{V}_{2l-1} = E_{l,l+1} \cup V_l, \qquad \hat{V}_{2l} = V_{l+1} \cup E_{l,l+1}$$

and

$$\hat{E}_{2l-1,2l} = \{(f, u) \in E_{l,l+1} \times V_{l+1} | u = r(f)\} \cup \{(v, e) \in V_l \times E_{l,l+1} | v = s(e)\},$$
$$\hat{E}_{2l,2l+1} = \{(v, e) \in V_{l+1} \times E_{l+1,l+2} | v = s(e)\} \cup \{(f, u) \in E_{l,l+1} \times V_{l+1} | u = r(f)\}.$$

The source maps $\hat{s}_{2l-1,2l} : \hat{E}_{2l-1,2l} \to \hat{V}_{2l-1}$ and $\hat{s}_{2l,2l+1} : \hat{E}_{2l,2l+1} \to \hat{V}_{2l}$ are defined as follows:

$$\hat{s}_{2l-1,2l}(f, u) = f \in E_{l,l+1}, \qquad \hat{s}_{2l-1,2l}(v, e) = v \in V_l,$$
$$\hat{s}_{2l,2l+1}(v, e) = v \in V_{l+1}, \qquad \hat{s}_{2l,2l+1}(f, u) = f \in E_{l,l+1}.$$

The range maps $\hat{r}_{2l-1,2l} : \hat{E}_{2l-1,2l} \to \hat{V}_{2l}$ and $\hat{r}_{2l,2l+1} : \hat{E}_{2l,2l+1} \to \hat{V}_{2l+1}$ are defined as follows:

$$\hat{r}_{2l-1,2l}(f, u) = u \in V_{l+1}, \qquad \hat{r}_{2l-1,2l}(v, e) = e \in E_{l,l+1},$$
$$\hat{r}_{2l,2l+1}(v, e) = e \in E_{l+1,l+2}, \qquad \hat{r}_{2l,2l+1}(f, u) = u \in V_{l+1}.$$

The maps $\hat{\iota}_{2l,2l-1} : \hat{V}_{2l} \to \hat{V}_{2l-1}$ and $\hat{\iota}_{2l+1,2l} : \hat{V}_{2l+1} \to \hat{V}_{2l}$ are defined as follows:

$$\hat{\iota}_{2l,2l-1}(u) = \iota(u) \text{ for } u \in V_{l+1}, \quad \hat{\iota}_{2l,2l-1}(f) = f \text{ for } f \in E_{l,l+1},$$
$$\hat{\iota}_{2l+1,2l}(e) = \iota(e) \text{ for } e \in E_{l+1,l+2}, \quad \hat{\iota}_{2l+1,2l}(v) = v \text{ for } v \in V_{l+1}$$

where $\iota(e) \in E_{l,l+1}$ is naturally defined for $e \in E_{l+1,l+2}$. Put $D_\Sigma = \{D_\alpha | \alpha \in \Sigma\}$, $C_\Sigma = \{C_\alpha | \alpha \in \Sigma\}$ and $\hat{\Sigma} = D_\Sigma \cup C_\Sigma$. The labeling $\hat{\lambda}$ is defined as a map from $\hat{E}$ to the alphabet $\hat{\Sigma}$ as follows: For $(f,u),(v,e)$ in $\hat{E}_{2l-1,2l} = \{(f,u) \in E_{l,l+1} \times V_{l+1} | u = r(f)\} \cup \{(v,e) \in V_l \times E_{l,l+1} | v = s(e)\}$, we set

$$\hat{\lambda}(f,u) = C_{\lambda(f)}, \quad \hat{\lambda}(v,e) = D_{\lambda(e)}.$$

For $(v,e),(f,u)$ in $\hat{E}_{2l,2l+1} = \{(v,e) \in V_{l+1} \times E_{l+1,l+2} | v = s(e)\} \cup \{(f,u) \in E_{l,l+1} \times V_{l+1} | u = r(f)\}$, we set

$$\hat{\lambda}(v,e) = D_{\lambda(e)}, \quad \hat{\lambda}(f,u) = C_{\lambda(f)}.$$

Then it is routine to check that $(\hat{V}, \hat{E}, \hat{\lambda}, \hat{\iota})$ is a bipartite λ-graph system over alphabet $\hat{\Sigma}$. Through the specifications $\varphi : \Sigma \to D_\Sigma \cdot C_\Sigma$ and $\phi : \Sigma^{[2]} \to C_\Sigma \cdot D_\Sigma$ defined by

$$\varphi(\alpha) = D_\alpha \cdot C_\alpha \quad \text{ and } \quad \phi(\alpha,\beta) = D_\alpha \cdot C_\beta,$$

we know that the symbolic matrix system for $(\hat{V}, \hat{E}, \hat{\lambda}, \hat{\iota})$ gives rise to a properly strong shift equivalence in 1-step between $(\mathcal{M}, I)$ and $(\mathcal{M}^{[2]}, I^{[2]})$.

Since $(\mathcal{M}^{[N+1]}, I^{[N+1]})$ is isomorphic to $((\mathcal{M}^{[N]})^{[2]}, (I^{[N]})^{[2]})$, we have

COROLLARY 5.3. *For any symbolic matrix system $(\mathcal{M}, I)$, we have*

$$(\mathcal{M}^{[N]}, I^{[N]}) \underset{N-st}{\approx} (\mathcal{M}, I)$$

a properly strong shift equivalence in N-step.

6. SHIFT EQUIVALENCE

By the discussions of Section 4, the topological conjugacy classes of subshifts are completely characterized by the strong shift equivalence classes of the associated canonical symbolic matrix systems. However, even for topological Markov shifts, there is no general algorithm known for deciding whether two nonnegative matrices are strong shift equivalent. R. F. Williams introduced the notion of shift equivalence between two nonnegative matrices that is weaker but easier to treat than the notion of strong shift equivalence ([Wi]). The formulation of shift equivalence between nonnegative matrices is described by certain algebraic relations between the matrices that determine a crucial invariant called the dimension group ([Kr], [Kr2]). The notion of shift equivalence has been generalized to symbolic matrices by Boyle-Krieger and studied as a topological conjugacy invariant for sofic subshifts in [BK].

We in this section introduce the notion of shift equivalence between two symbolic matrix systems as a generalization of Williams's notion for nonnegative matrices and Boyle-Krieger's notion for symbolic matrices. Let $(\mathcal{M}, I), (\mathcal{M}', I')$ be two symbolic matrix systems over alphabets Σ, Σ' respectively. For $N \in \mathbb{N}$, we put $(\Sigma)^N = \Sigma \cdots \Sigma, (\Sigma')^N = \Sigma' \cdots \Sigma'$: the N-times products.

Definition. For $N \in \mathbb{N}$, two symbolic matrix systems $(\mathcal{M}, I), (\mathcal{M}', I')$ are said to be *shift equivalent of lag* N if there exist alphabets C_N, D_N and specifications

$$\varphi_1 : \Sigma \cdot C_N \to C_N \cdot \Sigma', \qquad \varphi_2 : \Sigma' \cdot D_N \to D_N \cdot \Sigma$$

and

$$\psi_1 : (\Sigma)^N \to C_N \cdot D_N, \qquad \psi_2 : (\Sigma')^N \to D_N \cdot C_N$$

such that for each $l \in \mathbb{N}$, there exist an $m(l) \times m'(l+N)$ matrix $\mathcal{H}_l$ over C_N and an $m'(l) \times m(l+N)$ matrix $\mathcal{K}_l$ over D_N satisfying the following equations:

$$\mathcal{M}_{l,l+1}\mathcal{H}_{l+1} \overset{\varphi_1}{\simeq} \mathcal{H}_l\mathcal{M}'_{l+N,l+N+1}, \qquad \mathcal{M}'_{l,l+1}\mathcal{K}_{l+1} \overset{\varphi_2}{\simeq} \mathcal{K}_l\mathcal{M}_{l+N,l+N+1},$$

$$I_{l,l+N}\mathcal{M}_{l+N,l+2N} \overset{\psi_1}{\simeq} \mathcal{H}_l\mathcal{K}_{l+N}, \qquad I'_{l,l+N}\mathcal{M}'_{l+N,l+2N} \overset{\psi_2}{\simeq} \mathcal{K}_l\mathcal{H}_{l+N}$$

and

$$I_{l,l+1}\mathcal{H}_{l+1} = \mathcal{H}_l I'_{l+N,l+N+1}, \qquad I'_{l,l+1}\mathcal{K}_{l+1} = \mathcal{K}_l I_{l+N,l+N+1}.$$

We denote this situation by

$$(\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I') \quad \text{or} \quad (\mathcal{H}, \mathcal{K}) : (\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I')$$

and simply call it a *shift equivalence.*

Similarly to the case of nonnegative matrices and symbolic matrices, we can see the following lemma.

Lemma 6.1.

(i) $(\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I')$ *implies* $(\mathcal{M}, I) \underset{lagL}{\sim} (\mathcal{M}', I')$ *for all* $L \geq N$.

(ii) $(\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I')$ *and* $(\mathcal{M}', I') \underset{lagN'}{\sim} (\mathcal{M}'', I'')$ *implies* $(\mathcal{M}, I) \underset{lagN+N'}{\sim} (\mathcal{M}'', I'')$. *Hence shift equivalence is an equivalence relation on symbolic matrix systems.*

Proof. (i) Suppose that $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are shift equivalent of lag N. It suffices to show that they are shift equivalent of lag $N+1$. We use the same notation as above. Set the alphabets

$$C_{N+1} = C_N, \qquad D_{N+1} = D_N \cdot \Sigma.$$

Put the specification $\varphi'_1 = \varphi_1 : \Sigma \cdot C_{N+1} \to C_{N+1} \cdot \Sigma'$. Through the specification φ_2, we have a natural specification $\varphi'_2 : \Sigma' \cdot D_{N+1} \to D_{N+1} \cdot \Sigma$. Similarly, through the specifications $\psi_1, \psi_2, \varphi_1$, we have natural specifications

$$\psi'_1 : (\Sigma)^{N+1} \to C_{N+1} \cdot D_{N+1}, \qquad \psi'_2 : (\Sigma')^{N+1} \to D_{N+1} \cdot C_{N+1}.$$

Put the matrices

$$\mathcal{H}'_l = \mathcal{H}_l I'_{l+N,l+N+1}, \quad \mathcal{K}'_l = \mathcal{K}_l \mathcal{M}_{l+N,l+N+1}.$$

Then it is straightforward to see that they give a shift equivalence of lag $N+1$ between $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$.
(ii) Assume that

$$(\mathcal{H}, \mathcal{K}) : (\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I'), \qquad (\mathcal{H}', \mathcal{K}') : (\mathcal{M}', I') \underset{lagN'}{\sim} (\mathcal{M}'', I'').$$

Then it is routine to check that

$$(\mathcal{H}\mathcal{H}', \mathcal{K}'\mathcal{K}) : (\mathcal{M}, I) \underset{lagN+N'}{\sim} (\mathcal{M}'', I'').$$

Similarly to the case of matrices, we have

THEOREM 6.2. *Strong shift equivalence in N-step implies shift equivalence of lag N.*

Proof. Suppose that $(\mathcal{M}, I) \underset{N-st}{\approx} (\mathcal{M}', I')$ a strong shift equivalence in N-step. There exist symbolic matrix systems $(\mathcal{M}^{(i)}, I^{(i)})$ for $i = 1, \dots, N-1$ such that

$$\begin{aligned}(\mathcal{M}, I) = (\mathcal{M}^{(0)}, I^{(0)}) \underset{1-st}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-st}{\approx} (\mathcal{M}^{(2)}, I^{(2)}) \underset{1-st}{\approx} \\ \cdots \underset{1-st}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (\mathcal{M}^{(N)}, I^{(N)}) = (\mathcal{M}', I').\end{aligned}$$

Let $\mathcal{H}_l^{(i)}, \mathcal{K}_l^{(i)}$ be rectangular symbolic matrices that give a strong shift equivalence between $(\mathcal{M}^{(i-1)}, I^{(i-1)})$ and $(\mathcal{M}^{(i)}, I^{(i)})$ where $\mathcal{H}_l^{(i)}$ is an $m^{(i-1)}(l-1) \times m^{(i)}(l)$ matrix over alphabet $C(i)$ and $\mathcal{K}_l^{(i)}$ is an $m^{(i)}(l-1) \times m^{(i-1)}(l)$ matrix over alphabet $D(i)$ for each $l \in \mathbb{N}$ and $i = 1, \dots, N$. Set the alphabets

$$C_N = C(1) \cdots C(N), \qquad D_N = D(1) \cdots D(N).$$

Put the matrices

$$\mathcal{P}_l = \mathcal{H}_{l+2}^{(1)} \mathcal{H}_{l+3}^{(2)} \cdots \mathcal{H}_{l+N+1}^{(N)}, \qquad \mathcal{Q}_l = \mathcal{K}_{l+2}^{(1)} \mathcal{K}_{l+3}^{(2)} \cdots \mathcal{K}_{l+N+1}^{(N)}$$

an $m(l) \times m'(l+N)$ matrix over C_N, an $m'(l) \times m(l+N)$ matrix over D_N respectively. We then have the following natural specifications

$$\varphi_1 : \Sigma \cdot C_N \to C_N \cdot \Sigma', \qquad \varphi_2 : \Sigma' \cdot D_N \to D_N \cdot \Sigma$$

and

$$\psi_1 : (\Sigma)^N \to C_N \cdot D_N, \qquad \psi_2 : (\Sigma')^N \to D_N \cdot C_N$$

that yield a shift equivalence of lag N between $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$.

