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{\scriptstyle \rho_*}\big\downarrow & & {\scriptstyle \rho_*}\big\downarrow & & {\scriptstyle \rho_*}\big\downarrow \\
K_1(B_{\mathbb{R}}) & \xrightarrow{\mathrm{nr}_{B_{\mathbb{R}}}} & \zeta(B_{\mathbb{R}})^{\times} & \xrightarrow{\hat{\delta}^1_{\mathfrak{B},\mathbb{R}}} & \mathrm{Cl}(\mathfrak{B},\mathbb{R}).
\end{array}
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Diagram from

"Tamagawa Numbers for Motives with Coefficients"

see pages 501–570

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Stability of the Relativistic Electron-Positron Field of Atoms in Hartree-Fock Approximation: Heavy Elements [1]

Raymond Brummelhuis, Norbert Röhrl, Heinz Siedentop

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Abstract. We show that the modulus of the Coulomb Dirac operator with a sufficiently small coupling constant bounds the modulus of the free Dirac operator from above up to a multiplicative constant depending on the product of the nuclear charge and the electronic charge. This bound sharpens a result of Bach et al [2] and allows to prove the positivity of the relativistic electron-positron field of an atom in Hartree-Fock approximation for all elements occurring in nature.

2000 Mathematics Subject Classification: 35Q40, 81Q10
Keywords and Phrases: Dirac operator, stability of matter, QED, generalized Hartree-Fock states

1. Introduction

A complete formulation of quantum electrodynamics has been an elusive topic to this very day. In the absence of a mathematically and physically complete model various approximate models have been studied. A particular model which is of interest in atomic physics and quantum chemistry is the the electron-positron field (see, e.g., Chaix et al [4, 5]). The Hamiltonian of the electron-positron field in the Furry picture is given by

$$\mathbb{H} := \int d^3x : \Psi^*(x) D_{g,m} \Psi(x) : + \frac{\alpha}{2} \int d^3x \int d^3y \frac{: \Psi^*(x)\Psi(y)^*\Psi(y)\Psi(x) :}{|\mathbf{x} - \mathbf{y}|},$$

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where the normal ordering and the definition of the meaning of electrons and positrons is given by the splitting of $L^2(\mathbb{R}^2)\otimes\mathbb{C}^4$ into the positive and negative spectral subspaces of the atomic Dirac operator

$$D_{g,m} = \frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + m\beta - \frac{g}{|\mathbf{x}|}.$$

This model agrees up to the complete normal ordering of the interaction energy and the omission of all magnetic field terms with the standard Hamiltonian as found, e.g., in the textbook of Bjorken and Drell [3, (15.28)]. (Note that we freely use the notation of Thaller [8], Helffer and Siedentop [6], and Bach et al [2].)

From a mathematical point of view the model has been studied in a series of papers [2, 1, 7]. The first paper is of most interest to us. There it is shown that the energy $\mathcal{E}(\rho) := \rho(\mathbb{H})$ is nonnegative, if ρ is a generalized Hartree-Fock state provided that the fine structure constant $\alpha := e^2$ is taken to be its physical value $1/137$ and the atomic number Z does not exceed 68 (see Bach et al [2, Theorem 2]). This pioneering result is not quite satisfying from a physical point of view, since it does not allow for all occurring elements in nature, in particular not for the heavy elements for which relativistic mechanics ought to be most important. The main result of the present paper is

Theorem 1. *The energy $\mathcal{E}(\rho)$ is nonnegative in Hartree-Fock states ρ, if $\alpha \leq (4/\pi)(1-g^2)^{1/2}(\sqrt{4g^2+9}-4g)/3$.*

We use g instead of the nuclear number $Z = g/\alpha$ as the parameter for the strength of the Coulomb potential because this is the mathematically more natural choice. For the physical value of $\alpha \approx 1/137$ the latter condition is satisfied, if the atomic number Z does not exceed 117.

Our main technical result to prove Theorem 1 is

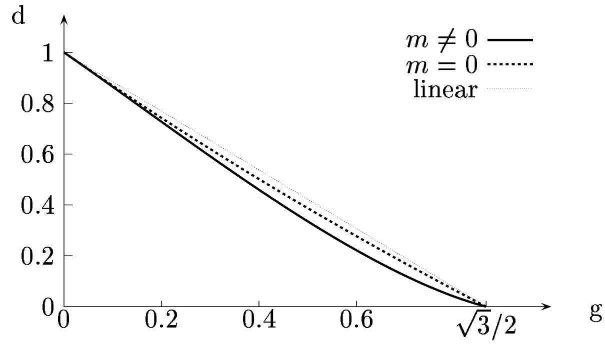
Lemma 1. *Let $g \in [0, \sqrt{3}/2]$ and*

$$d = \begin{cases} \frac{1}{3}(\sqrt{4g^2+9}-4g) & m = 0 \\ \sqrt{1-g^2}\,\frac{1}{3}(\sqrt{4g^2+9}-4g) & m > 0 \end{cases}.$$

Then we have for $m \geq 0$

$$|D_{g,m}| \geq d|D_{0,0}|. \tag{1}$$

The following graph gives an overview of the dependence of d on the coupling constant g



Our paper is organized as follows: in Section 2 we show how Lemma 1 proves our stability result. Section 3 contains the technical heart of our result. Among other things we will prove Theorem 1 in that section. Eventually, Section 4 contains some additional remarks on the optimality of our result.

2. Positivity of the Energy

As mentioned in the introduction, a first – but non-satisfactory result as far as it concerns heavy elements – is due to Bach et al [2]. Their proof consists basically of three steps:
(i) They show that positivity of the energy $\mathcal{E}(\rho)$ in generalized Hartree-Fock states ρ is equivalent to showing positivity of the Hartree-Fock functional

$$\mathcal{E}^{HF} : X \to \mathbb{R},$$
$$\mathcal{E}^{HF}(\gamma) = \mathrm{tr}(D_{g,m}\gamma) + \alpha D(\rho_\gamma, \rho_\gamma) - \frac{\alpha}{2}\int dxdy \frac{|\gamma(x,y)|^2}{|\mathbf{x}-\mathbf{y}|}$$

where $D(f,g) := (1/2)\int_{\mathbb{R}^6} d\mathbf{x}d\mathbf{y}\overline{f(\mathbf{x})}g(\mathbf{y})|\mathbf{x}-\mathbf{y}|^{-1}$ is the Coulomb scalar product, X is the set of trace class operators γ for which $|D_{0,m}|\gamma$ is also trace class and which fulfills $-P_- \leq \gamma \leq P_+$, and $\rho_\gamma(\mathbf{x}) := \sum_{\sigma=1}^4 \gamma(x,x)$. (See [2], Section 3.)
(ii) They show, that the positivity of $\mathcal{E}^{HF}$ follows from the inequality

$$|D_{g,m}| \geq d|D_{g,0}|$$

(Inequality (1)), if $\alpha \leq 4d/\pi$ (see [2], Theorem 2).
(iii) They show this inequality for $d = 1 - 2g$ implying then the positivity of $\mathcal{E}(\rho)$ in Hartree-Fock states ρ, if $\alpha \approx 1/137$ and $Z \leq 68$.
From the first two steps, the proof of Theorem 1 follows using Lemma 1. – Step (iii) indicates that it is essential to improve (1) which we shall accomplish in the next section.

3. Inequality Between Moduli of Dirac Operators

We now start with the main technical task, namely the proof of the key Lemma 1. We will first prove Inequality (1) in the massless case. Then we will roll back the "massive" case to the massless one.

Because there is no easy known way of writing down $|D_{g,0}|$ explicitly, we prove the stronger inequality

$$D_{g,0}^2 \geq d^2 D_{0,0}^2 \tag{2}$$

again following Bach et al [2]. However, those authors proceeded just using the triangular inequality. In fact this a severe step. Instead we shall show (2) with the sharp constant $d^2 = (\sqrt{4g^2+9} - 4g)^2/9$ in the massless case. Since the Coulomb Dirac operator is essentially selfadjoint on $\mathcal{D} := C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \otimes \mathbb{C}^4$ for $g \leq \sqrt{3}/2$, (2) is equivalent to showing

$$\|D_{g,0}f\|_2^2 - d^2\|D_{0,0}f\|_2^2 \geq 0$$

for all $f \in \mathcal{D}$.

Since the Coulomb Dirac operator – and thus also its square – commutes with the total angular momentum operator, we use a partial wave decomposition. The Dirac operator $D_{g,m}$ in channel κ equals to

$$h_{g,m,\kappa} := \begin{pmatrix} m - \frac{g}{r} & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -m - \frac{g}{r} \end{pmatrix}.$$

It suffices to show (2) for the squares of $h_{g,0,\kappa}$ and $h_{0,0,\kappa}$ for $\kappa = \pm 1, \pm 2, \ldots$.

Notice that $h_{g,0,\kappa}$ is homogeneous of degree -1 under dilations. Therefore it becomes – up to a shift – a multiplication operator under (unitary) Mellin transform. The unitary Mellin transform $\mathcal{M} : L^2(0,\infty) \to L^2(\mathbb{R}), f \mapsto f^\#$ used here is given by

$$f^\#(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-1/2-is} f(r) dr.$$

Unitarity can be seen by considering the isometry

$$\begin{array}{cccc} \iota : & L^2(0,\infty) & \longrightarrow & L^2(-\infty,\infty) \\ & f : r \mapsto f(r) & \mapsto & h : z \mapsto \mathrm{e}^{z/2} f(\mathrm{e}^z) \end{array}.$$

The Mellin transform is just the composition of the Fourier transform and ι. We recall the following two rules for $f^\# = \mathcal{M}(f)$ on smooth functions of compact support in $(0,\infty)$.

$$\left(r^\alpha f\right)^\# (s) = f^\#(s + i\alpha)$$

$$\left(\frac{d}{dr} f\right)^\# (s) = (is + \frac{1}{2}) f^\#(s - i)$$

These two rules give

$$\mathcal{M} h_{g,0,\kappa} \begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} -g & -is - \frac{1}{2} + \kappa \\ +is + \frac{1}{2} + \kappa & -g \end{pmatrix} \begin{pmatrix} \mathcal{M}f^+(s - \mathrm{i}) \\ \mathcal{M}f^-(s - \mathrm{i}) \end{pmatrix}.$$

If we denote above matrix by $h^{\mathcal{M}}_{g,0,\kappa}$, we see that (2) is equivalent to

$$(h^{\mathcal{M}}_{g,0,\kappa})^* h^{\mathcal{M}}_{g,0,\kappa} - d^2 (h^{\mathcal{M}}_{0,0,\kappa})^* h^{\mathcal{M}}_{0,0,\kappa} = \begin{pmatrix} g^2 + (1-d^2)(s^2 + (\kappa+\frac{1}{2})^2) & -2(\kappa - is)g \\ -2(\kappa+is)g & g^2 + (1-d^2)(s^2+(\kappa-\frac{1}{2})^2) \end{pmatrix} \geq 0, \tag{3}$$

where $\kappa = \pm1, \pm2, \ldots$. This is true if and only if the eigenvalues of the matrix on the left hand side of (3) are nonnegative for all $s \in \mathbb{R}$ and $\kappa = \pm1, \pm2, \ldots$. The eigenvalues are the solutions of the quadratic polynomial

$$\lambda^2 - 2\lambda\big(g^2 + (1-d^2)(s^2+\kappa^2+\frac{1}{4})\big) + \big(g^2+(1-d^2)(s^2+\kappa^2+\frac{1}{4})\big)^2 - (1-d^2)^2\kappa^2 - 4g^2(s^2+\kappa^2).$$

Hence the smaller one equals

$$\lambda_1 = g^2 + (1-d^2)(s^2+\kappa^2+\frac{1}{4}) - \sqrt{(1-d^2)^2\kappa^2 + 4g^2(s^2+\kappa^2)}.$$

Here we can already see that d may not exceed 1, and that $d = 1$ is only possible for $g = 0$. It the following we therefore restrict d to the interval $[0, 1)$. At first we look at the necessary condition $\lambda_1(s = 0) \geq 0$. Now,

$$\lambda_1(s=0) = g^2 + (1-d^2)(\kappa^2+\frac{1}{4}) - |\kappa|\sqrt{(1-d^2)^2+4g^2}$$

is positive, if $|\kappa|$ not in between the two numbers

$$\frac{\sqrt{(1-d^2)^2+4g^2} \pm \sqrt{(1-d^2)^2+4g^2-4(1-d^2)(g^2+(1-d^2)/4)}}{2(1-d^2)} = \frac{\sqrt{(1-d^2)^2+4g^2} \pm 2gd}{2(1-d^2)}.$$

But since we are only interested in integer $|\kappa| \geq 1$, we want to get the critical interval below 1 (to get the interval above 1 would require $g > \sqrt{3}/2$), i.e.,

$$\frac{\sqrt{(1-d^2)^2+4g^2}+2gd}{2(1-d^2)} \leq 1,$$

or – equivalently –

$$\sqrt{(1-d^2)^2+4g^2} \leq 2(1-d^2) - 2gd.$$

Since by definition of d we have $g \leq (1-d^2)/d$, the right hand side of above inequality is non-negative. Hence, the above line is equivalent to

$$4g^2 + 8dg - 3(1-d^2) \leq 0. \tag{4}$$

Solving (4) for d yields

$$d \leq 1/6\big(-8g + \sqrt{16g^2+36}\big) = 1/3\left(\sqrt{4g^2+9} - 4g\right). \tag{5}$$

We also need the solution for g:

$$g \leq \frac{1}{2}(\sqrt{3+d^2} - 2d) = \frac{3}{2}\frac{1-d^2}{\sqrt{3+d^2}+2d}. \tag{6}$$

We now compute the derivative

$$\frac{\partial\lambda_1}{\partial s} = 2s[1 - d^2 - 2g^2\big((1-d^2)^2\kappa^2 + 4g^2(s^2+\kappa^2)\big)^{-1/2}].$$

The possible extrema are $s = 0$ and the zeros of $[\dots]$. We will show below that under condition (5) only $s = 0$ is an extremum. It is necessarily a minimum, since $\lambda(s = \pm\infty) = \infty$, which concludes the proof. Now we show $[\dots] > 0$. The expression obviously reaches the smallest value if we choose $\kappa^2 = 1$ and $s = 0$. In this case we get the inequality

$$4g^4 - (1-d^2)^2((1-d^2)^2 + 4g^2) < 0,$$

which implies

$$g^2 < \frac{1+\sqrt{2}}{2}(1-d^2)^2. \tag{7}$$

By the necessary condition (6) we get a sufficient condition for (7) to hold

$$\frac{3}{2}\frac{1-d^2}{\sqrt{3+d^2}+2d} < \sqrt{\frac{1+\sqrt{2}}{2}}(1-d^2).$$

Because $d < 1$ this is equivalent to

$$3 < \sqrt{2}\sqrt{1+\sqrt{2}}(\sqrt{3+d^2}+2d)$$

and the right hand side is bigger than 3 for all d.

Before we proceed to the massive case, we note that we did not loose anything in the above computation, i.e., our value of d^2 is sharp for Inequality (2).

Next, we reduce the massive inequality to the already proven massless one. We have the following relation between the squares of the massive and massless Dirac operator

$$D_{g,m}^2 = D_{g,0}^2 + m^2 - 2m\beta g/|x|.$$

The above operator is obviously positive, but we will show in the following that we only need a fraction of the massless Dirac to control the mass terms.

To implement this idea, we show

$$\epsilon D_{g,0}^2 + m^2 - 2m\beta g/|x| \geq 0, \tag{8}$$

if and only if $\epsilon \geq g^2$.

To show (8), we note that from the known value of the least positive eigenvalue of the Coulomb Dirac operator (see, e.g., Thaller [8]) we have $D_{g,m}^2 \geq m^2(1 - g^2)$. Scaling the mass with $1/\epsilon$ and multiplying the equation by ϵ yields

$$\epsilon\frac{m^2(1-g^2)}{\epsilon^2} \leq \epsilon D_{g,m/\epsilon}^2 = \epsilon D_{g,0}^2 + \frac{1}{\epsilon}m^2 - 2m\beta g/|x|.$$

It follows that

$$\epsilon D_{g,0}^2 + m^2 - 2m\beta g/|x| \geq \left(1 - 1/\epsilon + \frac{1-g^2}{\epsilon}\right) m^2 = \left(1 - \frac{g^2}{\epsilon}\right) m^2,$$

showing (8), if $\epsilon \geq g^2$. This is also necessary, since all inequalities in the proof are sharp for f equal to the ground state eigenfunction.
With (8) the massive inequality follows in a single line:

$$D_{g,m}^2 = (1-g^2)D_{g,0}^2 + g^2 D_{g,0}^2 + m^2 - 2m\beta g/|x| \geq (1-g^2)d^2 D_{0,0}^2.$$

4. Supplementary Remarks on the Necessity of the Hypothesis $g < \sqrt{3}/2$

We wish to shed some additional light, on why g in our lemma does not exceed $\sqrt{3}/2$. In this section we will show again that for the "squared" inequality

$$D_{g,m}^2 \geq d^2 D_{0,m}^2 \tag{9}$$

we inevitably get $d^2 \leq 0$ for $g = \sqrt{3}/2$. This is because there are elements of the domain of $D_{\sqrt{3}/2,m}$ whose derivatives are not square integrable. One example is the eigenfunction of the lowest eigenvalue.
For general $g \in [0, \sqrt{3}/2]$ this function is given in channel $\kappa = -1$ as

$$n_g \begin{pmatrix} -g \\ 1-s \end{pmatrix} r^s \mathrm{e}^{-gmr},$$

where $s = \sqrt{1-g^2}$ and n_g is the normalization constant for the L^2-norm. Its derivative is square integrable, if and only if $s > 1/2$ or equivalently $g < \sqrt{3}/2$. To make the argument precise, we compute the L^2-norm of $h_{\sqrt{3}/2,m,-1}\Psi_\beta$ and $h_{0,m,-1}\Psi_\beta$ with $\beta \in (1,2]$, $g = \sqrt{3/2}$, $s = 1/2$, $m' > 0$, and

$$\Psi_\beta := n_\beta \begin{pmatrix} -g \\ -(s-1) \end{pmatrix} r^{\beta s} \mathrm{e}^{-gm'r}$$

with the normalization constant n_β. We will see that as $\beta \to 1$, the first one stays finite and the second one tends to infinity. This only leaves $d^2 \leq 0$ for $g = \sqrt{3}/2$ in (9). The value of m' is not relevant; it is just necessary to take $m \neq m'$ if $m = 0$ to keep Ψ_β square integrable. Now,

$$\begin{aligned} h_{g,m,-1}\Psi_\beta &= n_\beta \begin{pmatrix} -gm + g^2/r + (s-1)\frac{d}{dr} + (s-1)/r \\ -g\frac{d}{dr} + g/r + (s-1)m + (s-1)g/r \end{pmatrix} r^{\beta s} \mathrm{e}^{-gm'r} \\ &= n_\beta \begin{pmatrix} g^2 + (\beta s+1)(s-1) + r(-gm - (s-1)gm') \\ -g\beta s + g + (s-1)g + r(g^2 m' + (s-1)m) \end{pmatrix} r^{\beta s - 1} \mathrm{e}^{-gm'r}. \end{aligned}$$

Writing the above function as

$$n_\beta \begin{pmatrix} f_1(\beta) + r \cdot h_1 \\ f_2(\beta) + r \cdot h_2 \end{pmatrix} r^{\beta/2-1} \mathrm{e}^{-gm'r},$$

we get the following expression for its norm

$$n_\beta^2 \int_0^\infty ((f_1(\beta) + r \cdot h_1)^2 + (f_2(\beta) + r \cdot h_2)^2) r^{\beta-2} \mathrm{e}^{-2gm'r} dr.$$

The potentially unbounded terms are those involving f_i^2. Now, $f_1(\beta) = (1 - \beta)/4$, $f_2(\beta) = (1-\beta)\sqrt{3}/2$, and for $a \in (-1, 0), b > 0$ we have the straight forward inequality

$$\int_0^\infty r^a \mathrm{e}^{-br} dr \leq \frac{1}{a+1} + \frac{\mathrm{e}^{-b}}{b}.$$

Hence

$$(1-\beta)^2 \int_0^\infty r^{\beta-2} e^{-2gm'r}\, dr \to 0 \text{ for } \beta \to 1.$$

Proceeding as before we get in the free case

$$\begin{aligned} h_{0,m,-1}\Psi_\beta &= n_\beta \begin{pmatrix} -gm + (s-1)\frac{d}{dr} + (s-1)/r \\ -g\frac{d}{dr} + g/r + (s-1)m \end{pmatrix} r^{\beta s} \mathrm{e}^{-gm'r} \\ &= n_\beta \begin{pmatrix} (\beta s + 1)(s-1) + r(-gm - (s-1)gm') \\ -g\beta s + g + r(g^2 m' + (s-1)m) \end{pmatrix} r^{\beta s - 1} \mathrm{e}^{-gm'r}. \end{aligned}$$

But now the terms that depend on r like $r^{\beta s-1}$ do not vanish for $\beta \to 1$. Therefore the L^2-norm is unbounded.

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Compact Moduli Spaces of Stable Sheaves over Non-Algebraic Surfaces

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Abstract. We show that under certain conditions on the topological invariants, the moduli spaces of stable bundles over polarized non-algebraic surfaces may be compactified by allowing at the border isomorphy classes of stable non-necessarily locally-free sheaves. As a consequence, when the base surface is a primary Kodaira surface, we obtain examples of moduli spaces of stable sheaves which are compact holomorphically symplectic manifolds.

2000 Mathematics Subject Classification: 32C13

1 Introduction

Moduli spaces of stable vector bundles over polarized projective complex surfaces have been intensively studied. They admit projective compactifications which arise naturally as moduli spaces of semi-stable sheaves and a lot is known on their geometry. Apart from their intrinsic interest, these moduli spaces also provided a series of applications, the most spectacular of which being to Donaldson theory.

When one looks at non-algebraic complex surfaces, one still has a notion of stability for holomorphic vector bundles with respect to Gauduchon metrics on the surface and one gets the corresponding moduli spaces as open parts in the moduli spaces of simple sheaves. In order to compactify such a moduli space one may use the Kobayashi-Hitchin correspondence and the Uhlenbeck compactification of the moduli space of Hermite-Einstein connections. But the spaces one obtains in this way have a priori only a real-analytic structure. A different compactification method using isomorphy classes of vector bundles on blown-up surfaces is proposed by Buchdahl in [5] in the case of rank two vector

bundles or for topological invariants such that no properly semi-stable vector bundles exist.
In this paper we prove that under this last condition one may compactify the moduli space of stable vector bundles by considering the set of isomorphy classes of stable sheaves inside the moduli space of simple sheaves. See Theorem 4.3 for the precise formulation. In this way one gets a complex-analytic structure on the compactification. The idea of the proof is to show that the natural map from this set to the Uhlenbeck compactification of the moduli space of anti-self-dual connections is proper. We have restricted ourselves to the situation of anti-self-dual connections, rather than considering the more general Hermite-Einstein connections, since our main objective was to construct compactifications for moduli spaces of stable vector bundles over non-Kählerian surfaces. (In this case one can always reduce oneself to this situation by a suitable twist). In particular, when X is a primary Kodaira surface our compactness theorem combined with the existence results of [23] and [1] gives rise to moduli spaces which are holomorphically symplectic compact manifolds. Two ingredients are needed in the proof: a smoothness criterion for the moduli space of simple sheaves and a non-disconnecting property of the border of the Uhlenbeck compactification which follows from the gluing techniques of Taubes.
Acknowledgments I'd like to thank N. Buchdahl, P. Feehan and H. Spindler for valuable discussions.

2 Preliminaries

Let X be a compact (non-singular) complex surface. By a result of Gauduchon any hermitian metric on X is conformally equivalent to a metric g with $\partial\bar{\partial}$-closed Kähler form ω. We call such a metric a Gauduchon metric and fix one on X. We shall call the couple (X, g) or (X, ω) a polarized surface and ω the polarization. One has then a notion of stability for torsion-free coherent sheaves.

Definition 2.1 A torsion-free coherent sheaf $\mathcal{F}$ on X is called reducible if it admits a coherent subsheaf $\mathcal{F}'$ with $0 < \operatorname{rank} \mathcal{F}' < \operatorname{rank} \mathcal{F}$, (and irreducible otherwise). A torsion-free sheaf $\mathcal{F}$ on X is called stably irreducible if every torsion-free sheaf $\mathcal{F}'$ with

$$\operatorname{rank}(\mathcal{F}') = \operatorname{rank}(\mathcal{F}),\ c_1(\mathcal{F}') = c_1(\mathcal{F}),\ c_2(\mathcal{F}') \leq c_2(\mathcal{F})$$

is irreducible.

Remark that if X is algebraic (and thus projective), every torsion-free coherent sheaf $\mathcal{F}$ on X is reducible. But by [2] and [22] there exist irreducible rank-two holomorphic vector bundles on any non-algebraic surface. Moreover stably irreducible bundles have been constructed on 2-dimensional tori and on primary Kodaira surfaces in [23], [24] and [1].

We recall that on a non-algebraic surface the DISCRIMINANT of a rank r torsion-free coherent sheaf which is defined by

$$\Delta(\mathcal{F}) = \frac{1}{r}\Big(c_2(\mathcal{F}) - \frac{(r-1)}{2r}c_1(\mathcal{F})^2\Big)$$

is non-negative [2].

Let $\mathcal{M}^{st}(E, L)$ denote the moduli space of stable holomorphic structures in a vector bundle E of rank $r > 1$, determinant $L \in \mathrm{Pic}(X)$ and second Chern class $c \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. We consider the following condition on $(r, c_1(L), c)$:

(*) *every semi-stable vector bundle* $\mathcal{E}$ *with* $\mathrm{rank}(\mathcal{E}) = r$, $c_1(\mathcal{E}) = c_1(L)$ *and* $c_2(\mathcal{E}) \leq c$ *is stable.*

Under this condition Buchdahl constructed a compactification of $\mathcal{M}^{st}(E, L)$ in [5]. We shall show that under this same condition one can compactify $\mathcal{M}^{st}(E, L)$ allowing simple coherent sheaves in the border. For simplicity we shall restrict ourselves to the case $\deg_\omega L = 0$. When $b_1(X)$ is odd we can always reduce ourselves to this case by a suitable twist with a topologically trivial line bundle; (see the following Remark).

The condition (*) takes a different aspect according to the parity of the first Betti number of X or equivalently, according to the existence or non-existence of a Kähler metric on X.

Remark 2.2 (a) When $b_1(X)$ is odd (*) is equivalent to: "every torsion free sheaf $\mathcal{F}$ on X with $\mathrm{rank}(\mathcal{F}) = r$, $c_1(\mathcal{F}) = c_1(L)$ and $c_2(\mathcal{F}) \leq c$ is irreducible", i.e. $(r, c_1(L), c)$ describes the topological invariants of a stably irreducible vector bundle.

(b) When $b_1(X)$ is even and $c_1(L)$ is not a torsion class in $H^2(X, \mathbb{Z}_r)$ one can find a Kähler metric g such that $(r, c_1(L), c)$ satisfies (*) for all c.

(c) When $b_1(X)$ is odd or when $\deg L = 0$, (*) implies $c < 0$.

(d) If $b_2(X) = 0$ then there is no torsion-free coherent sheaf on X whose invariants satisfy (*).

Proof It is clear that the stable irreducibility condition is stronger than (*). Now if a sheaf $\mathcal{F}$ is not irreducible it admits some subsheaf $\mathcal{F}'$ with $0 < \mathrm{rank}\,\mathcal{F}' < \mathrm{rank}\,\mathcal{F}$. When $b_1(X)$ is odd the degree function $\deg_\omega : \mathrm{Pic}^0(X) \longrightarrow \mathbb{R}$ is surjective, so twisting by suitable invertible sheaves $L_1, L_2 \in \mathrm{Pic}^0(X)$ gives a semi-stable but not stable sheaf $(L_1 \otimes \mathcal{F}') \oplus (L_2 \otimes (\mathcal{F}/\mathcal{F}'))$ with the same Chern classes as $\mathcal{F}$. Since by taking double-duals the second Chern class decreases, we get a locally free sheaf

$$(L_1 \otimes (\mathcal{F}')^{\vee\vee}) \oplus (L_2 \otimes (\mathcal{F}/\mathcal{F}')^{\vee\vee})$$

which contradicts (*) for $(\mathrm{rank}(\mathcal{F}), c_1(\mathcal{F}), c_2(\mathcal{F}))$. This proves (a).

For (b) it is enough to take a Kähler class ω such that

$$\omega(r' \cdot c_1(L) - r \cdot \alpha) \neq 0 \text{ for all } \alpha \in NS(X)/\mathrm{Tors}(NS(X))$$

and integers r' with $0 < r' < r$. This is possible since the Kähler cone is open in $H^{1,1}(X)$.
For (c) just consider $(L \otimes L_1) \oplus \mathcal{O}_X^{\otimes(r-1)}$ for a suitable $L_1 \in \mathrm{Pic}^0(X)$ in case $b_1(X)$ odd. Finally, suppose $b_2(X) = 0$. Then X admits no Kähler structure hence $b_1(X)$ is odd. If $\mathcal{F}$ were a coherent sheaf on X whose invariants satisfy (*) we should have

$$\Delta(\mathcal{F}) = \frac{1}{r}\Big(c_2 - \frac{(r-1)}{2r} c_1(L)^2\Big) = \frac{1}{r} c_2 < 0$$

contradicting the non-negativity of the discriminant. □

3 The moduli space of simple sheaves

The existence of a coarse moduli space Spl_X for simple (torsion-free) sheaves over a compact complex space has been proved in [12] ; see also [19]. The resulting complex space is in general non-Hausdorff but points representing stable sheaves with respect to some polarization on X are always separated.
In order to give a better description of the base of the versal deformation of a coherent sheaf $\mathcal{F}$ we need to compare it to the deformation of its determinant line bundle $\det \mathcal{F}$. We first establish

Proposition 3.1 *Let X be a nonsingular compact complex surface, $(S, 0)$ a complex space germ, $\mathcal{F}$ a coherent sheaf on $X \times S$ flat over S and $q : X \times S \to X$ the projection. If the central fiber $\mathcal{F}_0 := \mathcal{F}|_{X \times \{0\}}$ is torsion-free then there exists a locally free resolution of $\mathcal{F}$ over $X \times S$ of the form*

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

where G is a locally free sheaf on X.

Proof In [20] it is proven that a resolution of $\mathcal{F}_0$ of the form

$$0 \longrightarrow G \longrightarrow E_0 \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

exists on X with G and E_0 locally free on X as soon as the rank of G is large enough and

$$H^2(X, \mathcal{H}om(\mathcal{F}_0, G)) = 0.$$

We only have to notice that when $\mathcal{F}_0$ and G vary in some flat families over S then one can extend the above exact sequence over $X \times S$. We choose S to be Stein and denote by $p : X \times S \to S$ the projection.

From the spectral sequence relating the relative and global Ext-s we deduce the surjectivity of the natural map

$$\mathrm{Ext}^1(X \times S; \mathcal{F}, q^*G) \longrightarrow H^0(S, \mathcal{E}xt^1(p; \mathcal{F}, q^*G)).$$

We can apply the base change theorem for the relative Ext^1 sheaf if we know that $\mathrm{Ext}^2(X; \mathcal{F}_0, G) = 0$ (cf. [3] Korollar 1). But in the spectral sequence

$$H^p(X, \mathcal{E}xt^q(\mathcal{F}_0, G)) \Longrightarrow \mathrm{Ext}^{p+q}(X; \mathcal{F}_0, G)$$

relating the local Ext $-s$ to the global ones, all degree two terms vanish since $H^2(X; \mathcal{H}om(\mathcal{F}_0, G)) = 0$ by assumption. Thus by base change

$$\mathrm{Ext}^1(X; \mathcal{F}_0, G) \cong \mathcal{E}xt^1(p; \mathcal{F}, q^*G)_0 \Big/ \mathfrak{m}_{S,0} \cdot \mathcal{E}xt^1(p; \mathcal{F}, q^*G)$$

and the natural map

$$\mathrm{Ext}^1(X \times S; \mathcal{F}, q^*G) \longrightarrow \mathrm{Ext}^1(X; \mathcal{F}_0, G)$$

given by restriction is surjective. $\square$

Let X, S and $\mathcal{F}$ be as above. One can use Proposition 3.1 to define a morphism

$$\det : (S, 0) \longrightarrow (\mathrm{Pic}(X), \det \mathcal{F}_0)$$

by associating to $\mathcal{F}$ its determinant line bundle $\det \mathcal{F}$.

The tangent space at the isomorphy class $[\mathcal{F}] \in Spl_X$ of a simple sheaf $\mathcal{F}$ is $\mathrm{Ext}^1(X; \mathcal{F}, \mathcal{F})$ since Spl_X is locally around $[\mathcal{F}]$ isomorphic to the base of the versal deformation of $\mathcal{F}$. The space of obstructions to the extension of a deformation of $\mathcal{F}$ is $\mathrm{Ext}^2(X; \mathcal{F}, \mathcal{F})$.

In order to state the next theorem which compares the deformations of $\mathcal{F}$ and $\det \mathcal{F}$, we have to recall the definition of the trace maps

$$tr^q : \mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X).$$

When $\mathcal{F}$ is locally free one defines $tr_{\mathcal{F}} : \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{O}_X$ in the usual way by taking local trivializations of $\mathcal{F}$. Suppose now that $\mathcal{F}$ has a locally free resolution $F^\bullet$. (See [21] and [10] for more general situations.) Then one defines

$$tr_{F^\bullet} : \mathcal{H}om^\bullet(F^\bullet, F^\bullet) \longrightarrow \mathcal{O}_X$$

by

$$tr_{F^\bullet} |_{\mathcal{H}om(F^i, F^j)} = \begin{cases} (-1)^i tr_{F^i} \ , & \text{for } i = j \\ 0 \ , & \text{for } i \neq j. \end{cases}$$

Here we denoted by $\mathcal{H}om^\bullet(F^\bullet, F^\bullet)$ the complex having $\mathcal{H}om^n(F^\bullet, F^\bullet) = \bigoplus_i \mathcal{H}om(F^i, F^{i+n})$ and differential

$$d(\varphi) = d_{F^\bullet} \circ \varphi - (-1)^{\deg \varphi} \cdot \varphi \circ d_{F^\bullet}$$

for local sections $\varphi \in \mathcal{H}om^n(F^\bullet, F^\bullet)$. $tr_{F^\bullet}$ becomes a morphism of complexes if we see $\mathcal{O}_X$ as a complex concentrated in degree zero. Thus $tr_{F^\bullet}$ induces morphisms at hypercohomology level. Since the hypercohomology groups of $\mathrm{Hom}^\bullet(F^\bullet, F^\bullet)$ and of $\mathcal{O}_X$ are $\mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F})$ and $H^q(X, \mathcal{O}_X)$ respectively, we get our desired maps

$$tr^q : \mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X).$$

Using tr^0 over open sets of X we get a sheaf homomorphism $tr : \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{O}_X$. Let $\mathcal{E}nd_0(\mathcal{F})$ be its kernel. If one denotes the kernel of $tr^q : \mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X)$ by $\mathrm{Ext}^q(X, \mathcal{F}, \mathcal{F})_0$ one gets natural maps $H^q(X, \mathcal{E}nd_0(\mathcal{F})) \longrightarrow \mathrm{Ext}^q(X, \mathcal{F}, \mathcal{F})_0$, which are isomorphisms for $\mathcal{F}$ locally free.
This construction generalizes immediately to give trace maps

$$tr^q : \mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F} \otimes N) \longrightarrow H^q(X, N)$$

for locally free sheaves N on X or for sheaves N such that $\mathcal{T}or_i^{\mathcal{O}_X}(N, \mathcal{F})$ vanish for $i > 0$.
The following Lemma is easy.

Lemma 3.2 *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X allowing finite locally free resolutions and $u \in \mathrm{Ext}^p(X; \mathcal{F}, \mathcal{G})$, $v \in \mathrm{Ext}^q(X; \mathcal{G}, \mathcal{F})$ then*

$$tr^{p+q}(u \cdot v) = (-1)^{p \cdot q} tr^{p+q}(v \cdot u).$$

Theorem 3.3 *Let X be a compact complex surface, $(S, 0)$ be a germ of a complex space and $\mathcal{F}$ a coherent sheaf on $X \times S$ flat over S such that $\mathcal{F}_0 := \mathcal{F}\Big|_{X \times \{0\}}$ is torsion-free. The following hold.*

(a) *The tangent map of $\det : S \to \mathrm{Pic}(X)$ in 0 factorizes as*

$$T_0 S \xrightarrow{KS} \mathrm{Ext}^1(X; \mathcal{F}, \mathcal{F}) \xrightarrow{tr^1} H^1(X, \mathcal{O}_X) = T_{[\det \mathcal{F}_0]}(\mathrm{Pic}(X)).$$

(b) *If T is a zero-dimensional complex space such that $\mathcal{O}_{S,0} = \mathcal{O}_{T,0}/I$ for an ideal I of $\mathcal{O}_{T,0}$ with $I \cdot \mathfrak{m}_{T,0} = 0$, then the obstruction $ob(\mathcal{F}, T)$ to the extension of $\mathcal{F}$ to $X \times T$ is mapped by*

$$\begin{aligned} tr^2 \otimes_{\mathbb{C}} id_I : &\mathrm{Ext}^2(X; \mathcal{F}_0, \mathcal{F}_0 \otimes_{\mathbb{C}} I) \cong \mathrm{Ext}^2(X; \mathcal{F}_0, \mathcal{F}_0) \otimes_{\mathbb{C}} I \longrightarrow \\ &\longrightarrow H^2(X, \mathcal{O}_X) \otimes_{\mathbb{C}} I \cong \mathrm{Ext}^2(X; \det \mathcal{F}_0, (\det \mathcal{F}_0) \otimes_{\mathbb{C}} I) \end{aligned}$$

to the obstruction to the extension of $\det \mathcal{F}$ to $X \times T$ which is zero.

Proof (a) We may suppose that S is the double point $(0, \mathbb{C}[\epsilon])$. We define the Kodaira-Spencer map by means of the Atiyah class (cf. [9]).
For a complex space Y let $p_1, p_2 : Y \times Y \to Y$ be the projections and $\Delta \subset Y \times Y$ the diagonal. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \longrightarrow \mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^2 \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

by $p_2^*\mathcal{F}$ for $\mathcal{F}$ locally free on Y and applying $p_{1,*}$ gives an exact sequence on Y

$$0 \longrightarrow \mathcal{F} \otimes \Omega_Y \longrightarrow p_{1,*}(p_2^*\mathcal{F} \otimes (\mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^2)) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The class $A(\mathcal{F}) \in \mathrm{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ of this extension is called the Atiyah class of $\mathcal{F}$. When $\mathcal{F}$ is not locally free but admits a finite locally free resolution $F^\bullet$ one gets again a class $A(\mathcal{F})$ in $\mathrm{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ seen as first cohomology group of $\mathcal{H}om^\bullet(F^\bullet, F^\bullet \otimes \Omega_Y)$.
Consider now $Y = X \times S$ with X and S as before, $p : Y \to S$, $q : Y \to X$ the projections and $\mathcal{F}$ as in the statement of the theorem.
he decomposition $\Omega_{X\times S} = q^*\Omega_X \oplus p^*\Omega_S$ induces

$$\begin{aligned}&\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes \Omega_{S\times X}) \cong\\ &\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes q^*\Omega_X) \oplus \mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes p^*\Omega_S).\end{aligned}$$

The component $A_S(\mathcal{F})$ of $A(\mathcal{F})$ lying in $\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes p^*\Omega_S)$ induces the "tangent vector" at 0 to the deformation $\mathcal{F}$ through the isomorphisms

$$\begin{aligned}&\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes p^*\Omega_S) \cong \mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes p^*\mathfrak{m}_{S,0}) \cong\\ &\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F}_0) \cong \mathrm{Ext}^1(X; \mathcal{F}_0, \mathcal{F}_0).\end{aligned}$$

Applying now $tr^1 : \mathrm{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y) \to H^1(Y; \Omega_Y)$ to the Atiyah class $A(\mathcal{F})$ gives the first Chern class of $\mathcal{F}$, $c_1(\mathcal{F}) := tr^1(A(\mathcal{F}))$, (cf. [10], [21]).
It is known that

$$c_1(\mathcal{F}) = c_1(\det \mathcal{F}), \text{ i.e. } tr^1(A(\mathcal{F})) = tr^1(A(\det \mathcal{F})).$$

Now $\det \mathcal{F}$ is invertible so

$$tr^1 : \mathrm{Ext}^1(Y, \det \mathcal{F}, (\det \mathcal{F}) \otimes \Omega_Y)) \longrightarrow H^1(Y, \Omega_Y)$$

is just the canonical isomorphism. Since tr^1 is compatible with the decomposition $\Omega_{X\times S} = q^*\Omega_X \oplus p^*\Omega_S$ we get $tr^1(A_S(\mathcal{F})) = A_S(\det \mathcal{F})$ which proves (a).
(b) In order to simplify notation we drop the index 0 from $\mathcal{O}_{S,0}, \mathfrak{m}_{S,0}$, $\mathcal{O}_{T,0}$, $\mathfrak{m}_{T,0}$ and we use the same symbols $\mathcal{O}_S, \mathfrak{m}_S, \mathcal{O}_T, \mathfrak{m}_T$ for the respective pulled-back sheaves through the projections $X \times S \to S$, $X \times T \to T$.
There are two exact sequences of $\mathcal{O}_S$-modules:

$$\begin{aligned}&(1) \qquad 0 \longrightarrow \mathfrak{m}_S \longrightarrow \mathcal{O}_S \longrightarrow \mathbb{C} \longrightarrow 0,\\ &(2) \qquad 0 \longrightarrow I \longrightarrow \mathfrak{m}_T \longrightarrow \mathfrak{m}_S \longrightarrow 0.\end{aligned}$$

(Use $I \cdot \mathfrak{m}_T = 0$ in order to make $\mathfrak{m}_T$ an $\mathcal{O}_S$-module.)

Let $j : \mathbb{C} \to \mathcal{O}_S$ be the $\mathbb{C}$-vector space injection given by the $\mathbb{C}$-algebra structure of $\mathcal{O}_S$. j induces a splitting of (1). Since $\mathcal{F}$ is flat over S we get exact sequences over $X \times S$

$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0$$
$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \longrightarrow 0$$

which remain exact as sequences over $\mathcal{O}_X$. Thus we get elements in $\mathrm{Ext}^1(X; \mathcal{F}_0, \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S)$ and $\mathrm{Ext}^1(X; \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S, \mathcal{F} \otimes_{\mathbb{C}} I)$ whose Yoneda composite $ob(\mathcal{F}, T)$ in $\mathrm{Ext}^2(X; \mathcal{F}_0, \mathcal{F} \otimes_{\mathbb{C}} I)$ is represented by the 2-fold exact sequence

$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

and is the obstruction to extending $\mathcal{F}$ from $X \times S$ to $X \times T$, as is well-known.

Consider now a resolution

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

of $\mathcal{F}$ as provided by Proposition 3.1, i.e. with G locally free on X and E locally free on $X \times S$. Our point is to compare $ob(\mathcal{F}, T)$ to $ob(E, T)$.

Since $\mathcal{F}$ is flat over S we get the following commutative diagrams with exact rows and columns by tensoring this resolution with the exact sequences (1) and (2):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_S \arrow[r] \arrow[d] & q^*G \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \otimes_{\mathcal{O}_S} \mathfrak{m}_S \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \qquad (1')$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & q^*G \otimes_{\mathbb{C}} I \arrow[r] \arrow[d] & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_T \arrow[r] \arrow[d] & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \otimes_{\mathcal{O}_S} I \arrow[r] \arrow[d] & E \otimes_{\mathcal{O}_S} \mathfrak{m}_T \arrow[r] \arrow[d] & E \otimes_{\mathcal{O}_S} \mathfrak{m}_S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \otimes_{\mathcal{O}_S} I \arrow[r] \arrow[d] & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T \arrow[r] \arrow[d] & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2'}$$

Using the section $j : \mathbb{C} \to \mathcal{O}_S$ we get an injective morphism of $\mathcal{O}_X$ sheaves

$$G_0 \xrightarrow{id_{q^*G} \otimes j} q^*G \otimes_{\mathbb{C}} \mathfrak{m}_T \longrightarrow E \otimes_{\mathcal{O}_S} \mathfrak{m}_T$$

which we call j_G.
From $(1')$ we get a short exact sequence over X in the obvious way

$$0 \longrightarrow (E \otimes_{\mathcal{O}_S} \mathfrak{m}_S) \oplus j_G(G_0) \longrightarrow E \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

Combining this with the middle row of $(2')$ we get a 2-fold extension

$$0 \longrightarrow (E \otimes_{\mathcal{O}_S} I) \oplus G_0 \longrightarrow (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 \longrightarrow E \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

whose class in $\mathrm{Ext}^2(X; \mathcal{F}_0, (E \otimes_{\mathcal{O}_S} I) \otimes G_0)$ we denote by u.
Let v be the surjection $E \to \mathcal{F}$ and

$$v' := \begin{pmatrix} v \otimes id_I \\ 0 \end{pmatrix} : (E \otimes_{\mathcal{O}_S} I) \oplus G_0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I,$$
$$v'' = \begin{pmatrix} v_0 \\ 0 \end{pmatrix} : E_0 \oplus G_0 \longrightarrow \mathcal{F}_0,$$

the $\mathcal{O}_X$-morphisms induced by v.
The commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 \arrow[r] \arrow[d, "v'"] & (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 \arrow[r] \arrow[d, "\left(\begin{smallmatrix} v \otimes id_{\mathfrak{m}_T} \\ 0 \end{smallmatrix}\right)"] & E \arrow[r] \arrow[d] & \mathcal{F}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \otimes_{\mathcal{O}_S} I \arrow[r] & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}_0 \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{array}{ccccccccc} 0 \longrightarrow & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 & \longrightarrow & E \oplus G_0 & \longrightarrow & E_0 \oplus G_0 & \longrightarrow 0 \\ & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \binom{id}{j_G}} & & \downarrow{\scriptstyle v''} & \\ 0 \longrightarrow & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 & \longrightarrow & (E \otimes \mathfrak{m}_T) \oplus G_0 & \longrightarrow & E & \longrightarrow & \mathcal{F}_0 & \longrightarrow 0 \end{array}$$

show that $ob(\mathcal{F}, T) = v' \cdot u$ and

$$(ob(E,T), 0) = u \cdot v'' \in \operatorname{Ext}^2(X; E_0 \oplus G_0, (E \otimes_{\mathcal{O}_S} I) \oplus G_0).$$

We may restrict ourselves to the situation when I is generated by one element. Then we have canonical isomorphisms of $\mathcal{O}_X$-modules $E_0 \cong E \otimes_{\mathcal{O}_S} I$ and $\mathcal{F}_0 \cong \mathcal{F} \otimes_{\mathcal{O}_S} I$. By these one may identify v' and v''. Now the Lemma 3.2 on the graded symmetry of the trace map with respect to the Yoneda pairing gives $tr^2(ob(\mathcal{F}, T)) = tr^2(ob(E, T))$.
But E is locally free and the assertion (b) of the theorem may be proved for it as in the projective case by a cocycle computation.
Thus $tr^2(ob(E,T)) = ob(\det E)$ and since $\det(E) = (\det \mathcal{F}) \otimes q^*(\det G)$ and $q^*(\det G)$ is trivially extendable, the assertion (b) is true for $\mathcal{F}$ as well. □
The theorem should be true in a more general context. In fact the proof of (a) is valid for any compact complex manifold X and flat sheaf $\mathcal{F}$ over $X \times S$. Our proof of (b) is in a way symmetric to the proof of Mukai in [17] who uses a resolution for $\mathcal{F}$ of a special form in the projective case.
Notation For a compact complex surface X and an element L in $\operatorname{Pic}(X)$ we denote by $Spl_X(L)$ the fiber of the morphism $\det : Spl_X \to \operatorname{Pic}(X)$ over L.

Corollary 3.4 *For a compact complex surface X and $L \in \operatorname{Pic}(X)$ the tangent space to $Spl_X(L)$ at an isomorphy class $[F]$ of a simple torsion-free sheaf F with $[\det F] = L$ is $\operatorname{Ext}^1(X; F, F)_0$. When $\operatorname{Ext}^2(X; F, F)_0 = 0$, $Spl_X(L)$ and Spl_X are smooth of dimensions*

$$\dim \operatorname{Ext}^1(X; F, F)_0 = 2 \operatorname{rank}(F)^2 \Delta(F) - (\operatorname{rank}(F)^2 - 1)\chi(\mathcal{O}_X)$$

and

$$\dim \operatorname{Ext}^1(X; F, F) = \dim \operatorname{Ext}^1(X; F, F)_0 + h^1(\mathcal{O}_X)$$

respectively.

We end this paragraph by a remark on the symplectic structure of the moduli space Spl_X when X is symplectic.
Recall that a complex manifold M is called holomorphically symplectic if it admits a global nondegenerate closed holomorphic two-form ω. For a surface X, being holomorphically symplectic thus means that the canonical line bundle K_X is trivial. For such an X, Spl_X is smooth and holomorphically symplectic

as well. The smoothness follows immediately from the above Corollary and a two-form ω is defined at $[F]$ on Spl_X as the composition:

$$T_{[F]}Spl_X \times T_{[F]}Spl_X \cong \mathrm{Ext}^1(X;F,F) \times \mathrm{Ext}^1(X;F,F) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^2(X;F,F) \xrightarrow{tr^2} H^2(X,\mathcal{O}_X) \cong H^2(X,K_X) \cong \mathbb{C}.$$

It can be shown exactly as in the algebraic case that ω is closed and nondegenerate on Spl_X (cf. [17], [9]). Moreover, it is easy to see that the restriction of ω to the fibers $Spl_X(L)$ of $\det : Spl_X \to \mathrm{Pic}(X)$ remains nondegenerate, in other words that $Spl_X(L)$ are holomorphically symplectic subvarieties of Spl_X.

4 The moduli space of ASD connections and the comparison map

4.1 The moduli space of anti-self-dual connections

In this subsection we recall some results about the moduli spaces of anti-self-dual connections in the context we shall need. The reader is referred to [6], [8] and [14] for a thorough treatment of these questions.

We start with a compact complex surface X equipped with a Gauduchon metric g and a differential (complex) vector bundle E with a hermitian metric h in its fibers. The space of all C^∞ unitary connections on E is an affine space modeled on $\mathcal{A}^1(X, End(E,h))$ and the C^∞ unitary automorphism group $\mathcal{G}$, also called gauge-group, operates on it. Here $End(E,h)$ is the bundle of skew-hermitian endomorphisms of (E,h). The subset of anti-self-dual connections is invariant under the action of the gauge-group and we denote the corresponding quotient by

$$\mathcal{M}^{ASD} = \mathcal{M}^{ASD}(E).$$

A unitary connection A on E is called REDUCIBLE if E admits a splitting in two parallel sub-bundles.

We use as in the previous section the determinant map

$$\det : \mathcal{M}^{ASD}(E) \longrightarrow \mathcal{M}^{ASD}(\det E)$$

which associates to A the connection $\det A$ in $\det E$. This is a fiber bundle over $\mathcal{M}^{ASD}(\det E)$ with fibers $\mathcal{M}^{ASD}(E,[a])$ where $[a]$ denotes the gauge equivalence class of the unitary connection a in $\det E$. We denote by $\mathcal{M}^{st}(E) = \mathcal{M}^{st}_g(E)$ the moduli space of stable holomorphic structures in E and by $\mathcal{M}^{st}(E,L)$ the fiber of the determinant map $\det : \mathcal{M}^{st}(E) \longrightarrow \mathrm{Pic}(X)$ over an element L of $\mathrm{Pic}(X)$. Then one has the following formulation of the Kobayashi-Hitchin correspondence.

Theorem 4.1 *Let X be a compact complex surface, g a Gauduchon metric on X, E a differentiable vector bundle over X, a an anti-self-dual connection on* $\det E$ *(with respect to g) and L the element in* $\mathrm{Pic}(X)$ *given by $\bar\partial_a$ on* $\det E$. *Then $\mathcal{M}^{st}(E,L)$ is an open part of $Spl_X(L)$ and the mapping $A \mapsto \bar\partial_A$ gives* *rise to a real-analytic isomorphism between the moduli space* $\mathcal{M}^{ASD,*}(E,[a])$ *of irreducible anti-self-dual connections which induce* $[a]$ *on* $\det E$ *and* $\mathcal{M}^{st}(E,L)$.

We may also look at $\mathcal{M}^{ASD}(E,[a])$ in the following way. We consider all anti-self-dual connections inducing a fixed connection a on $\det E$ and factor by those gauge transformations in $\mathcal{G}$ which preserve a. This is the same as taking gauge transformations of (E,h) which induce a constant multiple of the identity on $\det E$. Since constant multiples of the identity leave each connection invariant, whether on $\det E$ or on E, we may as well consider the action of the subgroup of $\mathcal{G}$ inducing the identity on $\det E$. We denote this group by $S\mathcal{G}$, the quotient space by $\mathcal{M}^{ASD}(E,a)$ and by $\mathcal{M}^{ASD,*}(E,a)$ the part consisting of irreducible connections. There is a natural injective map

$$\mathcal{M}^{ASD}(E,a) \longrightarrow \mathcal{M}^{ASD}(E,[a])$$

which associates to an $S\mathcal{G}$-equivalence class of a connection A its $\mathcal{G}$-equivalence class. The surjectivity of this map depends on the possibility to lift any unitary gauge transformation of $\det E$ to a gauge transformation of E. This possibility exists if E has a rank-one differential sub-bundle, in particular when $r := \operatorname{rank} E > 2$, since then E has a trivial sub-bundle of rank $r-2$. In this case one constructs a lifting by putting in this rank-one component the given automorphism of $\det E$ and the identity on the orthogonal complement. A lifting also exists for all gauge transformations of $(\det E, \det h)$ admitting an r-th root. More precisely, denoting the gauge group of $(\det E, \det h)$ by $\mathcal{U}(1)$, it is easy to see that the elements of the subgroup $\mathcal{U}(1)^r := \{u^r \mid u \in \mathcal{U}(1)\}$ can be lifted to elements of $\mathcal{G}$. Since the obstruction to taking r-th roots in $\mathcal{U}(1)$ lies in $H^1(X,\mathbb{Z}_r)$, as one deduces from the corresponding short exact sequence, we see that $\mathcal{U}(1)^r$ has finite index in $\mathcal{U}(1)$. From this it is not difficult to infer that $\mathcal{M}^{ASD}(E,[a])$ is isomorphic to a topologically disjoint union of finitely many parts of the form $\mathcal{M}^{ASD}(E,a_k)$ with $[a_k]=[a]$ for all k.

4.2 The Uhlenbeck compactification

We continue by stating some results we need on the Uhlenbeck compactification of the moduli space of anti-self-dual connections. References for this material are [6] and [8].

Let (X,g) and (E,h) be as in 4.1. For each non-negative integer k we consider hermitian bundles (E_{-k},h_{-k}) on X with $\operatorname{rank} E_{-k} = \operatorname{rank} E =: r, (\det E_{-k}, \det h_{-k}) \cong (\det E, \det h)$, $c_2(E_{-k}) = c_2(E) - k$. Set

$$\bar{\mathcal{M}}^U(E) := \bigcup_{k\in\mathbb{N}} (\mathcal{M}^{ASD}(E_{-k}) \times S^kX)$$
$$\bar{\mathcal{M}}^U(E,[a]) := \bigcup_{k\in\mathbb{N}} (\mathcal{M}^{ASD}(E_{-k},[a]) \times S^kX)$$
$$\bar{\mathcal{M}}^U(E,a) := \bigcup_{k\in\mathbb{N}} (\mathcal{M}^{ASD}(E_{-k},a) \times S^kX)$$

where $S^k X$ is the k-th symmetric power of X. The elements of these spaces are called IDEAL CONNECTIONS. The unions are finite since the second Chern class of a hermitian vector bundle admitting an anti-self-dual connection is bounded below (by $\frac{1}{2}c_1^2$).
To an element $([A], Z) \in \bar{\mathcal{M}}^U(E)$ one associates a Borel measure

$$\mu([A], Z) := |F_A|^2 + 8\pi^2 \delta_Z$$

where δ_Z is the Dirac measure whose mass at a point x of X equals the multiplicity $m_x(Z)$ of x in Z. We denote by $m(Z)$ the total multiplicity of Z. A topology for $\bar{\mathcal{M}}^U(E)$ is determined by the following neighborhood basis for $([A], Z)$:

$$\begin{aligned} V_{U,N,\epsilon}([A], Z) = \{([A'], Z') \in \bar{\mathcal{M}}^U(E) \mid \mu([A'], Z') \in U \text{ and there is an} \\ L_3^2\text{ -isomorphism } \psi : E_{-m(Z)} \mid_{X\setminus N} \longrightarrow E_{-m(Z')} \mid_{X\setminus N} \\ \text{such that } \|A - \psi^*(A')\|_{L_2^2(X\setminus N)} < \epsilon\} \end{aligned}$$

where $\epsilon > 0$ and U and N are neighborhoods of $\mu([A], Z)$ and $\operatorname{supp}(\delta_Z)$ respectively. This topology is first-countable and Hausdorff and induces the usual topology on each $\mathcal{M}^{ASD}(E_{-k}) \times S^k X$. Most importantly, by the weak compactness theorem of Uhlenbeck $\bar{\mathcal{M}}^U(E)$ is compact when endowed with this topology, $\mathcal{M}^{ASD}(E)$ is an open part of $\bar{\mathcal{M}}^U(E)$ and its closure $\bar{\mathcal{M}}^{ASD}(E)$ inside $\bar{\mathcal{M}}^U(E)$ is called the UHLENBECK COMPACTIFICATION of $\mathcal{M}^{ASD}(E)$. Analogous statements are valid for $\mathcal{M}^{ASD}(E, [a])$ and $\mathcal{M}^{ASD}(E, a)$.
Using a technique due to Taubes, one can obtain a neighborhood of an irreducible ideal connection $([A], Z)$ in the border of $\mathcal{M}^{ASD}(E, a)$ by gluing to A "concentrated" $SU(r)$ anti-self-dual connections over S^4. One obtains "cone bundle neighborhoods" for each such ideal connection $([A], Z)$ when $H^2(X, \mathcal{E}nd_0(E_{\bar\partial_A})) = 0$. For the precise statements and the proofs we refer the reader to [6] chapters 7 and 8 and to [8] 3.4. As a consequence of this description and of the connectivity of the moduli spaces of $SU(r)$ anti-self-dual connections over S^4 (see [15]) we have the following weaker property which will suffice to our needs.

PROPOSITION 4.2 *Around an irreducible ideal connection $([A], Z)$ with $H^2(X, \mathcal{E}nd_0(E_{\bar\partial_A})) = 0$ the border of the Uhlenbeck compactification $\bar{\mathcal{M}}^{ASD}(E, a)$ is* LOCALLY NON-DISCONNECTING *in $\bar{\mathcal{M}}^{ASD}(E, a)$, i.e. there exist arbitrarily small neighborhoods V of $([A], Z)$ in $\bar{\mathcal{M}}^{ASD}(E, a)$ with $V \bigcap \mathcal{M}^{ASD}(E, a)$ connected.*

Note that for $SU(2)$ connections a lot more has been proved, [7], [18]. In this case the Uhlenbeck compactification is the completion of the space of anti-self-dual connections with respect to a natural Riemannian metric.

4.3 The comparison map

We fix (X, g) a compact complex surface together with a Gauduchon metric on it, (E, h) a hermitian vector bundle over X, a an unitary anti-self-dual connection on $(\det E, \det h)$ and denote by L the (isomorphy class of the) holomorphic line bundle induced by $\bar\partial_a$ on $\det E$. Let $c_2 := c_2(E)$ and $r := \operatorname{rank} E$. We denote by $\mathcal{M}^{st}(r, L, c_2)$ the subset of Spl_X consisting of isomorphy classes of non-necessarily locally free sheaves F (with respect to g) with $\operatorname{rank} F = r, \det F = L, c_2(F) = c_2$.
In 4.1 we have mentioned the existence of a real-analytic isomorphism between $\mathcal{M}^{st}(E, L)$ and $\mathcal{M}^{ASD,*}(E, [a])$. When X is algebraic, $\operatorname{rank} E = 2$ and a is the trivial connection this isomorphism has been extended to a continuous map from the Gieseker compactification of $\mathcal{M}^{st}(E, \mathcal{O})$ to the Uhlenbeck compactification of $\mathcal{M}^{ASD}(E, 0)$ in [16] and [13]. The proof given in [16] adapts without difficulty to our case to show the continuity of the natural extension

$$\Phi : \mathcal{M}^{st}(r, L, c_2) \longrightarrow \bar{\mathcal{M}}^U(E, [a]).$$

Φ is defined by $\Phi([F]) = ([A], Z)$, where A is the unique unitary anti-self-dual connection inducing the holomorphic structure on $F^{\vee\vee}$ and Z describes the singularity set of F with multiplicities $m_x(Z) := \dim_{\mathbb{C}}(F_x^{\vee\vee}/F_x)$ for $x \in X$. The main result of this paragraph asserts that under certain conditions for X and E this map is proper as well.

Theorem 4.3 *Let X be a non-algebraic compact complex surface which has either Kodaira dimension $\operatorname{kod}(X) = -\infty$ or has trivial canonical bundle and let g be a Gauduchon metric on X. Let (E, h) be a hermitian vector bundle over X, $r := \operatorname{rank} E$, $c_2 := c_2(E)$, a an unitary anti-self-dual connection on $(\det E, \det h)$ and L the holomorphic line bundle induced by $\bar\partial_a$ on $\det E$. If $(r, c_1(L), c_2)$ satisfies condition (*) from section 2 then the following hold:*

(a) *the natural map $\Phi : \mathcal{M}^{st}(r, L, c_2) \longrightarrow \bar{\mathcal{M}}^U(E, [a])$ is continuous and proper,*

(b) *any unitary automorphism of $(\det E, \det h)$ lifts to an automorphism of (E, h) and*

(c) *$\mathcal{M}^{st}(r, L, c_2)$ is a compact complex (Hausdorff) manifold.*

Proof
Under the Theorem's assumptions we prove the following claims.
Claim 1. Spl_X is smooth and of the expected dimension at points $[F]$ of $\mathcal{M}^{st}(r, L, c_2)$.
By Corollary 3.4 for such a stable sheaf F we have to check that $Ext^2(X; F, F)_0 = 0$. When K_X is trivial this is equivalent to $\dim(Ext^2(X; F, F)) = 1$ and by Serre duality further to $\dim(Hom(X; F, F)) = 1$ which holds since stable sheaves are simple. So let now X be non-algebraic and

$\mathrm{kod}(X) = -\infty$. By surface classification $b_1(X)$ must be odd and Remark 2.2 shows that F is irreducible. In this case we shall show that $Ext^2(X; F, F) = 0$. By Serre duality we have $Ext^2(X; F, F) \cong \mathrm{Hom}(X; F, F \otimes K_X)^*$. By taking double duals $\mathrm{Hom}(X; F, F \otimes K_X)$ injects into $\mathrm{Hom}(X; F^{\vee\vee}, F^{\vee\vee} \otimes K_X)$. Suppose φ is a non-zero homomorphism $\varphi : F^{\vee\vee} \to F^{\vee\vee} \otimes K_X$. Then $\det\varphi : \det F^{\vee\vee} \to (\det F^{\vee\vee}) \otimes K_X^{\otimes r}$ cannot vanish identically since F is irreducible. Thus it induces a non-zero section of $K_X^{\otimes r}$ contradicting $\mathrm{kod}(X) = -\infty$.

Claim 2. $\mathcal{M}^{st}(r, L, c_2)$ *is open in* Spl_X.

This claim is known to be true over the open part of Spl_X parameterizing simple locally free sheaves and holds possibly in all generality. Here we give an ad-hoc proof.

If b_1 is odd or if the degree function $\deg_g : Pic(X) \longrightarrow \mathbb{R}$ vanishes identically the assertion follows from the condition (*). Suppose now that X is non-algebraic with b_1 even and trivial canonical bundle. Let F be a torsion-free sheaf on X with $\mathrm{rank}\, F = r, \det F = L$ and $c_2(F) = c_2$. If F is not stable then F sits in a short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

with F_1, F_2 torsion-free coherent sheaves on X. Let $r_1 := \mathrm{rank}\, F_1, r_2 := \mathrm{rank}\, F_2$. We first show that the possible values for $\deg F_1$ lie in a discrete subset of $\mathbb{R}$. An easy computation gives

$$-\frac{c_1(F_1)^2}{r_1} - \frac{c_1(F_2)^2}{r_2} = -\frac{c_1(F)^2}{r} + 2r\Delta(F) - 2r_1\Delta(F_1) - 2r_2\Delta(F_2).$$

Since all discriminants are non-negative we get

$$-\frac{c_1(F_1)^2}{r_1} - \frac{c_1(F_2)^2}{r_2} \leq -\frac{c_1(F)^2}{r} + 2r\Delta(F).$$

In particular $c_1(F_1)^2$ is bounded by a constant depending only on $(r, c_1(L), c_2)$. Since X is non-algebraic the intersection form on $NS(X)$ is negative semi-definite. In fact, by [4] $NS(X)/Tors(NS(X))$ can be written as a direct sum $N \bigoplus I$ where the intersection form is negative definite on N, I is the isotropy subgroup for the intersection form and I is cyclic. We denote by c a generator of I. It follows the existence of a finite number of classes b in N for which one can have $c_1(F_1) = b + \alpha c$ modulo torsion, with $\alpha \in \mathbb{N}$. Thus $\deg F_1 = \deg b + \alpha \deg c$ lies in a discrete subset of $\mathbb{R}$.

Let now $b \in NS(X)$ be such that $0 < \deg b \leq |\deg F_1|$ for all possible subsheaves F_1 as above with $\deg F_1 \neq 0$. We consider the torsion-free stable central fiber $\mathcal{F}_0$ of a family of sheaves $\mathcal{F}$ on $X \times S$ flat over S. Suppose that $\mathrm{rank}(\mathcal{F}_0) = r, \det\mathcal{F}_0 = L, c_2(\mathcal{F}_0) = c_2$. We choose an irreducible vector bundle G on X with $c_1(G) = -b$. Then $H^2(X, \mathcal{H}om(\mathcal{F}_0, G)) = 0$, so if $\mathrm{rank}\, G$ is large enough we can apply Proposition 3.1 to get an extension

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

with E locally free on $X \times S$, for a possibly smaller S. (As in Proposition 3.1 we have denoted by q the projection $X \times S \longrightarrow S$.) It is easy to check that E_0 doesn't have any subsheaf of degree larger than $-\deg b$. Thus E_0 is stable. Hence small deformations of E_0 are stable as well. As a consequence we get that small deformations of $\mathcal{F}_0$ will be stable. Indeed, it is enough to consider for a destabilizing subsheaf F_1 of F_s, for $s \in S$, the induced extension

$$0 \longrightarrow G \longrightarrow E_1 \longrightarrow F_1 \longrightarrow 0.$$

Then E_1 is a subsheaf of E_s with $\deg E_1 = \deg G + \deg F_1 \geq 0$. This contradicts the stability of E_s.

Claim 3. Any neighborhood in Spl_X of a point $[F]$ of $\mathcal{M}^{st}(r, L, c_2)$ contains isomorphy classes of locally free sheaves.

The proof goes as in the algebraic case by considering the "double-dual stratification" and making a dimension estimate. Here is a sketch of it.

If one takes a flat family $\mathcal{F}$ of torsion free sheaves on X over a reduced base S, one may consider for each fiber $\mathcal{F}_s$, $s \in S$, the injection into the double-dual $\mathcal{F}_s^{\vee\vee} := \mathcal{H}om(\mathcal{H}om(\mathcal{F}_s, \mathcal{O}_{X\times\{s\}}), \mathcal{O}_{X\times\{s\}})$. The double-duals form a flat family over some Zariski-open subset of S. To see this consider first $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_{X\times S})$. Since $\mathcal{F}$ is flat over S, one gets $(\mathcal{F}_s)^\vee = \mathcal{F}_s^\vee$. $\mathcal{F}^\vee$ is flat over the complement of a proper analytic subset of S and one repeats the procedure to obtain $\mathcal{F}^{\vee\vee}$ and $\mathcal{F}^{\vee\vee}/\mathcal{F}$ flat over some Zariski open subset S' of S. Over $X \times S'$, $\mathcal{F}^{\vee\vee}$ is locally free and $(\mathcal{F}^{\vee\vee}/\mathcal{F})_s = \mathcal{F}_s^{\vee\vee}/\mathcal{F}_s$ for $s \in S'$. Take now S a neighborhood of $[F]$ in $\mathcal{M}^{st}(r, L, c_2)$. Suppose that

$$\operatorname{length}(\mathcal{F}_{s_0}^{\vee\vee}/\mathcal{F}_{s_0}) = k > 0$$

for some $s_0 \in S'$. Taking S' smaller around s_0 if necessary, we find a morphism ϕ from S' to a neighborhood T of $[\mathcal{F}_{s_0}^{\vee\vee}]$ in $\mathcal{M}^{st}(r, L, c_2 - k)$ such that there exists a locally free universal family $\mathcal{E}$ on $X \times T$ with $\mathcal{E}_{t_0} \cong \mathcal{F}_{s_0}^{\vee\vee}$ for some $t_0 \in T$ and $(\mathrm{id}_X \times \phi)^*\mathcal{E} = \mathcal{F}^{\vee\vee}$. Let D be the relative Douady space of quotients of length k of the fibers of $\mathcal{E}$ and let $\pi : D \longrightarrow T$ be the projection. There exists an universal quotient $\mathcal{Q}$ of $(\mathrm{id}_X \times \pi)^*\mathcal{E}$ on $X \times D$. Since $\mathcal{F}^{\vee\vee}/\mathcal{F}$ is flat over S', ϕ lifts to a morphism $\tilde{\phi} : S' \longrightarrow D$ with $(id_X \times \tilde{\phi})^*\mathcal{Q} = \mathcal{F}^{\vee\vee}/\mathcal{F}$. By the universality of S' there exists also a morphism (of germs) $\psi : D \longrightarrow S'$ with $(\mathrm{id}_X \times \psi)^*\mathcal{F} = \operatorname{Ker}((\mathrm{id}_X \times \pi)^*\mathcal{E} \longrightarrow \mathcal{Q})$. One sees now that $\psi \circ \tilde{\phi}$ must be an isomorphism, in particular $\dim S' \leq \dim D$. Since S' and T have the expected dimensions, it is enough to compute now the relative dimension of D over T. This is $k(r+1)$. On the other side by Corollary 3.4 $\dim S' - \dim T = 2kr$. This forces $r = 1$ which is excluded by hypothesis.

After these preparations of a relatively general nature we get to the actual proof of the Theorem. We start with (b).

If $b_2^-(X)$ denotes the number of negative eigenvalues of the intersection form on $H^2(X, \mathbb{R})$, then for our surface X we have $b_2^-(X) > 0$. This is clear when K_X is trivial by classification and follows from the index theorem and Remark 2.2 (d) when $b_1(X)$ is odd. In particular, taking $p \in H^2(X, \mathbb{Z})$ with $p^2 < 0$ one constructs topologically split rank two vector bundles F with given first Chern class l and arbitrarily large second Chern class: just consider $(L\otimes P^{\otimes n})\oplus(P^*)^{\otimes n}$ where L and P are line bundles with $c_1(L)=l$, $c_1(P)=p$ and $n\in\mathbb{N}$. If E has rank two we take F with $\det F\cong\det E$ and $c_2(F)\geq c_2(E)=c_2$. (When $r>2$ assertion (b) is trivial; cf. section 4.1). We consider an anti-self-dual connection A in E inducing a on $\det E$ and $Z\subset X$ consisting of $c_2(F)-c_2(E)$ distinct points. By the computations from the proof of Claim 1 we see that A is irreducible and $H^2_{A,0}=0$. Using the gluing procedure mentioned in section 4.2 , one sees that a neighborhood of $([A],Z)$ in $\bar{\mathcal{M}}^U(F,[a])$ contains classes of irreducible anti-self-dual connections in F. We have seen in section 4.1 that any unitary automorphism of $\det F$ lifts to an unitary automorphism u of F. If we take a sequence of anti-self-dual connections (A_n) in F with $\det A_n=a$ and $([A_n])$ converging to $([A],Z)$, we get by applying u a limit connection B for subsequence of $(u(A_n))$. Since $\bar{\mathcal{M}}^U(F,[a])$ is Hausdorff, there exists an unitary automorphism $\tilde{u}$ of E with $\tilde{u}^*(B)=A$. It is clear that $\tilde{u}$ induces the original automorphism u on $\det F\cong\det E$.

We leave the proof of the following elementary topological lemma to the reader.

Lemma 4.4 *Let $\pi:Z\longrightarrow Y$ be a continuous surjective map between Hausdorff topological spaces. Suppose Z locally compact, Y locally connected and that there is a locally non-disconnecting closed subset Y_1 of Y with $Z_1:=\pi^{-1}(Y_1)$ compact and $\overset{\circ}{Z}_1=\emptyset$. Suppose further that π restricts to a homeomorphism*

$$\pi\mid_{Z\setminus Z_1,Y\setminus Y_1}: Z\setminus Z_1\longrightarrow Y\setminus Y_1.$$

Then for any neighborhood V of Z_1 in Z, $\pi(V)$ is a neighborhood of Y_1 in Y. If in addition Y is compact, then Z is compact as well.

We complete now the proof of the Theorem by induction on c_2. For fixed r and $c_1(E)$, $c_2(E)$ is bounded below if E is to admit an anti-self-dual connection. If we take c_2 minimal, then $\mathcal{M}^{st}(r,L,c_2)=\mathcal{M}^{st}(E,L)$ and $\mathcal{M}^{ASD,*}(E,[a])=\mathcal{M}^{ASD}(E,[a])$ is compact. From Theorem 4.1 we obtain that Φ is a homeomorphism in this case.

Take now c_2 arbitrary but such that the hypotheses of the Theorem hold and assume that the assertions of the Theorem are true for any smaller c_2. We apply Lemma 4.4 to the following situation:

$$Z:=\mathcal{M}^{st}(r,L,c_2),\ Y:=\bar{\mathcal{M}}^U(E,[a])=\bar{\mathcal{M}}^{ASD}(E,[a])\cong\bar{\mathcal{M}}^{ASD}(E,a).$$

The last equalities hold according to Claim 3 and Claim 4. Let further Y_1 be the border $\bar{\mathcal{M}}^{ASD}(E,a)\setminus\mathcal{M}^{ASD}(E,a)$ of the Uhlenbeck compactification and Z_1 be the locus $\mathcal{M}^{st}(r,L,c_2)\setminus\mathcal{M}^{st}(E,L)$ of singular stable sheaves in Spl_X. Z is smooth by Claim 1 and Hausdorff, Y_1 is locally non-disconnecting by Proposition 4.2, $\overset{\circ}{Z}_1=\emptyset$ by Claim 3 and $\pi\mid_{Z\setminus Z_1,Y\setminus Y_1}$ is a homeomorphism by Theorem 4.1. In order to be able to apply Lemma 4.4 and thus close the proof we only need to check that Z_1 is compact. We want to reduce this to the compactness of $\mathcal{M}^{st}(r, L, c_2 - 1)$ which is ensured by the induction hypothesis. We consider a finite open covering (T_i) of $\mathcal{M}^{st}(r, L, c_2 - 1)$ such that over each $X \times T_i$ an universal family $\mathcal{E}_i$ exists. The relative Douady space D_i parameterizing quotients of length one in the fibers of $\mathcal{E}_i$ is proper over (T_i). In fact it was shown in [11] that $D_i \cong \mathbb{P}(\mathcal{E}_i)$. If $\pi_i : D_i \longrightarrow T_i$ are the projections, we have universal quotients $\mathcal{Q}_i$ of $\pi^*\mathcal{E}_i$ and $\mathcal{F}_i := \mathrm{Ker}(\pi^*\mathcal{E}_i \longrightarrow \mathcal{Q}_i)$ are flat over D_i. This induces canonical morphisms $D_i \longrightarrow Z_1$. It is enough to notice that their images cover Z_1, or equivalently, that any singular stable sheaf F over X sits in an exact sequence of coherent sheaves

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0$$

with $\mathrm{length}\, Q = 1$ and E torsion-free. Such an extension is induced from

$$0 \longrightarrow F \longrightarrow F^{\vee\vee} \longrightarrow F^{\vee\vee}/F \longrightarrow 0$$

by any submodule Q of length one of $F^{\vee\vee}/F$. (To see that such Q exist recall that $(F^{\vee\vee}/F)_x$ is artinian over $\mathcal{O}_{X,x}$ and use Nakayama's Lemma). The Theorem is proved. $\square$

Remark 4.5 As a consequence of this theorem we get that when X is a 2-dimensional complex torus or a primary Kodaira surface and (r, L, c_2) is chosen in the stable irreducible range as in [24], [23] or [1], then $\mathcal{M}^{st}(r, L, c_2)$ is a holomorphically symplectic compact complex manifold.

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How Frequent Are Discrete Cyclic Subgroups of Semisimple Lie Groups?

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Abstract. Let G be a non-compact semisimple Lie group. We investigate the asymptotic behaviour of the probability of generating a discrete subgroup.

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1 Main result

For a locally compact topological group G let us define Δ_G as the set of all $g \in G$ such that the cyclic subgroup $\{g^n : n \in \mathbb{Z}\}$ of G is discrete. If there is no danger of ambiguity, we write simply Δ instead of Δ_G.
Let G be a connected non-compact real semisimple Lie group and μ a Haar measure on G. In a preceding article ([3]) we proved that $\mu(\Delta_G) = \infty$ and that furthermore $\mu(G \setminus \Delta_G) = \infty$ if G contains a compact Cartan subgroup and $\mu(G \setminus \Delta_G) = 0$ otherwise.
During the "Colloquium on Lie Theory and Application" in Vigo in July 2000 K. H. Hofmann suggested to me to investigate the asymptotic behavior of the ratio of volumes of the respective intersections with balls.
This paper is concerned with establishing such an asymptotic description.
The first problem is to make precise what is meant "balls". What is a natural choice of "balls" to be considered here? The first idea would be to consider balls with respect to some Riemannian metric which should be a canonical as possible. However, a non-compact semisimple Lie group does not admit any Riemannian metric invariant under both left and right translations and there is no good reason to discriminate against left or right wingers.
Here we took a different approach. Let K be a maximal compact subgroup of G and consider the double quotient $X = K\backslash G/K$. For a continuous exhaustion

function ρ on X we define "balls" $B_r = \{\rho < r\}$. We demonstrate that with respect to such an exhaustion asymptotically the share of Δ tends to one.
Now let us proceed to a precise statement.
First we recall that an "exhaustion function" ρ on a topological space X is a continuous map $\rho : X \to \mathbb{R}^+ \cup \{0\}$ such that $\rho^{-1}([0,r])$ is compact for all $r \in \mathbb{R}^+$. If H is a subgroup of a topological group G, then an exhaustion function ρ on G is called "H-biinvariant" if $\rho(hgh') = \rho(g)$ for all $g \in G$, $h, h' \in H$.

Theorem. *Let G be a connected, non-compact real semisimple Lie group, Δ the set of all elements $g \in G$ for which the generated subgroup $\{g^n : n \in \mathbb{Z}\}$ is discrete in G, μ a Haar measure on G, K a maximal compact subgroup of G, and $\rho : G \to \mathbb{R}^+ \cup \{0\}$ a K-biinvariant exhaustion function.*
Let $B_r = \{g \in G : \rho(g) < r\}$.
Then

$$\lim_{r\to\infty} \frac{\mu(\Delta \cap B_r)}{\mu(B_r)} = 1.$$

Proof. Let Z denote the center of G. We distinguish three different cases, depending on the cardinality of Z.
Case 1. Here we assume that the center Z is trivial. Then G admits a faithful representation $\lambda : G \to GL(V)$ (for instance, the adjoint representation is faithful.) Note that $\{g^n : n \in \mathbb{Z}\}$ must be discrete if $g \in G$ with $|\operatorname{Tr}(\lambda(g))| > n$.
Let K be a maximal compact subgroup of G, $Lie(G) = Lie(K) + \mathbf{p}$ a Cartan decomposition, $\mathbf{a}$ a maximal Abelian subspace of $\mathbf{p}$ and A the corresponding connected Lie subgroup of G. Then (see e.g. [2]) A is a reductive connected and simply-connected Lie group and closed in G. It follows that, in suitably chosen coordinates on V, the image $\lambda(A)$ is a closed subset of the set D^+ of all diagonal matrices with all entries non-negative. This implies in particular that $g \mapsto \operatorname{Tr}(\lambda(g))$ defines an exhaustion function on the closed set A.
Next recall that $G = KAK$ by a result of É. Cartan ([1], see also [2], thm.7.39). We will consider the double coset space $X = K\backslash G/K$ and the natural projection $p : G \to X$. By results due to Cartan (see [2]) $X = K\backslash G/K \simeq A/W$ where $W = N_G(A)/A$ is the (restricted) Weyl group. Since the trace of an endomorphism is invariant under conjugation, $\operatorname{Tr} \circ \lambda|_A$ is W-invariant, and therefore there exists an exhaustion function τ on X such that $\operatorname{Tr} \circ \lambda$ and $\tau \circ p$ coincide on A.
Using the natural projection $p : G \to X \simeq K\backslash G/K$ we define a Borel measure η on X by setting $\eta(U) = \mu(p^{-1}(U))$ for every Borel set $U \subset X$. This is an infinite measure, $\eta(X) = \mu(G) = +\infty$, and for every compact set $C \subset X$ we have $\eta(C) < \infty$, because $p^{-1}(C)$ is compact, too. Let ξ denote the normalized Haar measure on $K \times K$. Then for all $f \in C_c(G)$

$$\int_G f(g)d\mu(g) = \int_X \int_{K\times K} f(kah)d\xi(k,h)d\eta(a).$$

Next we define a function

$$\zeta : \mathrm{End}(\mathbb{R}^n) \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$$

by

$$\zeta(a, R) = \xi\left(S(a, R)\right)$$

where

$$S(a, R) = \{(k, h) \in K : |\operatorname{Tr}(\lambda(k) \cdot a \cdot (\lambda(h)))| < R\}.$$

If $\operatorname{Tr}(a) \neq 0$, then

$$\{(k, h) \in K : \operatorname{Tr}(\lambda(k) \cdot a \cdot (\lambda(h))) = 0\}$$

is a nowhere dense real analytic subset of $K \times K$ and therefore of measure zero. It follows that

$$\lim_{t \to 0} \zeta(a, t) = 0$$

for all $a \in \mathrm{End}(\mathbb{R}^n)$ with $\operatorname{Tr}(a) \neq 0$.
We observe that

$$S(\tilde{a}, R) \subset S(a, R + \epsilon)$$

for $a, \tilde{a} \in \mathrm{End}(\mathbb{R}^n)$ with $\sum_{i=1}^n |\lambda(a)_{ii} - \lambda(\tilde{a})_{ii}| < \epsilon$. Using this, it follows that

$$\lim_{n \to \infty} \zeta(a_n, r_n) = 0$$

for all convergent sequences $(a_n)_n$ in $\mathrm{End}(\mathbb{R}^n)$, $(r_n)_n$ in $\mathbb{R}$ with $\lim r_n = 0$ and $\operatorname{Tr}(\lim a_n) \neq 0$.
This in turns implies, that if we have a compact subset $C \subset \mathrm{End}(\mathbb{R}^n)$ such that $\operatorname{Tr}(c) \neq 0$ for all $c \in C$, then

$$\lim_{t \to 0} \left(\sup_{c \in C} \zeta(c, t) \right) = 0.$$

We now define such a compact set. Let C be the set of all diagonal matrices $diag(d_1, \ldots, d_n)$ in $\mathrm{End}(\mathbb{R}^n)$ with $0 \leq d_i \leq 1$ for all i and $\sum_i d_i = 1$.
Now C is a compact set with $\operatorname{Tr}(c) = 1$ for all $c \in C$. By definition of C it is clear that for every $a \in A$ there is an element $c \in C$ such that $c \operatorname{Tr}(\lambda(a)) = \lambda(a)$.
We claim: For every $\epsilon > 0$ there exists a number $R_0 > 0$ such that $\zeta(\lambda(a), n + 1) < \epsilon$ for all $a \in D^+$ with $\operatorname{Tr}(a) \geq R_0$. Indeed, for every ϵ there is a number δ_0 such that

$$\zeta(c, \delta) < \epsilon$$

for all $c \in C$, $\delta \leq \delta_0$.
By the linearity of the trace operator, we have

$$\zeta(c,\delta) = \zeta(xc, x\delta)$$

for all $c \in C$ and $x \in \mathbb{R}^+$. Now let $v \in D^+$. Then $v = xc$ with $c \in C$, $x \in \mathbb{R}^+$ and $\mathrm{Tr}(v) = x$. This implies

$$\zeta(v,\delta) = \zeta(xc,\delta) \leq \zeta(\mathrm{Tr}(v)c,\delta) = \zeta(c, \delta/\mathrm{Tr}(v)).$$

Therefore $\zeta(v,\delta) < \epsilon$ whenever $\delta/\mathrm{Tr}(v) < \delta_0$. Thus

$$\zeta(v, n+1) < \epsilon$$

for all $v \in D^+$ with $\mathrm{Tr}(v) > R_0 = (n+1)/\delta_0$.
Now fix a number $\epsilon > 0$. We will demonstrate that there exists a number $R > 0$ such that

$$\frac{\mu(\Delta \cap B_r)}{\mu(B_r)} > 1 - \epsilon$$

for all $r \geq R$.
We start by choosing R_0 such that $\zeta(a, n+1) < \epsilon/2$ for all $a \in A^+ \subset D^+$ with $\mathrm{Tr}(a) \geq R_0$. Recall that $\eta(X) = +\infty$ and that $\theta : X \to \mathbb{R}_0^+$ is an exhaustion function. Hence we may choose a number $R_1 > R_0$ such that

$$\eta(\{x \in X : \theta(x) \leq R_0\}) < \frac{\epsilon}{2}\eta(\{x \in X : \theta(x) \leq R_1\}).$$

Finally choose R such that $\{\theta \leq R_1\} \subset p(B_R)$.
Now we have for $r > R$:

$$\begin{aligned}
\mu(B_r \setminus \Delta) &\leq \mu(\{g \in B_r : \mathrm{Tr}(\lambda(g)) < n+1\}) \\
&= \int_{x \in p(B_r)} \zeta(x, n+1) d\eta \\
&= \int_{\theta(x) \leq R_0} \zeta(x, n+1) d\eta + \int_{R_0 \leq \theta(x), x \in p(B_r)} \zeta(x, n+1) d\eta \\
&< \eta(\{x \in X : \theta(x) \leq R_0\}) + \int_{R_0 \leq \theta(x), x \in p(B_r)} \frac{\epsilon}{2} d\eta \\
&\leq \frac{\epsilon}{2}\eta(\{x \in X : \theta(x) \leq R_0\}) + \frac{\epsilon}{2}\eta(\{x \in p(B_r) : R_0 \leq \theta(x)\}) \\
&= \epsilon\eta(p(B_r)) = \epsilon\mu(B_r).
\end{aligned}$$

Case 2. Assume that Z is finite, but non-trivial. Let $p : G \to G/Z$ denote the natural projection. Since Z is compact and normal, it is contained in every maximal compact subgroup K. Therefore every K-biinvariant exhaustion functions on G is a pull-back of a $p(K)$-biinvariant exhaustion function on G/Z.