For a subshift (Λ, σ) over Σ, its n-higher power shift $(\Lambda^{(n)}, \sigma)$ is defined to be the subshift (Λ, σ^n) over $(\Sigma)^n$ (cf.[LM]). Two subshifts is called eventually conjugate if their n-higher power shifts are conjugate for all large enough n ([Wi], [KimR]). Williams and Kim-Roush showed that two square nonnegative matrices are shift equivalent if and only if the associated topological Markov shifts are eventually conjugate. Boyle-Krieger generalized their result to symbolic matrices and sofic subshifts ([BK]). W. Krieger kindly asked the author whether or not these results can be generalized to general subshifts. The author sincerely thanks him for his question.

Proposition 6.3. *If symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are shift equivalent, their associated subshifts $\Lambda_{(\mathcal{M},I)}$ and $\Lambda_{(\mathcal{M}',I')}$ are eventually conjugate.*

To show the proposition, we provide a lemma that is proved by a straightforward calculation.

Lemma 6.4. *For a symbolic matrix system $(\mathcal{M}, I)$, let Λ the associated subshift. We set for $n, l \in \mathbb{N}$,*

$$I^n_{l,l+1} = I_{nl,nl+1} I_{nl+1,nl+2} \cdots I_{nl+n-1,nl+n},$$
$$\mathcal{M}^n_{l,l+1} = \mathcal{M}_{nl,nl+1} \mathcal{M}_{nl+1,nl+2} \cdots \mathcal{M}_{nl+n-1,nl+n}.$$

Then $(\mathcal{M}^n, I^n)$ becomes a symbolic matrix system whose associated subshift is the n-higher power shift $\Lambda^{(n)}$ of Λ.

Proof of Proposition 6.3. Put $\Lambda = \Lambda_{(\mathcal{M},I)}$, $\Lambda' = \Lambda_{(\mathcal{M}',I')}$ over Σ, Σ' respectively. Assume that

$$(\mathcal{H}, \mathcal{K}) : (\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I').$$

For a number $K \in \mathbb{N}$, put $n = K + N$. We will see that $\Lambda^{(n)} \underset{1-st}{\approx} \Lambda'^{(n)}$. Let C_N, D_N be alphabets as in Definition of shift equivalence. Set $C = C_N, D = D_N \cdot (\Sigma)^K$. There are natural specifications

$$(\Sigma)^n \to C \cdot D, \qquad (\Sigma')^n \to D \cdot C$$

by using the specifications in the shift equivalence between Λ and Λ'. Put the matrices

$$\mathcal{P}_l = \mathcal{H}_{nl-n} I'_{nl-K,nl-K+1} I'_{nl-K+1,nl-K+2} \cdots I'_{nl-1,nl},$$
$$\mathcal{Q}_l = \mathcal{K}_{nl-n} \mathcal{M}_{nl-K,nl-K+1} \mathcal{M}_{nl-K+1,nl-K+2} \cdots \mathcal{M}_{nl-1,nl}.$$

They are an $m(nl-n) \times m'(nl)$ matrix over C and an $m'(nl-n) \times m(nl)$ matrix over D respectively. We see that they yield a strong shift equivalence in 1-step between $(\mathcal{M}^n, I^n)$ and $(\mathcal{M}'^n, I'^n)$ so that their associated subshifts are topologically conjugate by Theorem 4.15.

We will comment on the notion of properly shift equivalence between symbolic matrix systems. The following is the definition of properly shift equivalence that is a slightly stronger than shift equivalence and weaker than properly strong shift equivalence.
Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be symbolic matrix systems over alphabets Σ, Σ' respectively. Hence $\mathcal{M}_{l,l+1}, I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices and $\mathcal{M}'_{l,l+1}, I'_{l,l+1}$ are $m'(l) \times m'(l+1)$ matrices.
DEFINITION. $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly shift equivalent of lag* N if there exist alphabets C_N, D_N and specifications

$$\begin{aligned} \varphi_1 :&\Sigma \cdot C_N \to C_N \cdot \Sigma', \qquad & \varphi_2 : \Sigma' \cdot D_N \to D_N \cdot \Sigma, \\ \psi_1 :&(\Sigma)^N \to C_N \cdot D_N, \qquad & \psi_2 : (\Sigma')^N \to D_N \cdot C_N \end{aligned}$$

and increasing sequences $n(l), n'(l)$ on $l \in \mathbb{N}$ such that for each $l \in \mathbb{N}$, there exist an $n(k) \times n'(k+2N-1)$ matrix $\mathcal{P}_k$ over C_N, an $n'(k) \times n(k+2N-1)$ matrix $\mathcal{Q}_k$ over D_N for $k = 2l, 2l+2N-1$, an $n(l) \times n(l+1)$ matrix X_l over $\{0,1\}$ and an $n'(l) \times n'(l+1)$ matrix X'_l over $\{0,1\}$ satisfying the following equations:
(6.1)

$$\mathcal{M}_{l,l+N} I_{l+N,l+2N-1} \overset{\psi_1}{\simeq} \mathcal{P}_{2l}\mathcal{Q}_{2l+2N-1}, \quad \mathcal{M}'_{l,l+N} I'_{l+N,l+2N-1} \overset{\psi_2}{\simeq} \mathcal{Q}_{2l}\mathcal{P}_{2l+2N-1},$$

$$\begin{aligned} \mathcal{M}_{l,l+1}\mathcal{P}_{2(l+1)}X'_{2l+2N+1} &\overset{\varphi_1}{\simeq} \mathcal{P}_{2l}X'_{2l+2N-1}\mathcal{M}'_{l+N,l+N+1}, \\ \mathcal{M}'_{l,l+1}\mathcal{Q}_{2(l+1)}X_{2l+2N+1} &\overset{\varphi_2}{\simeq} \mathcal{Q}_{2l}X_{2l+2N-1}\mathcal{M}_{l+N,l+N+1}, \\ I_{l,l+1} = X_{2l}X_{2l+1}, &\qquad I'_{l,l+1} = X'_{2l}X'_{2l+1} \end{aligned}$$

and

$$X_l\mathcal{P}_{l+1} = \mathcal{P}_l X'_{l+2N-1}, \qquad X'_l\mathcal{Q}_{l+1} = \mathcal{Q}_l X_{l+2N-1}.$$

We denote this situation by

$$(\mathcal{M}, I) \underset{N-pr}{\sim} (\mathcal{M}', I').$$

It follows that by (6.1), $n(2l) = m(l)$ and $n'(2l) = m'(l)$ for $l \in \mathbb{N}$.
For $N = 1$, if we understand that the matrices $I_{l+1,l+1}$ and $I'_{l+1,l+1}$ are the $m(l+1) \times m(l+1)$ identity matrix and the $m'(l+1) \times m'(l+1)$ identity matrix respectively, the properly shift equivalence of lag 1 is exactly the same as the properly strong shift equivalence in 1-step.

This definition is also a generalization of Boyle-Krieger 's shift equivalence between symbolic matrices ([BK] see also [N2]).
The following proposition is routine.

Proposition 6.5.

(i) $(\mathcal{M}, I) \underset{N-pr}{\sim} (\mathcal{M}', I')$ *implies* $(\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I')$. *That is, properly shift equivalence implies shift equivalence.*

(ii) $(\mathcal{M}, I) \underset{N-pr}{\approx} (\mathcal{M}', I')$ *implies* $(\mathcal{M}, I) \underset{N-pr}{\sim} (\mathcal{M}', I')$. *That is, properly strong shift equivalence implies properly shift equivalence.*

We thus summarize as in the following way:

$$
\begin{array}{ccc}
(\mathcal{M}, I) \underset{N-pr}{\approx} (\mathcal{M}', I') & \Longrightarrow & (\mathcal{M}, I) \underset{N-pr}{\sim} (\mathcal{M}', I')) \\
\Downarrow & & \Downarrow \\
(\mathcal{M}, I) \underset{N-st}{\approx} (\mathcal{M}', I') & \Longrightarrow & (\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I').
\end{array}
$$

We may define strong shift equivalence and shift equivalence between subshifts as their corresponding properties for their canonical symbolic matrix systems. Hence we can say that two subshifts are topologically conjugate if and only if they are strong shift equivalence. The strong shift equivalence for subshifts imply the shift equivalence.

7. Nonnegative matrix systems

In this section, we will introduce the notion of nonnegative matrix system that is also a generalization of nonnegative matrices. We will then generalize strong shift equivalence and shift equivalence between nonnegative matrices to between nonnegative matrix systems. Let $(A_{l,l+1}, I_{l,l+1}), l \in \mathbb{N}$ be a pair of sequences of rectangular matrices such that the following four conditions for each $l \in \mathbb{N}$ are satisfied:

(1) $A_{l,l+1}$ is an $m(l) \times m(l+1)$ rectangular matrix with entries in nonnegative integers.

(2) $I_{l,l+1}$ is an $m(l) \times m(l+1)$ rectangular matrix with entries in $\{0, 1\}$ satisfying the following two conditions:

(2-a) For i, there exists j such that $I_{l,l+1}(i, j) \neq 0$.

(2-b) For j, there uniquely exists i such that $I_{l,l+1}(i, j) \neq 0$.

(3) $m(l) \leq m(l+1)$.

(4) $I_{l,l+1}A_{l+1,l+2} = A_{l,l+1}I_{l+1,l+2}$.

The pair (A, I) is called a *nonnegative matrix system*. For $i = 1, \dots, m(l), j = 1, \dots, m(l+1)$, we denote by $A_{l,l+1}(i, j)$, $I_{l,l+1}(i, j)$ the (i, j)-components of $A_{l,l+1}$, $I_{l,l+1}$ respectively. A nonnegative matrix system (A, I) is said to be *essential* if it satisfies the following further conditions

(5-i) For i, there exists j such that $A_{l,l+1}(i, j) \neq 0$.

(5-ii) For j, there exists i such that $A_{l,l+1}(i, j) \neq 0$.

We henceforth study essential nonnegative matrix systems and call them nonnegative matrix systems for simplicity.

The property "sofic "for nonnegative matrix systems are similarly defined to the cases of symbolic matrix systems. The following is basic.

LEMMA 7.1. *For a symbolic matrix system* $(\mathcal{M}, I)$*, let* $M_{l,l+1}$ *be the* $m(l) \times m(l+1)$ *rectangular matrix obtained from* $\mathcal{M}_{l,l+1}$ *by setting all the symbols equal to* 1*. Then the resulting pair* (M, I) *becomes a nonnegative matrix system.*

We write the matrices above as $\mathrm{supp}(\mathcal{M}_{l,l+1}) = M_{l,l+1}$ and call $M_{l,l+1}$ the support of $\mathcal{M}_{l,l+1}$. The pair (M, I) is called the nonnegative matrix system associated with $(\mathcal{M}, I)$. Conversely we see

PROPOSITION 7.2. *For a nonnegative matrix system* (A, I) *and a symbolic matrix* $\mathcal{M}_{1,2}$ *over alphabet* Σ *such that* $\mathrm{supp}(\mathcal{M}_{1,2}) = A_{1,2}$*, there exists a sequence* $\mathcal{M}_{l,l+1}, l \in \mathbb{N}$ *of symbolic matrices over* Σ *such that the pair* $(\mathcal{M}, I)$ *is a symbolic matrix system and* $\mathrm{supp}(\mathcal{M}_{l,l+1}) = A_{l,l+1}$ *for all* $l \in \mathbb{N}$*.*

Proof. We will prove the assertion by induction. Assume that a symbolic matrix $\mathcal{M}_{k,k+1}$ is determined. For $j = 1, \dots, m(k+2)$, take a unique index $j' = 1, \dots, m(k+1)$ such that $I_{k+1,k+2}(j', j) = 1$. For $i = 1, \dots, m(k)$, suppose that $\mathcal{M}_{k,k+1}(i, j') = \alpha_1 + \cdots + \alpha_n$. Let $l_1, \dots, l_p$ be the set of all numbers $l = 1, \dots, m(k+1)$ satisfying $I_{k,k+1}(i, l) = 1$. Hence we have $n = \sum_{r=1}^{p} A_{k+1,k+2}(l_r, j)$. Put $\xi_r = A_{k+1,k+2}(l_r, j)$. Now we define

$$
\begin{aligned}
\mathcal{M}_{k+1,k+2}(l_1, j) &= \alpha_1 + \cdots + \alpha_{\xi_1}, \\
\mathcal{M}_{k+1,k+2}(l_2, j) &= \alpha_{\xi_1+1} + \cdots + \alpha_{\xi_1+\xi_2}, \\
\mathcal{M}_{k+1,k+2}(l_3, j) &= \alpha_{\xi_1+\xi_2+1} + \cdots + \alpha_{\xi_1+\xi_2+\xi_3}, \\
&\cdots\cdots \\
\mathcal{M}_{k+1,k+2}(l_p, j) &= \alpha_{\xi_1+\cdots+\xi_{p-1}+1} + \cdots + \alpha_n.
\end{aligned}
$$

Since for any $l = 1, \dots, m(k+1)$, there uniquely exists $i = 1, \dots, m(k)$ such that $I_{k,k+1}(i, l) = 1$ we may define $\mathcal{M}_{k+1,k+2}(l, j)$ for all $l = 1, \dots, m(k+1)$ by the above way. The matrices satisfy $I_{k,k+1}\mathcal{M}_{k+1,k+2} = \mathcal{M}_{k,k+1}I_{k+1,k+2}$ and $\mathrm{supp}(\mathcal{M}_{k+1,k+2})(l, j) = A_{k+1,k+2}(l, j)$.

For nonnegative matrix systems we will formulate strong shift equivalence as follows.

DEFINITION. Two nonnegative matrix systems $(A, I), (A', I')$ are said to be *strong shift equivalent in 1-step* if for each $l \in \mathbb{N}$, there exist an $m(l-1) \times m'(l)$ matrix H_l with entries in nonnegative integers and an $m'(l-1) \times m(l)$ matrix K_l with entries in nonnegative integers satisfying the following equations:

$$
I_{l-1,l}A_{l,l+1} = H_l K_{l+1}, \qquad I'_{l-1,l}A'_{l,l+1} = K_l H_{l+1}
$$

and

$$
H_l I'_{l,l+1} = I_{l-1,l}H_{l+1}, \qquad K_l I_{l,l+1} = I'_{l-1,l}K_{l+1}.
$$

We write this situation as

$$
(A, I) \underset{1-st}{\approx} (A', I').
$$

Two nonnegative matrix systems (A, I) and (A', I') are said to be *strong shift equivalent in N-step* if there exist nonnegative matrix systems $(A^{(i)}, I^{(i)}), i = 1, 2, \dots, N-1$ such that

$$(A, I) \underset{1-st}{\approx} (A^{(1)}, I^{(1)}) \underset{1-st}{\approx} (A^{(2)}, I^{(2)}) \underset{1-st}{\approx} \cdots \underset{1-st}{\approx} (A^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (A', I').$$

We denote this situation by

$$(A, I) \underset{N-st}{\approx} (A', I')$$

and simply call it a *strong shift equivalence*.

This formulation is also a generalization of Williams's strong shift equivalence between nonnegative matrices ([Wi]). Similarly to symbolic matrix systems, strong shift equivalence is an equivalence relation on nonnegative matrix systems.

We directly have

Proposition 7.3. *If two symbolic matrix systems are strong shift equivalence (in N-step), then the associated nonnegative matrix systems are strong shift equivalent (in N-step).*

We will describe the matrix relations appearing in the formulation of strong shift equivalence between nonnegative matrix systems in terms of certain single homomorphisms between inductive limits of abelian groups. For a nonnegative matrix system (A, I), the transpose $I^t_{l,l+1}$ of the matrix $I_{l,l+1}$ naturally induces an ordered homomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$, where the positive cone $\mathbb{Z}^{m(l)}_+$ of the group $\mathbb{Z}^{m(l)}$ is defined by

$$\mathbb{Z}^{m(l)}_+ = \{(n_1, n_2, \dots, n_{m(l)}) \in \mathbb{Z}^{m(l)} | n_i \geq 0, i = 1, 2 \dots m(l)\}.$$

We put the inductive limits:

$$\mathbb{Z}_{I^t} = \varinjlim \{I^t_{l,l+1} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}\},$$
$$\mathbb{Z}^+_{I^t} = \varinjlim \{I^t_{l,l+1} : \mathbb{Z}^{m(l)}_+ \to \mathbb{Z}^{m(l+1)}_+\}.$$

The condition (2-a) for the matrix $I_{l,l+1}$ says the following lemma.

Lemma 7.4. *For each $l \in \mathbb{N}$, the homomorphism $I^t_{l,l+1} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}$ is injective. Hence the canonical homomorphism $\iota_l : \mathbb{Z}^{m(l)} \to \mathbb{Z}_{I^t}$ is injective.*

By the relation: $I_{l,l+1}A_{l+1,l+2} = A_{l,l+1}I_{l+1,l+2}$, the sequence of the transposed matrices $A^t_{l,l+1}, l \in \mathbb{N}$ of the matrices $A_{l,l+1}, l \in \mathbb{N}$ yields an endomorphism of the ordered group $\mathbb{Z}_{I^t}$. We write it as $\lambda_{(A,I)}$.

Definition. For nonnegative matrix systems $(A, I), (A', I')$ and $L \in \mathbb{N}$, a homomorphism ξ from the group $\mathbb{Z}_{I^t}$ to the group $\mathbb{Z}_{I'^t}$ is said to be *finite homomorphism of lag L* if it satisfies the condition

$$\xi(\mathbb{Z}^{m(l)}) \subset \mathbb{Z}^{m'(l+L)} \quad \text{for all } l \in \mathbb{N}$$

where $\mathbb{Z}^{m(l)}$ and $\mathbb{Z}^{m'(l)}$ are naturally imbedded into $\mathbb{Z}_{I^t}$ and $\mathbb{Z}_{I'^t}$ respectively. We then have

Proposition 7.5. *Two nonnegative matrix systems* (A, I) *and* (A', I') *are strong shift equivalence in 1-step if and only if there exist order preserving finite homomorphisms of lag* 1*:* $\xi : \mathbb{Z}_{I^t} \to \mathbb{Z}_{I'^t}$ *and* $\eta : \mathbb{Z}_{I'^t} \to \mathbb{Z}_{I^t}$ *such that*

$$\eta \circ \xi = \lambda_{(A,I)}, \qquad \xi \circ \eta = \lambda_{(A',I')}.$$

Proof. Suppose that (A, I) and (A', I') are strong shift equivalent in 1-step. Let H_l, K_l be sequences of matrices that give rise to a strong shift equivalence between them. Then by the condition $I'^t_{l,l+1} H^t_l = H^t_{l+1} I^t_{l-1,l}$, the family $H^t_l, l \in \mathbb{N}$ yields a homomorphism from $\mathbb{Z}_{I^t}$ to $\mathbb{Z}_{I'^t}$ which we denote by ξ. Similarly we define a homomorphism η from $\mathbb{Z}_{I'^t}$ to $\mathbb{Z}_{I^t}$ induced by the family $K_l, l \in \mathbb{N}$. It is easy to see that the homomorphisms ξ, η are order preserving and finite homomorphisms of lag 1. By the condition $A'^t_{l,l+1} I^t_{l-1,l} = K^t_{l+1} H^t_l$, we see $\eta \circ \xi = \lambda_{(A,I)}$. Similarly, we have $\xi \circ \eta = \lambda_{(A',I')}$.
The converse implication is also easy by using Lemma 7.4. We in fact see that the matrices H_l, K_l are given by the transposed matrices of the restrictions of the homomorphisms ξ to $\mathbb{Z}^{m(l)} (\hookrightarrow \mathbb{Z}_{I^t})$ and η to $\mathbb{Z}^{m'(l)} (\hookrightarrow \mathbb{Z}_{I'^t})$ respectively. They satisfy the required conditions of strong shift equivalence between (A, I) and (A', I').

We will next formulate shift equivalence between two nonnegative matrix systems. For a nonnegative matrix system (A, I), we set the $m(l) \times m(l+k)$ matrices:

$$I_{l,l+k} = I_{l,l+1} \cdot I_{l+1,l+2} \cdots I_{l+k-1,l+k},$$
$$A_{l,l+k} = A_{l,l+1} \cdot A_{l+1,l+2} \cdots A_{l+k-1,l+k}$$

for each $l, k \in \mathbb{N}$.
Definition. Two nonnegative matrix systems $(A, I), (A', I')$ are said to be *shift equivalent of lag* N if for each $l \in \mathbb{N}$, there exist an $m(l) \times m'(l+N)$ matrix H_l with entries in nonnegative integers and an $m'(l) \times m(l+N)$ matrix K_l with entries in nonnegative integers satisfying the following equations:

$$A_{l,l+1} H_{l+1} = H_l A'_{l+N,l+N+1}, \qquad A'_{l,l+1} K_{l+1} = K_l A_{l+N,l+N+1},$$
$$H_l K_{l+N} = I_{l,l+N} A_{l+N,l+2N}, \qquad K_l H_{l+N} = I'_{l,l+N} A'_{l+N,l+2N}$$

and

$$I_{l,l+1} H_{l+1} = H_l I'_{l+N,l+N+1}, \qquad I'_{l,l+1} K_{l+1} = K_l I_{l+N,l+N+1}.$$

We write this situation as

$$(A, I) \underset{lagN}{\sim} (A', I') \quad \text{or} \quad (H, K) : (A, I) \underset{lagN}{\sim} (A', I')$$

and simply call it a *shift equivalence.*

This formulation is a generalization of Williams's shift equivalence between square matrices with entries in nonnegative integers ([Wi] see also [BK]).
Similarly to the case of shift equivalence for nonnegative matrices and symbolic matrix systems, we have.

Lemma 7.6.

(i) $(A,I) \underset{lagN}{\sim} (A',I')$ *implies* $(A,I) \underset{lagL}{\sim} (A',I')$ *for all* $L \geq N$.

(ii) $(A,I) \underset{lagN}{\sim} (A',I')$ *and* $(A',I') \underset{lagN'}{\sim} (A'',I'')$ *implies* $(A,I) \underset{lagN+N'}{\sim} (A'',I'')$. *Hence shift equivalence is an equivalence relation on nonnegative matrix systems.*

Similarly to Theorem 6.2, we have

Proposition 7.7. *For nonnegative matrix systems, strong shift equivalence in N-step implies shift equivalence of lag N.*

As in the case of strong shift equivalence, we may describe the matrix relations appearing in the formulation of shift equivalence in terms of single homomorphisms between inductive limits of abelian groups.

Proposition 7.8. *Two nonnegative matrix systems (A,I) and $(A'I')$ are shift equivalent of lag N if and only if there exist order preserving finite homomorphisms of lag N: $\xi : \mathbb{Z}_{I^t} \to \mathbb{Z}_{I'^t}$ and $\eta : \mathbb{Z}_{I'^t} \to \mathbb{Z}_{I^t}$ such that*

$$\lambda_{(A',I')} \circ \xi = \xi \circ \lambda_{(A,I)}, \qquad \lambda_{(A,I)} \circ \eta = \eta \circ \lambda_{(A',I')}$$

and

$$\eta \circ \xi = \lambda^N_{(A,I)}, \qquad \xi \circ \eta = \lambda^N_{(A',I')}.$$

Let $(\mathcal{M},I),(\mathcal{M}',I')$ be symbolic matrix systems and $(M,I),(M',I')$ be their supports respectively. The following proposition is direct.

Proposition 7.9.

(i) $(\mathcal{M},I) \underset{n-st}{\approx} (\mathcal{M}',I')$ *implies* $(M,I) \underset{n-st}{\approx} (M',I')$.

(ii) $(\mathcal{M},I) \underset{lagN}{\sim} (\mathcal{M}',I')$ *implies* $(M,I) \underset{lagN}{\sim} (M',I')$.

8. Dimension groups

In this section, we will introduce the notions of dimension group and dimension triple for nonnegative matrix systems that is shown to be a shift equivalence invariant. It is a generalization of the notions of dimension group and dimension triple for nonnegative matrices defined by W. Krieger in [Kr], [Kr2]. The Krieger's idea to define dimension groups for nonnegative matrices is based on the K-theory for C^*-algebras (cf.[Ef]). The author considered the dimension groups for subshifts by using K_0-groups for certain C^*-algebras associated with subshifts as in [Ma2],[Ma3]. It is a generalization of the original idea of Krieger. We will in this section formulate the dimension groups and the dimension triples for nonnegative matrix systems.