Finiteness of Z furthermore implies that $p^{-1}(\Delta_{G/Z}) = \Delta_G$ and that the Haar measure on G/Z pulls back to a Haar measure on G. Therefore the statement of the theorem for this case follows from the proof for case 1.
Case 3. Assume that Z is infinite. In this case Z is not compact. Since Cartan subgroups are maximally nilpotent and therefore necessarily contain Z, this implies that G admits no compact Cartan subgroups. By the results of [3] it follows that in this case $\mu(G \setminus \Delta) = 0$, implying $\mu(\Delta \cap B_r) = \mu(B_r)$ for all $r \in \mathbb{R}^+$.

$\square$

2 Interpretation from a Lie algebra point of view

One may consider the projection $Lie(G) \to \mathbb{P}(Lie(G))$. In the projective space $\mathbb{P}(Lie(G))$ both the set corresponding to compact Cartan subgroups as well as the set corresponding to non-compact Cartan subgroups contains non-empty open sets, if we assume that G is a non compact semisimple Lie group containing a compact Cartan subgroup. In this sense it seems that one the Lie algebra level the set Δ and its complement look as having the same size. How does this reconcile with our result? The answer may be found in the following reasoning: The correspondence between Lie algebra and Lie group is given by the exponential map. However, the exponential map behaves quite differently for compact and non-compact Cartan subgroups: it is injective on non-compact Cartan subgroups and has infinite kernel for compact Cartan subgroups. Thus, multiplicities are quite different for Lie algebra and Lie groups. Taking these multiplicities into account, it appears only reasonable that on the Lie group Δ dominates if both sides have the same size in $\mathbb{P}(Lie(G))$.

3 Explicit calculations for $SL(2,\mathbb{R})$

In this section, we deduce explicit results for the special case $G = SL_2(\mathbb{R})$.
In this case the KAK-decomposition can be written as the map

$$F : S^1 \times \mathbb{R}^{\geq 1} \times S^1 \to SL_2(\mathbb{R})$$

given by

$$F : (\theta, s, \phi) \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} s & \\ & s^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Then

$$F^* d\mu = 2\pi \left(s - s^{-3}\right) \frac{d\theta}{2\pi} \wedge ds \wedge \frac{d\phi}{2\pi}$$

for a Haar measure $d\mu$ on G. We can define a K-biinvariant exhaustion function ρ on $SL_2(\mathbb{R})$ by

$$\rho(g) = \max_{v \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{||g(v)||}{||v||}.$$

Then $\rho(F(\theta,s,\phi)) = s$.
An element $g \in SL_2(\mathbb{R})$ generates a discrete subgroup if and only if it is diagonalizable or unipotent or a torsion element. It follows that $g \in \Delta$ iff $|\operatorname{Tr}(g)| \geq 2$ or $\frac{1}{2}\operatorname{Tr}(g) = \cos(\frac{q}{2\pi})$ for a rational number $q \in \mathbb{Q}$. Hence $\{g : |\operatorname{Tr}(g)| > 2\} \subset \Delta$ and

$$\mu\left(\Delta \setminus \{g : |\operatorname{Tr}(g)| > 2\}\right) = 0.$$

An easy calculation yields

$$\operatorname{Tr}(F(\theta,s,\phi)) \quad = \quad (s+s^{-1})\cos(\theta+\phi).$$

It follows that

$$1-\zeta(s,2) = \xi\left(\{(\theta,\phi) : |\operatorname{Tr}(F(\theta,s,\phi)| > 2\}\right) = 4\arccos\frac{2}{s+s^{-1}}.$$

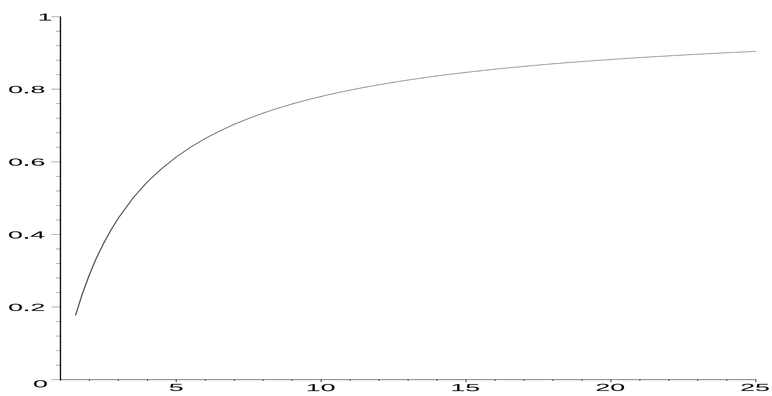
Therefore

$$\mu(B_r) = \int_{s=1}^{r} 2\pi(s-s^{-3})ds$$

and

$$\mu(B_r \cap \Delta) = \int_{s=1}^{r} 4\arccos\left(2/(s+s^{-1})\right)\ 2\pi(s-s^{-3})ds.$$

Using Maple, the graph of the function $f(r) = \mu(B_r \cap \Delta)/\mu(B_r)$ now appears as shown in the graphic below:



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Cyclic Projective Planes and Wada Dessins

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Abstract. Bipartite graphs occur in many parts of mathematics, and their embeddings into orientable compact surfaces are an old subject. A new interest comes from the fact that these embeddings give *dessins d'enfants* providing the surface with a unique structure as a Riemann surface and algebraic curve. In this paper, we study the (surprisingly many different) dessins coming from the graphs of finite cyclic projective planes. It turns out that all reasonable questions about these dessins — uniformity, regularity, automorphism groups, cartographic groups, defining equations of the algebraic curves, their fields of definition, Galois actions — depend on *cyclic orderings* of difference sets for the projective planes. We explain the interplay between number theoretic problems concerning these cyclic ordered difference sets and topological properties of the dessin like e.g. the *Wada property* that every vertex lies on the border of every cell.

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1 Finite projective planes and dessins d'enfants

1.1 Projective planes, bipartite graphs, and maps

It is well known that the incidence pattern of finite projective planes can be made visible by connected bipartite graphs using the following dictionary.

$$\begin{array}{rcl} \text{line} & \longleftrightarrow & \text{white vertex} \\ \text{point} & \longleftrightarrow & \text{black vertex} \\ \text{incidence} & \longleftrightarrow & \text{existence of a joining edge} \\ \text{flag} & \longleftrightarrow & \text{edge} \end{array}$$

Following this dictionary, the axioms of projective geometry translate into graph–theoretic properties like
For any two different black vertices there exists a unique white vertex as a common neighbor.
The same is true for elementary properties like
Every black vertex has precisely $q = n+1$ *white neighbors and every white vertex has precisely* $q = n+1$ *black neighbors. The graph has* $l = n^2+n+1$ *black and white vertices, respectively, and* ql *edges.*
As usual, we will call n the *order* of the projective plane. (Recall that up to now only finite projective planes of prime power order are known.) On the other hand, it is well known that connected graphs can be embedded as maps into oriented compact surfaces [Li].

1.2 Dessins d'enfants

Now, bipartite graphs embedded into orientable compact surfaces cutting these surfaces into simply connected cells represent a way to describe Grothendieck's *dessins d'enfants*.
Definition. A (p, q, r)–dessin is a bipartite graph on an orientable compact surface X with the following properties.

1. The complement of the graph is the disjoint union of simply connected open cells.
2. p is the l.c.m. of all valencies of the graph at the black points.
3. q is the l.c.m. of all valencies of the graph at the white points.
4. $2r$ is the l.c.m. of all valencies of the cells (i.e. the numbers of bordering edges; they have to be counted twice if they border the cell at both sides).

Dessins arise in a natural way on compact Riemann surfaces (non–singular complex projective algebraic curves) X if there is a non–constant meromorphic (= rational) Belyi function $\beta : X \to \overline{\mathbb{C}}$ ramified at most above $0, 1, \infty$. Then $\beta^{-1}\{0\}$, $\beta^{-1}\{1\}$ are the sets of white and black vertices respectively an the connected components of $\beta^{-1}]0,1[$ are the edges of the dessin. According to a theorem of Belyi such a function exists if and only if — as an algebraic curve — X can be defined over a number field. Moreover, for every dessin $\mathcal{D}$ on a compact orientable surface X there is a unique conformal structure on X such that $\mathcal{D}$ results from a corresponding Belyi function β on X. Therefore the combinatorics of dessins should encode all properties of curves definable over $\overline{\mathbb{Q}}$. For a survey on this topic, see [JS]. In the present paper, we concentrate on two aspects namely *uniformization theory* and *Galois actions*.
As a Riemann surface with a (p, q, r)–dessin, X is the quotient space of a subgroup Γ of the triangle group Δ of signature $\langle p, q, r\rangle$, acting discontinuously on $\overline{\mathbb{C}}$, $\mathbb{C}$ or the hyperbolic plane $\mathcal{H}$ if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1\,, \quad = 1 \quad \text{or} \quad < 1 \quad \text{respectively.}$$

The dessin is UNIFORM if all black points have equal valency p, all white points have equal valency q, and all cells have equal valency $2r$; equivalently, Γ has no torsion and is therefore the universal covering group of the Riemann surface. This is satisfied e.g. if the dessin is REGULAR, i.e. if its automorphism group G acts transitively on the edges; equivalently, Γ is a normal torsion–free subgroup of Δ (and then $\Delta/\Gamma \cong G$; for other reformulations of this condition see the first section of [StWo]). AUTOMORPHISM of the dessin means the restriction of an orientation–preserving topological — and automatically conformal — automorphism of X to the bipartite graph.

Recall that via the action of $\sigma \in \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ on the coefficients of the defining equations of the algebraic curve X — or of an extension of σ to $\mathrm{Aut}\,\mathbb{C}/\mathbb{Q}$ on the coordinates of their points — one has a Galois action on the set of Riemann surfaces. We can even speak of Galois actions on dessins in the following sense: for a dessins $\mathcal{D}$ on X consider the corresponding Belyi function β. Clearly, for every $\sigma \in \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ we have on the image curve X^σ a Belyi function β^σ defining a Galois conjugate dessin. (This Galois action is only the first step of Grothendieck's far reaching ideas for a better understanding of the structure of $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ via the so called Grothendieck–Teichmüller lego.)

1.3 The Fano plane. An easy observation

Concerning the embedding of the bipartite graph of a finite projective plane as a dessin on X, some immediate questions arise:

How does the structure of the Riemann surface depend on the choice of the embedding? Which additional structure of the projective plane (like e.g. $\mathrm{Aut}\,\mathbb{P}$*, the group of collineations) translates into a structure of the dessin and the Riemann surface?*

We are grateful to David Singerman who informed us about former work on these questions by himself [Si2], Fink and in particular Arthur White ([FiWh], [Wh]). In the following, we will take up their work under new topological and arithmetical aspects. Already the easiest example, i.e. the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$, shows the existence of different embeddings leading to different dessins.

Fig. 6.5 of [JS] shows one of two embeddings of the graph of the Fano plane as a regular $(3,3,3)$–dessin, consisting of 7 hexagons on a torus. The underlying Riemann surface is the torus $\mathbb{C}/\Lambda$ for the sublattice Λ of the hexagonal lattice $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$ corresponding to one of the two prime ideals of norm 7 in that ring of integers. The automorphism group G of the dessin is isomorphic to $Z_7 \rtimes Z_3$, in fact a subgroup of $\mathrm{PGL}_3(\mathbb{F}_2)$ (Z_m denotes the cyclic group of order m). This full group of collineations of the Fano plane contains elements not giving automorphisms of the dessin because an automorphism of the dessin fixing an edge is automatically the identity. There is another embedding of the Fano plane graph as a dessin to be discussed now which is better for generalizations to other projective planes: Identify $\mathbb{F}_2^3 - \{0\}$ with the multiplicative group $\mathbb{F}_8^*$

of order 7 and generator g. The exponents m of g give a bijection

$$\mathbb{P}^2(\mathbb{F}_2) \quad \longleftrightarrow \quad \mathbb{Z}/7\mathbb{Z}$$

and an analogous bijection — with g^{-1} as generator — for the lines of the Fano plane. To make the incidence structure visible we use the trace t of the field extension $\mathbb{F}_8/\mathbb{F}_2$ as a nondegenerate bilinear form

$$b \;:\; \mathbb{F}_8 \times \mathbb{F}_8 \;\to\; \mathbb{F}_2 \;:\; (x,y) \;\mapsto\; t(xy)\;.$$

Then the point x and the line y are incident if and only if $t(xy) = 0$. We may choose the generator g such that $t(g) = 0$; then a point g^m and a line g^{-k} are incident if and only

$$t(g^{m-k}) \;=\; 0 \quad \Longleftrightarrow \quad m-k \;\in\; \{1,2,4\}$$

what is easily seen using the Frobenius of $\mathbb{F}_8/\mathbb{F}_2$. Therefore, we may choose the local orientation of the Fano plane graph as given in Figure 1.

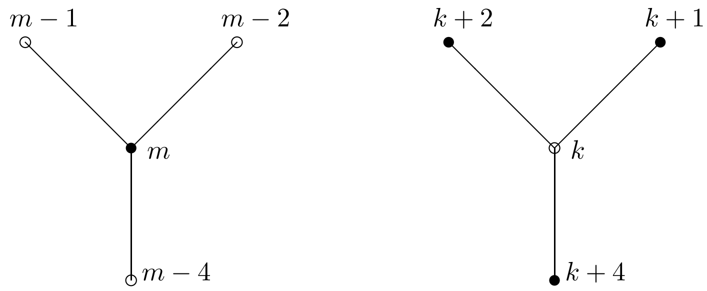

Figure 1: Local pattern of the Fano plane dessin

Then, the global dessin may be given as in Figure 2. To draw the picture on a Riemann surface, observe that every edge not incident with the white vertex 0 occurs twice. Identifying these edges, one obtains a $(3,3,7)$–dessin with 3 cells on a Riemann surface of genus 3. Here also, the automorphism group of the dessin is easily seen to be $Z_7 \rtimes Z_3$ which is a homomorphic image of the triangle group $\langle 3,3,7\rangle$ as well. Moreover, one may prove that the kernel Γ of this homomorphism is torsion free and a normal subgroup even in the triangle group $\langle 2,3,7\rangle$ with factor group $\mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{PGL}_3(\mathbb{F}_2)$. The Riemann surface is known to be uniquely determined by this property: it is Klein's quartic. One may vary Figure 1 by taking the mirror image on both sides: the global result will be a regular dessin looking like Figure 2 but with completely different identifications of the edges. Its automorphism group is again $Z_7 \rtimes Z_3$ and its Riemann surface is again Klein's quartic, and both dessins are Galois conjugate in the sense explained above, see Theorem 1.

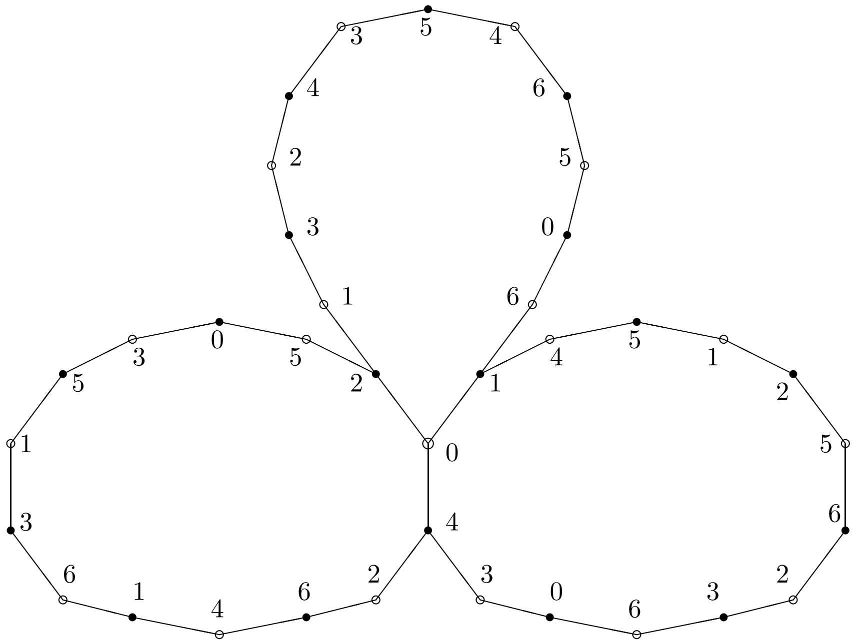

Figure 2: A $(3, 3, 7)$–dessin of the Fano plane

After all necessary identifications, we see that this Fano plane dessin has the remarkable property that every vertex lies on the border of every cell. Such phenomena occur even for subdivisions of the Euclidean plane into simply connected open domains, as was long time ago known to Kerékjártó and Brouwer ([Ke], p.120), and became popular more recently under the name *lakes of Wada* in the theory of dynamical systems [Ch]. Therefore, we propose the following *Definition.* A WADA DESSIN is characterized by the property that every vertex lies on the border of every cell.

(This property may be reformulated passing to a dual dessin by exchanging e.g. the white vertices with the cells: then we obtain a complete bipartite graph embedded in such a way that every white vertex lies on the border of every cell.) Comparing the four realizations of the Fano plane graph as dessins one may remark that the global picture depends heavily on the choice of the local orientation of the edges around the vertices, see the proof of Proposition 2. How typical are the Fano plane dessins for the general situation? An evident observation is

Proposition 1 *Let $\mathbb{P}$ be a finite projective plane of order n. Then any embedding of its graph as a dessin gives a (q,q,N)–dessin for some natural number N, where $q = n+1$ is the number of points on a line of $\mathbb{P}$. The automorphism group of the dessin corresponds to a subgroup of* $\mathrm{Aut}\,\mathbb{P}$ *acting fixed-point–free on the flags.*

To prove the last statement one has just to observe that only the identity automorphism of the dessin can fix an edge.

1.4 Main results

This first Proposition and the Fano plane example raise other questions:
is there a choice of the embedding such that $N = l$ is the number of points of $\mathbb{P}$? Is there a choice of the embedding such that the resulting dessin is a Wada dessin, uniform or even regular? Which subgroup of the collineation group of the projective plane becomes the automorphism group of the dessin? How does the absolute Galois group act on the corresponding set of algebraic curves? What is their field of definition?
It is not clear to us if these questions have a reasonable answer for very general embeddings of bipartite graphs coming from arbitrary finite projective planes. But it turns out that there is an interesting interplay between properties of $\mathbb{P}$ and the algebraic curve X if we concentrate on *cyclic* projective planes with an action of a *Singer group* Z_l and a *difference set* D — the definitions will be recalled in the beginning of the next section — and on embeddings compatible with the action of Z_l. First we (re)prove in Section 2

Theorem 1 *For the known cyclic projective planes $\mathbb{P}^2(\mathbb{F}_n)$ the graph has embeddings into regular dessins if and only if $n = 2$ or 8. For $n = 2$ these are*

- 2 *non–isomorphic but Galois conjugate regular $(3,3,7)$–dessins on Klein's quartic (defined over $\mathbb{Q}$),*
- 2 *non–isomorphic but Galois conjugate regular $(3,3,3)$–dessins on the elliptic curve with affine model $y^2 = x^3 - 1$.*

For $n = 8$ there are embeddings into

- 6 *non–isomorphic, Galois conjugate regular $(9,9,73)$–dessins of genus* 252*, defined over $\mathbb{Q}(\zeta_9)$, ζ_9 a 9–th primitive root of unity. Each pair of complex conjugate dessins lies on an algebraic curve defined over $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$.*
- 18 *non–isomorphic regular $(9,9,9)$–dessins of genus* 220 *defined over $\mathbb{Q}(\zeta_9)$ and forming* 3 *Galois orbits. They belong to* 18 *non–isomorphic algebraic curves definable over the same field and forming* 3 *Galois orbits as well.*

- 12 *non–isomorphic regular* $(9,9,3)$*–dessins of genus* 147 *lying on* 12 *non–isomorphic algebraic curves. The dessins and their curves form two Galois orbits and are defined over* $\mathbb{Q}(\zeta_9)$ *.*

The first sentence and the genera are known by [Si2], Sec. 5., 6., [FiWh], [Wh], Theorem 3.15, Theorem 5.3. Some results of [Wh], §3, Theorem 3.13, overlap also with

Theorem 2 *(and Definition). Let* $\mathbb{P}$ *be a cyclic projective plane with a fixed Singer group* Z_l *and a fixed difference set* D *. There is a bijection between*

- *pairs of cyclic orderings of* D *and*
- *embeddings of the graph of* $\mathbb{P}$ *as* (q,q,N)*–dessin* $\mathcal{D}$ *such that the automorphism group* $\operatorname{Aut}\mathcal{D}$ *contains* Z_l *.*

For special choices of these orderings, characterized by the fact that $\mathcal{D}/Z_l$ *is a genus* 0 *dessin,* $\mathcal{D}$ *becomes a* (q,q,l)*–dessin. We call these* $\mathcal{D}$ globe covering *dessins; they depend on only one cyclic ordering of* D *.*

(For the terminology *globe covering* see the proof, and for existence of isomorphisms between the resulting dessins see the Remark following the proof in Section 2.)

Theorem 3 *If* l *is prime, all globe covering dessins of a cyclic projective plane are uniform Wada dessins.*

It will turn out that such (q,q,l)–dessins are typical Wada dessins, see Section 5, Proposition 7. Regular Wada dessins can be completely characterized by group theoretical properties (see Proposition 8 and 9) but are in general very different from dessins coming from projective planes (Proposition 11).
For all other cyclic projective planes with the (possible) exception $n=4$ ($q=5$, $l=21$) there might exist embeddings onto uniform (q,q,l)–dessins as well. Some evidence for this conjecture — reformulated as a number theoretic question about cyclic orderings of difference sets — will follow from the proof of Theorem 3 (Section 3) and Proposition 4. Concerning the automorphism group of the dessin we will prove with similar methods as White [Wh], §3:

Theorem 4 *Let* $\mathbb{P}$ *be a cyclic projective plane of prime power order* $n=p^s\equiv 2 \bmod 3$ *, and suppose the number* $l=n^2+n+1$ *to be prime. Then the graph of* $\mathbb{P}$ *has embeddings as globe covering dessins with an automorphism group* $Z_l \rtimes Z_{3s}$ *.*

To explain how the subgroup Z_{3s} acts on the normal subgroup Z_l recall that p has the order $3s$ in the multiplicative group of prime residue classes $(\mathbb{Z}/l\mathbb{Z})^*$ [Wh], Lemma 3.3, hence acts by multiplication on $Z_l\cong\mathbb{Z}/l\mathbb{Z}$. As a special case, Theorem 4 contains the existence of regular dessins for the planes over

$\mathbb{F}_2, \mathbb{F}_8$. Section 6 gives a different proof of a more general result saying that for all l and all globe covering dessins, the automorphism group is of type $Z_l \rtimes Z_m$. Section 4 treats the explicit equations for the algebraic curves corresponding to the uniform globe covering dessins, in particular those of Theorem 3. We can give these equations in the relatively simple form

$$y^l \;=\; (x-\zeta^0)^{\overline{b_1}} \cdot \ldots \cdot (x-\zeta^{q-1})^{\overline{b_q}} \;,$$

where $\zeta = \zeta_q$ denotes a primitive q–th root of unity. The exponents $\overline{b_i}$ depend again on the ordering of the difference set of the projective plane, see Example 1 following Proposition 6. It will be shown that this equation can be replaced by another with coefficients in $\mathbb{Q}(\zeta+\zeta^{-1})$. Examples suggest that this field of definition is the smallest possible — Section 4 describes an effective procedure for the determination of the moduli field of the curve.
Even non–regular dessins have a description in terms of group theory, namely by their (hyper–) cartographic groups, i.e. the monodromy groups M of the Belyi function belonging to the dessin $\mathcal{D}$ (see the proof of Theorem 2 and Section 6). In the description given above using subgroups Γ of triangle groups Δ this monodromy group can be written as the quotient Δ/N by the maximal normal subgroup N of Δ contained in Γ . In other words, M is isomorphic to the automorphism group of the minimal regular cover R of $\mathcal{D}$. In particular, $M \cong \operatorname{Aut}\mathcal{D}$ for regular dessins. *How does M look like in the case of uniform dessins for cyclic projective planes?* In Section 6, we give the following partial answer:

Theorem 5 *Under the conditions of Theorem 3, the cartographic group of the dessin $\mathcal{D}$ is isomorphic to a semidirect product*

$$Z_l^r \rtimes Z_q$$

with an exponent $r < q$.

Again, we will prove a slightly more general version than stated here. Again, the ordered difference sets determine the precise nature of the cartographic group, i.e. the exponent r and the action of Z_q on Z_l^r .
It is a great pleasure for us to thank Gareth Jones for the many fruitful discussions on these subjects during the last Southampton–Frankfurt workshops on dessins and group actions.

2 Cyclic projective planes and difference sets

Recall that a finite projective plane $\mathbb{P}$ is called cyclic if there is a collineation a of order l generating a Singer subgroup of $\operatorname{Aut}\mathbb{P}$ acting sharply transitive on the points (and, by duality, on the lines) of $\mathbb{P}$. Fixing one point x and writing all points as $a^m(x)$ we may identify the points with the exponents $m \in \mathbb{Z}/l\mathbb{Z} \leftrightarrow Z_l$, hence read the cyclic automorphism group as the (additive) group Z_l acting by addition on Z_l. For the lines we adopt the same convention. In the case of the projective plane over the finite field $\mathbb{F}_n$ we may — as we did for the Fano plane — think of the exponents of some generator g of the multiplicative group $\mathbb{F}_{n^3}^*/\mathbb{F}_n^*$ and describe the incidence between points and lines using the trace t of $\mathbb{F}_{n^3}/\mathbb{F}_n$ as nondegenerate bilinear form. Locally, the embeddings in question will be chosen such that the incidence graph look as described in Figure 3,

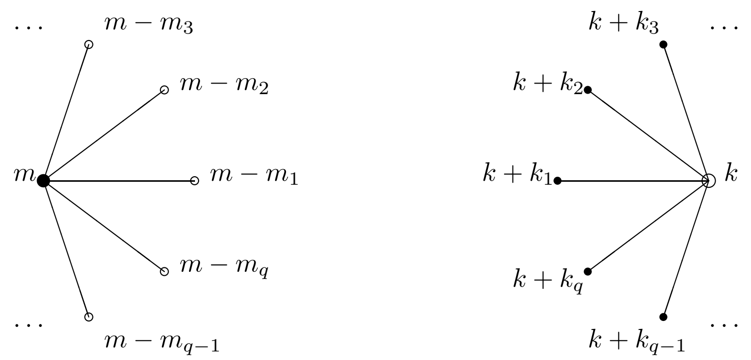

Figure 3: Local pattern of a dessin for a cyclic projective plane

for a fixed set $\{k_1, \dots, k_q\} = \{m_1, \dots, m_q\} \subset Z_l$ characterized by the property

$$t(g^{k_i}) = 0 \qquad \text{for all} \quad i = 1, \dots, q\,.$$

But we may use this figure for other cyclic projective planes as well (if there exist any) reading $\{k_1, \dots, k_q\} = \{m_1, \dots, m_q\} \subset Z_l$ as a DIFFERENCE SET D characterized by the property that for all $m \in Z_l$, $m \neq 0$, there are unique i and j with $m = k_i - k_j$. In any case, the cyclic collineation a of $\mathbb{P}$ can be identified with the shift

$$m \mapsto m + 1\,, \quad k \mapsto k + 1$$

proving graphically the *if* part of the following Proposition and the statement in Theorem 2 about the automorphism group as well.

PROPOSITION 2 *Let $\mathcal{D}$ be a dessin obtained by embedding the graph of a cyclic projective plane $\mathbb{P}$ with l points. Its Singer group Z_l becomes a subgroup of the automorphism group of $\mathcal{D}$ if and only if for all m and k the local orientation of the edges around the vertices are chosen as indicated in Figure 3. Such orientations correspond bijectively to the choice of a pair of orderings of a given difference set D for $\mathbb{P}$, both up to cyclic permutations.*

In fact, the translations $m \mapsto m+r$, $k \mapsto k+r$, $r \in Z_l$, preserve incidence and orientation and give the action of Z_l on the dessin. The *only if* part is true by the following reason: If a induces an automorphism of the dessin, the local orientations of the edges around the black vertices must show the same pattern as the left part of Figure 3, and an analogous statement is true for the white vertices.

Proof of Theorem 2. We know already by Proposition 2 that embeddings for the cyclic projective plane $\mathbb{P}$ as a dessin $\mathcal{D}$ with $Z_l \subseteq \mathrm{Aut}\,\mathcal{D}$ determine two orderings of the (fixed) difference set D. To prove the existence of such embeddings, choose a pair of orderings of D giving local orientations of the graph around all vertices as in Figure 3. These $2l$ drawings define in an obvious way local charts for an orientable surface into which the graph has to be embedded, and the numbering of the vertices gives the following unique prescription how to glue the local pieces together. Let Ω the set of all ql edges of $\mathcal{D}$ (flags of $\mathbb{P}$) and let M be the permutation group on Ω generated by b and w where b is the cyclic counterclockwise shift of all edges around the black vertices (i.e. sending the edge between m and $m-m_i$, for all $i \in Z_q$ and all $m \in Z_l$, to the edge between m and $m-m_{i+1}$ in the left part of Figure 3), and w is the corresponding counterclockwise shift of the edges around all white vertices. According to [JS], 5. Maps and Hypermaps, M and its generators b and w of order q define an *algebraic hypermap* on a unique compact Riemann surface X or — in the present terminology — a (q,q,N)–dessin on X where N is the order of wb in the *cartographic group* M. The surface X can be described explicitly as follows: there is an obvious homomorphism h of the triangle group $\Delta = \langle q,q,N\rangle$ onto M; let $H \subset M$ be the fixgroup of an arbitrary edge in Ω and let $\Gamma := h^{-1}(H)$, then we can define X as the quotient $\Gamma\backslash\mathcal{H}$.

For example, consider the case $n=4,\ q=5,\ l=21$ with the cyclic ordering of a difference set

$$(m_i)_{i \bmod 5} = (-3,0,1,6,8) \quad , \qquad (k_i)_{i \bmod 5} = (8,6,1,0,-3)\,.$$

Here one obtains a uniform $(5,5,5)$–dessin on a surface of genus 22 with 21 cells of valency 10 on which the Singer group Z_{21} acts fixed-point–free as cyclic permutation group of the set of cells. The quotient dessin $\mathcal{D}/Z_{21}$ has one cell, 5 edges, one black and one white vertex, hence genus 2.

For the last claim of the theorem suppose $\mathcal{D}$ to be globe covering. Since $\mathcal{D}/Z_l$ has genus 0 and q edges, one black and one white vertex (the poles), it has also q cells, and we can imagine the edges as meridians joining the poles and separating the cells. It is easy to see that this quotient dessin arises if and only if both orderings of D are the same, i.e. if in Figure 3 $m_i = k_i$ for all $i = 1,\dots,q$. Clearly, the globe covering dessins depend on only one cyclic ordering of D. Their cells look as indicated in Figure 4.

Then, the numbers corresponding to the vertices on the border of the cell form arithmetic progressions in Z_l and therefore this cell has $2l/c_i$ edges where c_i is the gcd of l and $k_{i+1}-k_i$. The resulting dessin is therefore a (q,q,N)–dessin where $2N = 2l/c$ is the lcm of the valencies of the cells and c the gcd of all

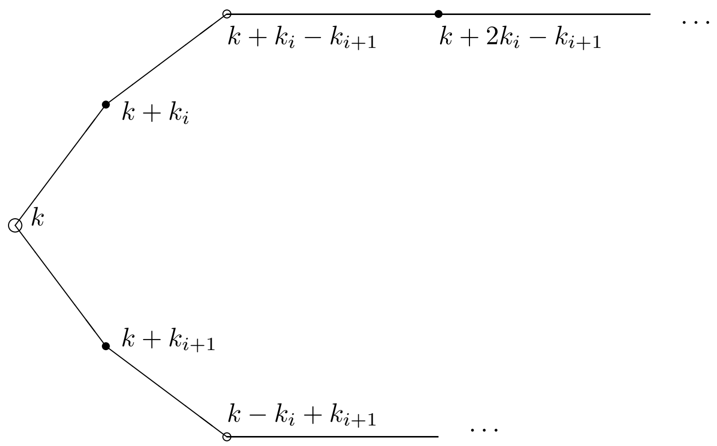

Figure 4: Cell of a globe covering dessin for a cyclic projective plane ($m_i = k_i$)

c_i . But $c > 1$ would imply that all differences $k_i - k_j$ were multiples of c in contradiction to the fundamental property of the difference set. Therefore, we have $N = l$ proving that globe covering dessins are (q, q, l)–dessins for $\mathbb{P}$.
Remark. Suppose $\mathcal{D}, \mathcal{D}'$ to be dessins resulting from two different pairs of orderings for D . Then there is no Z_l–equivariant isomorphism $i : \mathcal{D} \to \mathcal{D}'$, i.e. satisfying $i \circ a = a \circ i$, since in that case we could replace i by an isomorphism preserving the numbering of black and white vertices, hence also the local pattern of Figure 3. However, non–Z_l–equivariant isomorphisms may exist, related to multipliers of difference sets: for $n = 5,\ q = 6,\ l = 31$ take two different cyclic orderings of a fixed difference set D

$$(m_i) = (k_i) = (1,\ 5,\ 11,\ 25,\ 24,\ 27)\ , \quad (m_i') = (k_i') = (5,\ 25,\ 24,\ 1,\ 27,\ 11)$$

giving isomorphic dessins where the isomorphism is defined by $i : x \mapsto 5x$ mod l .
Exercise. Reverse the orientation in the right part of Figure 1 and show that this choice induces globally a $(3, 3, 3)$–Fano dessin. Reverse the orientation in the left part of Figure 1 to show that this choice induces globally another $(3, 3, 3)$–Fano dessin.
Proof of Theorem 1. That we can obtain regular dessins only for $n = 2$ and 8 follows directly from Proposition 1 and a theorem of Higman/McLaughlin [HML], Prop. 12, stating that for the planes $\mathbb{P}^2(\mathbb{F}_n)$ different from the Fano plane and $\mathbb{P}^2(\mathbb{F}_8)$, flag–transitive groups of collineations cannot act fixed-point–free on the flags. The converse direction is already verified for the Fano plane by giving two regular dessins in genus 1 and two in genus 3 . The genus 3 dessins belong to Klein's quartic which is known to be defined over $\mathbb{Q}$. As the two dessins on the elliptic curves they differ by their local orientation — see the exercise above (giving a *chiral pair of dessins*) — whence the dessins

have to be complex conjugate. For the genus 1 dessins, the underlying elliptic curve is the same for both dessins since it has an automorphism of order 3, hence uniquely determined with model $y^2 = x^3 - 1$. On the other hand, the dessins are not isomorphic: their vertices are obtained (with suitable coloring) by the points of $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$ on the two tori

$$\mathbb{C}/(2 \pm \sqrt{-3})\mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})] \, .$$

These two tori are of course isomorphic but there is no isomorphism mapping the two dessins onto each other since multiplication by $(2+\sqrt{-3})/(2-\sqrt{-3})$ does not give an automorphism of the elliptic curve.
The two dessins for the Fano plane on Klein's quartic are non–isomorphic since they correspond to two different normal subgroups of the triangle group $\langle 3, 3, 7\rangle$ which are conjugate in the index 2 extension $\langle 2, 3, 14\rangle$, compare also Lemma 1 and 2 below.
For the plane $\mathbb{P}^2(\mathbb{F}_8)$ one may verify that

$$k_i \;=\; 2^i \bmod 73 \, , \quad i \;=\; 0, \ldots, 8 \, ,$$

form a difference set. A cyclic order is provided by the cyclic order of the exponents $i \bmod 9$. Therefore, it is easy to verify that

$$b \;:\; m \;\mapsto\; 2m \;:\; Z_{73} \;\to\; Z_{73}$$

together with a generates an edge–transitive automorphism group $G \cong Z_{73} \rtimes Z_9$ of the $(9, 9, 73)$–dessin described by Figure 4.
For the proof of the statements about the different possible images under these embeddings recall that cocompact triangle groups Δ with signature $\langle p, q, r\rangle$ are presented by generators and relations

$$\gamma_0 \, , \; \gamma_1 \, , \; \gamma_\infty \, ; \quad \gamma_0^p \;=\; \gamma_1^q \;=\; \gamma_\infty^r \;=\; \gamma_0\gamma_1\gamma_\infty \;=\; 1 \, .$$

The following is well known and turns out to be very useful for the classification of regular dessins.

Lemma 1 *Let* $\Delta = \langle p, q, r\rangle$ *be a Fuchsian triangle group. Then there is a bijection between*

- *isomorphism classes of regular* (p, q, r)*–dessins with automorphism group* G*,*
- *normal torsion free subgroups* Γ *of* Δ *with* $\Delta/\Gamma \cong G$*,*
- *equivalence classes of epimorphisms* $h : \Delta \to G$*, with torsion–free kernel, i.e. mapping the generators* γ_i *of* Δ *onto generators of* G *of the same order. Two epimorphisms are equivalent if they result from each other by combination with an automorphism of* G*.*

(The next lemma and the following remark will explain in more detail why non–isomorphic dessins may however lie on isomorphic Riemann surfaces.) For the special G under consideration, it is easy to see that such epimorphisms exist only for the triangle groups $\langle 9,9,m\rangle$ with $m = 73$, 9 or 3. Following closely the method described in [StWo] we can select homomorphisms h (with pairwise different kernels) onto $G \cong Z_{73} \rtimes Z_9$ with generators a, b as above in the following way. For $\langle 9,9,73\rangle$ we may take

$$h(\gamma_0) \;=\; b^s \;,\; h(\gamma_1) \;=\; b^{-s}a^k \;,\; h(\gamma_\infty) \;=\; a^{-k} \;,\quad s \in (\mathbb{Z}/9\mathbb{Z})^*$$

(another choice of $k \in (\mathbb{Z}/73\mathbb{Z})^*$ changes h only by composition with an element of $\operatorname{Aut} G$). For $\langle 9,9,9\rangle$ we may take

$$h(\gamma_0) \;=\; b^s \;,\; h(\gamma_1) \;=\; b^t a^k \;,\; h(\gamma_\infty) \;=\; a^{-k} b^u \;,$$

$$s\,,\,t\,,\,u \in (\mathbb{Z}/9\mathbb{Z})^* \quad \text{with} \quad s + t + u \;\equiv\; 0 \mod 9$$

(same remark for the choice of k), and for $\langle 9,9,3\rangle$ we may take

$$h(\gamma_0) \;=\; b^s \;,\; h(\gamma_1) \;=\; b^t a^k \;,\; h(\gamma_\infty) \;=\; a^{-k} b^{3u} \;,$$

$$s\,,\,t\,,\,u \in (\mathbb{Z}/9\mathbb{Z})^* \quad \text{with} \quad s + t + 3u \;\equiv\; 0 \mod 9\,.$$

(same remark for the choice of k). The number of non–isomorphic dessins now follows from counting the possible parameter values s, t, u. The question if the underlying curves are isomorphic can be answered by another well known

Lemma 2 *Let Γ and N be two different torsion free normal subgroups of the Fuchsian triangle group Δ with isomorphic quotient $\Delta/\Gamma \cong G \cong \Delta/N$. The Riemann surfaces $\Gamma\backslash\mathcal{H}$ and $N\backslash\mathcal{H}$ are isomorphic if and only if the following equivalent conditions hold:*

- *Γ and N are $\mathrm{PSL}_2(\mathbb{R})$–conjugate.*
- *Γ and N are conjugate in some triangle group $\overline{\Delta} \supset \Delta$.*

(The two regular dessins corresponding to Γ and N are not isomorphic since the isomorphism of Riemann surfaces induced by the conjugation with $\overline{\Delta}$ permutes the different fix-point orbits of Δ, i.e. does not preserve at least the *color* of the vertices.) To apply this Lemma, one has to check if there are larger triangle groups and to control if the normal subgroups N remain normal in these larger triangle groups. Equivalently, one has to check if the homomorphisms found above are extendable to larger triangle groups than the original ones, see [StWo], Lemma 4. As an example, take the first case $m = 73$: here we obtain 6 different normal torsion–free subgroups N_s of Δ according to the 6 different choices of s. But Δ is contained with index 2 in the maximal triangle

group $\langle 2,9,146\rangle$ in which N_s and N_{-s} are conjugate. Therefore we obtain 6 non–isomorphic dessins but only 3 non–isomorphic Riemann surfaces.
The genera of the quotients of the upper half plane by these kernels can be computed by standard methods like Riemann–Hurwitz' theorem.
For the statements about fields of definition and Galois orbits recall first that curves X with many automorphisms can be defined over their field of moduli, i.e. the common fixed field of all σ with $X \cong X^\sigma$ [Wo1], Remark 4, [Wo2], Satz 3. For the determination of this field take again the example of the regular $(9,9,73)$–dessins. Let N_s be the kernel of the homomorphism h defined above by $h(\gamma_0) = b^s$, $h(\gamma_1) = b^{-s}a^k$ and let X_s be the quotient surface $N_s\backslash\mathcal{H}$. Recall that η is a MULTIPLIER of an automorphism α of X_s in some fixed point x if the action of α in a local coordinate z around x (corresponding to $z=0$) can be described by $z \mapsto \eta z$ (not to be confused with multipliers in the theory of difference sets!). Then it is easy to prove

Lemma 3 *On X_s the automorphism b has two fixed points with multipliers $\zeta_9^{\overline{s}}$ and $\zeta_9^{-\overline{s}}$ where $\zeta_9 = e^{2\pi i/9}$ and $\overline{s}s \equiv 1 \bmod 9$.*

Using the representation of the automorphism group on the canonical model or Belyi's cyclotomic character one may prove moreover

Lemma 4 *Let σ be $\in \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ and let b act as an automorphism of X with a multiplier η in the fixed point x. Then b acts in x^σ on X^σ with a multiplier $\sigma(\eta)$.*

Lemma 1, 2, 3 and the classification of the covering groups N_s show that the isomorphism class of X_s is uniquely determined among all surfaces with regular $(9,9,73)$–dessin and automorphism group G by the unordered pair of multipliers $\{\zeta_9^{\overline{s}}, \zeta_9^{-\overline{s}}\}$, and that the isomorphism class of dessins is uniquely determined by the ordered pair of multipliers. On the other hand, Lemma 4 shows that every σ fixing elementwise the cyclotomic field $\mathbb{Q}(\zeta_9)$ fixes the isomorphism class of dessin and curve. The Galois orbits are now easily determined by the action of $\mathrm{Gal}\,\mathbb{Q}(\zeta_9)/\mathbb{Q}$. The other cases can be treated in the same way.
It remains to prove that all the resulting bipartite graphs are isomorphic (as graphs, not as dessins) to the graph of $\mathbb{P}^2(\mathbb{F}_8)$. First, we observe that — by the freedom of choice of k — we may assume that all $h(\gamma_1)$ generate the same cyclic subgroup of G; the same observation holds trivially for all $h(\gamma_0)$. Then, from the first part of the proof we know that at least one resulting dessin has the desired property. Now, by the preceding classification of Riemann surfaces with a regular dessin and automorphism group G we obtain graph–isomorphic dessins what follows from a statement which might be of independent interest:

Proposition 3 *Let $\mathcal{D}_r, \mathcal{D}_l$ be regular (p,q,r)– and (m,n,l)–dessins with automorphism groups both isomorphic to G, induced by epimorphisms*

$$h_r : \langle p,q,r\rangle \to G\,, \quad h_l : \langle m,n,l\rangle \to G$$

with torsion–free kernels. The bipartite graphs of $\mathcal{D}_r$ *and* $\mathcal{D}_l$ *are graph–isomorphic if*

- $p = m$ *and* $q = n$,
- *by combination with a group automorphism,* h_r *and* h_l *can be chosen such that*

  1. $h_r(\gamma_0)$ *and* $h_l(\gamma_0)$ *generate the same subgroup* B *of* G ,
  2. $h_r(\gamma_1)$ *and* $h_l(\gamma_1)$ *generate the same subgroup* W *of* G .

Proof. A necessary condition for the existence of an isomorphism between both graphs is equality between the valencies in the vertices. Therefore we will suppose in the sequel that the first condition is satisfied. Since both dessins are regular with the same automorphism group, we can represent their edges by group elements $g \in G$ if we identify the edge 1 with the image of the hyperbolic line between the fixed points of γ_0 and γ_1 under the map of $\mathcal{H}$ onto its quotient by the kernels of h_r and h_l respectively. In order to describe the graph of $\mathcal{D}_r$ we have to describe incidence around black (white) vertices. Let B and W be the subgroups of G generated by $h_r(\gamma_0)$ and $h_r(\gamma_1)$ respectively. Then B and W consist of the edges incident with 1 in its black and white end–vertex, respectively. Using the G–action from the left, we see that the edge f is incident in its black end–vertex with all edges in fB and in its white end–vertex with all edges fW . Since this property does not depend on the choice of the generators of B and W , the conditions of Proposition 3 imply that the trivial and G–covariant application of edges $g \mapsto g$ induces an isomorphism of graphs.

Remarks. 1) The different non–isomorphic dessins for $P^2(\mathbb{F}_8)$ can be obtained as well by different pairs of orderings of the difference set. If one wants to obtain a regular dessin then only such permutations of D are admissible which are preserved by the multiplication with $2 \bmod 73$, and it is easy to see that there are precisely 6 such permutations. Applied independently to the incidence pattern around black and white vertices, this gives 36 different regular dessins as found in Theorem 1.
2) Which other cyclic projective planes besides the usual $P^2(\mathbb{F}_n)$ could exist, giving also a regular dessin? It is known that their order n has to be > 3600 ; furthermore, they should admit a sharply flag–transitive automorphism group, and by results of Kantor and Feit (see Theorem 8.18 of [Ju]) this could be possible only if a collection of exotic conditions holds: the order n of the plane must be a multiple of 8 but no power of 2 , the number l of points is a prime, and the difference D set can be chosen as set of powers $n^k \bmod (\mathbb{Z}/l\mathbb{Z})^*$ (for this point one may also consult Proposition 11). Furthermore, D is its own group of multipliers and contains all divisors of n .
3) Galois conjugate dessins are in general not necessarily graph isomorphic. Some non–regular examples can be found in [JSt], but there are also such

examples for regular dessins: the three regular $(2,3,7)$–dessins with automorphism group $G = \mathrm{PSL}_2(\mathbb{F}_{13})$ on three Macbeath–Hurwitz curves treated in [St] give three non–isomorphic but Galois conjugate dessins whose graphs are not isomorphic.

3 THE WADA PROPERTY

We mention first that the *proof of Theorem 3* is an almost trivial consequence of the proof of Theorem 2: For prime l and globe covering dessins, i.e. with $m_i = k_i$ for all i in Figure 3, every cell in Figure 4 has valency $2l$. Therefore, the dessin is uniform, and every vertex lies on the border of every cell.
According to standard conjectures of number theory, there should exist an infinity of prime powers n such that $l = n^2 + n + 1$ is a prime ($n = 2, 3, 5, 8, 9, 17, \dots$). But even for composite l, each difference $k_i - k_{i+1}$ defines a module for an arithmetic progression in $\mathbb{Z}/l\mathbb{Z}$ giving the sequence of black points in clockwise order around the cell, and similarly for the white points. However, the length of these arithmetic progressions (determining the valency of the cell) is in general a proper divisor of l. For example in the case $n = 4$, $q = 5$, $l = 21$ the difference set

$$D := \{-3, 0, 1, 6, 8\}$$

has no arrangement such that all differences $k_i - k_{i+1}$ are coprime to l (the indices i have to be considered $\bmod 5$, of course).
The question raised for composite l about the existence of uniform (q,q,l)–dessins for cyclic projective planes admitting Z_l as automorphism group may be reformulated now in the following way (note that for $n = 5$ we have $l = 31$ prime and that for $n = 6$ no difference set exists). *Let n be ≥ 7 and $l \geq 57$ be a composite number. Is it always possible to arrange a difference set*

$$D := \{k_i \mid i \bmod q\} \subset \mathbb{Z}/l\mathbb{Z}$$

in such a way that all successive differences $k_i - k_{i+1}$ are coprime to l ?
For small (prime powers) n the answer is positive thanks to the following Propositions.

PROPOSITION 4 *Let n be ≥ 7 and $l \geq 57$ be a composite number with prime divisor p. A difference set $D \subset \mathbb{Z}/l\mathbb{Z}$ can always be arranged in such a way that all successive differences of elements in D satisfy*

$$k_i - k_{i+1} \not\equiv 0 \mod p .$$

For the *proof* it is sufficient to show that no residue class $\bmod p$ contains $\geq q/2$ elements among the elements of D. This will follow from

PROPOSITION 5 *Let n be ≥ 7 and $l \geq 57$ be a composite number with prime divisor p and a difference set $D \subset \mathbb{Z}/l\mathbb{Z}$. Let a_r be the number of elements $d \in$*

D with $d \equiv r \bmod p$, $r = 1, \dots, p$. *Then the numbers* a_r *have the following properties.*

$$\sum_r a_r \;=\; q \;=\; n+1\,, \tag{1}$$

$$\sum_r a_r^2 \;=\; \frac{l}{p} + n \;=\; \frac{1}{p}\,(n^2 + (p+1)n + 1)\,, \tag{2}$$

$$\sum_r a_r a_{r+s} \;=\; \frac{l}{p} \;=\; \frac{1}{p}\,(n^2 + n + 1) \quad \textit{for all} \quad s \not\equiv 0 \bmod p\,, \tag{3}$$

$$||\,(a_1, \dots, a_p) - \frac{1}{p}(n+1, \dots, n+1)\,||^2 \;=\; \frac{p-1}{p}\,n \tag{4}$$

$$\mathit{Max}\,|\,a_r - \frac{n+1}{p}\,| \;<\; \sqrt{n} \tag{5}$$

$$a_r \;<\; \sqrt{n} + \frac{n+1}{p} \quad \textit{for all} \quad r\,. \tag{6}$$

(In (4), we use the Euclidean norm in $\mathbb{R}^p$ *).*

From the last inequality, $a_r < n/2$ follows for $n > 40$ and $p \geq 3$ or for $n > 8$ and $p \geq 7$ (note that the primes 2 and 5 never occur as divisors of l). Therefore, one has to check the truth of Proposition 4 by hand for some small n only. This can be done by giving the solutions of (1) to (3) for $p = 3$ and small n. These are, up to permutation of the coordinates

$$\begin{array}{rclcl} (a_1, a_2, a_3) & = & (4,3,1) & \text{for} & n = 7 \\ & & (7,4,3) & \text{for} & n = 13 \\ & & (9,7,4) & \text{for} & n = 19 \\ & & (12,7,7) & \text{for} & n = 25 \\ & & (13,12,7) & \text{for} & n = 31 \\ & & (16,13,9) & \text{for} & n = 37\,. \end{array}$$

Now we can explain the strategy how to construct uniform dessins for projective planes even in the case of composite l, e.g. for $n = 7$. Here we have two prime divisors $p = 3$, 19 dividing $l = 57$ and we have to arrange D in such a way that just every second $k_i \in D$ is congruent to $1 \bmod 3$. Then, Proposition 5 is satisfied for $p = 3$, and we have $2 \cdot 4! \cdot 4!$ possibilities for such arrangements. Among these possibilities, one has to find an arrangement satisfying Proposition 5 also for $p = 19$. This is obviously possible since for $p = 19$, equations (1) and (2) are satisfied with one $a_r = 2$, for other six indices m one has $a_m = 1$, and all other a_t vanish.

Proof of Proposition 5. Equation (1) just counts the number of elements in D. Equation (2) follows from the fact that precisely $\frac{l}{p}-1$ among the differences k_i-k_j, $i\neq j$, fall into the residue class $0 \bmod p$, and therefore

$$\sum_r a_r(a_r-1) \;=\; \frac{l}{p}-1\;,$$

Together with (1), this implies (2). Equation (3) follows by a similar consideration of the differences giving elements $\equiv d \not\equiv 0 \bmod p$. We may consider $(a_1,\ldots,a_p)$ as a point on the hyperplane given by equation (1). By Hesse's normal form, this hyperplane has square distance $\frac{1}{p}(n+1)^2$ from the origin, and the nearest point to the origin is of course $\frac{1}{p}(n+1,\ldots,n+1)$. Therefore, the square distance (2) and Pythagoras enable us to calculate the distance (4), and this implies (5).

Proof of Theorem 4. If $l=n^2+n+1$ is prime and $n=p^s$ a prime power, it is known [Wh], Lemma 3.3, that $p \bmod l$ has order $3s$ in the group $(\mathbb{Z}/l\mathbb{Z})^*$. Moreover, we may choose a difference set $D\subset Z_l=\mathbb{Z}/l\mathbb{Z}$ for the projective plane invariant under multiplication with $p \bmod l$. Therefore, as in the case of the plane $\mathbb{P}^2(\mathbb{F}_8)$ described in the proof of Theorem 1, we have at least a group of graph automorphisms isomorphic to $Z_l \rtimes Z_{3s}$. This group becomes an automorphism group of the globe covering dessin if and only if we can arrange D in such a way that the multiplication with p preserves this cyclic ordering of D.

LEMMA 5 *Under the hypotheses of Theorem 4, the action of Z_{3s} through multiplication by p^m on the p–invariant difference set D has orbits of length $3s$.*

Proof of the Lemma. Since l is prime and $p \bmod l$ has order $3s$, the orbits of the action on Z_l have length $3s$ or 1. Length 1 occurs for one orbit only, and this orbit cannot be contained in D since D has $q=n+1\equiv 0 \bmod 3$ elements.

Proof of Theorem 4, continued. Now let $k_1,\ldots,k_r\in D$ represent the Z_{3s}–orbits of the Z_{3s}–invariant difference set D, $r=l/3s$. Arrange D as

$$k_1,\ldots,k_r\,,\,pk_1,\ldots,pk_r,p^2k_1,\ldots\;\ldots,p^{3s-1}k_r\;.$$

Then it is easy to check that the multiplication with $p \bmod l$ preserves the cyclic order of the edges incident with black and white vertices as described in Figure 3. Since l is prime, this arrangement does not bother the property that every cell has valency $2l$, see the proof of Theorem 3 in the beginning of this section.

4 EQUATIONS

The aim of this section is the determination of explicit algebraic models for the curves corresponding to the uniform (q,q,l)–dessins $\mathcal{D}$ coming from cyclic projective planes as described in Theorem 3 and the last Section. We begin with a more general remark about globe covering (q,q,l)–dessins. The genus 0 quotient $\mathcal{D}/Z_l$ with its q cells, q edges and two vertices, both of valency q, belongs to a unique Fuchsian subgroup of the triangle group Δ of signature $\langle q,q,l\rangle$, its commutator subgroup Ψ of signature $<0;l^{(q)}>$, i.e. of genus 0 and with q inequivalent elliptic fixed points of order l such that

$$\Gamma \lhd \Psi \quad \text{with} \quad \Psi/\Gamma \cong Z_l\,, \tag{7}$$

$$\Psi \lhd \Delta \quad \text{with} \quad \Delta/\Psi \cong Z_q\,, \tag{8}$$

if as before $\mathcal{D}$ corresponds to the Fuchsian group Γ. In the cases studied in Theorem 3 and the last Section, i.e. for uniform dessins, Γ is the universal (torsion–free) covering group of the curve whose equation we want to determine, but with the exception of the cases studied in Theorem 1 we cannot suppose that Γ is normal in Δ.

Lemma 6 *Suppose $q>2$ and $-2+q(1-\frac{1}{l})>0$, and let Ψ be a Fuchsian group of signature $<0;l^{(q)}>$. Then the number of torsion–free normal subgroups of Ψ with cyclic factor group $\cong Z_l$ is a multiplicative function $f_q(l)$ of l. For p prime and integer exponents $a\geq 1$ we have*

$$f_q(p^a) \;=\; [(p-1)^{q-1}+1]\,p^{aq-2a-q+1}$$

if q is even, and for q odd we have

$$f_q(p^a) \;=\; [(p-1)^{q-1}-1]\,p^{aq-2a-q+1}\,.$$

Proof. In order to obtain a torsion–free normal subgroup of Ψ we have to map the generators $\gamma_1,\dots,\gamma_q$ onto $b_1,\dots,b_q\in(\mathbb{Z}/l\mathbb{Z})^*$ such that $\sum b_i\equiv 0 \bmod l$. By an obvious extension of Lemma 1 to $\Gamma\lhd\Psi$, the number of these congruence solutions is $\varphi(l)f_q(l)$ because two such epimorphisms $\Psi\to Z_l$ have the same kernel if and only if they result from each other by combination with one of the $\varphi(l)$ automorphisms of Z_l where φ denotes the Euler function. The multiplicativity of f_q is therefore a consequence of the Chinese remainder theorem and the multiplicativity of φ.
First, let $l=p$ be prime. Then we count the congruence solutions

$$\begin{aligned}
(p-1)\,f_q(p) \quad &= \quad \#\{\,(b_1,\dots,b_q)\,|\,b_i\in(\mathbb{Z}/p\mathbb{Z})^*\,,\,\sum b_i\equiv 0 \bmod p\,\} \;=\\
&= \quad \#\{\,(b_1,\dots,b_{q-2})\,|\,\sum^{q-2} b_i\equiv 0 \bmod p\,\}\,(p-1)\\
&\quad\quad +\#\{\,(b_1,\dots,b_{q-2})\,|\,\sum^{q-2} b_i\not\equiv 0 \bmod p\,\}\,(p-2) \;=\\
&= \quad \#\{\,(b_1,\dots,b_{q-2})\,|\,b_i\in(\mathbb{Z}/p\mathbb{Z})^*\}\,(p-1) \;-\; (p-1)^{q-2}\\
&\quad\quad +\#\{\,(b_1,\dots,b_{q-2})\,|\,\sum^{q-2} b_i\equiv 0 \bmod p\,\} \;=\\
&= \quad (p-1)^{q-1} \;-\; (p-1)^{q-2} \;+\; (p-1)\,f_{q-2}(p)
\end{aligned}$$

from which the formulae for $l = p$ follow easily by induction over q.
Now, let l be a prime power p^a, $a > 1$. Every solution of

$$\sum_{i=1}^{q} b_i \equiv 0 \bmod p^a , \quad b_i \in (\mathbb{Z}/p^a\mathbb{Z})^*$$

gives by reduction mod p a solution of $\sum b_i \equiv 0 \bmod p$, and conversely every solution of $\sum b_i \equiv 0 \bmod p$ comes from $p^{(a-1)(q-1)}$ solutions mod p^a since every $b_i \bmod p$ has p^{a-1} preimages in $(\mathbb{Z}/p^a\mathbb{Z})^*$, and we have a free choice for precisely $q-1$ of these preimages. Therefore Lemma 6 follows from

$$(p-1)\, p^{a-1}\, f_q(p^a) \;=\; p^{aq-a-q+1}\,(p-1)\, f_q(p) \,.$$

For another approach, in particular to the case of $l = p$ prime, see [J], p.500.

PROPOSITION 6 *Let Ψ of signature $< 0; l^{(q)} >$ be the unique normal subgroup of the triangle group Δ of signature $\langle q,q,l\rangle$, $q > 2$, $l > 3$ with factor group Z_q. Let Γ of signature $< (l-1)(q-2)/2; 0 >$ be the torsion–free kernel of the epimorphism $\Psi \to Z_l$ sending the canonical elliptic generators γ_i, $i = 1,\dots,q$, of Ψ onto $b_i \in (\mathbb{Z}/l\mathbb{Z})^*$ with $\sum b_i \equiv 0 \bmod l$ (w.l.o.g. we may normalize these epimorphisms by taking $b_1 = 1$). Let $\overline{b_i}$ be defined by $\overline{b_i} b_i \equiv 1 \bmod l$, and let $\zeta = \exp(2\pi i/q)$ be the multiplier of all γ_i. Then, as an algebraic curve, the quotient surface $\Gamma\backslash\mathcal{H}$ has a (singular, affine) model given by the equation*

$$y^l \;=\; (x-\zeta^0)^{\overline{b_1}} \cdot \ldots \cdot (x-\zeta^{q-1})^{\overline{b_q}} \,.$$

Proof. This curve defines a function field built up by two consecutive cyclic extensions

$$\mathbb{C}(x,y) \;\supset\; \mathbb{C}(x) \;\supset\; \mathbb{C}(x^q)$$

of orders l and q. The function x^q on this curve is a Belyi function whose ramification points lie above $0,1,\infty$ of orders q,l,q respectively. The condition $\sum b_i \equiv 0 \bmod l$ is necessary and sufficient to ensure that ∞ is unramified under the extension $\mathbb{C}(x,y)/\mathbb{C}(x)$. The choice of the exponents easily follows from a consideration of the local action of the automorphism group Z_l in its fixed points.

Example 1. For the globe covering dessins of Theorem 2, suppose that $k_1,\dots,k_q \in \mathbb{Z}/l\mathbb{Z}$ form the difference set D with $(k_i - k_{i+1}, l) = 1$ for all $i \in \mathbb{Z}/q\mathbb{Z}$. This is true for prime l (Theorem 3); the Propositions 4 and 5 give some evidence that there may exist orderings of D with that property as well for all other $q \neq 5$. Then the graph of the projective plane embeds into

$$y^l \;=\; (x-\zeta^0)^{\overline{k_2-k_1}} \cdot \ldots \cdot (x-\zeta^{q-1})^{\overline{k_1-k_q}} \,.$$

Remark. Cyclic permutations of the difference set in Example 1 give isomorphic dessins and should give therefore isomorphic curves. In Lemma 10 below, we will study these isomorphisms as coming from cyclic shifts of exponents.
Example 2. With $l = q$ and $b_1 = \ldots = b_q = 1$ the Fermat curves

$$y^q \;=\; (x-\zeta^0)\cdot\ldots\cdot(x-\zeta^{q-1}) \;=\; x^q - 1$$

fall under Proposition 6 as well.
Remark. Example 2 corresponds to a dessin for which — in the terminology of Proposition 6 — Γ is even normal in Δ. The dessin has therefore the larger automorphism group Z_q^2 inducing additional relations between the exponents (here: equality). Other examples of this type can be found in [StWo], Section 3.
The remaining part of this section is devoted to a determination of the moduli field for the curves treated in Proposition 6. To this aim, define

$$\mathbf{b} \;:=\; (b_1,\ldots,b_q) \;=\; (1,b_2,\ldots,b_q)$$

and $X_{\mathbf{b}} := \Gamma\backslash\mathcal{H}$ to be the curve with the affine equation arising in Proposition 6, i.e. with $b_i \in (\mathbb{Z}/l\mathbb{Z})^*$ for all i and $1+\sum_{i=2}^q b_i \equiv 0 \bmod l$. Clearly, the field of definition of $X_{\mathbf{b}}$ can be chosen as a subfield of the cyclotomic field $\mathbb{Q}(\zeta)$, hence also the field of moduli (recall that by the definition given in Section 2 between Lemma 2 and Lemma 3, the field of moduli is contained in any field of definition). We can give a slightly better result:

Lemma 7 *The curve $X_{\mathbf{b}}$ can be defined over $K = \mathbb{Q}(\zeta+\zeta^{-1}) = \mathbb{Q}(\cos 2\pi/q)$, and K contains the moduli field of $X_{\mathbf{b}}$.*

A direct *proof* is provided by a substitution $x = \mu(z)$ in the defining equation of $X_{\mathbf{b}}$ where μ denotes a fractional linear transformation defined over $\mathbb{Q}(\zeta)$ sending $\mathbb{R}\cup\{\infty\}$ onto the unit circle. Another way to prove the statement about the field of moduli relies on the fact that the complex conjugation on $X_{\mathbf{b}}$ corresponds on the one hand to the transformation

$$a \;:\; \mathbf{b} \;=\; (1,b_2,\ldots,b_q) \;\mapsto\; (1,b_q,\ldots,b_2)\,.$$

On the other hand, the same transformation of exponents corresponds to the isomorphism of curves given by

$$x \;\mapsto\; \frac{1}{x}\,, \quad y \;\mapsto\; \frac{y}{x}\,.$$

We know by Lemma 1 that there is a bijection between the normalized q–tuples $\mathbf{b}$ introduced above and the torsion–free normal subgroups N in this unique subgroup Ψ of $\Delta = \langle q,q,l\rangle$ with quotient $\Psi/N \cong Z_q$. The absolute Galois group does only permute the different curves $X_{\mathbf{b}}$. To determine their fields of moduli, one has therefore to determine the isomorphisms between these different $X_{\mathbf{b}}$, and by Lemma 2 we know that we have to determine all conjugacies between the different groups N in maximal triangle groups $\overline{\Delta}$.

Lemma 8 *Suppose* $q > 2$, $l > 3$, $q \neq l,\, 2l,\, 4l$ *and suppose that* $\langle q, q, l\rangle = \Delta$ *is a non–arithmetic triangle group. Then* $\overline{\Delta} = \langle 2, q, 2l\rangle$ *is the unique maximal triangle group containing* N, Ψ, Δ.

The uniqueness is a consequence of Margulis' characterization of non–arithmetic Fuchsian groups that the commensurator of N, Ψ, Δ is only a finite index supergroup of them. By work of Singerman [Si1], these supergroups are well known, and our hypotheses about q and l guarantee that $\langle 2, q, 2l\rangle$ is in fact *the* maximal triangle group to be considered here.
Remark. The Fermat curves give examples in which $\langle 2, q, 2q\rangle$ are not maximal — and for which the following determination of the moduli field needs an extra effort which is useless since we know that $K = \mathbb{Q}$. For the dessins arising from the embeddings of cyclic projective planes we have $l = q^2 - q + 1 > q$. The hypotheses of the Lemma are therefore violated only if Δ is an arithmetical triangle group. A look into Takeuchi's classification [Ta] shows that this is the case only for $\langle 3, 3, 7\rangle$. Since we already know that in this case $X_{\mathbf{b}}$ is isomorphic to Klein's quartic defined over $K = \mathbb{Q}$, we can concentrate on the cases satisfying the hypotheses of Lemma 8.
We continue with four rather obvious observations.

Lemma 9 *Under the hypotheses of Proposition 6,* Ψ *is a normal subgroup of* $\overline{\Delta} = \langle 2, q, 2l\rangle$*, and to the group inclusions* $\Psi \subset \Delta \subset \overline{\Delta}$ *correspond the normal function field extensions of their quotient spaces* $\mathbb{C}(x) \supset \mathbb{C}(x^q) \supset \mathbb{C}(x^q + x^{-q})$. *The quotient* $\overline{\Delta}/\Psi$ *is isomorphic to the dihedral group* $Z_q \rtimes Z_2$ *and acts as Galois group on the function field extension* $\mathbb{C}(x)/\mathbb{C}(x^q + x^{-q})$ *of degree* $2q$ *generated by*

$$a \,:\, x \,\mapsto\, \frac{1}{x} \quad , \quad b \,:\, x \,\mapsto\, \zeta x\,.$$

Lemma 10 *This group* $Z_q \rtimes Z_2$ *acts on the set of quotient curves* $X_{\mathbf{b}} \cong N\backslash\mathcal{H}$ *by*

$$a(\mathbf{b}) \;=\; a((1, b_2, \ldots, b_q)) \;=\; (1, b_q, b_{q-1}, \ldots, b_2)\,,$$

$$b(\mathbf{b}) \;=\; b((1, b_2, \ldots, b_q)) \;=\; (1, b_3 b_2^{-1}, \ldots, b_q b_2^{-1}, b_2^{-1})\,.$$

If the hypotheses of Lemma 8 are satisfied, the orbits under this group action form precisely the isomorphism classes among the curves $X_{\mathbf{b}}$.

Lemma 11 *For* $i \in \mathbb{Z}/q\mathbb{Z}$ *denote* $b^i(\mathbf{b}) =: (1, b_2', \ldots, b_q')$. *There is a* $k \equiv 1 - i \bmod q$ *such that the cyclic sequences of quotients*

$$1\,,\, b_2\,,\, \ldots\,,\, b_{q-1}\,,\, b_q \quad \textit{and} \quad 1\,,\, b_{k+1}'/b_k'\,,\, b_{k+2}'/b_k'\,,\, \ldots\,,\, b_{k-1}'/b_k'$$

coincide. The action of a *reverses the order of these sequences, i.e. replaces* b_{k+m}' *by* b_{k-m}'.

Lemma 12 *The action of the absolute Galois group* $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$ *on the set of curves* $X_{\mathbf{b}}$ *factorizes through* $G = \mathrm{Gal}\,\mathbb{Q}(\zeta)/\mathbb{Q}$ *. If we identify* G *in the usual way with the group of prime residue classes* $Z_q^* := (\mathbb{Z}/q\mathbb{Z})^*$ *, every* $r \in Z_q^*$ *acts on the set of* q*–tuples* $\mathbf{b}$ *by*

$$r \ : \ (1, b_2, \dots, b_q) \ \mapsto \ (1, b_{r^{-1}+1}, b_{2r^{-1}+1}, \dots, b_{(q-1)r^{-1}+1}) \ .$$

In particular, the action of $r = -1$ *coincides with the action of* a *.*

The last sentence again shows that the moduli field of every $X_{\mathbf{b}}$ is a real subfield of $\mathbb{Q}(\zeta)$. If $X_{\mathbf{b}}$ and $X_{\mathbf{b}'}$ are isomorphic curves, then their r–images are isomorphic, too, for all $r \in Z_q^*$. We obtain therefore

Lemma 13 *Under the hypotheses of Lemma 8 there is a well–defined action of the Galois group* $G = \mathrm{Gal}\,\mathbb{Q}(\zeta)/\mathbb{Q} = Z_q^*$ *on the* $Z_q \rtimes Z_2$*–orbits considered in Lemma 10. With this action of* G *, the moduli field of* $X_{\mathbf{b}}$ *is the fixed field of the stabilizer of* $(Z_q \rtimes Z_2)(\mathbf{b})$ *.*

Example. Let $n = 7$, $q = 8$, $l = 57$ and consider the uniform dessin for the plane $\mathbb{P}^2(\mathbb{F}_7)$ belonging to the cyclic ordered difference set

$$D \ = \ (0, 1, 3, 7, 21, -19, -24, -8)$$

satisfying in fact the condition $(k_i - k_{i+1}, l) \ = \ 1$, see the last section. Its algebraic curve $X_{\mathbf{b}}$ corresponds to the 8–tuple (of inverse exponents $\bmod 57$)

$$\mathbf{b} \ = \ (1, 2, 4, 14, 17, -5, 16, 8) \ .$$

For $r = 3$ and $r = 5 \in Z_8^* = G$ we obtain

$$3(\mathbf{b}) \ = \ (1, 14, 16, 2, 17, 8, 4, -5) \quad \text{and} \quad 5(\mathbf{b}) \ = \ (1, -5, 4, 8, 17, 2, 16, 14) \ .$$

Both do not belong to the $(Z_8 \rtimes Z_2)$–orbit of $\mathbf{b}$ what can easily be seen using Lemma 11: the cyclic sequence of the b_i contains the subsequent members $1, 2, 4$ which do not occur in any sequence of quotients 1 , $b'_{k\pm1}/b'_k$, $b'_{k\pm2}/b'_k$ for $3(\mathbf{b})$ and $5(\mathbf{b})$. Therefore, the field of moduli and the field of definition of $X_{\mathbf{b}}$ is in fact the fixed field $K = \cos 2\pi/8$ of the subgroup $\{1, -1\} \subset G$. Another interesting fact becomes visible in this example: Being the dessin of a projective plane is not a Galois invariant property because e.g. $5(\mathbf{b})$ does not consist of the successive differences of a difference set. By consequence, the existence or non–existence of quadrangle loops (see Prop. 10, next section) in a dessin is neither a Galois invariant.

5 Regular Wada dessins

We start with some more general remarks on the Wada property. Clearly, unicellular dessins are Wada dessins, and starting with unicellular dessins in

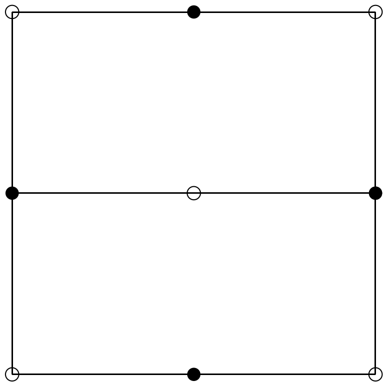

Figure 5: Non–uniform Wada dessin on a torus

positive genera, it is easy to construct Wada dessins by suitable subdivisions of the cell as in the following more general genus 1 example (Figure 5) in which the opposite borderlines have to be identified.
This $(4,4,3)$–dessin is not uniform since there are vertices of both colors with different valencies. The reason is the fact that on the border of each cell there are different vertices which have to be identified on the Riemann surface. In other words, one may draw a curve joining this vertex to itself in the cell but not null–homotopic in the (closed) cell. This turns out to be the only obstruction for Wada dessins with more than one cell to be uniform.
Definition. We call a dessin FLAT if the topological closure of all cells are simply connected.

PROPOSITION 7 *Let $\mathcal{D}$ be a flat Wada dessin with $q > 1$ cells. Then $\mathcal{D}$ is a uniform (q,q,l)–dessin where l denotes the number of black resp. white vertices.*

Proof. By definition, every vertex of $\mathcal{D}$ lies on the border of every cell, so the valencies of the vertices have to be at least q. On the other hand, no such valency can be $> q$: Otherwise there would exist a cell S having a vertex x twice on its border, more precisely one could join x with itself by a non–nullhomotopic curve in the cell, but — by hypothesis — null–homotopic in $\overline{S}$; therefore another vertex $y \neq x$ exists in the interior of this curve, lying on the border of (only) the cell S, hence $q = 1$, contradiction. By the same reason, every cell has precisely the valency $2l$.
Remark and Example. By Theorem 2, we know that the resulting dessin of the embedding of a cyclic projective plane's graph depends on the chosen orderings of the difference set. This is also true for the Wada property and for flatness: For $\mathbb{P}^2(\mathbb{F}_3)$ one has the difference set

$$D := \{0, 1, 3, 9\} \subset Z_{13} .$$

If we take the corresponding globe covering dessin, i.e. with $m_i = k_i$, we obtain a uniform flat $(4,4,13)$–dessin as in the proof of Theorem 3. If we change the cyclic orders of D into

$$(m_i)_{i=1,\dots,4} = (0,1,3,9) \quad \text{but} \quad (k_i)_{i=1,\dots,4} = (9,3,1,0)$$

we obtain a non–flat uniform $(4,4,26)$–Wada dessin with two cells. With the cyclic orders

$$(m_i)_{i=1,\dots,4} = (0,1,3,9) \quad \text{and} \quad (k_i)_{i=1,\dots,4} = (0,3,9,1)$$

we obtain a Wada dessin with two cells, one of valency $2 \cdot 13$ and the other of valency $2 \cdot 39$. In both examples the quotient by the Singer group Z_{13} is a dessin with one black vertex and one white vertex and $q = 4$ edges, but not with q cells in genus 0 (as for the globe covering dessins treated in Sections 3 and 4) but with two cells in genus 1.

In Section 2, we already met some special regular Wada dessins. Here we will characterize such dessins, give some more examples and explain why their underlying graphs do in general not come from finite projective planes even if the valencies q and l satisfy the necessary relation $l = n^2 + n + 1 = q^2 - q + 1$.

Proposition 8 *Let $\mathcal{D}$ be a regular (q,m,l)–dessin with automorphism group G, generated by elements b_0, b_1, b_∞ of respective orders q,m,l and generating cyclic subgroups B, W and C, respectively. Then $\mathcal{D}$ is a Wada dessin if and only if*

$$(G:B) = (C:C\cap B) \quad \textit{and} \quad (G:W) = (C:C\cap W)\,.$$

In that case, the number of cells of $\mathcal{D}$ is

$$(G:C) = (B:C\cap B) = (W:C\cap W)\,.$$

$\mathcal{D}$ is a flat Wada dessin if and only if moreover

$$C\cap B = C\cap W = \{1\}\,,$$

in other words if $\mathcal{D}$ has l black and l white vertices and $q = m$ cells.

Proof. There are a black vertex x fixed by B and a white vertex y fixed by W, both on the border of a cell fixed by C. Since $\mathcal{D}$ is a regular dessin, the automorphism group G acts transitively on all black (resp. white) vertices, and since B (resp. W) is the stabilizer subgroup of x (resp.y), the total number of black (resp. white) vertices is $(G:B)$ resp. $(G:W)$. Now, $\mathcal{D}$ is a Wada dessin if and only if all these black (resp. white) vertices form one orbit under the action of C. According to the class formula, this is the case if and only if

$$(G:B) = (C:C\cap B) \quad \text{and} \quad (G:W) = (C:C\cap W)\,.$$

The number of cells is deduced in a similar way by the action of G, B and W on the cells. Moreover, $\mathcal{D}$ is flat if and only if all the black (resp. white) vertices on the border of the cell fixed by C are pairwise different, i.e. if and only if l is the number of black (resp. white) vertices.
As a non–flat example, take the genus 3 curve with the affine model

$$y^2 \;=\; x^7 - x \,,$$

with $(4,4,6)$–dessin and an automorphism group G of order 12 (a semidirect product of a cyclic group of order 4 with a normal subgroup of order 3). Here we have two cells of valency 12 but only 3 different black (resp. white) vertices. It is not surprising that in the case of flat regular Wada dessins the structure of G can be determined rather precisely.

Proposition 9 *Let $\mathcal{D}$ be a regular (q,q,l)–Wada dessin with q cells and l black (resp. white) vertices. Then*

1. *$G = \mathrm{Aut}\,\mathcal{D}$ has order ql,*
2. *$G = BC = WC$ for the cyclic stabilizer subgroups B, W of a black and a white neighbor vertex and the cyclic stabilizer subgroup C of a cell,*
3. *$G'' = \{1\}$, i.e. G is metabelian.*
4. *If q is prime and $l > 1$, one has even $l \geq q$.*
5. *In the case $l = q$ prime, the dessin belongs to a Fermat curve of exponent q and with $G \cong Z_q^2$.*
6. *If l is prime $> q$ (arbitrary), q divides $l-1$ and $G \cong Z_l \rtimes Z_q$.*

Proof. 1) Clearly, $\mathcal{D}$ has ql edges. Since G acts sharply transitive, the number of edges is the order of G. A similar argument proves assertion 2). As we learned from Gareth Jones, 3) follows from 2) by a theorem of Itô [I]. 4) Because G is generated by two cyclic subgroups B and W of order q, they coincide if and only if $l = 1$. If not and q is prime, they satisfy moreover $B \cap W = \{1\}$, $\mathrm{ord}\,G \geq q^2$ and hence $l \geq q$. 5),6) Since G contains a cyclic subgroup of order l, the statements about the structure of G are standard consequences of Sylow's theorems. It is well known that regular (q,q,q)–dessins with automorphism group Z_q^2 belong to Fermat curves, see e.g. [JS],7. Examples 3. On the other hand: that these dessins are flat Wada dessins can easily be verified using Proposition 8.

Remark. If q is not prime, the statement 4) in general fails as the following example shows. On the elliptic curve $y^2 = x^4 - 1$ there is a regular $(4,4,2)$–dessin with 8 edges, $q = 4$ cells, $l = 2$ black resp. white points, automorphism group $G \cong Z_4 \times Z_2$ and disjoint generating subgroups $B \cong W \cong Z_4$, $C \cong Z_2$ (complete the Figure 5 dessin by two edges forming a vertical middle axis).

For the structure of G in these more general cases one may consult a paper of Huppert [Hu]. Theorem 2 of [I] gives the existence of normal subgroups $N \lhd G$ containing B, W or C and other normal subgroups contained in these cyclic subgroups, so it is possible to represent $\mathcal{D}$ by successive cyclic coverings of very simple genus 0 dessins.

In [StWo] we studied a series of regular (q, q, l)–dessins with q, l prime, $q | l-1$ and automorphism group $G \cong Z_l \rtimes Z_q$ giving examples for Proposition 9.6). For the purpose of the present paper, the hypothesis "q prime" is unnecessary, but we make the assumptions

$$q > 2\,, \quad l = q^2 - q + 1 \quad \text{prime}\,, \quad G \cong Z_l \rtimes Z_q$$

where Z_l is generated by a and Z_q by b satisfying the relation

$$b^{-1}\,a\,b = a^u$$

for some fixed prime residue class $u \in (\mathbb{Z}/l\mathbb{Z})^*$ of order q. Imitating the proof of Proposition 3, we generate the automorphism group of the dessin by a rotation b around a black vertex x and a rotation $b^{-1}a$ around a white vertex neighbor y. That all these dessins are flat Wada dessins follows again easily from Proposition 8.