Let (A,I) be a nonnegative matrix system. Recall that $\mathbb{Z}_{I^t}$ denotes the ordered group of the inductive limit of the sequence of the ordered abelian groups

$\mathbb{Z}^{m(l)}, l \in \mathbb{N}$ through the transposed matrices $I^t_{l,l+1}, l \in \mathbb{N}$. As seen in the previous discussion, the sequence of the transposed matrices $A^t_{l,l+1}$ naturally induces an order preserving endomorphism on the ordered group $\mathbb{Z}_{I^t}$ that is denoted by $\lambda_{(A,I)}$. We set $\mathbb{Z}_{I^t}(k) = \mathbb{Z}_{I^t}$ and $\mathbb{Z}^+_{I^t}(k) = \mathbb{Z}^+_{I^t}$ for $k \in \mathbb{N}$. We define an abelian group and its positive cone by the following inductive limits:

$$\Delta_{(A,I)} = \varinjlim_k \{\lambda_{(A,I)} : \mathbb{Z}_{I^t}(k) \to \mathbb{Z}_{I^t}(k+1)\},$$
$$\Delta^+_{(A,I)} = \varinjlim_k \{\lambda_{(A,I)} : \mathbb{Z}^+_{I^t}(k) \to \mathbb{Z}^+_{I^t}(k+1)\}.$$

We call the ordered group $(\Delta_{(A,I)}, \Delta^+_{(A,I)})$ the *dimension group for* (A,I). Since the map $\delta_{(A,I)} : \mathbb{Z}_{I^t}(k) \to \mathbb{Z}_{I^t}(k+1)$ defined by $\delta_{(A,I)}([X,k]) = ([X,k+1])$ for $X \in \mathbb{Z}_{I^t}$ yields an automorphism on $\Delta_{(A,I)}$ that preserves the positive cone $\Delta^+_{(A,I)}$. We also denote it by $\delta_{(A,I)}$ and call it the *dimension automorphism.* We call the triple $(\Delta_{(A,I)}, \Delta^+_{(A,I)}, \delta_{(A,I)})$ the *dimension triple for* (A,I) and the pair $(\Delta_{(A,I)}, \delta_{(A,I)})$ the *dimension pair for* (A,I).

PROPOSITION 8.1. *If two nonnegative matrix systems are shift equivalent, their dimension triples are isomorphic.*

Proof. Suppose that two nonnegative matrix systems (A,I) and (A',I') are shift equivalent of lag N. By Proposition 7.8, there exist order preserving finite homomorphisms $\xi : \mathbb{Z}_{I^t} \to \mathbb{Z}_{I'^t}$ and $\eta : \mathbb{Z}_{I'^t} \to \mathbb{Z}_{I^t}$ of lag N such that

$$\lambda_{(A',I')} \circ \xi = \xi \circ \lambda_{(A,I)}, \qquad \lambda_{(A,I)} \circ \eta = \eta \circ \lambda_{(A',I')}$$

and

$$\eta \circ \xi = \lambda^N_{(A,I)}, \qquad \xi \circ \eta = \lambda^N_{(A',I')}.$$

Define the maps $\Phi_\xi : \mathbb{Z}_{I^t}(k) \to \mathbb{Z}_{I'^t}(k)$ and $\Phi_\eta : \mathbb{Z}_{I'^t}(k) \to \mathbb{Z}_{I^t}(k)$ as $\Phi_\xi([X,k]) = ([\xi(X),k])$ and $\Phi_\eta([Y,k]) = ([\eta(Y),k])$ for $X \in \mathbb{Z}_{I^t}$, $Y \in \mathbb{Z}_{I'^t}$. It is easy to see that they induce homomorphisms from $\Delta_{(A,I)}$ to $\Delta_{(A',I')}$ and $\Delta_{(A',I')}$ to $\Delta_{(A,I)}$ respectively. We still denote them by Φ_ξ and Φ_η respectively. Since the homomorphisms ξ, η are order preserving, the maps Φ_ξ, Φ_η also preserve order structures of the dimension groups. It then follows that

$$\delta_{(A,I)} \circ \Phi_\eta = \Phi_\eta \circ \delta_{(A'I')}, \qquad \delta_{(A',I')} \circ \Phi_\xi = \Phi_\xi \circ \delta_{(A,I)}$$

and

$$\Phi_\eta \circ \Phi_\xi = \delta^{-N}_{(A,I)}, \qquad \Phi_\xi \circ \Phi_\eta = \delta^{-N}_{(A',I')}.$$

Therefore we see that the both maps Φ_ξ and Φ_η are isomorphisms and the corresponding dimension triples are isomorphic.

In particular we have (cf.[BK])

Proposition 8.2. *Two sofic nonnegative matrix systems are shift equivalent if and only if their dimension triples are isomorphic. Thus the dimension triple are complete invariants for shift equivalence of sofic nonnegative matrix systems.*

Proof. The only if part is from the preceding proposition. By a similar discussion to [Kr],[Kr2], we obtain the if part of the assertion.

We will define the dimension triples for symbolic matrix systems as the dimension triples for their supports. Namely let $(\mathcal{M}, I)$ be a symbolic matrix system and (M, I) its support. Then the dimension triple $(\Delta_{(\mathcal{M},I)}, \Delta^+_{(\mathcal{M},I)}, \delta_{(\mathcal{M},I)})$ is defined to be the dimension triple $(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)})$. We may also define dimension triples for subshifts as the dimension triple for their canonical symbolic matrix systems. Let Λ be a subshift and $(\mathcal{M}, I)$ its canonical symbolic matrix system for Λ. Then the *future dimension triple* $(\Delta_\Lambda, \Delta^+_\Lambda, \delta_\Lambda)$ for subshift Λ is defined to be the dimension triple $(\Delta_{(\mathcal{M},I)}, \Delta^+_{(\mathcal{M},I)}, \delta_{(\mathcal{M},I)})$. The *past dimension triple* for Λ is defined as the future dimension triple for the transposed subshift Λ^T for Λ.
Thus we have

Proposition 8.3. *The future dimension triples for subshifts are shift equivalence invariants and in particular topological conjugacy invariants.*

The notion of dimension pair $(\Delta_\Lambda, \delta_\Lambda)$ for subshifts has been also seen in [Le].

9. K-groups and Bowen-Franks groups

The Bowen-Franks groups for nonnegative matrices and hence for topological Markov shifts have been introduced by R. Bowen and J. Franks in [BF]. For an $n \times n$ nonnegative square matrix A, its Bowen-Franks group $BF(A)$ is defined by the group $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$. This group has discovered in a study of suspension flows of topological Markov shifts by Bowen and Franks (cf. [PS]). They showed that the groups are not only invariants under shift equivalence but also almost complete invariants under flow equivalence between nonnegative matrices.
We will in this section introduce and study the notion of Bowen-Franks groups for nonnegative matrix systems as a generalization of the original Bowen-Franks groups for nonnegative matrices. Our Bowen-Franks groups for a nonnegative matrix system consist of a pair of abelian groups. One corresponds to a generalization of the original Bowen-Franks group, called the Bowen-Franks group of degree zero, and the other one corresponds to its suspension, called the Bowen-Franks group of degree one. For matrices, the latter group is the torsion-free part of the original Bowen-Franks group. But in general nonnegative matrix systems the group of degree one is not necessarily the torsion-free part of the group of degree zero (see Section 10).
Before going to definition of the Bowen-Franks groups for nonnegative matrix systems, we introduce two abelian groups for nonnegative matrix systems, called K-groups, that will be proved to be invariant under shift equivalence.

Let (A, I) be a nonnegative matrix system. For $l \in \mathbb{N}$, we set the abelian groups

$$
\begin{aligned}
K_0^l(A,I) &= \mathbb{Z}^{m(l+1)}/(I_{l,l+1}^t - A_{l,l+1}^t)\mathbb{Z}^{m(l)},\\
K_1^l(A,I) &= \operatorname{Ker}(I_{l,l+1}^t - A_{l,l+1}^t) \text{ in } \mathbb{Z}^{m(l)}.
\end{aligned}
$$

Lemma 9.1. *The map* $I_{l,l+1}^t : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}$ *naturally induces homomorphisms between the following groups:*

$$
i_*^l : K_*^l(A,I) \to K_*^{l+1}(A,I) \qquad \textit{for} \quad * = 0, 1.
$$

The proof is straightforward by using the relations

$$
I_{l,l+1}A_{l+1,l+2} = A_{l,l+1}I_{l+1,l+2}.
$$

We now define the K-groups for nonnegative matrix system (A, I).

Definition. The *K-groups for* (A, I) are defined as the following inductive limits of the abelian groups:

$$
\begin{aligned}
K_0(A,I) &= \varinjlim_{l}\{i_0^l : K_0^l(A,I) \to K_0^{l+1}(A,I)\},\\
K_1(A,I) &= \varinjlim_{l}\{i_1^l : K_1^l(A,I) \to K_1^{l+1}(A,I)\}.
\end{aligned}
$$

For a symbolic matrix system $(\mathcal{M}, I)$, its K-groups $K_0(\mathcal{M}, I), K_1(\mathcal{M}, I)$ are defined to be the K-groups for the associated nonnegative matrix systems. It is easy to see that the groups $K_*(A, I)$ are also represented as in the following way

Proposition 9.2.

(i) $K_0(A,I) = \mathbb{Z}_{I^t}/(id - \lambda_{(A,I)})\mathbb{Z}_{I^t}$,
(ii) $K_1(A,I) = \operatorname{Ker}(id - \lambda_{(A,I)})$ *in* $\mathbb{Z}_{I^t}$.

We will see that the groups $K_*(A, I)$ are invariant under shift equivalence.

Lemma 9.3.

(i) $K_0(A,I) = \Delta_{(A,I)}/(id - \delta_{(A,I)})\Delta_{(A,I)}$,
(ii) $K_1(A,I) = \operatorname{Ker}(id - \delta_{(A,I)})$ *in* $\Delta_{(A,I)}$.

Proof. As the automorphism $\delta_{(A,I)}$ is given by $\lambda_{(A,I)} = \{A_{l,l+1}^t\}$ on $\Delta_{(A,I)}$, the assertions are easily proved.

Since the dimension triple $(\Delta_{(A,I)}, \Delta_{(A,I)}^+, \delta_{(A,I)})$ is invariant under shift equivalence of nonnegative matrix systems, we thus have

Proposition 9.4. *The groups* $K_i(A, I), i = 0, 1$ *are invariant under shift equivalence of nonnegative matrix systems.*

Set the abelian group

$$\mathbb{Z}_I = \varprojlim_l \{I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}\}$$

the projective limit of the system: $I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}, l \in \mathbb{N}$. The sequence $A_{l,l+1}, l \in \mathbb{N}$ naturally acts on $\mathbb{Z}_I$ as an endomorphism that we denote by A. The identity on $\mathbb{Z}_I$ is denoted by I. We now define the Bowen-Franks groups for (A, I) as follows:
Definition. For a nonnegative matrix system (A, I),

$$BF^0(A, I) = \mathbb{Z}_I/(I - A)\mathbb{Z}_I, \qquad BF^1(A, I) = \mathrm{Ker}(I - A) \text{ in } \mathbb{Z}_I.$$

We call $BF^0(A, I)$ the Bowen-Franks group for (A, I) of degree zero and $BF^1(A, I)$ the Bowen-Franks group for (A, I) of degree one. We see

Theorem 9.5. *The Bowen-Franks groups* $BF^i(A, I), i = 0, 1$ *are invariant under shift equivalence of nonnegative matrix systems.*

Proof. (i) Suppose that two nonnegative matrix systems (A, I) and (A', I') are shift equivalent of lag N. Let H_l, K_l be sequences of nonnegative matrices such that $(H, K) : (A, I) \underset{lagN}{\sim} (A', I')$. For $(x_i)_{i\in\mathbb{N}} \in \mathbb{Z}_I$, put $\Phi_K((x_i)_{i\in\mathbb{N}}) = (K_i(x_{N+i})_{i\in\mathbb{N}})$. It is easy to see that the Φ_K gives rise to a homomorphism from $\mathbb{Z}_I$ to $\mathbb{Z}_{I'}$. As we see the equality: $K_i \circ (I_{N+i,N+i+1} - A_{N+i,N+i+1}) = (I'_{i,i+1} - A'_{i,i+1}) \circ K_{i+1}$, the homomorphism induces a homomorphism from $\mathbb{Z}_I/(I - A)\mathbb{Z}_I$ to $\mathbb{Z}_{I'}/(I' - A')\mathbb{Z}_{I'}$. We denote it by $\bar{\Phi}_K$. We similarly have a homomorphism $\bar{\Phi}_H$ from $\mathbb{Z}_{I'}/(I' - A')\mathbb{Z}_{I'}$ to $\mathbb{Z}_I/(I - A)\mathbb{Z}_I$. Since we have $\Phi_H \circ \Phi_K = A^N$ on $\mathbb{Z}_I$ and $\Phi_K \circ \Phi_H = {A'}^N$ on $\mathbb{Z}_{I'}$, the homomorphisms $\bar{\Phi}_H$ and $\bar{\Phi}_K$ are inverses each other.
(ii) It is direct to see that the homomorphisms Φ_H and Φ_K induce isomorphisms between $\mathrm{Ker}(I - A)$ in $\mathbb{Z}_I$ and $\mathrm{Ker}(I' - A')$ in $\mathbb{Z}_{I'}$.

We will prove the following Universal Coefficient Theorem. It says that the Bowen-Franks groups are determined by the K-groups.

Theorem 9.6.

(i) *There exists a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A, I), \mathbb{Z}) \xrightarrow{\delta} BF^0(A, I) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(K_1(A, I), \mathbb{Z}) \longrightarrow 0$$

that splits unnaturally.

(ii)
$$BF^1(A, I) \cong \mathrm{Hom}_{\mathbb{Z}}(K_0(A, I), \mathbb{Z}).$$

In the theorem above, $\mathrm{Ext}^1_{\mathbb{Z}}$ is the derived functor of the Hom-functor in homological algebra. The formulations above come from the Universal Coefficient Theorem for K-theory of the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ ([Ma4]). General framework of the Universal Coefficient Theorem for K-theory of C^*-algebras have been proved in [Bro], [RS]. If an abelian group G is finitely generated, it is well known that

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{Z}) &= \text{ The torsion-free part of } G,\\ \mathrm{Ext}^1_{\mathbb{Z}}(G,\mathbb{Z}) &= \text{ The torsion part of } G.\end{aligned}$$

We provide some lemmas to prove Theorem 9.6.

LEMMA 9.7. $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}) = 0$.

Proof. It suffices to show that an extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \xrightarrow{\rho} \mathbb{Z}_{I^t} \longrightarrow 0$$

of abelian groups splits. For each $l \in \mathbb{N}$, let ι_l be the canonical inclusion of $\mathbb{Z}^{m(l)}$ into $\mathbb{Z}_{I^t}$. We will choose homomorphisms $\varphi_l : \mathbb{Z}^{m(l)} \to G$ such that

$$\rho \circ \varphi_l = \iota_l, \qquad \varphi_{l+1} \circ I^t_{l,l+1} = \varphi_l$$

as follows: Let $e^l_i, i = 1,\dots,m(l)$ be the standard basis of $\mathbb{Z}^{m(l)}$. We first take homomorphisms $\phi_l : \mathbb{Z}^{m(l)} \to G$ such that $\rho \circ \phi_l = \iota_l$ for $l \in \mathbb{N}$. Put $\varphi_1 = \phi_1$. Since we see $\rho((\phi_2 \circ I^t_{1,2} - \varphi_1)(e^1_i)) = 0$, we may regard the element $\phi_2 \circ I^t_{1,2}(e^1_i) - \varphi_1(e^1_i)$ as an integer m^1_i. For each $i = 1,\dots,m(1)$, take $r_i = 1,\dots,m(2)$ such that $I_{1,2}(i,r_i) = 1$. We set

$$\varphi_2(e^2_j) = \begin{cases} \phi_2(e^2_j) - m^1_i & \text{if } j = r_i\\ \phi_2(e^2_j) & \text{otherwise.}\end{cases}$$

Then it is easy to see that

$$\rho \circ \varphi_2 = \iota_2, \qquad \varphi_2 \circ I^t_{1,2} = \varphi_1.$$

By continuing these procedures, we can find a sequence of homomorphisms $\varphi_l, l \in \mathbb{N}$ that have the desired property. They give rise to a homomorphism $\varphi : \mathbb{Z}_{I^t} \to G$ such that $\rho \circ \varphi = id$.

LEMMA 9.8.
(i) $\mathrm{Ext}^1_{\mathbb{Z}}((id - \lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) = 0$.
(ii) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{Ker}(id - \lambda_{(A,I)})$ *in* $\mathbb{Z}_{I^t},\mathbb{Z}) = 0$.

Proof. Regard $(id-\lambda_{(A,I)})\mathbb{Z}_{I^t}$ and $\mathrm{Ker}(id-\lambda_{(A,I)})$ in $\mathbb{Z}_{I^t}$ as subgroups of $\mathbb{Z}_{I^t}$. Consider the following short exact sequences:

(9.1)
$$0 \longrightarrow (id-\lambda_{(A,I)})\mathbb{Z}_{I^t} \xrightarrow{\iota} \mathbb{Z}_{I^t} \xrightarrow{q} \mathbb{Z}_{I^t}/(id-\lambda_{(A,I)})\mathbb{Z}_{I^t} \longrightarrow 0,$$
(9.2)
$$0 \longrightarrow \mathrm{Ker}(id-\lambda_{(A,I)}) \xrightarrow{j} \mathbb{Z}_{I^t} \xrightarrow{id-\lambda_{(A,I)}} (id-\lambda_{(A,I)})\mathbb{Z}_{I^t} \longrightarrow 0$$

of abelian groups. They yield the following exact sequences respectively:

$$\begin{aligned}\cdots \longrightarrow &\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \\ &\longrightarrow \mathrm{Ext}^2_{\mathbb{Z}}(\mathbb{Z}_{I^t}/(id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \cdots\end{aligned}$$

and

$$\begin{aligned}\cdots \longrightarrow &\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{Ker}(id-\lambda_{(A,I)}),\mathbb{Z}) \\ &\longrightarrow \mathrm{Ext}^2_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \cdots\end{aligned}$$

As $\mathrm{Ext}^2_{\mathbb{Z}} = 0$, we have

$$\mathrm{Ext}^1_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{Ker}(id-\lambda_{(A,I)}),\mathbb{Z}) = 0$$

by the preceding lemma.

Lemma 9.9.

(i)
$$\begin{aligned}&\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{I^t}/(id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z})/\iota^*\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}).\end{aligned}$$

(ii)
$$\begin{aligned}&\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ker}(id-\lambda_{(A,I)}),\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})/(id-\lambda_{(A,I)})^*\mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}).\end{aligned}$$

Proof. The short exact sequences (9.1) and (9.2) make the following sequences exact:

(9.3)
$$\begin{aligned}0 \longrightarrow &\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t}/(id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \\ &\xrightarrow{q^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}) \xrightarrow{\iota^*} \mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \\ &\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{I^t}/(id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \cdots,\end{aligned}$$
(9.4)
$$\begin{aligned}0 \longrightarrow &\mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \xrightarrow{(id-\lambda_{(A,I)})^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}) \\ &\xrightarrow{j^*} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ker}(id-\lambda_{(A,I)}),\mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \cdots.\end{aligned}$$

Hence we get the desired isomorphisms.

Proof of Theorem 9.6. (i) By Proposition 9.2 and the previous lemmas, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(K_1(A,I),\mathbb{Z}) \cong& \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})/(id-\lambda_{(A,I)})^*\mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}),\\ \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A,I),\mathbb{Z}) \cong& \mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z})/\iota^*\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}).\end{aligned}$$

The exact sequence (9.4) says the map

$$(id-\lambda_{(A,I)})^* : \mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})$$

is injective. Hence we know that the group

$$\mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z})/\iota^*\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})$$

is isomorphic to the group

$$(id-\lambda_{(A,I)})^*\mathrm{Hom}_{\mathbb{Z}}(id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z})/(id-\lambda_{(A,I)})^*\iota^*\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}).$$

The map

$$(id-\lambda_{(A,I)})^*\iota^* : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})$$

is naturally regarded as the endomorphism

$$I - A : \mathbb{Z}_I \to \mathbb{Z}_I$$

through a natural identification between $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})$ and $\mathbb{Z}_I$. As there exists an short exact sequence

$$\begin{aligned}0 \longrightarrow& (id-\lambda_{(A,I)})^*\mathrm{Hom}_{\mathbb{Z}}(id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z})/(id-\lambda_{(A,I)})^*\iota^*\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})\\ \longrightarrow& \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})/(id-\lambda_{(A,I)})^*\iota^*\mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z})\\ \longrightarrow& \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z})/(id-\lambda_{(A,I)})^*\mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z})\\ \longrightarrow& 0,\end{aligned}$$

we obtain a short exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A,I),\mathbb{Z}) \xrightarrow{\delta} \mathbb{Z}_I/(I-A)\mathbb{Z}_I \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(K_1(A,I),\mathbb{Z}) \longrightarrow 0.$$

The short exact sequence above splits unnaturally, since the group $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mathbb{Z})$ is algebraically compact and the group $\mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Z})$ is torsion-free for any abelian groups G, H (cf. [KKS]).
(ii) By the exact sequence (9.3), we see

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(K_0(A,I),\mathbb{Z}) \cong& \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t}/(id-\lambda_{(A,I)})\mathbb{Z}_{I^t},),\mathbb{Z})\\ \cong& \mathrm{Ker}\ \iota^* : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t},\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}((id-\lambda_{(A,I)})\mathbb{Z}_{I^t},\mathbb{Z})\end{aligned}$$

By a natural identification between $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t}, \mathbb{Z})$ and $\mathbb{Z}_I$, we obtain Ker ι^* : $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{I^t}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}((id - \lambda_{(A,I)})\mathbb{Z}_{I^t}, \mathbb{Z})$ is regarded as $\mathrm{Ker}(I - A)$ in $\mathbb{Z}_I$. Thus we end the proof of the theorem.

Remark. Lemma 9.8 (ii) means $\mathrm{Ext}^1_{\mathbb{Z}}(K_1(A,I), \mathbb{Z}) = 0$. Hence the following short exact sequence clearly holds by Theorem 9.6 (ii):

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_1(A,I), \mathbb{Z}) \xrightarrow{\delta} BF^1(A,I) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(K_0(A,I), \mathbb{Z}) \longrightarrow 0.$$

Example. Let M be an $n \times n$ nonnegative matrix . Put for each $l \in \mathbb{N}$

$$A_{l,l+1} = M, \qquad I_{l,l+1} = \text{ the } n \times n \text{ identity matrix.}$$

Then (A, I) is a nonnegative matrix system. The K-groups are

$$K_0(A,I) = \mathbb{Z}^n/(1 - M^t)\mathbb{Z}^n, \qquad K_1((A,I) = \mathrm{Ker}(1 - M^t) \text{ in } \mathbb{Z}^n.$$

The Bowen-Franks groups are

$$BF^0(A,I) = \mathbb{Z}^n/(1 - M)\mathbb{Z}^n, \qquad BF^1((A,I) = \mathrm{Ker}(1 - M) \text{ in } \mathbb{Z}^n.$$

Hence we have

$$K_0(A,I) \cong BF^0(A,I) = BF(M) : \text{the original Bowen-Franks group for } M,$$
$$K_1(A,I) \cong BF^1(A,I) = \text{ the torsion-free part of } BF(M)$$

We will next define K-groups and Bowen-Franks groups for subshifts.
Definition. For a subshift Λ, let (A_Λ, I_Λ) be the canonical nonnegative matrix system associated with Λ. We define

$$K_i(\Lambda) = K_i(A_\Lambda, I_\Lambda), i = 0, 1 \quad : \text{the K-groups for } \Lambda$$
$$BF^i(\Lambda) = BF^i(A_\Lambda, I_\Lambda), i = 0, 1 \quad : \text{the Bowen-Franks groups for } \Lambda$$

We thus have

Theorem 9.10. *The K-groups $K_i(\Lambda)$ and the Bowen-Franks groups $BF^i(\Lambda)$ for subshift Λ are abelian groups that are invariant under shift equivalence of subshifts. In particular, they are topological conjugacy invariants of subshifts.*

Proposition 9.11. *Let Λ be a sofic subshift. We denote by $m(\Lambda)$ the cardinality of the vertices of the left Krieger cover graph for Λ and A_Λ its adjacency matrix. Then we have*

$$BF^0(\Lambda) = \mathbb{Z}^{m(\Lambda)}/(1 - A_\Lambda)\mathbb{Z}^{m(\Lambda)}, \qquad BF^1(\Lambda) = \mathrm{Ker}(1 - A_\Lambda) \textit{ in } \mathbb{Z}^{m(\Lambda)}.$$

Proof. As we see

$$K_0(\Lambda) = \mathbb{Z}^{m(\Lambda)}/(1 - A_\Lambda)\mathbb{Z}^{m(\Lambda)}, \qquad K_1(\Lambda) = \mathrm{Ker}(1 - A_\Lambda) \text{ in } \mathbb{Z}^{m(\Lambda)},$$

the assertion is clear.