Proposition 10 *Let $\mathcal{D}$ be a (q, q, m)–dessin with $l = q^2 - q + 1$ points, all vertices with valency $q > 2$. The underlying graph is the graph of a projective plane if and only if no quadrangle loop exists in $\mathcal{D}$, i.e. if there are no white vertices $y \neq y^*$, black vertices $x \neq x^*$ such that xyx^*y^*x are successive neighbors.*

Proof. If such a quadrangle loop exists, the uniqueness of the intersection points or joining lines is violated, whence we cannot have the graph of a projective plane. If no such quadrangle exists, counting neighbor vertices one easily shows that any two black vertices have a unique white neighbor in common, and that the respective statement is true for two white vertices. The existence of four points in general position follows easily from $q > 2$.

With Proposition 10 we can now see why e.g. the regular $(7, 7, 43)$–dessin with automorphism group $Z_{43} \rtimes Z_7$ has no underlying graph belonging to a projective plane:

Proposition 11 *Let $\mathcal{D}$ be a regular (q, q, l)–dessin with $l = q^2 - q + 1$ prime and automorphism group $G \cong Z_l \rtimes Z_q$ whose generators a, b of respective orders l, q satisfy*

$$b^{-1}\,a\,b = a^u\,,$$

$u \in (\mathbb{Z}/l\mathbb{Z})^*$ of order q. The underlying graph is a graph of a projective plane of order $n = q-1$ if and only if the powers u^k, $k = 1, \ldots, q$, form a difference set in $\mathbb{Z}/l\mathbb{Z}$.

Proof. Because the dessin is regular, we can start with any black vertex x and a white neighbor y , hence we will take the fixed points of b and $b^{-1}a$. Suppose there is a quadrangle loop as forbidden by Proposition 9, then

$$x^* \;=\; (b^{-1}a)^k(x)\,, \quad y^* \;=\; b^m(y)$$

with $k, m \not\equiv 0 \bmod q$, and the subgroups fixing these two points are generated by

$$(b^{-1}a)^k\, b\, (b^{-1}a)^{-k} \quad \text{and} \quad b^m(b^{-1}a)b^{-m}$$

respectively. An edge joining x^* with y^* exists if and only if it is the G–image of the edge joining y and x by a group element which can be written in two ways:

$$(b^m(b^{-1}a)b^{-m})^s\, b^m \;=\; ((b^{-1}a)^k\, b\,(b^{-1}a)^{-k})^r\,(b^{-1}a)^k$$

with $r, s \not\equiv 0 \bmod q$. This equation is equivalent to

$$(b^{-1}a)^s \;=\; b^{-m}\,(b^{-1}a)^k\, b^r$$

or, using the relation between a and b ,

$$a^{u+u^2+\ldots+u^s}\, b^{-s} \;=\; b^{-m}\, a^{u+u^2+\ldots+u^k}\, b^{-k+r} \;=\; a^{u^m(u+u^2+\ldots+u^k)}\, b^{-m-k+r}\;.$$

This relation holds if and only if

$$s+r \;\equiv\; m+k \mod q \quad \text{and} \quad u+\ldots+u^s \;\equiv\; u^m(u+\ldots+u^k) \mod l\;.$$

The second congruence is easily seen to be equivalent to

$$u^s\,-\,1 \;\equiv\; u^{m+k}\,-\,u^m$$

meaning that the powers of u do not form a difference set in $\mathbb{Z}/l\mathbb{Z}$. On the other hand, if the powers of u form a difference set, the last congruence is unsolvable for $s, m, k \not\equiv 0 \bmod q$, whence a quadrangle loop cannot exist.

6 The cartographic group

We prove the last theorem in the following more general form.

Proposition 12 *Let $\mathcal{D}$ be a globe covering dessin obtained by embedding the graph of a cyclic projective plane $\mathbb{P}$ of order $n = q-1$ with Singer group $Z_l \subseteq \operatorname{Aut}\mathcal{D}$. Then the cartographic group M of $\mathcal{D}$ is isomorphic to a semidirect product $A \rtimes Z_q$ for a quotient A of Z_l^n .*

As explained in the beginning of Section 4, the hypothesis *globe covering* says that $\mathcal{D}$ corresponds to a subgroup Γ of the triangle group $\Delta = \langle q,q,l\rangle$ with an intermediate normal subgroup $\Psi = \Delta'$ of signature $< 0; l^{(q)} >$ such that (7) and (8) hold. In contrast to Section 4, we can even admit the existence of torsion elements in Γ, in other words $\mathcal{D}$ is allowed to be a non–uniform dessin. The cartographic group of $\mathcal{D}$ can be introduced either as monodromy group of the corresponding Belyi function β or as a certain permutation group of the edges of $\mathcal{D}$ since these represent the sheets of the covering β. Here, the easiest way to determine M is the fact that M is isomorphic to the quotient Δ/N of Δ by its maximal normal subgroup N contained in Γ. Let Ψ' the commutator subgroup of Ψ. Since Ψ is normal in Δ, the same holds for Ψ'. The presentation of Ψ shows that

$$\Psi' \subseteq \Gamma \quad \text{with} \quad \Psi/\Psi' \cong Z_l^{q-1} .$$

Therefore, $\Psi' \subseteq N \subseteq \Gamma$, and if we denote the quotient Ψ/N by A, the result follows.

Remark. As in Section 4, the choice of the ordered difference set for $\mathbb{P}$ determines the homomorphism $\Psi \to Z_l$ with kernel Γ, and the action of Δ resp. Z_q on Z_l^{q-1} is also known. Using these data, it is in principle possible to determine A and the action of Z_q on A.

The same line of arguments as in the proof above gives a more general version of Theorem 4. Since the full automorphism group of $\mathcal{D}$ is isomorphic to $N_\Delta(\Gamma)/\Gamma$ where $N_\Delta(\Gamma)$ denotes the normalizer in Δ (containing Ψ, of course), we obtain

Proposition 13 *Under the hypotheses of Proposition 12 we have*

$$\mathrm{Aut}\,\mathcal{D} \cong Z_l \rtimes Z_m$$

for some divisor m of q.

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On the Γ-Factors of Motives II

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Abstract. Using an idea of C. Simpson we describe Serre's local Γ-factors in terms of a complex of sheaves on a simple dynamical system. This geometrizes our earlier construction of the Γ-factors. Relations of this approach with the ε-factors are also studied.

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1 Introduction

In [S] Serre defined local Euler factors $L_{\mathfrak{p}}(H^n(X), s)$ for the "motives" $H^n(X)$ where X is a smooth projective variety over a number field k. The definition at the finite places $\mathfrak{p}$ involves the Galois action on the l-adic cohomology groups $H^n_{\text{ét}}(X \otimes \overline{k}_{\mathfrak{p}}, \mathbb{Q}_l)$. At the infinite places the local Euler factor is a product of Gamma factors determined by the real Hodge structure on the singular cohomology $H^n_B(X \otimes \overline{k}_{\mathfrak{p}}, \mathbb{R})$. If $\mathfrak{p}$ is real then the Galois action induced by complex conjugation on $\overline{k}_{\mathfrak{p}}$ has to be taken into account as well.
Serre also conjectured a functional equation for the completed L-series, defined as the product over all places of the local Euler factors.
In his definitions and conjectures Serre was guided by a small number of examples and by the analogy with the case of varieties over function fields which is quite well understood. Since then many more examples over number fields notably from the theory of Shimura varieties have confirmed Serre's suggestions. The analogy between l-adic cohomology with its Galois action and singular cohomology with its Hodge structure is well established and the definition of the local Euler factors fits well into this philosophy. However in order to prove the functional equation in general, a deeper understanding than the one

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provided by an analogy is needed. First steps towards a uniform description of the local Euler factors were made in [D1], [D2], [D3]. There we constructed infinite dimensional complex vector spaces $\mathcal{F}_{\mathfrak{p}}(H^n(X))$ with a linear flow such that for all places:

$$L_{\mathfrak{p}}(H^n(X), s) = \det{}_{\infty}\Big(\frac{1}{2\pi}(s \cdot \mathrm{id} - \Theta) \,|\, \mathcal{F}_{\mathfrak{p}}(H^n(X))\Big)^{-1} . \tag{1}$$

Here Θ is the infinitesimal generator of the flow and $\det_{\infty}$ is the zeta-regularized determinant. Unfortunately the construction of the spaces $\mathcal{F}_{\mathfrak{p}}(H^n(X))$ was not really geometric. They were obtained by formal constructions from étale cohomology with its Galois action and from singular cohomology with its Hodge structure.

C. Consani [C] later developed a new infinite-dimensional cohomology theory $H^n_{\mathrm{Cons}}(Y)$ with operators N and Θ for varieties Y over $\mathbb{R}$ or $\mathbb{C}$ such that for infinite places $\mathfrak{p}$:

$$(\mathcal{F}_{\mathfrak{p}}(H^n(X)), \Theta) \cong (H^n_{\mathrm{Cons}}(X \otimes k_{\mathfrak{p}})^{N=0}, \Theta) . \tag{2}$$

Her constructions are inspired by the theory of degenerations of Hodge structure and her N has to be viewed as a monodromy operator. The formula for the archimedean local factors obtained by combining (1) and (2) is analogous to the expression for $L_{\mathfrak{p}}(H^n(X), s)$ at a prime $\mathfrak{p}$ of semistable reduction in terms of log-crystalline cohomology.

The conjectural approach to motivic L-functions outlined in [D7] suggests the following: It should be possible to obtain the spaces $\mathcal{F}_{\mathfrak{p}}(H^n(X))$ for archimedian $\mathfrak{p}$ together with their linear flow directly by some natural homological construction on a suitable non-linear dynamical system. Clearly, forming the intersection of the Hodge filtration with its complex conjugate and running the resulting filtration through a Rees module construction as in our first construction of $\mathcal{F}_{\mathfrak{p}}(H^n(X))$ in [D1] is not yet what we want: In this construction the linear flow appears only a posteriori on cohomology but it is not induced from a flow on some underlying space by passing to cohomology.

In the present paper in Theorems 4.2, 4.3, 4.4 we make a step towards this goal of a more direct dynamical description of the archimedian Gamma-factors. The approach is based on a result of Simpson which roughly speaking replaces the consideration of the Hodge filtration by looking at a relative de Rham complex with a deformed differential.

In our case, instead of the Hodge filtration $F^{\bullet}$ we require the non-algebraic filtration $F^{\bullet} \cap \overline{F^{\bullet}}$. This forces us to work in a real analytic context even for complex $\mathfrak{p}$. It seems difficult to carry Simpson's method over to this new context. However this is not necessary. By a small miracle – the splitting of a certain long exact sequence – his result can be brought to bear directly on our more complicated situation.

In the appendix to section 4 we explain a relation between Simpson's deformed complex and a relative de Rham complex on the deformation of X to the normal bundle of a base point. This observation probably holds the key for a complete dynamical understanding of the Gamma-factor.

Our main construction also provides a C^{ω}-vector bundle on $\mathbb{R}$ with a flow. Its fibre at zero can be used for a "dynamical" description of the contribution from $\mathfrak{p} \mid \infty$ in the motivic "explicit formulas" of analytic number theory.

In our investigation we encounter a torsion sheaf whose dimension is to some extent related to the ε-factor at $\mathfrak{p}$ of $H^n(X)$.

Using forms with logarithmic singularities one can probably deal more generally with the motives $H^n(X)$ where X is only smooth and quasiprojective.

It would also be of interest to give a construction for Consani's cohomology theory using the methods of the present paper.

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2 Preliminaries on the algebraic Rees sheaf

In this section we recall and expand upon a simple construction which to any filtered complex vector space attaches a sheaf on $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$ with a $\mathbb{G}_m$-action. We had used it in earlier work on the Γ-factors [D1], [D3] § 5. Later Simpson [Si] gave a more elegant treatment and proved some further properties. Most importantly for us he proved Theorem 5.1 below which was the starting point for the present paper. In the following we also extend his results to a variant of the construction where one starts from a filtered vector space with an involution. This is necessary later to deal not only with the complex places but with the real places as well.

Let $\mathcal{F}il_{\mathbb{C}}$ be the category of finite dimensional complex vector spaces V with a descending filtration $\mathrm{Fil}^r V$ such that $\mathrm{Fil}^{r_1} V = 0, \mathrm{Fil}^{r_2} V = V$ for some integers r_1, r_2. Let $\mathcal{F}il^{\pm}_{\mathbb{R}}$ be the category of finite dimensional complex vector spaces with a filtration as above and with an involution F_∞ which respects the filtration. Finally let $\mathcal{F}il_{\mathbb{R}}$ be the full subcategory of $\mathcal{F}il^{\pm}_{\mathbb{R}}$ consisting of objects where F_∞ induces multiplication by $(-1)^{\bullet}$ on $\mathrm{Gr}^{\bullet} V$.

These additive categories have $\otimes$-products and internal Hom's. We define Tate twists for every integer n by

$$(V, \mathrm{Fil}^{\bullet} V)(n) = (V, \mathrm{Fil}^{\bullet+n} V) \quad \text{in } \mathcal{F}il_{\mathbb{C}}$$

and by

$$(V, \mathrm{Fil}^{\bullet} V, F_\infty)(n) = (V, \mathrm{Fil}^{\bullet+n} V, (-1)^n F_\infty) \quad \text{in } \mathcal{F}il^{\pm}_{\mathbb{R}} \text{ and } \mathcal{F}il_{\mathbb{R}} \; .$$

Note that the full embedding:

$$i : \mathcal{F}il_{\mathbb{R}} \hookrightarrow \mathcal{F}il^{\pm}_{\mathbb{R}}$$

is split by the functor

$$s : \mathcal{F}il_{\mathbb{R}}^{\pm} \longrightarrow \mathcal{F}il_{\mathbb{R}}$$

which sends $(V, \widetilde{\mathrm{Fil}}^r V, F_\infty)$ to

$$(V, \mathrm{Fil}^r V = (\widetilde{\mathrm{Fil}}^r V)^{(-1)^r} + (\widetilde{\mathrm{Fil}}^{r+1} V)^{(-1)^{r+1}}, F_\infty)$$

i.e. $s \circ i = \mathrm{id}$. Here $W^{\pm 1}$ denotes the ± 1 eigenspace of F_∞ on W. For V in $\mathcal{F}il_{\mathbb{C}}$ following [Si] § 5 define a locally free sheaf $\xi_{\mathbb{C}}(V) = \xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)$ over $\mathbb{A}^1$ with action of $\mathbb{G}_m$ by

$$\xi_{\mathbb{C}}(V) = \sum_p \mathrm{Fil}^p V \otimes z^{-p} \mathcal{O}_{\mathbb{A}^1} \subset V \otimes j_* \mathcal{O}_{\mathbb{G}_m} .$$

Here $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ is the inclusion and z denotes a coordinate on $\mathbb{A}^1$ determined up to a scalar in $\mathbb{C}^*$. Unless stated otherwise the constructions in this paper are independent of z. The global sections of the "Rees sheaf" $\xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)$ form the "Rees module" over $\mathbb{C}[z]$:

$$\mathrm{Fil}^0(V \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]) = \sum_p \mathrm{Fil}^p V \otimes z^{-p} \mathbb{C}[z] \subset V \otimes \mathbb{C}[z, z^{-1}]$$

where $\mathrm{Fil}^p \mathbb{C}[z, z^{-1}] = z^p \mathbb{C}[z]$ for $p \in \mathbb{Z}$. The natural action of $\mathbb{G}_m$ on $\mathbb{A}^1$ induces a $\mathbb{G}_m$-action on $\xi_{\mathbb{C}}$ by pullback

$$\lambda^* : (\lambda)^{-1} \xi_{\mathbb{C}} \longrightarrow \xi_{\mathbb{C}} \, , \; v \otimes g(z) \longmapsto v \otimes g(\lambda z) \, . \tag{3}$$

Here $(\lambda)^{-1} \xi_{\mathbb{C}}$ denotes the inverse image of $\xi_{\mathbb{C}}$ under the multiplication by $\lambda \in \mathbb{C}^*$ map.

Let $sq : \mathbb{A}^1 \to \mathbb{A}^1$ be the squaring map $sq(z) = z^2$ and define $F_\infty : \mathbb{A}^1 \to \mathbb{A}^1$ as $F_\infty = -\mathrm{id}$. For V in $\mathcal{F}il_{\mathbb{R}}^{\pm}$ the actions of F_∞ on V and $\mathbb{G}_m \subset \mathbb{A}^1$ combine to an action

$$F_\infty^* : F_\infty^{-1}(V \otimes j_* \mathcal{O}_{\mathbb{G}_m}) \longrightarrow V \otimes j_* \mathcal{O}_{\mathbb{G}_m} \, .$$

Thus we get an involution F_∞ on the sheaf $sq_*(V \otimes j_* \mathbb{G}_m)$ and we define a locally free sheaf on $\mathbb{A}^1$ by:

$$\xi_{\mathbb{R}}(V) = \xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty) = (sq_* \xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V))^{F_\infty} \, .$$

The $\mathbb{G}_m$-action on $\xi_{\mathbb{C}}$ leads to an action

$$\lambda^* : (\lambda^2)^{-1} \xi_{\mathbb{R}} \longrightarrow \xi_{\mathbb{R}} \, .$$

The global sections of $\xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty)$ are given by

$$\Big(\sum_p \mathrm{Fil}^p V \otimes z^{-p} \mathbb{C}[z] \Big)^{F_\infty} \subset V \otimes \mathbb{C}[z, z^{-1}]$$

viewed as a $\mathbb{C}[z^2]$-module.

Remark 2.1 The global sections of ξ – with the action of the Lie algebra of $\mathbb{G}_m$ – were also considered in [D3] § 5

$$\begin{aligned}
\Gamma(\mathbb{A}^1, \xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)) &= \mathbb{D}^+(V, \mathrm{Fil}^\bullet V) \\
\Gamma(\mathbb{A}^1, \xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty)) &= \mathbb{D}^+(V, \mathrm{Fil}^\bullet V, F_\infty)
\end{aligned}$$

in the notation of that paper.

We denote by $E = E(V, \mathrm{Fil}^\bullet V)$ resp. $E = E(V, \mathrm{Fil}^\bullet V, F_\infty)$ the vector bundle on $\mathbb{A}^1$ corresponding to the locally free sheaf ξ. It has a contravariant $\mathbb{G}_m$-action with respect to the action of $\mathbb{G}_m$ on $\mathbb{A}^1$ by

$$\mathbb{G}_m \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \ , \ (\lambda, a) \longmapsto \lambda^{e_K} a \tag{4}$$

where $e_{\mathbb{C}} = 1$ and $e_{\mathbb{R}} = 2$.
For $K = \mathbb{C}$ resp. $\mathbb{R}$ let $\mathcal{D}_K$ be the category of locally free $\mathcal{O}_{\mathbb{A}^1}$-modules of finite rank with contravariant action by $\mathbb{G}_m$ with respect to the action (4) on $\mathbb{A}^1$. The category $\mathcal{D}_K$ has $\otimes$-products and internal Homs. For $\mathcal{M} \in \mathcal{D}_K$ set $\mathcal{M}(n) = z^n \mathcal{M}$ for any integer n. Thus $\mathcal{M}(n)$ is isomorphic to $\mathcal{M}$ as an $\mathcal{O}_{\mathbb{A}^1}$-module but with $\mathbb{G}_m$-action twisted as follows:

$$\lambda^*_{\mathcal{M}(n)} = \lambda^n \cdot \lambda^*_{\mathcal{M}} \ .$$

The following construction provides inverses to $\xi_{\mathbb{C}}$ and $\xi_{\mathbb{R}}$. For $\mathcal{M}$ in $\mathcal{D}_K$ set

$$\eta_K(\mathcal{M}) = \Gamma(\mathbb{G}_m, j^*\mathcal{M})^{\mathbb{G}_m} = (\Gamma(\mathbb{A}^1, \mathcal{M}) \otimes_{\mathbb{C}[z^{e_K}]} \mathbb{C}[z, z^{-1}])^{\mathbb{G}_m}$$

with the filtration (and in case $K = \mathbb{R}$ the involution) coming from the one on $\mathbb{C}[z, z^{-1}]$.
The main properties of ξ and η are contained in the following proposition.
Recall that a map $\varphi : V \to W$ of filtered vector spaces is called strict if $\varphi^{-1}\mathrm{Fil}^i W = \mathrm{Fil}^i V$ for all i.

Proposition 2.2 *a) The functor $\xi_K : \mathcal{F}il_K \to \mathcal{D}_K$ is an equivalence of additive categories with quasi-inverse η_K. It commutes with $\otimes$-products and internal Homs and we have that*

$$\dim V = \mathrm{rk} \xi_K(V) \quad \textit{and} \quad \dim \eta_K(\mathcal{M}) = \mathrm{rk}\mathcal{M}$$

for all V in $\mathcal{F}il_K$ and $\mathcal{M}$ in $\mathcal{D}_K$.
The functors $\xi_{\mathbb{C}}$ and $\xi_{\mathbb{R}} : \mathcal{F}il^{\pm}_{\mathbb{R}} \to \mathcal{D}_{\mathbb{R}}$ commute with Tate twists.
For $(V, \mathrm{Fil}^\bullet V, F_\infty) \in \mathcal{F}il^{\pm}_{\mathbb{R}}$ there is a canonical isomorphism:

$$\xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty)^* = \xi_{\mathbb{R}}(V^*, \mathrm{Fil}^{\bullet -1} V^*, F^*_\infty) \ .$$

Here $\mathrm{Fil}^p V^* := (\mathrm{Fil}^{1-p} V)^\perp$ *in* V^*.
b) The diagrams

$$\begin{tikzcd}
\mathcal{F}il^{\pm}_{\mathbb{R}} \arrow[r, "\xi_{\mathbb{R}}"] \arrow[d, "s"'] & \mathcal{D}_{\mathbb{R}} \\
\mathcal{F}il_{\mathbb{R}} \arrow[ur, "\xi_{\mathbb{R}}"'] &
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathcal{F}il^{\pm}_{\mathbb{R}} \arrow[r, "\xi_{\mathbb{R}}"] \arrow[dr, "s"'] & \mathcal{D}_{\mathbb{R}} \arrow[d, "\eta_{\mathbb{R}}"] \\
& \mathcal{F}il_{\mathbb{R}}
\end{tikzcd}$$

are commutative.
c) If $\varphi : U \to V$ *is a morphism in* $\mathcal{F}il_{\mathbb{C}}$ *resp.* $\mathcal{F}il_{\mathbb{R}}^{\pm}$ *then*
i) $\xi(\ker \varphi) = \ker(\xi(U) \to \xi(V))$
and
ii) $\xi(\operatorname{coker} \varphi) = \operatorname{coker}(\xi(U) \to \xi(V))/T$
where T *is the subsheaf of torsion elements. We have* $T = 0$ *if and only if* φ *resp.* $s(\varphi)$ *is strict.*

Proof a) is shown in [Si] § 5 for $K = \mathbb{C}$. Every object in $\mathcal{F}il_{\mathbb{R}}^{\pm}$ is the direct sum of objects $\mathbb{C}(n)^{\pm}$ defined as follows: The underlying vector space of $\mathbb{C}(n)^{\pm}$ is $\mathbb{C}$, the filtration is given by $\mathrm{Fil}^p\mathbb{C}(n)^{\pm} = \mathbb{C}(n)^{\pm}$ if $p \le -n$ and $= 0$ if $p > -n$. Finally F_∞ acts on $\mathbb{C}(n)^{\pm}$ by multiplication with ± 1. The objects of $\mathcal{F}il_{\mathbb{R}}$ are direct sums of objects $\mathbb{C}(n)^{(-1)^n}$ and we have that

$$s(\mathbb{C}(n)^{(-1)^n}) = \mathbb{C}(n)^{(-1)^n} \quad \text{and} \quad s(\mathbb{C}(n)^{(-1)^{n+1}}) = \mathbb{C}(n+1)^{(-1)^{n+1}} .$$

Using decompositions into $\mathbb{C}(n)^{(-1)^n}$'s, one checks that the natural maps

$$\xi_{\mathbb{R}}(V) \otimes \xi_{\mathbb{R}}(W) \longrightarrow \xi_{\mathbb{R}}(V \otimes W)$$

and

$$\underline{\mathrm{Hom}}(\xi_{\mathbb{R}}(V), \xi_{\mathbb{R}}(W)) \longrightarrow \xi_{\mathbb{R}}(\underline{\mathrm{Hom}}(V, W))$$

are isomorphisms for all V, W in $\mathcal{F}il_{\mathbb{R}}$. Moreover the rank assertions in a) follow. Commutation with Tate twists follows immediately from the definitions. The final isomorphism follows from the above and the first diagram in b) since a short calculation gives that:

$$(V, s\mathrm{Fil}^{\bullet} V, F_\infty)^* = (V^*, s(\mathrm{Fil}^{\bullet -1} V^*), F_\infty^*) .$$

The commutativities in b) can be seen using decompositions into $\mathbb{C}(n)^{\pm}$'s. In particular $\eta_{\mathbb{R}} \circ \xi_{\mathbb{R}} = \mathrm{id}$ for $\xi_{\mathbb{R}} : \mathcal{F}il_{\mathbb{R}} \to \mathcal{D}_{\mathbb{R}}$. The opposite isomorphism $\xi_{\mathbb{R}} \circ \eta_{\mathbb{R}} = \mathrm{id}$ follows as in Simpson [Si] § 5. Finally c) is stated in loc. cit. for $K = \mathbb{C}$ and remains true for $K = \mathbb{R}$. Part i) is straightforward. As for ii), by functoriality of ξ and the fact that $\xi(\operatorname{coker} \varphi)$ is torsion-free one is reduced to proving that the kernel of the natural surjection

$$\operatorname{coker}(\xi(U) \to \xi(V)) \twoheadrightarrow \xi(\operatorname{coker} \varphi)$$

is torsion. This can be checked using a suitable splitting of φ. □
The following facts about the structure of the Rees bundle were noted for $K = \mathbb{C}$ in [Si] § 5.

Proposition 2.3 *i) For all* V *in* $\mathcal{F}il_{\mathbb{C}}$ *resp.* $\mathcal{F}il_{\mathbb{R}}^{\pm}$ *there are canonical isomorphisms of vector bundles over* $\mathbb{G}_m$

$$j^* E(V, \mathrm{Fil}^{\bullet} V) \xrightarrow{\sim} V \times \mathbb{G}_m$$

resp.

$$sq^*j^*E(V,\mathrm{Fil}^\bullet V,F_\infty)\xrightarrow{\sim}V\times\mathbb{G}_m$$

functorial in V and compatible with the (contravariant) $\mathbb{G}_m$-action. Thus the local systems $F_\mathbb{C}$ and $sq^*F_\mathbb{R}$ are trivialized functorially in V. Under the isomorphism

$$E(V,\mathrm{Fil}^\bullet V,F_\infty)_1=(sq^*E(V,\mathrm{Fil}^\bullet V,F_\infty))_1\xrightarrow{\sim}V$$

the monodromy representation of $\pi_1(\mathbb{C}^*,1)\cong\mathbb{Z}$ maps n to F_∞^n.
ii) For V in $\mathcal{F}il_\mathbb{C}$ resp. $\mathcal{F}il_\mathbb{R}^\pm$ there are isomorphisms depending on the choice of a coordinate z on $\mathbb{A}^1$:

$$E(V,\mathrm{Fil}^\bullet V)_0\xrightarrow{\sim}\mathrm{Gr}^\bullet V$$

resp.

$$E(V,\mathrm{Fil}^\bullet V,F_\infty)_0\xrightarrow{\sim}\mathrm{Gr}^\bullet(sV)$$

functorial in V. They are compatible with the $\mathbb{G}_m$-action if $\mathbb{G}_m$ acts on Gr^pV resp. $\mathrm{Gr}^p(sV)$ by the character z^{-p}.

Proof i) We treat the case $K=\mathbb{R}$. It suffices to check that

$$sq^*j^*\xi(V,\mathrm{Fil}^\bullet V,F_\infty)\xrightarrow{\sim}V\otimes\mathcal{O}_{\mathbb{G}_m}$$

compatibly with the $\mathbb{G}_m$-action and functorially in V. This can be verified on global sections. The required maps

$$A=\Big(\sum_p\mathrm{Fil}^pV\otimes z^{-p}\mathbb{C}[z]\Big)^{F_\infty}\otimes_{\mathbb{C}[z^2]}\mathbb{C}[z,z^{-1}]\longrightarrow V\otimes\mathbb{C}[z,z^{-1}]$$

are obtained by composition:

$$\begin{array}{rcl}A&\longrightarrow&(V\otimes\mathbb{C}[z,z^{-1}])\otimes_{\mathbb{C}[z^2]}\mathbb{C}[z,z^{-1}]\\&\longrightarrow&(V\otimes\mathbb{C}[z,z^{-1}])\otimes_{\mathbb{C}[z]}\mathbb{C}[z,z^{-1}]=V\otimes\mathbb{C}[z,z^{-1}]\,.\end{array}$$

That they are isomorphisms needs to be checked on the generators $\mathbb{C}(n)^{(-1)^n}$ of $\mathcal{F}il_\mathbb{R}$ only. As for the second assertion it suffices to show that the diagram:

$$\begin{tikzcd}(sq^*F_\mathbb{R}(V))_1=F_\mathbb{R}(V)_1=(sq^*F_\mathbb{R}(V))_{-1} \arrow[d, "\alpha_1"', "\wr"] \arrow[r, phantom, ""] & \arrow[d, "\alpha_{-1}"', "\wr"] \\ V \arrow[r, "F_\infty"] & V\end{tikzcd}$$

is commutative where the vertical arrows come from the above trivialization. They are given by setting $z=1$ on the left and $z=-1$ on the right. The value of a global section of the form

$$\frac{1}{2}(v\otimes z^p+F_\infty(v\otimes z^p))=\frac{1}{2}(v+(-1)^pF_\infty(v))\otimes z^p$$

in $sq^* F_{\mathbb{R}}(V)_{\pm 1}$ is mapped by $\alpha_{\pm 1}$ to $(-1)^p \frac{1}{2}(v + (-1)^p F_\infty(v))$. Hence the composition $\alpha_{-1} \circ (\alpha_1)^{-1}$ maps $w = \frac{1}{2}(v + (-1)^p F_\infty(v))$ to $(-1)^p w = F_\infty(w)$. Since the w's generate V we have $\alpha_{-1} \circ (\alpha_1)^{-1} = F_\infty$ as claimed.
ii) This is a special case of Proposition 6.1. □

3 A real-analytic version of the Rees sheaf

A real structure on an object V of $\mathcal{F}il_{\mathbb{C}}$ leads to a real structure in the algebraic sense on the vector bundle $E_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)$ – it is then defined over $\mathbb{A}^1_{\mathbb{R}}$. For reasons explained in section 3 we are interested however in obtaining a real structure in the *topological* sense on the Rees bundle. There does not seem to be a natural real Rees bundle over $\mathbb{C} = \mathbb{A}^1(\mathbb{C})$. However over $\mathbb{R} \subset \mathbb{C}$ a suitable topologically real bundle can be constructed, and its properties will be important in section 4. We now proceed with the details.
Let $\mathcal{F}il^{\mathrm{real}}_{\mathbb{C}}$ etc. be categories defined as before but using real instead of complex vector spaces. Let $\mathcal{A}_Y$ denote the sheaf of real valued real-analytic functions on a real C^ω-manifold or more generally orbifold Y. For V in $\mathcal{F}il^{\mathrm{real}}_{\mathbb{C}}$ we set

$$\xi^\omega_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V) = \sum_p \mathrm{Fil}^p V \otimes r^{-p} \mathcal{A}_{\mathbb{R}} \subset V \otimes j_* \mathcal{A}_{\mathbb{R}^*}$$

where r denotes the coordinate on $\mathbb{R}$ and $j : \mathbb{R}^* \hookrightarrow \mathbb{R}$ is the inclusion. This is a free $\mathcal{A}_{\mathbb{R}}$-module. With respect to the flow $\phi^t_{\mathbb{C}}(r) = re^{-t}$ on $\mathbb{R}$ it is equipped with an action

$$\psi^t : (\phi^t_{\mathbb{C}})^{-1} \xi^\omega_{\mathbb{C}} \longrightarrow \xi^\omega_{\mathbb{C}}$$

which is induced by the pullback action:

$$\psi^t = (\phi^t_{\mathbb{C}})^* : (\phi^t_{\mathbb{C}})^{-1} j_* \mathcal{A}_{\mathbb{R}^*} \longrightarrow j_* \mathcal{A}_{\mathbb{R}^*} \; .$$

Let $sq : \mathbb{R} \to \mathbb{R}^{\ge 0}$ be the squaring map $sq(r) = r^2$ and consider the action of $\mu_2 = \{\pm 1\}$ on $\mathbb{R}$ by multiplication. Let $\rho : \mathbb{R} \to \mathbb{R}/\mu_2$ be the natural projection. If we view $\mathbb{R}^{\ge 0}$ as a C^ω-orbifold via the isomorphism

$$\overline{sq} : \mathbb{R}/\mu_2 \xrightarrow{\sim} \mathbb{R}^{\ge 0} \; , \; [r] \longmapsto r^2$$

we have

$$\mathcal{A}_{\mathbb{R}^{\ge 0}} = \overline{sq}_* (\rho_* \mathcal{A}_{\mathbb{R}})^{\mu_2} = (sq_* \mathcal{A}_{\mathbb{R}})^{\mu_2} \; .$$

In the previous situation over $\mathbb{C}$ the adjunction map:

$$sq^* : \mathcal{O}_{\mathbb{C}} \longrightarrow (sq_* \mathcal{O}_{\mathbb{C}})^{\mu_2} \; , \; f \longmapsto (z \mapsto f(z^2))$$

was an isomorphism and we used it to view $\xi_{\mathbb{R}} = (sq_* \xi_{\mathbb{C}})^{F_\infty}$ as an $\mathcal{O}_{\mathbb{C}}$-module. Over $\mathbb{R}$ however the corresponding map is not an isomorphism since $sq : \mathbb{R} \to \mathbb{R}$ is not even surjective and we will have to work with $\mathcal{A}_{\mathbb{R}^{\ge 0}}$ in the following.

For V in $\mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}}$ we set

$$\xi^{\omega}_{\mathbb{R}}(V, \mathrm{Fil}^{\bullet} V, F_{\infty}) = (sq_* \xi^{\omega}_{\mathbb{C}}(V, \mathrm{Fil}^{\bullet} V))^{F_{\infty}} \subset sq_*(V \otimes j_* \mathcal{A}_{\mathbb{R}^*})$$

viewed as a free $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-module on the orbifold $\mathbb{R}^{\geq 0}$. With respect to the flow $\phi^t_{\mathbb{R}}(r') = r' e^{-2t}$ on $\mathbb{R}^{\geq 0}$ where $r' = r^2$ we have an action

$$\psi^t : (\phi^t_{\mathbb{R}})^{-1} \xi^{\omega}_{\mathbb{R}} \longrightarrow \xi^{\omega}_{\mathbb{R}}$$

induced by the action ψ^t on $\xi^{\omega}_{\mathbb{C}}$.
Let $\mathcal{D}^{\omega}_K$ be the category of locally free $\mathcal{A}_{\mathbb{R}}$- resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-modules $\mathcal{M}$ with an action

$$\psi^t : (\phi^t_K)^{-1} \mathcal{M} \longrightarrow \mathcal{M} \; .$$

Then $\mathcal{D}^{\omega}_K$ has $\otimes$-products and internal Hom's and we define the Tate twist by an integer n as $\mathcal{M}(n) = r^n \mathcal{M}$. Then $\mathcal{M}(n)$ is canonically isomorphic to $\mathcal{M}$ as a module but equipped with the twisted action:

$$\psi^t_{\mathcal{M}(n)} = e^{-tn} \psi^t_{\mathcal{M}} \; .$$

As before $\xi^{\omega}_{\mathbb{C}}$ and $\xi^{\omega}_{\mathbb{R}} : \mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}} \to \mathcal{D}^{\omega}_{\mathbb{R}}$ commute with Tate twists.
The relation with the previous algebraic construction is the following. For Y as above set $\mathcal{O}_Y = \mathcal{A}_Y \otimes_{\mathbb{R}} \mathbb{C}$. Let $i : \mathbb{R} \hookrightarrow \mathbb{C}$ denote the inclusion. Then we have $\mathcal{O}_{\mathbb{R}} = i^{-1} \mathcal{O}_{\mathbb{C}}$ and $\mathcal{O}_{\mathbb{R}^{\geq 0}} = (sq_* \mathcal{O}_{\mathbb{R}})^{\mu_2} = i^{-1}(sq_* \mathcal{O}_{\mathbb{C}})^{\mu_2}$. Moreover:

$$\xi^{\omega}_K(V) \otimes_{\mathbb{R}} \mathbb{C} = i^{-1} \xi^{\mathrm{an}}_K (V \otimes \mathbb{C}) \; . \tag{5}$$

Here $\xi^{\mathrm{an}}_K(V \otimes \mathbb{C})$ is obtained from $\xi_K(V \otimes \mathbb{C})$ by analytification. It carries a natural involution J coming from the real structures V of $V \otimes \mathbb{C}$ and $\mathbb{R}[z, z^{-1}]$ of $\mathbb{C}[z, z^{-1}]$. The involution $\mathrm{id} \otimes c$ on the left of (5), where c is complex conjugation corresponds to $i^{-1}(J)$ on the right.
These facts can be used to see that the analytic version ξ^{ω}_K over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ of ξ_K has analogous properties as the algebraic ξ_K on $\mathbb{A}^1_{\mathbb{C}}$.
An object of $\mathcal{F}il_{\mathbb{C}}$ resp. $\mathcal{F}il^{\pm}_{\mathbb{R}}$ may be viewed as an object of $\mathcal{F}il_{\mathbb{C}}^{\mathrm{real}}$ resp. $\mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}}$ by considering the underlying $\mathbb{R}$-vector space. We write this functor as $V \mapsto V_{\mathbb{R}}$. It is clear from the definitions that

$$\xi^{\omega}_K(V_{\mathbb{R}}) = i^{-1} \xi^{\mathrm{an}}_K(V) \tag{6}$$

as $\mathcal{A}_{\mathbb{R}}$- resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-modules.
Looking at associated C^{ω}-vector bundles we get:

Corollary 3.1 *To every V in $\mathcal{F}il_{\mathbb{C}}^{\mathrm{real}}$ resp. $\mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}}$ there is functorially attached a real C^{ω}-bundle E^{ω} over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ together with a C^{ω}-action*

$$\psi^t : \phi^{t*}_K E^{\omega} \longrightarrow E^{\omega} \; .$$

The rank of E^{ω} equals the dimension of V and there are functorial isomorphisms:

$$E^{\omega}(V, \mathrm{Fil}^{\bullet} V)_0 \xrightarrow{\sim} \mathrm{Gr}^{\bullet} V \quad \textit{resp. } E^{\omega}(V, \mathrm{Fil}^{\bullet} V, F_{\infty})_0 \xrightarrow{\sim} \mathrm{Gr}^{\bullet}(sV)$$

such that ψ^t_0 corresponds to $e^{\bullet t}$.

4 The relation of Rees sheaves and Rees bundles with archimedian invariants of motives

In this section we first recall briefly the definition of the spaces $\mathcal{F}_{\mathfrak{p}}(M)$ for archimedian primes using Rees sheaves. We then describe the local contribution from $\mathfrak{p}$ in the motivic "explicit formulas" of analytic number theory in terms of a suitable Rees bundle. This formula is new. We also explain the motivation for considering C^{ω}-Rees bundles over $\mathbb{R}$ or $\mathbb{R}^{\geq 0}$ in the preceeding section.
Consider the category of (mixed) motives $\mathcal{M}_k$ over a number field k, for example in the sense of Deligne [De2] or Jannsen [J].
For an infinite place $\mathfrak{p}$ let $M_{\mathfrak{p}}$ be the real Hodge structure of $M \otimes_k k_{\mathfrak{p}}$. In case $\mathfrak{p}$ is real $M_{\mathfrak{p}}$ carries the action of an $\mathbb{R}$-linear involution $F_{\mathfrak{p}}$ which maps the Hodge filtration $F^{\bullet} M_{\mathfrak{p},\mathbb{C}}$ on $M_{\mathfrak{p},\mathbb{C}} = M_{\mathfrak{p}} \otimes_{\mathbb{R}} \mathbb{C}$ to $\overline{F}^{\bullet} M_{\mathfrak{p},\mathbb{C}}$. Consider the descending filtration

$$\gamma^{\nu} M_{\mathfrak{p}} = M_{\mathfrak{p}} \cap F^{\nu} M_{\mathfrak{p},\mathbb{C}} = M_{\mathfrak{p}} \cap F^{\nu} M_{\mathfrak{p},\mathbb{C}} \cap \overline{F}^{\nu} M_{\mathfrak{p},\mathbb{C}}$$

on $M_{\mathfrak{p}}$ and set

$$n_{\nu}(M_{\mathfrak{p}}) = \dim \mathrm{Gr}^{\nu}_{\gamma} M_{\mathfrak{p}} \; .$$

For real $\mathfrak{p}$ write

$$n^{\pm}_{\nu}(M_{\mathfrak{p}}) = \dim (\mathrm{Gr}^{\nu}_{\gamma} M_{\mathfrak{p}})^{\pm}$$

where $\pm$ denotes the ± 1 eigenspace of $F_{\mathfrak{p}}$.
Set $\mathcal{V}^{\nu} M_{\mathfrak{p}} = \gamma^{\nu} M_{\mathfrak{p}}$ if $\mathfrak{p}$ is complex and

$$\mathcal{V}^{\nu} M_{\mathfrak{p}} = (F^{\nu} M_{\mathfrak{p},\mathbb{C}} \cap M_{\mathfrak{p}})^{(-1)^{\nu}} \oplus (F^{\nu+1} M_{\mathfrak{p},\mathbb{C}} \cap M_{\mathfrak{p}})^{(-1)^{\nu+1}}$$

if $\mathfrak{p}$ is real. In other words:

$$(M_{\mathfrak{p}}, \mathcal{V}^{\bullet} M_{\mathfrak{p}}, F_{\infty}) = s(M_{\mathfrak{p}}, \gamma^{\bullet} M_{\mathfrak{p}}, F_{\infty}) \; .$$

In the real case there is an exact sequence

$$0 \longrightarrow (\mathrm{Gr}^{\nu+1}_{\gamma} M_{\mathfrak{p}})^{(-1)^{\nu}} \longrightarrow \mathrm{Gr}^{\nu}_{\mathcal{V}} M_{\mathfrak{p}} \longrightarrow (\mathrm{Gr}^{\nu}_{\gamma} M_{\mathfrak{p}})^{(-1)^{\nu}} \longrightarrow 0 \; .$$

We set $d_{\nu}(M_{\mathfrak{p}}) = \dim \mathrm{Gr}^{\nu}_{\mathcal{V}} M_{\mathfrak{p}}$ and $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$.
In [F-PR] the local Euler factors of M for the infinite places were defined as follows:

$$L_{\mathfrak{p}}(M,s) = \prod_{\nu} \Gamma_{\mathbb{C}}(s-\nu)^{n_{\nu}(M_{\mathfrak{p}})} \quad \text{if } \mathfrak{p} \text{ is complex}$$

and

$$L_{\mathfrak{p}}(M,s) = \prod_{\nu} \Gamma_{\mathbb{R}}(s+\varepsilon_{\nu}-\nu)^{n^{+}_{\nu}(M_{\mathfrak{p}})} \Gamma_{\mathbb{R}}(s+1-\varepsilon_{\nu}-\nu)^{n^{-}_{\nu}(M_{\mathfrak{p}})}$$

if $\mathfrak{p}$ is real. Here $\varepsilon_\nu \in \{0,1\}$ is determined by $\varepsilon_\nu \equiv \nu \bmod 2$.
Using the above exact sequence we get an alternative formula for real $\mathfrak{p}$:

$$L_{\mathfrak{p}}(M,s) = \prod_\nu \Gamma_{\mathbb{R}}(s-\nu)^{d_\nu(M_{\mathfrak{p}})} .$$

See also [D4] for background. It follows from remark 2.1 that for $\mathfrak{p} \,|\, \infty$ the space $\mathcal{F}_{\mathfrak{p}}(M)$ of [D3] § 5 is given as follows:

$$\mathcal{F}_{\mathfrak{p}}(M) = \Gamma(\mathbb{A}^1, \xi_{\mathbb{C}}(M_{\mathfrak{p}\mathbb{C}}, \gamma^\bullet M_{\mathfrak{p}\mathbb{C}})) \quad \text{if } \mathfrak{p} \text{ is complex}$$

and

$$\begin{aligned} \mathcal{F}_{\mathfrak{p}}(M) &= \Gamma(\mathbb{A}^1, \xi_{\mathbb{R}}(M_{\mathfrak{p}\mathbb{C}}, \gamma^\bullet M_{\mathfrak{p}\mathbb{C}}, F_\infty)) \\ &= \Gamma(\mathbb{A}^1, \xi_{\mathbb{R}}(M_{\mathfrak{p}\mathbb{C}}, \mathcal{V}^\bullet M_{\mathfrak{p}\mathbb{C}}, F_\infty)) \quad \text{if } \mathfrak{p} \text{ is real.} \end{aligned}$$

According to [D3] Cor. 6.5 we have:

$$L_{\mathfrak{p}}(M,s) = \det\nolimits_\infty \Big(\frac{1}{2\pi}(s \cdot \mathrm{id} - \Theta) \,|\, \mathcal{F}_{\mathfrak{p}}(M) \Big)^{-1} .$$

Here Θ is the infinitesimal generator of the $\mathbb{G}_m$-action of $\mathcal{F}_{\mathfrak{p}}(M)$, i.e. the induced action by $1 \in \mathbb{C} = \mathrm{Lie}\, \mathbb{G}_m$.
We define the real analytic version of $\mathcal{F}_{\mathfrak{p}}(M)$ as follows:

$$\mathcal{F}^\omega_{\mathfrak{p}}(M) = \Gamma(\mathbb{R}, \xi^\omega_{\mathbb{C}}(M_{\mathfrak{p}}, \gamma^\bullet M_{\mathfrak{p}})) \quad \text{if } \mathfrak{p} \text{ is complex}$$

and

$$\begin{aligned} \mathcal{F}^\omega_{\mathfrak{p}}(M) &= \Gamma(\mathbb{R}^{\ge 0}, \xi^\omega_{\mathbb{R}}(M_{\mathfrak{p}}, \gamma^\bullet M_{\mathfrak{p}}, F_\infty)) \\ &= \Gamma(\mathbb{R}^{\ge 0}, \xi^\omega_{\mathbb{R}}(M_{\mathfrak{p}}, \mathcal{V}^\bullet M_{\mathfrak{p}}, F_\infty)) \quad \text{if } \mathfrak{p} \text{ is real.} \end{aligned}$$

It follows from the above formula for $L_{\mathfrak{p}}(M,s)$ in terms of $\mathcal{F}_{\mathfrak{p}}(M)$ and the relation between ξ^ω_K and ξ_K that we have for all $\mathfrak{p} \,|\, \infty$:

$$L_{\mathfrak{p}}(M,s) = \det\nolimits_\infty \Big(\frac{1}{2\pi}(s \cdot \mathrm{id} - \Theta) \,|\, \mathcal{F}^\omega_{\mathfrak{p}}(M) \Big)^{-1} . \tag{7}$$

Here Θ denotes the infinitesimal generator of the flow ψ^{t*} induced on $\mathcal{F}^\omega_{\mathfrak{p}}(M)$ by the actions ψ^t and $\phi^t_{k_{\mathfrak{p}}}$ which were defined in section 2.
In the next section we will express $\mathcal{F}^\omega_{\mathfrak{p}}(H^n(X))$ for smooth projective varieties X/k in "dynamical" terms. Via formula (7) we then get formulas for the archimedian L-factors $L_{\mathfrak{p}}(H^n(X),s)$ which come from the geometry of a simple dynamical system.
Let us now turn to the motivic "explicit formulas" of analytic number theory. To every motive M in $\mathcal{M}_k$ one can attach local Euler factors $L_{\mathfrak{p}}(M,s)$ for all the places $\mathfrak{p}$ in k and global L-functions:

$$L(M,s) = \prod_{\mathfrak{p} \nmid \infty} L_{\mathfrak{p}}(M,s) \quad \text{and} \quad \hat{L}(M,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M,s) ,$$

c.f. [F-PR], [D4]. Assuming standard conjectures about the analytical behaviour of $L(M,s)$ and $L(M^*,s)$ proved in many interesting cases the following explicit formula in the analytic number theory of motives holds for every φ in $\mathcal{D}(\mathbb{R}^+) = C_0^\infty(\mathbb{R}^+)$ c.f. [D-Sch] (2.2.1):

$$-\sum_\rho \Phi(\rho)\mathrm{ord}_{s=\rho}\hat{L}(M,s) = \sum_{\mathfrak{p}} W_{\mathfrak{p}}(\varphi)\;. \tag{8}$$

Here $\Phi(s) = \int_{\mathbb{R}} \varphi(t)e^{ts}\,dt$ and $\mathfrak{p}$ runs over all places of k.
For finite $\mathfrak{p}$ we have

$$W_{\mathfrak{p}}(\varphi) = \log N\mathfrak{p}\sum_{k=1}^\infty \mathrm{Tr}(\mathrm{Fr}_{\mathfrak{p}}^k \mid M_l^{I_{\mathfrak{p}}})\varphi(k\log N\mathfrak{p}) \tag{9}$$

where $\mathrm{Fr}_{\mathfrak{p}}$ denotes a geometric Frobenius at $\mathfrak{p}$ and $M_l^{I_{\mathfrak{p}}}$ is the fixed module under inertia of the l-adic realization of M with $\mathfrak{p}\nmid l$.

The terms $W_{\mathfrak{p}}$ for the infinite places are given as follows: For complex $\mathfrak{p}$ we have:

$$W_{\mathfrak{p}}(\varphi) = \sum_\nu n_\nu(M_{\mathfrak{p}})\int_0^\infty \varphi(t)\frac{e^{\nu t}}{1-e^{-t}}\,dt \tag{10}$$

whereas for real $\mathfrak{p}$:

$$W_{\mathfrak{p}}(\varphi) = \sum_\nu d_\nu(M_{\mathfrak{p}})\int_0^\infty \varphi(t)\frac{e^{\nu t}}{1-e^{-2t}}\,dt\;. \tag{11}$$

The distributions $W_{\mathfrak{p}}$ for $\mathfrak{p}\mid\infty$ can be rewritten as follows:

$$W_{\mathfrak{p}} = \frac{\mathrm{Tr}(e^{\bullet t}\mid \mathrm{Gr}^\bullet_{\mathcal{V}} M_{\mathfrak{p}})}{1-e^{-\kappa_{\mathfrak{p}} t}} \tag{12}$$

where $e^{\bullet t}$ is the map $e^{\nu t}$ on Gr^ν and $\kappa_{\mathfrak{p}} = 2$ resp. 1 according to whether $\mathfrak{p}$ is real or complex.
In terms of our conjectural cohomology theory c.f. [D3] § 7, equation (8) can thus be reformulated as an equality of distributions on $\mathbb{R}^+$:

$$\begin{aligned}&\sum_i (-1)^i \mathrm{Tr}(\psi^*\mid H^i(\text{“}\overline{\mathrm{spec}\,\mathfrak{o}_k}\text{”},\mathcal{F}(M)))_{\mathrm{dis}} \\ &= \sum_{\mathfrak{p}\nmid\infty}\log N\mathfrak{p}\sum_{k=1}^\infty \mathrm{Tr}(\mathrm{Fr}_{\mathfrak{p}}^k\mid M_l^{I_{\mathfrak{p}}})\delta_{k\log N\mathfrak{p}} + \sum_{\mathfrak{p}\mid\infty}\frac{\mathrm{Tr}(e^{\bullet t}\mid\mathrm{Gr}^\bullet_{\mathcal{V}} M_{\mathfrak{p}})}{1-e^{-\kappa_{\mathfrak{p}} t}}\;.\end{aligned} \tag{13}$$

Compare [D-Sch] (3.1.1) for the elementary notion of distributional trace used on cohomology here. In the rest of this section we will be concerned with a deeper understanding of the function $\mathrm{Tr}(e^{\bullet t}\mid \mathrm{Gr}^\bullet_{\mathcal{V}} M_{\mathfrak{p}})$.

Certain dynamical trace formulas for vector bundles E over a manifold X with a flow ϕ^t and an action $\psi^t : \phi^{t*}E \to E$ involve local contributions at the fixed points x of the form:

$$\frac{\mathrm{Tr}(\psi_x^t \mid E_x)}{1 - e^{-\kappa_x t}} \qquad \text{some } \kappa_x > 0 \; .$$

This is explained in [D7] § 4. These formulas bear a striking resemblance to the "explicit formulas" and they suggest that infinite places correspond to fixed points of a flow. Incidentially the finite places would correspond to the periodic orbits. This analogy suggests that for the infinite places it should be possible to attach to M a real vector bundle E in the topological sense over a dynamical system with fixed points. If 0 denotes the fixed point corresponding to $\mathfrak{p}$, we should have:

$$\mathrm{Tr}(\psi_0^t \mid E_0) = \mathrm{Tr}(e^{\bullet t} \mid \mathrm{Gr}^\bullet_{\mathcal{V}} M_{\mathfrak{p}}) \; .$$

At least over one flowline this is achieved by Corollary 3.1 as follows.
Define as follows a real C^ω-bundle $E^\omega_{\mathfrak{p}}(M)$ over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ together with a C^ω-action

$$\psi^t : \phi_K^{t*} E^\omega_{\mathfrak{p}}(M) \longrightarrow E^\omega_{\mathfrak{p}}(M) \; .$$

Set

$$E^\omega_{\mathfrak{p}}(M) = E^\omega(M_{\mathfrak{p}}, \gamma^\bullet M_{\mathfrak{p}}) \quad \text{if } \mathfrak{p} \text{ is complex}$$

and

$$\begin{aligned} E^\omega_{\mathfrak{p}}(M) &= E^\omega(M_{\mathfrak{p}}, \gamma^\bullet M_{\mathfrak{p}}, F_\infty) \\ &= E^\omega(M_{\mathfrak{p}}, \mathcal{V}^\bullet M_{\mathfrak{p}}, F_\infty) \quad \text{if } \mathfrak{p} \text{ is real.} \end{aligned}$$

Note that this is just the C^ω-bundle corresponding to the locally free sheaf $\mathcal{F}^\omega_{\mathfrak{p}}(M)$ defined earlier. According to Corollary 3.1 we then have:

Proposition 4.1 *There are functorial isomorphisms*

$$E^\omega_{\mathfrak{p}}(M)_0 \xrightarrow{\sim} \mathrm{Gr}^\bullet_{\mathcal{V}} M_{\mathfrak{p}}$$

for all $\mathfrak{p} \mid \infty$ such that ψ_0^t corresponds to $e^{\bullet t}$. In particular we find:

$$\begin{aligned} \mathrm{Tr}(\psi_0^t \mid E^\omega_{\mathfrak{p}}(M)_0) &= \mathrm{Tr}(e^{\bullet t} \mid \mathrm{Gr}^\bullet_{\mathcal{V}} M_{\mathfrak{p}}) \\ &= (1 - e^{-\kappa_{\mathfrak{p}} t}) W_{\mathfrak{p}} \; . \end{aligned}$$

5 A geometrical construction of $\mathcal{F}^\omega_{\mathfrak{p}}(M)$ and $E^\omega_{\mathfrak{p}}(M)$ for $M = H^n(X)$

In this section we express the locally free sheaf $\mathcal{F}^\omega_{\mathfrak{p}}(H^n(X))$ over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ of section 3 in terms of higher direct image sheaves modulo torsion. The construction is based on the following result of Simpson [Si] Prop. 5.1, 5.2. For a variety $X/\mathbb{C}$ we write X^{an} for the associated complex space.

Theorem 5.1 (Simpson) *Let $X/\mathbb{C}$ be a smooth proper variety and let $F^\bullet$ be the Hodge filtration on $H^n(X^{\mathrm{an}}, \mathbb{C})$. Then we have:*

$$\xi_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{C}), F^\bullet) = R^n\pi_*(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$$

where $\pi : X \times \mathbb{A}^1 \to \mathbb{A}^1$ denotes the projection.

Remarks (1) The $\mathbb{G}_m$-action on the deformed complex $(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$ given by sending a homogenous form ω to $\lambda^{-\deg\omega} \cdot \lambda^*(\omega)$ for $\lambda \in \mathbb{G}_m$ induces a $\mathbb{G}_m$-action on $R^n\pi_*(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$. Under the isomorphism of the theorem it corresponds to the $\mathbb{G}_m$-action on $\xi_{\mathbb{C}}(H^n(X, \mathbb{C}), F^\bullet)$ defined by formula (4).
(2) In the appendix to this section we relate the complex $(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$ to the complex of relative differential forms on a suitable deformation of $X \times \mathbb{A}^1$.

In the situation of the theorem consider the natural morphism from the spectral sequence:

$$E_1^{pq} = (R^q\pi_*(\Omega^p_{X\times\mathbb{A}^1/\mathbb{A}^1}))^{\mathrm{an}} \Longrightarrow (R^n\pi_*(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd))^{\mathrm{an}}$$

to the spectral sequence

$$E_1^{pq} = R^q\pi_*(\Omega^p_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}}) \Longrightarrow R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}}, zd) .$$

By *GAGA* it is an isomorphism on the E_1-terms and hence on the end terms as well. Thus we get a natural isomorphism of locally free $\mathcal{O}_{\mathbb{C}}$-modules:

$$\xi^{\mathrm{an}}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{C}), F^\bullet) = R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}}, zd) . \tag{14}$$

Let $\mathrm{id} \times i : X^{\mathrm{an}} \times \mathbb{R} \hookrightarrow X^{\mathrm{an}} \times \mathbb{C}$ be the inclusion and set

$$\Omega^p_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}} = (\mathrm{id} \times i)^{-1}\Omega^p_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}} .$$

It is the subsheaf of $\mathbb{C}$-valued smooth relative differential forms on $X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}$ which are holomorphic in the X^{an}-coordinates and real analytic in the $\mathbb{R}$-variable. We then have an equality of complexes

$$(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) = (\mathrm{id} \times i)^{-1}(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}}, zd) .$$

We define an action ψ^t of $\mathbb{R}$ on $(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd)$ by sending a homogenous form ω to $e^{t\deg\omega} \cdot (\mathrm{id} \times \phi^t_{\mathbb{C}})^*\omega$:

$$\psi^t : (\mathrm{id} \times \phi^t_{\mathbb{C}})^{-1}(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \longrightarrow (\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) .$$

This induces an action:

$$\psi^t : (\phi^t_{\mathbb{C}})^{-1}R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \longrightarrow R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd)$$

and hence an $\mathcal{A}_{\mathbb{R}}$-linear action

$$\psi^t : \phi_{\mathbb{C}}^{t*} R^n \pi_* (\Omega^\bullet_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd) \longrightarrow R^n \pi_* (\Omega^\bullet_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd) \ .$$

By proper base change we obtain from (14) that

$$i^{-1} \xi_{\mathbb{C}}^{\mathrm{an}} (H^n (X^{\mathrm{an}}, \mathbb{C}), F^\bullet) = R^n \pi_* (\Omega^\bullet_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd) \ . \tag{15}$$

According to (6) this gives an isomorphism

$$\xi_{\mathbb{C}}^{\omega} ((H^n (X^{\mathrm{an}}, \mathbb{C}), F^\bullet)_{\mathbb{R}}) = R^n \pi_* (\Omega^\bullet_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd) \tag{16}$$

of locally free $\mathcal{A}_{\mathbb{R}}$-modules which is compatible with the action ψ^t relative to $\phi_{\mathbb{C}}^t$.
Let $\mathcal{DR}_{X/\mathbb{C}}$ be the cokernel of the natural inclusion of complexes of $\pi^{-1}\mathcal{A}_{\mathbb{R}}$-modules on $X^{\mathrm{an}} \times \mathbb{R}$ with action ψ^t

$$\pi^{-1} \mathcal{A}_{\mathbb{R}} \longrightarrow (\Omega^\bullet_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd) \ .$$

Here $\pi^{-1}\mathcal{A}_{\mathbb{R}}$ is viewed as a complex concentrated in degree zero and on it ψ^t acts by pullback via $\mathrm{id} \times \phi_{\mathbb{C}}^t$. The projection formula gives us

$$R^n \pi_* (\pi^{-1} \mathcal{A}_{\mathbb{R}}) = H^n (X^{\mathrm{an}}, \mathbb{R}) \otimes \mathcal{A}_{\mathbb{R}} = \xi_{\mathbb{C}}^{\omega} (H^n (X^{\mathrm{an}}, \mathbb{R}), \mathrm{Fil}_0^\bullet) \tag{17}$$

where

$$\mathrm{Fil}_0^p H^n (X^{\mathrm{an}}, \mathbb{R}) = H^n (X^{\mathrm{an}}, \mathbb{R})$$

for $p \leq 0$ and $\mathrm{Fil}_0^p = 0$ for $p > 0$. We thus get a long exact ψ^t-equivariant sequence of coherent $\mathcal{A}_{\mathbb{R}}$-modules:

$$\begin{array}{cccccc} \ldots & \to & \xi_{\mathbb{C}}^{\omega} (H^n (X^{\mathrm{an}}, \mathbb{R}), \mathrm{Fil}_0^\bullet) & \to & \xi_{\mathbb{C}}^{\omega} ((H^n (X^{\mathrm{an}}, \mathbb{C}), F^\bullet)_{\mathbb{R}}) & \to \\ & \to & R^n \pi_* \mathcal{DR}_{X/\mathbb{C}} & \to & \xi_{\mathbb{C}}^{\omega} (H^{n+1} (X^{\mathrm{an}}, \mathbb{R}), \mathrm{Fil}_0^\bullet) & \to \quad \ldots \end{array} \tag{18}$$

For any n the natural map

$$\xi_{\mathbb{C}}^{\omega} (H^n (X^{\mathrm{an}}, \mathbb{R}), \mathrm{Fil}_0^\bullet) \longrightarrow \xi_{\mathbb{C}}^{\omega} ((H^n (X^{\mathrm{an}}, \mathbb{C}), F^\bullet)_{\mathbb{R}})$$

is injective by the $\xi_{\mathbb{C}}^{\omega}$-analogue of Prop. 2.2 c), part i) since it is induced by the inclusion of objects in $\mathcal{F}il_{\mathbb{C}}^{\mathrm{real}}$:

$$(H^n (X^{\mathrm{an}}, \mathbb{R}), \mathrm{Fil}_0^\bullet) \hookrightarrow (H^n (X^{\mathrm{an}}, \mathbb{C}), F^\bullet)_{\mathbb{R}} \ . \tag{19}$$

The injectivity can also be seen by noting that the fibres of the associated C^ω-vector bundles for $r \in \mathbb{R}^*$ are naturally isomorphic to $H^n(X^{\mathrm{an}}, \mathbb{R})$ resp. $H^n(X^{\mathrm{an}}, \mathbb{C})$, the map being the inclusion c.f. the $\xi_{\mathbb{C}}^{\omega}$-analogue of Proposition 2.3 i).

Therefore the long exact sequence (18) splits into the short exact sequences:

$$\begin{array}{rcl} 0 \to \xi^{\omega}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}), \mathrm{Fil}^{\bullet}_0) & \to & \xi^{\omega}_{\mathbb{C}}((H^n(X^{\mathrm{an}}, \mathbb{C}), F^{\bullet})_{\mathbb{R}}) \\ & \stackrel{\alpha}{\to} & R^n \pi_* \mathcal{DR}_{X/\mathbb{C}} \to 0 \; . \end{array} \tag{20}$$

Using the $\xi^{\omega}_{\mathbb{C}}$-version of Proposition 3.1 c) ii) we therefore get a ψ^t-equivariant isomorphism of $\mathcal{A}_{\mathbb{R}}$-modules:

$$R^n \pi_*(\mathcal{DR}_{X/\mathbb{C}})/\mathcal{A}_{\mathbb{R}}\text{-torsion} \xrightarrow{\sim} \xi^{\omega}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet})) \; . \tag{21}$$

Here we have used the exact sequence:

$$0 \longrightarrow H^n(X^{\mathrm{an}}, \mathbb{R}) \longrightarrow H^n(X^{\mathrm{an}}, \mathbb{C}) \xrightarrow{\pi_1} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \; ,$$

where $\pi_1(f) = \frac{1}{2}(f - \overline{f})$.
Let us now indicate the necessary amendments for the case $K = \mathbb{R}$. We consider a smooth and proper variety $X/\mathbb{R}$. Its associated complex manifold X^{an} is equipped with an antiholomorphic involution F_{∞}, which in turn gives rise to an involution $\overline{F}^*_{\infty}$ of $H^n(X^{\mathrm{an}}, \mathbb{R}(1))$ which maps the filtration $\pi_1(F^{\bullet})$ to itself. By definition of $\xi^{\omega}_{\mathbb{R}}$ we have

$$\xi^{\omega}_{\mathbb{R}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet}), \overline{F}^*_{\infty}) = (sq_* \xi^{\omega}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet})))^{F_{\infty}} \tag{22}$$

where $F_{\infty} \mathrel{\widehat{=}} \overline{F}^*_{\infty} \otimes (-\mathrm{id})^*$.
To deal with the other side of (21) consider the μ_2-action on $X^{\mathrm{an}} \times \mathbb{R}$ by $F_{\infty} \times (-\mathrm{id})$ and let

$$\lambda : X^{\mathrm{an}} \times \mathbb{R} \longrightarrow X^{\mathrm{an}} \times_{\mu_2} \mathbb{R} = (X^{\mathrm{an}} \times \mathbb{R})/\mu_2$$

be the canonical projection.
The map

$$\lambda_*(\pi^{-1}\mathcal{A}_{\mathbb{R}}) \longrightarrow \lambda_*(\Omega^{\bullet}_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd)$$

becomes μ_2-equivariant if $-1 \in \mu_2$ acts by $(F_{\infty} \times (-\mathrm{id}))^*$ on the left and by sending a homogenous form ω to $(-1)^{\deg \omega}(F_{\infty} \times (-\mathrm{id}))^* \overline{\omega}$ on the right. We set

$$\Omega^{\bullet}_{X^{\mathrm{an}} \times_{\mu_2} \mathbb{R}/(\mathbb{R}/\mu_2)} = \Big(\lambda_*(\Omega^{\bullet}_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd)\Big)^{\mu_2}$$

and

$$\mathcal{DR}_{X/\mathbb{R}} = (\lambda_* \mathcal{DR}_{X/\mathbb{C}})^{\mu_2} \; .$$

Let π be the composed map

$$\pi : X^{\mathrm{an}} \times_{\mu_2} \mathbb{R} \longrightarrow \mathbb{R}/\mu_2 \xrightarrow[\sim]{\overline{sq}} \mathbb{R}^{\geq 0} \; .$$

Combining the isomorphisms (21) and (22) we obtain an isomorphism of free $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-modules on $\mathbb{R}^{\geq 0}$:

$$R^n\pi_*(\mathcal{DR}_{X/\mathbb{R}})/\mathcal{A}_{\mathbb{R}^{\geq 0}}\text{-torsion} \xrightarrow{\sim} \xi^\omega_{\mathbb{R}}(H^n(X^{\mathrm{an}},\mathbb{R}(1)),\pi_1(F^\bullet),\overline{F}^*_\infty) \ . \tag{23}$$

The left hand side carries a natural action ψ^t with respect to the flow $\phi^t_{\mathbb{R}}$ on $\mathbb{R}^{\geq 0}$ and the isomorphism (23) is ψ^t-equivariant.
As before we have a short exact sequence:

$$\begin{aligned} 0 \to \xi^\omega_{\mathbb{R}}(H^n(X^{\mathrm{an}},\mathbb{R}),\mathrm{Fil}^\bullet_0,F^*_\infty) &\to \xi^\omega_{\mathbb{R}}((H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet,\overline{F}^*_\infty)_{\mathbb{R}}) \\ &\overset{\alpha}{\to} R^n\pi_*\mathcal{DR}_{X/\mathbb{R}} \to 0 \ . \end{aligned} \tag{24}$$

The first main result of this section is the following:

Theorem 5.2 *Fix a smooth and proper variety X/K of dimension d where $K=\mathbb{C}$ or $\mathbb{R}$. Assume that $n+m=2d$. Then we have natural isomorphisms:*

1) $\xi_{\mathbb{C}}(H^m(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet) = (2\pi i)^{1-d}\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(R^n\pi_*\mathcal{DR}_{X/\mathbb{C}},\mathcal{A}_{\mathbb{R}}(-d))$
in case $K=\mathbb{C}$ and

2) $\xi_{\mathbb{R}}(H^m(X^{\mathrm{an}},\mathbb{R}),\mathcal{V}^\bullet,F_\infty)$
$= (2\pi i)^{1-d}\underline{\mathrm{Hom}}_{\mathcal{A}^{\geq 0}_{\mathbb{R}}}(R^n\pi_*\mathcal{DR}_{X/\mathbb{R}},\mathcal{A}_{\mathbb{R}^{\geq 0}}(1-d))$
if $K=\mathbb{R}$.

These isomorphisms respect the $\mathcal{A}_{\mathbb{R}}$-resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-module structure and the flow ψ^t.

Proof Consider the perfect pairing of $\mathbb{R}$-Hodge structures:

$$\langle,\rangle : H^n(X^{\mathrm{an}}) \times H^m(X^{\mathrm{an}}) \xrightarrow{\cup} H^{2d}(X^{\mathrm{an}}) \overset{\mathrm{tr}}{\xrightarrow{\sim}} \mathbb{R}(-d) \tag{25}$$

given by $\cup$-product followed by the trace isomorphism

$$\mathrm{tr}(c) = \frac{1}{(2\pi i)^d}\int_{X^{\mathrm{an}}} c \ .$$

It says in particular that

$$F^iH^n(X^{\mathrm{an}},\mathbb{C})^\perp = F^{d+1-i}H^m(X^{\mathrm{an}},\mathbb{C}) \ . \tag{26}$$

Moreover it leads to a perfect pairing of $\mathbb{R}$-vector spaces:

$$\langle,\rangle : H^n(X^{\mathrm{an}},\mathbb{R}(1)) \times H^m(X^{\mathrm{an}},\mathbb{R}(d-1)) \longrightarrow \mathbb{R} \ . \tag{27}$$

Now according to the ω-version of Proposition 2.2 a) we have:

$$\begin{aligned} &\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}\left(\xi^\omega_{\mathbb{C}}(H^n(X^{\mathrm{an}},\mathbb{R}(1)),\pi_1(F^\bullet)),\mathcal{A}_{\mathbb{R}}\right) \\ &\quad = \xi^\omega_{\mathbb{C}}\left(H^n(X^{\mathrm{an}},\mathbb{R}(1))^*,\pi_1(F^{1-\bullet})^\perp\right) \\ &\quad \overset{(27)}{=} \xi^\omega_{\mathbb{C}}\left(H^m(X^{\mathrm{an}},\mathbb{R}(d-1)),\mathrm{Fil}^\bullet\right) \end{aligned}$$

where Fil^p consists of those elements $u \in H^m(X^{\mathrm{an}}, \mathbb{R}(d-1))$ with

$$\langle \pi_1(F^{1-p}), u\rangle = \pi_d \langle F^{1-p}, u\rangle = 0$$

i.e. with

$$\langle F^{1-p}, u\rangle = 0 \; .$$

Using (26) we find:

$$\begin{array}{rcl} \mathrm{Fil}^p & = & H^m(X^{\mathrm{an}}, \mathbb{R}(d-1)) \cap F^{p+d} H^m(X^{\mathrm{an}}, \mathbb{C}) \\ & = & (2\pi i)^{d-1} \gamma^{p+d} H^m(X^{\mathrm{an}}, \mathbb{R}) \end{array}$$

and therefore:

$$\begin{array}{rl} & \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(\xi^{\omega}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet})), \mathcal{A}_{\mathbb{R}}) \\ = & (2\pi i)^{d-1} \xi^{\omega}_{\mathbb{C}}((H^m(X^{\mathrm{an}}, \mathbb{R}), \gamma^{\bullet})(d)) \\ = & (2\pi i)^{d-1} \xi^{\omega}_{\mathbb{C}}(H^m(X^{\mathrm{an}}, \mathbb{R}), \gamma^{\bullet})(d) \; . \end{array}$$

Combining this with the isomorphism (21) we get the first assertion. As for the second note that by Proposition 2.2 a) we have:

$$\begin{array}{rl} & \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}\ge 0}}(\xi^{\omega}_{\mathbb{R}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet}), \overline{F}^*_{\infty}), \mathcal{A}_{\mathbb{R}\ge 0}) \\ = & \xi^{\omega}_{\mathbb{R}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1))^*, \pi_1(F^{2-\bullet})^{\perp} \;, \text{ dual of } \overline{F}^*_{\infty}) \; . \end{array}$$

Since for $X/\mathbb{R}$ the pairing (25) is $\overline{F}^*_{\infty}$-equivariant this equals

$$\xi^{\omega}_{\mathbb{R}}(H^m(X^{\mathrm{an}}, \mathbb{R}(d-1)), \mathrm{Fil}^{\bullet}, \overline{F}^*_{\infty})$$

where Fil^p consists of those elements u with:

$$\langle \pi_1(F^{2-p}), u\rangle = 0 \; .$$

Thus

$$\mathrm{Fil}^p = (2\pi i)^{d-1} \gamma^{p+d-1} H^m(X^{\mathrm{an}}, \mathbb{R})$$

in $\mathcal{F}il^{\pm \mathrm{real}}_{\mathbb{R}}$. Hence:

$$\begin{array}{rl} & \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}\ge 0}}(\xi^{\omega}_{\mathbb{R}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet}), \overline{F}^*_{\infty}), \mathcal{A}_{\mathbb{R}\ge 0}) \\ = & (2\pi i)^{d-1} \xi^{\omega}_{\mathbb{R}}((H^m(X^{\mathrm{an}}, \mathbb{R}), \gamma^{\bullet}, F^*_{\infty})(d-1)) \\ = & (2\pi i)^{d-1} \xi^{\omega}_{\mathbb{R}}(H^m(X^{\mathrm{an}}, \mathbb{R}), \gamma^{\bullet}, F^*_{\infty})(d-1) \; . \end{array}$$

Since we can replace $\gamma^{\bullet}$ by $\mathcal{V}^{\bullet} = s\gamma^{\bullet}$ in the last expression the second formula of the theorem now follows by invoking the isomorphism (23). □

If X/K is projective, fixing a polarization defined over K the hard Lefschetz theorem together with Poincaré duality provides an isomorphism of $\mathbb{R}$-Hodge structures over K:

$$H^n(X^{\mathrm{an}})^* = H^n(X^{\mathrm{an}})(1) \ . \tag{28}$$

Similar arguments as before based on (28) instead of (25) then give the following result:

Corollary 5.3 *Fix a smooth projective variety X/K together with the class of a hyperplane section over K. There are canonical isomorphisms:*
1) $\xi_{\mathbb{C}}(H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet) = \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(R^n\pi_*\mathcal{DR}_{X/\mathbb{C}},\mathcal{A}_{\mathbb{R}}(-1))$
in case $K=\mathbb{C}$ and
2) $\xi_{\mathbb{R}}(H^n(X^{\mathrm{an}},\mathbb{R}),\mathcal{V}^\bullet,F_\infty^*) = \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}\geq 0}}(R^n\pi_*\mathcal{DR}_{X/\mathbb{R}},\mathcal{A}_{\mathbb{R}^{\geq 0}})$
if $K=\mathbb{R}$. These isomorphisms respect the $\mathcal{A}_{\mathbb{R}}$-resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-module structure and the action of the flow.

A consideration of the sequence

$$\begin{array}{rcl} 0 \longrightarrow (H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet) & \longrightarrow & (H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet)_{\mathbb{R}} \\ & \xrightarrow{\pi_1} & (H^n(X^{\mathrm{an}},\mathbb{R}(1)),\pi_1(F^\bullet)) \longrightarrow 0 \end{array}$$

in $\mathcal{F}il_{\mathbb{C}}^{\mathrm{real}}$ and of

$$\begin{array}{rcl} 0 \longrightarrow (H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet,F_\infty^*) & \longrightarrow & (H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet,\overline{F}_\infty^*)_{\mathbb{R}} \\ & \xrightarrow{\pi_1} & (H^n(X^{\mathrm{an}},\mathbb{R}(1)),\pi_1(F^\bullet),\overline{F}_\infty^*) \longrightarrow 0 \end{array}$$

in $\mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}}$ leads to the following expressions for ξ_K of $(H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet,(F_\infty^*))$ which are not based on duality:

Theorem 5.4 *Let X be a smooth and proper variety over K. Then we have for $K=\mathbb{C}$*

1) $\xi_{\mathbb{C}}(H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet)$
$= \mathrm{Ker}\left(R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}},rd) \xrightarrow{\alpha} R^n\pi_*\mathcal{DR}_{X/\mathbb{C}}/\mathcal{A}_{\mathbb{R}}\text{-}torsion\right)$
= inverse image in $R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}},rd)$ of the maximal $\mathcal{A}_{\mathbb{R}}$-submodule of $R^n\pi_*\mathcal{DR}_{X/\mathbb{C}}$ with support in $0\in\mathbb{R}$.

For $K=\mathbb{R}$ we find similarly:

2) $\xi_{\mathbb{R}}((H^n(X^{\mathrm{an}},\mathbb{R}),\mathcal{V}^\bullet,F_\infty^*)$
$= \mathrm{Ker}\left(R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times_{\mu_2}\mathbb{R}/(\mathbb{R}/\mu_2)},rd) \xrightarrow{\alpha} R^n\pi_*\mathcal{DR}_{X/\mathbb{R}}/\mathcal{A}_{\mathbb{R}^{\geq 0}}\text{-}torsion\right)$
= inverse image in $R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times_{\mu_2}\mathbb{R}/(\mathbb{R}/\mu_2)},rd)$ of the maximal $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-submodule of $R^n\pi_*\mathcal{DR}_{X/\mathbb{R}}$ with support in $0\in\mathbb{R}^{\geq 0}$.

By passing to the associated C^ω-vector bundles over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ the preceeding theorems and corollary give a geometric construction of the C^ω-bundle $E_{\mathfrak{p}}^\omega(M)$ attached to a motive M in section 3. The Hodge theoretic notions previously required for its definition have been replaced by using suitably deformed complexes and their dynamics.

Appendix

In this appendix we relate the deformed complex $(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$ in Simpson's theorem 4.1 to the ordinary complex of relative differential forms on a suitable space.
Let X be a variety over a field k. For a closed subvariety $Y \subset X$ let $M = M(Y,X)$ denote the deformation to the normal bundle c.f. [V1] § 2. Let $I \subset \mathcal{O}_X$ be the ideal corresponding to Y. Filtering $\mathcal{O}_X$ by the powers I^i for $i \in \mathbb{Z}$ with $I^i = \mathcal{O}_X$ for $i \le 0$ we have:

$$\begin{aligned} M &= \operatorname{spec} \mathrm{Fil}^0(k[z,z^{-1}] \otimes_k \mathcal{O}_X) \\ &= \operatorname{spec} \left(\bigoplus_{i\in\mathbb{Z}} z^{-i} I^i \right) . \end{aligned}$$

Here spec denotes the spectrum of a quasi-coherent $\mathcal{O}_X$-algebra. By construction M is equipped with a flat map

$$\pi_M : M \longrightarrow \mathbb{A}^1$$

and an affine map

$$\rho : M \longrightarrow X .$$

They combine to a map:

$$h = (\rho, \pi_M) : M \longrightarrow X \times \mathbb{A}^1$$

such that the diagram

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "\pi_M"'] & & X \times \mathbb{A}^1 \arrow[dl, "\pi"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

commutes.
The map π_M is equivariant with respect to the natural $\mathbb{G}_m$-actions on M and $\mathbb{A}^1$ defined by $\lambda \cdot z = \lambda z$ for $\lambda \in \mathbb{G}_m$. The map h becomes equivariant if $\mathbb{G}_m$ acts on $X \times \mathbb{A}^1$ via the second factor.
It is immediate from the definitions that if $f : X' \to X$ is a flat map of varieties and $Y' = Y \times_X X'$ then

$$M(Y', X') = M(Y,X) \times_X X' . \tag{29}$$

Moreover the diagram

$$\begin{tikzcd} M(Y',X') \arrow[r, "h"] \arrow[d, "f_M"'] & X' \times \mathbb{A}^1 \arrow[d, "f \times \mathrm{id}"] \\ M(Y,X) \arrow[r, "h"] & X \times \mathbb{A}^1 \end{tikzcd} \tag{30}$$

is commutative and cartesian.
From now on let X be a smooth variety over an algebraically closed field k and fix a base point $* \in X$. Set $M = M(*, X)$ and consider the natural map

$$h : M \longrightarrow X \times \mathbb{A}^1 .$$

Pullback of differential forms induces a map :

$$\mu : h^*\Omega^p_{X\times\mathbb{A}^1/\mathbb{A}^1} \longrightarrow \Omega^p_{M/\mathbb{A}^1} \ , \ \mu(\omega) = h^*(\omega) .$$

We can now formulate the main observation of this appendix:

Theorem 5.5 *For every $p \geq 0$ the sheaf $\Omega^p_{M/\mathbb{A}^1}$ has no z-torsion and we have that*

$$\operatorname{Im}\mu = z^p\Omega^p_{M/\mathbb{A}^1} .$$

The map of $\mathcal{O}_M$-modules:

$$\alpha : h^*\Omega^p_{X\times\mathbb{A}^1/\mathbb{A}^1} \longrightarrow \Omega^p_{M/\mathbb{A}^1} \ , \ \alpha(\omega) = z^{-p}h^*(\omega)$$

which is well defined by the preceeding assertions is an isomorphism. Hence we get an isomorphism of complexes:

$$\alpha : h^*(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd) \xrightarrow{\sim} \Omega^\bullet_{M/\mathbb{A}^1} \ , \ \alpha(\omega) = z^{-\deg\omega}h^*(\omega) .$$

Remarks. 1) Under the isomorphism α the $\mathbb{G}_m$-action on the left, as defined after theorem 5.1, corresponds to the natural $\mathbb{G}_m$-action on $\Omega^\bullet_{M/\mathbb{A}^1}$ by pullback $\lambda \cdot \omega = \lambda^*(\omega)$.
2) By a slightly more sophisticated construction one can get rid of the choice of base point: The spaces $M(*, X)$ define a family $\mathcal{M} \to X$. The maps $h : M(*, X) \to X \times \mathbb{A}^1$ lead to a map $\mathcal{M} \to X \times X \times \mathbb{A}^1$. Replace M by the inverse image in $\mathcal{M}$ of $\Delta \times \mathbb{A}^1$ where $\Delta \subset X \times X$ is the diagonal. This is independent of the choice of base point.

Proof of 4.5 We first check the assertions for the pair $(0, \mathbb{A}^n), n \geq 1$. In this case $M = M(0, \mathbb{A}^n)$ is the spectrum of the ring

$$B = k[z, x_1, \dots, x_n, y_1, \dots, y_n]/(zy_1 - x_1, \dots, zy_n - x_n) .$$

The maps $\mathbb{A}^1 \xleftarrow{\pi_M} M \xrightarrow{\rho} \mathbb{A}^n$ are induced by the natural inclusions

$$k[z] \hookrightarrow B \hookleftarrow k[x_1, \dots, x_n] .$$

The B-module $\Omega^1_{B/k[z]}$ is generated by $d\overline{x}_i, d\overline{y}_i$ for $1 \leq i \leq n$ modulo the relations $zd\overline{y}_i = d\overline{x}_i$. Hence it is freely generated by the $d\overline{y}_i$ and in particular z-torsion free. The B-module

$$\Omega^1_{k[z,x_1,\dots,x_n]/k[z]} \otimes_{k[z,x_1,\dots,x_n]} B$$

is free on the generators dx_i. The map μ corresponds to the natural inclusion of this free B-module into $\Omega^1_{B/k[z]}$ which sends dx_i to $d\overline{x}_i = z d\overline{y}_i$. The map α which sends dx_i to $d\overline{y}_i$ is an isomorphism. Hence the theorem for the pair $(0, \mathbb{A}^n)$.
In the general case choose an open subvariety $U \subset X$ containing $* \in U$ and an étale map

$$f : U \longrightarrow \mathbb{A}^n$$

such that $f^{-1}(0) = *$. By (29) and (30) we then have a cartesian diagram:

$$\begin{tikzcd}
M(0,\mathbb{A}^n) \times_{\mathbb{A}^n} U \arrow[r, equal] \arrow[dr, "\text{proj.}"'] & M(*,U) \arrow[r, "h"] \arrow[d, "f_M"'] & U \times \mathbb{A}^1 \arrow[d, "f \times \mathrm{id}"] \\
& M(0,\mathbb{A}^n) \arrow[r, "h"] & \mathbb{A}^n \times \mathbb{A}^1 .
\end{tikzcd}$$

Since $f \times \mathrm{id}$ and hence f_M are étale we know by [M] Theorem 25.1 (2) that

$$\Omega^p_{M(*,U)/\mathbb{A}^1} = f_M^* \Omega^p_{M(0,\mathbb{A}^n)/\mathbb{A}^1} \tag{31}$$

and

$$\Omega^p_{U \times \mathbb{A}^1/\mathbb{A}^1} = (f \times \mathrm{id})^* \Omega^p_{\mathbb{A}^n \times \mathbb{A}^1/\mathbb{A}^1} .$$

As we have seen, $\Omega^p_{M(0,\mathbb{A}^n)/\mathbb{A}^1}$ has no z-torsion. Since f_M is flat the same is true for $\Omega^p_{M(*,U)/\mathbb{A}^1}$ by (31). Applying f_M^* to the isomorphism

$$\alpha : h^* \Omega^p_{\mathbb{A}^n \times \mathbb{A}^1/\mathbb{A}^1} \xrightarrow{\sim} \Omega^p_{M(0,\mathbb{A}^n)/\mathbb{A}^1}$$

it follows from the above that

$$\alpha : h^* \Omega^p_{U \times \mathbb{A}^1/\mathbb{A}^1} \xrightarrow{\sim} \Omega^p_{M(*,U)/\mathbb{A}^1}$$

is an isomorphism as well.
We now choose an open subvariety $V \subset X$ not containing the point $*$ and such that $U \cup V = X$. Then $M(*,U)$ and $M(\emptyset, V)$ are open subvarieties of $M(*,X)$ and we have that

$$M(*,X) = M(*,U) \cup M(\emptyset, V) .$$

As we have seen the map α for $M(*,X)$ is an isomorphism over $M(*,U)$. Over $M(\emptyset,V)$ it is an isomorphism as well since

$$M(\emptyset, V) = V \times \mathbb{G}_m$$

canonically. Hence the theorem follows. □

6 The torsion of $\mathbf{R}^n\pi_*\mathcal{DR}_{X/K}$

In this section we describe the $\mathcal{A}_{\mathbb{R}}$-resp. $\mathcal{A}_{\mathbb{R}\geq 0}$-torsion $\mathcal{T}_{X/\mathbb{C}}$ resp. $\mathcal{T}_{X/\mathbb{R}}$ of the sheaves $R^n\pi_*\mathcal{DR}_{X/\mathbb{C}}$ resp. $R^n\pi_*\mathcal{DR}_{X/\mathbb{R}}$ which were introduced in the last section. For this we first have to extend Proposition 2.3 ii) somewhat.
For a filtered vector space $V \in \mathcal{F}il_{\mathbb{C}}$ and any $N \geq 1$ define a graded vector space by

$$ {}^N\mathrm{Gr}^\bullet V = \bigoplus_{p\in\mathbb{Z}} \mathrm{Fil}^p V/\mathrm{Fil}^{p+N} V \ . $$

It becomes a $\mathbb{C}[z]/(z^N)$-module by letting z act as the one-shift to the left: For v in $\mathrm{Fil}^p V/\mathrm{Fil}^{p+N} V$ set

$$ z \cdot v = \text{image of } v \text{ in } \mathrm{Fil}^{p-1}V/\mathrm{Fil}^{p+N-1}V \ . $$

This action depends on the choice of z. For $N = 1$ we have ${}^N\mathrm{Gr}^\bullet V = \mathrm{Gr}^\bullet V$.
To V in $\mathcal{F}il^{\pm}_{\mathbb{R}}, N \geq 1$ we attach the graded vector space:

$$ {}^{2N}_{\mathbb{R}}\mathrm{Gr}^\bullet V := ({}^{2N}\mathrm{Gr}^\bullet V)^{F_\infty = (-1)^\bullet} \ . $$

It is a $\mathbb{C}[z^2]/(z^{2N})$-module and for $N = 1$ and V in $\mathcal{F}il_{\mathbb{R}}$ we have:

$$ {}^{2}_{\mathbb{R}}\mathrm{Gr}^\bullet V = \mathrm{Gr}^\bullet V \ . \tag{32} $$

With these notations the following result holds:

Proposition 6.1 *a) For V in $\mathcal{F}il_{\mathbb{C}}, N \geq 1$ there are functorial isomorphisms of free $\mathbb{C}[z]/(z^N)$-modules:*

$$ i_0^{-1}(\xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V) \otimes \mathcal{O}_{\mathbb{A}^1}/z^N\mathcal{O}_{\mathbb{A}^1}) = {}^N\mathrm{Gr}^\bullet V \ . $$

Here $i_0 : 0 \hookrightarrow \mathbb{A}^1$ denotes the inclusion of the origin.

b) For V in $\mathcal{F}il^{\pm}_{\mathbb{R}}, N \geq 1$ there are functorial isomorphisms of free $\mathbb{C}[z^2]/(z^{2N})$-modules:

$$ i_0^{-1}(\xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty) \otimes \mathcal{O}_{\underset{\sim}{\mathbb{A}}^1}/z^{2N}\mathcal{O}_{\underset{\sim}{\mathbb{A}}^1}) = {}^{2N}_{\mathbb{R}}\mathrm{Gr}^\bullet V \ . $$

Here, $\underset{\sim}{\mathbb{A}}^1 = \mathrm{spec}\,\mathbb{C}[z^2]$ and $i_0 : 0 \hookrightarrow \underset{\sim}{\mathbb{A}}^1$ is the inclusion.

The isomorphisms in a) and b) are compatible with the $\mathbb{G}_m$-action if $\mathbb{G}_m$ acts on the right in degree p by the character z^{-p}. They depend on the choice of z.

Proof For $V \in \mathcal{F}il_{\mathbb{C}}$ the map:

$$ \mathrm{Fil}^p V/\mathrm{Fil}^{p+N} V \longrightarrow \Big(\sum_i \mathrm{Fil}^i V \otimes z^{-i}\mathbb{C}[z]\Big) \otimes \mathbb{C}[z]/(z^N) $$

sending $v + \mathrm{Fil}^{p+N} V$ to $v \otimes z^{-p} \bmod z^N$ is well defined. The induced map

$$ {}^N\mathrm{Gr}^\bullet V \longrightarrow \Gamma(\mathbb{A}^1, \xi_\mathbb{C}(V, \mathrm{Fil}^\bullet V)) \otimes \mathbb{C}[z]/(z^N) $$

is surjective and $\mathbb{C}[z]/(z^N)$-linear by construction. Since

$$ \dim \, {}^N\mathrm{Gr}^\bullet V = N \dim V $$

both sides have the same $\mathbb{C}$-dimension and hence a) follows.
Given $V \in \mathcal{F}il_\mathbb{R}^\pm$ we may view it as an object of $\mathcal{F}il_\mathbb{C}$ and we get an isomorphism of $\mathbb{C}[z]/(z^{2N})$-modules

$$ \begin{array}{ccc} {}^{2N}\mathrm{Gr}^\bullet V & \longrightarrow & \Big(\sum_i \mathrm{Fil}^i V \otimes z^{-i}\mathbb{C}[z]\Big) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]/(z^{2N}) \\ & & \| \\ & & \Big(\sum_i \mathrm{Fil}^i V \otimes z^{-i}\mathbb{C}[z]\Big) \otimes_{\mathbb{C}[z^2]} \mathbb{C}[z^2]/(z^{2N}) \,. \end{array} $$

Passing to invariants under $F_\infty \otimes (-\mathrm{id})^*$ on the right corresponds to taking invariants under $(-1)^\bullet F_\infty$ on the left. Hence assertion b). The claim about the $\mathbb{G}_m$-action is clear. □
As before there is an ω-version of this proposition over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ which we will use in the sequel.
For a proper and smooth variety $X/\mathbb{C}$ consider the exact sequence of $\mathbb{R}$-vector spaces:

$$ 0 \longrightarrow H^n(X^{\mathrm{an}}, \mathbb{R}) \longrightarrow H^n(X^{\mathrm{an}}, \mathbb{C}) \xrightarrow{\pi_1} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \,. \tag{33} $$

It leads to a complex of $\mathbb{R}[r]/(r^N)$-modules:

$$ \begin{array}{rll} 0 & \longrightarrow & {}^N\mathrm{Gr}^\bullet_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) \xrightarrow{\iota_N} {}^N\mathrm{Gr}^\bullet_F H^n(X^{\mathrm{an}}, \mathbb{C}) \\ & \xrightarrow{\pi_1} & {}^N\mathrm{Gr}^\bullet_{\pi_1(F)} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \end{array} \tag{34} $$

which is right exact but not exact in the middle or on the left in general. Denote by ${}^N\mathcal{H}^\bullet_{X/\mathbb{C}}$ its middle cohomology.
For a proper and smooth variety $X/\mathbb{R}$ we obtain from (34) equipped with the action of $\overline{F}^*_\infty$ a complex of $\mathbb{R}[r^2]/(r^{2N})$-modules

$$ \begin{array}{rll} 0 & \longrightarrow & {}^{2N}_{\mathbb{R}}\mathrm{Gr}^\bullet_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) \xrightarrow{\iota_{2N}} {}^{2N}_{\mathbb{R}}\mathrm{Gr}^\bullet_F H^n(X^{\mathrm{an}}, \mathbb{C}) \\ & \xrightarrow{\pi_1} & {}^{2N}_{\mathbb{R}}\mathrm{Gr}^\bullet_{\pi_1(F)} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \,. \end{array} \tag{35} $$

It is again right exact and we denote its middle cohomology by ${}^{2N}\mathcal{H}^\bullet_{X/\mathbb{R}}$. As $\mathbb{R}$-vector spaces both ${}^N\mathcal{H}^\bullet_{X/\mathbb{C}}$ and ${}^{2N}\mathcal{H}^\bullet_{X/\mathbb{R}}$ are naturally graded.
We can now describe the torsion sheaves $\mathcal{T}_{X/K}$ for $K = \mathbb{C}, \mathbb{R}$:

Theorem 6.2 *For $N \gg 0$ the map α in (20) resp. (24) induces isomorphisms of $\mathcal{A}_{\mathbb{R}}$- resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-modules:*

$$\alpha_0 : i_{0*}({}^N\mathcal{H}^{\bullet}_{X/\mathbb{C}}) \xrightarrow{\sim} \mathcal{T}_{X/\mathbb{C}}$$

resp.

$$\alpha_0 : i_{0*}({}^{2N}\mathcal{H}^{\bullet}_{X/\mathbb{R}}) \longrightarrow \mathcal{T}_{X/\mathbb{R}} \ .$$

Here the operation ψ_0^t on $\mathcal{T}_K$ corresponds to multiplication by $e^{\bullet t}$ on the left.

Proof For any $N \geq 1$ the exact sequence:

$$0 \longrightarrow \mathcal{T}_{X/\mathbb{C}} \longrightarrow R^n\pi_*\mathcal{DR}_{X/\mathbb{C}} \xrightarrow{(21)} \xi^{\omega}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet})) \longrightarrow 0$$

remains exact after tensoring with $\mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}$ since $\xi^{\omega}_{\mathbb{C}}$ is $\mathcal{A}_{\mathbb{R}}$-torsion free. Together with the short exact sequence (34) and the ω-version of Proposition 6.1 a) we obtain the following exact and commutative diagram of $\mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}$-modules:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & i_0^{-1}(\mathcal{T}_{X/\mathbb{C}} \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}) \arrow[d] & \\
{}^N\mathrm{Gr}^{\bullet}_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) \arrow[r] & {}^N\mathrm{Gr}^{\bullet}_F H^n(X^{\mathrm{an}}, \mathbb{C}) \arrow[r, "\alpha_0"] \arrow[d, "\pi_1"'] & i_0^{-1}(R^n\pi_*\mathcal{DR}_{X/\mathbb{C}} \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}) \arrow[r] \arrow[d] & 0 \\
 & {}^N\mathrm{Gr}^{\bullet}_{\pi_1(F)} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \arrow[r, equal] & i_0^{-1}(\xi^{\omega}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet})) \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}) \arrow[d] & \\
 & & 0 \ . &
\end{tikzcd}$$

This shows that α_0 induces an isomorphism of $\mathcal{A}_{\mathbb{R}}$-modules

$$\alpha_0 : \ {}^N\mathcal{H}^{\bullet}_{X/\mathbb{C}} \xrightarrow{\sim} i_0^{-1}(\mathcal{T}_{X/\mathbb{C}} \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}) \ .$$

Since $\mathcal{T}_{X/\mathbb{C}}$ is a coherent torsion sheaf with support in $0 \in \mathbb{R}$ we have

$$\mathcal{T}_{X/\mathbb{C}} = \mathcal{T}_{X/\mathbb{C}} \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}$$

for $N \gg 0$ which gives the first assertion. The remark on ψ_0^t follows from Proposition 6.1 since the map α in the exact sequence (20) is ψ^t-equivariant. The assertion over $\mathbb{R}$ follows similarly. □

In the next result we will view $i_0^{-1}\mathcal{T}_{X/K}$ simply as a finite dimensional $\mathbb{R}$-vector space with a linear flow ψ_0^t. Let Θ be its infinitesimal generator i.e. $\psi_0^t = \exp t\Theta$ on $i_0^{-1}\mathcal{T}_{X/K}$.

Proposition 6.3 *The endomorphism Θ of $i_0^{-1}\mathcal{T}_{X/K}$ is diagonalizable over $\mathbb{R}$. For $\alpha = p \in \{1, \dots, n\}$ the dimension of its α-eigenspace is* $\dim \gamma^p H^n(X^{\mathrm{an}}, \mathbb{R})$

if $K=\mathbb{C}$ *and* $\dim(\gamma^p H^n(X^{\mathrm{an}},\mathbb{R})^{(-1)^p})$ *if* $K=\mathbb{R}$. *For all other values of* α *the* α*-eigenspace is zero. In particular we have*

$$\begin{aligned}\det_{\mathbb{R}}(s-\Theta\,|\,i_0^{-1}\mathcal{T}_{X/\mathbb{C}}) &= \prod_{0<p\le n}(s-p)^{\dim\gamma^p}\\ \dim_{\mathbb{R}}(i_0^{-1}\mathcal{T}_{X/\mathbb{C}}) &= \sum_{p\in\mathbb{Z}} p\dim \mathrm{Gr}^p_\gamma H^n(X^{\mathrm{an}},\mathbb{R})\end{aligned}$$

and

$$\begin{aligned}\det_{\mathbb{R}}(s-\Theta\,|\,i_0^{-1}\mathcal{T}_{X/\mathbb{R}}) &= \prod_{0<p\le n}(s-p)^{\dim(\gamma^p)^{(-1)^p}}\\ \dim_{\mathbb{R}}(i_0^{-1}\mathcal{T}_{X/\mathbb{R}}) &= \frac{1}{2}\dim H^n(X^{\mathrm{an}},\mathbb{R})^- + \frac{1}{2}\sum_{p\in\mathbb{Z}} p\dim\mathrm{Gr}^p_{\mathcal{V}} H^n(X^{\mathrm{an}},\mathbb{R})\ .\end{aligned}$$

Remark: According to the proposition the torsion $\mathcal{T}_{X/K}$ is zero iff $\gamma^1=0$ in case $K=\mathbb{C}$ and $(\gamma^1)^-=0=(\gamma^2)^+$ in case $K=\mathbb{R}$. These conditions are equivalent to the strictness of the inclusion (19) if $K=\mathbb{C}$ and to the strictness of

$$(H^n(X^{\mathrm{an}},\mathbb{R}), s\mathrm{Fil}_0^\bullet)\hookrightarrow (H^n(X^{\mathrm{an}},\mathbb{C}), sF^\bullet)_{\mathbb{R}}$$

if $K=\mathbb{R}$. Here s is formed with respect to $\overline{F}^*_\infty$. This is as it must be according to proposition 2.2 c) ii). More explicitly $\mathcal{T}_{X/\mathbb{C}}$ is zero iff H^n has Hodge type $(n,0),(0,n)$ whereas $\mathcal{T}_{X/\mathbb{R}}$ is zero iff H^n has Hodge type either $(n,0),(0,n)$ or $(2,0),(1,1),(0,2)$ with F_∞ acting trivially on H^{11}.

Proof of 6.3: We assume that $K=\mathbb{C}$, the case $K=\mathbb{R}$ being similar. According to theorem 6.2 the operator Θ is diagonalizable on $i_0^{-1}\mathcal{T}_{X/\mathbb{C}}$ the possible eigenvalues being integers. For $p\in\mathbb{Z}$ and $N\gg 0$ we have:

$$\begin{aligned}&\dim\mathrm{Ker}\,(p-\Theta\,|\,i_0^{-1}\mathcal{T}_{X/\mathbb{C}}) = \dim\,{}^N\mathcal{H}^p_{\mathbb{C}}\\ &\overset{(34)}{=}\quad \dim\mathrm{Ker}\,\iota^p_N - \dim\,{}^N\mathrm{Gr}^p_{\mathrm{Fil}_0}H^n(X^{\mathrm{an}},\mathbb{R})\\ &\qquad + \dim_{\mathbb{R}}\,{}^N\mathrm{Gr}^p_F H^n(X^{\mathrm{an}},\mathbb{C}) - \dim\,{}^N\mathrm{Gr}^p_{\pi_1(F)}H^n(X^{\mathrm{an}},\mathbb{R}(1))\ .\end{aligned}$$

Using the exact sequence:

$$\begin{aligned}0 \longrightarrow\ & {}^N\mathrm{Gr}^p_\gamma H^n(X^{\mathrm{an}},\mathbb{R}) \longrightarrow {}^N\mathrm{Gr}^p_F H^n(X^{\mathrm{an}},\mathbb{C})\\ \xrightarrow{\pi_1}\ & {}^N\mathrm{Gr}^p_{\pi_1(F)}H^n(X^{\mathrm{an}},\mathbb{R}(1)) \longrightarrow 0\end{aligned}$$

we see that this is equal to:

$$\dim\mathrm{Ker}\,\iota^p_N + \dim\,{}^N\mathrm{Gr}^p_\gamma H^n(X^{\mathrm{an}},\mathbb{R}) - \dim\,{}^N\mathrm{Gr}^p_{\mathrm{Fil}_0}H^n(X^{\mathrm{an}},\mathbb{R})\ .$$

Since

$$^N\mathrm{Gr}^{\bullet}_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) = \bigoplus_{-N<p\le 0} H^n(X^{\mathrm{an}}, \mathbb{R})$$

we find

$$\begin{aligned}\mathrm{Ker}\,\iota_N &= \mathrm{Ker}\left(\bigoplus_{-N<p\le 0} H^n(X^{\mathrm{an}}, \mathbb{R}) \longrightarrow \bigoplus_{-N<p\le 0} F^p/F^{p+N}\right)\\ &= \bigoplus_{-N<p\le n-N} \gamma^{p+N} H^n(X^{\mathrm{an}}, \mathbb{R})\end{aligned}$$

if $N \ge n$. A short calculation now gives the result. □

Remark. I cannot make the idea rigorous at present but it seems to me that the complexes $R^n\pi_*\mathcal{DR}_{X/\mathbb{C}}$ and $R^n\pi_*\mathcal{DR}_{X/\mathbb{R}}$ should have an interpretation in terms of a suitable perverse sheaf theory. Let us look at an analogy:
Consider a possibly singular variety Y over $\mathbb{F}_p$ and let $j : U \subset Y$ be a smooth open subvariety. If $\pi : X \to U$ is smooth and proper the intermediate extension $\mathcal{F} = j_{!*}R^n\pi_*\mathbb{Q}_l$ for $l \neq p$ is a pure perverse sheaf. We have the L-function

$$\begin{aligned}L_Y(H^n(X), t) &:= \prod_{y\in|Y|} \det_{\mathbb{Q}_l}(1 - t\mathrm{Fr}_{\overline{y}} \mid \mathcal{F}_{\overline{y}})^{-1}\\ &= \prod_i \det_{\mathbb{Q}_l}(1 - t\mathrm{Fr}_p \mid H^i(Y \otimes \overline{\mathbb{F}}_p, \mathcal{F}))^{(-1)^{i+1}} .\end{aligned}$$

By perverse sheaf theory and Deligne's work on the Weil conjectures it satisfies a functional equation and the Riemann hypotheses.
For varieties over number fields Y corresponds to the "curve" $\overline{\mathrm{spec}\,\mathfrak{o}_k}$ and for U we can take e.g. $\mathrm{spec}\,\mathfrak{o}_k$. Hypothetically a better analogue for Y (or more precisely for $Y \otimes \overline{\mathbb{F}}_p$) is the dynamical system ("$\overline{\mathrm{spec}\,\mathfrak{o}_k}$", ϕ^t) whose existence is conjectured in [D7]. For U we would take the subsystem ("$\mathrm{spec}\,\mathfrak{o}_k$", ϕ^t) which has no fixed points of the flow i.e. singularities. This is one motivation for the above idea. Another comes from the discussion in sections 5 and 9 of [D5].
Incidentally the appendix to the preceeding section was motivated by the use of the deformation to the normal cone in perverse sheaf theory [V2].
We would also like to point out that there is an exact triangle in the derived category of $\mathcal{A}_{\mathbb{R}}$-modules with a flow:

$$\xi_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}), \gamma^{\bullet}) \longrightarrow P \longrightarrow \mathcal{T}^*_{\mathbb{C}}(-1)[-1] \longrightarrow \dots$$

where

$$P = R\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(R^n\pi_*\mathcal{D}_{X/\mathbb{C}}, \mathcal{A}_{\mathbb{R}}(-1)) .$$

Here $\xi_{\mathbb{C}}$ sits in degree zero with one-dimensional support and $\mathcal{T}^*_{\mathbb{C}}(-1)[-1]$ sits in degree one with zero-dimensional support. This follows by applying $R\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(\_, \mathcal{A}_{\mathbb{R}}(-1))$ to the exact sequence:

$$0 \longrightarrow \mathcal{T}_{X/\mathbb{C}} \longrightarrow R^n\pi_*\mathcal{D}_{X/\mathbb{C}} \longrightarrow R^n\pi_*(\mathcal{D}_{X/\mathbb{C}})/\mathcal{T}_{X/\mathbb{C}} \longrightarrow 0$$

and noting that

$$\begin{aligned} \underline{\mathrm{Ext}}^1_{\mathcal{A}_{\mathbb{R}}}(\mathcal{T}_{X/\mathbb{C}}, \mathcal{A}_{\mathbb{R}}(-1)) &= \mathcal{T}^*_{X/\mathbb{C}}(-1) \\ &:= \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(\mathcal{T}_{X/\mathbb{C}}, \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}(-1)) \quad \text{for } N \gg 0 \,. \end{aligned}$$

A similar exact triangle exists for $K = \mathbb{R}$ of course.

Remark. One may wonder whether the torsion $\mathcal{T}_{X/K}$ is also relevant for the L-function. It seems to be partly responsible for the ε-factor at infinity as follows: Let X/K be as usual a smooth and proper variety over $K = \mathbb{R}$ or $\mathbb{C}$. With normalizations as in [De1] 5.3 the ε-factor of $H^n(X)$ is given by:

$$\varepsilon = \exp(i\pi\mathcal{D}) \quad \text{where } \mathcal{D} = \frac{1}{e_K}\sum_{p\in\mathbb{Z}} p(h_p - d_p) \,.$$

Here:

$$h_p = \dim_{\mathbb{C}} \mathrm{Gr}^p_F H^n(X, \mathbb{C}) \quad \text{and} \quad d_p = \dim \mathrm{Gr}^p_{\mathcal{V}} H^n(X, \mathbb{R}) \,.$$

This description of the ε-factor can be checked directly. Alternatively it can be found in a more general context in the proof of [D6] Prop. 2.7.
With these notations we have by 6.3:

$$\dim(i_0^{-1}\mathcal{T}_{X/\mathbb{C}}) = \sum_{p\in\mathbb{Z}} pd_p$$

and

$$\dim(i_0^{-1}\mathcal{T}_{X/\mathbb{R}}) = \frac{1}{2}\sum_{p\in\mathbb{Z}} pd_p + \frac{1}{2}\dim H^n(X^{\mathrm{an}}, \mathbb{R})^- \,.$$

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Involutions and Trace Forms on Exterior Powers of a Central Simple Algebra

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Abstract. For A a central simple algebra of degree $2n$, the nth exterior power algebra $\lambda^n A$ is endowed with an involution which provides an interesting invariant of A. In the case where A is isomorphic to $Q \otimes B$ for some quaternion algebra Q, we describe this involution quite explicitly in terms of the norm form for Q and the corresponding involution for B.

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The classification of irreducible representations of a split semisimple simply connected algebraic group G over an arbitrary field F is well-known: they are in one-to-one correspondence with the cone of dominant weights of G. Furthermore, one can tell whether or not an irreducible representation is orthogonal or symplectic (= supports a G-invariant bilinear form which is respectively symmetric or skew-symmetric) by inspecting the corresponding dominant weight [11, §3.11]. (Throughout this paper, we only consider fields of characteristic $\neq 2$, cf. 1.8.) A G-invariant bilinear form on an irreducible representation is necessarily unique up to a scalar multiple.

If the assumption that G is split is dropped, then the Galois group Γ of a separable closure F_s of F over F acts on the cone of dominant weights (via the so-called "$*$-action"), and this action may be nontrivial. Those irreducible representations corresponding to dominant weights which are not fixed by Γ are not defined over F. Although an irreducible representation ρ whose dominant weight is fixed by Γ may not be F-defined, there is always some central simple

F-algebra A and a map $G \to SL_1(A)$ defined over F which is an appropriate descent of ρ, see [14] or [12, p. 230, Prop. 1] for details. The algebra A is uniquely determined up to F-isomorphism. If ρ is orthogonal or symplectic over F_s, then it is easy to show that A supports a unique G-invariant involution γ of the first kind which is adjoint to the G-invariant bilinear form over every extension of F where A is split and hence ρ is defined.

It is of interest to determine γ. For example, invariants of γ in turn provide invariants of G. All involutions γ have been implicitly determined for $F = \mathbb{Q}_p$ and $F = \mathbb{R}$ in [4] and [5], but over an arbitrary field the problem is much more difficult since involutions are no longer classified by their classical invariants [2]. We restrict our attention to simply connected groups of type ${}^1A_{2n-1}$; that is, to the case $G = SL_1(A)$ for A a central simple F-algebra of degree $2n$. Moreover, we will focus on the fundamental irreducible representation corresponding to the middle vertex of the Dynkin diagram of G, which supports a G-invariant involution γ.

For any nonnegative integer $k \leq 2n$, there is a central simple F-algebra $\lambda^k A$ attached to A called the *kth exterior power of A*, and the appropriate analogues of the fundamental representations of $SL_1(A)$ are the natural maps $SL_1(A) \to SL_1(\lambda^k A)$ for $1 \leq k < 2n$. The representation we will study, which corresponds to the middle vertex of the Dynkin diagram, is the $k = n$ case. In general, $\lambda^k A$ is of degree $\binom{2n}{k}$ and is Brauer-equivalent to $A^{\otimes k}$, see [7, 10.A]. It is defined so that when A is the split algebra $A = \mathrm{End}_F(W)$, this $\lambda^k \,\mathrm{End}_F(W)$ is naturally isomorphic to $\mathrm{End}_F(\wedge^k W)$.

The nth exterior power $\lambda^n A$ is endowed with a canonical involution γ such that when A is split, γ is adjoint to the bilinear form θ defined on $\wedge^n W$ by the equation $\theta(x_1 \wedge \ldots \wedge x_n, y_1 \wedge \ldots \wedge y_n)e = x_1 \wedge \ldots \wedge x_n \wedge y_1 \wedge \ldots \wedge y_n$, where e is any basis of the 1-dimensional vector space $\wedge^{2n} W$. This involution is preserved by the image of G in $SL_1(\lambda^n A)$ and is the one we wish to describe. If n is even and $A^{\otimes n}$ is split, then γ is orthogonal and $\lambda^n A$ is split, so our fundamental representation of G is defined over F and orthogonal. For example, for A a biquaternion algebra over an arbitrary field F, γ is adjoint to an Albert form of A [3, 6.2]. In this paper, we provide a complete description of γ for G of type ${}^1A_{2n-1}$ when n is odd (see 1.1) or when n is even and A is isomorphic to $B \otimes Q$ where Q is a quaternion algebra (in 1.4 and 1.5). In particular, until now a description of γ has not been known for any algebra A of index ≥ 8. If A is a tensor product of quaternion algebras, we provide (in 1.6 below) a formula that gives γ in terms of the norm forms of the quaternion algebras.

Describing this particular involution γ is also interesting from the point of view of groups of type ${}^1D_{2n}$. Such a group is isogenous to $G = \mathrm{Spin}(E, \sigma)$ for E a central simple algebra of degree $4n$ and σ an orthogonal involution with trivial discriminant. If σ is hyperbolic, then E is isomorphic to $M_2(A)$ for some algebra A of degree $2n$. The analogue of the direct sum of the two half-spin representations for $\mathrm{Spin}(M_2(A), \sigma)$ over F is the map $G \to SL_1(C(M_2(A), \sigma))$ where $C(M_2(A), \sigma)$ denotes the even Clifford algebra of $(M_2(A), \sigma)$. This alge-

bra is endowed with a canonical involution $\underline{\sigma}$ which is G-invariant; it is mostly hyperbolic but contains a nontrivial piece which is isomorphic to $(\lambda^n A, \gamma)$. Please see [3] for a precise statement and [10] for a rational proof.
This relationship between representations of D_{2n} and A_{2n-1} as well as the results in this paper hint at a general theory of orthogonal representations of semisimple algebraic groups over arbitrary fields. We hope to study this in the future.

1 Statement of the main results

We will always assume that our base field F has characteristic $\neq 2$ and that A is a central simple F-algebra of degree $2n$. (See 1.8 for a discussion of the characteristic 2 case.) We assume moreover that A is isomorphic to a tensor product $A = Q \otimes B$, where Q is a quaternion algebra over F, and B is a central simple F-algebra, necessarily of degree n. Note that this is always the case when n is odd. We write γ_Q for the canonical symplectic involution on Q and n_Q for the norm form.
If n is odd, the main result is the following, proven in Section 4:

Theorem 1.1. *If n is odd, the algebra with involution $(\lambda^n(Q \otimes B), \gamma)$ is Witt-equivalent to $(Q, \gamma_Q)^{\otimes n}$.*

Witt-equivalence for central simple algebras is the natural generalization of Witt-equivalence for quadratic forms, see [1] for a definition.
Assume now that n is even, $n = 2m$. Then $\lambda^n A$ is split and the involution γ is orthogonal. We fix some quadratic form q_A to which γ is adjoint. It is only defined up to similarity.
The algebra $\lambda^m B$ is endowed with a canonical involution which we denote by γ_m. For $k = 0, \ldots, n$, we let $t_k : \lambda^k B \to F$ be the reduced trace quadratic form defined by

$$t_k(x) = \mathrm{Trd}_{\lambda^k B}(x^2). \tag{1.2}$$

This form also has a natural description from the representation-theoretic viewpoint: The group $SL_1(B)$ acts on the vector space $\lambda^k B$, and when B is split $\lambda^k B$ is isomorphic to a tensor product of an irreducible representation with its dual, see Section 2. Consequently, there is a canonical $SL_1(B)$-invariant quadratic form on $\lambda^k B$; it is t_k.
We let t_m^+ and t_m^- denote the restrictions of t_m to the subspaces $\mathrm{Sym}(\lambda^m B, \gamma_m)$ and $\mathrm{Skew}(\lambda^m B, \gamma_m)$ of elements of $\lambda^m B$ which are respectively symmetric and skew-symmetric under γ_m, so that $t_m = t_m^+ \oplus t_m^-$. The forms thus defined are related by the following equation, proven in 5.5:

Theorem 1.3. *In the Witt ring of F, the following equality holds:*

$$\langle 2 \rangle \cdot \sum_{k=0}^{m-1} (-1)^k t_k = \begin{cases} -t_m^- & \textit{if } m \textit{ is even,} \\ t_m^+ & \textit{if } m \textit{ is odd.} \end{cases}$$

The similarity class of q_A is determined by the following theorem, proven in 5.7:

Theorem 1.4. *If n is even, $n = 2m$, the similarity class of q_A contains the quadratic form:*

$$t_m^+ - t_m^- + n_Q \cdot \Big(t_m^- + \sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}} \langle 2\rangle t_k\Big) \qquad \text{if } m \text{ is even,}$$
$$t_m^- - t_m^+ + n_Q \cdot \Big(\sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}} \langle 2\rangle t_k\Big) \qquad \text{if } m \text{ is odd.}$$

The Witt class of this quadratic form can be described more precisely under some additional assumptions (see Proposition 6.1 for precise statements). We just mention here a particular case in which the formula reduces to be quite nice.

Assume that m is even and B is of exponent at most 2. Then $\lambda^m B$ is split, and its canonical involution is adjoint to a quadratic form q_B. Even though this form is only defined up to a scalar factor, its square is actually defined up to isometry. We then have the following, proven in 5.8:

Corollary 1.5. *If m is even (i.e., $\deg B \equiv 0 \mod 4$) and B is of exponent at most 2, then the similarity class of q_A contains a form whose Witt class is $q_B^2 + n_Q\left(2^{n-2} - \frac{1}{2}\binom{n}{m} - \wedge^2 q_B\right)$.*

Some of the notation needs an explanation. For a quadratic form q on a vector space W with associated symmetric bilinear form b so that $q(w) = b(w,w)$, we have an induced quadratic form on $\wedge^2 W$ which we denote by $\wedge^2 q$. For $x_1, x_2, y_1, y_2 \in W$, its associated symmetric bilinear form $\wedge^2 b$ is defined by

$$(\wedge^2 b)(x_1 \wedge x_2, y_1 \wedge y_2) = b(x_1,y_1)b(x_2,y_2) - b(x_1,y_2)b(x_2,y_1).$$

Thus if $q = \langle \alpha_1, \dots, \alpha_n\rangle$, we have

$$\wedge^2 q \simeq \oplus_{1\le i<j\le n}\langle \alpha_i\alpha_j\rangle.$$

From this, one sees that even if q is just defined up to similarity, $\wedge^2 q$ is well-defined up to isometry. (The form $\wedge^2 q$ also admits a representation-theoretic description: It is isomorphic to a scalar multiple of the Killing form on the Lie algebra $\mathfrak{o}(q)$, where the scalar factor depends only on the dimension of q.)

From Corollary 1.5, we also get the following, which is proven in 6.3:

Corollary 1.6. *Let $A_r = Q_1 \otimes \cdots \otimes Q_r$ be a tensor product of r quaternion F-algebras, where $r \ge 3$, and let T_{A_r} be the reduced trace quadratic form on A_r. The similarity class of q_{A_r} contains a quadratic form whose Witt class is*

$$2^{n-1} - \frac{2^{n-2}}{n}\langle 2^r\rangle \cdot T_{A_r} = 2^{f(r)}\big(2^r - (2-n_{Q_1})\cdots(2-n_{Q_r})\big),$$

where $n = 2^{r-1} = \frac{1}{2}\deg A$ and $f(r) = 2^{r-1} - r - 1$.

In particular, for $r = 3$, we get the quadratic form

$$4(n_{Q_1} + n_{Q_2} + n_{Q_3}) - 2(n_{Q_1}n_{Q_2} + n_{Q_1}n_{Q_3} + n_{Q_2}n_{Q_3}) + n_{Q_1}n_{Q_2}n_{Q_3}.$$

Adrian Wadsworth had casually conjectured a description of q_{A_3} in [3, 6.8], and we now see that his conjecture was not quite correct in that it omitted the $n_{Q_1}n_{Q_2}n_{Q_3}$ term.

As a consequence of Corollary 1.6, we can show that the form q_A lies in the nth power of the fundamental ideal of the Witt ring WF for many central simple algebras A of degree $2n$; the following result is proven in 6.4:

Corollary 1.7. *Suppose that A is a central simple algebra of degree $2n \equiv 0 \mod 4$ which is isomorphic to matrices over a tensor product of quaternion algebras. Then the form q_A lies in I^nF.*

The first author conjectured [3, 6.6] that q_A lies in I^nF for *all* central simple F-algebras A of degree $2n \equiv 0 \mod 4$ and such that $A^{\otimes 2}$ is split. Corollary 1.7 fails to prove the full conjecture because for every integer $r \geq 3$ there exists a division algebra A of degree 2^r and exponent 2 such that A doesn't decompose as $A' \otimes A''$ for any nontrivial division algebras A' and A'' [6, 3.3], so such an A doesn't satisfy the hypotheses of Corollary 1.7.
If A is a tensor product of two quaternion algebras, the form q_A is an Albert form of A, and the Witt index of q_A determines the Schur index of A, as Albert has shown (see for instance [7, (16.5)]). Corollary 1.6 shows that one cannot expect nice results relating the Witt index of q_{A_r} and the Schur index of A_r for $r \geq 3$. As pointed out to us by Jan van Geel, the difficulty is that Merkurjev has constructed in [9, §3] algebras of the form A_r for $r \geq 3$ (i.e., tensor products of at least 3 quaternion algebras) which are skew fields but whose center, F, has $I^3F = 0$. By Corollary 1.7, the forms q_{A_r} are then hyperbolic.

Remark 1.8 (characteristic 2). One might hope that results concerning representations of algebraic groups would not involve the restriction that the characteristic is not 2. However, removing this restriction for the results in this paper would necessarily dramatically change their nature. For example, the trace forms t_k occurring here are degenerate in characteristic 2. Also, our methods require the ability to take tensor products of quadratic forms and to scale by a factor of $\langle 2 \rangle$, neither of which are available in characteristic 2. These restrictions may be avoidable, but we have chosen not to attempt to do so because such an attempt would almost certainly make this paper so technical that it would be nearly unreadable.

2 Description of $\lambda^n M_2(B)$

In order to prove these results, we have to describe the algebra with involution $(\lambda^n(Q \otimes B), \gamma)$, which we will do by Galois descent. Hence we first give a description of $\lambda^n M_2(B)$, see Theorem 2.5 below.

Assume $B = \mathrm{End}_F(V)$ for some n-dimensional vector space V. For $0 \le k \le n$, we have $\lambda^k B = \mathrm{End}_F(\wedge^k V)$. We identify $M_2(B) \simeq \mathrm{End}_F(V \oplus V)$ by mapping $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(B)$ to the endomorphism

$$(x, y) \mapsto \big(a(x) + b(y), c(x) + d(y)\big).$$

The distinguished choice of embedding of B in $M_2(B)$ corresponds with the obvious choice of direct sum decomposition of $V \oplus V$. (There are many others.) This gives an identification $\lambda^n M_2(B) = \mathrm{End}_F(\wedge^n(V \oplus V))$. For all integers k, ℓ, this decomposition determines $\wedge^k V \otimes \wedge^\ell V$ as a vector subspace of $\wedge^{k+\ell}(V \oplus V)$ by mapping $(x_1 \wedge \cdots \wedge x_k) \otimes (y_1 \wedge \cdots \wedge y_\ell)$ to

$$(x_1, 0) \wedge \cdots \wedge (x_k, 0) \wedge (0, y_1) \wedge \cdots \wedge (0, y_\ell) \in \wedge^{k+\ell}(V \oplus V).$$

In particular, we have

$$\wedge^n (V \oplus V) = \oplus_{k=0}^n \left(\wedge^k V \otimes \wedge^{n-k} V\right). \tag{2.1}$$

For each k, the space $\wedge^k V \otimes \wedge^{n-k} V$ can be identified to $\mathrm{End}_F(\wedge^k V)$ as follows. Fix a nonzero element (hence a basis) e of $\wedge^n V$ and define a bilinear form

$$\theta_k \colon \; \wedge^k V \times \wedge^{n-k} V \to F$$

by the equation

$$\theta_k(x_k, x_{n-k})\, e = x_k \wedge x_{n-k} \text{ for } x_\ell \in \wedge^\ell V.$$

This form is nonsingular, so it provides the identification mentioned above

$$\wedge^k V \otimes \wedge^{n-k} V = \mathrm{End}_F(\wedge^k V) \tag{2.2}$$

by sending $x_k \otimes x_{n-k}$ to the map $y \mapsto x_k \theta_{n-k}(x_{n-k}, y)$. The product in $\mathrm{End}_F(\wedge^k V)$ then corresponds in $\wedge^k V \otimes \wedge^{n-k} V$ to

$$(x_k \otimes x_{n-k})(y_k \otimes y_{n-k}) = \theta_{n-k}(x_{n-k}, y_k)\, x_k \otimes y_{n-k}.$$

From (2.1) and (2.2), we deduce an identification of the corresponding endomorphism rings

$$\lambda^n M_2(B) = \mathrm{End}_F(\oplus_{k=0}^n \lambda^k B). \tag{2.3}$$

This remains true in the case when B is non split, as we will prove by Galois descent. First, we must introduce some maps on $\oplus_{k=0}^n \lambda^k B$.
Since the bilinear form θ_k is nonsingular, for any $f \in \mathrm{End}_F(\wedge^k V)$, we have a unique element $\gamma_k(f) \in \mathrm{End}_F(\wedge^{n-k} V)$ such that

$$\theta_k\left(f(x), y\right) = \theta_k\left(x, \gamma_k(f)(y)\right),$$

for every $x \in \wedge^k V$ and $y \in \wedge^{n-k} V$. This defines a canonical anti-isomorphism (not depending on the choice of e)

$$\gamma_k \colon \operatorname{End}_F(\wedge^k V) \to \operatorname{End}_F(\wedge^{n-k} V)$$

such that

$$\gamma_k(x \otimes y) = (-1)^{k(n-k)} y \otimes x \tag{2.4}$$

for x and y as before. One may easily verify that $\gamma_{n-k} \circ \gamma_k = \mathrm{Id}_{\operatorname{End}_F(\wedge^k V)}$ for all $k = 0, \dots, n$. By Galois descent, the maps γ_k are defined even when B is nonsplit, i.e., we have anti-isomorphisms $\gamma_k \colon \lambda^k B \to \lambda^{n-k} B$ such that $\gamma_k \circ \gamma_{n-k} = \mathrm{Id}_{\lambda^k B}$ (see [7, Exercise 12, p. 147] for a rational definition). In the particular case where n is even, by definition of the bilinear form $\theta_{n/2}$, the map $\gamma_{n/2}$ is actually the canonical involution on $\lambda^{n/2} B$.

Theorem 2.5. *Whether or not B is split, there is a canonical isomorphism*

$$\Phi \colon \lambda^n M_2(B) \to \operatorname{End}_F(\lambda^0 B \oplus \dots \oplus \lambda^n B)$$

which in the split case is the identification (2.3) *above. The canonical involution γ on $\lambda^n M_2(B)$ induces via Φ an involution on $\operatorname{End}_F(\oplus_{k=0}^n \lambda^k B)$ which is adjoint to the bilinear form T defined on $\lambda^0 B \oplus \dots \oplus \lambda^n B$ by*

$$T(u,v) = \begin{cases} (-1)^\ell \operatorname{Trd}_{\lambda^k B}\big(u\gamma_\ell(v)\big) & \text{if } k + \ell = n, \\ 0 & \text{if } k + \ell \neq n, \end{cases}$$

for any $u \in \lambda^k B$ and $v \in \lambda^\ell B$.

Proof. We prove this by Galois descent. Fix a separable closure F_s of F and let $\Gamma := \operatorname{Gal}(F_s/F)$ be the absolute Galois group. We fix a vector space V over F such that $\dim_F V = \deg B = n$ and let $V_s = V \otimes_F F_s$. We fix also an F_s-algebra isomorphism $\varphi \colon B \otimes_F F_s \xrightarrow{\sim} \operatorname{End}_F(V) \otimes_F F_s$. Every $\sigma \in \Gamma$ acts canonically on V_s and $\operatorname{End}_{F_s}(V_s) = \operatorname{End}_F(V) \otimes_F F_s$; we denote again by σ these canonical actions, so that $\sigma(f) = \sigma \circ f \circ \sigma^{-1}$ for $f \in \operatorname{End}_{F_s}(V_s)$. On the other hand, the canonical action of Γ on $B \otimes_F F_s$ corresponds under φ to some twisted action $*$ on $\operatorname{End}_{F_s}(V_s)$. Since every F_s-linear automorphism of $\operatorname{End}_{F_s}(V_s)$ is inner, we may find $g_\sigma \in \mathrm{GL}(V_s)$ such that

$$\sigma * f = g_\sigma \circ \sigma(f) \circ g_\sigma^{-1} = \operatorname{Int}(g_\sigma) \circ \sigma(f) \qquad \text{for all } f \in \operatorname{End}_{F_s}(V_s).$$

Then φ induces an F-algebra isomorphism from B onto the F-subalgebra

$$\big\{ f \in \operatorname{End}_{F_s}(V_s) \mid g_\sigma \circ \sigma(f) \circ g_\sigma^{-1} = f \text{ for all } \sigma \in \Gamma \big\}.$$

The $*$-action of Γ on $\operatorname{End}_{F_s}(V_s)$ induces twisted actions on $\operatorname{End}_{F_s}\big(\wedge^n(V_s \oplus V_s)\big)$ and on $\operatorname{End}_{F_s}\big(\oplus_{k=0}^n \operatorname{End}_{F_s}(\wedge^k V_s)\big)$ such that the F-algebras of Γ-invariant elements are $\lambda^n\big(M_2(B)\big)$ and $\mathrm{End}_F(\oplus_{k=0}^n \lambda^k B)$ respectively. To prove the first assertion of the theorem, we will show that these actions correspond to each other under the isomorphism

$$\mathrm{End}_{F_s}\big(\wedge^n(V_s \oplus V_s)\big) \xrightarrow{\sim} \mathrm{End}_{F_s}\big(\oplus_{k=0}^n \mathrm{End}_{F_s}(\wedge^k V_s)\big)$$

derived from (2.1) and (2.2).
For $\sigma \in \Gamma$ and $k = 0, \ldots, n$, define $\wedge^k g_\sigma \in \mathrm{GL}(\wedge^k V_s)$ by

$$\wedge^k g_\sigma(x_1 \wedge \ldots \wedge x_k) = g_\sigma(x_1) \wedge \ldots \wedge g_\sigma(x_k).$$

Then φ induces an F-algebra isomorphism from $\lambda^k B$ onto the F-subalgebra

$$\big\{f \in \mathrm{End}_{F_s}(\wedge^k V_s) \mid \wedge^k g_\sigma \circ \sigma(f) \circ (\wedge^k g_\sigma)^{-1} = f \text{ for all } \sigma \in \Gamma\big\},$$

hence also from $\mathrm{End}_F(\oplus_{k=0}^n \lambda^k B)$ to

$$\Big\{f \in \mathrm{End}_{F_s}\big(\oplus_{k=0}^n \mathrm{End}_{F_s}(\wedge^k V_s)\big) \mid \\ \big(\oplus_k \mathrm{Int}(\wedge^k g_\sigma)\big) \circ \sigma(f) = f \circ \big(\oplus_k \mathrm{Int}(\wedge^k g_\sigma)\big) \text{ for all } \sigma \in \Gamma\Big\}.$$

Similarly, define $\wedge^n(g_\sigma \oplus g_\sigma) \in \mathrm{GL}\big(\wedge^n(V_s \oplus V_s)\big)$ by

$$\wedge^n\,(g_\sigma \oplus g_\sigma)\big((x_1, y_1) \wedge \ldots \wedge (x_n, y_n)\big) = \\ \big(g_\sigma(x_1), g_\sigma(y_1)\big) \wedge \ldots \wedge \big(g_\sigma(x_n), g_\sigma(y_n)\big),$$

so that $\lambda^n\big(M_2(B)\big)$ can be identified through φ with

$$\Big\{f \in \mathrm{End}_{F_s}\big(\wedge^n(V_s \oplus V_s)\big) \mid \\ \wedge^n\,(g_\sigma \oplus g_\sigma) \circ \sigma(f) = f \circ \wedge^n(g_\sigma \oplus g_\sigma) \text{ for all } \sigma \in \Gamma\Big\}.$$

Certainly, $\wedge^n(g_\sigma \oplus g_\sigma) = \oplus_{k=0}^n(\wedge^k g_\sigma \otimes \wedge^{n-k} g_\sigma)$ under (2.1), and computation shows that $\wedge^k g_\sigma \otimes \wedge^{n-k} g_\sigma = (\det g_\sigma)\,\mathrm{Int}(\wedge^k g_\sigma)$ under (2.2). Therefore, (2.1) and (2.2) induce an isomorphism of F-algebras

$$\Phi\colon\ \lambda^n\big(M_2(B)\big) \xrightarrow{\sim} \mathrm{End}_F(\oplus_{k=0}^n \lambda^k B).$$

To complete the proof of the theorem, we show that the canonical involution γ on $\lambda^n\big(M_2(B)\big)$ corresponds to the adjoint involution with respect to T under Φ. In order to do so, we view $\lambda^n\big(M_2(B)\big)$ and $\mathrm{End}_F(\oplus_{k=0}^n \lambda^k B)$ as the fixed subalgebras of $\mathrm{End}_{F_s}\big(\wedge^n(V_s \oplus V_s)\big)$ and $\mathrm{End}_{F_s}\big(\oplus_{k=0}^n \mathrm{End}_{F_s}(\wedge^k V_s)\big)$, and show that the canonical involution γ on $\mathrm{End}_{F_s}\big(\wedge^n(V_s \oplus V_s)\big)$ corresponds to the adjoint involution with respect to T (extended to F_s) under the isomorphism induced by (2.1) and (2.2).

Taking any nonzero element $e \in \wedge^n V_s$, the identification $\wedge^{2n}(V_s \oplus V_s) = \wedge^n V_s \otimes \wedge^n V_s$ allows us to write $e \otimes e$ for a nonzero element of $\wedge^{2n}(V_s \otimes V_s)$. Then γ is adjoint to the bilinear form

$$\Theta\colon \ \wedge^n (V_s \oplus V_s) \times \wedge^n (V_s \oplus V_s) \to F_s$$

given by

$$\Theta(x,y)\, e \otimes e = x \wedge y \text{ for } x, y \in \wedge^n (V_s \oplus V_s)$$

as was mentioned in the introduction. Using the identification of $\wedge^k V_s \otimes \wedge^{n-k} V_s$ as a subspace of $\wedge^n(V_s \oplus V_s)$, we have that for $x_i, y_i \in \wedge^i V_s$,

$$\Theta(x_k \otimes x_{n-k}, y_\ell \otimes y_{n-\ell}) = \begin{cases} (-1)^\ell \theta_k(x_k, y_\ell)\theta_{n-k}(x_{n-k}, y_{n-\ell}) & \text{if } k+\ell = n, \\ 0 & \text{if } k+\ell \neq n. \end{cases}$$

We translate this into terms involving B, using the isomorphism φ to identify $\lambda^k B_s := (\lambda^k B) \otimes_F F_s$ with $\mathrm{End}_{F_s}(\wedge^k V_s)$. In particular, we know that

$$\mathrm{Trd}_{\lambda^k B_s}(x_k \otimes x_{n-k}) = \theta_{n-k}(x_{n-k}, x_k)$$

for Trd the reduced trace, and that

$$\theta_k(x_k, x_{n-k}) = (-1)^{k(n-k)} \theta_{n-k}(x_{n-k}, x_k).$$

So for $x = x_k \otimes x_{n-k} \in \lambda^k B_s$ and $y = y_\ell \otimes y_{n-\ell} \in \lambda^\ell B_s$,

$$\Theta(x,y) = \begin{cases} (-1)^\ell \,\mathrm{Trd}_{\lambda^k B_s}(\gamma_\ell(y)x) & \text{if } k+\ell = n, \\ 0 & \text{if } k+\ell \neq n. \end{cases}$$

Of course, in the $k+\ell = n$ case we could just as easily have taken

$$\Theta(x,y) = (-1)^\ell \,\mathrm{Trd}_{\lambda^\ell B_s}(\gamma_k(x)y).$$

So, the vector space isomorphism derived from (2.1) and (2.2) is an isometry of Θ and T, and it follows that the canonical involution γ adjoint to Θ corresponds to the adjoint involution to T under Φ. □

For later use, we prove a little bit more about this isomorphism Φ. Let us consider the elements $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in M_2(B)$, and let t be an indeterminate over F. We write λ^n for the map $M_2(B) \to \lambda^n M_2(B)$ defined in [7, 14.3], which is a homogeneous polynomial map of degree n. In the split case where $M_2(B)$ is identified with $\mathrm{End}_F(V \oplus V)$ and $\lambda^n M_2(B)$ with $\mathrm{End}_F\big(\wedge^n (V \oplus V)\big)$, the map is given by

$$(\lambda^n f)(w_1 \wedge \cdots \wedge w_n) = f(w_1) \wedge \cdots \wedge f(w_n)$$

for $f \in \mathrm{End}_F(V \oplus V)$ and $w_1, \ldots, w_n \in V \oplus V$. Whether or not B is split, there exist $\ell_0, \ldots, \ell_n \in \lambda^n M_2(B)$ such that

$$\lambda^n(e_1 + te_2) = t^n \ell_0 + t^{n-1}\ell_1 + \cdots + t\ell_{n-1} + \ell_n.$$

We then have

Lemma 2.6. *For $k = 0, \dots, n$, the image of ℓ_k under Φ is the projection on $\lambda^k B$. Moreover, we have $\gamma(\ell_k) = \ell_{n-k}$.*

Proof. It is enough to prove it in the split case. Hence, we may assume $B = \mathrm{End}_F(V)$, and use identification (2.2) of the previous section. An element of $\lambda^k B = \mathrm{End}_F(\wedge^k V)$ can be written as $(x_1 \wedge \cdots \wedge x_k) \otimes (y_1 \wedge \cdots \wedge y_{n-k})$, where $x_1, \dots, x_k, y_1, \dots, y_{n-k} \in V$. The endomorphism $\lambda^n(e_1 + te_2)$ acts on this element as follows:

$$\begin{aligned}\lambda^n(e_1+te_2)\,&((x_1\wedge\cdots\wedge x_k)\otimes(y_1\wedge\cdots\wedge y_{n-k}))\\ &= (x_1,0)\wedge\cdots\wedge(x_k,0)\wedge(0,ty_1)\wedge\cdots\wedge(0,ty_{n-k})\\ &= t^{n-k}(x_1\wedge\cdots\wedge x_k)\otimes(y_1\wedge\cdots\wedge y_{n-k}).\end{aligned}$$

Hence, the image under ℓ_i of this element is itself if $i = k$ and 0 otherwise. This proves the first assertion of the lemma. By Theorem 2.5, to prove the second one, one has to check that for any $u, v \in \lambda^0 B \oplus \cdots \oplus \lambda^n B$, we have $T(\ell_i(u), v) = T(u, \ell_{n-i}(v))$, which follows easily from the description of T given in that theorem. □

Remark 2.7. By the previous lemma, the elements $\ell_0, \dots, \ell_n \in \lambda^n M_2(B)$ are orthogonal idempotents. Hence, the fact that $\gamma(\ell_k) = \ell_{n-k}$ for all $k = 0, \dots, n$ implies that the involution γ is hyperbolic if n is odd and Witt-equivalent to its restriction to $\ell_m \lambda^n M_2(B) \ell_m$ if $n = 2m$.

We will also use the following:

Lemma 2.8. *For any $b \in F^\times$, consider $g_0 := \left(\begin{smallmatrix} 0 & b \\ 1 & 0 \end{smallmatrix}\right) \in M_2(B)$, and set $g := \lambda^n(g_0)$. We have:*

1. *for any $u \in \lambda^k B$, $\Phi(g)(u) = b^{n-k}\gamma_k(u) \in \lambda^{n-k} B$;*
2. *$g^2 = b^n$ and $\gamma(g) = (-1)^n g$;*
3. *For any $k = 0, \dots, n$, $g\ell_k = \ell_{n-k} g$.*

Proof. Again, it is enough to prove it in the split case. A direct computation then shows that for any $x \otimes y \in \wedge^k V \otimes \wedge^{n-k} V = \lambda^k B$, we have

$$g(x \otimes y) = (-1)^{k(n-k)} b^{n-k} (y \otimes x),$$

which combined with (2.4) gives (1), which in turn easily implies (3). The first part of (2) is because λ^n restricts to be a group homomorphism on $M_2(B)^*$ [7, 14.3], and the second part then follows since $\gamma(g)g = \mathrm{Nrd}_{\lambda^n M_2(B)}(g) = (-b)^n$ by [7, 14.4]. □

3 Description of $\lambda^n(Q \otimes B)$

We suppose that $Q = (a,b)_F$ is a quaternion F-algebra and B is an arbitrary central simple F-algebra of degree n. We will describe $\lambda^n(Q \otimes B)$ by Galois descent from $K = F(\alpha)$, where $\alpha \in F_s$ is a fixed square root of a. More precisely, let us identify Q with the F-subalgebra of $M_2(K)$ generated by $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)$ and $g_0 = \left(\begin{smallmatrix} 0 & b \\ 1 & 0 \end{smallmatrix}\right)$, i.e.,

$$Q = \{x \in M_2(K) \mid g_0 \bar{x} g_0^{-1} = x\},$$

where $\bar{\ }$ denotes the non-trivial automorphism of K/F. We also have

$$Q \otimes B = \{x \in M_2(B_K) \mid g_0 \bar{x} g_0^{-1} = x\},$$

where $B_K = B \otimes_F K$, and g_0 is now viewed as an element of $M_2(B_K)$.
The canonical map $\lambda^n \colon A \to \lambda^n A$ restricts to be a group homomorphism on A^* [7, 14.3]. Moreover, when $\deg A = 2n$, for $a \in A^*$, $\mathrm{Int}(\lambda^n(a))$ preserves the canonical involution γ on $\lambda^n A$ [7, 14.4], and so we get a map

$$\lambda^n \colon \ \mathrm{Aut}(A) \to \mathrm{Aut}(\lambda^n A, \gamma).$$

In particular this holds for $A = M_2(B_K)$. This induces a map on Galois cohomology

$$H^1(K/F, \mathrm{Aut}(M_2(B_K))) \xrightarrow{H^1(\lambda^n)} H^1(K/F, \mathrm{Aut}(\lambda^n M_2(B_K), \gamma)).$$

The image under this map of the 1-cocycle $\bar{\ } \mapsto \mathrm{Int}(g_0)$ is the 1-cocycle $\bar{\ } \mapsto \mathrm{Int}(\lambda^n g_0)$, as in the preceding section. Since the former 1-cocycle corresponds to $Q \otimes B$, the latter corresponds to $\lambda^n(Q \otimes B)$, so

$$\lambda^n(Q \otimes B) = \{x \in \lambda^n M_2(B_K) \mid g \bar{x} g^{-1} = x\} \tag{3.1}$$

for $g := \lambda^n(g_0)$. We fix this definition of g for the rest of the paper.

4 The n odd case

This section is essentially the proof of Theorem 1.1.
We set $\lambda^{\mathrm{even}} B := \oplus_{\substack{0 \le k < n \\ k \text{ even}}} \lambda^k B$. For $0 \le k \le n$, we let t_k be the reduced trace quadratic form on $\lambda^k B$ as in (1.2). We then have the following:

Lemma 4.1. *When $n = \deg B$ is odd, the algebra with involution $(\lambda^n(Q \otimes B), \gamma)$ is isomorphic to $(Q, \gamma_Q) \otimes (C, \sigma)$, where (C, σ) is isomorphic to $\mathrm{End}_F(\lambda^{\mathrm{even}} B)$ endowed with the adjoint involution with respect to $\sum_{\substack{0 \le k < n \\ k \text{ even}}} t_k$.*

Proof. If $i, j \in Q$ satisfy $i^2 = a$, $j^2 = b$ and $ij = -ji$, then since λ^n restricts to be a group homomorphism on $(Q \otimes B)^*$, $\lambda^n(i \otimes 1)$ and $\lambda^n(j \otimes 1) \in \lambda^n(Q \otimes B)$ anticommute and satisfy

$$\lambda^n(i \otimes 1)^2 = a^n, \qquad \lambda^n(j \otimes 1)^2 = b^n,$$
$$\gamma(\lambda^n(i \otimes 1)) = -\lambda^n(i \otimes 1), \qquad \gamma(\lambda^n(j \otimes 1)) = -\lambda^n(j \otimes 1).$$

(For the bottom two equations, see [7, (14.4)].) Hence, these two elements generate a copy of Q in $\lambda^n(Q \otimes B)$ on which γ restricts to be γ_Q and we have $(\lambda^n(Q\otimes B),\gamma) \simeq (Q,\gamma_Q)\otimes(C,\sigma)$, where C is the centralizer of Q in $\lambda^n(Q\otimes B)$ and σ denotes the restriction of γ to C [7, 1.5].
To describe C, we take $i = \alpha(e_1 - e_2)$ and $j = g_0$, as in the beginning of the previous section, so that $\lambda^n(j \otimes 1) = g$ and

$$\lambda^n(i\otimes 1) = \alpha^n((-1)^n\ell_0 + (-1)^{n-1}\ell_1 + \cdots + \ell_n) = -\alpha^n(\ell_{\text{even}} - \ell_{\text{odd}}),$$

where $\ell_{\text{even}} = \sum_{\substack{0\le k\le n\\ k \text{ even}}} \ell_k$ and $\ell_{\text{odd}} = \sum_{\substack{0\le k\le n\\ k \text{ odd}}} \ell_k$.
Let us consider the map $\Psi\colon\ \ell_{\text{even}}\lambda^n(M_2(B))\ell_{\text{even}} \to \lambda^n(M_2(B_K))$ defined by $\Psi(x) = x + gxg^{-1}$. A direct computation shows that Ψ is an F-algebra homomorphism, amazingly. Clearly, $\overline{\Psi(x)} = \Psi(x)$ and since $g^2 = b^n$ is central (see Lemma 2.8), $g\Psi(x) = \Psi(x)g$ for all x. Hence, the image of Ψ is contained in $\lambda^n(Q\otimes B)$ and is centralized by g. Moreover,

$$\lambda^n(i\otimes 1)\Psi(x) = -\alpha^n(x - gxg^{-1}) = \Psi(x)\lambda^n(i\otimes 1).$$

Hence, the image of Ψ also centralizes $\lambda^n(i \otimes 1)$, and is therefore contained in C. Now, since ℓ_{even} is an idempotent of $\lambda^n(M_2(B))$, the algebra $\ell_{\text{even}}\lambda^n(M_2(B))\ell_{\text{even}}$ is simple, hence Ψ is injective. By dimension count it follows that its image is exactly C.
Since $\gamma(\Psi(x)) = \Psi(g^{-1}\gamma(x)g)$, the involution σ on C corresponds via Ψ to $\operatorname{Int}(g^{-1})\circ\gamma$ on $\ell_{\text{even}}\lambda^n(M_2(B))\ell_{\text{even}}$. Note that if $x \in \ell_{\text{even}}\lambda^n(M_2(B))\ell_{\text{even}}$, then $\gamma(x) \in \ell_{\text{odd}}\lambda^n(M_2(B))\ell_{\text{odd}}$ and $g^{-1}\gamma(x)g \in \ell_{\text{even}}\lambda^n(M_2(B))\ell_{\text{even}}$. By Theorem 2.5, we get that (C,σ) is isomorphic to $\operatorname{End}_F(\lambda^{\text{even}}B)$ endowed with the involution adjoint to the quadratic form T' defined by $T'(u,v) = T(u,\Phi(g)(v))$. Using the description of T given in Theorem 2.5 and Lemma 2.8(1), it is easy to check that the $\lambda^k B$ are pairwise orthogonal for T' and that T' restricts to be $\langle(-b)^{n-k}\rangle t_k$ on $\lambda^k B$. Thus T' is similar to $\sum_{\substack{0\le k<n\\ k \text{ even}}} t_k$. □

Let us now prove Theorem 1.1. If $n = 2m+1$, then the algebra with involution $(Q,\gamma_Q)^{\otimes n}$ is isomorphic to $(Q,\gamma_Q)\otimes(\operatorname{End}_F(Q),\operatorname{ad}_{n_Q})^{\otimes m}$, where ad_{n_Q} denotes the adjoint involution with respect to the quadratic form n_Q. Indeed, one may easily check that $(Q\otimes Q,\gamma_Q\otimes\gamma_Q)$ is isomorphic to $\big(\operatorname{End}_F(Q),\operatorname{ad}_{T_{(Q,\gamma_Q)}}\big)$, where $T_{(Q,\gamma_Q)}$ is the quadratic form defined by $T_{(Q,\gamma_Q)}(x) = \operatorname{Trd}_Q(x\gamma_Q(x))$. Since for any $x \in Q$, we have $x\gamma_Q(x) = n_Q(x) \in F$, $T_{(Q,\gamma_Q)} = \langle 2\rangle n_Q$, and $(Q^{\otimes 2},\gamma_Q^{\otimes 2}) \simeq (\operatorname{End}_F(Q),\operatorname{ad}_{n_Q})$. Therefore, to prove Theorem 1.1, it suffices to show that the algebras with involution $(Q,\gamma_Q)\otimes(C,\sigma)$ and $(Q,\gamma_Q)\otimes(\operatorname{End}_F(Q),\operatorname{ad}_{n_Q})^{\otimes m}$ are Witt-equivalent. We will use the following lemma:

Lemma 4.2. *Let (U,q) and (U',q') be two quadratic spaces over F. There exists an isomorphism*

$$(Q,\gamma_Q)\otimes(\operatorname{End}_F(U),\operatorname{ad}_q) \simeq (Q,\gamma_Q)\otimes(\operatorname{End}_F(U'),\operatorname{ad}_{q'})$$

if and only if the quadratic forms $n_Q\otimes q$ and $n_Q\otimes q'$ are similar.

Proof. Consider the right Q-vector space $U_Q = U \otimes_F Q$. The quadratic form q on U induces a hermitian form $h\colon U_Q \times U_Q \to Q$ (with respect to γ_Q) such that

$$h(u \otimes x, u' \otimes x') = \frac{1}{2}\big(q(u+u') - q(u) - q(u')\big)\gamma_Q(x)x'$$

for u, $u' \in U$ and x, $x' \in Q$. The adjoint involution ad_h satisfies

$$(\mathrm{End}_Q(U_Q), \mathrm{ad}_h) = (\mathrm{End}_F(U), \mathrm{ad}_q) \otimes (Q, \gamma_Q). \tag{4.3}$$

The trace form of h, which is by definition the quadratic form

$$U \otimes_F Q \to F,\ x \mapsto h(x,x),$$

is $q \otimes n_Q$. Similarly, we denote by h' the hermitian form induced by q'. By a theorem of Jacobson [13, 10.1.7], the hermitian modules (U_Q, h) and (U'_Q, h') are isomorphic if and only if their trace forms are isometric. Hence, if the quadratic forms $q \otimes n_Q$ and $q' \otimes n_Q$ are similar, i.e., $q \otimes n_Q \simeq \langle\mu\rangle q' \otimes n_Q$ for some $\mu \in F^*$, then the hermitian forms h and $\langle\mu\rangle h'$ are isomorphic, which proves that

$$(Q, \gamma_Q) \otimes (\mathrm{End}_F(U), \mathrm{ad}_q) \simeq (Q, \gamma_Q) \otimes (\mathrm{End}_F(U'), \mathrm{ad}_{q'}).$$

Conversely, if there is such an isomorphism, then equation (4.3) shows that the hermitian forms h and h' are similar, hence their trace forms $q \otimes n_Q$ and $q' \otimes n_Q$ also are similar. $\square$

These two lemmas reduce the proof of Theorem 1.1 to showing that the quadratic forms $n_Q \otimes \sum_{\substack{0 \le k < n \\ k \text{ even}}} t_k$ and $n_Q^{\otimes(m+1)}$ are Witt-equivalent, up to a scalar factor.
On the one hand, we have $n_Q^{\otimes(m+1)} = 4^m n_Q$, since $n_Q^{\otimes 2} = 4n_Q$. On the other hand, since the algebra B is split by an odd-degree field extension, Springer's Theorem [8, VII.2.3] shows that t_k is isometric to the trace form of

$$\lambda^k(M_n(F)) = M_{\binom{n}{k}}(F)$$

which is Witt-equivalent to $\binom{n}{k}\langle 1\rangle$. Hence the Witt class of $n_Q \otimes \sum_{\substack{0 \le k < n \\ k \text{ even}}} t_k$ is

$$\sum_{\substack{0 \le k < n \\ k \text{ even}}} \binom{n}{k} n_Q = 2^{n-1} n_Q = 4^m n_Q,$$

which completes the proof of Theorem 1.1.

5 The n even case

In this section, we prove Theorems 1.3, 1.4, and Corollary 1.5.

Assume from now on that n is even and write $n = 2m$. Consider the element of $\lambda^n(M_2(B_K))$

$$h = \alpha(1 - b^{-m}g)(\ell_0 + \cdots + \ell_{m-1} + \tfrac{1}{2}\ell_m) + (1 + b^{-m}g)(\tfrac{1}{2}\ell_m + \ell_{m+1} + \cdots + \ell_n).$$

One can check that

$$h^{-1} = \tfrac{1}{2}\left((\alpha^{-1} + b^{-m}g)(\ell_0 + \cdots + \ell_m) + (1 - b^{-m}g\alpha^{-1})(\ell_m + \cdots + \ell_n)\right)$$

and $g = b^m\, h\overline{h}^{-1}$.
Therefore, it follows from (3.1) that

$$\lambda^n(Q \otimes B) = h\lambda^n M_2(B)h^{-1} \subset \lambda^n M_2(B)_K.$$

Using the isomorphism Φ of Theorem 2.5 as an identification, we then have

$$\lambda^n(Q \otimes B) = \operatorname{End}_F\left(h(\lambda^0 B) \oplus \cdots \oplus h(\lambda^n B)\right),$$

and the canonical involution on $\lambda^n(Q \otimes B)$ is adjoint to the restriction of the bilinear form T_K to the F-subspace $h(\lambda^0 B) \oplus \cdots \oplus h(\lambda^n B)$. This restriction is given by the following formula:

Lemma 5.1. *The F-subspaces $h(\lambda^k B)$ are pairwise orthogonal. Moreover, for u, $v \in \lambda^k B$ we have*

$$T_K\left(h(u), h(v)\right) = \begin{cases} -2a(-1)^k b^{m-k} \operatorname{Trd}_{\lambda^k B}(uv) & \text{if } k < m, \\ (-1)^m \Big(\frac{1+a}{2} \operatorname{Trd}_{\lambda^m B}\big(\gamma_m(u)v\big) + \frac{1-a}{2} \operatorname{Trd}_{\lambda^m B}(uv)\Big) & \text{if } k = m, \\ 2(-1)^k b^{m-k} \operatorname{Trd}_{\lambda^k B}(uv) & \text{if } k > m. \end{cases}$$

Proof. Using Lemmas 2.6 and 2.8(1), one may easily check that for any $u \in \lambda^k B$, we have

$$h(u) = \begin{cases} \alpha(u - b^{m-k}\gamma_k(u)) & \text{if } k < m, \\ \frac{1}{2}\big[(1+\alpha)u + (1-\alpha)\gamma_k(u)\big] & \text{if } k = m, \\ u + b^{m-k}\gamma_k(u) & \text{if } k > m. \end{cases}$$

The claim then follows from the description of T given in Theorem 2.5 and Lemma 2.8(1) by some direct computations. For instance, if $u, v \in \lambda^m B$, we get

$$T_K\big(h(u), h(v)\big) = (-1)^m \operatorname{Trd}_{\lambda^m B_K}\big[h(u)\gamma_m\big(h(v)\big)\big] \tag{5.2}$$

by Theorem 2.5, and

$$\begin{aligned} &h(u)\gamma_m\big(h(v)\big) = \\ &\tfrac{1}{4}\big[(1+\alpha)^2 u\gamma_m(v) + (1-a)\big(uv + \gamma_m(u)\gamma_m(v)\big) + (1-\alpha)^2\gamma_m(u)v\big]. \end{aligned} \tag{5.3}$$

Since $\operatorname{Trd}_{\lambda^m B}\big(u\gamma_m(v)\big) = \operatorname{Trd}_{\lambda^m B}\big(\gamma_m(u)v\big)$ and $\operatorname{Trd}_{\lambda^m B}\big(\gamma_m(u)\gamma_m(v)\big) = \operatorname{Trd}_{\lambda^m B}(uv)$, it follows that

$$\operatorname{Trd}_{\lambda^m B_K}\big[(1+\alpha)^2 u\gamma_m(v) + (1-\alpha)^2\gamma_m(u)v\big] = 2(1+a)\operatorname{Trd}_{\lambda^m B}\big(\gamma_m(u)v\big)$$

and

$$\operatorname{Trd}_{\lambda^m B_K}\big[(1-a)(uv+\gamma_m(u)\gamma_m(v))\big] = 2(1-a)\operatorname{Trd}_{\lambda^m B}(uv).$$

Therefore, (5.2) and (5.3) yield

$$T_K\big(h(u),h(v)\big) = (-1)^m\frac{1+a}{2}\operatorname{Trd}_{\lambda^m B}\big(\gamma_m(u)v\big) + (-1)^m\frac{1-a}{2}\operatorname{Trd}_{\lambda^m B}(uv).$$

□

This lemma provides a first description of the similarity class of q_A:

Proposition 5.4. *If n is even, the similarity class of q_A contains the quadratic form:*

$$\big(\oplus_{0\le k<m}\langle 2(-1)^k b^{m-k}\rangle\langle 1,-a\rangle t_k\big)\oplus\langle(-1)^m\rangle(t_m^+\oplus\langle -a\rangle t_m^-).$$

Proof. Since the anti-isomorphism γ_k defines an isometry $t_k\simeq t_{n-k}$, the restriction of T_K to $h(\lambda^k B\oplus\lambda^{n-k}B)$, for all $k<m$, is

$$\langle 2(-1)^k b^{m-k}\rangle\langle 1,-a\rangle t_k.$$

Moreover, we have

$$\frac{1+a}{2}\operatorname{Trd}_{\lambda^m B}\big(\gamma_m(u)v\big) + \frac{1-a}{2}\operatorname{Trd}_{\lambda^m B}(uv) = \begin{cases}\operatorname{Trd}_{\lambda^m B}(uv) & \text{if } u\in\operatorname{Sym}(\lambda^m B,\gamma_m),\\ -a\operatorname{Trd}_{\lambda^m B}(uv) & \text{if } u\in\operatorname{Skew}(\lambda^m B,\gamma_m).\end{cases}$$

Hence, the proposition clearly follows from the lemma. □

5.5. Proof of Theorem 1.3.

Theorem 1.3 is a consequence of the preceding results in the special case where $Q=(a,b)_F$ is split. In that case, we may take $b=1$ so that the matrix $g_0=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ then decomposes as $g_0=f_0\bar f_0^{-1}$, where $f_0=\left(\begin{smallmatrix}1&-\alpha\\1&\alpha\end{smallmatrix}\right)$. Hence, if we let $f=\lambda^n f_0$, we have $g=f\bar f^{-1}$. On the other hand, we also have $g=h\bar h^{-1}$, for h as in the preceding section, hence $f^{-1}h=\overline{f^{-1}h}$, which means that $f^{-1}h\in\lambda^n\big(M_2(B)\big)$. Considering the isomorphism Φ of Theorem 2.5 as an identification as we did in the preceding section, we get that $f^{-1}h\in\operatorname{End}_F(\lambda^0 B\oplus\cdots\oplus\lambda^n B)$, hence

$$h(\lambda^0 B\oplus\cdots\oplus\lambda^n B) = f(\lambda^0 B\oplus\cdots\oplus\lambda^n B).$$

To prove Theorem 1.3, we compute the restriction of T_K to this F-subspace in two different ways. First, we use [7, (14.4)], which says that f is a similarity for T_K with similarity factor $\mathrm{Nrd}_{M_2(B_K)}(f_0) = (-2\alpha)^n = 2^n a^m$. Hence, for any $u, v \in \lambda^0 B \oplus \cdots \oplus \lambda^n B$, we have

$$T_K(f(u), f(v)) = 2^n a^m T(u, v).$$

By Remark 2.7 and Theorem 2.5, the form T is Witt-equivalent to its restriction to $\lambda^m B$, which is isometric to $\langle(-1)^m\rangle(t_m^+ \oplus \langle -1\rangle t_m^-)$.
Second, the restriction of T_K to $h(\lambda^0 B \oplus \cdots \oplus \lambda^n B)$ has been computed in Lemma 5.1 and the proof of Proposition 5.4. Comparing the results, we get that the quadratic forms

$$\left(\oplus_{0\le k<m}\langle 2(-1)^k\rangle\langle 1, -a\rangle t_k\right) \oplus \langle(-1)^m\rangle(t_m^+ \oplus \langle -a\rangle t_m^-)$$

and

$$\langle 2^n a^m\rangle\langle(-1)^m\rangle(t_m^+ \oplus \langle -1\rangle t_m^-)$$

are Witt-equivalent. If m is even, we get that the following equality holds in the Witt ring:

$$\left(\sum\nolimits_{0\le k<m}\langle 2(-1)^k\rangle\langle 1, -a\rangle t_k\right) + t_m^+ + \langle -a\rangle t_m^- = t_m^+ - t_m^-,$$

from which we deduce

$$\langle 1, -a\rangle \left(\left(\sum\nolimits_{0\le k<m}\langle 2(-1)^k\rangle t_k\right) + t_m^-\right) = 0.$$

To finish the proof, we may assume a is an indeterminate over the base field F. The previous equality then implies that the quadratic form

$$\left(\bigoplus\nolimits_{0\le k<m}\langle 2(-1)^k\rangle t_k\right) \oplus t_m^-$$

is hyperbolic, which proves the theorem in this case. A similar argument finishes the proof for the m odd case.

Remark 5.6. Let $t_{(\lambda^m B, \gamma_m)} \colon \lambda^m B \to F$ be the quadratic form

$$t_{(\lambda^m B, \gamma_m)}(x) = \mathrm{Trd}_{\lambda^m B}(\gamma_m(x)x).$$

Using Theorem 1.3, together with the facts that $t_{n-k} = t_k$, $t_{(\lambda^m B, \gamma_m)} = t_m^+ - t_m^-$, and that $2q \simeq 2\langle 2\rangle q$ for an arbitrary quadratic form q since $2\langle 2\rangle = 2\langle 1\rangle$, we obtain the following memorable formula:

$$\sum_{k=0}^{n}(-1)^k t_k = t_{(\lambda^m B, \gamma_m)} \qquad \text{in } WF.$$

5.7. Proof of Theorem 1.4. Consider first the case where m is even. In that case, Theorem 1.3 yields

$$\sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}}\langle 2\rangle t_k + t_m^- = \sum\nolimits_{\substack{0\le k<m\\ k \text{ odd}}}\langle 2\rangle t_k.$$

Substituting in the formula given in Proposition 5.4, we get that the similarity class of q_A contains a quadratic form whose Witt class is

$$\begin{aligned}\sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}}\langle 2,-2a\rangle t_k + \sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}}\langle -2b, 2ab\rangle t_k + \langle -a,-b,ab\rangle t_m^- + t_m^+ \\ = \sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}}\langle 2\rangle n_Q t_k + t_m^+ - t_m^- + n_Q t_m^-.\end{aligned}$$

Now, suppose m is odd. Multiplying by $\langle a\rangle$ the quadratic form given in Proposition 5.4 does not change its similarity class, and shows that the similarity class of q_A contains a quadratic form whose Witt class is

$$\langle 1,-a\rangle\cdot\Big(t_m^+ + \sum\nolimits_{0\le k<m}\langle 2(-b)^{k+1}\rangle t_k\Big) + t_m^- - t_m^+.$$

Substituting for t_m^+ the formula of Theorem 1.3 simplifies the expression in brackets to $\langle 1,-b\rangle\cdot\Big(\sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}}\langle 2\rangle t_k\Big)$ and completes the proof.

5.8. Proof of Corollary 1.5. Let us assume that B is of exponent at most 2. Then, for any even k, the algebra $\lambda^k B$ is split. Hence, its trace form t_k is Witt-equivalent to $\binom{n}{k}$. Since m is even, $\lambda^m B$ is also split, and its canonical involution γ_m is adjoint to a quadratic form q_B. This form is only defined up to a scalar factor, but its square is defined up to isometry. Now [7, 11.4] gives relationships between q_B and the forms t_m^+ and t_m^-:

$$t_m^+ - t_m^- \simeq q_B^2 \qquad \text{and} \qquad -t_m^- \simeq \langle 1/2\rangle \wedge^2 q_B.$$

Hence, by Theorem 1.4, the similarity class of q_A contains a form whose Witt class is

$$q_B^2 + n_Q\Big(\langle -2\rangle(\wedge^2 q_B) + \sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}}\tbinom{n}{k}\langle 2\rangle\Big).$$

One may easily check that, since $\langle 2,2\rangle \simeq \langle 1,1\rangle$ and q_B is even-dimensional, $q_B^2 \simeq \langle 2\rangle q_B^2$. Since we are concerned only with the similarity class of q_A, we may therefore forget the factors $\langle 2\rangle$ throughout. Moreover, since m is even, $\sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}}\binom{n}{k} = 2^{n-2} - \frac{1}{2}\binom{n}{m}$, and Corollary 1.5 follows.

6 Another approach to the n even case

Let us decompose $B = B_0 \otimes B_1$, where $\deg B_0 = 2m_0$ is a power of 2 and $\deg B_1 = m_1$ is odd. We have $m = m_0 m_1$, and m is even if and only if $m_0 > 1$.

We write T_0 for the trace form of B_0. Under the assumption that $B_0^{\otimes 2}$ is split (which is automatic if m is odd), we will give a different characterization of q_A for $A = Q \otimes B$ than the one in Theorem 1.4. Corollaries 1.6 and 1.7 will follow from this.

Proposition 6.1. *Suppose that $B_0^{\otimes 2}$ is split. Then the similarity class of q_A contains a form whose Witt class is*

$$2^{n-1} + \frac{2^{n-3}}{m_0} T_0(n_Q - 2) \text{ if } m \text{ is even}$$

and

$$2^{n-2}(n_Q - n_{B_0}) \text{ if } m \text{ is odd.}$$

(Note that B_0 is a quaternion algebra if m is odd.)

This result is already known for m odd: If A is a biquaternion algebra (i.e., $m = 1$) it is [3, 6.2], and in general it follows from [3, 6.4] by a straightforward computation, using the fact that for any integer $k \geq 1$, one has $n_Q^k = 2^{2(k-1)} n_Q$. However, the results from [3] make use of Clifford algebras, which seems a long way to go. So we include a direct proof.
We start with a lemma.

Lemma 6.2. *Suppose that $B_0^{\otimes 2}$ is split. Then the quadratic form t_k is Witt-equivalent to $\binom{n}{k}$ if k is even and $\frac{1}{2m_0}\binom{n}{k} T_0$ if k is odd. Moreover, we have:*

$$t_m^- = \frac{2^{n-3}}{m_0} \langle 2 \rangle T_0 - \Big(2^{n-2} - \tfrac{1}{2}\tbinom{n}{m}\Big) \langle 2 \rangle \text{ if } m \text{ is even,}$$

and

$$t_m^+ = 2^{n-2} \langle 2 \rangle - \Big(2^{n-3} - \tfrac{1}{4}\tbinom{n}{m}\Big) \langle 2 \rangle T_0 \quad \text{if } m \text{ is odd.}$$

This lemma actually specifies t_m^+ and t_m^- whatever the parity of m since in both cases $t_m = t_m^+ + t_m^-$, and t_m is known.