In the final section, we will see an example of a nonsofic subshift Λ for which $BF^1(\Lambda)$ is no longer the torsion-free part of the group $BF^0(\Lambda)$.

Remark. In [Ma], the author introduced the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ as a generalization of the construction of the Cuntz-Krieger algebra $\mathcal{O}_A$ associated with the topological Markov shift Λ_A determined by a matrix A with entries in $\{0, 1\}$. Cuntz-Krieger proved in [CK] that the Ext-group $\mathrm{Ext}(\mathcal{O}_A)$ of the C^*-algebra $\mathcal{O}_A$ is $\mathbb{Z}^n/(1 - A)\mathbb{Z}^n$: the Bowen-Franks group of the matrix A. The author in [Ma4] generalized the notion of the Bowen-Franks group to the subshifts as:

$$BF(\Lambda) := \mathrm{Ext}(\mathcal{O}_\Lambda).$$

From the view point of the K-theory for C^*-algebras, the invariants $K_i, BF^i, i = 0, 1$ introduced in this section appear as

$$K_0(\Lambda) = K_0(\mathcal{O}_\Lambda), \qquad K_1(\Lambda) = K_1(\mathcal{O}_\Lambda)$$

and

$$BF^0(\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda), \qquad BF^1(\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda \otimes C_0(\mathbb{R})).$$

The formulations in Theorem 9.6 come from the Universal Coefficients Theorem for C^*-algebras ([Bro], [RS]).

As the K-groups and the Ext-groups for C^*-algebras are stably isomorphic invariant and the stable isomorphism class of the C^*-algebra $\mathcal{O}_\Lambda$ with gauge action is invariant under topological conjugacy class of subshifts ([Ma5]), we know that the dimension triple, the K-groups and the Bowen-Franks groups for subshifts are topological conjugacy invariants without using discussions of this paper under some mild conditions for subshifts.

The Bowen-Franks group for nonnegative matrix was first invented for use as an invariant of flow equivalence of the associated topological Markov shift rather than topological conjugacy ([BF],[Fr],[PS]). We can prove that the K-groups $K_*(\Lambda)$ and hence the Bowen-Franks groups $BF^*(\Lambda)$ for subshift are also invariant under flow equivalence of subshift by using a result of Parry-Sullivan [PS]. The proof, that we do not give in this paper, will appear in a forthcoming paper (cf.[Ma4],[Ma5]).

We will finally present another candidate of Bowen-Franks groups for subshifts. For a topological Markov shift Λ_A determined by an $n \times n$ matrix A with entries in $\{0, 1\}$, the group $BF(\Lambda_A)$ is isomorphic to the K_0-group for the subshift Λ_{A^t} determined by the transpose of the matrix A. The subshift Λ_{A^t} is the transpose Λ_A^T of Λ_A as a subshift. From this point of view, it seems to be one way to define the Bowen-Franks group for canonical symbolic matrix systems as the K-groups for their transpose.

Let $(\mathcal{M}, I)$ be a canonical symbolic matrix system and $\Lambda_{(\mathcal{M},I)}$ the associated subshift. We define the transpose $(\mathcal{M}^T, I^T)$ of $(\mathcal{M}, I)$ as the canonical symbolic matrix system for the transpose $\Lambda^T_{(\mathcal{M},I)}$ of the subshift $\Lambda_{(\mathcal{M},I)}$. We will define another pair of Bowen-Franks groups as in the following way.

Definition. For a canonical symbolic matrix system $(\mathcal{M}, I)$, we define

$$BF^i_K(\mathcal{M}, I) = K_i(\mathcal{M}^T, I^T), \qquad i = 0, 1$$

where $K_i(\mathcal{M}^T, I^T)$ is defined as the K_i-groups for the nonnegative matrix system associated with $(\mathcal{M}^T, I^T)$. We call them the *Bowen-Franks groups from K for* $(\mathcal{M}, I)$. For a subshift Λ, let $(\mathcal{M}, I)$ be its canonical symbolic matrix system. We will then define Bowen-Franks groups (from K) for subshift as follows:

$$BF^i_K(\Lambda) = BF^i_K(\mathcal{M}, I), \qquad i = 0, 1.$$

We thus have

Proposition 9.12. *The Bowen-Franks groups $BF^i_K(\Lambda), i = 0, 1$ from K for subshift Λ are topological conjugacy invariants of subshifts.*

Proof. Suppose that two subshifts Λ, Λ' are topologically conjugate. We denote by $(\mathcal{M}, I), (\mathcal{M}', I')$ their canonical symbolic matrix systems respectively. Hence their transposed subshifts Λ^T, Λ'^T are topologically conjugate so that their canonical symbolic matrix systems $(\mathcal{M}^T, I^T), (\mathcal{M}'^T, I'^T)$ are strong shift equivalent and hence shift equivalent. As their corresponding nonnegative matrix systems $(M^T, I^T), (M'^T, I'^T)$ are shift equivalent, we have $K_i(M^T, I^T) = K_i(M'^T, I'^T)$ for $i = 0, 1$.

Proposition 9.13. *For a topological Markov shift Λ_A determined by an $n \times n$ square matrix A with entries in $\{0, 1\}$, we have*

$$BF^0_K(\Lambda_A) = \mathbb{Z}^n/(1-A)\mathbb{Z}^n = BF(\Lambda_A), \quad BF^1_K(\Lambda_A) = \mathrm{Ker}(1-A) \text{ in } \mathbb{Z}^n.$$

Hence the group $BF^1_K(\Lambda)$ is the torsion-free part of the group $BF^0_K(\Lambda)$.

We will finally present the calculation formulae for the Bowen-Franks groups from K. For a subshift Λ, let X^-_Λ be the set of all left-infinite sequences appearing in Λ. That is

$$X^-_\Lambda = \{(..., z_{-2}, z_{-1}, z_0) \in \prod_{i=-\infty}^{0} \Sigma_i | (z_i)_{i\in\mathbb{Z}} \in \Lambda\}.$$

We will define l-future equivalence in the space X^-_Λ in a symmetric way to the previous l-past equivalence. Namely, for $z \in X^-_\Lambda$ and $l \in \mathbb{N}$, put

$$\Lambda^{-l}(z) = \{\mu \in \Lambda^l | z\mu \in X^-_\Lambda\}.$$

Two points $z, w \in X_\Lambda^-$ are said to be *l-future equivalent* if $\Lambda^{-l}(z) = \Lambda^{-l}(w)$. We write this equivalence as $x \sim_{-l} y$. For a fixed $l \in \mathbb{N}$, let $P_i^l, i = 1, 2, \dots, n(l)$ be the set of all l-future equivalence classes of X_Λ^-. We define two rectangular $n(l) \times n(l+1)$ matrices $J_{l,l+1}, B_{l,l+1}$ with entries in $\{0,1\}$ and entries in nonnegative integers similarly to the matrix $I_{l,l+1}, A_{l,l+1}$. Namely, we define

$$J_{l,l+1} \text{ for } \Lambda = I_{l,l+1} \text{ for } \Lambda^T, \qquad B_{l,l+1} \text{ for } \Lambda = A_{l,l+1} \text{ for } \Lambda^T.$$

By [Ma2;Theorem 4.9], we have

THEOREM 9.14.

(i) $BF_K^0(\Lambda) = \varinjlim_l \{J_{l,l+1}^t : \mathbb{Z}^{n(l)}/(J_{l,l+1}^t - B_{l,l+1}^t)\mathbb{Z}^{n(l)}\}$.

(ii) $BF_K^1(\Lambda) = \varinjlim_l \{J_{l,l+1}^t : \mathrm{Ker}(J_{l,l+1}^t - B_{l,l+1}^t) \textit{ in } \mathbb{Z}^{n(l)}\}$.

We similarly obtain by Lemma 9.2,

THEOREM 9.15. *The past dimension pair $(\Delta_{\Lambda^T}, \delta_{\Lambda^T})$ for subshift Λ determines the Bowen-Franks group $BF_K^i(\Lambda), i = 0, 1$ from K for Λ,*

10. SPECTRUM

It is well-known that the set of all nonzero eigenvalues of a nonnegative matrix M is a shift equivalence invariant. The set of M is called the nonzero spectrum of M and plays an important rôle for studying dynamical properties of the associated topological Markov shift (cf.[LM],[Ki]). In this section, we introduce the notion of spectrum of nonnegative matrix system (A, I). It is an eigenvalue of (A, I) in the sense stated bellow. We denote by $Sp(A, I)$ the set of all eigenvalues of (A, I). As the sequence of the sizes of matrices $A_{l,l+1}, I_{l,l+1}, l \in \mathbb{N}$ are increasing, it seems to be natural to deal with eigenvalues of (A, I) with a certain boundedness condition defined bellow on the corresponding eigenvectors. We denote by $Sp_b(A, I)$ the set of all eigenvalues of (A, I) with the boundedness condition on the corresponding eigenvectors. We will prove that the both of the sets of nonzero spectrum of $Sp(A, I)$ and $Sp_b(A, I)$ are invariant under shift equivalence of (A, I).

We fix a nonnegative matrix system (A, I) throughout this section.

DEFINITION. A sequence $\{v^l\}_{l\in\mathbb{N}}$ of vectors $v^l = (v_1^l, \dots, v_{m(l)}^l) \in \mathbb{C}^{m(l)}, l \in \mathbb{N}$ is called an *I-compatible vector* if it satisfies the conditions:

$$v^l = I_{l,l+1} v^{l+1} \qquad \text{for all} \quad l \in \mathbb{N}. \tag{10.1}$$

An I-compatible vector $\{v^l\}_{l\in\mathbb{N}}$ is said to be nonzero if v^l is a nonzero vector for some l. If $v_i^l \geq 0$ (resp. $v_i^l > 0$) for all $i = 1, \dots, m(l)$ and $l \in \mathbb{N}$, $\{v^l\}_{l\in\mathbb{N}}$ is said to be nonnegative (resp. positive). If there exists a number M such that $\sum_{i=1}^{m(l)} |v_i^l| \leq M$ for all $l \in \mathbb{N}$, $\{v^l\}_{l\in\mathbb{N}}$ is said to be bounded. We remark that, for an I-compatible vector $\{v^l\}_{l\in\mathbb{N}}$, $v^N \neq 0$ for some N implies $v^l \neq 0$ for all $l \geq N$.

Definition. For a complex number β, a nonzero I-compatible vector $\{v^l\}$ is called an *eigenvector* of (A, I) for *eigenvalue* β if it satisfies the conditions:

$$A_{l,l+1}v^{l+1} = \beta v^l \qquad \text{for all} \quad l \in \mathbb{N}. \tag{10.2}$$

An eigenvalue β is said to be bounded if it is an eigenvalue for a bounded eigenvector.
Remark. If a sequence v^l of vectors satisfies the above conditions (10.1),(10.2) for $l = N, N+1, \ldots$ for some N, we may extendedly define vectors v^l for $l = 1, \ldots, N-1$ for which $\{v^l\}_{l\in\mathbb{N}}$ satisfy the conditions (10.1),(10.2) for all $l \in \mathbb{N}$ by using the condition (10.1).
Definition. Let $Sp^\times(A, I)$ be the set of all nonzero eigenvalues of (A, I) and $Sp_b^\times(A, I)$ the set of all nonzero bounded eigenvalues of (A, I). We call them the *nonzero spectrum* of (A, I) and the *nonzero bounded spectrum* of (A, I) respectively.
We will prove

Theorem 10.1. *If two nonnegative matrix systems are shift equivalent, their nonzero spectrum coincide.*

Proof. Suppose that two nonnegative matrix systems (A, I) and (A', I') are shift equivalent of lag N. Let H_l, K_l be sequences of nonnegative matrices such that $(H, K) : (A, I) \underset{lagN}{\sim} (A', I')$. We will show $Sp^\times(A, I) \subset Sp^\times(A', I')$. For $\beta \in Sp^\times(A, I)$ with nonzero eigenvector v^l, we set $u^l = K_l v^{l+N}$ for $l \in \mathbb{N}$. It is direct to see that

$$u^l = I'_{l,l+1}u^{l+1}, \qquad A'_{l,l+1}u^{l+1} = \beta u^l.$$

Now if the vectors u^l are zero for all $l \geq l_0$ for some l_0, by the equality $H_l K_{l+N} v^{l+2N} = I_{l,l+N} A_{l+N,l+2N} v^{l+2N}$, it follows that

$$0 = A_{l,l+N} I_{l+N,l+2N} v^{l+2N} = A_{l,l+N} v^{l+N} = \beta v^l.$$

Thus $v^l = 0$ for all $l \geq l_0$ and hence for all $l \in \mathbb{N}$, a contradiction. Therefore β is a nonzero eigenvalue of (A', I').

We will next show that the nonzero bounded spectrum of (A, I) is also invariant under shift equivalence. We must provide some lemmas.