Proof. Since B_1 is split by an odd-degree field extension, Springer's Theorem shows that t_k is isometric to the trace form of $\lambda^k\big(B_0 \otimes M_{m_1}(F)\big)$. If k is even, this algebra is split, and the result is clear. If k is odd, the algebra is Brauer-equivalent to B_0, hence isomorphic to $M_p(F) \otimes B_0$, where $p = \frac{1}{2m_0}\binom{n}{k}$. The form of t_k for k odd then follows from the fact that the trace form of a tensor product of central simple algebras is isometric to the product of the trace forms of each factor.
We have $m = m_0 m_1$, and m is odd if and only if $m_0 = 1$. Recall that

$$\sum_{\substack{0 \leq k < m \\ k \text{ even}}} \binom{n}{k} = \begin{cases} 2^{n-2} & \text{if } m \text{ is odd,} \\ 2^{n-2} - \frac{1}{2}\binom{n}{m} & \text{if } m \text{ is even,} \end{cases}$$

and

$$\sum\nolimits_{\substack{0\le k<m\\ k \text{ odd}}} \binom{n}{k} = \begin{cases} 2^{n-2} - \frac{1}{2}\binom{n}{m} & \text{if } m \text{ is odd,} \\ 2^{n-2} & \text{if } m \text{ is even.} \end{cases}$$

The second part of the lemma then follows from Theorem 1.3 by a direct computation. □

Let us now prove Proposition 6.1. Assume first that m is even. The preceding lemma yields

$$t_m^- + \sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}} \langle 2\rangle t_k = \frac{2^{n-3}}{m_0}\langle 2\rangle T_0$$

and

$$t_m^+ - t_m^- = \binom{n}{m} - 2t_m^- = 2^{n-1}\langle 2\rangle - \frac{2^{n-2}}{m_0}\langle 2\rangle T_0 + \binom{n}{m}\langle 1,-2\rangle.$$

Since $\binom{n}{m}$ is even, the last term on the right side vanishes, hence the quadratic form given by Theorem 1.4 is

$$\langle 2\rangle\Big(2^{n-1} - \frac{2^{n-2}}{m_0}T_0 + \frac{2^{n-3}}{m_0}n_Q T_0\Big).$$

This finishes the m even case.

Assume now that m is odd. Then, B_0 is a quaternion algebra, and $T_0 = \langle 2\rangle(2 - n_{B_0})$. The preceding lemma yields

$$\sum\nolimits_{\substack{0\le k<m\\ k \text{ even}}} \langle 2\rangle t_k = 2^{n-2}\langle 2\rangle$$

and

$$t_m^- - t_m^+ = \tfrac{1}{2}\tbinom{n}{m}T_0 - 2t_m^+ = \tfrac{1}{2}\tbinom{n}{m}T_0 - 2^{n-1}\langle 2\rangle + \Big(2^{n-2} - \tfrac{1}{2}\tbinom{n}{m}\Big)\langle 2\rangle T_0.$$

If $m = 1$, then this reduces to $t_m^- - t_m^+ = -\langle 2\rangle n_{B_0}$, and Theorem 1.4 gives the desired result. Otherwise, since m is odd and $m \ge 3$, the integer $2^{n-2} - \frac{1}{2}\binom{n}{m}$ is even, by [7, (10.29)], hence $\big(2^{n-2} - \frac{1}{2}\binom{n}{m}\big)\langle 2\rangle = 2^{n-2} - \frac{1}{2}\binom{n}{m}$ and the right side of the last displayed equation simplifies to yield

$$t_m^- - t_m^+ = -2^{n-2}\langle 2\rangle n_{B_0}.$$

Therefore, the quadratic form given by Theorem 1.4 is $2^{n-2}\langle 2\rangle(n_Q - n_{B_0})$, which is isometric to $2^{n-2}(n_Q - n_{B_0})$ since $2^{n-2}\langle 2\rangle = 2^{n-2}$, and the proof of Proposition 6.1 is complete.

6.3. Proof of Corollary 1.6. Corollary 1.6 can be proved by induction, using the formula given in Corollary 1.5, but it can also be directly deduced from Proposition 6.1. Indeed, let us assume $A = A_r = Q_1 \otimes \cdots \otimes Q_r$ is a product of $r \geq 3$ quaternion algebras. We let $B = Q_2 \otimes \cdots \otimes Q_r$. Its degree $n = 2^{r-1}$ is a power of 2, and since $r \geq 3$, $m = 2^{r-2}$ is even. In the notation from earlier in this previous section, we have $B_0 = B$ and $B_0^{\otimes 2}$ is split. Hence, we may apply Proposition 6.1. The form T_0 is the trace form of B, that is the tensor product of the trace forms of the quaternion algebras Q_i for $i = 2, \ldots, r$. Hence, we have $T_0 = \langle 2^{r-1}\rangle(2 - n_{Q_2}) \cdots (2 - n_{Q_r})$, and Proposition 6.1 tells us that the similarity class of q_A contains a form whose Witt class is

$$\begin{aligned} 2^{n-1} + \frac{2^{n-3}}{2^{r-2}} \langle 2^{r-1}\rangle (n_{Q_1} - 2)(2 - n_{Q_2}) \cdots (2 - n_{Q_r}) = \\ = 2^{n-1}\langle 2^{r-1}\rangle - 2^{n-r-1}\langle 2^{r-1}\rangle (2 - n_{Q_1})(2 - n_{Q_2}) \cdots (2 - n_{Q_r}) \\ = \langle 2^{r-1}\rangle 2^{n-r-1}\big(2^r - (2 - n_{Q_1}) \cdots (2 - n_{Q_r})\big), \end{aligned}$$

which proves the corollary.

6.4. Proof of Corollary 1.7. Let us now consider a central simple algebra A as in the statement of Corollary 1.7. Then A is isomorphic to $M_k(A_r)$, where $A_r = Q_1 \otimes \cdots \otimes Q_r$ is a product of r quaternion algebras. If A is split then q_A is hyperbolic and the result is clear, so we may assume that $r \neq 0$. Because $\deg A \equiv 0 \mod 4$ by hypothesis, we may further assume that $r \neq 1$ (so that $r \geq 2$), with perhaps some of the Q_i being split.
We first treat the $k = 1$ case. If $r = 2$, then A is biquaternion algebra and q_A is an Albert form, which lies in I^2F. If $r \geq 3$, then by Corollary 1.6 we have to prove that

$$2^{n-1} - 2^{n-r-1}(2 - n_{Q_1}) \cdots (2 - n_{Q_r})$$

lies in I^nF. When we expand this product, the terms of the form 2^{n-1} cancel, and we are left with a sum of terms of the form $\pm 2^{n-\ell-1} n_{Q_{i_1}} \cdots n_{Q_{i_\ell}}$, where $\ell \geq 1$. Since for any i the form n_{Q_i} lies in I^2F, $2^{n-\ell-1} n_{Q_{i_1}} \cdots n_{Q_{i_\ell}}$ belongs to $I^{n-\ell-1+2\ell}F = I^{n+\ell-1}F$, and hence to I^nF.
Now suppose that $k \geq 2$. Since $r \geq 2$, we have $\deg(A_r) \equiv 0 \mod 4$ and we can apply [3, 6.3(1)]. Hence, the similarity class of q_A contains a form which is Witt-equivalent to $q_{A_r}^{\otimes k}$. Since the result holds for A_r by the $k = 1$ case, we are done.

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On Neeman's Well Generated Triangulated Categories

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Abstract. We characterize Neeman's well generated triangulated categories and discuss some of its basic properties.

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In his recent book [4], Neeman introduces a class of triangulated categories which he calls *well generated*. Although Neeman's definition is not easily stated, it becomes quite clear that for triangulated categories 'well generated' is the appropriate generalization of 'compactly generated' [3]. The well generated categories share many important properties with the compactly generated categories. In addition, the class of well generated categories is closed under various natural constructions, for instance, passing to appropriate localizing subcategories and localizations. Our aim in this note is to provide an equivalent definition for well generated categories which seems to be more natural.

Theorem A. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Then $\mathcal{T}$ is well generated in the sense of* [4] *if and only if there exists a set $\mathcal{S}_0$ of objects satisfying:*

(G1) *an object $X \in \mathcal{T}$ is zero provided that $(S, X) = 0$ for all $S \in \mathcal{S}_0$;*

(G2) *for every set of maps $X_i \to Y_i$ in $\mathcal{T}$ the induced map $(S, \coprod_i X_i) \to (S, \coprod_i Y_i)$ is surjective for all $S \in \mathcal{S}_0$ provided that $(S, X_i) \to (S, Y_i)$ is surjective for all i and $S \in \mathcal{S}_0$;*

(G3) *the objects in $\mathcal{S}_0$ are α-small for some cardinal α.*

Here, (X, Y) denotes the maps $X \to Y$. In addition, we recall that an object S is *α-small* if every map $S \to \coprod_{i \in I} X_i$ factors through $\coprod_{i \in J} X_i$ for some $J \subseteq I$ with card $J < \alpha$. Conditions (G1) and (G3) are fairly natural to consider; (G2) is taken from [2] where it is shown that Brown's Representability Theorem holds for a triangulated category $\mathcal{T}$ whenever there is a set $\mathcal{S}_0$ of objects satisfying (G1) – (G2).

Neeman's definition and our characterization of well generated triangulated categories are based on the concept of compactness. Let us explain this. We fix a triangulated category $\mathcal{T}$ with arbitrary coproducts and a cardinal α. Clearly, there exists a unique maximal class $\mathcal{S}$ of α-small objects in $\mathcal{T}$ such that the following holds:

(G4) every map $S \to \coprod_i X_i$ from $S \in \mathcal{S}$ into a coproduct in $\mathcal{T}$ factors through a map $\coprod_i \phi_i \colon \coprod_i S_i \to \coprod_i X_i$ with $S_i \in \mathcal{S}$ for all i.

Simplifying Neeman's terminology, we call the objects in $\mathcal{S}$ *α-compact* and write $\mathcal{T}^\alpha$ for the full subcategory of α-compact objects. We have now the following more explicit description of such compact objects.

Theorem B. *Let $\mathcal{T}$ be a well generated triangulated category and let $\mathcal{S}_0$ be a set of objects satisfying* (G1) – (G2). *Then there exists for every cardinal α a cardinal $\beta \geq \alpha$ such that $X \in \mathcal{T}$ is β-compact if and only if* $\operatorname{card}(S, X) < \beta$ *for all $S \in \mathcal{S}_0$.*

The characterization of well generated triangulated categories uses a result which relates condition (G2) and (G4) to another condition. To state this, let $\mathcal{C}$ be a small additive category and fix a *regular cardinal* α, that is, α is not the sum of fewer than α cardinals, all smaller than α. An *α-product* is a product of less than α factors, and we suppose that α-products exist in $\mathcal{C}$. We denote by $\operatorname{Prod}_\alpha(\mathcal{C}, \mathrm{Ab})$ the category of functors $\mathcal{C} \to \mathrm{Ab}$ into the category of abelian groups which preserve α-products; the morphisms between two functors are the natural transformations.

Theorem C. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and α be a regular cardinal. Let $\mathcal{S}_0$ be a set of objects in $\mathcal{T}$ and denote by $\mathcal{S}$ the full subcategory of α-coproducts of objects in $\mathcal{S}_0$. Then the following are equivalent:*

(1) (G2) *holds for $\mathcal{S}_0$ and every object in $\mathcal{S}_0$ is α-small.*
(2) (G4) *holds for $\mathcal{S}$ and every object in $\mathcal{S}$ is α-small.*
(3) *The functor $\mathcal{T} \to \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto (-, X)|_{\mathcal{S}}$, preserves arbitrary coproducts.*

Proofs

Let us start with some preparations. Throughout we fix a triangulated category $\mathcal{T}$ with arbitrary coproducts. Let $\mathcal{C}$ be an additive category $\mathcal{C}$. A functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups is *coherent* if there exists an exact sequence

$$(-, X) \longrightarrow (-, Y) \longrightarrow F \longrightarrow 0.$$

The natural transformations between two coherent functors form a set, and the coherent functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ form an additive category with cokernels which we denote by $\widehat{\mathcal{C}}$. A basic tool is the *Yoneda functor*

$$\mathcal{C} \longrightarrow \widehat{\mathcal{C}}, \quad X \mapsto H_X = (-, X).$$

Note that $\widehat{\mathcal{C}}$ is a cocomplete category if $\mathcal{C}$ has arbitrary coproducts; in this case the Yoneda functor preserves all coproducts. Recall from [1] that an object

X in a cocomplete category is *α-presentable* if the functor $(X,-)$ preserves α-directed colimits.

Lemma 1. *Let $\mathcal{C}$ be an additive category with arbitrary coproducts. Then an object $X \in \mathcal{C}$ is α-small if and only if $(-,X)$ is α-presentable in $\widehat{\mathcal{C}}$.*

Proof. Straightforward. $\square$

Suppose that $\mathcal{C}$ has kernels and α-products. Then we denote by $\operatorname{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$ the category of left exact functors $\mathcal{C} \to \mathrm{Ab}$ which preserve α-products. Given a class $\mathcal{S}$ of objects in $\mathcal{T}$, we denote by $\operatorname{Add}\mathcal{S}$ the closure of $\mathcal{S}$ in $\mathcal{T}$ under all coproducts and direct factors.

Lemma 2. *Let $\mathcal{S}$ be a small additive subcategory of $\mathcal{T}$ and let α be a regular cardinal. Suppose that every $X \in \mathcal{S}$ is α-small and that $\mathcal{S}$ is closed under α-coproducts in $\mathcal{T}$. Then the assignment $F \mapsto F|_{\mathcal{S}}$ induces an equivalence $f\colon \widehat{\operatorname{Add}\mathcal{S}} \to \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$.*

Proof. First observe that H_X is α-presentable in $\widehat{\operatorname{Add}\mathcal{S}}$ for every $X \in \mathcal{S}$ by Lemma 1. The inclusion $i\colon \mathcal{S} \to \operatorname{Add}\mathcal{S}$ induces a right exact functor $i^*\colon \widehat{\mathcal{S}} \to \widehat{\operatorname{Add}\mathcal{S}}$ which sends H_X to H_{iX}. This functor identifies $\widehat{\mathcal{S}}$ with the full subcategory of all α-colimits of objects in $\{H_X \mid X \in \mathcal{S}\}$. It follows from Satz 7.8 in [1] that i^* induces a fully faithful functor $j\colon \operatorname{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab}) \to \widehat{\operatorname{Add}\mathcal{S}}$ which sends a representable functor $(-,X)$ to i^*X and identifies $\operatorname{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab})$ with the full subcategory of all colimits of objects in $\{H_X \mid X \in \mathcal{S}\}$. Thus j is an equivalence. Now consider the Yoneda functor $h\colon \mathcal{S} \to \widehat{\mathcal{S}}$. It is easily checked that the restriction functor

$$h_*\colon \operatorname{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab}) \longrightarrow \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab}), \quad F \mapsto F \circ h,$$

is an equivalence. We have $f \circ j \cong h_*$ and conclude that f is an equivalence. $\square$

Lemma 3. *Let $\mathcal{S}_0$ be a set of objects in $\mathcal{T}$ and let $\mathcal{S} = \operatorname{Add}\mathcal{S}_0$. Then the functor*

$$\mathcal{T} \longrightarrow \widehat{\mathcal{S}}, \quad X \mapsto (-,X)|_{\mathcal{S}},$$

preserves coproducts if and only if (G2) *holds for $\mathcal{S}_0$.*

Proof. See Lemma 3 in [2]. $\square$

Lemma 4. *Let $\mathcal{S}$ be a set of α-small objects in $\mathcal{T}$ which is closed under α-coproducts. Then* (G2) *and* (G4) *are equivalent for $\mathcal{S}$.*

Proof. It is clear that (G4) implies (G2). To prove the converse, let $S \to \coprod_i X_i$ be a map in $\mathcal{T}$ with $S \in \mathcal{S}$. Choose for every i a map $\psi_i\colon \coprod_j S_{ij} \to X_i$ with $S_{ij} \in \mathcal{S}$ for all j such that every map $X \to X_i$ with $X \in \mathcal{S}$ factors through ψ_i. Then (G2) implies that the map $S \to \coprod_i X_i$ factors through $\amalg_i \psi_i\colon \coprod_i \coprod_j S_{ij} \to \coprod_i X_i$. Using the fact that S is α-small and that $\mathcal{S}$ has α-coproducts, we can replace for each i the coproduct $\coprod_j S_{ij}$ by some $S_i \in \mathcal{S}$. $\square$

Proof of Theorem C. The equivalence of (1) and (2) is Lemma 4 and it remains to show that (1) and (3) are equivalent. We fix a set of objects $\mathcal{S}_0$ in $\mathcal{T}$ and a regular cardinal α. The full subcategory of α-coproducts of objects in $\mathcal{S}_0$ is denoted by $\mathcal{S}$. We can write

$$f\colon \mathcal{T} \longrightarrow \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto (-,X)|_{\mathcal{S}},$$

as composite

$$f\colon \mathcal{T} \xrightarrow{g} \widehat{\operatorname{Add}\mathcal{S}_0} = \widehat{\operatorname{Add}\mathcal{S}} \xrightarrow{h} \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$$

where $gX = (-,X)|_{\operatorname{Add}\mathcal{S}_0}$ and $hF = F|_{\mathcal{S}}$. Now suppose that every object in $\mathcal{S}_0$ is α-small and that (G2) holds. Then it follows from Lemma 3 that g preserves coproducts and Lemma 2 implies that h is an equivalence. We conclude that f preserves coproducts.
Conversely, suppose that f preserves coproducts. It follows that the right exact functor $f^*\colon \widehat{\mathcal{T}} \to \operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ which sends H_X to fX preserves colomits. Now identify $\operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ with $\operatorname{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab})$ as in the proof of Lemma 2. Using Satz 5.5 in [1], it is not hard to see that f^* has a left adjoint which sends $(-,X)$ in $\operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ to H_X. A left adjoint of a functor which preserves α-directed colimits, sends α-presentable objects to α-presentable objects. But the representable functors are α-presentable in $\operatorname{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$. We conclude from Lemma 1 that every $X \in \mathcal{S}$ is α-small. The first part of the proof shows that (G2) holds as well. □

Remark. The proof of Theorem C does not use the triangulated structure of $\mathcal{T}$. One needs that $\mathcal{T}$ is an additive category with arbitrary coproducts and that weak kernels exist in $\mathcal{T}$.

Recall that a full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is *localizing* if $\mathcal{S}$ is closed under arbitrary coproducts. Given a regular cardinal α, we call $\mathcal{S}$ *α-localizing* if $\mathcal{S}$ is closed under α-coproducts and direct factors. For example, the full subcategory $\mathcal{T}_\alpha$ of α-small objects in $\mathcal{T}$ is α-localizing. The full subcategory $\mathcal{T}^\alpha$ of α-compact objects is α-localizing as well.

LEMMA 5. *Let $\mathcal{S}_0$ be a set of α-small objects satisfying* (G1) – (G2) *and denote by $\mathcal{S}$ the smallest α-localizing subcategory containing $\mathcal{S}_0$. Then $\mathcal{S} = \mathcal{T}^\alpha$. Moreover:*

(1) *The objects in $\mathcal{S}$ form, up to isomorphism, a set of α-small objects satisfying* (G2).
(2) *Every set of α-small objects satisfying* (G2) *is contained in $\mathcal{S}$.*

Proof. First we prove (1) and (2); the proof for $\mathcal{S} = \mathcal{T}^\alpha$ is given at the end.
(1) Using the equivalent condition (G4) from Lemma 4, it is straightforward to check that (G2) is preserved if we pass to the closure with respect to forming triangles and α-coproducts. The closure $\mathcal{S}$ can be constructed explicitly, and this shows that the isomorphism classes of objects form a set.
(2) Let $\mathcal{S}_1$ be a set of α-small objects satisfying (G2). We denote by $\mathcal{S}'$ the smallest α-localizing subcategory containing $\mathcal{S}_0 \cup \mathcal{S}_1$ and claim that $\mathcal{S}' = \mathcal{S}$.

Consider the full subcategory $\mathcal{T}'$ of objects $Y \in \mathcal{T}$ such that every map $X \to Y$ with $X \in \mathcal{S}'$ factors through some object in $\mathcal{S}$. Using the fact that (G4) holds for $\mathcal{S}'$ by Lemma 4, it is straightforward to check that $\mathcal{T}'$ is a localizing subcategory containing $\mathcal{S}_0$. The Corollary of Theorem A in [2] shows that $\mathcal{T}$ has no proper localizing subcategory containing $\mathcal{S}_0$. Thus $\mathcal{T}' = \mathcal{T}$ and id_X factors through some object in $\mathcal{S}$ for every $X \in \mathcal{S}'$. We conclude that $\mathcal{S}' = \mathcal{S}$. The proof for the equality $\mathcal{S} = \mathcal{T}^\alpha$ is the same as in (2) if we replace $\mathcal{S}'$ by $\mathcal{T}^\alpha$. □

Lemma 6. *Suppose that the isomorphism classes of objects in $\mathcal{T}^\alpha$ form a set. Then an object $X \in \mathcal{T}$ belongs to $\mathcal{T}^\alpha$ if and only if X is α-compact in the sense of* [4].

Proof. It is automatic from the Definition 1.9 of an α-compact object in [4] that $X \in \mathcal{T}$ belongs to $\mathcal{T}^\alpha$ if X is α-compact in the sense of [4]. Theorem 1.8 in [4] shows that the objects in $\mathcal{T}$ which are α-compact in the sense of [4], form the unique maximal subcategory $\mathcal{S} \subseteq \mathcal{T}_\alpha$ such that the canonical functor $\mathcal{T} \to \mathrm{Prod}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ preserves coproducts. On the other hand, Theorem C shows that $\mathcal{T}^\alpha$ has precisely this property. □

We are now in a position to prove our main result. To this end recall from Definition 1.15 in [4] that $\mathcal{T}$ is *well generated* if there is a regular cardinal α such that condition (2) in the following theorem holds.

Theorem A. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Then the following are equivalent for a regular cardinal α:*

(1) *There exists a set of α-small objects satisfying* (G1) – (G2).
(2) *The isomorphism classes of objects in $\mathcal{T}^\alpha$ form a set, and $\mathcal{T}$ is the smallest localizing subcategory containing $\mathcal{T}^\alpha$.*

Proof. (1) ⇒ (2) Let $\mathcal{S}_0$ be a set of α-small objects satisfying (G1) – (G2). It follows from Lemma 5 that the isomorphism classes in $\mathcal{T}^\alpha$ form a set. The Corollary of Theorem A in [2] shows that $\mathcal{T}$ has no proper localizing subcategory containing $\mathcal{S}_0$.
(2) ⇒ (1) Choose a representative set $\mathcal{S}_0$ of objects in $\mathcal{T}^\alpha$. It follows from Lemma 4 that (G2) holds for $\mathcal{S}_0$. To check (G1) let $\mathcal{Y}$ be the class of objects $Y \in \mathcal{T}$ satisfying $(S, Y) = 0$ for all $S \in \mathcal{S}_0$. Then the objects $X \in \mathcal{T}$ satisfying $(X, Y) = 0$ for all $Y \in \mathcal{Y}$ form a localizing subcategory $\mathcal{X}$ containing $\mathcal{S}_0$. Thus $\mathcal{X} = \mathcal{T}$ and $\mathcal{Y} = \{0\}$. We conclude that $\mathcal{S}_0$ is a set of α-small objects satisfying (G1) – (G2). □

The following immediate consequence of Theorem A is due to Neeman [4].

Corollary. *Let $\mathcal{T}$ be a well generated triangulated category. Then $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$ where α runs through all cardinals.*

We end this note with a proof of Theorem B.

Proof of Theorem B. Let $\mathcal{S}_0$ be a set of objects satisfying (G1) – (G2) and fix a cardinal α. We suppose that $\mathcal{T}$ is well generated. Therefore the objects in $\mathcal{S}_0$ are α'-small for some cardinal α'. It follows from Theorem C in [2] that there is a cardinal $\beta \geq \alpha + \alpha'$ such that an object $X \in \mathcal{T}$ belongs to the smallest β-localizing subcategory containing $\mathcal{S}_0$ if and only if $\operatorname{card}(S, X) < \beta$ for all $S \in \mathcal{S}_0$. The assertion now follows from Lemma 5. □

Remark. There is an explicit description for the cardinal β in Theorem B; see Theorem C in [2].

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Permanence Properties of the Baum-Connes Conjecture

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Abstract. In this paper we study the stability of the Baum-Connes conjecture with coefficients under various natural operations on the groups. We show that the class of groups satisfying this conjecture is stable under taking subgroups, Cartesian products, and more generally, under certain group extensions. In particular, we show that a group satisfies the conjecture if it has an amenable normal subgroup such that the associated quotient group satisfies the conjecture. We also study a natural induction homomorphism between the topological K-theory of a subgroup H of G and the topological K-theory of G with induced coefficient algebra, and show that this map is always bijective. Using this, we are also able to present new examples of groups which satisfy the conjecture with trivial coefficients.

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0 Introduction.

Given a locally compact group G and a C^*-algebra B equipped with a pointwise continuous action of G by $*$-automorphisms, Baum, Connes and Higson constructed in [2] the topological K-theory $\mathrm{K}_*^{\mathrm{top}}(G;B)$ of G with coefficients in B and an assembly map

$$\mu_{G,B} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*(B \rtimes_r G).$$

The Baum-Connes conjecture (with coefficients, cf. [2, §9]) asserts that $\mu_{G,B}$ is an isomorphism for all G and for every G-C^*-algebra B. For convenience, we will use the following

Notation. We say that G *satisfies BCC* (Baum-Connes conjecture with coefficients) if $\mu_{G,B}$ is an isomorphism for every G-algebra B. Moreover, we say that G satisfies BCI (resp. BCS) if the assembly map is injective (resp. surjective) for all B. In case we want to specify the coefficient algebra, we simply say that G satisfies BCC (resp. BCI, BCS) for B.

Although BCC has been shown to be true for many groups (for a general overview of recent results we recommend the surveys [24, 28]), it seems now to be clear that there exist examples of groups for which the assembly map is not always surjective (there are counterexamples due to Higson, Lafforgue, Osawa, Skandalis and Yu, which base on a recent announcement by Gromov on the existence of finitely presented groups with certain graph-theoretic properties). However, knowing that the conjecture fails in some cases makes it even more important to be able to describe the class of groups which do satisfy the conjecture. A natural problem in this direction is to investigate how the conjecture behaves under certain standard operations on the group, like passing to (closed) subgroups or taking group extensions.
Partial answers to the extension problem were given in [5], for the case of semi-direct products by a totally disconnected or almost connected group. The argument in [5] is based on the construction of a partial assembly map associated to a semi-direct product, which generalizes and factors the assembly map of the Baum-Connes conjecture.
In [6] we extended the definition of this partial assembly map in order to decompose the assembly map for arbitrary (non-split) group extension: If N is any closed normal subgroup of G and B is a G-algebra, we constructed a natural homomorphism (the partial assembly map)

$$\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N),$$

which factorizes the assembly map for G in the sense that

$$\mu_{G,B} = \mu_{G/N,B\rtimes_r N} \circ \mu_{N,B}^{G,N},$$

where $\mu_{G/N,B\rtimes_r N}$ denotes the assembly map for G/N with (twisted) coefficient algebra $B \rtimes_r N$. For this construction, we had to use Green's notion of twisted actions which allows to decompose $B\rtimes_r G$ as an iterated twisted crossed product $B\rtimes_r N\rtimes_r G/N$. To make sense of topological K-theory with twisted coefficients, we had to adapt Kasparov's equivariant KK-theory to cover twisted group actions on C^*-algebras. The main results on extensions obtained in [6] are the following: Assume that G has a γ-element (see Definition 1.7 below), and that G/N is either almost connected or totally disconnected. Then G satisfies BCC if G/N and any compact extensions of N in G satisfy BCC.

In this article we want to generalize these results in two directions: Remove the assumption on the topology of the quotient group and lift the requirement that the group has a γ-element. We reached the latter objective in full generality in the case when G/N is totally disconnected (see Theorem 3.3 below), inspired from some ideas exposed in [22], where Oyono-Oyono obtains quite similar results for discrete G. We were also able to reach the other objectives to a very far extend (see the discussion below).
For the investigation of the subgroup problem we study a natural induction homomorphism

$$\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H; B) \to \mathrm{K}_*^{\mathrm{top}}(G; \mathrm{Ind}_H^G B),$$

which provides a link between the assembly map for a subgroup H of a group G, with coefficient algebra B and the assembly map for G with coefficients in the induced G-algebra $\mathrm{Ind}_H^G B$ (see Proposition 2.3 below). For discrete G, this map has been studied by Guentner, Higson and Trout in [11] (in the frame of E-theory), where they showed that it is an isomorphism if H is finite. Later, in [21], H. Oyono-Oyono was able to prove the bijectivity of the induction map for arbitrary subgroups of discrete groups. Here we prove that

- The induction homomorphism $\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H; B) \to \mathrm{K}_*^{\mathrm{top}}(G; \mathrm{Ind}_H^G B)$ is ALWAYS bijective (Theorem 2.2).

As a direct consequence we get

- If G satisfies BCC (resp. BCI, BCS), the same is true for every closed subgroup H of G (Theorem 2.5).

Combining this with our previous results on group extensions we are able to make further progress in this direction. In fact we show

- Suppose that N is a closed normal subgroup of G such that N satisfies the Haagerup property (in particular, if N is amenable). Then, if G/N satisfies BCC (resp. BCS), the same is true for G (Corollary 3.14; but see Theorem 3.12 for a more general statement).
- A direct product $G_1 \times G_2$ satisfies BCC if and only if G_1 and G_2 satisfy BCC (Theorem 3.17).

Another application of the bijectivity of the induction homomorphism is given in [7], where it is shown that the generalized Green-Julg theorem (i.e., BCC for *proper* G-algebras) holds for all (second countable) locally compact groups G. Further, in §4 below, we apply the induction isomorphism in a specific example, which hints into the direction of a more general "Mackey-Machine" for the investigation of the Baum-Connes conjecture.
The outline of the paper is as follows: After a short preliminary section (§1), we give a detailed discussion on the induction homomorphism in §2, where we prove all relevant results, except of the bijectivity of this map. In §3 we briefly

discuss the partial assembly map and prove most of our results on group extensions, except of our main technical result on extensions by totally disconnected groups. In §4 we present an example, which illustrates how our results can be used towards a Mackey-Machine approach to the Baum-Connes conjecture. Using some general ideas, we show that for $K=\mathbb{R}$ or $\mathbb{C}$ the Baum-Connes conjecture with trivial coefficients is true for the groups $K^n \rtimes SL_n(K)$, a result which has been known, so far, only for the cases $n \leq 2$.
The most difficult (and technical) results of this paper are the proofs of Theorem 2.2 and Theorem 3.3 on the bijectivity of the induction homomorphism and the bijectivity of the partial assembly map for totally disconnected quotients, respectively. For this reason we decided to devote two extra sections (§5 and §6) to the proofs of these results. There are some substantial similarities in the proofs of these theorems: Both depend deeply on a certain realization of the universal example $\mathcal{E}(G)$ for proper actions of G (which is an important ingredient in the computation of topological K-theory), using the fact that $\mathcal{E}(G)$ can be realized as a simplicial complex if G is totally disconnected. Since the proof of Theorem 3.3 seemed a bit easier (and perhaps more illustrative), we decided to do this result first (§5). Note that the approach via our special realization of $\mathcal{E}(G)$ seems to have a bunch of other important consequences. So, as a further example for the usefulness of this approach, we show in our final section, §7, that the topological K-theory of a group G is continuous in the coefficient algebras, i.e.,

$$\mathrm{K}_*^{\mathrm{top}}(G; \lim_i A_i) = \lim_i \mathrm{K}_*^{\mathrm{top}}(G; A_i)$$

for any inductive limit $\lim_i A_i$ of G-algebras A_i. This result plays an important role in the proof of the generalized Green-Julg theorem given in [7].
In order to avoid unnecessary repetitions, we have chosen to make the following general conventions: All C^*-algebras (except of multiplier algebras) are supposed to be separable and by group we mean a locally compact second countable Hausdorff topological group.

1 Some preliminaries

Let G be a group. By a *proper G-space* we shall always understand a locally compact space X endowed with an action of G such that the map $G \times X \to X \times X, \quad (g,x) \mapsto (gx,x)$ is continuous and proper (inverse images of compact sets are compact). A *universal example for the proper actions of G, $\mathcal{E}(G)$*, is a proper G-space such that for any other proper G-space Z, there is a continuous and G-equivariant map $F: Z \to \mathcal{E}(G)$ which is unique up to G-equivariant homotopy. Note that $\mathcal{E}(G)$ is uniquely defined up to G-homotopy. The existence of universal proper spaces is shown in [17].
Now let N denote a closed normal subgroup of G. A *twisted action* of (G,N) on a C^*-algebra D (in the sense of Green, [12]) consists of a strongly continuous action by $*$-automorphisms $\alpha: G \to \mathrm{Aut}(D)$ together with a strictly continuous

homomorphism $\tau : N \to UM(D)$ of N into the group of unitaries of the multiplier algebra $M(D)$ of D, such that

$$\alpha_n(d) = \tau_n d \tau_n^* \quad \text{and} \quad \alpha_s(\tau_n) = \tau_{sns^{-1}}, \quad \text{for all } d \in D, s \in G, n \in N.$$

If equipped with such a twisted action, D will be called a (G, N)-algebra. Note that a twisted action of (G, N) should be viewed as a generalization of a G/N-action. In particular, every G/N-algebra can be regarded as a (G, N)-algebra by inflating a given action β of G/N to the twisted action $(\operatorname{Inf}\beta, 1_N)$ of (G, N), and the corresponding twisted crossed products by (G, N) coincide with the ordinary crossed products by G/N. The main advantage of working with twisted actions is, that they allow to decompose crossed products: If B is a G-algebra, then $B \rtimes_r N$ becomes a (G, N)-algebra in a canonical way, so that the iterated (twisted) crossed product $B \rtimes_r N \rtimes_r (G, N)$ is canonically isomorphic to $B \rtimes_r G$. We refer to [6] for more details on these facts and for the construction of the bifunctor $\mathrm{KK}_*^{G,N}(D_1, D_2)$ for pairs (D_1, D_2) of (G, N)-algebras, which extends Kasparov's equivariant $\mathrm{KK}^{G/N}$-theory for G/N-algebras.

Definition 1.1. Let D be a (G, N)-algebra. The *topological* K*-theory of* G/N with coefficient algebra D is

$$\mathrm{K}_*^{\mathrm{top}}(G/N; D) = \lim_{Y} \mathrm{KK}_*^{G,N}\big(C_0(Y), D\big),$$

where the limit is taken over the directed system of all G/N-compact subspaces Y (i.e., $(G/N)\backslash Y$ is compact) of a given universal example $\mathcal{E}(G/N)$ for the proper actions of G/N.

Remark 1.2. In this work we are using a notion of proper G-spaces (resp. G/N-spaces) which differs from the notion used in [2]. This leads to different notions of universal proper G-spaces (resp. G/N-spaces). However, it is shown in [7] that both notions of properness lead to equivalent definitions of the topological K-theory of G (resp. G/N).
If D is a G/N-algebra (viewed as a (G, N)-algebra as explained above), then the above definition of the topological K-theory of G/N with coefficient algebra D coincides with the usual definition of the topological K-theory of G/N with untwisted coefficient algebra D (this follows from [6, Corollary 3.14]). In particular, if $N = \{e\}$ in the above definition, we recover the usual topological K-theory of G with coefficients in the G-algebra D.

For any proper G-space Z, $C_c(Z)$ carries a canonical $C_0(G\backslash Z) - C_c(G \times Z)$ bimodule structure, where we regard $C_c(G \times Z)$ as a dense subalgebra of $C_0(Z) \rtimes G$. This bimodule structure extends to give a $C_0(G\backslash Z) - C_0(Z) \rtimes G$ Hilbert bimodule $\Lambda_{Z,G}$. For reference, the module operations are given on the dense subspaces by

$$\begin{aligned}(\varphi\cdot\xi)(z) &= \varphi(Gz)\xi(z)\\ \langle\xi,\eta\rangle(s,z) &= \Delta_G(s)^{-1/2}\overline{\xi(z)}\eta(s^{-1}z)\\ (\xi\cdot f)(z) &= \int_G \xi(s^{-1}z)f(s^{-1},s^{-1}z)\Delta_G(s)^{-1/2}ds,\end{aligned}\tag{1.1}$$

with $\varphi\in C_0(G\backslash Z)$, ξ, $\eta\in C_c(Z)$, and $f\in C_c(G\times Z)$. Together with the zero operator, we obtain an element (also denoted $\Lambda_{Z,G}$) of the Kasparov group $\mathrm{KK}_0\big(C_0(G\backslash Z), C_0(Z)\rtimes G\big)$ (see [6, §5] for more details). Moreover, if Z is G-compact, i.e., $G\backslash Z$ is compact, then we can pair $\Lambda_{Z,G}$ with the unital homomorphism $\mathbb{C}\to C(G\backslash Z)$ to obtain a canonical element $\lambda_{Z,G}\in \mathrm{K}_0\big(C_0(Z)\rtimes G\big)$. We now recall the definition of the twisted Baum-Connes assembly map as introduced in [6]:

DEFINITION 1.3. Let D be a (G,N)-algebra. The *twisted assembly map* for G/N with coefficients in D, $\mu_{G/N,D}:\ \mathrm{K}_*^{\mathrm{top}}(G/N;D)\to \mathrm{K}_*\big(D\rtimes_r(G,N)\big)$, is defined inductively by the maps

$$\mu_{G/N,D}[Y]:\mathrm{KK}_*^{G,N}\big(C_0(Y),D\big)\to \mathrm{K}_*\big(D\rtimes_r(G,N)\big),$$

where Y runs through the G/N-compact subspaces of a given realization of $\mathcal{E}(G/N)$ and $\mu_{G/N,D}[Y]$ is defined via the composition of maps

$$\begin{tikzcd}
\mathrm{KK}_*^{G,N}\big(C_0(Y),D\big) \arrow[rr, dashed] \arrow[drr, "j^G_{N,r}"'] & & \mathrm{K}_*\big(D\rtimes_r(G,N)\big)\\
& & \mathrm{KK}_*\big(C_0(Y)\rtimes(G,N), D\rtimes_r(G,N)\big) \arrow[u, "\lambda_{Y,G/N}\otimes\cdot"']
\end{tikzcd}$$

Here $j^G_{N,r}$ denotes the descent in twisted equivariant KK-theory as described in [6, §4].

REMARK 1.4. For a G/N-algebra D, viewed as a (G,N)-algebra via inflation, the assembly map of the above definition coincides with the usual Baum-Connes assembly map for G/N with coefficient algebra D. This follows directly from [6, Corollary 3.14]. Of course, if $N=\{e\}$, we get the usual assembly map for G.
It is important to note that by a result of [10], any twisted action of (G,N) is Morita equivalent, and hence $\mathrm{KK}^{G,N}$-equivalent, to an untwisted action of G/N, so that bijectivity, injectivity or surjectivity of the assembly map of Definition 1.3 is equivalent to the corresponding properties of the usual Baum-Connes assembly map for G/N with the corresponding G/N-algebra as coefficient (see [6, 5.6]).

The introduction of twisted coefficients enabled us in [6] to define a partial assembly map for (G,N), which will also play a central role in this paper. We will recall the precise definition of this partial assembly map in §3 below. For its construction we shall need to work with a kind of fundamental class

$$\Lambda_{X,N}^{G,N} \in \mathrm{KK}_0^{G,N}\big(C_0(N\backslash X), C_0(X) \rtimes N\big), \tag{1.2}$$

associated to a proper G-space X, which will play a similar role in the definition of the partial assembly map as the class $\lambda_{Y,G/N}$ in Definition 1.3. We briefly recall its construction: If X is a proper G-space, the given G-action restricts to a proper action of N on X. Thus we can form the $C_0(N\backslash X) - C_0(X) \rtimes N$ bimodule $\Lambda_{X,N}$ as described above. As was shown in [6, §5], there exists a canonical (twisted) action of (G,N) on $\Lambda_{X,N}$, which (again together with the zero operator) provides the element $\Lambda_{X,N}^{G,N}$ of (1.2).

Recall that for two locally compact spaces X and Y, any $*$-homomorphism $\psi : C_0(X) \to C_0(Y)$ is a composition

$$C_0(X) \xrightarrow{\psi_1} C_0(Z) \xrightarrow{\psi_2} C_0(Y),$$

where Z is an open subset of Y, ψ_2 is the canonical inclusion and ψ_1 is induced by a continuous proper map, say $\varphi : Z \to X$, via $\psi_1(f)(z) = f\big(\varphi(z)\big)$, $f \in C_0(X)$. In fact, Z is the open subset of Y corresponding to the ideal $\psi\big(C_0(X)\big)\cdot C_0(Y) \subseteq C_0(Y)$ (note that by an easy application of Cohen's factorization theorem, $\psi\big(C_0(X)\big)\cdot C_0(Y) = \{\psi(f)\cdot g \mid f \in C_0(X), g \in C_0(Y)\}$ is a closed ideal of $C_0(Y)$), and φ is the proper map induced from the non-degenerate $*$-homomorphism $C_0(X) \to \psi\big(C_0(X)\big)\cdot C_0(Y) = C_0(Z); f \mapsto \psi(f)$.

If X and Y are G-spaces and ψ is G-equivariant, then all maps in the above decomposition (and also the map $\varphi : Z \to X$) are G-equivariant. If, moreover, G acts properly on X and Y, and N is a closed normal subgroup of G, then there exist canonical maps

$$C_0(N\backslash X) \xrightarrow{\psi_{1,N}} C_0(N\backslash Z) \xrightarrow{\psi_{2,N}} C_0(N\backslash Y),$$

where the first homomorphism is induced by the proper map

$$\varphi_N : N\backslash Z \to N\backslash X, \quad \varphi_N(Nz) = N\varphi(z),$$

and the second map is induced via the inclusion of the open set $N\backslash Z$ into $N\backslash Y$. Note that the composition $\psi_N := \psi_{2,N} \circ \psi_{1,N}$ satisfies the equation

$$\psi(g\cdot f) = \psi_N(g)\cdot\psi(f), \qquad g \in C_0(N\backslash X), f \in C_0(X).$$

The following lemma will be used frequently throughout this work.

Lemma 1.5 (cf. [6, Lemma 5.13]). *Let $\psi : C_0(X) \to C_0(Y)$ and $\psi_N : C_0(N\backslash X) \to C_0(N\backslash Y)$ be as above. Then*

$$[\Lambda_{X,N}^{G,N}] \otimes j_{\{e\},r}^N([\psi]) = [\psi_N] \otimes [\Lambda_{Y,N}^{G,N}] \quad \text{in } \mathrm{KK}_0^{G,N}\big(C_0(N\backslash X), C_0(Y) \rtimes N\big),$$

where $j_{\{e\},r}^N : \mathrm{KK}^G\big(C_0(X), C_0(Y)\big) \to \mathrm{KK}^{G,N}\big(C_0(X) \rtimes N, C_0(Y) \rtimes N\big)$ denotes the partial descent of [6, §4]. (Note that by the properness of the N-actions, the maximal crossed products coincide with the reduced crossed products.)

Proof. By the decomposition argument presented above, it is sufficient to prove each of the following special cases:

(1) ψ is induced by a continuous and proper G-map $\varphi : Y \to X$ (as explained above), or

(2) X is an open subset of Y and $\psi : C_0(X) \to C_0(Y)$ is the inclusion.

Since all operators in the Kasparov triples defining the KK-elements of the lemma are the zero operators, it is enough to show that the two $C_0(N\backslash X) - C_0(Y) \rtimes N$ Hilbert bimodules $C_0(N\backslash X) \otimes_{C_0(N\backslash Y)} \Lambda_{Y,N}^{G,N}$ and $\Lambda_{X,N}^{G,N} \otimes_{C_0(X)\rtimes N} \big(C_0(Y) \rtimes N\big)$ are equivariantly isomorphic. Using the formulas for the module operations as given in Equation (1.1) above, we see that $C_0(N\backslash X) \otimes_{C_0(N\backslash Y)} \Lambda_{Y,N}^{G,N}$ is just the closure of $\psi_N\big(C_c(N\backslash X)\big) \cdot C_c(Y) \subseteq \Lambda_{Y,N}^{G,N}$ (pointwise multiplication). Now consider the map

$$\Phi : C_c(X) \odot C_c(N \times Y) \to C_c(Y)$$

defined by

$$\Phi(\xi \otimes f)(y) = \int_N \psi(\xi)(ny) f(n, ny) \Delta_N(n)^{-1/2} dn.$$

A lengthy but straightforward computation shows that Φ is an isometry with respect to the right inner products on $\Lambda_{X,N}^{G,N} \otimes_{C_0(X)\rtimes N} \big(C_0(Y) \rtimes N\big)$ and $\Lambda_{Y,N}^{G,N}$, respectively, and therefore extends to an isometry

$$\Phi : \Lambda_{X,N}^{G,N} \otimes_{C_0(X)\rtimes N} \big(C_0(Y) \rtimes N\big) \to \Lambda_{Y,N}^{G,N}.$$

Factoring $C_c(X)$ as $C_c(N\backslash X) \cdot C_c(X)$, it follows directly from the formula for Φ that it has its image in $\psi_N\big(C_0(N\backslash X)\big) \cdot \Lambda_{Y,N}^{G,N}$. Another short computation shows that Φ respects the module actions and that it is equivariant for the given (G, N)-actions on the modules and algebras (see [6, §5] for the precise formulas of those actions).

Thus the only thing which remains to be checked is the surjectivity of Φ, at least if ψ satisfies either (1) or (2). This is trivial in the case of (2) and we restrict ourselves to (1).

Consider any functions $h \in C_c(N\backslash X)$ and $\eta \in C_c(Y)$. We want to construct $\xi \in C_c(X)$ and $f \in C_c(N \times Y)$ such that $\Phi(\xi \otimes f) = \psi_N(h) \cdot \eta$. For this we choose a function $c : X \to \mathbb{R}$ such that c^2 is a cut-off function for the action of N on X (i.e., the restriction of c^2 to any N-compact subset of X has compact support and $\int_N c^2(nx)dn = 1$ for all $x \in X$). We define $\xi \in C_c(X)$ by $\xi(x) = h(Nx)c(x)$, and we define $f \in C_c(N\times Y)$ by $f(n, ny) = \Delta_N(n)^{1/2} c\big(n\varphi(y)\big)\eta(y)$. Then

$$\begin{aligned}
\Phi(\xi \otimes f)(y) &= \int_N \xi\big(n\varphi(y)\big) f(n, ny) \Delta_N(y)^{-1/2} dn \\
&= \int_N h\big(N\varphi(y)\big) c^2\big(n\varphi(y)\big)\eta(y)\, dn \\
&= \big(\psi_N(h) \cdot \eta\big)(y).
\end{aligned}$$

This finishes the proof. □

In what follows, we will frequently have to work with the notion of a γ-element, which goes back to the pioneering work of Kasparov [15, 14], and which turned out to be the most important tool for the investigation of the Baum-Connes conjecture. We first have to introduce the notion of proper G-algebras:

Definition 1.6. A G-C^*-algebra $\mathcal{A}$ is called a *proper G-algebra*, if there exists a proper G-space X and a non-degenerate G-equivariant homomorphism $\Phi : C_0(X) \to ZM(\mathcal{A})$, the center of the multiplier algebra of $\mathcal{A}$.

We can now recall the abstract definition of a γ-element:

Definition 1.7 (cf. [25, §5], [14, §3 - 5]). Let G be a group. An element $\gamma_G \in \mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$ is called a *γ-element* for G if

(1) there exists a proper G-algebra $\mathcal{A}$ and (Dirac and dual-Dirac) elements $D \in \mathrm{KK}_0^G(\mathcal{A},\mathbb{C})$, $\eta \in \mathrm{KK}_0^G(\mathbb{C},\mathcal{A})$ such that $\gamma_G = \eta \otimes_{\mathcal{A}} D$;

(2) for any proper G-space Z we have $p^*(\gamma_G) = 1_Z \in \mathcal{R}\mathrm{KK}_0^G\big(Z; C_0(Z), C_0(Z)\big)$, where p maps Z to the one-point set $\{\mathrm{pt}\}$ (see [14, Proposition 2.20]).

Remark 1.8. If G has a γ-element, then it follows from the work of Kasparov and Tu (see [14, 25]) that G satisfies BCI (i.e., the assembly map $\mu_{G,B}$ is injective for any coefficient algebra B). Moreover, if $\gamma_G = 1 \in \mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$, then G satisfies BCC (we refer to [27] for a concise proof of this result). By a result of Higson and Kasparov ([13], but see also [26]), every group G which satisfies the Haagerup property (in particular every amenable group G) has $1 \in \mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$ as a γ-element, and hence all such groups satisfy BCC for every coefficient algebra B. Moreover, by the work of Kasparov, [15, 14], every group which can be embedded as a closed subgroup of an almost connected group (i.e., a group with compact component group G/G_0) has a γ-element. We refer to [6, §6] for a slightly more detailed account on γ-elements.

2 Induction and the Baum-Connes conjecture for subgroups

Let H be a closed subgroup of the group G and let B be an H-algebra. In this section we want to discuss the induction homomorphism

$$\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H;B) \to \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$$

between the topological K-theory of H with coefficients in B and the topological K-theory of G with coefficients in the induced algebra $\mathrm{Ind}_H^G B$. We will then use this homomorphism to show that BCC passes to closed subgroups.
Recall that the induced algebra $\mathrm{Ind}_H^G B$ is defined as

$$\left\{ f \in C_b(G,B) \;\middle|\; \begin{array}{l} h(f(s)) = f(sh^{-1}) \text{ for all } s \in G, h \in H \\ \text{and } sH \mapsto \|f(s)\| \in C_0(G/H) \end{array} \right\}$$

together with the pointwise operations and the supremum norm. If $B = C_0(X)$ is abelian, then $\operatorname{Ind}_H^G C_0(X)$ is canonically isomorphic to $C_0(G \times_H X)$, where $G \times_H X = (G \times X)/H$ (with respect to the diagonal action $h(g,x) = (gh^{-1}, hx)$) denotes the classical induced G-space.

If A and B are two H-algebras, then Kasparov constructed a natural induction homomorphism

$$i_H^G : \mathrm{KK}_*^H(A,B) \to \mathrm{KK}_*^G(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B)$$

(see [15, §5] and [14, §3]). Let us briefly recall its construction: Suppose that $x \in \mathrm{KK}_*^H(A,B)$ is represented by a Kasparov triple (E, Φ, T). Similar to the construction of the induced algebras we can form the induced $\operatorname{Ind}_H^G B$-Hilbert module $\operatorname{Ind}_H^G E$ as the set

$$\left\{ \xi \in C_b(G,E) \;\middle|\; \begin{array}{l} h(\xi(s)) = \xi(sh^{-1}) \text{ for all } s \in G, h \in H \\ \text{and } sH \mapsto \|\xi(s)\| \in C_0(G/H) \end{array} \right\},$$

equipped with the pointwise actions and inner products. Pointwise action on the left provides an obvious induced representation $\operatorname{Ind}_H^G \Phi : \operatorname{Ind}_H^G A \to \mathcal{L}(\operatorname{Ind}_H^G E)$. Using a cut-off function $c : G \to [0,\infty[$ for the right translation action of H on G, Kasparov constructs an operator $\tilde{T} \in \mathcal{L}(\operatorname{Ind}_H^G E)$ by the formula:

$$\tilde{T}\xi(g) = \int_H c(gh)h(T(\xi(gh)))dh, \quad \xi \in \operatorname{Ind}_K^G \mathcal{E} \tag{2.1}$$

(see [15, Lemma 2 of §5]), to obtain the Kasparov triple $(\operatorname{Ind}_H^G E, \operatorname{Ind}_H^G \Phi, \tilde{T})$ which represents the element $i_H^G(x) \in \mathrm{KK}_*^G(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B)$.

Now suppose that X is an H-compact proper H-space. Then $G \times_H X$ is a G-compact proper G-space and, therefore, there exists a continuous G-map $F : G \times_H X \to \mathcal{E}(G)$ with G-compact image $Y \subseteq \mathcal{E}(G)$. The composition

$$\begin{tikzcd}
\mathrm{KK}_*^H\big(C_0(X), B\big) \arrow[r, dashed] \arrow[dr, "i_H^G"'] & \mathrm{KK}_*^G(C_0(Y), \operatorname{Ind}_H^G B) \\
& \mathrm{KK}_*^G\big(\operatorname{Ind}_H^G C_0(X), \operatorname{Ind}_H^G B\big) \arrow[u, "F^*"']
\end{tikzcd}$$

provides a well defined homomorphism

$$\operatorname{Ind}_H^G[X] : \mathrm{KK}_*^H(C_0(X), B) \to \mathrm{K}_*^{\mathrm{top}}(G; \operatorname{Ind}_H^G B), \tag{2.2}$$

and it is straightforward to check (using a special case of Lemma 1.5) that the maps $\operatorname{Ind}_H^G[X]$ are compatible with taking inclusions $i : X_1 \to X_2$ (i.e., that $\operatorname{Ind}_H^G[X_2] \circ i^* = \operatorname{Ind}_H^G[X_1]$). Thus, if we let X run through the H-compact subsets of $\mathcal{E}(H)$ we obtain a well defined homomorphism

$$\operatorname{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H; B) \to \mathrm{K}_*^{\mathrm{top}}(G; \operatorname{Ind}_H^G B).$$

Definition 2.1. The homomorphism $\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H;B) \to \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$ is called the *induction homomorphism* between $\mathrm{K}_*^{\mathrm{top}}(H;B)$ and $\mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$.

The following theorem is one of the main results of this paper. Since the proof is rather complex and technical, we postpone it to §6 below. For discrete G and finite subgroups H, a similar result (using E-theory) was obtained by Guentner, Higson and Trout in [11], and they asked the question whether the result could be true in more generality. In [21], Oyono-Oyono proves a similar result for arbitrary subgroups of discrete groups.

Theorem 2.2. *Let H be a closed subgroup of G, and let B be an H-algebra. Then the induction map $\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H;B) \to \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$ is an isomorphism.*

The above theorem has many interesting consequences. It provides a connection between the Baum-Connes conjectures for G and H, as we shall study in more details below. It also allows to prove the fact that the Baum-Connes assembly map

$$\mu_{G,\mathcal{A}} : \mathrm{K}_*^{\mathrm{top}}(G;\mathcal{A}) \to K_*(\mathcal{A} \rtimes_r G)$$

is an isomorphism whenever $\mathcal{A}$ is a proper G-algebra, as is worked out in more detail in [7]. Note that this was an open question for quite some time, and was only known for discrete groups by the work of Guentner, Higson and Trout in [11]. Another use of this isomorphism theorem will be indicated in §4 below.

Proposition 2.3. *Let H be a closed subgroup of the group G, and let B be an H-algebra. Let $x \in \mathrm{KK}_0\big(B \rtimes_r H, (\mathrm{Ind}_H^G B) \rtimes_r G\big)$ be defined by the canonical Morita equivalence between $(\mathrm{Ind}_H^G B) \rtimes_r G$ and $B \rtimes_r H$ (e.g., see [12, Theorem 17]). Then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{top}}(H;B) \arrow[r, "\mu_{H,B}"] \arrow[d, "\mathrm{Ind}_H^G"'] & \mathrm{K}_*(B \rtimes_r H) \arrow[d, "\cong"', "\cdot\otimes x"] \\
\mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B) \arrow[r, "\mu_{G,\mathrm{Ind}_H^G B}"] & \mathrm{K}_*\big((\mathrm{Ind}_H^G B) \rtimes_r G\big).
\end{tikzcd}$$

For the proof we need

Lemma 2.4. *Let H be a closed subgroup of G. If $\mathcal{E}(G)$ is a universal example for the proper actions of G, then, by restricting the action to H, it is also a universal example for the proper actions of H.*

Proof. Since $\mathcal{E}(G)$ is unique up to G-homotopy (which certainly implies H-homotopy), it is sufficient to show that the result holds for one particular realization of $\mathcal{E}(G)$. By [17], a realization can be constructed as follows: Choose any proper G-space Z and let $\mathcal{E}(G)$ be the set of positive Radon-measures on Z with total mass in the half open interval $]\frac{1}{2}, 1]$, equipped with the weak$*$-topology and the canonical G-action. Now since the action of G on Z restricts to a proper action of H on Z, the same set of Radon measures provides a realization of $\mathcal{E}(H)$. $\square$

Proof of Proposition 2.3. By the definition of Ind_H^G and Lemma 2.4, it is enough to show that, for a given realization of $\mathcal{E}(G)$, the diagram

$$\begin{tikzcd}
\mathrm{KK}_*^H\big(C_0(X), B\big) \arrow[r, "{[\Lambda_{X,H}]\otimes j^H_{\{e\},r}(\cdot)}"] \arrow[d, "\mathrm{Ind}_H^G"'] & \mathrm{K}_*(B \rtimes_r H) \arrow[d, "\cdot\otimes x"] \\
\mathrm{KK}_*^G\big(C_0(G\times_H X), \mathrm{Ind}_H^G B\big) \arrow[r, "{[\Lambda_{G\times_H X,G}]\otimes j^G_{\{e\},r}(\cdot)}"] \arrow[d, "F^*"'] & \mathrm{K}_*\big((\mathrm{Ind}_H^G B)\rtimes_r G\big) \arrow[d, "="] \\
\mathrm{KK}_*^G\big(C_0(G\cdot X), \mathrm{Ind}_H^G B\big) \arrow[r, "{[\Lambda_{G\cdot X,G}]\otimes j^G_{\{e\},r}(\cdot)}"] & \mathrm{K}_*\big((\mathrm{Ind}_H^G B)\rtimes_r G\big)
\end{tikzcd} \tag{2.3}$$

commutes, where X is any H-compact subset X of $\mathcal{E}(G)$ (also serving as a universal example for proper actions of H). An easy application of Lemma 1.5 implies that the bottom square commutes, so we may restrict our attention to the upper square.

Let y denote the invertible element of $\mathrm{KK}_0\big(C_0(G\times_H X)\rtimes G, C_0(X)\rtimes H\big)$ which is implemented by the canonical Morita equivalence between $C_0(G\times_H X)\rtimes G$ and $C_0(X)\rtimes H$. Then it follows from [14, corollary on p. 176]) that the square

$$\begin{tikzcd}
\mathrm{KK}_*^H\big(C_0(X), B\big) \arrow[r, "j^H_{\{e\},r}"] \arrow[d, "\mathrm{Ind}_H^G"'] & \mathrm{KK}_*\big(C_0(X)\rtimes H, B\rtimes_r H\big) \arrow[d, "y\otimes\cdot\otimes x"] \\
\mathrm{KK}_*^G\big(C_0(G\times_H X), \mathrm{Ind}_H^G B\big) \arrow[r, "j^G_{\{e\},r}"] & \mathrm{KK}_*\big(C_0(G\times_H X)\rtimes G, (\mathrm{Ind}_H^G B)\rtimes_r G\big)
\end{tikzcd}$$

commutes. So the commutativity of (2.3) will follow if we can show that

$$\begin{tikzcd}
\mathrm{KK}_*\big(C_0(X)\rtimes H, B\rtimes_r H\big) \arrow[r, "y\otimes\cdot\otimes x"] \arrow[d, "{[\Lambda_{X,H}]\otimes\cdot}"'] & \mathrm{KK}_*\big(C_0(G\times_H X)\rtimes G, (\mathrm{Ind}_H^G B)\rtimes_r G\big) \arrow[d, "{[\Lambda_{G\times_H X,G}]\otimes\cdot}"] \\
\mathrm{KK}_*\big(C_0(H\backslash X), B\rtimes_r H\big) \arrow[r, "\cdot\otimes x"'] & \mathrm{KK}_*\big(C_0(H\backslash X), (\mathrm{Ind}_H^G B)\rtimes_r G\big)
\end{tikzcd}$$

commutes. For this it is enough to prove that

$$[\Lambda_{G\times_H X,G}]\otimes y = [\Lambda_{X,H}] \qquad \text{in } \mathrm{KK}_0\big(C_0(H\backslash X), C_0(X)\rtimes H\big), \tag{2.4}$$

where we identify $G\backslash(G\times_H X)$ with $H\backslash X$ via $G[x,s]\mapsto Hx$. All KK-classes appearing in this equation are given by a Hilbert bimodule together with the zero operator: $\Lambda_{G\times_H X,G}$ (resp. $\Lambda_{X,H}$) is the Hilbert module obtained by taking the completion of $C_c(G\times_H X)$ (resp. $C_c(X)$) with respect to the inner product, the right action of $C_0(G\times_H X)\rtimes G$ (resp. $C_0(X)\rtimes H$) and the left action of $C_0\big(G\backslash(G\times_H X)\big)$ (resp. $C_0(H\backslash X)$) as given in (1.1). The underlying module M for y is obtained by taking the completion of the $C_c\big(G, C_0(G\times_H X)\big)$ – $C_c\big(H, C_0(X)\big)$ bimodule $C_c\big(G, C_0(X)\big)$ with respect to the formulas:

$$(\varphi\cdot\eta)(s)(x) = \int_G \varphi(t)(s,x)\eta(t^{-1})(x)dt$$

$$\langle\eta_1,\eta_2\rangle(u)(x) = \int_G \Delta_G(t)^{1/2}\Delta_H(t)^{-1/2}\overline{\eta_1(t)(x)}\eta_2(tu)(u^{-1}x)dt$$

$$(\eta\cdot f)(s)(x) = \Delta_G(s)^{1/2}\Delta_H(s)^{-1/2}\int_H \eta(su)(u^{-1}x)f(u^{-1})(u^{-1}x)du,$$

where $\varphi\in C_c\big(G, C_0(G\times_H X)\big)$, $\eta_1,\eta_2\in C_c\big(G, C_0(X)\big)$ and $f\in C_c\big(H, C_0(X)\big)$. Consider the assignment $\Phi:\ C_c(G\times_H X)\odot C_c\big(G, C_0(X)\big)\to C_c(X)$ defined by

$$\Phi(\xi\otimes\eta)(x) = \int_G \xi(t^{-1},x)\eta(t^{-1})(x)\Delta_G(t)^{-1/2}dt,$$

$\xi\in C_c(G\times_H X)$, $\eta\in C_c\big(G, C_0(X)\big)$. One can check that this map extends to a well defined morphism $\Phi:\ \Lambda_{G\times_H X,G}\otimes_{C_0(G\times_H X)\rtimes G} M\to\Lambda_{X,H}$ which respects the corresponding left and right actions and which is isometric with respect to the $C_0(X)\rtimes H$-valued inner products. To see that it is also surjective let $c: G\cdot X\to[0,\infty[$ be a continuous function such that c^2 is a cut-off function for the proper G-space $G\cdot X$ (cf. proof of Lemma 1.5). Set $\xi(s,x)=c(sx)$. For any $\zeta\in C_c(X)$, set $\eta(s)(x)=c(sx)\zeta(x)\Delta_G(s)^{1/2}$. Then $\zeta=\Phi(\xi\otimes\eta)\in\Lambda_{X,H}$. This proves (2.4). □

As a direct consequence of Theorem 2.2 and Proposition 2.3 we get

Theorem 2.5. *Let H be a closed subgroup of G and let B be an H-algebra. Then the following statements are equivalent:*

(i) H satisfies BCC (resp. BCI, resp. BCS) for B;

(ii) G satisfies BCC (resp. BCI, resp. BCS) for $\operatorname{Ind}_H^G B$.

In particular, if G satisfies BCC (resp. BCI, resp. BCS) for all coefficients, the same is true for H.

We say that a group G satisfies the Baum-Connes conjecture with *abelian coefficients* if the assembly map

$$\mu_{G,A}:\mathrm{K}_*^{\mathrm{top}}(G;A)\to K_*(A\rtimes_r G)$$

is an isomorphism for every commutative C^*-algebra A. Since any commutative H-algebra induces to a commutative G-algebra, we get the following direct corollary of Theorem 2.5:

Corollary 2.6. *Let H be a closed subgroup of G. If G satisfies the Baum-Connes conjecture with abelian coefficients, then the same is true for H.*

3 The Baum-Connes conjecture for group extensions

In this section we want to present our new results on the stability of the Baum-Connes conjecture for group extensions. For this we have to recall from [6] the definition of the partial assembly map

$$\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N),$$

where $B \rtimes_r N$ is equipped with the decomposition twisted action of (G,N). Let B be any G-algebra and let X be a G-compact proper G-space. Let

$$\Lambda_{X,N}^{G,N} \in \mathrm{KK}_0^{G,N}\big(C_0(N\backslash X), C_0(X) \rtimes N\big)$$

be the fundamental class associated to X as described in §1, (1.2). The composition of maps

$$\begin{tikzcd} \mathrm{KK}_*^G\big(C_0(X),B\big) \arrow[r, dashed] \arrow[dr, "j_{\{e\},r}^N"'] & \mathrm{KK}_*^{G,N}\big(C_0(N\backslash X), B \rtimes_r N\big) \\ & \mathrm{KK}_*^{G,N}\big(C_0(X) \rtimes N, B \rtimes_r N\big) \arrow[u, "\Lambda_{X,N}^{G,N}\otimes\cdot"'] \end{tikzcd}$$

determines a map

$$\nu[X] : \mathrm{KK}_*^G\big(C_0(X),B\big) \to \mathrm{KK}_*^{G,N}\big(C_0(N\backslash X), B \rtimes_r N\big). \tag{3.1}$$

Now observe that if X is a proper G-space, then $N\backslash X$ is a proper G/N-space, and therefore there exists a homotopically unique continuous G/N-equivariant map $F : N\backslash X \to \mathcal{E}(G/N)$. In particular, there exists a homotopically unique continuous G/N-map $F : N\backslash\mathcal{E}(G) \to \mathcal{E}(G/N)$.

Definition 3.1. Let $F : N\backslash\mathcal{E}(G) \to \mathcal{E}(G/N)$ be as above. For each G-compact subset $X \subseteq \mathcal{E}(G)$ let

$$\mu_{N,B}^{G,N}[X] = F^* \circ \nu[X] : \ \mathrm{KK}_*^G\big(C_0(X),B\big) \to \mathrm{KK}_*^{G,N}\Big(C_0\big(F(N\backslash X)\big), B \rtimes_r N\Big).$$

Then it follows from Lemma 1.5 that the maps $\mu_{N,B}^{G,N}[X]$ are compatible with respect to taking inclusions, and, therefore, they determine a well defined homomorphism

$$\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N).$$

$\mu_{N,B}^{G,N}$ is called the *partial assembly map* for (G,N) with coefficient algebra B.

The following result was one of the main outcomes of [6], and it is central for the investigations in this section. Recall that if B is a G-algebra and N is a closed normal subgroup of G, then $B \rtimes_r G$ is canonically isomorphic to $B \rtimes_r N \rtimes_r (G,N)$.

Proposition 3.2 (cf. [6, Proposition 5.15]). *The diagram*

$$\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(G;B) & \xrightarrow{\mu_{G,B}} & \mathrm{K}_*(B \rtimes_r G) \\
\mu_{N,B}^{G,N}\Big\downarrow & & \Big\uparrow\cong \\
\mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N) & \xrightarrow{\mu_{G/N, B\rtimes_r N}} & \mathrm{K}_*\big((B \rtimes_r N) \rtimes_r (G,N)\big)
\end{array}$$

commutes. Thus, if the partial assembly map

$$\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N)$$

is bijective, then G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.

Proposition 3.2 gives a strong motivation to study the conditions under which the partial assembly map is an isomorphism. The main technical result in this direction is the following theorem. The proof will be given in §5 below.

Theorem 3.3. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N has a compact open subgroup. Let B be any G-algebra and assume that for every compact open subgroup $\dot{K}$ of G/N, the subgroup $q^{-1}(\dot{K})$ of G satisfies BCC with coefficients in B. Then the partial assembly map $\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N)$ is bijective.*

As a direct corollary of the theorem and of Proposition 3.2 we get:

Corollary 3.4. *Assume that G, N, G/N and B satisfy all assumptions of Theorem 3.3. Then G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.*

Note that we obtained a similar result in [6, Proposition 7.8] under the additional assumption that G has a γ-element. Although the proof of that special case is easier than the proof of the above result, it can be quite difficult to check the existence of a γ-element in practice. For discrete G, a similar result (without requiring a γ-element) has been obtained by Oyono-Oyono in [22], and the proof of Theorem 3.3, as presented in §5 below, is partly inspired by the ideas of [22].

In [8, Example 3, §5.1] it is shown that if N is a normal subgroup of a group K such that K/N is compact, and if N satisfies the Haagerup property, then K satisfies the Haagerup property. We mentioned earlier (see Remark 1.8) that it follows from the work of Higson, Kasparov and Tu [13, 25] that such groups satisfy BCC. Thus we get

Corollary 3.5. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N has a compact open subgroup. Suppose further that N satisfies the Haagerup property (e.g., if N is amenable). Then, if B is a G-algebra, G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.*

Let G be a group and let G_0 denote the connected component of the identity of G. Then G_0 is a closed normal subgroup of G and G/G_0 is totally disconnected. Thus we may apply the above results to the extension $1 \to G_0 \to G \to G/G_0 \to 1$. In particular, we get

Corollary 3.6. *Assume that G_0 satisfies the Haagerup property and let B be a G-algebra. Then G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/G_0 satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r G_0$.*

In what follows we want to get rid of the condition of G/N having a compact open subgroup. It turns out that, at least if we restrict our attention to property BCC or BCS, we can indeed obtain very far reaching generalizations. A very important tool for this is the use of Theorem 2.2 and its consequences as described in the previous section. The main idea is to reduce to the two cases where the quotient group is either totally disconnected or almost connected (i.e., $(G/N)/(G/N)_0$ is compact). The first case is the one treated above, and the second case was treated in [6] (under the assumption that G has a γ-element). In fact, combining [6, (2) of Proposition 6.7] (note that the injectivity condition in that statement is satisfied by Remark 1.8 if G/N is almost connected) with [6, Proposition 7.6], we get

Theorem 3.7. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N is almost connected and such that G has a γ-element. Let B be a G-algebra, and assume that for the maximal compact subgroup $\dot{K}$ of G/N, the group $q^{-1}(\dot{K})$ satisfies BCC for B. Then the partial assembly map $\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G; B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N)$ of Definition 3.1 is bijective. It then follows that G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.*

As indicated above, we want to combine Theorem 3.7 with Theorem 3.3 in order to cover arbitrary quotients G/N. But before we do this, we want to weaken the assumption on the γ-element in the above theorem. This is done in Lemma 3.9 below, where we show that it is actually enough to assume the existence of a γ-element for the inverse image $K = q^{-1}(\dot{K}) \subseteq G$ of the maximal compact subgroup $\dot{K}$ of G/N. But for the proof of this, we first need another lemma.

For notation: If A is a C^*-algebra and X is a locally compact space, we will write $A(X) := A \otimes C_0(X)$. If A is a G-algebra and X is a G-space, then $A(X)$ carries the diagonal action. Recall that if X is a G-space, K is a closed subgroup of G, and A is a K-algebra, then $\mathrm{Ind}_K^G(A(X)) \cong \big(\mathrm{Ind}_K^G A\big)(X)$ (cf. [14, 3.6]). In fact, both algebras can be viewed as a subalgebra of $C_b(G \times X, A)$: The elements $F \in \mathrm{Ind}_K^G(A(X))$ satisfy the equation $F(gk, x) = k^{-1}\big(F(g, kx)\big)$ and the elements $G \in \big(\mathrm{Ind}_K^G A\big)(X)$ satisfy the equation $G(gk, x) = k^{-1}\big(G(g, x)\big)$. It is then easy to check that

$$\Phi : \mathrm{Ind}_K^G(A(X)) \to \big(\mathrm{Ind}_K^G A\big)(X);\ \Phi(F)(g, x) = F(g, g^{-1}x) \tag{3.2}$$

is the desired isomorphism.

Lemma 3.8. *Let K be a closed subgroup of G, and let A and B be two K-algebras. Let X be a locally compact G-space. Then the following diagram commutes*

$$\begin{tikzcd}
\mathrm{KK}_*^K(A,B) \arrow[r, "i_K^G"] \arrow[d, "p_X^*"'] & \mathrm{KK}_*^G(\mathrm{Ind}_K^G A, \mathrm{Ind}_K^G B) \arrow[d, "p_X^*"] \\
\mathcal{R}\mathrm{KK}_*^K\big(X; A(X), B(X)\big) \arrow[d, "i^{K,G}"'] & \mathcal{R}\mathrm{KK}_*^G\big(X; (\mathrm{Ind}_K^G A)(X), (\mathrm{Ind}_K^G B)(X)\big) \\
\mathcal{R}\mathrm{KK}_*^G\big(G \times_K X; \mathrm{Ind}_K^G(A(X)), \mathrm{Ind}_K^G(B(X))\big) \arrow[ur] &
\end{tikzcd}$$

where p_X^* is induced by the map $p_X : X \to \{pt\}$ (see [14, Proposition 2.20]), i_K^G is the induction morphism defined in [15, Theorem 1 of §5] (see §2 above), $i^{K,G}$ is the induction morphism defined in [14, §3.6], and the bottom slant arrow is obtained by first identifying $G \times_K X$ with $G/K \times X$ via $[g,x] \mapsto (gK, gx)$, then forgetting the action of $C_0(G/K)$, and eventually identifying $\mathrm{Ind}_K^G(A(X))$ with $(\mathrm{Ind}_K^G A)(X)$ (resp. $\mathrm{Ind}_K^G(B(X))$ with $(\mathrm{Ind}_K^G B)(X)$) via the isomorphism of (3.2) above.

Proof. Let $(\mathcal{E}, \Phi, T)$ be any cycle in $\mathbb{E}^K(A,B)$. By construction, the image of the class of this cycle by $p_X^* \circ i_K^G$ is the class in $\mathcal{R}\mathrm{KK}_*^G\big(X; (\mathrm{Ind}_K^G A)(X), (\mathrm{Ind}_K^G B)(X)\big)$ of the cycle

$$P = \big((\mathrm{Ind}_K^G \mathcal{E}) \otimes C_0(X), (\mathrm{Ind}_K^G \Phi) \otimes 1, \tilde{T} \otimes 1\big), \tag{3.3}$$

where the action of $C_0(X)$ is given via the natural inclusions

$$C_0(X) \to M\big((\mathrm{Ind}_K^G A) \otimes C_0(X)\big),\ M\big((\mathrm{Ind}_K^G B) \otimes C_0(X)\big); \quad f \mapsto 1 \otimes f,$$

and where the operator $\tilde{T}$ on $\mathrm{Ind}_K^G \mathcal{E}$ is given by Equation (2.1).
On the other hand, the composition $i^{K,G} \circ p_X^*$ maps the class of $(\mathcal{E}, \Phi, T)$ to the class of the triple

$$Q = \big(\mathrm{Ind}_K^G(\mathcal{E} \otimes C_0(X)), \mathrm{Ind}_K^G(\Phi \otimes 1), \widetilde{T \otimes 1}\big),$$

where the $C_0(G \times_K X)$-actions on the algebras $\mathrm{Ind}_K^G(A \otimes C_0(X))$ (resp. $\mathrm{Ind}_K^G(B \otimes C_0(X))$) are given by

$$(\varphi \cdot F)(g,x) = \varphi([g,x])F(g,x), \text{ for all } \varphi \in C_0(G \times_K X),\ F \in \mathrm{Ind}_K^G(A \otimes C_0(X))$$

(resp. $F \in \mathrm{Ind}_K^G(B \otimes C_0(X))$). We now apply the isomorphism (3.2) to the algebras $\mathrm{Ind}_K^G(A \otimes C_0(X))$ and $\mathrm{Ind}_K^G(B \otimes C_0(X))$. The same formula provides an isomorphism of Hilbert modules $\Psi : \mathrm{Ind}_K^G(\mathcal{E} \otimes C_0(X)) \to (\mathrm{Ind}_K^G \mathcal{E}) \otimes C_0(X)$. If we now identify $G \times_K X$ with $G/K \times X$ as in the lemma, and if we then forget the $C_0(G/K)$ action on the algebras, then a straightforward computation shows that these isomorphisms turn Q into the cycle P of (3.3).

∎

Lemma 3.9. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N is almost connected. Let $\dot{K}$ be the maximal compact subgroup of G/N and let $K = q^{-1}(\dot{K})$. Assume that K has a γ-element. Then G has a γ-element.*

Proof. Let $\mathcal{A}_K$, (resp. D_K, η_K) be the proper algebra (resp. the Dirac, dual-Dirac element) associated to γ_K, the γ-element of K. By Remark 1.8, G/N also has a γ-element, and we write $\gamma_{G/N}$, $\mathcal{A}_{G/N}$, $D_{G/N}$ and $\eta_{G/N}$ accordingly. Let V be the tangent space of G/K at the point eK and let C_V be the Clifford algebra of V. When viewed as a G-algebra (with trivial N-action), the algebra $\mathcal{A}_{G/N}$ is given by $\mathcal{A}_{G/N} = \operatorname{Ind}_K^G C_V$ (denoted $C_\tau(G/K)$ in [14]).
Define

$$D_G := i_K^G\big(\sigma_{C_V}(D_K)\big) \otimes_{\mathcal{A}_{G/N}} D_{G/N} \in \mathrm{KK}_0^G\left(\operatorname{Ind}_K^G(\mathcal{A}_K \otimes C_V), \mathbb{C}\right) \quad \text{and}$$
$$\eta_G := \eta_{G/N} \otimes_{\mathcal{A}_{G/N}} i_K^G\big(\sigma_{C_V}(\eta_K)\big) \in \mathrm{KK}_0^G\left(\mathbb{C}, \operatorname{Ind}_K^G(\mathcal{A}_K \otimes C_V)\right),$$

where $D_{G/N}$ and $\eta_{G/N}$ are viewed as elements of the respective KK^G-groups by inflating the actions of G/N to G, and $i_K^G : \mathrm{KK}_*^K(A,B) \to \mathrm{KK}_*^G(\operatorname{Ind}_K^G A, \operatorname{Ind}_K^G B)$ denotes Kasparov's induction homomorphism. Since $\mathcal{A}_K$ is K-proper, the algebra $\operatorname{Ind}_K^G(\mathcal{A}_K \otimes C_V)$ is G-proper.
Thus, to see that $\gamma_G = \eta_G \otimes D_G \in \mathrm{KK}_0^G(\mathbb{C},\mathbb{C})$ is a γ-element for G (cf. Definition 1.7), it suffices to check that $p_X^*(\gamma_G) = 1 \in \mathcal{R}\mathrm{KK}_0^G\big(X; C_0(X), C_0(X)\big)$ for every proper G-space X, where p_X denotes the map from X to the one-point set.
Let $z = \sigma_{C_V}(\gamma_K) \in \mathrm{KK}_0^K(C_V, C_V)$. Since X is a proper K-space, it follows that $p_X^*(\gamma_K) = 1$, and, therefore, that $p_X^*(z) = 1 \in \mathcal{R}\mathrm{KK}^K\big(X; C_V(X), C_V(X)\big)$. It then follows that $i^{K,G} \circ p_X^*(z) = 1$, and hence, by Lemma 3.8, that

$$p_X^* \circ i_K^G\big(\sigma_{C_V}(\gamma_K)\big) = p_X^* \circ i_K^G(z) = 1 \in \mathcal{R}\mathrm{KK}^G\big(X; (\operatorname{Ind}_K^G C_V)(X), (\operatorname{Ind}_K^G C_V)(X)\big).$$

On the other hand, the map $p_X : X \to \{\mathrm{pt}\}$ factors G/N-equivariantly through $N\backslash X$, which is a proper G/N-space. Thus, $p_X^*(\gamma_{G/N}) = 1 \in \mathcal{R}\mathrm{KK}_0^G\big(X; C_0(X), C_0(X)\big)$. But this implies

$$\begin{aligned} p_X^*(\gamma_G) &= p_X^*\big(\eta_{G/N} \otimes i_K^G\big(\sigma_{C_V}(\gamma_K)\big) \otimes D_{G/N}\big) \\ &= p_X^*\big(\eta_{G/N}\big) \otimes p_X^*\big(i_K^G(\sigma_{C_V}(\gamma_K))\big) \otimes p_X^*\big(D_{G/N}\big) \\ &= p_X^*\big(\eta_{G/N}\big) \otimes p_X^*\big(D_{G/N}\big) = p_X^*(\gamma_{G/N}) = 1. \end{aligned}$$

□

As a consequence of the above lemma and of Theorem 3.7 we obtain

Corollary 3.10. *Assume that $1 \to N \to G \xrightarrow{q} G/N \to 1$ is a group extension such that G/N is almost connected. Assume further that the inverse image $K = q^{-1}(\dot{K}) \subseteq G$ of the maximal compact subgroup $\dot{K}$ of G/N has a γ-element and satisfies BCC for the given G-algebra B (which is always true if N has the Haagerup property). Then G satisfies BCC (resp. BCI, resp. BCS) for B if and only if G/N satisfies BCC (resp. BCI, resp. BCS) for $B \rtimes_r N$.*

The following proposition is the main step for linking our previous results in order to cover general quotients G/N:

Proposition 3.11. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups and let B be a G-algebra. Denote by $(G/N)_0$ the connected component of the identity in G/N and let $M := q^{-1}\big((G/N)_0\big) \subseteq G$. If $\ddot{H}$ is a compact subgroup of G/M, we denote by $\dot{H} \subset G/N$ the inverse image of $\ddot{H}$ in G/N, and we let H denote the inverse image of $\ddot{H}$ in G. Then the following are true:*

(i) If for every $\ddot{H}$ as above, the group H satisfies BCC for B, then the partial assembly map for (G,M) with coefficients in B is bijective.

(ii) If G/N satisfies BCC (resp. BCS) for every coefficient algebra, then G/M satisfies BCC (resp. BCS) for $B \rtimes_r M$.

Proof. For (i), let us consider the extension $1 \to M \to G \xrightarrow{p} G/M \to 1$. Note that G/M is isomorphic to $(G/N)/(M/N) \cong (G/N)/(G/N)_0$, and hence it is totally disconnected. The condition in (i) is then precisely what we need to apply Theorem 3.3 to this extension.
For (ii), we write G/N as an extension of M/N by G/M:

$$1 \to M/N \to G/N \xrightarrow{r} G/M \to 1. \tag{3.4}$$

Note that G/M is totally disconnected and that the crossed product $B \rtimes_r M$ is isomorphic to $(B \rtimes_r N) \rtimes_r (M,N)$. Thus, applying Proposition 3.2 to extension (3.4), the result will follow if the partial assembly map corresponding to (3.4) with coefficients in $B \rtimes_r N$ is a bijection: If G/N satisfies BCC (resp. BCS) for the coefficient algebra $B \rtimes_r N$, G/M will satisfy BCC (resp. BCS) for the crossed product $B \rtimes_r M \cong (B \rtimes_r N) \rtimes_r (M,N)$.
To see that the partial assembly map for (3.4) with coefficients in $B \rtimes_r N$ is bijective, we apply Theorem 3.3 to this extension. It then follows that it is enough to check that, whenever $\ddot{H}$ is a compact subgroup of G/M, the group $\dot{H} \subseteq G/N$ satisfies BCC for $B \rtimes_r N$. We do this by using the hereditarity result of Theorem 2.5: $\dot{H}$ is a subgroup of G/N which is assumed to satisfy (at least) BCS with arbitrary coefficients. Thus, Theorem 2.5 implies that $\dot{H}$ satisfies (at least) BCS, too. Since $\dot{H}$ is almost connected (as a compact extension of the connected group $M/N \cong (G/N)_0$), it also satisfies BCI for arbitrary coefficient algebras by Remark 1.8. □

We now formulate and prove our extension result for arbitrary quotients G/N.

Theorem 3.12. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups and let B be any G-algebra. Assume that for every compact subgroup $\dot{C}$ of G/N, the group $C = q^{-1}(\dot{C})$ has a γ-element and satisfies BCC for B. Then, if G/N satisfies BCC (resp. BCS) for* ARBITRARY *coefficients, then G satisfies BCC (resp. BCS) for B.*

Proof. We are going to use the reduction argument of Proposition 3.11: Denote by $(G/N)_0$ the connected component of the identity in G/N and let $M = q^{-1}\big((G/N)_0\big) \subseteq G$. Let $\ddot{H}$ be any compact subgroup of G/M, let $\dot{H} \subset G/N$ be the inverse image of $\ddot{H}$ in G/N, and let H denote the inverse image of $\ddot{H}$ in G. Note that $\dot{H}$ is an almost connected group. Let $\dot{K}$ be its maximal compact subgroup. Then $\dot{K}$ is a compact subgroup of G/N, so, by assumption, $K = q^{-1}(\dot{K})$ has a γ-element and satisfies BCC for B. Lemma 3.9 now implies that H itself has a γ-element. Applying Theorem 3.7 to the extension $1 \to N \to H \to \dot{H} \to 1$, it follows that H satisfies BCC for B: Since G/N satisfies BCS for all coefficients, the same is true for $\dot{H} \subseteq G/N$ by Theorem 2.5. Since $\dot{H}$ is almost connected, it also satisfies BCI. Thus Theorem 3.7 applies.
We can now apply Proposition 3.11: By (i), the partial assembly map for (G, M) with coefficients in B in bijective, and, by (ii), G/M satisfies BCC (resp. BCS) for $B \rtimes_r M$. Thus, Proposition 3.2 implies that G satisfies BCC (resp. BCS) for B. $\square$

REMARK 3.13. Note that the statement of Theorem 3.12 is a bit weaker than the statements of Theorems 3.3 and 3.7 above, since it requires that G/N satisfies BCC (resp. BCS) for ALL coefficients, while the previous results only required that G/N satisfies BCC (resp. BCI, BCS) for $B \rtimes_r N$. Also, Theorem 3.12 does not give any information on condition BCI.
If we could show that Theorem 3.7 holds without requiring a γ-element for G, then no reference to γ-elements would be needed in Theorem 3.12 above. However, note that the assumption on the existence of γ-elements for the compact extensions of N in G is much easier to check than the assumption on the existence of a γ-element for G, as we did in [6]. A particularly nice application is given when N satisfies the Haagerup property. As mentioned earlier (see the discussions before Corollary 3.5) the Haagerup property for N implies the Haagerup property for every compact extension of N in G. Thus, all compact extensions of N in G have a γ-element and satisfy BCC (see Remark 1.8). Thus, the following is a direct corollary of Theorem 3.12:

COROLLARY 3.14. *Let N be a closed normal subgroup of G such that N satisfies the Haagerup property (e.g., if N is amenable). Then, if G/N satisfies BCC (resp. BCS), the same is true for G.*

In what follows next, we want to look at the consequences of the above results on the stability of the Baum-Connes conjecture under taking direct products of groups. We need:

LEMMA 3.15. *Let G_1 and G_2 be groups. Suppose that G_1 has a compact open subgroup, or is an almost connected group, and that G_2 has a compact open subgroup. Let B be a $G_1 \times G_2$-algebra, and assume that for every compact subgroup K_2 of G_2, G_1 satisfies BCC for $B \rtimes K_2$. Then the partial assembly map $\mu_{G_1,B}^{G_1\times G_2, G_1} : \mathrm{K}_*^{\mathrm{top}}(G_1 \times G_2; B) \to \mathrm{K}_*^{\mathrm{top}}(G_2; B \rtimes_r G_1)$ of Definition 3.1 is bijective.*

Proof. Take any compact subgroup K_2 of G_2. If G_1 has a compact open subgroup, we can apply Theorem 3.3 to the extension $K_2 \times G_1$ of K_2 by G_1. Since G_1 satisfies BCC for $B \rtimes K_2$ by assumption, it is enough to check that for every compact subgroup K_1 of G_1, the group $K_2 \times K_1$ satisfies BCC (which is clear). It follows that for every compact subgroup K_2 of G_2, $G_1 \times K_2$ satisfies BCC for B. Thus the result follows from applying Theorem 3.3 to $G_1 \times G_2$.
If G_1 is almost connected, the same is true for $G_1 \times K_2$ (since K_2 is compact), so $G_1 \times K_2$ has a γ-element by Remark 1.8. Replacing Theorem 3.3 by Theorem 3.7 in the above argument gives the result. □

Remark 3.16. In the prove of the theorem below, we shall also need a twisted version of the above lemma, i.e., a version in which the quotient group G/N has a product structure as above. However, this extension follows from the above lemma by the result of [10] that every (G, N)-algebra is Morita equivalent to some G/N-algebra.

Theorem 3.17. *Let G_1 and G_2 be two groups. Then the following statements are true:*

(i) The product group $G = G_1 \times G_2$ satisfies BCC if and only if G_1 and G_2 satisfy BCC.

(ii) Suppose that G_1 satisfies BCC. Then $G = G_1 \times G_2$ satisfies BCS if and only if G_2 satisfies BCS.

(iii) Suppose that G_1 has a compact open subgroup, or is almost connected. Suppose further that G_2 has a compact open subgroup. If G_1 satisfies BCC and G_2 satisfies BCI, then $G = G_1 \times G_2$ satisfies BCI.

Proof. We first prove (i) and (ii). If $G = G_1 \times G_2$ satisfies BCC (resp. BCS), the same is true for G_1 and G_2 by Theorem 2.5. Assume now that G_1 satisfies BCC. Let G_0 (resp. $G_{1,0}$, $G_{2,0}$) denote the connected component of G (resp. G_1, G_2). It is clear that $G_0 = G_{1,0} \times G_{2,0}$. Consider the extension $1 \to G_0 \to G \to G/G_0 \to 1$. The quotient group is totally disconnected. Let B be any G-algebra. By Corollary 3.4, to see that G satisfies BCS (resp. BCC) for B, it is enough to show that

(a) any compact extension of G_0 satisfies BCC, and

(b) G/G_0 satisfies BCS (resp. BCC) for $B \rtimes_r G_0$.

For (a), note that if L is a compact extension of G_0, L is contained in a direct product $L_1 \times L_2$, where L_1 is a compact extension of $G_{1,0}$ and L_2 is a compact extension of $G_{2,0}$. Being a subgroup of G_1 (resp. G_2), L_1 (resp. L_2) satisfies BCS by Theorem 2.5. Both groups being almost connected, they also satisfy BCI, whence BCC. Consider the extension $1 \to L_1 \times \{e\} \to L_1 \times L_2 \to L_2 \to 1$. By Theorem 3.7, to see that $L_1 \times L_2$ satisfies BCC (and hence that L satisfies BCC by Theorem 2.5), it suffices to check that $L_1 \times K_2$ satisfies BCC whenever K_2 is a compact subgroup of G_2. To see this, we use again Theorem 3.7 to reduce to the group $K_1 \times K_2$, where K_1 is an arbitrary compact subgroup of G_1. But compact groups satisfy BCC.

In order to check (b), first observe that we just saw in particular, that compact extensions of $G_{1,0}$ in G_1 (resp. of $G_{2,0}$ in G_2) satisfy BCC. Hence it follows from Theorem 3.3 and Proposition 3.2 that $G_1/G_{1,0}$ satisfies BCC with twisted coefficients in $A \rtimes_r G_{1,0}$, where A_1 is any G_1-algebra. A similar result holds for $G_2/G_{2,0}$.

In particular, it follows that $G_1/G_{1,0}$ satisfies BCC with (twisted) coefficients in

$$B \rtimes_r G_0 \rtimes_r \dot{K}_2 = B \rtimes_r G_{2,0} \rtimes_r \dot{K}_2 \rtimes_r G_{1,0} = B \rtimes_r K_2 \rtimes_r G_{1,0},$$

where $\dot{K}_2$ is any compact subgroup of $G_2/G_{2,0}$ and K_2 denotes its inverse image in G_2. Note that in the above formula we took the freedom to write the twisted crossed products by the pairs $(K_2 \times G_{1,0}, G_0)$ (in the first crossed product) and $(K_2, G_{2,0})$ (in the second crossed product) simply as crossed products by the common quotient $\dot{K}_2$. Using the definition of the twisted crossed products (see [6]), it is fairly straightforward (but tedious) to check that all three crossed products in the above formula do coincide.

A similar argument shows that $G_2/G_{2,0}$ satisfies BCS (resp. BCC) with coefficients in the algebra

$$B \rtimes_r G_0 \rtimes_r G_1/G_{1,0} = B \rtimes_r G_1 \rtimes_r G_{2,0}.$$

Using the twisted version of Lemma 3.15 (replacing G_1 by $G_1/G_{1,0}$, G_2 by $G_2/G_{2,0}$ and B by $B \rtimes_r G_0 = B \rtimes_r G_{1,0} \rtimes_r G_{2,0}$), we see that the partial assembly map for the extension of $G_1/G_{1,0}$ by $G_2/G_{2,0}$ and with coefficients in $B \rtimes_r G_0$ is bijective. Composing this with the assembly map for $G_2/G_{2,0}$ with coefficient algebra $B \rtimes_r G_0 \rtimes_r G_1/G_{1,0} = B \rtimes_r G_1 \rtimes_r G_{2,0}$, we get (b).

We have now completed the proofs of (i) and (ii). (iii) is a direct consequence of Lemma 3.15 and Proposition 3.2.

□

4 An example for the Baum-Connes conjecture with trivial coefficients

In this section we want to show how the results of the previous sections may be combined in order to produce new examples for the validity of the Baum-Connes conjecture without coefficients. The methods we use here give a hint into a direction of a more general “Mackey-machine” for computing the topological K-theory of group extensions via an induction process.

The basic idea is to use our partial assembly map to write $\mathrm{K}_*^{\mathrm{top}}(G;\mathbb{C})$ as $\mathrm{K}_*^{\mathrm{top}}(G/N; C_r^*(N))$, where N is a closed normal subgroup of G (see the main results of §3 above). In good cases, we might be able to decompose $C_r^*(N)$ into finitely many pieces (i.e., G-invariant subquotients) which are induced from smaller groups, which satisfy the conjecture for the respective coefficient algebras. The bijectivity of the induction homomorphism (see Theorem 2.2) then gives the conjecture for the original pieces, and, using excision, we end up with the desired result for G. Below, we will give some explicit examples for this procedure.

But before we present the examples, we need to mention some results on the functorial properties of the topological K-theory of a fixed group G, viewed as a functor on the category of G-C^*-algebras.

By a result of Kasparov and Skandalis (see [17, Appendix]), it is known that for any proper G-algebra D the functor $A \mapsto \mathrm{KK}^G(D, A)$ is half exact. Replacing D by $C_0(X)$, for X a G-compact subspace of $\mathcal{E}(G)$, and taking the limit over X, implies that the topological K-theory functor $A \mapsto \mathrm{K}_*^{\mathrm{top}}(G; A)$ is half exact, too. Since this functor is also homotopy invariant and satisfies Bott-periodicity, it follows from some general arguments (which, for instance, are outlined in [4, Chapter IX]) that it satisfies excision in the sense that every short exact sequence

$$0 \to I \to A \to A/I \to 0$$

of G-algebras induces a natural six-term exact sequence

$$\begin{array}{ccccc}
\mathrm{K}_0^{\mathrm{top}}(G; I) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(G; A) & \longrightarrow & \mathrm{K}_0^{\mathrm{top}}(G; A/I) \\
{\scriptstyle\partial}\big\uparrow & & & & \big\downarrow{\scriptstyle\partial} \\
\mathrm{K}_1^{\mathrm{top}}(G; A/I) & \longleftarrow & \mathrm{K}_1^{\mathrm{top}}(G; A) & \longleftarrow & \mathrm{K}_1^{\mathrm{top}}(G; I).
\end{array}$$

If G satisfies BCC, then it follows from the half exactness of $\mathrm{K}_*^{\mathrm{top}}(G, \cdot)$ and the naturality of the assembly map that the functor

$$A \mapsto \mathrm{K}_*(A \rtimes_r G)$$

has to be half exact, too. Thus we see that BCC can only hold for G if G is K-*exact* in the sense that for every short exact sequence $0 \to I \to A \to A/I \to 0$ of G-algebras, the natural sequence

$$\mathrm{K}_*(I \rtimes_r G) \to \mathrm{K}_*(A \rtimes_r G) \to \mathrm{K}_*(A/I \rtimes_r G)$$

is exact in the middle term. By the same general arguments as used above, K-exactness of G implies that every short exact sequence of G-algebras induces a natural six-term exact sequence for the K-theories of the reduced crossed products.

Proposition 4.1. *Assume that G is* K*-exact and that $0 \to I \to A \to A/I \to 0$ is a short exact sequence of G-algebras. Let $\partial : \mathrm{K}_{*+1}^{\mathrm{top}}(G; A/I) \to \mathrm{K}_*^{\mathrm{top}}(G; I)$ and $\delta : \mathrm{K}_{*+1}(A/I \rtimes_r G) \to \mathrm{K}_*(I \rtimes_r G)$ denote the boundary maps in the respective* *six-term exact sequences. Then the diagram*

$$\begin{tikzcd} \mathrm{K}^{\mathrm{top}}_{*+1}(G;A/I) \arrow[r, "\partial"] \arrow[d, "\mu_{G,A/I}"'] & \mathrm{K}^{\mathrm{top}}_{*}(G;I) \arrow[d, "\mu_{G,I}"] \\ \mathrm{K}_{*+1}(A/I \rtimes_r G) \arrow[r, "\delta"'] & \mathrm{K}_*(I \rtimes_r G) \end{tikzcd}$$

commutes. In fact, the assembly map commutes with all maps in the respective six-term exact sequences.

Proof. This result follows easily from the naturality of the assembly map and the general construction of the boundary maps: By [4, Theorem 21.4.3], it follows that the boundary maps can be factorized via K-theory maps coming from $*$-homomorphisms $\Phi : C_0(\mathbb{R}) \otimes A/I \to C_q$ and $e : I \to C_q$, where C_q denotes the mapping cone for the quotient map $q : A \to A/I$. More precisely, using suspension, the above diagram splits into the diagram

$$\begin{tikzcd} \mathrm{K}^{\mathrm{top}}_{*}\big(G;C_0(\mathbb{R})\otimes A/I\big) \arrow[r, "\Phi_*"] \arrow[d, "\mu_{G,C_0(\mathbb{R})\otimes A/I}"'] & \mathrm{K}^{\mathrm{top}}_{*}(G;C_q) \arrow[d, "\mu_{G,C_q}"'] & \mathrm{K}^{\mathrm{top}}_{*}(G;I) \arrow[l, "e_*"', "\cong"] \arrow[d, "\mu_{G,I}"] \\ \mathrm{K}_*\big((C_0(\mathbb{R})\otimes A/I) \rtimes_r G\big) \arrow[r, "\Phi_*"'] & \mathrm{K}_*(C_q \rtimes_r G) & \mathrm{K}_*(I \rtimes_r G), \arrow[l, "\cong", "e_*"'] \end{tikzcd}$$

which commutes by the naturality of the assembly map. The result then follows from the fact that the assembly map commutes with Bott periodicity. □

Using the above observations, an easy application of the Five Lemma gives the following general principle:

PROPOSITION 4.2. *Suppose that G is* K*-exact and let $0 \to I \to A \to A/I \to 0$ be a short exact sequence of G-algebras. If G satisfies BCC for two of the algebras I, A and A/I, then it satisfies BCC for all three algebras.*

We are now ready to present our example.

EXAMPLE 4.3. Let $K = \mathbb{R}$ or $\mathbb{C}$. The semi-direct product groups $K^n \rtimes SL_n(K)$, $n \in \mathbb{N}^*$, (where the action of $SL_n(K)$ on K^n is by matrix multiplication) satisfy the Baum-Connes conjecture *without* coefficients (i.e., with coefficient algebra $\mathbb{C}$). We believe that this was known before only for the cases $n \leq 2$.

The proof is by induction on n. For short, let us write $H_n = SL_n(K)$ and $G_n = K^n \rtimes H_n$. For $n = 1$, the conclusion holds. By induction, take $n > 1$ and let us assume that the conclusion holds for $n - 1$.
Since K^n is abelian, hence amenable, it follows from Theorem 3.7 that G_n satisfies the Baum-Connes conjecture for $\mathbb{C}$ if and only H_n satisfies BCC for

$C_r^*(K^n) \cong C_0(K^n)$. Moreover, the Gelfand transform carries the decomposition action of H_n on $C^*(K^n)$ to the action of H_n on $C_0(K^n)$ given by the formula

$$(x \cdot f)(\eta) = f(x^* \cdot \eta), \quad x \in SL_n(K), \eta \in K^n,$$

where x^* denotes the adjoint of the matrix x.
There are two orbits under this action of H_n on K^n: $\{0\}$ and $K^n \smallsetminus \{0\}$. Let $\eta = (1, 0, \ldots, 0)^t \in K^n$. Then a short computation shows that the stabilizer of η under the above action is isomorphic to $K^{n-1} \rtimes SL_{n-1}(K) = G_{n-1}$, and therefore we get

$$C_0(K^n \setminus \{0\}) \cong C_0(H_n/G_{n-1}) \cong \mathrm{Ind}_{G_{n-1}}^{H_n} \mathbb{C}.$$

By [19], we know that the semi-simple group $SL_n(K)$ satisfies BCC for $\mathbb{C}$ and by the induction assumption we know that G_{n-1} also satisfies BCC for $\mathbb{C}$. Using Theorem 2.5 it follows that H_n satisfies BCC for $C_0(K^n \setminus \{0\})$. Thus, applying Proposition 4.2 to the short exact sequence $0 \to C_0(K^n \setminus \{0\}) \to C_0(K^n) \to \mathbb{C} \to 0$ gives the result.

Remark 4.4. A similar argument can be used to show that $\mathbb{Q}_p^n \rtimes GL_n(\mathbb{Q}_p)$ satisfies BCC with coefficients in $\mathbb{C}$. Since $GL_n(\mathbb{Q}_p)$ can be written as the product $\mathbb{Q}_p 1 \cdot GL_n(\mathcal{O})$, where $GL_n(\mathcal{O})$ is the compact group of invertible matrices with p-adic integer entries, it is a (non-direct) product of an amenable group and a compact group, and therefore exact by a standard argument (e.g., see [18]). Moreover, by results of Baum, Higson and Plymen [3] and Lafforgue [19], it is known that $GL_n(\mathbb{Q}_p)$ satisfies BCC for $\mathbb{C}$. Now the same procedure as used in the above example, using Theorem 3.3 instead of Theorem 3.7, gives the result.

5 Proof of Theorem 3.3

In this section we give the proof of Theorem 3.3. As indicated in the introduction, some of the main ideas (and intermediate results) used in this proof will also be applied in the proofs of the bijectivity of the induction homomorphism as given in §6, and the continuity of topological K-theory with respect to the coefficients as presented in §7. First we recall the statement of the theorem.

Theorem 3.3. *Let $1 \to N \to G \xrightarrow{q} G/N \to 1$ be an extension of groups such that G/N has a compact open subgroup. Let B be a G-algebra and assume that for every compact open subgroup $\dot{K}$ of G/N, the subgroup $q^{-1}(\dot{K})$ of G satisfies BCC for B. Then the partial assembly map $\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G; B) \to \mathrm{K}_*^{\mathrm{top}}(G/N; B \rtimes_r N)$ of Definition 3.1 is bijective.*

As mentioned in the introduction, the proof relies on a special realization of a universal example for the proper actions of G. In order to obtain this special realization, we start with the following easy observation:

LEMMA 5.1. *Let G_1 and G_2 be groups. If $\mathcal{E}(G_1)$ (resp. $\mathcal{E}(G_2)$) is a realization of the universal example for the proper actions of G_1 (resp. G_2), then $\mathcal{E}(G_1) \times \mathcal{E}(G_2)$, equipped with the product action of $G_1 \times G_2$, is a realization of the universal example for the proper actions of $G_1 \times G_2$.*

Proof. The action of $G_1 \times G_2$ on $\mathcal{E}(G_1) \times \mathcal{E}(G_2)$ is clearly proper. To show the universal property, take any space X endowed with a proper action of $G_1 \times G_2$. This action restricts to proper actions of G_1 and G_2 on X, so there exist continuous G_i-maps $F_i : X \to \mathcal{E}(G_i)$, $i = 1, 2$. Then $F : X \to \mathcal{E}(G_1) \times \mathcal{E}(G_2); F(x) = (F_1(x), F_2(x))$ is a continuous $G_1 \times G_2$-equivariant map. Conversely, every continuous $G_1 \times G_2$-map is of this form. Thus the uniqueness (up to homotopy), follows from the uniqueness of $\mathcal{E}(G_1)$ and $\mathcal{E}(G_2)$. $\square$

If a group G has at least one compact open subgroup, the constructions in [17] provide a realization of $\mathcal{E}(G)$ as a simplicial complex (the constructions in [17] are given for discrete G; the adaptations for the more general case of groups with compact open subgroup are given in the discussion following [6, Lemma 7.10]). Summing up the results of [17] and [6, §7], we obtain

PROPOSITION 5.2. *Let G be a group having a compact open subgroup K. Then there exists a realization $\mathcal{E}(G)$ of the universal example for proper actions of G, such that*

(i) $\mathcal{E}(G)$ is the geometric realization of a locally finite simplicial complex on which G acts properly and simplicially,

(ii) If S is any simplex of $\mathcal{E}(G)$, $\overset{o}{S}$ its interior, and $g \in G$, then either g acts as the identity on S or $g\overset{o}{S} \cap \overset{o}{S} = \emptyset$.

In this section we will from now on assume that N is a closed normal subgroup of G such that G/N has a compact open subgroup. Thus, in what follows next, we can always assume that $\mathcal{E}(G/N)$ has the structure of a simplicial complex as described in Proposition 5.2 above. The following lemma shows how this provides a special realization for the universal example for the proper actions of G:

LEMMA 5.3. *Let $\mathcal{E}(G)$ be a universal example for the proper actions of G. Then the Cartesian product $\mathcal{E}(G) \times \mathcal{E}(G/N)$, endowed with the diagonal action $g(x, y) = (gx, \dot{g}y)$, is also a universal example for the proper actions of G.*

Proof. $\mathcal{E}(G) \times \mathcal{E}(G/N)$ is a universal example for $G \times G/N$ by Lemma 5.1. Because G can be seen as a closed subgroup of $G \times G/N$ via the map $g \mapsto (g, \dot{g})$, the result follows from Lemma 2.4. $\square$

The main advantage of taking $\mathcal{E}(G) \times \mathcal{E}(G/N)$ as a universal example for the proper actions of G comes from the fact that the simplicial structure of $\mathcal{E}(G/N)$ allows us to use induction arguments on the dimension of simplices and to "compress" to smaller subgroups.

Remark 5.4. In what follows we will denote by π_2 the projection of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ onto the second factor $\mathcal{E}(G/N)$. Moreover, if we restrict the diagonal G-action on $\mathcal{E}(G) \times \mathcal{E}(G/N)$ to N, the quotient space $N\backslash(\mathcal{E}(G) \times \mathcal{E}(G/N))$ can be identified with $(N\backslash\mathcal{E}(G)) \times \mathcal{E}(G/N)$, and we will always denote by p_2 the second projection of $(N\backslash\mathcal{E}(G)) \times \mathcal{E}(G/N)$ onto $\mathcal{E}(G/N)$. Note that by the universal property of $\mathcal{E}(G/N)$, $p_2 : N\backslash(\mathcal{E}(G) \times \mathcal{E}(G/N)) \to \mathcal{E}(G/N)$ is *the* unique (up to G/N-equivariant homotopy) continuous G/N-equivariant map.

We are now using $\mathcal{E}(G) \times \mathcal{E}(G/N)$ to compute the topological K-theory of G. We start with

Definition 5.5. Let Y be a G/N-compact subset of $\mathcal{E}(G/N)$. Then we define $\mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B)$ to be the inductive limit

$$\mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B) := \lim_{X} \mathrm{KK}_*^G\big(C_0(X), B\big),$$

over all G-compact subspaces X of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ which satisfy $\pi_2(X) = Y$.

Lemma 5.6. *Assume that G/N has a compact open subgroup and that $\mathcal{E}(G/N)$ has the simplicial structure described in Proposition 5.2. Let $\mathcal{F}$ denote the family of subsets Y of $\mathcal{E}(G/N)$ such that Y is the G/N-saturation of a finite union of simplices of $\mathcal{E}(G/N)$. Then the topological* K*-theory of G with coefficients in the G-algebra B can be computed by the formula*

$$\mathrm{K}_*^{\mathrm{top}}(G;B) = \lim_{Y\in\mathcal{F}} \mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B).$$

Proof. Using $\mathcal{E}(G) \times \mathcal{E}(G/N)$ as a realization of the universal example for the proper actions of G, $\mathrm{K}_*^{\mathrm{top}}(G;B)$ can be computed as $\mathrm{K}_*^{\mathrm{top}}(G;B) = \lim_Z \mathrm{KK}_*^G\big(C_0(Z), B\big)$, where Z runs through the family of G-compact subsets of $\mathcal{E}(G) \times \mathcal{E}(G/N)$. Clearly, any such Z is contained in a G-compact subset X of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ satisfying $\pi_2(X) = Y$ for some $Y \in \mathcal{F}$: Choose any $Y \in \mathcal{F}$ such that $\pi_2(Z) \subseteq Y$. Y can be written as $Y = G/N \cdot K$, where K is a compact subset of $\mathcal{E}(G/N)$. Take any point x in $\mathcal{E}(G)$ and put $X = Z \cup G \cdot \big(\{x\} \times K\big) \subseteq \mathcal{E}(G) \times \mathcal{E}(G/N)$. $\square$

To each piece of the above decomposition of $\mathrm{K}_*^{\mathrm{top}}(G;B)$ via the elements $Y \in \mathcal{F}$ corresponds a piece of the partial assembly map for (G,N): If Y is a G-compact subset of $\mathcal{E}(G/N)$ and X is a G-compact subset of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ (viewed as a universal example for the proper actions of G) such that $\pi_2(X) = Y$, we obtain from Definition 3.1 a well defined morphism

$$\mu_{N,B}^{G,N}[X] : \mathrm{KK}_*^G(C_0(X), B) \to \mathrm{KK}_*^{G,N}(C_0(Y), B \rtimes_r N)$$

(note that the map $F : N\backslash\mathcal{E}(G) \to \mathcal{E}(G/N)$ of Definition 3.1 is given by the projection $p_2 : N\backslash\big(\mathcal{E}(G) \times \mathcal{E}(G/N)\big) \to \mathcal{E}(G/N)$, and that $p_2(X) = Y$). It follows from Lemma 1.5 that the maps $\mu_{N,B}^{G,N}[X]$ commute with the maps induced by taking inclusions. Thus we may define

DEFINITION 5.7. For any G/N-compact subset Y of $\mathcal{E}(G/N)$ we define

$$\mu_{N,B}^{G,N}\langle Y\rangle : \mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B) \to \mathrm{KK}_*^{G,N}\big(C_0(Y), B \rtimes_r N\big)$$

as the map which is obtained inductively from the morphisms

$$\mathrm{KK}_*^{G}\big(C_0(X), B\big) \xrightarrow{\mu_{N,B}^{G,N}[X]} \mathrm{KK}_*^{G,N}\big(C_0(Y), B \rtimes_r N\big),$$

where X runs through all G-compact subspaces of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ which satisfy $\pi_2(X) = Y$.

We observe that not only $\mathrm{K}_*^{\mathrm{top}}(G;B)$ but also the partial assembly map associated to (G,N) can be recovered from the decomposition described above. This follows directly from the definitions.

LEMMA 5.8. *Let N, G, $\mathcal{E}(G/N)$, and $\mathcal{F}$ be as in Lemma 5.6. Then the partial assembly map $\mu_{N,B}^{G,N} : \mathrm{K}_*^{\mathrm{top}}(G;B) \to \mathrm{K}_*^{\mathrm{top}}(G;B \rtimes_r N)$ of Definition 3.1 can be computed inductively from the maps*

$$\mu_{N,B}^{G,N}\langle Y\rangle : \mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B) \to \mathrm{KK}_*^{G,N}\big(C_0(Y), B \rtimes_r N\big), \quad Y \in \mathcal{F}.$$

In view of Lemma 5.8, the proof of Theorem 3.3 reduces to the proof of:

PROPOSITION 5.9. *Let N, G, $\mathcal{E}(G/N)$ and $\mathcal{F}$ be as in Lemma 5.6. Let $q : G \to G/N$ be the quotient map and let B be any G-algebra. Assume further that for any compact open subgroup $\dot{K}$ of G/N, the subgroup $q^{-1}(\dot{K})$ of G satisfies BCC for B. Then, for any $Y \in \mathcal{F}$, the map*

$$\mu_{N,B}^{G,N}\langle Y\rangle : \mathrm{K}_*^{\mathrm{top}}\langle Y\rangle(G;B) \to \mathrm{KK}_*^{G,N}\big(C_0(Y), B \rtimes_r N\big)$$

is bijective.

To prove Proposition 5.9, we use two ingredients (which goes back to some ideas used in [11, Chapter 12] and [21, §5]). The first is to make an induction on the maximal dimension of the simplices generating Y, using a Mayer-Vietoris argument. For this we need a relative version of Definition 5.7:

DEFINITION 5.10. Let Y be a G/N-compact subset of $\mathcal{E}(G/N)$ and let Y_0 be an open (in the relative topology) G/N-equivariant subset of Y. For any G-compact set $X \subseteq \mathcal{E}(G) \times \mathcal{E}(G/N)$ satisfying $\pi_2(X) = Y$ we put $X_0 = X \cap \pi_2^{-1}(Y_0)$. Consider the composition of maps

$$\begin{tikzcd}
\mathrm{KK}_*^{G}\big(C_0(X_0), B\big) \arrow[r, dashed] \arrow[dr, "{\nu[X_0]}"'] & \mathrm{KK}_*^{G,N}\big(C_0(Y_0), B \rtimes_r N\big) \\
& \mathrm{KK}_*^{G,N}\big(C_0(N\backslash X_0), B \rtimes_r N\big) \arrow[u, "p_2^*"']
\end{tikzcd}$$

where $p_2 : N\backslash(\mathcal{E}(G) \times \mathcal{E}(G/N)) \to \mathcal{E}(G/N)$ denotes the projection onto the second factor and $\nu[X_0]$ is as in Equation (3.1). Using Lemma 1.5 we see that these maps induce a well defined map

$$\mu_{N,B}^{G,N}\langle Y_0\rangle : \lim_X \mathrm{KK}_*^G\big(C_0(X_0), B\big) \to \mathrm{KK}_*^{G,N}\big(C_0(Y_0), B \rtimes_r N\big),$$

on the inductive limit, where X runs through all G-compact subsets of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ which project onto Y.

Remark 5.11. It is important to note that both, the limit $\lim_X \mathrm{KK}_*^G\big(C_0(X_0), B\big)$, and the map $\mu_{N,B}^{G,N}\langle Y_0\rangle$, only depend on the space Y_0 and not on the particular choice of the G-compact set $Y \supseteq Y_0$. To see this, it is enough to observe that, if $Y_0 \subseteq Y \subseteq Y_1$ such that Y and Y_1 are G/N-compact and Y_0 is open in Y_1, then for any G-compact set $X \subseteq \mathcal{E}(G) \times \mathcal{E}(G/N)$ with $\pi_2(X) = Y$, there exists a G-compact set $X_1 \subseteq \mathcal{E}(G) \times \mathcal{E}(G/N)$ with $\pi_2(X_1) = Y_1$ and $X \subseteq X_1$ (then $X_0 \subseteq X_{1,0}$, and, conversely, if X_1 is given, and if we put $X = X_1 \cap \pi_2^{-1}(Y)$, we get $X_{1,0} \subseteq X_0$).

We will make use of Definition 5.10 in our Mayer-Vietoris argument. The second ingredient for the proof of Proposition 5.9 is a reduction argument based on an isomorphism in KK-theory, which makes it possible to simplify the group $\mathrm{KK}_*^{G,N}\big(C_0(Y_0), B \rtimes_r N\big)$, if Y_0, as a G-space, is induced from an open subgroup of G. The following characterization of induced spaces is taken from [9]:

Proposition 5.12 (cf. [9, Corollary 2]). *Let Y be a locally compact G-space and let C be a closed subgroup of G. Then Y is G-homeomorphic to an induced space $G \times_C T$, for some C-space T, if and only if there exists a continuous G-equivariant map $p : Y \to G/C$. In that case, the C-space T can be chosen as $T = p^{-1}(\{eC\}) \subseteq Y$, and a G-homeomorphism is given by the mapping $G \times_C T \to Y; [(s, x)] \mapsto sx$.*

Remark 5.13. Assume that G/N has a compact open subgroup and that $\mathcal{E}(G/N)$ has the simplicial structure of Proposition 5.2. By part (ii) of that proposition, the above characterization of induced spaces shows immediately that any subspace Z of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ which projects onto the G-saturation of the interior $\mathring{S}$ of a simplex S of $\mathcal{E}(G/N)$ is an induced space $Z = G \times_C T$, where C is the stabilizer of S under the action of G and $T = \pi_2^{-1}(\mathring{S}) \cap Z$. Since the action of G/N on $\mathcal{E}(G/N)$ is continuous, simplicial, and proper, C is an open subgroup of G and C/N is a compact subgroup of G/N.

One of the basic ideas of the proof of Theorem 3.3 is to “compress” to the open subgroup $C \subseteq G$ of the above remark. For this we want to use a more general compression isomorphism in twisted equivariant KK-theory, which we are now going to describe.

So assume that C is an open subgroup of G containing the closed normal subgroup N of G, and let A be a (C, N)-algebra. Let $\mathrm{Ind}_C^G A$ denote the induced algebra of A. Then $\mathrm{Ind}_C^G A$ is a (G,N)-algebra in a canonical way: If $\tau : N \to UM(A)$ is the twist for the original C-action on A, then the twist $\mathrm{Ind}\,\tau : N \to UM(\mathrm{Ind}_C^G A)$ for the induced G-action on $\mathrm{Ind}_C^G A$ is given by the formula

$$\big(\mathrm{Ind}\,\tau_n \cdot F\big)(g) = \tau_{g^{-1}ng} \cdot F(g), \quad F \in \mathrm{Ind}_C^G A.$$

Since C is open in G, there exists a canonical (C,N)-equivariant embedding $i_A : A \to \mathrm{Ind}_C^G A$, given by

$$\big(i_A(a)\big)(g) = \begin{cases} g^{-1}(a) & \text{if } g \in C; \\ 0 & \text{if } g \notin C. \end{cases}$$

For any (G,N)-algebra D, the compression homomorphism

$$\mathrm{comp}_C^G : \mathrm{KK}_*^{G,N}\left(\mathrm{Ind}_C^G A, D\right) \to \mathrm{KK}_*^{C,N}(A,D),$$

is then defined as the composition

$$\mathrm{KK}_*^{G,N}\left(\mathrm{Ind}_C^G A, D\right) \overset{\mathrm{res}_C^G}{\to} \mathrm{KK}_*^{C,N}\left(\mathrm{Ind}_C^G A, D\right) \overset{i_A^*}{\to} \mathrm{KK}_*^{C,N}(A,D).$$

It is shown in [5] (extending earlier results of [11] and [21]) that the compression map is an isomorphism if $N = \{e\}$ and C is a compact open subgroup of G. But for our purposes it is necessary to get rid of these assumptions. Thanks to a recent result of Ralf Meyer, this is indeed possible:

PROPOSITION 5.14. *The map*

$$\mathrm{comp}_C^G : \mathrm{KK}_*^{G,N}\left(\mathrm{Ind}_C^G A, D\right) \to \mathrm{KK}_*^{C,N}(A,D)$$

is an isomorphism.

Proof. We first note that the result is invariant under passing to Morita equivalent twisted actions in both variables: First, if we replace D by a Morita equivalent (G,N)-algebra D', say, and if $y \in \mathrm{KK}_0^{G,N}(D,D')$ is the corresponding invertible element, then the statement follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{KK}_*^{G,N}\left(\mathrm{Ind}_C^G A, D\right) & \xrightarrow{\mathrm{comp}_C^G} & \mathrm{KK}_*^{C,N}(A,D) \\ {\scriptstyle \cdot\otimes y}\downarrow\cong & & \cong\downarrow{\scriptstyle \cdot\otimes \mathrm{res}_C^G(y)} \\ \mathrm{KK}_*^{G,N}\left(\mathrm{Ind}_C^G A, D'\right) & \xrightarrow{\mathrm{comp}_C^G} & \mathrm{KK}_*^{C,N}(A,D'). \end{array}$$

Secondly, if we replace A by a Morita equivalent (C,N)-algebra A', and if $x \in \mathrm{KK}_0^{C,N}(A',A)$ denotes the corresponding invertible element, the statement follows from the commutativity of

$$\begin{array}{ccc} \mathrm{KK}_*^{G,N}\big(\mathrm{Ind}_C^G A, D\big) & \xrightarrow{\mathrm{comp}_C^G} & \mathrm{KK}_*^{C,N}(A,D) \\ {\scriptstyle i_C^G(x)\otimes\cdot}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle x\otimes\cdot} \\ \mathrm{KK}_*^{G,N}\big(\mathrm{Ind}_C^G A', D\big) & \xrightarrow{\mathrm{comp}_C^G} & \mathrm{KK}_*^{C,N}(A',D), \end{array}$$

which follows from the equation $[i'_A]\otimes \mathrm{res}_C^G\big(i_C^G(x)\big) = x\otimes[i_A]$ in $\mathrm{KK}_0^{C,N}(A',\mathrm{Ind}_C^G A)$.

Since every twisted action of (G,N) (resp. (C,N)) is Morita equivalent to an ordinary action of G/N (resp. C/N) by [10, Theorem 1], it follows from the results of [6, §3] that we may assume without loss of generality that $N=\{e\}$ and all actions are untwisted. Moreover, since an action $\alpha : C\to \mathrm{Aut}(A)$ is Morita equivalent to the stabilized action $\alpha\otimes\mathrm{Ad}\,\lambda : C\to\mathrm{Aut}\big(A\otimes\mathcal{K}(L^2(C))\big)$, where λ denotes the left regular representation of C (a Morita equivalence is given by $(A\otimes L^2(C),\alpha\otimes\lambda)$), we can use [20, Proposition 3.2] in order to assume without loss of generality that every element $\alpha\in\mathrm{KK}^C(A,D)$ can be represented by a Kasparov triple $(\mathcal{E},\Phi,T)$, such that $\Phi(A)\mathcal{E}=\mathcal{E}$ and such that T is a C-equivariant operator on $\mathcal{E}$. Moreover, by [20, Proposition 3.4], we can also assume that the homotopies between equivalent triples have the same properties.

Using these reductions, we can now follow the constructions of [5, Lemma 4.11] (see also [21]) to build an inverse

$$\mathrm{inf}_C^G : \mathrm{KK}_*^C(A,D)\to\mathrm{KK}_*^G(\mathrm{Ind}_C^G A, D)$$

for the compression homomorphism comp_C^G: Let $\alpha\in\mathrm{KK}_*^C(A,D)$ be represented by a Kasparov triple $(\mathcal{E},\Phi,T)$ with the properties as described above. Consider the complex vector space E consisting of all continuous functions $\xi : G\to\mathcal{E}$ such that

- $\xi(gc)=c^{-1}(\xi(g))$ for all $g\in G$, $c\in C$;
- the map $gC\mapsto\|\xi(g)\|$ has finite support in G/C.

Then E becomes a G-equivariant pre-Hilbert D-module by defining the D-valued inner product, the right D-action on E, and the action of G on E by

$$\langle\xi,\eta\rangle_D=\sum_{\dot g\in G/C} g(\langle\xi(g),\eta(g)\rangle_D),\quad (\xi\cdot d)(g)=\xi(g)\cdot g^{-1}(d),\quad\text{and}$$

$$(g\cdot\xi)(g')=\xi(g^{-1}g'),$$

for all $g,g'\in G$, $\xi,\eta\in E$ and $d\in D$. Let $\tilde{\mathcal{E}}$ denote the completion of E and define $\tilde\Phi:\mathrm{Ind}_C^G A\to\mathcal{L}(\tilde{\mathcal{E}})$ and an operator $\tilde T\in\mathcal{L}(\tilde{\mathcal{E}})$ by

$$\big(\tilde\Phi(F)\cdot\xi\big)(g)=\Phi\big(F(g)\big)\cdot\big(\xi(g)\big)\quad\text{and}\quad\big(\tilde T\xi\big)(g)=T\big(\xi(g)\big),$$

for $F \in \operatorname{Ind}_C^G A$ and $\xi \in E$. We want to define

$$\inf_C^G\big([(\mathcal{E},\Phi,T)]\big) = [(\tilde{\mathcal{E}},\tilde{\Phi},\tilde{T})] \in \mathrm{KK}^G\big(\operatorname{Ind}_C^G A, D\big). \tag{5.1}$$

For this we first have to show that $(\tilde{\mathcal{E}},\tilde{\Phi},\tilde{T})$ is a Kasparov triple in $\mathbb{E}^G(\operatorname{Ind}_C^G A, D)$. Since $\tilde{T}$ is clearly G-equivariant, it is enough to check that $[\tilde{T},\tilde{\Phi}(F)]$, $(\tilde{T}^2-1)\tilde{\Phi}(F)$, and $(\tilde{T}^*-\tilde{T})\tilde{\Phi}(F)$ are compact operators on $\tilde{\mathcal{E}}$ for all $F \in \operatorname{Ind}_C^G A$. Since $\tilde{T}$ is G-equivariant, we may replace F by any translate of it, and since the finite sums of the translates of the elements of the form $i_A(a)$, $a \in A$, are dense in $\operatorname{Ind}_C^G A$, we may even assume that $F = i_A(a)$ for some $a \in A$. Now observe that $\mathcal{E}$ embeds (C-equivariantly) as a direct summand of $\tilde{\mathcal{E}}$ via

$$\big(i_{\mathcal{E}}(w)\big)(g) = \begin{cases} g^{-1}(w) & \text{if } g \in C,\\ 0 & \text{if } g \notin C.\end{cases}$$

This induces a corresponding embedding $i_{\mathcal{K}(\mathcal{E})} : \mathcal{K}(\mathcal{E}) \to \mathcal{K}(\tilde{\mathcal{E}})$, and it follows directly from the formulas that $[\tilde{T},\tilde{\Phi}(i_A(a))] = i_{\mathcal{K}(\mathcal{E})}\big([T,\Phi(a)]\big)$, $(\tilde{T}^2 - 1)\tilde{\Phi}(i_A(a)) = i_{\mathcal{K}(\mathcal{E})}\big((T^2-1)\Phi(a)\big)$, and $(\tilde{T}^*-\tilde{T})\tilde{\Phi}(i_A(a)) = i_{\mathcal{K}(\mathcal{E})}\big((T^*-T)\Phi(a)\big)$, and hence all three elements are in $\mathcal{K}(\tilde{\mathcal{E}})$. Since the assignment $(\mathcal{E},\Phi,T) \mapsto (\tilde{\mathcal{E}},\tilde{\Phi},\tilde{T})$ preserves homotopy (we just apply the same construction to a homotopy), it is now clear that (5.1) determines a well defined map in KK-theory.
It is easy to check that $\operatorname{comp}_C^G \circ \inf_C^G$ is the identity on $\mathrm{KK}^C(A,D)$: Write $\tilde{\mathcal{E}} = \mathcal{E} \bigoplus \mathcal{F}$, with respect to the C-equivariant embedding $i_{\mathcal{E}} : \mathcal{E} \to \tilde{\mathcal{E}}$ considered above. Then show that $\tilde{\Phi} \circ i_A$ decomposes as $\Phi \bigoplus 0$ under the above decomposition of $\tilde{\mathcal{E}}$, from which it follows that $\operatorname{comp}_C^G \circ \inf_C^G\big([\mathcal{E},\Phi,T]\big) = [(\mathcal{E},\Phi,T)] \bigoplus [(\mathcal{F},0,\tilde{T})] = [(\mathcal{E},\Phi,T)] \in \mathrm{KK}^C(A,D)$.
Conversely, to see that $\inf_C^G \circ \operatorname{comp}_C^G = \mathrm{id}_{\mathrm{KK}^G(\operatorname{Ind}_C^G A, D)}$, we start with a Kasparov triple $(\mathcal{F},\Psi,S)$ representing a class in $\mathrm{KK}^G(\operatorname{Ind}_C^G A, D)$. Passing to the stabilization $A \otimes \mathcal{K}(L^2(G))$, if necessary (equipped with action $\alpha \otimes \operatorname{Ad}\lambda_G$), we can use the equation $\operatorname{Ind}_C^H\big(A \otimes \mathcal{K}(L^2(G))\big) = \big(\operatorname{Ind}_H^G A\big) \otimes \mathcal{K}(L^2(G))$ in order to apply Meyer's result [20, Proposition 3.2] to the induced algebra $\operatorname{Ind}_C^G A$. Thus we may assume without loss of generality that

(1) $\Psi(\operatorname{Ind}_C^G A)\mathcal{F} = \mathcal{F}$, and

(2) the operator $S \in \mathcal{L}(\mathcal{F})$ is G-equivariant.

We can use (1) to define a family $\{p_{\dot{g}} \mid \dot{g} \in G/C\}$ of projections on $\mathcal{F}$ by $p_{\dot{g}}\big(\Psi(F)\xi\big) = \Psi(F|_{gC})\xi$, for $F \in \operatorname{Ind}_C^G A$. We may then assume additionally that

(3) $p_{\dot{g}}S = Sp_{\dot{g}}$ for all $\dot{g} \in G/C$.

In fact, if S does not satisfy this condition, then we pass to the compact perturbation $S' = \sum_{\dot{g} \in G/C} p_{\dot{g}} S p_{\dot{g}}$ of S, which then satisfies (1)–(3) (to see that S' is a compact perturbation of S, i.e., that $(S-S')\Psi(F) \in \mathcal{K}(\mathcal{F})$ for all $F \in \operatorname{Ind}_C^G A$, one first observes that $(S-S')\Psi(F) = \sum_{G/C}(S - p_{\dot{g}}S)\Psi(F|_{gC})$, where the sum converges in the norm topology, and then one uses the compactness of $[S, \Psi(F|_{gC})]$ to see that each summand is a compact operator). Using these properties, we easily check that $(\mathcal{E}, \Phi, T) := (p_{\dot{e}}\mathcal{F}, p_{\dot{e}}\Psi p_{\dot{e}}, p_{\dot{e}}Sp_{\dot{e}})$ is a representative for $\mathrm{comp}_C^G([\mathcal{F}, \Psi, S])$. A straightforward computation then shows that

$$\Theta : \tilde{\mathcal{E}} \to \mathcal{F}; \quad \Theta(\xi) = \sum_{\dot{g} \in G/C} g(\xi(g))$$

is an isomorphism which intertwines $\tilde{\Phi}$ with Ψ and $\tilde{T}$ with S. □

The main reduction argument for the proof of Proposition 5.9 is contained in the following lemma. We resume the situation of Lemma 5.6, i.e., we assume that G/N has a compact open subgroup, and $\mathcal{E}(G/N)$ has the structure of a simplicial complex as in Proposition 5.2. Moreover, for a simplex S of $\mathcal{E}(G/N)$ we let $\mathring{S}$ denote its interior, n its dimension and C the open subgroup of G which stabilizes S (with respect to the inflated action of G on $\mathcal{E}(G/N)$).

Lemma 5.15. *For any simplex S of $\mathcal{E}(G/N)$, let $Y \in \mathcal{F}$ be the G-saturation of S in $\mathcal{E}(G/N)$, and let Y_0 be the open subset of Y generated by $\mathring{S}$ under the action of G. Let X be a G-compact subspace of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ such that $\pi_2(X) = Y$, and let X_0 be the open subset of X defined by $X_0 = X \cap \pi^{-1}(Y_0)$. Then:*

- *(i) After enlarging X, if necessary, we may assume that there exists a C-compact subset T of $\mathcal{E}(G)$ such that X_0 is G-homeomorphic to the induced space $G \times_C (T \times \mathring{S})$.*
- *(ii) For every G-algebra B, the diagram*

$$\begin{array}{ccc}
\mathrm{KK}_i^G\big(C_0(X_0), B\big) & \xrightarrow{\mu_{N,B}^{G,N}[X_0]} & \mathrm{KK}_i^{G,N}\big(C_0(Y_0), B \rtimes_r N\big) \\
{\scriptstyle \mathrm{comp}_C^G}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \mathrm{comp}_C^G} \\
\mathrm{KK}_i^C\big(C_0(T \times \mathring{S}), B\big) & \xrightarrow{\mu_{N,B}^{C,N}[T \times \mathring{S}]} & \mathrm{KK}_i^{C,N}\big(C_0(\mathring{S}), B \rtimes_r N\big) \\
{\scriptstyle \beta\otimes.}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \beta\otimes.} \\
\mathrm{KK}_{i+n}^C\big(C_0(T), B\big) & \xrightarrow{\mu_{N,B}^{C,N}[T]} & \mathrm{KK}_{i+n}^{C,N}(\mathbb{C}, B \rtimes_r N)
\end{array}$$

  commutes, where $\beta \in \mathrm{KK}_n\big(\mathbb{C}, C_0(\mathbb{R}^n)\big)$ denotes the Bott element.

Proof. Since S generates Y as a G-space, we can choose a compact subset $L \subseteq \mathcal{E}(G) \times S$ such that L generates X as a G-space and such that $\pi_2(L) = S$. Let $T = C \cdot \pi_1(L)$, where $\pi_1 : \mathcal{E}(G) \times \mathcal{E}(G/N) \to \mathcal{E}(G)$ denotes the projection on the first factor, and let $X' = G \cdot (T \times S)$. Then $X \subseteq X'$, X' is G-compact, and $X_0' \cong G \times_C (T \times \mathring{S})$ by Remark 5.13. Thus, replacing X by X' gives (i).

For the proof of (ii), first note that β can be seen as an element of $\mathrm{KK}_n^{C,N}\big(\mathbb{C}, C_0(\overset{\circ}{S})\big)$ because the action of C on $\overset{\circ}{S}$ is trivial and $\overset{\circ}{S}$ is homeomorphic to $\mathbb{R}^n$. To see the commutativity of the upper square of the diagram, we first have to introduce some notation:

- $i_1 : C_0(T\times \overset{\circ}{S}) \to C_0(X_0)$ and $i_2 : C_0(\overset{\circ}{S}) \to C_0(Y_0)$ denote the canonical inclusions (recall that because C is open in G, $\overset{\circ}{S}$ and $T\times \overset{\circ}{S}$ are open subsets of Y_0 and X_0 respectively).

- $q_1 : C_0(Y_0) \to C_0(N\backslash X_0)$ and $q_2 : C_0(\overset{\circ}{S}) \to C_0\big(N\backslash(T\times \overset{\circ}{S})\big)$ are the homomorphisms induced by the second projection $p_2 : N\backslash\big(\mathcal{E}(G)\times \mathcal{E}(G/N)\big) \to \mathcal{E}(G/N)$ (note that the restrictions of p_2 to $N\backslash X_0 \to Y_0$ and to $N\backslash(T\times \overset{\circ}{S}) \to\overset{\circ}{S}$ are proper maps).

The corresponding elements in the various equivariant KK-groups are denoted by the same letters. Using the definitions of $\mu_{N,B}^{C,N}[X_0]$, $\mu_{N,B}^{C,N}[T\times \overset{\circ}{S}]$ and comp_C^G (see Definition 5.10 and Proposition 5.14 above), we get for all $\alpha \in \mathrm{KK}_*^G\big(C_0(X_0), B\big)$:

$$\mu_{N,B}^{C,N}[T\times \overset{\circ}{S}] \circ \mathrm{comp}_C^G(\alpha) = q_2 \otimes \Lambda^{C,N}_{T\times\overset{\circ}{S},N} \otimes j^N_{\{e\},r}(i_1 \otimes \mathrm{res}_C^G(\alpha)).$$

On the other hand we have

$$\mathrm{comp}_C^G \circ\mu_{N,B}^{G,N}[X_0](\alpha) = i_2 \otimes \mathrm{res}_C^G\left(q_1 \otimes \Lambda^{G,N}_{X_0,N} \otimes j^N_{\{e\},r}(\alpha)\right).$$

But it is clear from Equation (1.1) that $\mathrm{res}_C^G(\Lambda^{G,N}_{X_0,N})$ is nothing but $\Lambda^{C,N}_{X_0,N}$. Using the fact that $\mathrm{res}_C^G(j^N_{\{e\},r}(\alpha)) = j^N_{\{e\},r}(\mathrm{res}_C^G(\alpha))$ (cf. [6, (2) Remark 4.6]), we note that the commutativity of the upper square of the diagram reduces to the equality:

$$i_2 \otimes \mathrm{res}_C^G(q_1) \otimes \Lambda^{C,N}_{X_0,N} = q_2 \otimes \Lambda^{C,N}_{T\times\overset{\circ}{S},N} \otimes j^N_{\{e\},r}(i_1),$$

which follows from Lemma 1.5.
To see the commutativity of the lower square of the diagram, we first observe that, since N (as a subgroup of C) acts trivially on $\overset{\circ}{S}$, we have

$$p_2^*(\Lambda^{C,N}_{T\times\overset{\circ}{S},N}) = \sigma_{C_0(\overset{\circ}{S})}\big(p_2^*(\Lambda^{C,N}_{T,N})\big) \quad\text{and}\quad j^N_{\{e\},r}\big(\sigma_{C(T)}(\beta)\big) = \sigma_{C(T)\rtimes N}(\beta),$$

where for any (C,N)-algebra D, $\sigma_D : \mathrm{KK}^{C,N}(A,B) \to \mathrm{KK}^{C,N}(A\otimes D, B\otimes D)$ denotes the external tensor product operator. Using this and the commutativity of the Kasparov product over $\mathbb{C}$, we compute for $\alpha \in \mathrm{KK}_i^{C,N}(C_0(T\times \overset{\circ}{S}), B)$:

$$\begin{aligned}
\mu_{N,B}^{C,N}[T](\beta\otimes\alpha) &= p_2^*(\Lambda_{T,N}^{C,N}) \otimes_{C(T)\rtimes N} j_{\{e\},r}^N\Big(\sigma_{C_0(T)}(\beta)\otimes_{C_0(T\times\overset{\circ}{S})}\alpha\Big)\\
&= p_2^*(\Lambda_{T,N}^{C,N}) \otimes_{C(T)\rtimes N} \big(\sigma_{C_0(T)\rtimes N}(\beta)\big)\otimes_{C_0(T\times\overset{\circ}{S})\rtimes N}\big(j_{\{e\},r}^N(\alpha)\big)\\
&= \Big(p_2^*(\Lambda_{T,N}^{C,N})\otimes_{\mathbb{C}}\beta\Big)\otimes_{C_0(T\times\overset{\circ}{S})\rtimes N}\big(j_{\{e\},r}^N(\alpha)\big)\\
&= \Big(\beta\otimes_{\mathbb{C}} p_2^*(\Lambda_{T,N}^{C,N})\Big)\otimes_{C_0(T\times\overset{\circ}{S})\rtimes N}\big(j_{\{e\},r}^N(\alpha)\big)\\
&= \Big(\beta\otimes_{C_0(\overset{\circ}{S})}\sigma_{C_0(\overset{\circ}{S})}\big(p_2^*(\Lambda_{T,N}^{C,N})\big)\Big)\otimes_{C_0(T\times\overset{\circ}{S})\rtimes N}\big(j_{\{e\},r}^N(\alpha)\big)\\
&= \beta\otimes_{C_0(\overset{\circ}{S})}\Big(p_2^*(\Lambda_{T\times\overset{\circ}{S},N}^{C,N})\otimes_{C_0(T\times\overset{\circ}{S})\rtimes N}\big(j_{\{e\},r}^N(\alpha)\big)\Big)\\
&= \beta\otimes\mu_{N,B}^{C,N}[T\times\overset{\circ}{S}](\alpha).
\end{aligned}$$

□

In what follows next, we still assume that G/N has a compact open subgroup, and that $\mathcal{E}(G/N)$ has the structure of a simplicial complex. As before, $\mathcal{F}$ denotes the family of G-saturations of a finite union of simplices in $\mathcal{E}(G/N)$.

Proposition 5.16. *Let B be a G-algebra such that for every compact subgroup $\dot{K}$ of G/N the group $K = q^{-1}(\dot{K}) \subseteq G$ satisfies BCC for B. Let $Y \in \mathcal{F}$ and let W be a finite set of simplices whose union generates Y as a G/N-space. Define* $\dim(Y)$ *to be the highest dimension of simplices in W and let $Y_0 \subseteq \mathcal{E}(G/N)$ be the G/N-saturation of the interiors of the simplices of dimension* $\dim(Y)$ *in W.*

(i) Assume that $\dim(Y) = n > 0$*. Then the partial assembly map*

$$\mu_{N,B}^{G,N}\langle Y_0\rangle :\ \lim_X \mathrm{KK}_*^G\big(C_0(X_0), B\big) \to \mathrm{KK}_*^{G,N}\big(C_0(Y_0), B\rtimes_r N\big)$$

of Definition 5.10 is bijective (recall that the limit is taken over the G-compact subsets X of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ which satisfy $\pi_2(X) = Y$ and $X_0 = X\cap\pi_2^{-1}(Y_0)$).

(ii) Assume $\dim(Y) = 0$*. Then the partial assembly map*

$$\mu_{N,B}^{G,N}\langle Y\rangle :\ \lim_X \mathrm{KK}_*^G\big(C_0(X), B\big) \to \mathrm{KK}_*^{G,N}\big(C_0(Y), B\rtimes_r N\big)$$

of Definition 5.7 is bijective.

Proof. To show that (i) holds, note first that Y_0 is a finite union of disjoint spaces, each being the G-saturation of the interior $\overset{\circ}{S}$ of only one simplex S in $\mathcal{E}(G/N)$. We can therefore assume that Y is the G-saturation of S. It is not hard to check that the diagram of Lemma 5.15 is compatible with taking the inductive limit over the G-compact subsets X of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ which satisfy

$\pi_2(X) = Y$. Moreover, by part (i) of Lemma 5.15, it follows that taking the limits over all $X_0 = X \cap \pi_2^{-1}(Y_0)$ such that X projects onto Y is the same as taking the limit over the sets $G \times_C (T \times \overset{\circ}{S})$, where T runs through the C-compact subsets of $\mathcal{E}(G)$. Thus, taking the limit over T of the diagram in part (ii) of Lemma 5.15 gives:

$$\begin{array}{ccc}
\lim_X \mathrm{KK}_i^G\big(C_0(X_0), B\big) & \xrightarrow{\mu_{N,B}^{G,N}\langle Y\rangle} & \mathrm{KK}_i^{G,N}\big(C_0(G/C), B \rtimes_r N\big) \\
\cong\big\downarrow & & \big\downarrow\cong \\
\lim_T \mathrm{KK}_{i+n}^C(C_0(T), B) & & \mathrm{KK}_{i+n}^{C,N}(\mathbb{C}, B \rtimes_r N) \\
=\big\downarrow & & \big\downarrow= \\
\mathrm{K}_{i+n}^{\mathrm{top}}(C; B) & \xrightarrow[\mu_{N,B}^{C,N}]{} & \mathrm{K}_{i+n}^{\mathrm{top}}(C/N; B \rtimes_r N),
\end{array}$$

where the first upper vertical arrows are given by the compositions $\mathrm{Bott} \circ \mathrm{comp}_C^G$. We now use the assumption that C satisfies BCC for B: Since C/N is compact (and thus satisfies BCC), Proposition 3.2 implies that the partial assembly map $\mu_{N,B}^{C,N}$ is a bijection. The above diagram then completes the proof of part (i) of the proposition.
For (ii), the same argument applies, starting from the fact that Y is a finite union of disjoint "induced spaces" $G/N \cdot x \cong G/C$, where x denotes a vertex of $\mathcal{E}(G/N)$ and C its stabilizer under the action of G. No Bott map (and thus no dimension shift) is required to get the analogue of the above diagram in this case. ☐

As we have already suggested, we are going to use Proposition 5.16 for an induction argument on the maximal dimension of the simplices involved. To do this, we need to be able to put the above maps into a six-term exact sequence in KK-theory, namely the Mayer-Vietoris sequence associated to the inclusion $Y_0 \to Y$.

Lemma 5.17. *Let Y, $n = \dim(Y)$, and Y_0 be as in Proposition 5.16. Assume further that $n > 0$. Then Y_0 is a nonempty open subset of Y and $Y_1 = Y \smallsetminus Y_0$ is an element of $\mathcal{F}$ and satisfies $\dim(Y_1) = n - 1$. Furthermore, for any G-compact subset X of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ such that $\pi_2(X) = Y$, we write $X_0 = X \cap \pi_2^{-1}(Y_0)$ and $X_1 = X \cap \pi_2^{-1}(Y_1)$. Then we get two equivariant exact sequences of commutative C^*-algebras:*

$$\begin{array}{ll}
\delta: & 0 \to C_0(X_0) \to C_0(X) \to C_0(X_1) \to 0 \quad \textit{and} \\
d: & 0 \to C_0(Y_0) \to C_0(Y) \to C_0(Y_1) \to 0
\end{array}$$

which determine elements $[\delta] \in \mathrm{KK}_1^G\big(C_0(X_1), C_0(X_0)\big)$ and $[d] \in \mathrm{KK}_1^G\big(C_0(Y_1), C_0(Y_0)\big)$ such that

$$[p_2^*] \otimes [\Lambda_{X_1,N}^{G,N}] \otimes j_{\{e\},r}^N([\delta]) = [d] \otimes [p_2^*] \otimes [\Lambda_{X_0,N}^{G,N}] \in \mathrm{KK}_1^{G,N}\big(C_0(Y_1), C_0(X_0) \rtimes N\big).$$

Proof. The existence of $[\delta] \in \mathrm{KK}_1^G\big(C_0(X_1), C_0(X_0)\big)$ and $[d] \in \mathrm{KK}_1^G\big(C_0(Y_1), C_0(Y_0)\big)$ is a particular case of [16, Corollary of Proposition 6.2]. For the equation, we first consider the extensions

$$\begin{array}{ccccccccc} d: & 0 & \longrightarrow & C_0(Y_0) & \longrightarrow & C_0(Y) & \longrightarrow & C_0(Y_1) & \longrightarrow 0, \\ & & & \Big\downarrow{\scriptstyle p_2^*} & & \Big\downarrow{\scriptstyle p_2^*} & & \Big\downarrow{\scriptstyle p_2^*} & \\ \delta_N: & 0 & \longrightarrow & C_0(N\backslash X_0) & \longrightarrow & C_0(N\backslash X) & \longrightarrow & C_0(N\backslash X_1) & \longrightarrow 0. \end{array}$$

Applying [23, Lemma 1.5] to this diagram implies that $[p_2^*] \otimes [\delta_N] = [d] \otimes [p_2^*] \in \mathrm{KK}_1^{G,N}\big(C_0(Y_1), C_0(N\backslash X_0)\big)$. Thus, it is enough to check that

$$[\Lambda_{X_1,N}^{G,N}] \otimes j_{\{e\},r}^N([\delta]) = [\delta_N] \otimes [\Lambda_{X_0,N}^{G,N}] \in \mathrm{KK}_1^{G,N}\big(C_0(N\backslash X_1), C_0(X_0) \rtimes N\big). \tag{5.2}$$

According to [1, Remarque 7.5, (2)], $[\delta_N]$ and $[\delta]$ are obtained from the Bott element $\beta \in \mathrm{KK}_1\big(\mathbb{C}, C_0(]0,1[)\big)$. To be more precise, recall from [4, §19.5] that if

$$c: 0 \to J \to A \to A/J \to 0$$

is a semi-split short exact sequence of C^*-algebras (i.e., there exists a completely positive section $q: A/J \to A$), then the canonical embedding

$$e: J \to C_q := C_0\big([0,1[, A\big)/C_0\big(]0,1[, J\big)$$

determines a KK-equivalence $[e] \in \mathrm{KK}_0(J, C_q)$. The same computations show that if the above short exact sequence is equivariant with respect to an action of a group G and if q can be chosen to be equivariant as well, then $e: J \to C_q := C_0\big([0,1[, A\big)/C_0\big(]0,1[, J\big)$ determines a KK-equivalence $[e] \in \mathrm{KK}_0^G(J, C_q)$ (where G acts trivially on $[0,1]$). Moreover, if we also consider the canonical inclusion

$$i: C_0\big(]0,1[, A/J\big) \to C_q,$$

then it follows from [1, Remarque 7.5, (2)] that the element $[c] \in \mathrm{KK}_1^G(A/J, J)$ coming from any equivariantly semi-split short exact sequence as above satisfies the equation

$$[e] \otimes [c] = \sigma_{A/J}(\beta) \otimes [i]. \tag{5.3}$$

We want to apply this to the short exact sequences δ and δ_N. For this define Z by $Z = \big(X \times [0,1[\big) \smallsetminus \big(X_0 \times]0,1[\big)$. Then $C_0(Z)$ is the algebra C_q corresponding to the extension δ, and $C_0(N\backslash Z)$ becomes the substitute for C_q with respect to the extension δ_N. Let $e: C_0(X_0) \to C_0(Z)$ and $e_N: C_0(N\backslash X_0) \to C_0(N\backslash Z)$ denote the canonical inclusions (which, by the above discussion, are KK-equivalences) and let i and i_N denote the canonical inclusions of $C_0\big(X_1 \times]0,1[\big)$

and $C_0\big(N\backslash(X_1\times]0,1[)\big)$ into $C_0(Z)$ and $C_0(N\backslash Z)$, respectively. Using this notation we now compute

$$\begin{aligned}
&\Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}([\delta]) = [\delta_N]\otimes\Lambda^{G,N}_{X_0,N}\\
\Leftrightarrow\quad &\Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}([\delta]\otimes[e]) = [\delta_N]\otimes\Lambda^{G,N}_{X_0,N}\otimes j^N_{\{e\},r}([e])\\
\Leftrightarrow\quad &\Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}([\delta]\otimes[e]) = [\delta_N]\otimes[e_N]\otimes\Lambda^{G,N}_{Z,N}\ , \quad \text{by Lemma 1.5}\\
\Leftrightarrow\quad &\Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}\big(\sigma_{C_0(X_1)}(\beta)\otimes[i]\big) = \sigma_{C_0(N\backslash X)}(\beta)\otimes[i_N]\otimes\Lambda^{G,N}_{Z,N}\ , \text{ by (5.3)}\\
\Leftrightarrow\quad &\Lambda^{G,N}_{X_1,N}\otimes j^N_{\{e\},r}\big(\sigma_{C_0(X_1)}(\beta)\otimes[i]\big) = \sigma_{C_0(N\backslash X)}(\beta)\otimes\Lambda^{G,N}_{X_1\times]0,1[,N}\otimes j^N_{\{e\},r}([i])
\end{aligned}$$

where the last line uses Lemma 1.5. Since G acts trivially on $]0,1[$, it follows that

$$\Lambda^{G,N}_{X_1\times]0,1[,N} = \sigma_{C_0\left(]0,1[\right)}(\Lambda^{G,N}_{X_1,N}). \tag{5.4}$$

On the other hand, since $\beta\in \mathrm{KK}_1\left(\mathbb{C}, C_0\big(]0,1[\big)\right)$ (inflated to the various equivariant KK-groups), it follows that

$$j^N_{\{e\},r}\big(\sigma_{C_0(X_1)}(\beta)\big) = \sigma_{C_0(X_1)\rtimes N}(\beta). \tag{5.5}$$

Using (5.4) and (5.5), the above computation shows that it is enough to prove that $\big(\Lambda^{G,N}_{X_1,N}\otimes\sigma_{C_0(X_1)\rtimes N}(\beta)\big)\otimes j^N_{\{e\},r}([i])$ is equal to $\big(\sigma_{C_0(N\backslash X)}(\beta)\otimes\sigma_{C_0(]0,1[)}(\Lambda^{G,N}_{X_1,N})\big)\otimes j^N_{\{e\},r}([i])$ to get Equation (5.2). Using Kasparov's notations, this becomes

$$\big(\Lambda^{G,N}_{X_1,N}\otimes_{\mathbb{C}}\beta\big)j^N_{\{e\},r}([i]) = \big(\beta\otimes_{\mathbb{C}}\Lambda^{G,N}_{X_1,N}\big)j^N_{\{e\},r}([i]),$$

which is a consequence of the commutativity of the Kasparov product over $\mathbb{C}$ (see [14, Theorem 2.14]). This finishes the proof. □

We are now able to complete the proof of Proposition 5.9. This will also complete the proof of Theorem 3.3 since, as noted earlier, the theorem is a consequence of Proposition 5.9 and Lemma 5.8.

Proof of Proposition 5.9. We are going to make an induction on the dimension of $Y\in\mathcal{F}$. Let $Y\in\mathcal{F}$ such that $\dim(Y)=0$. Then $\mu^{G,N}_{N,B}\langle Y\rangle$ is bijective by (ii) of Proposition 5.16.
Let n be an arbitrary non-negative integer, and assume that $\mu^{G,N}_{N,B}\langle Z\rangle$ is bijective for all $Z\in\mathcal{F}$ such that $\dim(Z)\le n$.
Take $Y\in\mathcal{F}$ such that $\dim(Y)=n+1$, and let W be a finite set of simplices in $\mathcal{E}(G/N)$ which generate Y under the action of G. Define Y_0 to be the G-saturation of the union of the interiors of the simplices of dimension $n+1$ in W. Then Y_0 is open in Y and $Y_1 = Y\smallsetminus Y_0$ is an element of $\mathcal{F}$ which has dimension less or equal to n.
Consider any G-compact subset X of $\mathcal{E}(G)\times\mathcal{E}(G/N)$ which satisfies $\pi_2(X)=Y$ and put $X_0 = X\cap\pi_2^{-1}(Y_0)$ and $X_1 = X\cap\pi_2^{-1}(Y_1)$. Using Lemma 5.17, we

obtain two long exact sequences in equivariant KK-theory, where the boundary maps are given by Kasparov product with the elements $[\delta]$ and $[d]$, respectively. Using [23, Lemma 1.5] we see that the sequence for X is compatible with taking inclusions of G-compact sets. Thus we can form the inductive limit over the G-compact subsets X of $\mathcal{E}(G) \times \mathcal{E}(G/N)$ which satisfy $\pi_2(X) = Y$, to obtain a diagram

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i+1}\big(C_0(X_1), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y_1\rangle} & \mathrm{KK}^{G,N}_{i+1}\big(C_0(Y_1), B \rtimes_r N\big) \\
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i}\big(C_0(X_0), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y_0\rangle} & \mathrm{KK}^{G,N}_{i}\big(C_0(Y_0), B \rtimes_r N\big) \\
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i}\big(C_0(X), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y\rangle} & \mathrm{KK}^{G,N}_{i}\big(C_0(Y), B \rtimes_r N\big) \\
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i}\big(C_0(X_1), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y_1\rangle} & \mathrm{KK}^{G,N}_{i}\big(C_0(Y_1), B \rtimes_r N\big) \\
\uparrow & & \uparrow \\
\lim_X \mathrm{KK}^G_{i+1}\big(C_0(X_0), B\big) & \xrightarrow{\mu^{G,N}_{N,B}\langle Y_0\rangle} & \mathrm{KK}^{G,N}_{i+1}\big(C_0(Y_0), B \rtimes_r N\big) \\
\uparrow & & \uparrow
\end{array}$$

in which the vertical sequences are exact. Using Lemma 5.17, it follows from the definition of the horizontal maps that the diagram commutes. By the induction hypothesis, the two horizontal arrows corresponding to Y_1 are bijective, and part (i) of Proposition 5.16 ensures that those corresponding to Y_0 are also bijective. Thus, it follows from the Five Lemma that $\mu^{G,N}_{N,B}\langle Y\rangle$ is bijective, too. □

6 Proof of the induction isomorphism

In this section we give the proof of the bijectivity of the induction homomorphism as stated in Theorem 2.2. For convenience, let's restate the theorem:

Theorem 2.2. *Let H be a closed subgroup of a group G, and let B be an H-algebra. Then the map $\mathrm{Ind}_H^G : \mathrm{K}^{\mathrm{top}}_*(H; B) \to \mathrm{K}^{\mathrm{top}}_*(G; \mathrm{Ind}_H^G B)$ is an isomorphism.*

As in the proof of Theorem 3.3 (see §5), we will use a special realization of the universal proper space for G to obtain a certain simplicial structure which allows an induction argument based on excision. In fact, if G_0 denotes the connected component of the identity in G, then G/G_0 is a totally disconnected group, and therefore has a compact open subgroup. Thus, by Proposition

5.2, there exists a realization of $\mathcal{E}(G/G_0)$ as a simplicial complex. If $\mathcal{E}(G)$ is any realization of the universal proper G-space, then, by Lemma 5.3, $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$ equipped with the diagonal G-action is also a universal proper G-space, which we will use throughout this section to compute the topological K-theories of G and H.

The strategy used in the previous section will allow us to reduce the problem of the bijectivity of the induction homomorphism to the case of almost connected groups. But by Kasparov's work, every almost connected group has a γ-element (see Definition 1.7 and Remark 1.8). So, as a first step, we start with giving the proof under the extra condition that G has a γ-element. For this we have to use the following general lemma about the image of the assembly map in the presence of a γ-element for G. Note that this lemma is well known to the experts (it is implicitly contained in the work of Kasparov and Tu [15, 14, 25]). However, it seems that there exist no direct references. Thus, for the reader's convenience, we present a short argument building on [25, Proposition 5.23].

Lemma 6.1. *Assume that G has a γ-element $\gamma = \eta \otimes_{\mathcal{A}} D \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ (see Definition 1.7). Then, for every G-algebra B, the assembly map induces an isomorphism between $\mathrm{K}_*^{\mathrm{top}}(G; B)$ and the γ-part*

$$\gamma\big(\mathrm{K}_*(B \rtimes_r G)\big) := \mathrm{K}_*(B \rtimes_r G) \otimes j_r^G(\sigma_B(\gamma)) \subseteq \mathrm{K}_*(B \rtimes_r G)$$

of $\mathrm{K}_*(B \rtimes_r G)$.

Proof. We will use the facts that the assembly map $\mu_{G,B}$ is injective, whenever G has a γ-element ([25, Proposition 5.23]) and that $\mu_{G,D}$ is surjective if D is a proper G-algebra (which follows from the descent isomorphism of [17]). It follows from part (2) of Definition 1.7 (see [6, Remark 6.4]) that the right Kasparov product with $\sigma_B(\gamma)$ determines the identity map on $\mathrm{KK}_*^G(C_0(X), B)$ for every proper G-space X. Thus, γ acts as the identity on $\mathrm{K}_*^{\mathrm{top}}(G; B)$ via right Kasparov product. This easily implies that the image of the assembly map $\mu_{G,B}$ lies in the γ-part of $\mathrm{K}_*(B \rtimes_r G)$, and we get a commutative diagram

$$\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(G; B) & \xrightarrow{\mu_{G,B}(\cdot)} & \gamma_G\big(\mathrm{K}_*(B \rtimes_r G)\big) \\
{\scriptstyle \cdot\otimes\eta}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\otimes j_r^G(\sigma_B(\eta))} \\
\mathrm{K}_*^{\mathrm{top}}(G; B \otimes \mathcal{A}) & \xrightarrow{\mu_{G,B\otimes\mathcal{A}}(\cdot)} & \gamma_G\Big(\mathrm{K}_*\big((B \otimes \mathcal{A}) \rtimes_r G\big)\Big) \\
{\scriptstyle \cdot\otimes D}\big\downarrow & & \big\downarrow{\scriptstyle \cdot\otimes j_r^G(\sigma_B(D))} \\
\mathrm{K}_*^{\mathrm{top}}(G; B) & \xrightarrow{\mu_{G,B}(\cdot)} & \gamma_G\big(\mathrm{K}_*(B \rtimes_r G)\big).
\end{array}$$

Since $B \otimes \mathcal{A}$ is a proper G-algebra, the middle horizontal row is a bijection, and, by the above discussion, the composition of the left-hand side vertical rows is the identity on $\mathrm{K}_*^{\mathrm{top}}(G; B)$. Finally, since γ is an idempotent in $\mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$ by [25, Proposition 5.20], the composition of the right-hand side vertical arrows is the identity of $\gamma\big(\mathrm{K}_*(B \rtimes_r G)\big)$. Now a straightforward diagram chase gives the result. □

Lemma 6.2. *Let H be a closed subgroup of G and assume that G has a γ-element. Then, for every G-algebra B, the induction homomorphism $\mathrm{Ind}_H^G : \mathrm{K}_*^{\mathrm{top}}(H;B) \to \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B)$ is bijective.*

Proof. Let γ_G be the γ-element of G. Then $\gamma_H = \mathrm{res}_H^G(\gamma_G)$ is the γ-element of H by [6, Remark 6.4]. Let $x \in \mathrm{KK}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G, B \rtimes_r H\right)$ denote the invertible element implementing the canonical Morita equivalence between $(\mathrm{Ind}_H^G B) \rtimes_r G$ and $B \rtimes_r H$. As was already noted for the proof of [6, Proposition 6.9], the corollary on page 176 of [14] and item (2) of [15, Theorem 1 of §5] imply:

$$j_r^G\big(\sigma_{\mathrm{Ind}_H^G B}(\gamma_G)\big) = x \otimes_{B\rtimes_r H} j_r^H\big(\sigma_B(\gamma_H)\big) \otimes_{B\rtimes_r H} x^{-1}. \tag{6.1}$$

Together with Proposition 2.3, this implies that the two squares of the following diagram are commutative:

$$\begin{tikzcd}
\mathrm{K}_*^{\mathrm{top}}(H;B) \arrow[r, "\mathrm{Ind}_H^G"] \arrow[d, "\mu_{H,B}"'] & \mathrm{K}_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G B) \arrow[d, "\mu_{G,\mathrm{Ind}_H^G B}"] \\
\mathrm{K}_*(B \rtimes_r H) \arrow[r, "\cdot\otimes x^{-1}"] \arrow[d, "\cdot\otimes j_r^H\left(\sigma_B(\gamma_H)\right)"'] & \mathrm{K}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G\right) \arrow[d, "\cdot\otimes j_r^G\left(\sigma_{\mathrm{Ind}_H^G B}(\gamma_G)\right)"] \\
\gamma_H\big(\mathrm{K}_*(B \rtimes_r H)\big) \arrow[r, "\cdot\otimes x^{-1}"'] & \gamma_G\Big(\mathrm{K}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G\right)\Big),
\end{tikzcd}$$

where $\gamma_H\big(\mathrm{K}_*(B \rtimes_r H)\big)$ (resp. $\gamma_G\Big(\mathrm{K}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G\right)\Big)$) denotes the γ-part of $\mathrm{K}_*(B \rtimes_r H)$ (resp. $\mathrm{K}_*\left((\mathrm{Ind}_H^G B) \rtimes_r G\right)$). But Lemma 6.1 implies that the compositions of the vertical arrows are isomorphisms. Further, since the middle row of the above diagram is an isomorphism, Equation 6.1 also implies that the bottom arrow is an isomorphism. But then the top arrow has to be an isomorphism, too. □

As noted above, our aim is to reduce the proof of the general result of Theorem 2.2 to the special case where G is almost connected, in which case the result follows from Lemma 6.2. We start the reduction argument with some preliminaries:

Lemma 6.3. *Let H be a closed subgroup of G, let C be an open subgroup of G, and let B be an H-algebra. For each $\ddot{g}$ in the double coset space $H\backslash G/C$ (which is a discrete countable space) we put $C_H^g = C \cap g^{-1}Hg \subseteq C$, and we view B as a C_H^g-algebra by putting $g^{-1}hg \cdot b := h \cdot b$, $h \in H$, $b \in B$. Then the induced algebra $\mathrm{Ind}_H^G B$ is C-equivariantly isomorphic to $\bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{Ind}_{C_H^g}^C B$.*

Similarly, if $\mathcal{E}$ is an H-equivariant B-Hilbert module, there is a C-equivariant isomorphism between $\operatorname{Ind}_H^G \mathcal{E}$, viewed as a C-equivariant $\operatorname{Ind}_H^G B$-Hilbert module, and the $\bigoplus \operatorname{Ind}_{C_H^g}^C B$-Hilbert module $\bigoplus_{\ddot{g}\in H\backslash G/C} \operatorname{Ind}_{C_H^g}^C \mathcal{E}$.

Proof. Chose a set of representatives $\Gamma = \{g_0,\ g_1,\ ..., g_n,\ ...\}$ for $H\backslash G/C$ in G. By definition, $\operatorname{Ind}_H^G B$ is the subalgebra of $C_b(G,B)$ (the C^*-algebra of continuous bounded functions on G with values in B) consisting of all functions f which satisfy the conditions:

(i) $f(st) = t^{-1}\cdot f(s)$ for any $s\in G$ and $t\in H$,

(ii) $sH \mapsto \|f(s)\|$ is an element of $C_0(G/H)$.

The G action is given by $(s\cdot f)(t) = f(s^{-1}t)$, for $s,t\in G$.
For f in $\operatorname{Ind}_H^G B$ and $g\in\Gamma$, we define $\phi_{\ddot{g}} : \operatorname{Ind}_H^G B \to \operatorname{Ind}_{C_H^g}^C B$ by

$$\big(\phi_{\ddot{g}}(f)\big)(s) = g^{-1}\big(f(sg^{-1})\big), \quad \text{for all } s\in C.$$

It is straightforward to check that $\phi_{\ddot{g}}$ is a well defined C-equivariant $*$-homomorphism. Note that for any f in $\operatorname{Ind}_H^G B$ and any $\epsilon > 0$, there is a compact set $K\subseteq G/H$ such that, for all s in G, s belongs to K if $||f(s)|| \geq \epsilon/2$. Thus, if we denote by $\ddot{K}$ the image of K^{-1} in $H\backslash G/C$, we see that $||\phi_{\ddot{g}}(f)|| < \epsilon$ whenever $\ddot{g}$ does not belong to the compact set $\ddot{K}$. Hence the sequence $\Phi(f) = \big(\phi_{\ddot{g}}(f)\big)_{g\in\Gamma}$ belongs to $\bigoplus_{\ddot{g}\in H\backslash G/C} \operatorname{Ind}_{C_H^g}^C B$. It is readily seen that this defines an isomorphism Φ between $\operatorname{Ind}_H^G B$ and $\bigoplus_{\ddot{g}\in H\backslash G/C} \operatorname{Ind}_{C_H^g}^C B$: If $\lambda = (\lambda_g)_{g\in\Gamma} \in \bigoplus_{\ddot{g}\in H\backslash G/C} \operatorname{Ind}_{C_H^g}^C B$, we define $\Psi(\lambda)\in C_b(G,B)$ by $\Psi(\lambda)(s) = h^{-1}g_i^{-1}\lambda_{g_i}(c)$, whenever s is equal to cg_ih with $g_i\in\Gamma$. Then Ψ takes values in $\operatorname{Ind}_H^G B$ and Ψ is inverse to Φ.
A similar computation implies the decomposition $\operatorname{Ind}_H^G \mathcal{E} \cong \bigoplus \operatorname{Ind}_{C_H^g}^C \mathcal{E}$. □

LEMMA 6.4. *Let H be a closed subgroup of G, let C be an open subgroup of G, and let A and B be two H-algebras. For $g\in G$ let*

$$\lambda_g : \operatorname{Ind}_{C_H^g}^C A \to \bigoplus_{\ddot{g}'\in H\backslash G/C} \operatorname{Ind}_{C_H^{g'}}^C A \quad \textit{and} \quad \rho_g : \bigoplus_{\ddot{g}'\in H\backslash G/C} \operatorname{Ind}_{C_H^{g'}}^C B \to \operatorname{Ind}_{C_H^g}^C B$$

denote the canonical C-equivariant inclusions and projections, respectively. Then, using the direct sum decomposition provided by Lemma 6.3, the diagram

$$\begin{tikzcd}
\mathrm{KK}_*^H(A,B) \arrow[r, "i_H^G"] \arrow[d, "\mathrm{res}^H_{C_H^g}"'] & \mathrm{KK}_*^G(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B) \arrow[d, "\mathrm{res}^G_C"] \\
\mathrm{KK}_*^{C_H^g}(A,B) \arrow[d, "i^C_{C_H^g}"'] & \mathrm{KK}_*^C(\bigoplus \operatorname{Ind}_{C_H^{g'}}^C A, \bigoplus \operatorname{Ind}_{C_H^{g''}}^C B) \arrow[d, "\rho_g^*"] \\
\mathrm{KK}_*^C(\operatorname{Ind}_{C_H^g}^C A, \operatorname{Ind}_{C_H^g}^C B) & \mathrm{KK}_*^C(\bigoplus \operatorname{Ind}_{C_H^{g'}}^C A, \operatorname{Ind}_{C_H^g}^C B) \arrow[l, "\lambda_{g,*}"]
\end{tikzcd}$$

commutes (here the restriction $\operatorname{res}^H_{C^g_H} : \mathrm{KK}^H_*(A,B) \to \mathrm{KK}^{C^g_H}_*(A,B)$ *is defined by first restricting to* $H \cap gCg^{-1} = gC^g_H g^{-1}$ *and then* identifying *this group with* C^g_H *via conjugation). Moreover, if* $Hg'C \neq HgC$*, then* $[\lambda_g] \otimes \operatorname{res}^G_C \circ i^G_H(\alpha) \otimes [\rho_{g'}] = 0$ *for all* $\alpha \in \mathrm{KK}^H_*(A,B)$.