Lemma 10.2. *Put* $N_A^l = \max_j \sum_{i=1}^{m(l)} A_{l,l+1}(i,j)$ *for* $l \in \mathbb{N}$. *We have* $N_A^l = N_A^{l+1}$. *That is, the value* N_A^l *does not depend on the choice of* $l \in \mathbb{N}$.

Proof. We note that $\sum_{i=1}^{m(l)} I_{l,l+1}(i,j) = 1$ for each j. It follows that

$$\begin{aligned}\sum_{j=1}^{m(l+1)} A_{l+1,l+2}(j,k) &= \sum_{i=1}^{m(l)} \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) A_{l+1,l+2}(j,k)\\ &= \sum_{i=1}^{m(l)} \sum_{p=1}^{m(l+1)} A_{l,l+1}(i,p) I_{l+1,l+2}(p,k).\end{aligned}$$

Hence for $k = 1, \dots, m(l+2)$, there uniquely exists $p_k = 1, \dots, m(l+1)$ such that

$$\sum_{j=1}^{m(l+1)} A_{l+1,l+2}(j,k) = \sum_{i=1}^{m(l)} A_{l,l+1}(i,p_k).$$

This implies the inequality $N_A^{l+1} \le N_A^l$. For $p = 1, \dots, m(l+1)$, take $k_p = 1, \dots, m(l+2)$ with $I_{l+1,l+2}(p,k_p) = 1$. It follows that

$$\begin{aligned}
\sum_{i=1}^{m(l)} A_{l,l+1}(i,p) &= \sum_{i=1}^{m(l)} A_{l,l+1}(i,p) I_{l+1,l+2}(p,k_p) \\
&= \sum_{i=1}^{m(l)} \sum_{q=1}^{m(l+1)} A_{l,l+1}(i,q) I_{l+1,l+2}(q,k_p) \\
&= \sum_{i=1}^{m(l)} \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) A_{l+1,l+2}(j,k_p) \\
&= \sum_{j=1}^{m(l+1)} A_{l+1,l+2}(j,k_p).
\end{aligned}$$

This implies the inequality $N_A^l \le N_A^{l+1}$.

Set $N_A = \max_j \sum_{i=1}^{m(l)} A_{l,l+1}(i,j)$ that is independent of the choice of $l \in \mathbb{N}$. For an I-compatible vector $\{v^l\}_{l\in\mathbb{N}}$, we put $\|v^l\| = \sum_{i=1}^{m(l)} |v_i^l|$.

LEMMA 10.3. *The sequence $\{\|v^l\|\}_{l\in\mathbb{N}}$ is increasing. If $\{v^l\}_{l\in\mathbb{N}}$ is nonnegative, $\{\|v^l\|\}_{l\in\mathbb{N}}$ is constant and hence $\{v^l\}_{l\in\mathbb{N}}$ is bounded.*

Proof. We know $\sum_{i=1}^{m(l)} |I_{l,l+1}(i,j) v_j^{l+1}| = |v_j^{l+1}|$ and

$$\|v^l\| \le \sum_{i=1}^{m(l)} \sum_{j=1}^{m(l+1)} |I_{l,l+1}(i,j) v_j^{l+1}| \le \sum_{j=1}^{m(l+1)} |v_j^{l+1}| = \|v^{l+1}\|.$$

If $\{v^l\}_{l\in\mathbb{N}}$ is nonnegative, both of the inequalities above go to equalities.

For a bounded I-compatible vector $v = \{v^l\}_{l\in\mathbb{N}}$, we put

$$\|v\|_1 = \sup_{l\to\infty} \|v^l\|.$$

PROPOSITION 10.4. $Sp_b^\times(A,I) \subset \{z \in \mathbb{C} | |z| \le N_A\}$.

Proof. For $\beta \in Sp(A,I)$ with a bounded eigenvector $\{v^l\}_{l\in\mathbb{N}}$, we have

$$\beta \sum_{i=1}^{m(l)} |v_i^l| \le \sum_{j=1}^{m(l+1)} (\max_j \sum_{i=1}^{m(l)} A_{l,l+1}(i,j)) |v_j^{l+1}|.$$

Hence we obtain the inequality

$$\beta\|v^l\| \le N_A\|v^{l+1}\|.$$

As $\{v^l\}$ is bounded, the limit $\lim_{l\to\infty}\|v^l\| = \|v\|_1$ exists so that we have a desired assertion.

We denote by $\mathfrak{B}_I$ the set of all bounded I-compatible vectors. It is a complex Banach space with norm $\|\cdot\|_1$. A nonnegative I-compatible vector $v = \{v^l\}_{l\in\mathbb{N}}$ is called a *state* for I if $\|v\|_1 = 1$. Let $\mathfrak{S}_I$ be the set of all states for I. It is a convex subset of $\mathfrak{B}_I$.

Lemma 10.5. *For $v = \{v^l\}_{l\in\mathbb{N}} \in \mathfrak{B}_I$, put*

$$|v|_i^l = \sup_{N\ge l}\sum_{j=1}^{m(N)} I_{l,N}(i,j)|v_j^N| \qquad \text{for} \quad i = 1,\dots,m(l), \quad l\in\mathbb{N}.$$

We then have

(i) $|v|_i^l < \infty$.

(ii) *The vectors defined by $|v|^l = (|v|_1^l, |v|_2^l, \dots, |v|_{m(l)}^l)$ for $l\in\mathbb{N}$ give rise to a nonnegative I-compatible vector.*

Proof. (i) By the inequality $\sum_{j=1}^{m(N)} I_{l,N}(i,j)|v_j^N| \le \sum_{j=1}^{m(N)} |v_j^N| = \|v^N\|$, we get $|v|_i^l \le \|v\|_1$.

(ii) As we easily see

$$\sum_{k=1}^{m(N)} I_{l,N}(i,k)|v_k^N| \le \sum_{j=1}^{m(N+1)} I_{l,N+1}(i,j)|v_j^{N+1}|,$$

the sequence of sums $\sum_{j=1}^{m(N)} I_{l,N}(i,j)|v_j^N|$ is increasing on N so that we have

$$|v|_i^l = \lim_{N\to\infty}\sum_{j=1}^{m(N)} I_{l,N}(i,j)|v_j^N|.$$

Hence the following equalities hold

$$\begin{aligned}
\sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j)|v|_j^{l+1} &= \sum_{j=1}^{m(l+1)} \lim_{N\to\infty}(\sum_{k=1}^{m(N)} I_{l,l+1}(i,j)I_{l+1,N}(j,k)|v_k^N|) \\
&= \lim_{N\to\infty}\sum_{k=1}^{m(N)}\sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j)I_{l+1,N}(j,k)|v_k^N| \\
&= \lim_{N\to\infty}\sum_{k=1}^{m(N)} I_{l,N}(i,k)|v_k^N| = |v|_i^l
\end{aligned}$$

so that the vectors $\{|v|^l\}_{l\in\mathbb{N}}$ yield an I-compatible vector.

The I-compatible vector $|v|$ for $v\in\mathfrak{B}_I$ is called the *total variation* of v. A bounded I-compatible vector $v\in\mathfrak{B}_I$ is said to be real if all elements v_i^l of the vectors $v^l, l\in\mathbb{N}$ are real numbers. Thus we obtain

COROLLARY 10.6. *For a real bounded I-compatible vector $v \in \mathfrak{B}_I$, there exist nonnegative bounded I-compatible vectors $v^+, v^- \in \mathfrak{B}_I$ such that*

$$v = v^+ - v^-, \qquad |v| = v^+ + v^-.$$

This decomposition is called the Jordan decomposition of v.

Proof. As $|v|_i^l \geq |v_i^l|$ for each i, l, by putting

$$v^+ = \frac{1}{2}(|v| + v), \qquad v^- = \frac{1}{2}(|v| - v)$$

we get the desired assertions.

COROLLARY 10.7. *For a bounded I-compatible vector $v \in \mathfrak{B}_I$, there exist states $v_j \in \mathfrak{S}_I$ and nonnegative real numbers $c_j \in \mathbb{R}$ such that*

$$v = c_1 v_1 - c_2 v_2 + i(c_3 v_3 - c_4 v_4).$$

PROPOSITION 10.8. *For a bounded I-compatible vector $v \in \mathfrak{B}_I$, we put*

$$(L_A v)_i^l = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,j) v_j^{l+1} \qquad \text{for} \quad i = 1, \dots, m(l), \quad l \in \mathbb{N}.$$

Then L_A gives rise to a bounded linear operator on the Banach space $\mathfrak{B}_I$ that satisfies $\|L_A\| = N_A$, where the norm of L_A is given by $\|L_A\| = \sup_{v \neq 0} \frac{\|L_A v\|_1}{\|v\|_1}$.

To prove the proposition above, we note the following lemma.

LEMMA 10.9. *For an arbitrary fixed $l \in \mathbb{N}$ and nonnegative real numbers c_i for $i = 1, \dots, m(l)$, there exists a nonnegative I-compatible vector $v \in \mathfrak{B}_I$ such that $v_i^l = c_i$ for $i = 1, \dots, m(l)$.*

Proof. Put $v_i^l = c_i^l$ for $i = 1, \dots, m(l)$. For $k \leq l$, we put $v^k = I_{k,l} v^l$. For $k = l+1$, we can choose nonnegative real numbers $v_j^{l+1}, j = 1, \dots, m(l+1)$ such that $v_i^l = \sum_{j=1}^{m(l+1)} v_j^{l+1}$ because for each j there uniquely exists i satisfying $I_{l,l+1}(i,j) = 1$ and $I_{l,l+1}(i',j) = 0$ for other i'. Hence we may get a nonnegative I-compatible vector v by induction such that $v_i^l = c_i, i = 1, \dots, m(l)$.

Proof of Proposition 10.8. We first show that $L_A v$ is a bounded I-compatible vector. By the relation $I_{l,l+1} A_{l+1,l+2} = A_{l,l+1} I_{l+1,l+2}$, it is direct to see that $L_A v$ is an I-compatible vector. We have $\|(L_A v)^l\| \leq N_A \|v^{l+1}\|$ so that $\|L_A v\|_1 \leq N_A \|v\|_1$. Hence $L_A v$ is bounded and $\|L_A\| \leq N_A$. Fix $l \in \mathbb{N}$. Take i_0 such that $\max_i \sum_{h=1}^{m(l-1)} A_{l-1,l}(h,i) = \sum_{h=1}^{m(l-1)} A_{l-1,l}(h,i_0)$. By the previous lemma, there exists a nonnegative I-compatible vector $v \in \mathfrak{B}_I$ such that $v_{i_0}^l = 1$ and $v_i^l \neq 0$ for $i \neq i_0$. It then follows that

$$\|(L_A v)^{l-1}\| = \sum_{h=1}^{m(l-1)} A_{l-1,l}(h, i_0) = N_A.$$

Thus we get $\|(L_A v)\|_1 = N_A$. As $\|v\|_1 = \|v^l\| = 1$, we conclude $\|L_A\| \geq N_A$ so that $\|L_A\| = N_A$.

Therefore we have

Corollary 10.10. *For a complex number β, it belongs to $Sp_b(A, I)$ if and only if it satisfies $L_A v = \beta v$ for some $v \in \mathfrak{B}_I$. That is, the bounded spectrum of (A, I) are nothing but the eigenvalues of the bounded positive operator L_A on the Banach space $\mathfrak{B}_I$.*

Corresponding to Theorem 10.1, we have

Theorem 10.11. *If two nonnegative matrix systems are shift equivalent, their nonzero bounded spectrum coincide.*

Proof. Suppose that two nonnegative matrix systems (A, I) and (A', I') are shift equivalent of lag N. Let H_l, K_l be sequences of nonnegative matrices such that $(H, K) : (A, I) \underset{lagN}{\sim} (A', I')$. Following the proof of Theorem 10.1, it suffices to show that for a bounded vector $v \in \mathfrak{B}_I$, the vectors defined by $u^l = K_l v^{l+N}, l \in \mathbb{N}$ give rise to a bounded vector. As the equalities $I'_{l,l+1} K_{l+1} = K_l I_{l+N,l+N+1}$ hold, the boundedness of the vector $\{u^l\}_{l \in \mathbb{N}}$ is shown by a similar manner to the proof of the boundedness of the vector $L_A v$ as in the proof of Proposition 10.8. Hence we know $Sp_b^{\times}(A, I) = Sp_b^{\times}(A', I')$.

We will next see that the set $Sp_b^{\times}(A, I)$ is not empty. We will consider another topology on $\mathfrak{B}_I$. The topology is defined from the subbases of open sets of the form:

$$U_l(v, i, \epsilon) = \{u \in \mathfrak{B}_I \mid |v_i^l - u_i^l| < \epsilon\} \quad \text{for} \quad v \in \mathfrak{B}_I, i = 1, \dots, m(l), \epsilon > 0, l \in \mathbb{N}.$$

We call it the weak topology on $\mathfrak{B}_I$. It is straightforward to see that the state space $\mathfrak{S}_I$ is compact in the topology. Let $\sigma(L_A)$ be the set of all spectrum of L_A as a bounded linear operator on the Banach space $\mathfrak{B}_I$. General theory of bounded linear operators tells us that the set $\sigma(L_A)$ is not empty. Let r_A be the spectral radius of the operator L_A on $\mathfrak{B}_I$, that is, $r_A = \sup\{|r| : r \in \sigma(L_A)\}$.

Proposition 10.12. *There exists a state $v \in \mathfrak{S}_I$ such that $L_A v = r_A v$. Hence we have $r_A \in Sp_b^{\times}(A, I)$.*

Our proof is completely similar to the proof of [MWY;Lemma 4.1]. We will give a proof for the sake of completeness.

Proof. Let $R_A(z)$ be the resolvent of L_A that is defined by $R_A(z)v = (z - L_A)^{-1} v$ for $z \in \mathbb{C}$ with $|z| > r_A$ and $v \in \mathfrak{B}_I$. For $z \in \mathbb{C}$ with $|z| > r_A$, we see $R_A(z)v = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} L_A^k(v)$ and

$$|(L_A^k(v)_i^l| \le \sum_{j=1}^{m(l+k)} A_{l,l+k}(i, j) |v_j^{l+k}|.$$

As $|v_j^{l+k}| \le |v|_j^{l+k}$, it follows that $|(R_A(z)v)_i^l| \le (R_A(|z|)|v|)_i^l$ and hence

$$\|R_A(z)v\|_1 \le \|R_A(|z|)|v|\|_1. \tag{10.3}$$

Since $\{R_A(z)\}_{|z|>r_A}$ can not be uniformly bounded in the set $\mathcal{L}(\mathfrak{B}_I)$ of all bounded linear operators on $\mathfrak{B}_I$, by the inequality (10.3) we may find $v_0 \in \mathfrak{S}_I$ so that $\|R_A(t)v_0\|_1$ is unbounded for $t \downarrow r_A$. Put

$$v_n = \frac{R_A(r_A + \frac{1}{n})v_0}{\|R_A(r_A + \frac{1}{n})v_0\|} \qquad \text{for} \quad n = 1, 2, \dots.$$

As L_A is a positive operator on $\mathfrak{B}_I$, the operator $R_A(t)$ is also positive so that the vectors $v_n, n = 1, 2, \dots$ are states. Hence there exists a limit point v_∞ of the sequence $\{v_n\}$ in $\mathfrak{S}_I$ in the weak topology of $\mathfrak{S}_I$. The following identity

$$(r_A - L_A)v_n = -\frac{1}{n}v_n + \frac{v_0}{\|R_A(r_A + \frac{1}{n})v_0\|}$$

implies $r_A v_\infty = L_A v_\infty$. As (A, I) is essential, the vector $L_A v_\infty$ can not be zero. Hence we have $r_A > 0$ and $r_A \in Sp_b^\times(A, I)$.

The author would like to thank Yasuo Watatani for pointing out an inaccuracy of a proof of the proposition above given in an earlier version of this paper.

We finally show that the spectrum are majorizied by topological entropy of the associated subshift. It is well-known that topological entropy $h_{\text{top}}(\Lambda)$ for subshift Λ is given by

$$h_{\text{top}}(\Lambda) = \lim_{k\to\infty} \frac{1}{k} \log \sharp|\Lambda^k|$$

where $\sharp|\Lambda^k|$ denotes the cardinality of the set of all admissible words of length k in the subshift Λ (cf.[LM],[Ki]).
We say a symbolic matrix system $(\mathcal{M}, I)$ to be left resolving if a symbol appearing in $\mathcal{M}(i,j)$ can not appear in $\mathcal{M}(i', j)$ for other $i' \neq i$, equivalently, its λ-graph system is left resolving. As in Proposition 3.8, a canonical symbolic matrix system is left resolving.

PROPOSITION 10.13. *Let $(\mathcal{M}, I)$ be a left resolving symbolic matrix system and (M, I) its associated nonnegative matrix system. For any $\beta \in Sp_b(M, I)$, we have the inequalities:*

$$\log|\beta| \le \log r_M \le h_{\text{top}}(\Lambda_{(\mathcal{M},I)})$$

where r_M is the spectral radius of the operator L_M on $\mathfrak{B}_I$ and $\Lambda_{(\mathcal{M},I)}$ is the associated subshift with $(\mathcal{M}, I)$.

Proof. The inequality $\log|\beta| \le \log r_M$ is clear. By the previous lemma, take $v \in \mathfrak{S}_I$ such that $L_M v = r_M v$. We have for $k \in \mathbb{N}$,

$$r_M^k v_i^1 = \sum_{j=1}^{m(k+1)} M_{1,k+1}(i,j) v_j^{k+1}.$$

As $\sum_{i=1}^{m(1)} v_i^1 = 1$, it follows that

$$r_M^k \le (\max_j \sum_{i=1}^{m(1)} M_{1,k+1}(i,j)) \sum_{j=1}^{m(k+1)} v_j^{k+1} = \|L_M^k\|.$$

We may find j_0 such that $\|L_M^k\| = \sum_{i=1}^{m(1)} M_{1,k+1}(i,j_0)$. Since $(\mathcal{M}, I)$ is left resolving, the number $\sum_{i=1}^{m(1)} M_{1,k+1}(i,j_0)$ is majorized by the cardinality $\sharp|\Lambda^k_{(\mathcal{M},I)}|$ of the set of all admissible words of length k in the subshift $\Lambda_{(\mathcal{M},I)}$. Thus we obtain the inequalities

$$r_M^k \le \|L_M^k\| \le \sharp|\Lambda^k_{(\mathcal{M},I)}|.$$

As $\|L_M^k\|^{\frac{1}{k}} \to r_M$ for $k \to \infty$, we have desired inequalities.

For subshift (Λ, σ), let (M, I) be its canonical nonnegative matrix system. We define the nonzero spectrum $Sp^\times(\Lambda)$ and the nonzero bounded spectrum $Sp_b^\times(\Lambda)$ of Λ by the nonzero spectrum and the nonzero bounded spectrum of (M, I) respectively. We have thus proved

Theorem 10.14. *Both the sets $Sp^\times(\Lambda)$ and $Sp_b^\times(\Lambda)$ are not empty and topological conjugacy invariants of subshifts. In particular, $Sp_b^\times(\Lambda)$ is bounded by the topological entropy of the subshift (Λ, σ).*

11. Example

We will give an example of the canonical symbolic matrix system, the K-groups and the Bowen-Franks groups for a certain nonsofic subshift, that is called the context free shift in [LM]. Let Σ be the set of symbols $\{a, b, c\}$. The nonsofic subshift is defined to be the subshift Z over Σ whose forbidden words are

$$\mathcal{F}_Z = \{ab^m c^k a | m \ne k\}$$

where the word $ab^m c^k a$ means $a \underbrace{b \cdots b}_{m \text{ times}} \underbrace{c \cdots c}_{k \text{ times}} a$ (cf.[LM]). In [Ma6], the C^*-algebra $\mathcal{O}_Z$ associated with the subshift Z has been studied so that its K-groups has been calculated. By using discussions of the computation of the K-groups, we may write the canonical symbolic matrix system for Z. Let X_Z be the corresponding one-sided subshift for Z. Define sequences of subsets of X_Z in the following way.

$$P_0 = \{c^k b^\infty | k \ge 0\} \cup \{b^k c^m b y \in X_Z | k \ge 0, m \ge 1, y \in X_Z\}$$

and for $n, j = 0, 1, \dots,$

$$\begin{aligned} E_j &= \{c^j a y \in X_Z | y \in X_Z\}, \\ Q_n &= \cup_{j>n} E_j, \\ F_j &= \{b^m c^{m+j} a y \in X_Z | m \ge 1, y \in X_Z\}, \\ R_n &= \{b^m c^k a y \in X_Z | m \ge 1, k \ge 0, m + j \ne k \text{ for } j = 0, 1, \dots, n\}. \end{aligned}$$

Lemma 11.1([Ma6;Lemma 4.3]). *For each $l \in \mathbb{N}$, the space X_Z is decomposed into the disjoint union:*

$$X_Z = P_0 \cup_{j=0}^{l-1} E_j \cup Q_{l-1} \cup_{j=0}^{l-1} F_j \cup R_{l-1}.$$

This decomposition of X_Z into $2l+3$-components corresponds to the l-past equivalence classes of X_Z.

The canonical symbolic matrix systems $\mathcal{M}_{l,l+1}, I_{l,l+1}$ for Z are $m(l)(=2l+3) \times m(l+1)(=2l+5)$ matrices that are written as follows:

$\mathcal{M}_{l,l+1} =$

$$\begin{bmatrix}
b+c & & c & & c & & c & & c & & c & & \cdots & & \cdots & & c \\
a & a & a & & & & & & & & & & & & & & \\
 & & & b & b & & & & & & & & & & & & \\
 & c & & & & & & & & & & & & & & & \\
 & & & & & b & b & & & & & & & & & & \\
 & & & c & & & & & & & & & & & & & \\
 & & & & & & & b & b & & & & & & & & \\
 & & & & & c & & & & & & & & & & & \\
 & & & & & & & & & \ddots & \ddots & & & & & & \\
 & & & & & & & \ddots & & & & & & & & & \\
 & & & & & & & & & & & \ddots & \ddots & & & & \\
 & & & & & & & & & \ddots & & & & & & & \\
 & & & & & & & & & & & & & b & b & & \\
 & & & & & & & & & & & c & & c & & c & \\
 & b & b & & & & & & & & & & & & & b & b
\end{bmatrix},$$

$$I_{l,l+1} = \begin{bmatrix}
1 & & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & & \\
 & & & & & & 1 & & & & & & & & & & \\
 & & & & & & & 1 & & & & & & & & & \\
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & & & 1 & & & & & & & \\
 & & & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & & & & 1 & & 1 & \\
 & & & & & & & & & & & & & & 1 & & 1
\end{bmatrix}$$

along the following ordered basis

$$P_0, E_0, F_0, E_1, F_1, \dots, E_{l-1}, F_{l-1}, Q_{l-1}, R_{l-1}$$

where in the matrices above, blanks denote zeros. The transposed matrices of its nonnegative matrix systems are written as:

$$A^t_{l,l+1} = \begin{bmatrix}
2 & 1 & & & & & & & & & & & & & \\
 & 1 & & 1 & & & & & & & & & & & 1 \\
1 & 1 & & & & & & & & & & & & & 1 \\
 & & 1 & & & 1 & & & & & & & & & \\
1 & & 1 & & & & & & & & & & & & \\
 & & & & 1 & & & 1 & & & & & & & \\
1 & & & & 1 & & & & & & & & & & \\
 & & & & & & 1 & & & \ddots & & & & & \\
1 & & & & & & 1 & & & & & & & & \\
 & & & & & & & & 1 & & & \ddots & & & \\
1 & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & & \ddots & & & 1 & \\
\vdots & & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & & & 1 & 1 & \\
\vdots & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & & 1 & 1 \\
1 & & & & & & & & & & & & & & 1
\end{bmatrix},$$

$$I^t_{l,l+1} = \begin{bmatrix}
1 & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & \\
 & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & & & & & & \\
 & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & & 1
\end{bmatrix}.$$

Hence we have

Proposition 11.2.

$$K_0(Z) = \mathbb{Z}, \qquad K_1(Z) = 0 \quad \textit{and} \quad BF^0(Z) = 0, \qquad BF^1(Z) = \mathbb{Z}.$$

Since the subshift Z is conjugate to its transpose Z^T and by the formula for the Bowen-Franks groups from K for subshifts, we obtain

Proposition 11.3.

$$BF^0_K(Z) = \mathbb{Z}, \qquad BF^1_K(Z) = 0.$$

Hence these types of the Bowen-Franks groups can not be realized in sofic subshifts because $BF^1(Z)$ (resp. $BF^1_K(Z)$) is not the torsion-free part of $BF^0(Z)$ (resp. $BF^0_*(Z)$). We finally see

Proposition 11.4([Ma6:Theorem 6.9]). *The spectral radius of the operator L_A is $1+\sqrt{1+\sqrt{3}} = 2.65289\cdots$ that is the topological entropy for the subshift Z. Hence the maximum value of $Sp_b^\times(A, I)$ is $1+\sqrt{1+\sqrt{3}}$.*

In [KMW], the K-groups and the dimension groups for β-shifts have been calculated. The K-groups and the Bowen-Franks groups for the Dyck shifts are also calculated in [Ma7].

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