Proof. Note that for the proof of the commutativity of the above diagram we may replace A by any H-algebra A' which is H-equivariantly Morita equivalent to A (a similar statement holds for B, but we only need this for A). This follows easily from the fact that any H-equivariant Morita equivalence X between two H-algebras A and A' induces to a G-equivariant Morita equivalence $\operatorname{Ind}_H^G X$ between $\operatorname{Ind}_H^G A$ and $\operatorname{Ind}_H^G A'$, which by Lemma 6.3 also has a C-equivariant direct sum decomposition. Using this one checks that each map in the above diagram commutes with the respective Morita equivalences.
Replacing A by the Morita equivalent H-algebra $A \otimes \mathcal{K}(L^2(H))$, if necessary (with action given by $\alpha \otimes \operatorname{Ad}\lambda$, where $\alpha : H \to \operatorname{Aut}(A)$ denotes the given action on A and λ denotes the right regular representation of H), we can now use Meyer's result [20, Proposition 3.2] in order to assume that every $\alpha \in \mathrm{KK}^H_*(A,B)$ can be represented by a Kasparov triple $(\mathcal{E},\Phi,T) \in \mathbb{E}^H(A,B)$ such that $\Phi(A)\mathcal{E} = \mathcal{E}$ and such that T is H-equivariant. Using the formulas for the definitions of i^G_H and $i^C_{C^g_H}$, respectively (see §2), it follows from the decomposition of $\operatorname{Ind}_H^G \mathcal{E}$ as given in Lemma 6.3 and the H-equivariance of T, that

$$\operatorname{res}^G_C\left(\operatorname{Ind}_H^G \mathcal{E}, \operatorname{Ind}_H^G \Phi, \tilde{T}\right) \cong \Big(\bigoplus_{H\backslash G/C} \operatorname{Ind}^C_{C^g_H} \mathcal{E}_g, \bigoplus_{H\backslash G/C} \operatorname{Ind}^C_{C^g_H} \Phi_g, \bigoplus_{H\backslash G/C} \tilde{T}_g \Big)$$

in $\mathbb{E}^C(\operatorname{Ind}_H^G A, \operatorname{Ind}_H^G B)$, where $(\mathcal{E}_g, \Phi_g, T_g)$ denotes the cycle in $\mathbb{E}^{C^g_H}_*(A,B)$ obtained by first restricting the H action to $gCg^{-1} \cap H$, and then identifying $gCg^{-1} \cap H$ with C^g_H via the isomorphism given by conjugation with g. The result now follows immediately from this decomposition. □

The next result will be extended to arbitrary groups in §7 below (see Proposition 7.1). We only need here the weaker version where we assume that G is an almost connected group. As for the induction homomorphism, the proof of the general case will be done by a reduction to this case where G is almost connected.

Lemma 6.5. *Let G be an almost connected group and let $B = \lim_i B_i$ be an inductive limit of G-algebras B_i, $i \in I$ (with G-equivariant structure maps). Then*

$$\mathrm{K}^{\mathrm{top}}_*(G;B) \cong \lim_i \mathrm{K}^{\mathrm{top}}_*(G;B_i),$$

where the isomorphism is obtained from the morphisms $f_{i,*} : \mathrm{K}^{\mathrm{top}}_*(G;B_i) \to \mathrm{K}^{\mathrm{top}}_*(G;B)$*, which are induced by the canonical maps* $f_i : B_i \to B$.

Proof. Because G is almost connected, the functor which associates to a G-algebra the corresponding reduced crossed-product algebra is continuous with respect to taking inductive limits (in fact this holds whenever G is an exact group in the sense of [18]). We also know that G has a γ-element. So the lemma is an immediate consequence of the isomorphism $\mathrm{K}_*^{\mathrm{top}}(G;B) \cong \gamma\big(\mathrm{K}_*(B \rtimes_r G)\big)$ of Lemma 6.1 and of the continuity of K-theory ([4]). $\square$

We now come back to the proof of Theorem 2.2. As noted before, we use $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$ as a universal example for the proper actions of G and H (see Lemmas 5.3 and 2.4 above), and assume that $\mathcal{E}(G/G_0)$ has the simplicial structure described in Proposition 5.2. As mentioned before, we can do this because G/G_0 is totally disconnected and, therefore, it has a compact open subgroup. In the following, we denote by $\mathcal{F}_G$ the family of all subsets of $\mathcal{E}(G/G_0)$ which are G-saturation of finite unions of simplices of $\mathcal{E}(G/G_0)$ (cf. Lemma 5.6, the subscript here is to prevent confusion between G- and H-actions).
As shown in Lemma 5.6, we can use $\mathcal{F}_G$ to compute $\mathrm{K}_*^{\mathrm{top}}(G; \mathrm{Ind}_H^G B)$ in the following way:

$$\mathrm{K}_*^{\mathrm{top}}(G; \mathrm{Ind}_H^G B) = \varinjlim_{Z \in \mathcal{F}_G} \varinjlim_{\substack{Y \subset \mathcal{E}(G) \times Z \\ Y\ G\text{–compact}}} \mathrm{KK}_*^G\big(C_0(Y), \mathrm{Ind}_H^G B\big). \tag{6.2}$$

But since $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$ (with action restricted to H) serves also as a realization of the universal example for proper H-actions, we can use $\mathcal{F}_G$ also for the computation of $\mathrm{K}_*^{\mathrm{top}}(H;B)$:

$$\mathrm{K}_*^{\mathrm{top}}(H;B) = \varinjlim_{Z \in \mathcal{F}_G} \varinjlim_{\substack{Y \subset \mathcal{E}(G) \times Z \\ Y\ G\text{–compact}}} \varinjlim_{\substack{X \subset Y \\ X\ H\text{–compact} \\ G\cdot X = Y}} \mathrm{KK}_*^H\big(C_0(X), B\big). \tag{6.3}$$

The above formulas correspond to those of Lemma 5.6. Although these formulas look a bit complicated, they offer the advantage of breaking down the computation of the two topological K-theories into pieces which correspond to each other via the induction morphism and on which we can do an induction argument on the dimension $\dim(Z)$ of the elements $Z \in \mathcal{F}_G$.
In effect, using the above notations, note that $C_0(Y) = F^* \circ \mathrm{Ind}_H^G\big(C_0(X)\big)$, where $F : G \times_H X \to \mathcal{E}(G) \times \mathcal{E}(G/G_0)$ is defined by $F([s,x]) = s \cdot x$. Thus, it follows from the definition of the induction homomorphism that, on $\mathrm{KK}_*^H(C_0(X), B)$, it factorizes via the diagram

$$\begin{array}{ccc}
\mathrm{KK}_*^H(C_0(X), B) & \xrightarrow{F^* \circ i_H^G} & \mathrm{KK}_*^G(C_0(Y), \mathrm{Ind}_H^G B) \\
\downarrow & & \downarrow \\
\mathrm{K}_*^{\mathrm{top}}(H;B) & \xrightarrow[\mathrm{Ind}_H^G]{} & \mathrm{K}_*^{\mathrm{top}}(G; \mathrm{Ind}_H^G B).
\end{array}$$

In what follows next we will simply write Ind_H^G for the map $F^* \circ i_H^G$ in the above diagram. Hence, in view of formula (6.2) and (6.3), the conclusion of Theorem 2.2 will follows from:

PROPOSITION 6.6. *For each $Z \in \mathcal{F}_G$, the induction map Ind_H^G induces an isomorphism*

$$\varinjlim_{\substack{Y\subset\mathcal{E}(G)\times Z\\ Y\ G\text{-compact}}} \quad \varinjlim_{\substack{X\subset Y\\ X\ H\text{-compact}\\ G\cdot X=Y}} \mathrm{KK}_*^H\big(C_0(X),B\big) \xrightarrow{\mathrm{Ind}_H^G} \varinjlim_{\substack{Y\subset\mathcal{E}(G)\times Z\\ Y\ G\text{-compact}}} \mathrm{KK}_*^G\big(C_0(Y),\mathrm{Ind}_H^G B\big).$$

As mentioned earlier, we want to use induction on $n = \dim(Z)$. As in the proof of Proposition 5.9, we introduce the following notations: Z_0 is the G/G_0-saturation of the interiors of the simplices of dimension n generating Z, $Z_1 = Z \smallsetminus Z_0$; and we put

$$\begin{aligned} Y_0 &= Y\cap(\mathcal{E}(G)\times Z_0); & X_0 &= X\cap(\mathcal{E}(G)\times Z_0);\\ Y_1 &= Y\cap(\mathcal{E}(G)\times Z_1); & X_1 &= X\cap(\mathcal{E}(G)\times Z_1). \end{aligned}$$

Note that

$$C_0(Y) = F^*\circ\mathrm{Ind}_H^G\big(C_0(X)\big), \qquad C_0(Y_0) = F^*\circ\mathrm{Ind}_H^G\big(C_0(X_0)\big),$$

$$\text{and}\quad C_0(Y_1) = F^*\circ\mathrm{Ind}_H^G\big(C_0(X_1)\big),$$

and that we have the exact sequences:

$$\begin{aligned} \delta:&\quad 0 \longrightarrow C_0(X_0) \longrightarrow C_0(X) \longrightarrow C_0(X_1) \longrightarrow 0, \quad \text{and}\\ d:&\quad 0 \longrightarrow C_0(Y_0) \longrightarrow C_0(Y) \longrightarrow C_0(Y_1) \longrightarrow 0. \end{aligned}$$

Each of these two short exact sequences gives rise to a long exact sequence in equivariant KK-theory, which are linked by the induction homomorphisms Ind_H^G:

$$\begin{tikzcd}
{} & {} \\
\mathrm{KK}^H_{i+1}\big(C_0(X_1),B\big) \arrow[u] \arrow[r, "\mathrm{Ind}_H^G"] & \mathrm{KK}^{G,N}_{i+1}\big(C_0(Y_1),\mathrm{Ind}_H^G B\big) \arrow[u] \\
\mathrm{KK}^H_{i}\big(C_0(X_0),B\big) \arrow[u] \arrow[r, "\mathrm{Ind}_H^G"] & \mathrm{KK}^{G,N}_{i}\big(C_0(Y_0),\mathrm{Ind}_H^G B\big) \arrow[u] \\
\mathrm{KK}^H_{i}\big(C_0(X),B\big) \arrow[u] \arrow[r, "\mathrm{Ind}_H^G"] & \mathrm{KK}^{G,N}_{i}\big(C_0(Y),\mathrm{Ind}_H^G B\big) \arrow[u] \\
\mathrm{KK}^H_{i}\big(C_0(X_1),B\big) \arrow[u] \arrow[r, "\mathrm{Ind}_H^G"] & \lim_Y \mathrm{KK}^{G,N}_{i}\big(C_0(Y_1),\mathrm{Ind}_H^G B\big) \arrow[u] \\
\mathrm{KK}^H_{i-1}\big(C_0(X_0),B\big) \arrow[u] \arrow[r, "\mathrm{Ind}_H^G"] & \mathrm{KK}^{G,N}_{i-1}\big(C_0(Y_0),\mathrm{Ind}_H^G B\big) \arrow[u] \\
{} \arrow[u] & {} \arrow[u]
\end{tikzcd} \tag{6.4}$$

We need

LEMMA 6.7. *The above diagram commutes.*

Proof. The only slight difficulty arises at the square

$$\begin{tikzcd}
\mathrm{KK}^H_{i+1}(C_0(X_1),B) \arrow[r, "\mathrm{Ind}_H^G"] & \mathrm{KK}^G_{i+1}(C_0(Y_1),\mathrm{Ind}_H^G B) \\
\mathrm{KK}^H_{i}(C_0(X_0),B) \arrow[u, "{[\delta]\otimes}"] \arrow[r, "\mathrm{Ind}_H^G"] & \mathrm{KK}^G_{i}(C_0(Y_0),\mathrm{Ind}_H^G B). \arrow[u, "{[d]\otimes}"']
\end{tikzcd} \tag{6.5}$$

By the naturality of the boundary maps, we may assume without loss of generality that $Y = G\times_H X$ (and then $Y_i = G\times_H X_i$, $i=0,1$), and that Ind_H^G coincides with Kasparov's induction i_H^G. We then follow the constructions in the proof of Lemma 5.17: Define the spaces

$$T = \big(X\times[0,1[\big)\smallsetminus\big(X_0\times]0,1[\big) \text{ and } W = \big((G\times_H X)\times[0,1[\big)\smallsetminus\big((G\times_H X_0)\times]0,1[\big).$$

Let $e_X : C_0(X_0)\to C_0(T)$ and $e_{G\times_H X} : C_0(G\times_H X_0)\to C_0(W)$ denote the canonical inclusions, and let i_X and $i_{G\times_H X}$ denote the canonical inclusions of $C_0\big(X_1\times]0,1[\big)$ and $C_0\big((G\times_H X_1)\times]0,1[\big)$ into $C_0(T)$ and $C_0(W)$, respectively. A short computation shows that $[e_{G\times_H X}] = i_H^G([e_X])$ and $[i_{G\times_H X}] = i_H^G([i_X])$ (where H and G act trivially on $[0,1]$). Moreover, we know from the discussion in the proof of Lemma 5.17 that $[e_X]$ and $[e_{G\times_H X}]$ are KK-equivalences and that

$$[e_X]\otimes[\delta] = \sigma_{C_0(X_1)}(\beta)\otimes[i_X] \quad\text{and}\quad [e_{G\times_H X}]\otimes[d] = \sigma_{C_0(G\times_H X_1)}(\beta)\otimes[i_{G\times_H X}],$$

where $\beta \in \mathrm{KK}_1\big(\mathbb{C}, C_0(]0,1[)\big)$ denotes the Bott-element, viewed as an element of the equivariant KK-groups with respect to the trivial group actions. Using the fact that i_H^G preserves Kasparov products, we now get

$$\begin{aligned}
[e_{G\times_H X}] \otimes [d] &= \sigma_{C_0(G\times_H X_1)}(\beta) \otimes [i_{G\times_H X}] \\
&= i_H^G\big(\sigma_{C_0(X_1)}(\beta) \otimes [i_X]\big) \\
&= i_H^G\big([e_X] \otimes \delta\big) \\
&= [e_{G\times_H X}] \otimes i_H^G([\delta]).
\end{aligned}$$

Since $[e_{G\times_H X}]$ is a KK-equivalence, it follows that $[d] = i_H^G([\delta])$, which easily implies the commutativity of (6.5). $\square$

We are now taking limits of Diagram (6.4): First we are taking the inductive limit over the H-compact sets X such that $X \subset Y$ and $G \cdot X = Y$, and then we take the limit over the G-compact subsets Y of $\mathcal{E}(G) \times Z$. As a result, we obtain the commutative diagram

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}^G_{i+1}\big(C_0(X_1), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}^{G,N}_{i+1}\big(C_0(Y_1), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}^G_{i}\big(C_0(X_0), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}^{G,N}_{i}\big(C_0(Y_0), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}^G_{i}\big(C_0(X), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}^{G,N}_{i}\big(C_0(Y), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}^G_{i}\big(C_0(X_1), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}^{G,N}_{i}\big(C_0(Y_1), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow \\
\lim_Y \lim_X \mathrm{KK}^G_{i-1}\big(C_0(X_0), B\big) & \xrightarrow{\mathrm{Ind}_H^G} & \lim_Y \mathrm{KK}^{G,N}_{i-1}\big(C_0(Y_0), \mathrm{Ind}_H^G B\big) \\
\uparrow & & \uparrow
\end{array}$$

Using the same induction argument as in the proof of Proposition 5.16 of the previous section (based on the Five Lemma), the demonstration of Proposition 6.6, and hence the proof of Theorem 2.2 reduces to show

Lemma 6.8. *Let Z be an element of the family $\mathcal{F}_G$.*

(i) If $\dim(Z) > 0$, then the map

$$\varinjlim_{\substack{Y\subset\mathcal{E}(G)\times Z\\ Y\ G\text{-compact}}} \ \varinjlim_{\substack{X\subset Y\\ X\ H\text{-compact}\\ G\cdot X=Y}} \mathrm{KK}_i^H\big(C_0(X_0), B\big) \ \overset{\mathrm{Ind}_H^G}{\longrightarrow} \ \varinjlim_{\substack{Y\subset\mathcal{E}(G)\times Z\\ Y\ G\text{-compact}}} \mathrm{KK}_i^G\big(C_0(Y_0), \mathrm{Ind}_H^G B\big)$$

is a bijection.

(ii) If $\dim(Z) = 0$*, then the map*

$$\varinjlim_{\substack{Y\subset\mathcal{E}(G)\times Z\\ Y\ G\text{-compact}}}\ \varinjlim_{\substack{X\subset Y\\ X\ H\text{-compact}\\ G\cdot X=Y}} \mathrm{KK}_i^H\big(C_0(X),B\big) \overset{\mathrm{Ind}_H^G}{\longrightarrow} \varinjlim_{\substack{Y\subset\mathcal{E}(G)\times Z\\ Y\ G\text{-compact}}} \mathrm{KK}_i^G\big(C_0(Y),\mathrm{Ind}_H^G B\big)$$

is a bijection.

Proof. We will only show part (i), since part (ii) follows by almost the same (but somewhat easier) arguments. So assume that $\dim(Z) > 0$. By the definition of $\mathcal{F}_G$, the space Z_0 is a disjoint union of finitely many spaces Z_0^i, $i = 1, 2, ..., k$, each of the form $Z_0^i = G/G_0\cdot \overset{o}{S}_i$, where the S_i are simplices of dimensions $\dim(Z)$ of $\mathcal{E}(G/G_0)$. Setting $Z^i = G/G_0 \cdot S_i$, $Y_0^i = Y \cap (\mathcal{E}(G/G_0) \times Z_0^i)$ and $X_0^i = X \cap Y_0^i$, we obtain finite partitions of Y_0 and of X_0:

$$C_0(Y) = \bigoplus_{i=1}^{k} C_0(Y_0^i) \quad \text{and} \quad C_0(X) = \bigoplus_{i=1}^{k} C_0(X_0^i).$$

Note that these decompositions are compatible with the morphism Ind_H^G, so it is enough to give a proof of Lemma 6.8 in the case where Z is the G/G_0-saturation of a single simplex S of $\mathcal{E}(G/G_0)$. Further, the inductive limits over Y are taken over the G-compact subspaces of $\mathcal{E}(G) \times Z$. But any such space can be embedded in a G-compact set of the special form $Y = G \cdot (K \times S)$, where K is a compact subset of $\mathcal{E}(G)$. Hence, we can assume that every set Y which appears in the formula of the inductive limit is of this special kind.
Denote by $\dot{C}$ the stabilizer of S under the action of G/G_0, and let $C := q^{-1}(\dot{C}) \subseteq G$. Then

$$Y = G\cdot(K\times S) = G\cdot\big((C\cdot K)\times S\big) \quad \text{and} \quad Y_0 = G\times_C\big((C\cdot K)\times \overset{o}{S}\,\big).$$

For every double coset $\ddot{g} \in H\backslash G/C$, we consider the space

$$Y^{\ddot{g}} = Hg\cdot(C\cdot K\times S).$$

It is a closed H-invariant subspace of Y, and any H-compact subspace X of Y can be written as

$$X = \cup_{\ddot{g}\in F_X} X^{\ddot{g}}, \quad \text{with } X^{\ddot{g}} := X\cap Y^{\ddot{g}},$$

where F_X is a *finite* subset of $H\backslash G/C$. Put $X_0^{\ddot{g}} = X^{\ddot{g}} \cap Y_0$. We record the fact that each $X^{\ddot{g}}$ is an H-compact subspace of $Y^{\ddot{g}}$, and that X_0 is the disjoint union of the $X_0^{\ddot{g}}$, $\ddot{g} \in F_X$. As a consequence, we get

$$\mathrm{KK}_*^H\big(C_0(X_0),B\big) = \bigoplus_{\ddot{g}\in F_X} \mathrm{KK}_*^H\big(C_0(X_0^{\ddot{g}}),B\big).$$

Moreover, any H-compact subset of $Y^{\ddot{g}} = Hg \cdot (C \cdot K \times S)$ can be realized as a subset of an H-compact set of the form $Hg \cdot (L \times S)$, for L a compact subset of $C \cdot K$ satisfying $C \cdot L = C \cdot K$. Thus, when taking the inductive limit over the H-compact sets X which satisfy $G \cdot X = Y$, we can always enlarge X in order to assume that for every $\ddot{g} \in F_X$, $X^{\ddot{g}} = Hg \cdot (L \times S)$, for some compact subset $L \subseteq C \cdot K$ such that $C \cdot L = C \cdot K$. Moreover, it follows from this that $X_0^{\ddot{g}} = Hg \cdot (L \times \overset{\circ}{S})$. Thus, we obtain:

$$\varinjlim_{\substack{X \subset Y = G\cdot(K\times S)\\ X\ H\text{–compact}\\ GX=Y}} \mathrm{KK}_i^H \big(C_0(X_0), B\big) = \bigoplus_{\ddot{g}\in H\backslash G/C} \varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L = C\cdot K}} \mathrm{KK}_i^H \big(C_0(Hg\cdot(L\times \overset{\circ}{S})), B\big).$$

Hence, in order to prove the first part of the lemma, it is enough to show the bijectivity of

$$\varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}} \bigoplus_{\ddot{g}} \varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}} \mathrm{KK}_i^H\big(C_0(Hg\cdot(L\times\overset{\circ}{S})), B\big) \xrightarrow{\mathrm{Ind}_H^G} \varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}} \mathrm{KK}_i^G\big(C_0(G\cdot(K\times\overset{\circ}{S})), \mathrm{Ind}_H^G B\big). \tag{6.6}$$

We already noticed that $Y_0 = G \cdot (K\times \overset{\circ}{S})$ is canonically G-homeomorphic to the induced space $Y_0 = G \times_C (C\cdot K\times \overset{\circ}{S})$. Correspondingly, we now check that $X_0^{\ddot{g}} = Hg \cdot (L\times \overset{\circ}{S})$ is also an induced space. The composition

$$\begin{array}{rcccl} X_0^{\ddot{g}} = Hg\cdot(L\times\overset{\circ}{S}) & \xrightarrow{\pi_2} & Hg\cdot\overset{\circ}{S} & \to & H/(gCg^{-1}\cap H)\\ hg\cdot(l,s) & \mapsto & hgs & \mapsto & h(gCg{-}1\cap H) \end{array}$$

is an H-equivariant map, and the pre-image of the coset $gCg{-}1 \cap H$ of the identity is $\pi_2^{-1}(g\cdot \overset{\circ}{S}) = g\cdot (C_H^g \cdot L\times \overset{\circ}{S})$, where C_H^g is the group $C\cap g^{-1}Hg$. Applying Proposition 5.12, we see that $X_0^{\ddot{g}}$ is H-homeomorphic to the induced space $H\times_{gCg^{-1}\cap H}\big(g\cdot(C_H^g\cdot L\times\overset{\circ}{S})\big)$, with H-homeomorphism given by

$$\begin{array}{rcl} X_0^{\ddot{g}} = Hg\cdot(L\times\overset{\circ}{S}) & \to & H\times_{gCg^{-1}\cap H}\big(g\cdot(C_H^g\cdot L\times\overset{\circ}{S})\big)\\ hg\cdot(l,s) & \mapsto & \big(h, g\cdot(l,s)\big) \end{array}$$

Let $\varphi_H := \mathrm{Bott}\circ \mathrm{comp}_{C_H^g}^H$ be the composition of the sequence of isomorphisms

$$\begin{array}{rcl} \mathrm{KK}_i^H\big(C_0(Hg\cdot(L\times\overset{\circ}{S})), B\big) & \xrightarrow{\mathrm{comp}_{C_H^g}^H} & \mathrm{KK}_i^{C_H^g}\big(C_0(C_H^g\cdot L\times\overset{\circ}{S}), B\big)\\ & \xrightarrow{\mathrm{Bott}} & \mathrm{KK}_{i+n}^{C_H^g}\big(C_0(C_H^g\cdot L), B\big), \end{array}$$

where $n = \dim(S)$. Here the compression isomorphism $\mathrm{comp}_{C_H^g}^H$ has to be understood as the composition of the compression

$$\mathrm{comp}_{H\cap gCg^{-1}}^H : \mathrm{KK}_*^H\big(C_0(X_0^{\ddot{g}}), B\big) \to \mathrm{KK}_*^{H\cap gCg^{-1}}\big(C_0(g\cdot(C_H^g\cdot L\times\overset{\circ}{S}), B\big),$$

and then making the identification

$$\mathrm{KK}_*^{H\cap gCg^{-1}}\big(C_0(g\cdot(C_H^g\cdot L\times \mathring{S}),B\big)\cong \mathrm{KK}_*^{C_H^g}\big(C_0(C_H^g\cdot L\times \mathring{S}),B\big),$$

which comes from identifying C_H^g with $H\cap gCg^{-1}$ via conjugation with g (please compare with the definition of the restriction map in Lemma 6.4). In particular, we regard B as a C_H^g-algebra by setting $g^{-1}hg\cdot b:=h\cdot b$ for $h\in gCg^{-1}\cap H$. By first taking the direct limit over the compact subsets $L\subseteq C\cdot K$, then taking the algebraic direct sum over the double cosets of $H\backslash G/C$, and eventually taking the inductive limit over the compact subsets $K\subseteq\mathcal{E}(G)$, we then obtain an isomorphism

$$\varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}} \bigoplus_{\ddot g} \varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}} \mathrm{KK}_i^H\big(C_0(X_0^{\ddot g}),B\big) \xrightarrow{\varphi_H} \varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}} \bigoplus_{\ddot g} \varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}} \mathrm{KK}_{i+n}^{C_H^g}\big(C_0(C_H^G\cdot L),B\big).$$

Note that the direct limit over K and the direct sum over $\ddot g\in H\backslash G/C$ can be permuted in the right-hand side term. Thus we get

$$\varinjlim_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}} \bigoplus_{\ddot g\in H\backslash G/C} \varinjlim_{\substack{L\subset C\cdot K\\ \text{compact}\\ C\cdot L=C\cdot K}} \mathrm{KK}_{i+n}^{C_H^g}\big(C_0(C_H^G\cdot L),B\big) = \bigoplus_{\ddot g\in H\backslash G/C} \mathrm{K}_{i+n}^{\mathrm{top}}(C_H^g;B).$$

In the end, we see that the left-hand side of (6.6) is isomorphic to $\bigoplus_{\ddot g\in H\backslash G/C} \mathrm{K}_{i+n}^{\mathrm{top}}(C_H^g;B)$.

On the right-hand side of (6.6) we have a corresponding sequence of isomorphisms: We first consider the composition $\varphi_G:=\mathrm{comp}_C^G\circ\mathrm{Bott}$ of the sequence of isomorphisms

$$\begin{aligned}\mathrm{KK}_i^G\big(C_0(Y_0),\mathrm{Ind}_H^G B\big) &\xrightarrow{\mathrm{comp}_C^G} \mathrm{KK}_i^C\big(C_0(C\cdot K\times\mathring{S}),\mathrm{Ind}_H^G B\big)\\ &\xrightarrow{\mathrm{Bott}} \mathrm{KK}_{i+n}^C\big(C_0(C\cdot K),\mathrm{Ind}_H^G B\big).\end{aligned}$$

Exactly as above, taking the direct limit over the compact subsets K of $\mathcal{E}(G)$, we obtain an isomorphism for the right-hand side of (6.6):

$$\begin{tikzcd} \varinjlim\limits_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}} \mathrm{KK}_i^G\big(C_0(G\cdot(K\times\mathring{S})),\mathrm{Ind}_H^G B\big) \arrow[r, "\varphi_G"] & \varinjlim\limits_{\substack{K\subset\mathcal{E}(G)\\ \text{compact}}} \mathrm{KK}_{i+n}^C\big(C_0(C\cdot K),\mathrm{Ind}_H^G B\big) \arrow[d, "\cong"] \\ & \mathrm{K}_{i+n}^{\mathrm{top}}(C;\mathrm{Ind}_H^G B) \end{tikzcd}$$

On the other hand, we can use Lemma 6.3 to see that $\mathrm{Ind}_H^G B$ is C-isomorphic to the direct sum $\bigoplus\limits_{\ddot{g}\in H\backslash G/C} \mathrm{Ind}_{C_H^g}^C B$. But C is a compact extension of G_0, so it is almost connected . Therefore, Lemma 6.5 implies that

$$\mathrm{K}_*^{\mathrm{top}}(C;\mathrm{Ind}_H^G B) \cong \mathrm{K}_*^{\mathrm{top}}(C; \bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{Ind}_{C_H^g}^C B) \cong \bigoplus_{\ddot{g}\in H\backslash G/C} \mathrm{K}_*^{\mathrm{top}}(C;\mathrm{Ind}_{C_H^g}^C B).$$

Let us now consider the following diagram, where the top line is the map (6.6)

$$\begin{tikzcd}
\varinjlim\limits_{Y} \bigoplus\limits_{\ddot{g}} \varinjlim\limits_{X^{\ddot{g}}\subset Y^{\ddot{g}}} \mathrm{KK}_i^H\big(C_0(X_0^{\ddot{g}}),B\big) \arrow[r, "\mathrm{Ind}_H^G"] \arrow[dd, "\varphi_H"'] & \varinjlim\limits_{Y} \mathrm{KK}_i^G\big(C_0(Y_0),\mathrm{Ind}_H^G B\big) \arrow[d, "\varphi_G"'] \\
 & \mathrm{K}_{i+n}^{\mathrm{top}}(C;\mathrm{Ind}_H^G B) \arrow[d, "\cong"'] \\
\bigoplus\limits_{\ddot{g}\in H\backslash G/C} \mathrm{K}_{i+n}^{\mathrm{top}}(C_H^g;B) \arrow[r, "\bigoplus \mathrm{Ind}_{C_H^g}^C"'] & \bigoplus\limits_{\ddot{g}\in H\backslash G/C} \mathrm{K}_{i+n}^{\mathrm{top}}(C;\mathrm{Ind}_{C_H^g}^C B).
\end{tikzcd} \tag{6.7}$$

The columns of (6.7) are bijections. The bottom line (obtained from the induction morphisms from C_H^g to C) is an isomorphism by Lemma 6.2, since C has a γ-element. Hence, to obtain that the map in (6.6) is an isomorphism, and thus to conclude the proof of the lemma, we just have to show that Diagram (6.7) commutes.

For this let g be any element in G, let K be a compact subset of $\mathcal{E}(G)$, and let L be a compact subset of $C\cdot K$ such that $C\cdot L = C\cdot K$. Let

$$X_0^{\ddot{g}} = Hg\cdot(L\times \mathring{S}) \quad \text{and} \quad Y_0 = G\times_C (C\cdot L\times \mathring{S}).$$

For each $g''\in G$ let $\rho_{g''} : \bigoplus \mathrm{Ind}_{C_H^{g'}}^C B \to \mathrm{Ind}_{C_H^{g''}}^C B$ denote the canonical projection. To see that (6.7) commutes, we have to verify the following two statements:

(i) The diagram

$$\begin{tikzcd}
\mathrm{KK}_i^H\big(C_0(X_0^{\ddot{g}}), B\big) \arrow[r, "\mathrm{Ind}_H^G"] \arrow[d, "\mathrm{comp}^H_{C_H^g}"'] & \mathrm{KK}_i^G\big(C_0(Y_0), \mathrm{Ind}_H^G B\big) \arrow[d, "\mathrm{comp}_C^G"] \\
\mathrm{KK}_i^{C_H^g}\big(C_0(C_H^g \cdot L \times \mathring{S}), B\big) \arrow[dd, "\otimes\,\mathrm{Bott}"'] & \mathrm{KK}_i^C\big(C_0(C \cdot K \times \mathring{S}), \mathrm{Ind}_H^G B\big) \arrow[d, "\otimes\,\mathrm{Bott}"] \\
 & \mathrm{KK}_{i+n}^C\big(C_0(C \cdot K), \mathrm{Ind}_H^G B\big) \arrow[d, "\rho_g^*"] \\
\mathrm{KK}_{i+n}^{C_H^g}\big(C_0(C_H^g \cdot L), B\big) \arrow[r, "\mathrm{Ind}^C_{C_H^g}"'] & \mathrm{KK}_{i+n}^C\big(C_0(C \cdot K), \mathrm{Ind}^C_{C_H^g} B\big)
\end{tikzcd}$$

commutes, and

(ii) The composition

$$\begin{aligned}
\mathrm{KK}_i^H\big(C_0(X_0^{\ddot{g}}), B\big) &\xrightarrow{\mathrm{Ind}_H^G} \mathrm{KK}_i^G\big(C_0(Y_0), \mathrm{Ind}_H^G B\big) \\
&\xrightarrow{\mathrm{comp}_C^G} \mathrm{KK}_i^C\big(C_0(C \cdot K \times \mathring{S}), \mathrm{Ind}_H^G B\big) \\
&\xrightarrow{\mathrm{Bott}} \mathrm{KK}_{i+n}^C\big(C_0(C \cdot K), \mathrm{Ind}_H^G B\big) \\
&\xrightarrow{\rho_{g''}} \mathrm{KK}_{i+n}^C\big(C_0(C \cdot K), \mathrm{Ind}^C_{C_H^{g''}} B\big)
\end{aligned}$$

is the zero homomorphism whenever $g'' \notin HgC$.

If $\beta \in \mathrm{KK}_n(\mathbb{C}, C_0(\mathring{S}))$ denotes the Bott-element, then the first condition is just the equation

$$\beta \otimes \mathrm{comp}_C^G \circ \mathrm{Ind}_H^G(\alpha) \otimes [\rho_g] = \mathrm{Ind}^C_{C_H^g}(\beta \otimes \mathrm{comp}^H_{C_H^g}(\alpha)),$$

for all $\alpha \in \mathrm{KK}_i^H\big(C_0(X_0^{\ddot{g}}), B\big)$. Since C acts trivially on $\mathring{S}$, we can permute $\mathrm{Ind}^C_{C_H^g}$ and the product with the Bott element β. Thus, the problem reduces to showing that

$$\mathrm{comp}_C^G \circ \mathrm{Ind}_H^G(\alpha) \otimes [\rho_g] = \mathrm{Ind}^H_{C_H^g} \circ \mathrm{comp}^H_{C_H^g}(\alpha). \tag{6.8}$$

In order to check this equation, it is useful to introduce the following notations:

- F_1 is the G-equivariant map appearing in the definition of Ind_H^G:

$$\begin{array}{rccc} F_1: & G \times_H X_0^{\ddot{g}} & \to & Y_0 \subset \mathcal{E}(G) \times \mathcal{E}(G/N) \\ & [g_1, hg \cdot (l, s)] & \mapsto & g_1 hg \cdot (l, s), \end{array}$$

where we used the equation $X_0^{\ddot{g}} = Hg(L \times \mathring{S})$.

- i_1 is the C-equivariant inclusion used to define comp_C^G

$$i_1: \quad C \cdot L \times \overset{\circ}{S} \quad \to \quad G \times_C (C \cdot L \times \overset{\circ}{S}) \cong Y_0,$$

- i_2 is the C_H^g-equivariant inclusion in the definition of $\mathrm{comp}_{C_H^g}^H$

$$\begin{array}{rccc} i_2: & C_H^g \cdot L \times \overset{\circ}{S} & \to & X_0^{\ddot{g}} = Hg \cdot (L \times \overset{\circ}{S}) \\ & (c \cdot l, s) & \mapsto & gc \cdot (l, s), \end{array}$$

- F_2 is the C-equivariant map in the definition of $\mathrm{Ind}_{C_H^g}^C$

$$\begin{array}{rccc} F_2: & C \times_{C_H^g} (C_H^g \cdot L \times \overset{\circ}{S}) & \to & C \cdot L \times \overset{\circ}{S} \\ & [c, (c' \cdot l, s)] & \mapsto & (c'c \cdot l, s). \end{array}$$

We will also use that i_2 induces a C-equivariant injection

$$\begin{array}{rccc} I_2: & C \times_{C_H^g} (C_H^g \cdot L \times \overset{\circ}{S}) & \to & C \times_{C_H^g} (X_0^{\ddot{g}}) = C \times_{C_H^g} \left(Hg \cdot (L \times \overset{\circ}{S})\right) \\ & [c_1, (c_2 \cdot l, s)] & \mapsto & [c_1, gc_2 \cdot (l, s)]. \end{array}$$

and we will denote by i_3 the C-equivariant inclusion

$$\begin{array}{rccc} i_3: & C \times_{C_H^g} X_0^{\ddot{g}} & \to & G \times_H (X_0^{\ddot{g}}) \\ & [c, x] & \mapsto & [cg^{-1}, x]. \end{array}$$

Writing the KK-classes defined by these maps with the same letters, Equation (6.8) becomes:

$$[i_1] \otimes \mathrm{res}_C^G([F_1] \otimes i_H^G(\alpha)) \otimes [\rho_g] = [F_2] \otimes i_{C_H^g}^C([i_2] \otimes \mathrm{res}_{C_H^G}^H(\alpha)).$$

Since $[I_2] = \mathrm{Ind}_C^{C_H^g}([i_2])$, this is equivalent to:

$$[i_1] \otimes \mathrm{res}_C^G([F_1]) \otimes \mathrm{res}_C^G(\mathrm{Ind}_H^G(\alpha)) \otimes [\rho_g] = [F_2] \otimes [I_2] \otimes \mathrm{Ind}_{C_H^g}^C(\mathrm{res}_{C_H^G}^H(\alpha)). \tag{6.9}$$

A short computation shows that $[i_1] \otimes \mathrm{res}_C^G([F_1]) = [F_2] \otimes [I_2] \otimes [i_3]$ (just compute the compositions of the associated $*$-homomorphisms). Thus, Equation (6.9) reduces to:

$$[i_3] \otimes \mathrm{res}_C^G(i_H^G(\alpha)) \otimes [\rho_g] = i_{C_H^g}^C(\mathrm{res}_{C_H^G}^H(\alpha)).$$

Note now that $i_{3,*}$ coincides with the canonical inclusion

$$\lambda_g: \mathrm{Ind}_{C_H^g}^C C_0(X_0^{\ddot{g}}) \to \bigoplus_{g'} \mathrm{Ind}_{C_H^{g'}}^C C_0(X_0^{\ddot{g}}) \cong \mathrm{res}_C^G(\mathrm{Ind}_H^G C_0(X_0^{\ddot{g}})),$$

of Lemma 6.4, so that the last equality follows from the statement of that lemma.

We now have verified statement (i). Using the above computations, the proof of statement (ii) follows from the equation $[i_3] \otimes \mathrm{res}_C^G(i_H^G(\alpha)) \otimes [\rho_g''] = 0$, for $g'' \notin HgC$, which is also a consequence of Lemma 6.4. □

7 Continuity of topological K-theory

The aim of this short section is to state and prove a generalization of Lemma 6.5 which is used in [7]. In a similar way as in §5-6 above, we obtain the result by using $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$ as a universal example for the proper actions of G, where we assume that $\mathcal{E}(G/G_0)$ is a simplicial complex (compare with the discussions preceding Proposition 6.6).

Proposition 7.1. *Let G be a group, let (B_i, f_{ij}) be an inductive system of G-algebras, and let $B = \lim B_i$. Then*

$$\mathrm{K}_*^{\mathrm{top}}(G; B) \cong \lim_i \mathrm{K}_*^{\mathrm{top}}(G; B_i),$$

where the isomorphism is obtained from the morphisms $f_{i,*} : \mathrm{K}_*^{\mathrm{top}}(G; B_i) \to \mathrm{K}_*^{\mathrm{top}}(G; B)$ induced by the canonical maps $f_i : B_i \to B$.

Proof. Let $f^* : \lim_i \mathrm{K}_*^{\mathrm{top}}(G; B_i) \to \mathrm{K}_*^{\mathrm{top}}(G; B)$ be the homomorphism induced by the morphisms $f_i : B_i \to B$, using the covariance of the topological K-theory groups as a functor on the category of G-C^*-algebras and the universal property of the inductive limit. We want to show that f^* is an isomorphism. For every proper G-space X, let

$$f_X^* : \lim_i \mathrm{KK}_*^G\big(C_0(X), B_i\big) \to \mathrm{KK}_*^G\big(C_0(X), B\big)$$

denote the canonical morphism on the level of X. Since the structure maps for taking the limits over X are given by left Kasparov products and the structure maps for taking limits over the B_i are given by right Kasparov products, it follows from the associativity of the Kasparov product that the limits can be permuted. Thus, the map f^* can be computed via the maps f_X^* by

$$\lim_X \Big(\lim_i \mathrm{KK}_*^G(C_0(X), B_i) \Big) \xrightarrow{f_X^*} \lim_X \mathrm{KK}_*^G(C_0(X), B), \tag{7.1}$$

where X runs through the G-compact subsets of $\mathcal{E}(G) \times \mathcal{E}(G/G_0)$, which we use as a realization of the universal example for the proper actions of G.
As before, let $\mathcal{F}$ denote the family of all G-saturations Z of finite unions of simplices in $\mathcal{E}(G/G_0)$. It follows then from Lemma 5.6 that

$$\begin{aligned} \mathrm{K}_*^{\mathrm{top}}(G; B_i) &= \lim_{Z \in \mathcal{F}} \lim_X \mathrm{KK}_*^G\big(C_0(X), B_i\big) \quad \text{and} \\ \mathrm{K}_*^{\mathrm{top}}(G; B) &= \lim_{Z \in \mathcal{F}} \lim_X \mathrm{KK}_*^G\big(C_0(X), B\big), \end{aligned}$$

where X runs through the G-compact subsets of $\mathcal{E}(G) \times Z$ such that $\pi_2(X) = Z$, and where $\pi_2 : \mathcal{E}(G) \times \mathcal{E}(G/G_0) \to \mathcal{E}(G/G_0)$ denotes the projection onto the second factor. Combining these formulas with (7.1), the result will follow if we can show that for each $Z \in \mathcal{F}$ the map

$$\lim_{\substack{X \subseteq \mathcal{E}(G) \times Z \\ X\ G-\mathrm{compact} \\ \pi_2(X) = Z}} \lim_i \mathrm{KK}_*^G(C_0(X), B_i) \xrightarrow{f_X^*} \lim_{\substack{X \subseteq \mathcal{E}(G) \times Z \\ X\ G-\mathrm{compact} \\ \pi_2(X) = Z}} \mathrm{KK}_*^G(C_0(X), B) \tag{7.2}$$

is an isomorphism. We write Z_0 for the union of the interiors $\overset{\circ}{S}$ of simplices S in Z of maximal dimension, and we put $Z_1 = Z \smallsetminus Z_0$,

$$X_0 = X \cap \big(\mathcal{E}(G) \times Z_0\big) \qquad \text{and} \qquad X_1 = X \cap \big(\mathcal{E}(G) \times Z_1\big).$$

Doing a similar, but much easier Five-Lemma argument as in the previous sections (compare with the discussions preceding Lemma 6.8), the result will follow if we can show that the following two statements are true:

(i) Assume that Z is generated by a single simplex S in $\mathcal{E}(G/G_0)$ with $\dim(S) > 0$. Then

$$\lim_{\substack{X\subseteq\mathcal{E}(G)\times Z\\ X\ G-\text{compact}\\ \pi_2(X)=Z}} \lim_i \mathrm{KK}_*^G(C_0(X_0), B_i) \xrightarrow{f_{X_0}^*} \lim_{\substack{X\subseteq\mathcal{E}(G)\times Z\\ X\ G-\text{compact}\\ \pi_2(X)=Z}} \mathrm{KK}_*^G(C_0(X_0), B)$$

is bijective.

(ii) Assume that Z is the orbit of a single vertex in $\mathcal{E}(G/G_0)$. Then the map in (7.2) is bijective.

Again, the proof of (ii) is slightly easier than the proof of (i) (because we don't have to deal with the Bott-map), so we concentrate on (i). By the structure of Z, we have

$$Z_0 = G/G_0 \cdot \overset{\circ}{S} \cong (G/G_0) \times_{\dot{C}} \overset{\circ}{S},$$

where $\overset{\circ}{S}$ denotes the interior of the single simplex S generating Z and $\dot{C} \subseteq G/G_0$ denotes the stabilizer of S. Thus, if X is a G-compact subset of $\mathcal{E}(G) \times Z$ such that $\pi_2(X) = Z$, then it follows from Proposition 5.12 that X_0 is G-homeomorphic to the induced space $G \times_C \big(X \cap \pi_2^{-1}(\overset{\circ}{S})\big)$, where $C := q^{-1}(\dot{C}) \subseteq G$. Enlarging X, if necessary, we may further assume that $X \cap \pi_2^{-1}(\overset{\circ}{S}) = C \cdot K \times \overset{\circ}{S}$ for some compact subset $K \subseteq \mathcal{E}(G)$. Thus, using compression and Bott-periodicity, and taking the limit over X, we obtain the following commutative diagram

$$\begin{array}{ccc}
\lim\limits_{K,i} \mathrm{KK}_*^G\big(C_0\big(G \times_C (C{\cdot}K \times \overset{\circ}{S})\big), B_i\big) & \xrightarrow{f_{X_0}^*} & \lim\limits_{K} \mathrm{KK}_*^G\big(C_0\big(G \times_C (C{\cdot}K \times \overset{\circ}{S})\big), B\big) \\
{\scriptstyle \mathrm{Bott}\circ\mathrm{comp}_C^G}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle \mathrm{Bott}\circ\mathrm{comp}_C^G} \\
\lim\limits_{K} \lim\limits_{i} \mathrm{KK}_*^C\big(C_0(C \cdot K), B_i\big) & \xrightarrow{f_{C\cdot K}^*} & \lim\limits_{K} \mathrm{KK}_*^C\big(C_0(C \cdot K), B\big) \\
{\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\
\lim\limits_{i} \mathrm{K}_*^{\mathrm{top}}(C; B_i) & \xrightarrow{f^*} & \mathrm{K}_*^{\mathrm{top}}(C; B),
\end{array}$$

where K runs through the compact subsets of $\mathcal{E}(G)$. Note that the left-hand lower vertical isomorphism is given by permuting the limits. The top horizontal line coincides with the map in Item (i) above. Thus, since the bottom horizontal row is an isomorphism by Lemma 6.5 (again, here C is almost connected), the result follows. □

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Rolling Factors Deformations and Extensions of Canonical Curves

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Abstract. A tetragonal canonical curve is the complete intersection of two divisors on a scroll. The equations can be written in 'rolling factors' format. For such homogeneous ideals we give methods to compute infinitesimal deformations. Deformations can be obstructed. For the case of quadratic equations on the scroll we derive explicit base equations. They are used to study extensions of tetragonal curves.

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An easy dimension count shows that not all canonical curves are hyperplane sections of $K3$ surfaces. A surface with a given curve as hyperplane section is called an extension of the curve. With this terminology, the general canonical curve has only trivial extensions, obtained by taking a cone over the curve. In this paper we concentrate on extensions of tetragonal curves.

The extension problem is related to deformation theory for cones. This is best seen in terms of equations. Suppose we have coordinates $(x_0 : \cdots : x_n : t)$ on $\mathbb{P}^{n+1}$ with the special hyperplane section given by $t = 0$. We describe an extension W of a variety $V\colon f_j(x_i) = 0$ by a system of equations $F_j(x_i, t) = 0$ with $F_j(x_i, 0) = f_j(x_i)$. We write $F_j(x_i, t) = f_j(x_i) + tf'_j(x_i) + \cdots + a_j t^{d_j}$, where d_j is the degree of F_j. Considering $(x_0, \ldots, x_n, t)$ as affine coordinates on $\mathbb{C}^{n+1} \times \mathbb{C}$ we can read the equations in a different way. The equations $f_j(x_i) = 0$ define the affine cone $C(V)$ over V and $F_j(x_i, t) = 0$ describes a 1-parameter deformation of $C(V)$. The corresponding infinitesimal deformation is $f_j(x_i) \mapsto f'_j(x_i)$, which is a deformation of weight -1. Conversely, given a 1-parameter deformation $F_j(x_i, t) = 0$ of $C(V)$, with F_j homogeneous of degree d_j, we get an extension W of V. For most of the cones considered here the only infinitesimal deformations of negative weight have weight -1 and in that case the versal deformation in negative weight gives a good description of all possible extensions.

As the number of equations typically is much larger than the codimension one needs good ways to describe them. A prime example is a determinantal scheme X: its ideal is generated by the $t \times t$ minors of an $r \times s$ matrix, which gives a compact description of the equations. Following Miles Reid we call this a format. Canonical curves are not themselves determinantal, but they do lie on scrolls: a k-gonal curve lies on a $(k-1)$-dimensional scroll, which is given by the minors of a $2 \times (g-k+1)$ matrix. For $k=3$ the curve is a divisor on the scroll, given by one bihomogeneous equation, and for $k=4$ it is a complete intersection, given by two bihomogeneous equations. In these cases there is a simple procedure (*'rolling factors'*) to write out one resp. two sets of equations on $\mathbb{P}^{g-1}$ cutting out the curve on the scroll.

Powerful methods exist to compute infinitesimal deformations without using explicit equations. We used them for the extension problem for hyperelliptic curves of high degree [Stevens 1996] and trigonal canonical curves [Drewes–Stevens 1996]. In these papers also several direct computations with the equations occur. They seem unavoidable for tetragonal curves, the subject of a preprint by James N. Brawner [Brawner 1996]. The results of these computations do not depend on the particular way of choosing the equations cutting out the curve on the scroll. This observation was the starting point of this paper.

We distinguish between different types of deformations and extensions. If only the equations on the scroll are deformed, but not the scroll itself we speak of *pure rolling factors deformations*. A typical extension lies then on the projective cone over the scroll. Such a cone is a special case of a scroll of one dimension higher. If the extension lies on a scroll which is not a cone, the equations of the scroll are also deformed. We have a *rolling factors deformation*. Finally if the extension does not lie on a scroll of one dimension higher we are in the situation of a *non-scrollar* deformation. Non-scrollar extensions of tetragonal curves occur only in connection with Del Pezzo surfaces. Not every infinitesimal deformation of a scroll gives rise to a deformation of complete intersections on it. One needs certain lifting conditions, which are linear equations in the deformation variables of the scroll. Our first main result describes them, depending only on the coefficients of the equations on the scroll.

The next problem is to extend the infinitesimal deformations to a versal deformation. Here we restrict ourselves to the case that all defining equations are quadratic. Our methods thus do not apply to trigonal curves, but we can handle tetragonal curves. Rolling factors obstructions arise. Previously we observed that one can write them down, given explicit equations on $\mathbb{P}^n$ [Stevens 1996, Prop. 2.12]. Here we give formulas depending only on the coefficients of the equations on the scroll. As first application we study base spaces for hyperelliptic cones. The equations have enough structure so that explicit solutions can be given.

Surfaces with canonical hyperplane sections are a classical subject. References to the older literature can be found in Epema's thesis [Epema 1983], which is especially relevant for our purposes. His results say that apart from $K3$ surfaces only rational surfaces or birationally ruled surfaces can occur. Furthermore he describes a construction of such surfaces. Extensions of pure rolling factors type of tetragonal curves fit very well in this description. A general rolling factors extension is a complete intersection on a nonsingular four-dimensional scroll. The classification of such surfaces [Brawner 1997], which we recall below, shows that surfaces with isolated singularities and in particular $K3$s can only occur if the degrees of the equations on the scroll differ at most by 4. A tetragonal curve of high genus with general discrete invariants has no pure rolling factors deformations. Extensions exist if the base equations have a solution. For low genus we have more variables than equations. For the maximal genus where almost all curves have a $K3$ extension we find:

Proposition. *The general tetragonal curve of genus* 15 *is hyperplane section of* 256 *different* $K3$ *surfaces.*

We also look at examples with genus 16 and 17. It is unclear to us which property of a curve makes it have an extension (apart from the property of being a hyperplane section).
The contents of this paper is as follows. First we describe the rolling factors format and explain in detail the equations and relations for the complete intersection of two divisors on a scroll. Next we recall how canonical curves fit into this pattern. In particular we describe the discrete invariants for tetragonal curves. The same is done for $K3$ surfaces. The second section is devoted to the computation of infinitesimal deformations. First non-scrollar deformations are treated, followed by rolling factors deformations. The main result here describes the lifting matrix. As application the dimension of T^1 is determined for tetragonal cones. In the third section the base equations for complete intersections of quadrics on scrolls are derived. As examples base spaces for hyperelliptic cones are studied. The final section describes extensions of tetragonal curves.

1. Rolling factors format.

A subvariety of a determinantal variety can be described by the determinantal equations and additional equations obtained by 'rolling factors' [Reid 1989]. A typical example is the case of divisors on scrolls.
We start with a k-dimensional rational normal scroll $S \subset \mathbb{P}^n$ (for the theory of scrolls we refer to [Reid 1997]). The classical construction is to take k complementary linear subspaces L_i spanning $\mathbb{P}^n$, each containing a parametrized rational normal curve $\phi_i\colon \mathbb{P}^1 \to C_i \subset L_i$ of degree $d_i = \dim L_i$, and to take for each $p \in \mathbb{P}^1$ the span of the points $\phi_i(p)$. The degree of S is $d = \sum d_i = n-k+1$. If all $d_i > 0$ the scroll S is a $\mathbb{P}^{d-1}$-bundle over $\mathbb{P}^1$. We allow however that $d_i = 0$ for some i. Then S is the image of $\mathbb{P}^{d-1}$-bundle $\widetilde{S}$ over $\mathbb{P}^1$ and $\widetilde{S} \to S$ is a rational resolution of singularities.
To give a coordinate description, we take homogeneous coordinates $(s:t)$ on $\mathbb{P}^1$, and $(z^{(1)} : \cdots : z^{(k)})$ on the fibres. Coordinates on $\mathbb{P}^n$ are $z_j^{(i)} = z^{(i)} s^{d_i - j} t^j$,

with $0 \le j \le d_i$, $1 \le i \le k$. We give the variable $z^{(i)}$ the weight $-d_i$. The scroll S is given by the minors of the matrix

$$\Phi = \begin{pmatrix} z_0^{(1)} & \dots & z_{d_1-1}^{(1)} & \dots & z_0^{(k)} & \dots & z_{d_k-1}^{(k)} \\ z_1^{(1)} & \dots & z_{d_1}^{(1)} & \dots & z_1^{(k)} & \dots & z_{d_k}^{(k)} \end{pmatrix}.$$

We now consider a divisor on $\widetilde{S}$ in the linear system $|aH - bR|$, where the hyperplane class H and the ruling R generate the Picard group of $\widetilde{S}$. When we speak of degree on $\widetilde{S}$ this will be with respect to H. The divisor can be given by one bihomogeneous equation $P(s,t,z^{(i)})$ of degree a in the $z^{(i)}$, and total degree $-b$. By multiplying $P(s,t,z^{(i)})$ with a polynomial of degree b in $(s:t)$ we obtain an equation of degree 0, which can be expressed as polynomial of degree a in the $z_j^{(i)}$; this expression is not unique, but the difference of two expressions lies in the ideal of the scroll. By the obvious choice, multiplying with $s^{b-m}t^m$, we obtain $b+1$ equations P_m. In the transition from the equation P_m to P_{m+1} we have to increase by one the sum of the lower indices of the factors $z_j^{(i)}$ in each monomial, and we can and will always achieve this by increasing exactly one index. This amounts to replacing a $z_j^{(i)}$, which occurs in the top row of the matrix, by the element $z_{j+1}^{(i)}$ in the bottom row of the same column. This is the procedure of '*rolling factors*'.

Example 1.1. Consider the cone over $2d-b$ points in $\mathbb{P}^d$, lying on a rational normal curve of degree d, with $b < d$. Let the polynomial $P(s,t) = p_0 s^{2d-b} + p_1 s^{2d-b-1}t + \cdots + p_{2d-b}t^{2d-b}$ determine the points on the rational curve. We get the determinantal

$$\begin{vmatrix} z_0 & z_1 & \dots & z_{d-1} \\ z_1 & z_2 & \dots & z_d \end{vmatrix}$$

and additional equations P_m. To be specific we assume that $b = 2c$:

$$\begin{aligned} P_0 &= p_0 z_0^2 + p_1 z_0 z_1 + \cdots + p_{2d-2c-1} z_{d-c-1} z_{d-c} + p_{2d-2c} z_{d-c}^2 \\ P_1 &= p_0 z_0 z_1 + p_1 z_1^2 + \cdots + p_{2d-2c-1} z_{d-c}^2 + p_{2d-2c} z_{d-c} z_{d-c+1} \\ &\vdots \\ P_{2c} &= p_0 z_c^2 + p_1 z_c z_{c+1} + \cdots + p_{2d-2c-1} z_{d-1} z_d + p_{2d-2c} z_d^2 \,. \end{aligned}$$

The 'rolling factors' phenomenon can also occur if the entries of the matrix are more general.

Example 1.2. Consider a non-singular hyperelliptic curve of genus 5, with a half-canonical line bundle $L = g_1^2 + P_1 + P_2$ where the P_i are Weierstrass points. According to [Reid 1989], Thm. 3, the ring $R(C,L) = \bigoplus H^0(C,nL)$ is $k[x_1,x_2,y_1,y_2,z_1,z_2]/I$ with I given by the determinantal

$$\begin{vmatrix} x_1 & y_1 & x_2^2 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{vmatrix}$$

and the three rolling factors equations

$$\begin{aligned} z_1^2 &= x_1^2 h + y_1^3 + x_2^4 y_2 \\ z_1 z_2 &= x_1 x_2 h + y_1^2 x_1^2 + x_2^2 y_2^2 \\ z_2^2 &= x_2^2 h + y_1 x_1^4 + y_2^3 \end{aligned}$$

where h is some quartic in x_1, x_2, y_1, y_2.

The description of the syzygies of a subvariety V of the scroll S proceeds in two steps. First one constructs a resolution of $\mathcal{O}_{\widetilde{V}}$ by vector bundles on $\widetilde{S}$ which are repeated extensions of line bundles. Schreyer describes, following Eisenbud, Eagon-Northcott type complexes $\mathcal{C}^b$ such that $\mathcal{C}^b(a)$ is the minimal resolution of $i_*\big(\mathcal{O}_{\widetilde{S}}(-aH+bR)\big)$ as $\mathcal{O}_{\mathbb{P}^n}$-module, if $b \geq -1$ [Schreyer 1986]. Here $i\colon \widetilde{S} \to \mathbb{P}^n$ is the map defined by H. The resolution of $\mathcal{O}_V$ is then obtained by taking an (iterated) mapping cone.
The matrix Φ defining the scroll can be obtained intrinsically from the multiplication map

$$H^0\mathcal{O}_{\widetilde{S}}(R) \otimes H^0\mathcal{O}_{\widetilde{S}}(H-R) \longrightarrow H^0\mathcal{O}_{\widetilde{S}}(H)\ .$$

In general, given a map $\Phi\colon F \to G$ of locally free sheaves of rank f and g respectively, $f \geq g$, on a variety one defines Eagon-Northcott type complexes $\mathcal{C}^b$, $b \geq -1$, in the following way:

$$\mathcal{C}^b_j = \begin{cases} \bigwedge^j F \otimes S_{b-j} G, & \text{for } 0 \leq j \leq b \\ \bigwedge^{j+g-1} F \otimes D_{j-b-1} G^* \otimes \bigwedge^g G^*, & \text{for } j \geq b+1 \end{cases}$$

with differential defined by multiplication with $\Phi \in F^* \otimes G$ for $j \neq b+1$ and $\bigwedge^g \Phi \in \bigwedge^g F^* \otimes \bigwedge^g G$ for $j = b+1$ in the appropriate term of the exterior, symmetric or divided power algebra.
In our situation $F \cong \mathcal{O}^d_{\mathbb{P}^n}(-1)$ and $G \cong \mathcal{O}^2_{\mathbb{P}^n}$ with Φ given by the matrix of the scroll. Then $\mathcal{C}^b(-a)$ is for $b \geq -1$ the minimal resolution of $\mathcal{O}_{\widetilde{S}}(-aH+bR)$ as $\mathcal{O}_{\mathbb{P}}$-module [Schreyer 1986, Cor. 1.2].
Now let $V \subset S \subset \mathbb{P}^n$ be a 'complete intersection' of divisors $Y_i \sim a_iH - b_iR$, $i = 1, \ldots, l$, on a k-dimensional rational scroll of degree d with $b_i \geq 0$. The resolution of $\mathcal{O}_V$ as $\mathcal{O}_S$-module is a Koszul complex and the iterated mapping cone of complexes $\mathcal{C}^b$ is the minimal resolution [Schreyer 1986, Sect. 3, Example].
To make this resolution more explicit we look at the case $l = 2$, which is relevant for tetragonal curves. The iterated mapping cone is

$$\big[\mathcal{C}^{b_1+b_2}(-a_1-a_2) \longrightarrow \mathcal{C}^{b_1}(-a_1) \oplus \mathcal{C}^{b_2}(-a_2)\big] \longrightarrow \mathcal{C}^0$$

To describe equations and relations we give the first steps of this complex. We first consider the case that $b_1 \geq b_2 > 0$. We write $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^n}$. We get the double complex

$$\begin{array}{ccccc}
\mathcal{O} & \longleftarrow & \bigwedge^2 \mathcal{O}^d(-1) & \longleftarrow & \bigwedge^3 \mathcal{O}^d(-1)\otimes\mathcal{O}^2 \\
\uparrow & & \uparrow & & \\
S_{b_1}\mathcal{O}^2(-a_1)\oplus S_{b_2}\mathcal{O}^2(-a_2) & \longleftarrow & \mathcal{O}^d(-1)\otimes S_{b_1-1}\mathcal{O}^2(-a_1)\oplus\mathcal{O}^d(-1)\otimes S_{b_2-1}\mathcal{O}^2(-a_2) & & \\
\uparrow & & & & \\
S_{b_1+b_2}\mathcal{O}^2(-a_1-a_2) & & & &
\end{array}$$

The equations for V consist of the determinantal ones plus two sets of additional equations obtained by rolling factors: the two equations $P^{(1)}$, $P^{(2)}$ defining V on the scroll give rise to b_1+1 equations $P^{(1)}_m$ and b_2+1 equations $P^{(2)}_m$.
To describe the relations we introduce the following notation. A column in the matrix Φ has the form $(z^{(i)}_j, z^{(i)}_{j+1})$. We write symbolically $(z_\alpha, z_{\alpha+1})$, where the index α stands for the pair ${}^{(i)}_{j}$ and $\alpha+1$ means adding 1 to the lower index. More generally, if $\alpha = {}^{(i)}_{j}$ and $\alpha' = {}^{(i')}_{j'}$ then the sum $\alpha+\alpha' := j+j'$ only involves the lower indices. To access the upper index we say that α is of type i. The rolling factors assumption is that two consecutive additional equations are of the form

$$P_m = \sum_\alpha p_{\alpha,m} z_\alpha,$$
$$P_{m+1} = \sum_\alpha p_{\alpha,m} z_{\alpha+1}.$$

where the polynomials $p_{\alpha,m}$ depend on the z-variables and the sum runs over all possible pairs $\alpha = {}^{(i)}_{j}$. To roll from P_{m+1} to P_{m+2} we collect the 'coefficients' in the equation P_{m+1} in a different way: we also have $P_{m+1} = \sum_\alpha p_{\alpha,m+1} z_\alpha$.
We write the scrollar equations as $f_{\alpha\beta} = z_\alpha z_{\beta+1} - z_{\alpha+1} z_\beta$. The relations between them are

$$R_{\alpha,\beta,\gamma} = f_{\alpha,\beta} z_\alpha - f_{\alpha,\gamma} z_\beta + f_{\beta,\gamma} z_\alpha,$$
$$S_{\alpha,\beta,\gamma} = f_{\alpha,\beta} z_{\alpha+1} - f_{\alpha,\gamma} z_{\beta+1} + f_{\beta,\gamma} z_{\alpha+1},$$

which corresponds to the term $\bigwedge^3 \mathcal{O}^d(-1)\otimes\mathcal{O}^2$ in Schreyer's resolution. The second line yields relations involving the two sets of $P^{(n)}_m$:

$$R^n_{\beta,m} = P^{(n)}_{m+1} z_\beta - P^{(n)}_m z_{\beta+1} - \sum_\alpha f_{\beta,\alpha} p^{(n)}_{\alpha,m},$$

where $n = 1,2$ and $0 \le m < b_i$. We note the following relation:

$$R^n_{\beta,m} z_\gamma - R^n_{\beta,m} z_\beta - \sum R^n_{\beta,\gamma,\alpha} p^{(n)}_{\alpha,m} = P^{(n)}_m f_{\beta,\gamma} - f_{\beta,\gamma} P^{(n)}_m.$$

The right hand side is a Koszul relation; the second factor in each product is considered as coefficient. There are similar expressions involving $z_{\gamma+1}$, $z_{\beta+1}$

and $S_{\beta,\gamma,\alpha}$. Finally by multiplication with suitable powers of s and t the Koszul relation $P^{(1)}P^{(2)} - P^{(2)}P^{(1)}$ gives rise to $b_1 + b_2 + 1$ relations — this is the term $S_{b_1+b_2}\mathcal{O}^2(-a_1 - a_2)$.
In case $b_1 > b_2 = 0$ the resolution is

$$\begin{array}{ccccc}
\mathcal{O} & \leftarrow & \bigwedge^2 \mathcal{O}^d(-1) & \leftarrow & \bigwedge^3 \mathcal{O}^d(-1)\otimes\mathcal{O}^2 \\
\uparrow & & \uparrow & & \\
S_{b_1}\mathcal{O}^2(-a_1)\oplus\mathcal{O}^2(-a_2) & \leftarrow & \mathcal{O}^d(-1)\otimes S_{b_1-1}\mathcal{O}^2(-a_1)\oplus\bigwedge^2\mathcal{O}^d(-1-a_2) & & \\
\uparrow & & & & \\
S_{b_1}\mathcal{O}^2(-a_1-a_2) & & & &
\end{array}$$

The new term expresses the Koszul relations between the one equation $P^{(2)}$ and the determinantal equations (which had previously been expressible in terms of rolling factors relations). For the computation of deformations these relations may be ignored.
Finally, if $b_2 = -1$, the equations change drastically.

(1.3) *Canonical curves* [Schreyer 1986].
A k-gonal canonical curve lies on a $(k-1)$-dimensional scroll of degree $d = g - k + 1$. We write D for the divisor of the g^1_k. To describe the type $S(e_1, ..., e_{k-1})$ of the scroll we introduce the numbers

$$f_i = h^0(C, K - iD) - h^0(C, K - (i+i)D) = k + h^0(iD) - h^0((i+1)D)$$

for $i \geq 0$ and set

$$e_i = \#\{j \mid f_j \geq i\} - 1 \; .$$

In particular, e_1 is the minimal number i such that $h^0((i+1)D) - h^0(iD) = k$ and it satisfies therefore $e_1 \leq \frac{2g-2}{k}$.
A trigonal curve lies on a scroll of type $S(e_1, e_2)$ and degree $d = e_1 + e_2 = g - 2$ with

$$\frac{2g-2}{3} \geq e_1 \geq e_2 \geq \frac{g-4}{3}$$

as a divisor of type $3H - (g-4)R$. The minimal resolution of $\mathcal{O}_C$ is given by the mapping cone

$$\mathcal{C}^{d-2}(-3) \longrightarrow \mathcal{C}^0 \; .$$

Introducing bihomogeneous coordinates $(x : y; s : t)$ and coordinates $x_i = xs^{e_1-i}t^i$, $y_i = ys^{e_2-i}t^i$ we obtain the scroll

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{e_1-1} & y_0 & y_1 & \dots & y_{e_2-1} \\ x_1 & x_2 & \dots & x_{e_1} & y_1 & y_2 & \dots & y_{e_2} \end{pmatrix}$$

and a bihomogeneous equation for C

$$P = A_{2e_1-e_2+2}x^3 + B_{e_1+2}x^2y + C_{e_2+2}xy^2 + D_{2e_2-e_1+2}y^3$$

where $A_{2e_1-e_2+2}$ is a polynomial in $(s:t)$ of degree $2e_1-e_2+2$ and similarly for the other coefficients. By rolling factors P gives rise to $g-3$ extra equations. The inequality $e_1 \leq \frac{2g-2}{3}$ can also be explained from the condition that the curve C is nonsingular, which implies that the polynomial P is irreducible, and therefore the degree $2e_2-e_1+2 = 2g-2-3e_1$ of the polynomial $D_{2e_2-e_1+2}$ is nonnegative. The other inequality follows from this one because $e_1 = g-e_2-2$, but also by considering the degree of $A_{2e_1-e_2+2}$.
A tetragonal curve of genus $g \geq 5$ is a complete intersection of divisors $Y \sim 2H-b_1R$ and $Z \sim 2H-b_2R$ on a scroll of type $S(e_1,e_2,e_3)$ of degree $d = e_1+e_2+e_3 = g-3$, with $b_1+b_2 = d-2$, and

$$\frac{g-1}{2} \geq e_1 \geq e_2 \geq e_3 \geq 0$$

We introduce bihomogeneous coordinates $(x:y:z;s:t)$. Then Y is given by an equation

$$P = P_{1,1}x^2 + P_{1,2}xy + \cdots + P_{3,3}z^2$$

with P_{ij} (if nonzero) a polynomial in $(s:t)$ of degree $e_i+e_j-b_1$ and likewise Z has equation

$$Q = Q_{1,1}x^2 + Q_{1,2}xy + \cdots + Q_{3,3}z^2$$

with $\deg Q_{ij} = e_i+e_j-b_2$.
The minimal resolution is of type discussed above, because the condition $-1 \leq b_2 \leq b_1 \leq d-1$ is satisfied: the only possibility to have a divisor of type $2H-bR$ with $b \geq d$ is to have $e_1 = e_2 = d/2$, $e_3 = 0$ and $b = d$, but then the equation P is of the form $\alpha x^2 + \beta xy + \gamma y^2$ with constant coefficients, so reducible. If $b_2 = -1$ also cubics are needed to generate the ideal, so the curve admits also a g^1_3 or g^2_5; this happens only up to $g = 6$. We exclude these cases and assume that $b_2 \geq 0$.

LEMMA 1.4. *We have $b_1 \leq 2e_2$ and $b_2 \leq 2e_3$.*

Proof. If $b_1 > 2e_2$ the polynomials P_{22}, P_{23} and P_{33} vanish so P is reducible and therefore C. If $b_2 > 2e_3$ then P_{33} and Q_{33} vanish. This means that the section $x = y = 0$ is a component of $Y \cap Z$ on the $\mathbb{P}^2$-bundle whose image in $\mathbb{P}^{g-1}$ is the scroll (if $e_3 > 0$ the scroll is nonsingular, but for $e_3 = 0$ it is a cone). As the arithmetic genus of $Y \cap Z$ is g and its image has to be the nonsingular curve C of genus g, the line cannot be a component. $\square$

This Lemma is parts 2 – 4 in [Brawner 1997, Prop. 3.1]. Its last part is incorrect. It states that $b_1 \leq e_1+e_3$ if $e_3 > 0$, and builds upon the fact that Y has only isolated singularities. However the discussion in [Schreyer 1986] makes clear that this need not be the case.
The surface Y fibres over $\mathbb{P}^1$. There are now two cases, first that the general fibre is a non-singular conic. In this case one of the coefficients P_{13}, P_{23} or P_{33} is nonzero, giving indeed $b_1 \leq e_1+e_3$.

The other possibility is that each fibre is a singular conic. Then Y is a birationally ruled surface over a (hyper)elliptic curve E with a rational curve $\overline{E}$ of double points, the canonical image of E, and C does not intersect $\overline{E}$. This means that the section $\overline{E}$ of the scroll does not intersect the surface Z, so if one inserts the parametrisation of $\overline{E}$ in the equation of Z one obtains a non-zero constant. Let the section be given by polynomials in $(s:t)$, which if nonzero have degree d_s-e_1, d_s-e_2, d_s-e_3. Inserting them in the polynomial Q gives a polynomial of degree $2d_s-b_2$. So b_2 is even and $2d_s=b_2\leq 2e_3$. On the other hand $d_s-e_3\geq 0$ so $d_s=e_3$ and $b_2=2e_3$. The genus of E satisfies $p_a(E)=b_2/2+1$. If $b_1>e_1+e_3$, then Y is singular along the section $x=y=0$. An hyperelliptic involution can also occur if $b_1\leq e_1+e_3$.
We have shown:

LEMMA 1.5. *If Y is singular, in particular if $b_1>e_1+e_3$, then $b_2=2e_3$.*

Finally we analyze the case $b_2=0$ (cf. [Brawner 1996]).

LEMMA 1.6. *A nonsingular tetragonal curve is bi-elliptic or lies on a Del Pezzo surface if and only if $b_2=0$. The first case occurs for $e_3=0$, and the second for the values $(2,0,0)$, $(1,1,0)$, $(2,1,0)$, $(1,1,1)$, $(3,1,0)$, $(2,2,0)$, $(2,1,1)$, $(3,2,0)$, $(2,2,1)$, $(4,2,0)$, $(3,2,1)$ or $(2,2,2)$ of the triple (e_1,e_2,e_3).*

Proof. If the curve is bielliptic or lies on a Del Pezzo, the g^1_4 is not unique, which implies that the scroll is not unique. This is only possible if $b_2=0$ by [Schreyer 1986], p. 127. Then C is the complete intersection of a quadric and a surface Y of degree $g-1$, which is uniquely determined by C.
The inequality $e_1+e_2+e_3-2=b_1\leq 2e_2$ shows that $e_3\leq e_2-e-1+2\leq 2$. If the general fibre of Y over $\mathbb{P}^1$ is non-singular we have $b_1\leq e_1+e_3$. This gives $e_2\leq 2$ and $b_1\leq 4$. The possible values are now easily determined. If the general fibre of Y is singular then $e_3=b_2/2=0$ and Y is an elliptic cone. □

(1.7) *$K3$ surfaces.*
Let X be a $K3$ surface (with at most rational double point singularities) on a scroll. If the scroll is nonsingular the projection onto $\mathbb{P}^1$ gives an elliptic fibration on X, whose general fibre is smooth. This is even true if the scroll is singular: the strict transform $\widetilde{X}$ on $\widetilde{S}$ has only isolated singularities.
We start with the case of divisors. A treatment of such scrollar surfaces with an elliptic fibration can be found in [Reid 1997, 2.11]. One finds:

LEMMA 1.8. *For the general $F\in|3H-kR|$ on a scroll $S(e_1,e_2,e_3)$ the general fibre of the elliptic fibration is a nonsingular cubic curve if and only if $k\leq 3e_2$ and $k\leq e_1+2e_3$.*

If one fixes k and $e_1+e_2+e_3$ these conditions limit the possible distribution of the integers (e_1,e_2,e_3). By the adjunction formula one has $k=e_1+e_2+e_3-2$ for a $K3$ surface. In this case we obtain 12 solutions, which fall into 3 deformation types of scrolls, according to $\sum e_i \pmod 3$:

$$(e+2,e,e-2) \to (e+1,e,e-1) \to (e,e,e)$$
$$(e+3,e,e-2) \to (e+2,e,e-1) \to (e+1,e+1,e-1) \to (e+1,e,e)$$
$$(e{+}4,e,e{-}2) \to (e{+}3,e,e{-}1) \to (e{+}2,e{+}1,e{-}1) \to (e{+}2,e,e) \to (e{+}1,e{+}1,e)$$

The general element of the linear system can only have singularities at the base locus. The base locus is the section $(0{:}0{:}1)$ if and only if $k > 3e_3$ and there is a singularity at the points $(s{:}t)$ where both $A_{e_1+2e_3-k}$ and $A_{e_2+2e_3-k}$ vanish. The assumption that the coefficients are general implies now that $\deg A_{e_2+2e_3-k} < 0$ and $\deg A_{e_1+2e_3-k} > 0$.
In the 12 cases above this occurs only for $(e+3,e,e-1)$ and $(e+2,e,e-1)$. In the first case the term y^2z is also missing, yielding that there is an A_2-singularity at the only zero of $A_{e_2+2e_3-k}$, whereas the second case gives an A_1. The scroll $S_{e+4,e,e-2}$ deforms into $S_{e+3,e,e-1}$, but the general $K3$-surface on it does not deform to a $K3$ on $S_{e+3,e,e-1}$, but only those with an A_2-singularity. These results hold if all $e_i > 0$; we leave the modifications in case $e_3 = 0$ to the reader.
The tetragonal case is given as exercise in [Reid 1997] and the complete solution (modulo some minor mistakes) can be found in [Brawner 1997]. We give the results:

LEMMA 1.9. *For the general complete intersection of divisors of type $2H - b_1R$ and $2H - b_2R$ on a scroll S_{e_1,e_2,e_3,e_4} the general fibre of the elliptic fibration is a nonsingular quartic curve if and only if either*
α: $b_1 \le e_1 + e_3$, $b_1 \le 2e_2$ and $b_2 \le 2e_4$, or
β: $b_1 \le e_1 + e_4$, $b_1 \le 2e_2$, $2e_4 < b_2 \le 2e_3$ and $b_2 \le e_2 + e_4$.

PROPOSITION 1.10. *The general element is singular at a point of the section $(0:0:0:1)$ if the invariants satisfy in addition one of the following conditions:*
1α: $b_2 < 2e_4$, $b_1 > e_1 + e_4$.
$1\beta i$: $b_2 \le e_3 + e_4$, $e_2 + e_4 < b_1 < e_1 + e_4$.
$1\beta ii$: $b_2 > e_3 + e_4$, and $e_1 + e_2 + 2e_4 > b_1 + b_2$.
There is a singularity with $z \ne 0$ if
2α: $b_1 > e_2 + e_3$, $e_1 + e_3 > b_1 > e_1 + e_4$.
$2\alpha\beta$: $e_1 + e_4 \ge b_1 > e_2 + e_3$ and
i: if $b_2 \le 2e_4$ then $2(e_1 + e_3 + e_4) > 2b_1 + b_2$
ii: if $2e_4 < b_2 \le e_3 + e_4$ then $e_1 + 2e_3 + e_4 > b_1 + b_2$
iii: $e_3 + e_4 < b_2 < 2e_3$

For $K3$ surfaces we need $b_1 + b_2 = e_1 + e_2 + e_3 + e_4 - 2$. We give a table listing the possibilities under this assumption, cf. [Brawner 1997, Table A.1–A.4].
The table lists the possible values for (b_1, b_2) and gives for each pair the invariants (e_1, e_2, e_3, e_4) of the scrolls on which the curve can lie. These form one deformation type with adjacencies going vertically, except $S_{e+2,e,e,e}$ and

(b_1, b_2)	(e_1, e_2, e_3, e_4)	$\mathcal{M}$	base	sings
$(2e, 2e-2)$	$(e+3, e+1, e-1, e-3)$	17	B_{e-1}	$--$
	$(e+3, e, e-1, e-2)$	15	B_{e-1}	A_3
	$(e+2, e+1, e-1, e-2)$	16	B_{e-1}	A_1
	$(e+2, e, e, e-2)$	16	B_{e-2}	$--$
	$(e+2, e, e-1, e-1)$	15	B_{e-1}	$2A_1$
	$(e+1, e+1, e-1, e-1)$	16	B_{e-1}	$--$
	$(e+1, e, e, e-1)$	17	B_{e-1}	$--$
	(e, e, e, e)	17	$\emptyset$	$--$
$(2e-1, 2e-1)$	$(e+1, e+1, e, e-2)$	17	B_{e-2}	$--$
	$(e+1, e, e, e-1)$	17	B_{e-1}	$--$
	(e, e, e, e)	18	$\emptyset$	$--$
$(2e+1, 2e-2)$	$(e+4, e+1, e-1, e-3)$	17	B_{e-1}	$--$
	$(e+3, e+1, e-1, e-2)$	16	B_{e-1}	A_1
	$(e+2, e+1, e-1, e-1)$	16	B_{e-1}	$--$
	$(e+1, e+1, e, e-1)$	17	B_e	$--$
$(2e, 2e-1)$	$(e+2, e+1, e, e-2)$	17	B_{e-2}	$--$
	$(e+2, e, e, e-1)$	15	B_{e-1}	A_1
	$(e+1, e+1, e, e-1)$	17	B_{e-1}	$--$
	$(e+1, e, e, e)$	18	$\emptyset$	$--$
$(2e+2, 2e-2)$	$(e+5, e+1, e-1, e-3)$	18	B_{e-1}	$--$
	$(e+4, e+1, e-1, e-2)$	17	B_{e-1}	A_1
	$(e+3, e+1, e-1, e-1)$	17	B_{e-1}	$--$
	$(e+2, e+1, e, e-1)$	18	B_e	$--$
	$(e+1, e+1, e+1, e-1)$	18	B_{e-1}	$--$
$(2e+1, 2e-1)$	$(e+3, e+1, e, e-2)$	16	B_e	$--$
	$(e+2, e+1, e, e-1)$	16	B_e	$--$
	$(e+1, e+1, e, e)$	17	B_e	$--$
$(2e, 2e)$	$(e+2, e+2, e, e-2)$	17	B_{e-2}	$--$
	$(e+2, e+1, e, e-1)$	16	B_{e-1}	A_1
	$(e+1, e+1, e+1, e-1)$	17	B_{e-1}	$--$
	$(e+2, e, e, e)$	15	$\emptyset$	$--$
	$(e+1, e+1, e, e)$	17	$\emptyset$	$--$
$(2e+2, 2e-1)$	$(e+4, e+1, e, e-2)$	16	B_e	A_1
	$(e+3, e+1, e, e-1)$	16	B_e	A_1
	$(e+2, e+1, e, e)$	17	B_e	A_1
	$(e+1, e+1, e+1, e)$	17	B_e	A_1
$(2e+1, 2e)$	$(e+3, e+2, e, e-2)$	17	B_e	$--$
	$(e+3, e+1, e, e-1)$	15	B_e	A_2
	$(e+2, e+2, e, e-1)$	16	B_e	A_1
	$(e+2, e+1, e+1, e-1)$	17	B_{e-1}	$--$
	$(e+2, e+1, e, e)$	16	B_e	$--$
	$(e+1, e+1, e+1, e)$	18	B_e	$--$

$S_{e+1,e+1,e+1,e-1}$ which do not deform into each other but are both deformations of $S_{e+2,e+1,e,e-1}$ and both deform to $S_{e+1,e+1,e,e}$. Furthermore we give the number of moduli for each family, in the column $\mathcal{M}$.
In the table we also list the base locus of $|2H - b_1R|$ (which contains that of $|2H - b_2R|$). The base locus is a subscroll, for which we use the following notation [Reid 1997, 2.8]: we denote by B_a the subscroll corresponding to the subset of all e_i with $e_i \le a$, defined by the equations $z^{(j)} = 0$ for $e_j > a$. We give the number and type of the singularities of the general element; the number given in the second half of [Brawner 1997, Table A.2] is not correct.
As example of the computations we look at $(e+3, e, e-1, e-2)$ with $(b_1, b_2) = (2e, 2e-2)$. The two equations have the form

$$p_1xw + p_2xz + p_0y^2 + p_3yx + p_6x^2$$
$$q_0z^2 + q_0yw + q_3xw + q_1yz + q_4xz + q_2y^2 + q_5yx + q_8x^2 \ ,$$

where the index denotes the degree in $(s:t)$. We first use coordinate transformations to simplify these equations. By replacing y, z and w by suitable multiples we may assume that the three constant polynomials are 1. Now replacing z by $z - \frac{1}{2}q_1y - \frac{1}{2}q_4x$ removes the yz and xz terms. We then replace y by $y - q_3x$ to get rid of the xw term. By changing w we finally achieve the form $z^2 + yw + q_8x^2$. By a change in $(s:t)$ we may assume that $p_1 = s$. We now look at the affine chart $(w = 1, t = 1)$ and find $y = -z^2 - q_8x^2$, which we insert in the other equation to get an equation of the form $x(s + p_2z + \cdots) + z^4$, which is an A_3.
We leave it again to the reader to analyze which further singularities can occur if $e_4 = 0$.

2. Infinitesimal deformations.

Deformations of cones over complete intersections on scrolls need not preserve the rolling factors format. We shall study in detail those who do. Many deformations of negative weight are of this type.

Definition 2.1. A *pure rolling factors deformation* is a deformation in which the scroll is undeformed and only the equations on the scroll are perturbed.

This means that the deformation of the additional equations can be written with the rolling factors. Such deformations are always unobstructed. However this is not the only type of deformation for which the scroll is not changed. In weight zero one can have deformations inside the scroll, where the type $(b_1, \dots, b_l)$ changes.

Definition 2.2. A (general) *rolling factors deformation* is a deformation in which the scroll is deformed and the additional equations are written in rolling factors with respect to the deformed scroll.

The equations for the total space of a 1-parameter rolling factors deformation describe a scroll of one dimension higher, containing a subvariety of the same codimension, again in rolling factors format. Deformations over higher dimensional base spaces may be obstructed. Again in weight zero one can have deformations of the scroll, where also the type $(b_1, \dots, b_l)$ changes.

Finally there are *non-scrollar deformations*, where the perturbation of the scrollar equations does not define a deformation of the scroll. Examples of this phenomenon are easy to find (but difficult to describe explicitly). A trigonal canonical curve is a divisor in a scroll, whereas the general canonical curve of the same genus g is not of this type: the codimension of the trigonal locus in moduli space is $g-4$.

Example 2.3. To give an example of a deformation inside a scroll, we let C be a tetragonal curve in $\mathbb{P}^9$ with invariants $(2,2,2;3,1)$. Then there is a weight 0 deformation to a curve of type $(2,2,2;2,2)$. To be specific, let C be given by $P = sx^2 + ty^2 + (s+t)z^2$, $Q = t^3x^2 + s^3y^2 + (s^3 - t^3)z^2$. We do not deform the scroll, but only the additional equations:

$$
\begin{array}{l}
x_0^2 + y_0y_1 + z_0^2 \quad + \varepsilon(z_1^2 - x_1^2) \\
x_0x_1 + y_1^2 + z_0z_1 + \varepsilon(z_1z_2 - x_1x_2) \\
x_1^2 + y_1y_2 + z_1^2 \quad + \varepsilon(y_0y_1 + z_0z_1) \\
x_1x_2 + y_2^2 + z_1z_2 + \varepsilon(y_1^2 + z_1^2) \\[4pt]
x_1x_2 + y_0^2 + z_0^2 - z_1z_2 \\
x_2^2 + y_0y_1 + z_0z_1 - z_2^2
\end{array}
$$

For $\varepsilon \neq 0$ we can write the ideal as

$$
\begin{array}{l}
x_0^2 + y_0y_1 + z_0^2 \quad + \varepsilon(z_1^2 - x_1^2) \\
x_0x_1 + y_1^2 + z_0z_1 + \varepsilon(z_1z_2 - x_1x_2) \\
x_1^2 + y_1y_2 + z_1^2 \quad + \varepsilon(z_2^2 - x_2^2) \\[4pt]
x_0x_1 + y_1^2 + z_0z_1 + \varepsilon(y_0^2 + z_0^2) \\
x_1^2 + y_1y_2 + z_1^2 \quad + \varepsilon(y_0y_1 + z_0z_1) \\
x_1x_2 + y_2^2 + z_1z_2 + \varepsilon(y_1^2 + z_1^2)
\end{array}
$$

We can describe this deformation in the following way. Write $Q = sQ_s + tQ_t$. The two times three equations above are obtained by rolling factors from $sP - \varepsilon Q_t$ and $tP + \varepsilon Q_s$. We may generalize this example.

Lemma 2.4. *Let V be a complete intersection of divisors of type $aH - b_1R$, $aH - b_2R$, given by equations P, Q. If it is possible to write $Q = sQ_s + t^{b_1-b_2-1}Q_t$ then the equations $sP - \varepsilon Q_t$, $t^{b_1-b_2-1}P + \varepsilon Q_s$ give a deformation to a complete intersection of type $aH - (b_1-1)R$, $aH - (b_2+1)R$.*

In general one has to combine such a deformation with a deformation of the scroll.

(2.5) *Non-scrollar deformations.*

Example 2.6. As mentioned before such deformations must exist in weight zero for trigonal cones. We proceed with the explicit computation of embedded deformations. We start from the normal bundle exact sequence

$$0 \longrightarrow N_{S/C} \longrightarrow N_C \longrightarrow N_S \otimes \mathcal{O}_C \longrightarrow 0\ .$$

As C is a curve of type $3H-(g-4)R$ on S we have that $C\cdot C=3g+6$ and $H^1(C,N_{S/C})=0$. So we are interested in $H^0(C,N_S\otimes\mathcal{O}_C)$, and more particularly in the cokernel of the map $H^0(S,N_S)\longrightarrow H^0(C,N_S\otimes\mathcal{O}_C)$, as $H^0(S,N_S)$ gives deformations of the scroll.

Proposition 2.7. *The cokernel of the map $H^0(S,N_S)\longrightarrow H^0(C,N_S\otimes\mathcal{O}_C)$ has dimension $g-4$.*

Proof. An element of $H^0(C,N_S\otimes\mathcal{O}_C)$ is a function φ on the equations of the scroll such that the generators of the module of relations map to zero in $\mathcal{O}_C$ and it lies in the image of $H^0(S,N_S)$ if the function values can be lifted to $\mathcal{O}_S$ such that the relations map to $0\in\mathcal{O}_S$. Therefore we perform our computations in $\mathcal{O}_S$.
We have to introduce some more notation. Using the equations described in (1.3) we have three types of scrollar equations, $f_{i,j}=x_ix_{j+1}-x_{i+1}x_j$, $g_{i,j}=y_iy_{j+1}-y_{i+1}y_j$ and mixed equations $h_{i,j}=x_iy_{j+1}-x_{i+1}y_j$. The scrollar relations come from doubling a row in the matrix and there are two ways to do this. The equations resulting from doubling the top row can be divided by s, and the other ones by t, so the result is the same.
A relation involving only equations of type $f_{i,j}$ gives the condition

$$xs^{e_1-i-1}t^i\varphi(f_{j,k})-xs^{e_1-j-1}t^j\varphi(f_{i,k})+xs^{e_1-k-1}t^k\varphi(f_{i,j})=0\in\mathcal{O}_C$$

which may be divided by x. As the image $\varphi(f_{i,j})$ is quadratic in x and y the resulting left hand side cannot be a multiple of the equation of C, so we have

$$s^{e_1-i-1}t^i\varphi(f_{j,k})-s^{e-1-j-1}t^j\varphi(f_{i,k})+s^{e_1-k-1}t^k\varphi(f_{i,j})=0\in\mathcal{O}_S$$

and the analogous equation involving only the $g_{i,j}$ equations.
For the mixed equations we get

$$xs^{e_1-i-1}t^i\varphi(h_{j,k})-xs^{e_1-j-1}t^j\varphi(h_{i,k})+ys^{e_2-k-1}t^k\varphi(f_{i,j})=\psi_{i,j;k}P\in\mathcal{O}_S$$

with $\psi_{i,j;k}$ of degree $e_1+e_2-3=g-5$ and analogous ones involving $g_{i,j}$ with coefficients $\psi_{i;j,k}$. These coefficients are not independent, but satisfy a systems of equations coming from the syzygies between the relations. They can also be verified directly. We obtain

$$s^{e_1-i-1}t^i\psi_{j,k;l}-s^{e_1-j-1}t^j\psi_{i,k;l}+s^{e_1-k-1}t^k\psi_{i,j;l}=0\in\mathcal{O}_S$$

and

$$xs^{e_1-i-1}t^i\psi_{j;k,l} - xs^{e_1-j-1}t^j\psi_{i;k,l} + ys^{e_2-k-1}t^k\psi_{i,j;l} - ys^{e_2-l-1}t^l\psi_{i,j;k} = 0$$

The last set of equations shows that $s^{e_2-k-1}t^k\psi_{i,j;l} = s^{e_2-l-1}t^l\psi_{i,j;k}$ (rolling factors!) and therefore $\psi_{i,j;k} = s^{e_2-k-1}t^k\psi_{i,j;}$ with $\psi_{i,j;}$ of degree e_1-2. This yields the equations

$$s^{e_1-i-1}t^i\psi_{j,k;} - s^{e_1-j-1}t^j\psi_{i,k;} + s^{e_1-k-1}t^k\psi_{i,j;} = 0$$

Our next goal is to express all $\psi_{i,j;}$ in terms of the $\psi_{i,i+1;}$ (where $0 \le i \le e_1-2$). First we observe by using the last equation for the triples $(0,i,i+1)$ and $(i,i+1,e_1-1)$ that $\psi_{i,i+1;}$ is divisible by t^i and by s^{e_1-i-2} so $\psi_{i,i+1;} = s^{e_1-i-2}t^ic_i$ for some constant c_i. By induction it then follows that $\psi_{i,j;} = s^{e_1-i-2}t^ic_{j-1} + s^{e_1-i-3}t^{i+1}c_{j-2} + \cdots + s^{e_1-j-1}t^{j-1}c_i$, so the solution of the equations depends on e_1-1 constants. Similarly one finds e_2-1 constants d_i for the $\psi_{i;j,k}$ so altogether $e_1+e_2-2 = g-4$ constants.
Finally we can solve for the perturbations of the equations. We give the formulas in the case that all d_i and all c_i but one are zero, say $c_\gamma = 1$. This implies that $\psi_{i,j;} = 0$ if $\gamma \notin [i,j)$ and $\psi_{i,j;} = s^{e_1-i-j+\gamma-1}t^{i+j-\gamma-1}$ if $\gamma \in [i,j)$; under the last assumption $\psi_{i,j;k} = s^{g-4-i-j-k+\gamma}t^{i+j+k-\gamma-1}$. We take $\varphi(f_{i,j}) = 0$ if $\gamma \notin [i,j)$. It follows that for a fixed k the $\varphi(h_{i,k})$ with $i \le \gamma$ are related by rolling factors, as are the $\varphi(h_{i,k})$ with $i > \gamma$. This reduces the mixed equations with fixed k to one, which can be solved for in a uniform way for all k. To this end we write the equation P as

$$(s^{2e_1-e_2-\gamma+2}A^+_\gamma + t^{\gamma+1}A^-_{2e_1-e_2-\gamma+1})x^3 + y(B_{e_1+2}x^2 + C_{e_2+2}xy + D_{2e_2-e_1+2}y^2)$$

which we will abbreviate as $(s^{2e_1-e_2-\gamma+2}A^+ + t^{\gamma+1}A^-)x^3 + yE$. We set

$$\begin{array}{rcll}
\varphi(f_{i,j}) & = & 0, & \text{if } \gamma \notin [i,j) \\
\varphi(f_{i,j}) & = & s^{e_1-i-j-1+\gamma}t^{i+j-\gamma-1}E, & \text{if } \gamma \in [i,j) \\
\varphi(g_{i,j}) & = & 0, & \\
\varphi(h_{i,k}) & = & -s^{e_2-1-k-i+\gamma}t^{i+k}A^-x^2, & \text{if } i \le \gamma \\
\varphi(h_{i,k}) & = & s^{2e_1+1-i-k}t^{i+k-\gamma-1}A^+x^2, & \text{if } i > \gamma
\end{array}$$

This is well defined, because all exponents of s and t are positive. □

A similar computation can be used to show that all elements of $T^1(\nu)$ with $\nu > 0$ can be written rolling factors type. However, even more is true, they can be represented as pure rolling factors deformations, see [Drewes–Stevens 1996], where a direct argument is given.

We generalize the above discussion to the case of a complete intersection of divisors of type $aH - b_iR$ (with the same $a \ge 2$) on a scroll

$$\begin{pmatrix} z^{(1)}_0 & \dots & z^{(1)}_{d_1-1} & \dots & z^{(k)}_0 & \dots & z^{(k)}_{d_k-1} \\ z^{(1)}_1 & \dots & z^{(1)}_{d_1} & \dots & z^{(k)}_1 & \dots & z^{(k)}_{d_k} \end{pmatrix}.$$

We have equations $f_{ij}^{(\alpha\beta)} = z_i^{(\alpha)} z_{j+1}^{(\beta)} - z_{i+1}^{(\alpha)} z_j^{(\beta)}$. The lowest degree in which non rolling factors deformations can occur is $a-3$. We get the conditions

$$z^{(\alpha)} s^{d_\alpha - i-1} t^i \varphi\big(f_{jk}^{(\beta\gamma)}\big) - z^{(\beta)} s^{d_\beta - j-1} t^j \varphi\big(f_{ik}^{(\alpha\gamma)}\big) + z^{(\gamma)} s^{d_\gamma - k-1} t^k \varphi\big(f_{ij}^{(\alpha\beta)}\big) = \sum_l \psi_{ijk;l}^{(\alpha\beta\gamma)} P^{(l)}$$

with the $\psi_{ijk;l}^{(\alpha\beta\gamma)}$ homogeneous polynomials in $(s{:}t)$ of degree $b_l - 1$. The relations between these polynomials come from the syzygies of the scroll: we add four of these relations, multiplied with a term linear in the $z^{(\alpha)}$; then the left hand side becomes zero, leading to a relation (in $\mathcal{O}_S$) between the $P^{(l)}$. As we are dealing with a complete intersection, the relations are generated by Koszul relations. Because the coefficients of the relation obtained are linear in the $z^{(\alpha)}$, they cannot lie in the ideal generated by the $P^{(l)}$ (as $a \geq 2$), so they vanish and we obtain for each l equations

$$z^{(\alpha)} s^{d_\alpha - i-1} t^i \psi_{jkm;l}^{(\beta\gamma\delta)} - z^{(\beta)} s^{d_\beta - j-1} t^j \psi_{ikm;l}^{(\alpha\gamma\delta)} + z^{(\gamma)} s^{d_\gamma - k-1} t^k \psi_{ijm;l}^{(\alpha\beta\delta)} - z^{(\delta)} s^{d_\delta - m-1} t^m \psi_{ijk;l}^{(\alpha\beta\gamma)} = 0 \,.$$

Here some of the $\alpha, \dots, \delta$ may coincide. If e.g. δ is different from α, β and γ, then $\psi_{ijk;l}^{(\alpha\beta\gamma)} = 0$. If there are at least four different indices (e.g. if the scroll is nonsingular of dimension at least four) then δ can always be chosen in this way, so all coefficients vanish and every deformation of degree $a-3$ is of rolling factors type.
Suppose now the scroll is a cone over a nonsingular 3-dimensional scroll, i.e. we have three different indices at our disposal. Then every $\psi_{ijk;l}^{(\alpha\beta\gamma)}$ with at most two different upper indices vanishes, and the ones with three different indices satisfy rolling factors equations. We conclude that for pairwise different α, β, γ

$$\psi_{ijk;l}^{(\alpha\beta\gamma)} = s^{d-i-j-k-3} t^{i+j+k} \psi'_l$$

with $d = d_\alpha + d_\beta + d_\gamma$ the degree of the scroll.
Finally, for the cone over a 2-dimensional scroll we get similar computations as in the trigonal example above.

Proposition 2.8. *A tetragonal cone (with $g > 5$) has non-scrollar deformations of degree -1 if and only if $b_2 = 0$. If the canonical curve lies on a Del Pezzo surface then the dimension is 1. If the curve is bielliptic then the dimension is $b_1 = g-5$.*

Proof. First suppose $e_3 > 0$. Then the only possibly non zero coefficients are the ψ'_l, which have degree $b_l + 2 - \sum e_i$. As $b_1 + b_2 = \sum e_i - 2$ they do not vanish iff $b_2 = 0$. In this case the computation yields one non rolling factors deformation of the Del Pezzo surface on which the curve lies.

If $e_3 = 0$, then $b_2 = 0$. For a bielliptic curve the methods above yield $(e_1 - 1) + (e_2 - 1) = b_1 = g - 5$ non-scrollar deformations (a detailed computation is given in [Brawner 1996]). Suppose now that the curve lies on a (singular) Del Pezzo surface. If $b_1 = e_1 > e_2 = 2$ then the equation P contains the monomial xz with nonzero coefficient, which we take to be 1, while there is no monomial yz. After a coordinate transformation we may assume that the same holds in case $e_1 = e_2 = 2$. Let $\varphi(h_{i,k}) \equiv \zeta_{i,k} z \bmod (x, y)$. In the equation

$$xs^{e_1-i-1}t^i\varphi(g_{j,k}) - ys^{e_2-j-1}t^j\varphi(h_{i,k}) + ys^{e_2-k-1}t^k\varphi(h_{i,j}) = \psi_{i;j,k}P$$

holding in $\mathcal{O}_S$ the monomial yz occurs only on the left hand side, which shows that the $\zeta_{i,k}$ are of rolling factors type in the first index. Being constants, they vanish. This means that in the equation

$$xs^{e_1-i-1}t^i\varphi(h_{j,k}) - xs^{e_1-j-1}t^j\varphi(h_{i,k}) + ys^{e_2-k-1}t^k\varphi(f_{i,j}) = \psi_{i,j;k}P$$

the monomial xz does not occur on the left hand side and therefore $\psi_{i,j;k} = 0$. We find only $e_2 - 1 = 1$ non rolling factors deformation. If $e_1 = 2$, $e_2 = 1$ we find one deformation. Finally, if $e_1 = 3$, $e_2 = 1$ then there is only one type of mixed equation. We have two constants c_0 and c_1. Let the coefficient of xz in P be $p_0 s + p_1 t$. We obtain the equations

$$\begin{aligned} s\zeta_{1,0} - t\zeta_{0,0} &= c_0(p_0 s + p_1 t) \\ s\zeta_{2,0} - t\zeta_{1,0} &= c_1(p_0 s + p_1 t) \end{aligned}$$

from which we conclude that $p_0c_0 + p_1c_1 = 0$, giving again only one non rolling factors deformation. $\square$

(2.9) *Rolling factors deformations of degree* -1.
We look at the miniversal deformation of the scroll:

$$\begin{pmatrix} z_0^{(1)} & \dots & z_{d_1-2}^{(1)} & z_{d_1-1}^{(1)} & z_0^{(2)} & \dots & z_{d_k-2}^{(k)} & z_{d_k-1}^{(k)} \\ z_1^{(1)} + \zeta_1^{(1)} & \dots & z_{d_1-1}^{(1)} + \zeta_{d_1-1}^{(1)} & z_{d_1}^{(1)} & z_1^{(2)} & \dots & z_{d_k-1}^{(k)} + \zeta_{d_k-1}^{(k)} & z_{d_k}^{(k)} \end{pmatrix}$$

To compute which of those deformations can be lifted to deformations of a complete intersection on the scroll we have to compute perturbations of the additional equations.
We assume that we have a complete intersection of divisors of type $aH - b_iR$ (with the same $a \geq 2$).
Extending the notation introduced before we write the columns in the matrix symbolically as $(z_\alpha, z_{\alpha+1} + \zeta_{\alpha+1})$. In order that this makes sense for all columns we introduce dummy variables $\zeta_0^{(i)}$ and $\zeta_{d_i}^{(i)}$ with the value 0.
The Koszul type relations give no new conditions, but the relation

$$P_{m+1}z_\beta - P_m z_{\beta+1} - \sum_\alpha p_{\alpha,m} f_{\beta\alpha} = 0$$

gives as equation in the local ring for the perturbations P'_m of P_m:

$$P'_{m+1}z_\beta - P'_m z_{\beta+1} - \sum_\alpha p_{\alpha,m}(\zeta_{\alpha+1}z_\beta - z_\alpha\zeta_{\beta+1}) = 0\ .$$

In particular we see that we can look at one equation on the scroll at a time. As $\sum p_{\alpha,m}z_\alpha = P_m$ the coefficient of $\zeta_{\beta+1}$ vanishes. Because $tz_\beta - sz_{\beta+1} = 0$ we get a condition which is independent of β:

$$sP'_{m+1} - tP'_m - s\sum_\alpha p_{\alpha,m}\zeta_{\alpha+1} = 0$$

This has to hold in the local ring, but as the degree of the $p_{\alpha,m}$ is lower than that of the equations defining the complete intersection on the scroll (here we use the assumption that all degrees a are equal), it holds on the scroll. From it we derive the equation

$$s^bP'_b - t^bP'_0 = \sum_{m=0}^{b-1}\sum_\alpha s^{m+1}t^{b-m-1}p_{\alpha,m}\zeta_{\alpha+1} \qquad (S)$$

which has to be solved with P'_b and P'_0 polynomials in the z_α of degree $a-1$. We determine the monomials on the right hand side.
The result depends on the chosen equations, but only on P_0 and P_b and not on the intermediate ones, provided they are obtained by rolling factors.

Example 2.10. Let $b = 4$. We take variables $y_i = s^{3-i}t^iy$, $z_i = s^{3-i}t^iz$ with deformations η_i, ζ_i, and roll from y_0z_0 to y_2z_2 in two different ways:

$$y_0z_0 \to y_1z_0 \to y_1z_1 \to y_2z_1 \to y_2z_2$$
$$y_0z_0 \to y_0z_1 \to y_0z_2 \to y_1z_2 \to y_2z_2$$

This gives as right-hand side of the equation (S) in the two cases

$$s^4t^3z\eta_1 + s^4t^3y\zeta_1 + s^5t^2z\eta_2 + s^5t^2y\zeta_2$$
$$s^4t^3y\zeta_1 + s^5t^2y\zeta_2 + s^4t^3z\eta_1 + s^5t^2z\eta_2$$

which is the same expression. Similarly, if we roll from z_0^2 to z_2^2 we get

$$2s^4t^3z\zeta_1 + 2s^5t^2z\zeta_2$$

However, if we roll in the last step from y_1z_2 to y_1z_3 we get

$$s^4t^3y\zeta_1 + s^5t^2y\zeta_2 + s^4t^3z\eta_1$$

(remember that we have no deformation parameter ζ_3).

To analyze the general situation it is convenient to use multi-index notation. The equation P of a divisor in $|aH - bR|$ may then be written as

$$P = \sum_{|I|=a} \sum_{j=0}^{\langle e,I\rangle - b} p_{I,j} s^{\langle e,I\rangle - b - j} t^j z^I \ .$$

Here $e = (e_1, \ldots, e_k)$ is the vector of degrees and z^I stands for $(z^{(1)})^{i_1} \cdot \ldots \cdot (z^{(k)})^{i_k}$.

PROPOSITION 2.11. *The lifting condition for the equations P_m is that for each I with $|I| = a-1$ and $\langle e, I\rangle < b-1$ the following $b - \langle e, I\rangle - 1$ linear equations hold:*

$$\sum_{l=1}^{k} \sum_{j=0}^{\langle e,I+\delta_l\rangle - b} (i_l + 1) p_{I+\delta_l,j} \zeta^{(l)}_{j+n} = 0 \ ,$$

where $0 < n < b - \langle e, I\rangle$

Proof. We look at a monomial $s^{\langle e,I'\rangle - b - j} t^j z^{I'}$. In rolling from P_0 to P_m we go from z_A to z_{A+B}. Here we write a monomial as product of a factors: $z_{\alpha_1} \cdots z_{\alpha_a}$ with i'_l factors of type l. Let $I' = I + \delta_l$ with δ_l the lth unit vector. The monomial leads to an expression in which the coefficient of z^I is

$$\sum_{\{q|\alpha_q \text{ of type } l\}} \sum_{r=1}^{\beta_q} s^{\langle e,I\rangle + r - j + \alpha_q} t^{b-r+j-\alpha_q} \zeta_{\alpha_q + r}$$

We stress that the choice of α_q can be very different for different j.
We collect all contributions and look at the coefficient of $s^{\langle e,I\rangle + n} t^{b-n} z^I$ with $0 < n < b - \langle e, I\rangle$. This cannot be realized as left-hand side of equation (S). Because $b - n = b - r + j - \alpha_q$ this coefficient is

$$\sum_{l=1}^{k} \sum_{j=0}^{\langle e,I+\delta_l\rangle - b} (i_l + 1) p_{I+\delta_l,j} \zeta^{(l)}_{j+n} = 0 \ ,$$

We note that all terms really occur: in rolling from z_A to z_{A+B} we have to increase the qth factor sufficiently many times, because $\langle e, I\rangle < b - 1$. □

Example 2.12: trigonal cones. Let the curve be given by the bihomogeneous equation

$$F = A_{2a-m+2} z^3 + B_{a+2} z^2 w + C_{m+2} z w^2 + D_{2m-a+2} w^3$$

then there are only conditions for $I = (0, 2)$, i.e. for w^2, as $a+m = g-2 > b-1$. So if $2m < b - 1 = g - 5$ we get $b - 2m - 1 = a - m - 3$ equations on the deformation variables $\zeta_1, \ldots, \zeta_{a-1}, \omega_1, \ldots, \omega_{m-1}$

$$\sum_{j=0}^{m+2} c_j \zeta_{j+n} + 3 \sum_{j=0}^{2m-a+2} d_j \omega_{j+n} = 0 \ ,$$

as stated in [Drewes–Stevens 1996, 3.11]. We have a system of linear equations so we can write the coefficient matrix. It consists of two blocks $(\,\mathcal{C} \mid \mathcal{D}\,)$ with $\mathcal{C}$ of the form

$$\begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{m+2} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & c_0 & c_1 & \dots & c_{m+1} & c_{m+2} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & c_0 & \dots & c_m & c_{m+1} & c_{m+2} & \dots & 0 & 0 & 0 \\ & & & \ddots & & & & \ddots & & & \\ 0 & 0 & 0 & \dots & c_0 & c_1 & c_2 & \dots & c_{m+2} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & c_0 & c_1 & \dots & c_{m+1} & c_{m+2} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & c_0 & \dots & c_m & c_{m+1} & c_{m+2} \end{pmatrix}$$

and $\mathcal{D}$ similarly. Obviously this system has maximal rank.

The Proposition gives a system of linear equations and we call the coefficient matrix *lifting matrix*. It was introduced for tetragonal cones in [Brawner 1996]. In general the lifting matrix will have maximal rank, but it is a difficult question to decide when this happens.

Example 2.13: trigonal $K3$s. We take the invariants (e,e,e), $b = 3e-2$ with $e \geq 3$. The $K3$ lies on $\mathbb{P}^2 \times \mathbb{P}^1$ and is given by an equation of bidegree $(3,2)$. Now there are six Is with $|I| = 2$ each giving rise to $e-3$ equations in $3(e-1)$ deformation variables. In general the matrix has maximal rank, but for special surfaces the rank can drop. Consider an equation of type $p_1x^3+p_2y^3+p_3z^3$ with the p_i quadratic polynomials in $(s:t)$ without common or multiple zeroes. Then the surface is smooth. The lifting equations corresponding to the quadratic monomials xy, xz and yz vanish identically and the lifting matrix reduces to a block-diagonal matrix of rank $3(e-3)$. The kernel has dimension 6, but the corresponding deformations are obstructed: an extension of the $K3$ would be a Fano 3-fold with isolated singularities lying as divisor of type $3H - bR$ on a scroll $S(e_1,e_2,e_3,e_4)$, and a computation reveals that such a Fano can only exist for $\sum e_i \leq 8$.

We can say something more for the lifting conditions coming from one quadratic equation.

Proposition 2.14. *The lifting matrix for one quadratic equation has dependent rows if and only if the generic fibre has a singular point on the subscroll B_{b-1}.*

Proof. The equation P on the scroll can be written in the form ${}^t z\Pi z$ with Π a symmetric $k \times k$ matrix with polynomials in $(s:t)$ as entries. The condition that there is a singular section of the form $z = (0,\dots,0,$ $z^{(l+1)}(s,t),\dots,z^{(k)}(s,t))$ with $e_l > b-1$ is that ${}^t z\Pi = 0$ or ${}^t z_{>l}\Pi_{>l} = 0$ where $z_{>l} = (z^{(l+1)}(s,t),\dots,z^{(k)}(s,t))$ and $\Pi_{>l}$ is the matrix consisting of the last $k-l$ rows of Π. The resulting system of equations for the coefficients of the polynomials $z^{(i)}(s,t)$ gives exactly the lifting matrix. $\square$

(2.15) *Tetragonal curves.* Most of the following results are contained in the preprint [Brawner 1996]. We have two equations on the scroll and the lifting matrix M can have rows coming from both equations. We first suppose that $b_2 > 0$. Then also $e_3 > 0$ and the number of columns of M is always $\sum(e_i - 1) = g - 6$, but the number of rows depends on the values of $(e_1, e_2, e_3; b_1, b_2)$: it is $\sum_{i,j} \max(0, b_i - e_j - 1)$.

Theorem 2.16. *Let X be the cone over a tetragonal canonical curve and suppose that $b_2 > 0$. Then $\dim T^1_X(-2) = 0$. Suppose that the g^1_4 is not composed with an involution of genus $\frac{b_2}{2} + 1$.*

1) If $b_1 < e_1 + 1$ or $b_2 < e_3 + 1$ or $g \leq 15$ then $\dim T^1_X(-1) = 9 + \dim \operatorname{Cork} M$.

2) If $b_1 \geq e_1 + 1$, $b_2 \geq e_3 + 1$ and $g > 15$ then $9 + \dim \operatorname{Cork} M \leq \dim T^1_X(-1) \leq \frac{g+3}{6} + 6 + \dim \operatorname{Cork} M$ and the maximum is obtained for g of the form $6n - 3$ and $(e_1, e_2, e_3; b_1, b_2) = (3n-2, 2n-2, n-2; 4n-4, 2n-4)$.

3) For generic values of the moduli $\dim \operatorname{Cork} M = 0$.

Proof. If $b_2 > 0$ there are only rolling factors deformations in negative degrees. In particular $\dim T^1_X(-2) = 0$. The number of pure rolling factors deformations is $\rho = \sum_{i,j} \max(e_j - b_i + 1, 0)$. The number of rows in the lifting matrix is $\sum_{i,j} \max(0, b_i - e_j - 1) = 3(b_1 + b_2) - 2(e_1 + e_2 + e_3 + 3) + \rho = g - 15 + \rho$. If $\rho > 9$ the number of rows exceeds the number of columns and $\dim T^1_X(-1) = \rho + \dim \operatorname{Cork} M$; otherwise it is $9 + \dim \operatorname{Cork} M$. So we have to estimate ρ.
As the g^1_4 is not composed we have $b_1 \leq e_1 + e_3$. Together with $b_1 \leq 2e_2$ we get $3b_1 \leq 2g - 6$ and $3b_2 \geq g - 9$; from $e_1 \leq \frac{g-1}{2}$ we now derive $e_1 - b_2 + 1 \leq \frac{g-1}{2} + 1 - \frac{g-9}{3}$. Also $b_2 = e_1 + e_2 + e_3 - 2 - b_1 \geq e_2 - 2$, so $e_2 - b_2 + 1 \leq 3$.
2) Suppose first that $b_1 \geq e_1 + 1$ and $b_2 \geq e_3 + 1$. Then $\rho = \max(0, e_1 - b_2 + 1) + \max(0, e_2 - b_2 + 1) \leq \frac{g+3}{6} + 6$. Equality is achieved iff $e_1 = (g-1)/2$, $b_2 = (g-9)/3$ and $e_2 = b_2 + 2$, so g has the form $6n - 3$ and $(e_1, e_2, e_3; b_1, b_2) = (3n-2, 2n-2, n-2; 4n-4, 2n-4)$.
1) In all other cases $\rho \leq 9$: if $b_1 \geq e_1 + 1$, but $b_2 < e_3 + 1$ then $\rho = (e_1 - b_2 + 1) + (e_2 - b_2 + 1) + (e_3 - b_2 + 1) = g - 3b_2 \leq 9$. If $e_2 + 1 \leq b_1 < e_1 + 1$ then $\rho = (e_1 - b_1 + 1) + (e_1 - b_2 + 1) + \max(0, e_2 - b_2 + 1) + \max(0, e_3 - b_2 + 1) = 2e_1 + 7 - g + \max(0, e_2 - b_2 + 1) + \max(0, e_3 - b_2 + 1)$. As $b_2 > 0$ we have that $\max(0, e_2 - b_2 + 1) + \max(0, e_3 - b_2 + 1) > \max(0, e_2 + e_3 - b_2 + 1)$. But from $b_1 \leq e_1$ it follows that $b_2 \geq e_2 + e_3 - 2$. So $\rho \leq (g-1) + 7 - g + 3 = 9$. If $b_1 < e_2 + 1$ then $\rho \leq 2e_1 + 2e_2 + 4 - 2b_1 - 2b_2 + 2e_3 = 8$.
3) It is easy to construct lifting matrices of maximal rank for all possible numbers of blocks occurring. □

Now we consider the case that the g^1_4 is composed with an involution of genus $g' = \frac{b_2}{2} + 1$. So if $b_2 > 0$, then $g' > 1$. After a coordinate transformation we may assume that the surface Y is singular along the section $x = y = 0$, so its equation depends only on x and y: $P = P(x, y; s, t)$. We may assume that Q has the form $Q = z^2 + Q'(x, y; s, t)$. Let M_{xy} be the submatrix of the lifting matrix consisting of the blocks coming from P and Q' and the ξ and η deformations.

THEOREM 2.17. *Let X be a tetragonal canonical cone such that the g^1_4 is composed with an involution of genus $g' > 1$. Then* $\dim T^1_X(-1) = e_1 + e_2 - 2e_3 + 6 + \operatorname{Cork} M_{xy}$.

Proof. The rows in the lifting matrix M coming from the first equation and the variable z vanish identically. The second equation gives a z-block which is an identity matrix of size $b_2 - e_3 - 1 = e_3 - 1$, so all ζ variables have to vanish. What remains is the matrix M_{xy} which has $e_1 + e_2 - 2$ columns. The number of rows is $\max(0, e_2 - e_3 - 3) + \max(0, e_1 - e_3 - 3) + \max(0, 2e_3 - e_1 - 1) + \max(0, 2e_3 - e_2 - 1)$. We estimate the last two terms with $e_3 - 1$ and the first two by $e_2 - e_3$, resp. $e_1 - e_3$. Therefore the number of rows is at most $e_1 + e_2 - 2$. For each term which contributes 0 to the sum we have pure rolling factors deformations, so if the matrix has maximal rank the dimension of $T^1_X(-1)$ is $e_1 + e_2 - 2 - (2e_3 - 8)$. □

Example 2.18. It is possible that the lifting matrix M does not have full rank even if the g^1_4 is not composed. An example with invariants $(6, 5, 5; 7, 7)$ is the curve given by the equations $(s^5 + t^5)x^2 + s^3y^2 + t^3z^2$, $s^5x^2 + (s^3 - t^3)(y - z)^2 + 2t^3z^2$. The matrix is

$$\left(\begin{array}{ccc|cccc|cccc}
0 & \dots & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\hline
0 & \dots & 0 & 2 & 0 & 0 & -2 & -2 & 0 & 0 & 2 \\
0 & \dots & 0 & -2 & 0 & 0 & 2 & 2 & 0 & 0 & 2
\end{array}\right).$$

Finally we mention the case $b_2 = 0$. Bielliptic curves ($e_3 = 0$) are treated in [Ciliberto–Miranda 1992], curves on a Del Pezzo in [Brawner 1996] (but he overlooks those with $e_3 = 0$). Now there is only one equation coming from Q, which can be perturbed arbitrarily. As the z variable does not enter the scroll, we have one coordinate transformation left. The lifting matrix involves only rows coming from the equation P. One checks that the matrix M resp. M_{xy} has maximal rank and the number of rows does not exceed the number of columns. Together with the number of non-scrollar deformations (Prop. 2.8) this yields the following result, where we have excluded the complete intersection case $g = 5$.

PROPOSITION 2.19. *Let X be the cone over a tetragonal canonical curve C with $b_2 = 0$ and $g > 5$. Then* $\dim T^1_X(-2) = 1$.
1) If C lies on a Del Pezzo surface then $\dim T^1_X(-1) = 10$.
2) If C is bielliptic ($e_3 = 0$), then $\dim T^1_X(-1) = 2g - 2$.

Remark 2.20. For all non-hyperelliptic canonical cones the dimension of $T^1_X(\nu)$ with $\nu \geq 0$ is the same. The Wahl map easily gives $\dim T^1_X(0) = 3g - 3$, $\dim T^1_X(1) = g$, $\dim T^1_X(2) = 1$ and $\dim T^1_X(\nu) = 0$ for $\nu \geq 3$ (see e.g. [Drewes–Stevens 1996], 3.3).

3. ROLLING FACTORS OBSTRUCTIONS.

Rolling factors deformations can be obstructed. We first give a general result on the dimension of T^2. For the case of quadratic equations on the scroll one can actually write down the base equations.

PROPOSITION 3.1. *Let X be the cone over a complete intersection of divisors of type $aH - b_iR$ with $b_i > 0$ (and the same $a \geq 2$) on a scroll. If $a > 2$, then* $\dim T^2_X(-a) = \sum(b_i - 1)$, *and* $\dim T^2_X(-a) \geq \sum(b_i - 1)$ *in case $a = 2$.*

Proof. Let $\psi \in \mathrm{Hom}(R/R_0, \mathcal{O}_X)$ be an homogeneous element of degree $-a$. The degree of $\psi(R_{\alpha,\beta,\gamma})$ is $3 - a$, so ψ vanishes on the scrollar relations, if $a > 2$. If $a = 2$ we can assert that the functions vanishing on the scrollar relations span a subspace of $T^2_X(-2)$.
As the degree of the relation $R^n_{\alpha,m}$ is $a + 1$, the image $\psi(R^n_{\alpha,m})$ is a linear function of the coordinates. The relations

$$R^n_{\alpha,m} z_\beta - R^n_{\beta,m} z_\alpha - \sum R^n_{j,k,\gamma} p^n_{\gamma,m} = P^{(n)}_m f_{\alpha,\beta} - f_{\alpha,\beta} P^{(n)}_m .$$

imply that the $\psi(R^n_{\alpha,m})$ are also in rolling factors form. A basis (of the relevant subspace) of $\mathrm{Hom}(R/R_0, \mathcal{O}_X)(-a)$ consists of the $2\sum b_i$ elements $\psi^i_{l,s}(R^j_{\alpha,m}) = \delta_{ij}\delta_{lm} z_\alpha$, $\psi^i_{l,t}(R^j_{\alpha,m}) = \delta_{ij}\delta_{lm} z_{\alpha+1}$, where $0 \leq m < b_j$. The image of $P^{(i)}_m$ in $\mathrm{Hom}(R/R_0, \mathcal{O}_X)(-a)$ is $\psi^i_{m-1,s} - \psi^i_{m,t}$, if $0 < m < b_i$, $-\psi^i_{0,t}$ for $m = 0$, and $\psi^i_{b_i-1,s}$ for $m = b_i$. The quotient has dimension $\sum(b_i - 1)$. □

For $a = 2$ only the rolling factors obstructions will contribute to the base equations. A more detailed study could reveal if there are other obstructions. Typically this can happen, if there exist non-scrollar deformations. As example we mention Wahl's result for tetragonal cones that $\dim T^2_X(-2) = g - 7 = b_1 + b_2 - 2$, if $b_2 > 0$, whereas for a curve on a Del Pezzo the dimension is $2(g - 6)$ [Wahl 1997, Thm. 5.6].
In the quadratic case we can easily write the base equations, given a first order lift of the scrollar deformations. We can consider each equation on the scroll separately, so we will suppress the upper index of the additional equations in our notation. We may assume that we have pure rolling factors deformations ρ_α and that the lifting conditions are satisfied. We can write the perturbation of the equation P_m as

$$P_m(z) + P'_m(z, \zeta, \rho) .$$

Note that P'_m is linear in z. Now we have the following result [Stevens 1996].

PROPOSITION 3.2. *The maximal extension of the infinitesimal deformation defined by the P'_m is given by the $b - 1$ base equations*

$$P'_m(\zeta, \zeta, \rho) - P_m(\zeta) = 0 ,$$

with $1 \leq m \leq b - 1$.

Proof. We also suppress ρ from the notation. We have to lift the relations $R_{\beta,m}$. As the lifting equations are satisfied we can write

$$P'_{m+1}(z, \zeta) z_\beta - P'_m(z, \zeta) z_{\beta+1} - \sum p_{\alpha,m}(z) \zeta_{\alpha+1} z_\beta = \sum f_{\beta\gamma} d_\gamma(\zeta) ,$$

because the left hand side lies in the ideal of the scroll. This identity involving quadratic monomials in the z-variables can be lifted to the deformation of the scroll. We write $\tilde f_{\beta\alpha}$ for the deformed equation $z_\beta(z_{\alpha+1}+\zeta_{\alpha+1})-(z_{\beta+1}+\zeta_{\beta+1})z_\alpha$. We get

$$P'_{m+1}(z+\zeta,\zeta)z_\beta-P'_m(z,\zeta)(z_{\beta+1}+\zeta_{\beta+1})-\sum p_{\alpha,m}(z+\zeta)z_{\alpha+1}z_\beta=\sum \tilde f_{\beta\gamma}d_\gamma(\zeta).$$

We now lift the relation $R_{\beta,m}$:

$$\begin{aligned}\big(P_{m+1}(z)+P'_{m+1}(z+\zeta,\zeta)-P_{m+1}(\zeta)\big)\,z_\beta-(P_m(z)+P'_m(z,\zeta))\,(z_{\beta+1}+\zeta_{\beta+1})\\ -\sum \tilde f_{\beta\alpha}p_{\alpha,m}(z)-\sum \tilde f_{\beta\gamma}d_\gamma(\zeta)=0.\end{aligned}$$

If $1\le m\le b-1$, then P_m occurs in a relation as first and as second term. Therefore $P'_m(z,\zeta)$ and $P'_m(z+\zeta,\zeta)-P_m(\zeta)$ have to be equal. These equations correspond to the $b-1$ elements of $T^2_X(-2)$, constructed above. □

Example 3.3. We continue with our rolling factors example 2.10. We look at two ways of rolling:

$$\begin{gathered}y_0z_0\to y_1z_0\to y_1z_1\to y_2z_1\to y_2z_2\\ y_0z_0\to y_0z_1\to y_0z_2\to y_1z_2\to y_1z_3\end{gathered}$$

The equation for P'_0 and P'_4 has a unique solution with $P'_0=0$. We get

$$\begin{array}{rcrr} P'_0 &=& 0, & 0\\ P'_1 &=& \eta_1z_0, & y_0\zeta_1\\ P'_2 &=& \eta_1z_1+y_1\zeta_1, & y_1\zeta_1+y_0\zeta_2\\ P'_3 &=& \eta_1z_2+y_2\zeta_1+\eta_2z_1, & y_2\zeta_1+y_1\zeta_2+\eta_1z_2\\ P'_4 &=& \eta_1z_3+y_3\zeta_1+\eta_2z_2+y_2\zeta_2, & y_3\zeta_1+y_2\zeta_2+\eta_1z_3\end{array}$$

The resulting base equations are in both cases

$$0,\eta_1\zeta_1,\eta_1\zeta_2+\eta_2\zeta_1$$

In general the quadratic base equations are not uniquely determined. They can be modified by multiples of the linear lifting equations, if such are present. The other source of non-uniqueness is the possibility of coordinate transformations using the pure rolling factors variables.

Theorem 3.4. *Let $P=\sum p_{I,k}s^{\langle e,I\rangle-b-k}t^kz^I$ define a divisor of type $2H-bR$. It leads to quadratic base equations $\pi_1,\ \dots,\ \pi_{b-1}$. The coefficient $p_{I,k}$ gives the following term in π_m. We write $z^I=xy$ and assume that $e_x\ge e_y$.*

I. If $e_x<b$ then for $m\le k$ the term is $-\sum_{l=m}^k\eta_{k-l+m}\xi_l$, while for $m>k$ it is

$$\sum_{l=\max(k+m-e_y+1,k+1)}^{\min(e_x-1,m-1)}\eta_{k-l+m}\xi_l\ .$$

II. *If $e_x \geq b$ then for $m \leq k+b-e_x$ the term is $-\sum_{l=m+e_x-b}^{k} \eta_{k-l+m}\xi_l$, while for $m > k+b-e_x$ it is*

$$\sum_{l=\max(k+m-e_y+1,k+1)}^{\min(e_x-b+m-1,k+m-1)} \eta_{k-l+m}\xi_l \ .$$

Furthermore, if $e_x \geq b$ the $e_x - b + 1$ pure rolling factors deformations involving x contribute $\rho_0\xi_m + \cdots + \rho_{e_x-b}\xi_{m+e_x-b}$ to π_m.

Proof. We need explicit equations P_m. The monomial $s^{e-x+e_y-b-k}t^k xy$ gives a rolling monomial $x_{i(m)}y_{j(m)}$, where $i(m)+j(m) = k+m$. Let $i(0) = i$, $i(b) = i'$, $j(0) = j$ and $j(b) = j'$. We have to compute P'_m. Equation (S) gives

$$s^b P'_b - t^b P'_0 = \sum_{l=1}^{j'-j} s^{e_x-k+j+l}t^{k+b-j-l}x\eta_{j+l} + \sum_{n=1}^{i'-i} s^{e_y-k+i+n}t^{k+b-i-n}y\xi_{i+n} \ ,$$

which we rewrite as

$$s^b P'_b - t^b P'_0 = \sum_{l=j+1}^{j'} s^{e_x-k+l}t^{k+b-l}x\eta_l + \sum_{l=i+1}^{i'} s^{e_y-k+l}t^{k+b-l}y\xi_l \ .$$

Case I: $e_x < b$. The condition $k+b \leq e_x+e_y$ implies $k < e_y$. We solve for P'_0:

$$P'_0 = -\sum_{l=j+1}^{k} x_{k-l}\eta_l - \sum_{l=i+1}^{k} y_{k-l}\xi_l \ .$$

For the P'_m we formally write the formula

$$P'_m = -\sum_{l=j+1}^{k} x_{k-l+m}\eta_l - \sum_{l=i+1}^{k} y_{k-l+m}\xi_l + \sum_{l=j+1}^{j(m)} x_{k-l+m}\eta_l + \sum_{l=i+1}^{i(m)} y_{k-l+m}\xi_l \ .$$

This expression can involve non-existing x or y variables: for y this happens if $k-l+m > e_y$, or $l < m+k-e_y$. The terms in the two sums involving y cancel. If $i(m) < k$, then the smallest non-canceling term has $l = i(m)+1$ and $i(m)+1 \geq i(m)+j(m)-e_y = k+m-e_y$. If $i(m) > k$ we a sum of positive terms starting with $k+1$. If $k < l < m+k-e_y$ then our monomial contributes to the lifting conditions, and we can leave out this term. The sum therefore now starts at $\max(k+1, m+k-e_y)$. Keeping this in mind we determine the term in the base equation π_m from the formal formula. To this end we change the summation variable in the sums containing x-variables and arrive, using $i(m)+j(m) = k+m$, at

$$-\sum_{l=m}^{m+i-1} \xi_l\eta_{k-l+m} - \sum_{l=i+1}^{k} \eta_{k-l+m}\xi_l + \sum_{l=i(m)}^{m+i-1} \xi_l\eta_{k-l+m} + \sum_{l=i+1}^{i(m)} \eta_{k-l+m}\xi_l - \xi_{i(m)}\eta_{j(m)}$$
$$= -\sum_{l=m}^{m+i-1} \xi_l\eta_{k-l+m} - \sum_{l=i+1}^{k} \eta_{k-l+m}\xi_l + \sum_{l=i+1}^{m+i-1} \eta_{k-l+m}\xi_l \ .$$

If $k \geq m$ the terms from $l = m$ to $l = k$ occur twice with a minus sign and once with a plus. Otherwise all negative terms cancel, but we have to take the lifting conditions into account.

Case II: $e_x \geq b$. Now there are $e_x - b + 1$ pure rolling factors deformations present: we can perturb P_m with $\rho_0 x_m + \cdots + \rho_{e_x-b} x_{m+e_x-b}$. These contribute $\rho_0 \xi_m + \cdots + \rho_{e_x-b}\xi_{m+e_x-b}$ to the equation π_m.
We can roll using only the x variable: $x_{i+m}y_j$, with $i + j = k$ and $i + b \leq e_x$. We take $i = k$ if $k + b \leq e_x$ and $i = e_x - b$ otherwise. We get

$$s^b P_b' - t^b P_0' = \sum_{l=i+1}^{i+b} s^{e_y-k+l} t^{k+b-l} y \xi_l \ .$$

We solve:

$$P_0' = -\sum_{l=i+1}^{k} p y_{k-l} \xi_l$$

and

$$P_m' = -\sum_{l=i+1}^{k} y_{k-l+m}\xi_l + \sum_{l=i+1}^{i+m} y_{k-l+m}\xi_l \ .$$

Again if $k < l < m + k - e_y$ our monomial contributes to the lifting conditions, and the sum starts at $\max(k + 1, m + k - e_y)$. We get as contribution to π_m

$$-\sum_{l=i+1}^{k} \eta_{k-l+m}\xi_l + \sum_{l=i+1}^{i+m-1} \eta_{k-l+m}\xi_l \ .$$

Taking the lifting conditions and our choice of i into account we get the statement of the theorem. $\square$

Example 3.5: Case I. Let $b = 7$, $e_x = 5$ and $e_y = 4$. Consider the equation $P = (p_0 s^2 + p_1 st + p_2 t^2)xy$. This leads to the following six equations:

$$\begin{array}{lllll}
\pi_1 = & & & & - p_2(\xi_1\eta_2 + \xi_2\eta_1) \\
\pi_2 = & p_0\xi_1\eta_1 & & & \qquad\qquad - p_2\xi_2\eta_2 \\
\pi_3 = & p_0(\xi_1\eta_2 + \xi_2\eta_1) & + p_1\xi_2\eta_2 & & \\
\pi_4 = & p_0(\xi_1\eta_3 + \xi_2\eta_2) + \xi_3\eta_1 & + p_1(\xi_2\eta_3 + \xi_3\eta_2) & + p_2\xi_3\eta_3 & \\
\pi_5 = & p_0(\xi_2\eta_3 + \xi_3\eta_2) + \xi_4\eta_1 & + p_1(\xi_3\eta_3 + \xi_4\eta_2) & + p_2\xi_4\eta_3 & \\
\pi_6 = & p_0(\xi_3\eta_3 + \xi_4\eta_2) & + p_1\xi_4\eta_3 & &
\end{array}$$

If we write a matrix with the coefficients of the p_i in the columns with rows coming from the equations π_m we find that the first $k + 1$ rows form a skew symmetric matrix. This is due to the specific choices made in the above proof. One can also get any other block to be skew symmetric by using the lifting conditions. In this example they are $p_0\eta_1 + p_1\eta_2 + p_2\eta_3 = 0$, $p_0\xi_1 + p_1\xi_2 + p_2\xi_3 = 0$ and $p_0\xi_2 + p_1\xi_3 + p_2\xi_4 = 0$. From the skew symmetry we can conclude:

Proposition 3.6. *If $e_y \le e_x < b$ then the $b-1$ equations π_m coming from the equation $P = (\sum_{j=0}^{k} p_j s^{k-j} t^k)xy$, where $b+k = e_x + e_y$, satisfy $b-k-1$ linear relations $\sum_{j=0}^{k} p_j \pi_{i+j} = 0$, for $0 < i < b-k$.*

Example 3.7: case II. Let $b = 4$, $e_x = 5$ and $e_y = 3$. Consider the equation $P = (p_0 s^4 + p_1 s^3 t + p_2 s^2 t^2 + p_3 s t^3 + p_4 t^4)xy$. This leads to the following three equations:

$$\begin{aligned}
\pi_1 &= \rho_0\xi_1 + \rho_1\xi_2 \quad - p_2\xi_2\eta_1 - p_3(\xi_2\eta_2 + \xi_3\eta_1)p_4(\xi_3\eta_2 + \xi_4\eta_1) \\
\pi_2 &= \rho_0\xi_2 + \rho_1\xi_3 \quad + p_0\xi_1\eta_1 + p_1\xi_2\eta_1 - p_3\xi_3\eta_2 - p_4\xi_4\eta_2 \\
\pi_3 &= \rho_0\xi_3 + \rho_1\xi_4 \quad + p_0(\xi_1\eta_2 + \xi_2\eta_1) + p_1(\xi_2\eta_2 + \xi_3\eta_1) + p_2\xi_3\eta_2
\end{aligned}$$

(3.8) *Hyperelliptic cones* (cf. [Stevens 1996]). Let X be the cone over a hyperelliptic curve C embedded with a line bundle L of degree $d \ge 2g+3$. Then $\dim T^1_X(-1) = 2g+2$. The curve lies on a scroll of degree $d-g-1$ as curve of type $2H-(d-2g-2)R$. The number of rolling factors equations is $d-2g-3$, so we have at least as many equations as variables if $d > 4g+4$. In that case only conical deformations exist, so all deformations in negative degree are obstructed.

The easiest case to describe is $L = ng^1_2$. The curve C has an affine equation $y^2 = \sum_{k=0}^{2g+2} p_k t^k$, which gives the bihomogeneous equation $(\sum_{k=0}^{2g+2} p_k s^{2g+2-k} t^k)x^2 - y^2 = 0$. The line bundle L embeds C in a scroll $S(n, n-g-1)$, and there are $2n-2g-1$ rolling factors equations P_m, coming from $p(s,t)x^2 - y^2$. The lifting matrix is a block diagonal matrix with the y-block equal to $-2I_{n-g-2}$, and the x-block a $(n-2g-3)\times(n-1)$ matrix, so the dimension of the space of lifting deformations of the scroll is $2g+2$ if $n \ge 2g+3$. If $n \le 2g+3$, the x-block is not present, and all $n-1$ ξ-deformations lift. Furthermore there are $2g+3-n$ pure rolling factors deformations. This shows again that $\dim T^1_X(-1) = 2g+2$.

Proposition 3.9. *If $n \ge 2g+3$ the base space in negative degrees is a zero-dimensional complete intersection of $2g+2$ quadratic equations.*

Proof. We may assume that the highest coefficient p_{2g+2} in $p(s,t)$ equals 1. The lifting equations allow now to eliminate the variables $\xi_{2g+3}, \dots, \xi_{n-1}$. The base equations π_m involve only the ξ_i and are therefore not linearly independent. Because $p_{2g+2} = 1$ we can discard all π_m with $m > 2g+2$. The first $2g+2$ equations involve only the first $2g+2$ variables. This shows that we have the same system of equations for all $n \ge 2g+3$. As we know that there are no deformations over a positive dimensional base, we conclude that the base space is a complete intersection of $2g+2$ equations. □

Remark 3.10. The fact that the system of equations above defines a complete intersection can also be seen directly. In fact we have the following result:

Lemma 3.11. *The system of $e = b-1$ equations π_m in $e-1$ variables ξ_i coming from one polynomial $P_{b-2}(s,t)x^2$ is a zero-dimensional complete intersection if and only if $P_{b-2}(s,t)$ has no multiple roots.*

Proof. First we note that there are only $b-2$ linearly independent equations. We put $\xi_i = s^{b-i-1}t^i$. Then

$$\begin{aligned}\pi_m &= \sum_{k=0}^{b-2}(m-k-1)p_k s^{2b-k-m-2}t^{k+m-2}\\ &= s^{b-m}t^{m-2}\big(\sum(b-2-k)p_k s^{b-2-k}t^k + (m+1-b)\sum p_k s^{b-2-k}t^k\big) \qquad .\end{aligned}$$

The form $P(s,t)$ has multiple roots if and only if $P(s,t)$ and $s\frac{\partial}{\partial s}P(s,t)$ have a common zero $(s_0 : t_0)$. Then $\xi_i = s_0^{b-i-1}t_0^i$ is a nontrivial solution to the system of equations.
We show the converse by induction. One first checks that a linear transformation in $(s:t)$ does not change the isomorphism type of the ideal. We apply a transformation such that $s=0$ is a single root of P, so $p_0 = 0$ but $p_1 \neq 0$. The equations $\pi_2, \dots, \pi_{b-1}$ now do not involve the variable ξ_1 and are by the induction hypotheses a complete intersection in $e-2$ variables, so their zero set is the ξ_1-axis with multiple structure. The equation π_1 has the form $-p_1\xi_1^2 + \dots$, so the whole system has a zero-dimensional solution set. □

Remark 3.12. For $\deg L = 4g+4$ the base space is a cone over 2^{2g+1} points in a very special position: there exist $2g+2$ hyperplanes $\{l_i = 0\}$ such that the base is given by $l_i^2 = l_j^2$ [Stevens 1996]. We can make this more explicit in the case $L = (2g+2)g_2^1$. Again the y-block of the lifting matrix is a multiple of the identity, but now there is also one rolling factors deformation parameter ρ. More generally, we look the equations coming from $p(s,t)x^2$ with $\deg p = b = e$. We get base equations $\Pi_m = \rho\xi_m + \pi_m$, where π_m is a quadratic equation in the ξ-variables only. One solution is clearly $\xi_i = 0$ for all i. To find the others we eliminate ρ:

$$\operatorname{Rank}\begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_{e-1} \\ \xi_1 & \xi_2 & \dots & \xi_{e-1}\end{pmatrix} \le 1\ . \qquad (**)$$

The equations Π_m can be changed by changing ρ, but this system is independent of such changes. Write inhomogeneously $p(t) = p_0 + p_1 t + \dots + p_{e-1}t^{e-1} + t^e = \prod(t-\alpha_i)$, where the α_i are the roots of $p(t)$.

LEMMA 3.13. *The e points $P_i = (1 : \alpha_i : \alpha_i^2 : \dots : \alpha_i^{e-2})$ are solutions to the system $(**)$.*

Proof. Let α be a root of p and insert $\xi_i = \alpha^{i-1}$ in the system $(**)$. We simplify the matrix by column operations: subtract α times the jth column from the $(j+1)$st column, starting at the end. The matrix has clearly rank 1, if $\pi_{j+1}(\alpha) - \alpha\pi_j(\alpha) = 0$, where $\pi_j(\alpha)$ is the result of substituting $\xi_i = \alpha^{i-1}$ in the equation π_j. The coefficient p_k occurs in $\pi_j(\alpha)$ in the term $lp_k\alpha^{j+k-2}$ for some integer l, and in the term $(l+1)p_k\alpha^{j+k-1}$ in $\pi_{j+1}(\alpha)$. Therefore $\pi_{j+1}(\alpha) - \alpha\pi_j(\alpha) = -\sum p_k\alpha^{j+k-1} = -\alpha^{j-1}p(\alpha) = 0$. □

The remaining solutions are found in the following way. Divide the set of roots into two subsets I and J. The points P_i lie on a rational normal curve. Therefore the points P_i with $i \in I$ span a linear subspace L_I of dimension $|I|-1$.

Claim. *The intersection point $P_I := L_I \cap L_J$ is a solution to $(**)$.*

The proof is a similar but more complicated computation. We determine here only the point P_I. The condition that the point $\sum_{i\in I} \lambda_i P_i$ lies in L_J is that

$$\operatorname{Rank}\begin{pmatrix} \sum \lambda_i & \dots & \sum \lambda_i \alpha_i^{e-2} \\ 1 & \dots & \alpha_{j_1}^{e-2} \\ \vdots & & \vdots \\ 1 & \dots & \alpha_{j_{|J|}}^{e-2} \end{pmatrix} = |J| \, .$$

We find the resulting linear equations on the λ_i by extending the matrix to a square matrix by adding $|I|-2$ rows of points on the rational normal curve, for which we take roots. Then only two λ_i survive, and they come with a Vandermonde determinant as coefficient. Upon dividing by common factors we get $(\prod_{i\neq i_1,i_2}(\alpha_{i_1}-\alpha_i))\lambda_{i_1} + (\prod_{i\neq i_1,i_2}(\alpha_{i_2}-\alpha_i))\lambda_{i_2} = 0$. We multiply with $\alpha_{i_1}-\alpha_{i_2}$. Noting that $\prod_{i\neq i_1}(\alpha_{i_1}-\alpha_i) = p'(\alpha_{i_1})$ (with $p'(t)$ the derivative of $p(t)$) we get $p'(\alpha_{i_1})\lambda_{i_1} = p'(\alpha_{i_2})\lambda_{i_2}$.
We write out the equations for $e=5$:

$$\begin{gathered} \rho\xi_1 - p_1\xi_1^2 - 2p_2\xi_1\xi_2 - p_3\xi_2^2 - 2p_4\xi_2\xi_3 - p_5(2\xi_2\xi_4 + \xi_3^2) \\ \rho\xi_2 + p_0\xi_1^2 - p_2\xi_2^2 - p_4\xi_3^2 - 2p_5\xi_3\xi_4 \\ \rho\xi_3 + 2p_0\xi_1\xi_2 + p_1\xi_2^2 - p_3\xi_3^2 - p_5\xi_4^2 \\ \rho\xi_4 + p_0(2\xi_1\xi_3 + \xi_2^2) + 2p_1\xi_2\xi_3 + p_2\xi_3^2 + 2p_3\xi_3\xi_4 + p_4\xi_4^2 \end{gathered}$$

Let α be a root of $p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4 + t^5$, and $\beta, \dots, \varepsilon$ the remaining roots. Write σ'_i for the ith symmetric function of these four roots. Then a solution is $\xi_i = \alpha^{i-1}$, $\rho = \alpha^4 - \alpha^3\sigma'_1 - \alpha^2\sigma'_2 - \alpha\sigma'_3 + \sigma'_4$. Given two roots α and β we get a solution $\xi_i = (\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)\alpha^i + (\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)\beta^i$. To write ρ we set $\mu = (\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)$, $\lambda = (\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)$ and σ''_i the ith symmetric function in γ, δ and ε. Then and $\rho = \mu(\alpha^4 - \alpha^2(\alpha+2\beta)\sigma''_1 - \alpha^2\sigma''_2 - \alpha\sigma''_3) + \lambda(\beta^4 - \beta^2(\alpha+2\alpha)\sigma''_1 - \beta^2\sigma''_2 - \beta\sigma''_3)$. The hyperplane through $(1{:}0{:}0{:}0{:}0)$, P_γ, P_δ and P_ε is $l^-_{\alpha\beta} = \sigma''_3\xi_1 - \sigma''_2\xi_2 + \sigma''_1\xi_3 - \xi_4$. In it lie also $P_{\gamma\delta}$, $P_{\gamma\varepsilon}$, $P_{\delta\varepsilon}$ and $P_{\alpha\beta}$. The hyperplane containing the remaining points is $l^+_{\alpha\beta} = \rho - (\alpha+\beta)l^-_{\alpha\beta} + 2\sigma''_3\xi_2 + 2\alpha\beta\xi_3$. We put $l_a = \rho - 2\sigma'_4\xi_1 + 2\sigma'_3\xi_2 + 2\alpha\sigma'_1\xi_3 - 2\alpha\xi_4$. Then $l_\alpha^2 - l_\beta^2 = 4(\alpha-\beta)l^-_{\alpha\beta}l^+_{\alpha\beta}$.

4. Tetragonal curves.

An extension of a canonical curve yields a surface with the given canonical curve as hyperplane section. Surfaces with canonical hyperplane sections were studied in Dick Epema's thesis [Epema 1983]. Only a limited list of surfaces can occur.

Theorem 4.1([Epema 1983], Cor. I.5.5 and Cor. II.3.3). *Let W be a surface with canonical hyperplane sections. Then one of the following holds:*
(a) *W is a $K3$ surface with at most rational double points as singularities,*
(b) *W is a rational surface with one minimally elliptic singularity and possibly rational double points,*
(c) *W is a birationally ruled surface over an elliptic curve Γ with as non-rational singularities either*
   i) *two simple elliptic singularities with exceptional divisor isomorphic to Γ, or*
   ii) *one Gorenstein singularity with $p_g = 2$,*
(d) *W is a birationally ruled surface over a curve Γ of genus $q \geq 2$ with one non-rational singularity with $p_g = q+1$, whose exceptional divisor contains exactly one non-rational curve isomorphic to Γ.*

Case (c) occurs for bi-elliptic curves (see below). If we exclude them and curves of low genus on Del Pezzo surfaces, then all extensions of tetragonal curves are of rolling factors type. The surface W has therefore to occur in our classification of complete intersection surfaces on scrolls. In particular, $K3$ surfaces can only occur if $b_1 \leq b_2 + 4$. This has consequences for deformations of tetragonal cones.

Proposition 4.2. *Pure rolling factors deformations are always unobstructed. If $e_3 > 0$ and $b_1 > b_2 + 4$ the remaining deformations are obstructed.*

Proof. The first statement follows directly from the form of the equations. For the second we note that the total space of a nontrivial one-parameter deformation of a scroll with $e_3 > 0$ is a scroll with $e_4 > 0$. □

By taking hyperplane sections of a general element in each of the families of the classification we obtain for all g tetragonal curves with $b_1 \leq b_2 + 4$ lying on $K3$ surfaces (with at most rational double points). To realize the other types of surfaces we give a construction, which goes back to [Du Val 1933]. His construction was generalized to the non-rational case in [Epema 1983]. In our situation we want a given curve to be a hyperplane section. A general construction for given hyperplane sections of regular surfaces is given in [Wahl 1998].

Construction 4.3. *Let Y be a surface containing the curve C and let $D \in |-K_Y|$ be an anticanonical divisor. Let $\widetilde{Y}$ be the blow up of Y in the scheme $Z = C \cap D$. If the linear subsystem $\mathcal{C}'$ of $|C|$ with base scheme Z has dimension g, it associated map contracts D and blows down $\widetilde{Y}$ to a surface $\overline{Y}$ with C as canonical hyperplane section.*

Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then we have the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{I}_Z\mathcal{O}_Y(C) \longrightarrow \mathcal{O}_C(C-Z) \longrightarrow 0$$

and by the adjunction formula $\mathcal{O}_C(C-Z) = K_C$. If $h^0(\mathcal{I}_Z\mathcal{O}_Y(C)) = g+1$ then the map $H^0(\mathcal{I}_Z\mathcal{O}_Y(C)) \longrightarrow H^0(K_C)$ is surjective, a condition which is

automatically satisfied if Y is a regular surface. This yields that the special hyperplane section is the curve C in its canonical embedding.
Suppose that Y is not regular. By Epema's classification Y is then a birationally ruled surface, over a curve Γ of genus q. Let $\widetilde{C}$ be the strict transform of C on $\widetilde{Y}$ and $\overline{C}$ its image on $\overline{Y}$. Then $H^0(\mathcal{I}_Z\mathcal{O}_Y(C)) = H^0(\mathcal{O}_{\widetilde{Y}}(\widetilde{C})) = H^0(\mathcal{O}_{\overline{Y}}(\overline{C}))$. We look at the exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{Y}} \longrightarrow \mathcal{O}_{\overline{Y}}(\overline{C}) \longrightarrow \mathcal{O}_{\overline{C}}(\overline{C}) = K_C \longrightarrow 0\ .$$

We compute $H^1(\mathcal{O}_{\overline{Y}})$ with the spectral sequence for the map $\pi\colon \widetilde{Y} \longrightarrow \overline{Y}$. This gives us the long exact sequence

$$0 \longrightarrow H^1(\mathcal{O}_{\overline{Y}}) \longrightarrow H^1(\mathcal{O}_{\widetilde{Y}}) \longrightarrow H^0(R^1\pi_*\mathcal{O}_{\widetilde{Y}}) \longrightarrow H^2(\mathcal{O}_{\overline{Y}}) \longrightarrow 0$$

in which $\dim H^1(\mathcal{O}_{\widetilde{Y}}) = q$. We choose D in such a way that the composed map $H^1(\mathcal{O}_\Gamma) \longrightarrow R^1\pi_*\mathcal{O}_{\widetilde{Y}} \longrightarrow H^1(\mathcal{O}_{\widetilde{D}})$, where $\widetilde{D}$ is the exceptional divisor of the map π, is injective. Then the map $H^0(\mathcal{I}_Z\mathcal{O}_Y(C)) \longrightarrow H^0(K_C)$ is surjective.
To apply the construction we need a surface on which the curve C lies. In the tetragonal case a natural candidate is the surface Y of type $2H - b_1R$ on the scroll.
We first assume that $e_1 < b_1$, so there are no pure rolling factors deformations coming from the first equation on the scroll. The canonical divisor of the scroll S is $-3H + (b_1 + b_2)R$ [Schreyer 1986, 1.7]. So an anticanonical divisor on Y is of type $H - b_2R$. Let $T = \tau_{e_1-b_2}(s,t)x + \tau_{e_2-b_2}(s,t)y + \tau_{e_3-b_2}(s,t)z$ be the equation of such a divisor. Sections of $\mathcal{I}_Z\mathcal{O}_Y(C)$ are Q (which defines C), and $x_iT = s^{e_1-i}t^ixT$, y_iT and z_iT. With coordinates $(t : x_i : y_i : z_i)$ on $\mathbb{P}^g$ we get by rolling factors $b_2 + 1$ equations $\widetilde{Q}_m$ from the relation $Q(\tau_{e_1-b_2}(s,t)x + \tau_{e_2-b_2}(s,t)y + \tau_{e_3-b_2}(s,t)z) = (Q_{1,1}x^2 + \cdots + Q_{3,3}z^2)T$. As t is also a coordinate on the four-dimensional scroll, which is the cone over S, we can write the equation on the scroll as

$$Q_{1,1}x^2 + \cdots + Q_{3,3}z^2 - (\tau_{e_1-b_2}x + \cdots + \tau_{e_3-b_2}z)t\ .$$

We analyze the resulting singularities. If Y is a rational surface, we have a anticanonical divisor D which has arithmetic genus 1, giving a minimally elliptic singularity on the total space of the deformation.
If Y is a ruled surface over a hyperelliptic curve Γ, then D passes through the double locus. This gives an exceptional divisor with Γ as only non-rational curve.

Example 4.4. Let $(e_1, e_2, e_3; b_1, b_2) = (3n-2, 2n-2, n-2; 4n-4, 2n-4)$. If the coefficient of xz does not vanish, we may bring the equation P onto the form $xz - y^2$. The second equation has the form $z^2 + q_nzy + q_{2n}zx + q_{3n}xy + q_{4n}x^2$ from which z may be eliminated to obtain a quartic equation for y. The case of a cyclic curve $y^4 + q_{4n}x^4$ is a special instance. The equation P gives a

square lifting matrix in which the antidiagonal blocks are square unit matrices. Therefore the only deformations are pure rolling factors deformations, coming from the second equation, in number $(n+3)+3=\frac{g+3}{6}+6$. We have $T=\tau_{n+2}x+\tau_2 y$. The section $(0:0:1)$ is always a component of D. If $t_2\not\equiv 0$ we have a cusp singularity, but if $\tau_2\equiv 0$ the section occurs with multiplicity 2 in D.
If however the coefficient of xz vanishes, the surface Y is singular. After a coordinate transformation its equation is $y^2+p_{2n}x^2$, the other equation being $z^2+q_{3n}xy+q_{4n}x^2$. In this case the lifting matrix has (up to a factor $\frac{1}{2}$) the following block structure

$$\begin{pmatrix} \Pi & 0 & 0\\ 0 & I & 0\\ 0 & 0 & 0\\ 0 & 0 & I \end{pmatrix}$$

so there are $2n$ ξ-deformations, on which we have $4n-5$ base equations coming from the equation P. Of these are only $2n$ linearly independent, defining a zero-dimensional complete intersection (see Lemma 3.11). These deformations are therefore obstructed, leaving us again with only the pure rolling factors deformations. The curve D consists of the double locus and in general $2n+4$ lines.

The same computation as above works for bielliptic cones. In that case one has a deformation of weight -2. The total space is a surface in weighted projective space $\mathbb{P}(1,\dots,1,2)$. Replacing the deformation parameter t by t^2 we get a surface in ordinary $\mathbb{P}^g$. This is a surface with two simple elliptic singularities. The most general surface of this type is the intersection of our elliptic cone with one dimensional vertex with the hypersurface given by

$$\widetilde{Q}=z^2+Q(x_i,y_i)+tl(x_i,y_i)+at^2\ ,$$

where $l(x_i,y_i)$ is a linear form in the coordinates x_i, y_i. If the coefficient a vanishes, we get a surface with one singularity with $p_g=2$. The construction above gives an equation of the form $\widetilde{Q}=z^2+\cdots+azt$, which after a coordinate transformation becomes $z^2+\cdots-\frac{1}{4}a^2t^2$.

PROPOSITION 4.5. *For bielliptic cones of genus $g>10$ the only deformations of negative weight are pure rolling factors deformations.*

Proof. Each infinitesimal deformation of the bielliptic cone induces an infinitesimal deformation of the cone over the projective cone over the elliptic curve. The same holds therefore for complete deformations of negative weight. It is well-known that the cone over an elliptic curve of degree at least 10 has only obstructed deformations of negative weight. Therefore the deformation of the elliptic cone is trivial and the only possibility is to deform the last quadratic equation. □

On the other hand, non-scrollar extension do occur for bielliptic curves with $g\le 10$ and for tetragonal curves on Del Pezzo surfaces.

Example 4.6. A bielliptic curve of genus 10 lies on the projective cone over an elliptic curve of degree 9. Such a cone is can be smoothed to the triple Veronese embedding of $\mathbb{P}^2$. Let W be a $K3$ surface of degree 2, a double cover of $\mathbb{P}^2$ branched along a sextic curve. We re-embed W with $|3L|$, where L is the pull-back of a line on $\mathbb{P}^2$. The image lies on the cone over the Verones embedding. A hyperplane section through the vertex of the cone is a bielliptic curve, whereas the general hyperplane section has a g^2_6. This example, due to [Donagi–Morrison 1989], is the only case where the gonality of smooth curves in a base-point-free ample linear system on a $K3$ surface is not constant [Ciliberto–Pareschi 1995].

Now we look at the case that also the first set of equations admit pure rolling factors deformations.

Lemma 4.7. *If $e_1 \geq b_1$ then $e_1 \leq b_1 + 2$ and $b_1 \leq b_2 + 4$.*

Proof. Under the assumption $e_1 \geq b_1$ we have $e_2 + e_3 - 2 \leq b_2 \leq 2e_3$ so $e_2 \leq e_3 + 2$ and $b_1 \leq 2e_2 \leq e_2 + e_3 + 2 \leq b_2 + 4$. Furthermore $e - 1 = b_1 + b_2 + 2 - e_2 - e_3 \leq b_1 + 2$. □

It is now easy to list all 18 possibilities, ranging from $(2e + 2, e, e; 2e, 2e)$ to $(2e + 4, e + 2, e; 2e + 4, 2e)$. A look at the table of tetragonal $K3$ surfaces reveals that all possibilities are realizable as special sections of $K3$-surfaces; e.g., the hyperplane section $x_{e+2} = y_0$ of a $K3$ with invariants $(e+2, e+2, e+2, e; 2e+4, 2e)$ yields the last case.
On the other hand, every family of $K3$ surfaces contains degenerate elements with singularities of higher genus. Those can be constructed with Epema's construction and in fact he gives rather complete results for quartic hypersurfaces [Epema 1983]. The classification of such surfaces is due to [Rohn 1884] and is quite involved. In those cases the rational or ruled surfaces on which the canonical curve lies are not evident. For pure rolling factors extensions the situation is better; in fact, we can make the following simple observation.

Proposition 4.8. *Let W be a pure rolling factors extension of tetragonal curve, which is not bi-elliptic. It lies on the cone over the 3-dimensional scroll S with vertex in $p = (0 : \ldots : 0 : 1)$ and the projection from the point p yields a surface $Y \subset \mathbb{P}^{g-1}$ on which C lies.*

Example 4.9. If $b_1 > e_1$ then X lies on the cone over the surface Y on the scroll and the projection is just this surface Y, so we get the construction described above.

Example 4.10. Consider the curve with invariants $(8, 4, 2; 8, 4)$. In general a pure rolling factors extension leads to a $K3$-surface (with an ordinary double point). It is the case $e = 3$ of $(e+5, e+1, e-1, e-3; 2e+2, 2e-1)$ from the table; the singularity appears because the section $(0:0:0:1)$ is contracted. To find the equation of Y on the scroll we have to eliminate the last coordinate w. The deformed equation $\widetilde{P}$ is $P + axw$, while $\widetilde{Q} = Q + byw + c_2(s,t)xw$ with a

and b nonzero constants. The equation of Y is therefore $(by+c_2(s,t)x)P-axQ$, which defines a divisor of type $3H-(b_1+b_2)R$ on the scroll.

Example 4.11: the case $(2e+2,e,e;2e,2e)$. We first derive a normal form for the equations P and Q. We start with the restriction to $x=0$. We have a pencil of quadrics so we may choose the first equation as y^2 and the second as z^2. We get:

$$\begin{aligned} &P\colon y^2+p_{e+2}xz+p_{2e+4}x^2\\ &Q\colon z^2+q_{e+2}xy+q_{2e+4}x^2\ . \end{aligned}$$

There are $3+3$ pure rolling factors deformations:

$$\begin{aligned} &\widetilde{P}\colon P+(\rho_0s^2+\rho_1st+\rho_2t^2)x\\ &\widetilde{Q}\colon Q+(\tau_0s^2+\tau_1st+\tau_2t^2)x\ . \end{aligned}$$

If the polynomials $\rho:=\rho_0s^2+\rho_1st+\rho_2t^2$ and $\tau:=\tau_0s^2+\tau_1st+\tau_2t^2$ are proportional, so $\lambda\rho+\mu\tau=0$, then the surface Y is the surface $\lambda P+\mu Q=0$ from the pencil. In general the anticanonical divisor D contains the two sections given by $x=0$, $\lambda y^2+\mu z^2=0$ and the singularity on the deformation is a cusp singularity. If ρ and τ have $0\le\gamma<2$ roots in common, the projected surface is a divisor of type $2H-(2e-2+\gamma)R$. In general we get a simple elliptic singularity.
To describe the remaining deformations we look at the lifting matrix, which is a block matrix

$$\begin{pmatrix} 0 & 2I & 0\\ \Pi_{e+2} & 0 & 0\\ \Xi_{e+2} & 0 & 0\\ 0 & 0 & 2I \end{pmatrix}$$

of size $(4e-4)\times(4e-1)$. Its rank is $2e-2$ if p_{e+2} and q_{e+2} both vanish identically, and lies between $3e-3$ and $4e-4$ otherwise. The solution space has dimension $\gamma\ge3$ with strict inequality iff the polynomials p_{e+2} and q_{e+2} have γ roots in common. The η and ζ deformations vanish. Therefore the base equations depend only on p_{2e+4} and q_{2e+4}. They are $2(2e-1)$ quadratic equations on $2e+1+6$ variables, which may or may not have solutions.

We now turn to the other deformations in general. A dimension count shows that the general tetragonal curve of genus $g>15$ cannot lie on a $K3$ surface, so the deformations are obstructed. For a general tetragonal cone we have that $\dim T^1_X=9$. There are $(b_1-1)+(b_2-1)=g-7$ quadratic base equations. Compare this with the dimension of T^2:

THEOREM 4.12([Wahl, Thm. 5.9]). *Let X be a tetragonal cone with $e_3>0$. Then $\dim T^2_X(-k)=0$ for $k>2$ and $\dim T^2_X(-k)=g-7$ if $b_2>0$. If $b_2=0$, then $\dim T^2_X(-k)=2(g-6)$.*

In particular, if $g>15$ we have more equations than variables and in general there are no solutions. For special moduli solutions do exist and one expects in general exactly one solution.

(4.13) *The case* $g = 15$. Consider the most general situation, of equal invariants: $e_1 = e_2 = e_3 = 4$, $b_1 = b_2 = 5$. In this case there are no pure rolling factor deformations and no lifting conditions.

Proposition 4.14. *The general tetragonal curve with* $e_1 = e_2 = e_3 = 4$, $b_1 = b_2 = 5$ *is hyperplane section of* 256 *different* $K3$ *surfaces.*

Proof. We have 8 homogeneous quadratic equations in 9 variables, which define a complete intersection of degree 2^8. We give an explicit example. Take the curve, given by the equations

$$(s^3+t^3)x^2+(s^3+2t^3)y^2+(s^3-2t^3)z^2$$
$$(s^2+t^2)(s-t)x^2+s^2(s+t)y^2+t^3z^2$$

on the scroll. The base equations are formed according to Thm. 3.4. One computes that indeed we have a complete intersection, which is non-singular. □

It is very difficult to find solutions to such equations, and I have not succeeded to do so in the specific example. Note that the absence of mixed terms in x, y and z on the scroll means that the automorphism group of the curve has order at least eight and it operates on the base space: given one solution one finds three other ones by multiplying all ξ_i or all ζ_i by -1.

Remark 4.15. Alternatively one can start with a $K3$ surface and take a general hyperplane section. Therefore we look at complete intersections of two surfaces of type $2H-5R$ on a scroll of type $(3,3,3,3)$. Such a $K3$ surface can have infinitesimal deformations of negative weight (which are always obstructed). The lifting matrix for the $K3$ has size 8×8. The equations P and Q on the scroll are pencils of quadrics. In general such a pencil has 4 singular fibres and by taking a suitable linear combination we may suppose that P has the form

$$sX^2+tY^2+(s+t)Z^2+(s-t)W^2\,.$$

The polynomial Q is then a general pencil with 20 coefficients, of which one can be made to vanish by subtracting a multiple of P. This shows that these $K3$ surfaces depend on 18 moduli. Let $Q=(a_{11}s+b_{11}t)X^2+2(a_{12}s+b_{12}t)XY+\dots+(a_{44}s+b_{44}t)W^2$. Then the lifting matrix is

$$2\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&-1\\a_{11}&b_{11}&a_{12}&b_{12}&a_{13}&b_{13}&a_{14}&b_{14}\\a_{12}&b_{12}&a_{22}&b_{22}&a_{23}&b_{23}&a_{24}&b_{24}\\a_{13}&b_{13}&a_{23}&b_{23}&a_{33}&b_{33}&a_{34}&b_{34}\\a_{14}&b_{14}&a_{24}&b_{24}&a_{34}&b_{34}&a_{44}&b_{44}\end{pmatrix}\,.$$

For nonsingular $K3$ surfaces this matrix has at least rank 5, and it is possible to write down examples with exactly rank 5. Rank 4 can be realized with surfaces with isolated singularities. An explicit example (with a slightly different basis for the pencil) is

$$P = sX^2 + tY^2 + (s+t)Z^2$$
$$Q = sX^2 - tY^2 + (s-t)W^2$$

with ordinary double points at $sX = tY = (s+t)Z = (s-t)W = 0$. The hyperplane section $X_3 + Z_2 + W_1 + Y_0 = t^3X + s^2tZ + st^2W + s^3Y$ does not pass through the singular points and defines a smooth tetragonal curve with $e_i = 4$. The base space for this curve is still a complete intersection, but the line corresponding to the singular $K3$ surface is a multiple solution.

(4.16) *The case* $g = 16$. The curves lying on a $K3$ form a codimension one subspace in the moduli space of tetragonal curves of genus $g = 16$. In terms of the coefficients of the equations of the scroll one gets an equation of high degree. It makes no sense to write it. We will not study the most general case $(5,4,4;6,5)$ but $(5,5,3;6,5)$. These curves form a codimension two subspace in moduli. The computations will show that the condition of being a hyperplane section has again codimension one. The lifting matrix need not have full rank. We have $b_1 = 2e_3$, and the g^1_4 can be composed.
Suppose that the coefficient of z^2 in the first equation on the scroll does not vanish. With a coordinate transformation we may assume that the equation has the form $z^2 + P_4(s,t;x,y)$ with P_4 of degree 4 in $(s:t)$ and quadratic in $(x:y)$. Then we can take Q to be without z^2 term. Let $q_{0;1}s^3 + \ldots + q_{3;1}t^3$ be the coefficient of xz and $q_{0;2}s^3 + \ldots + q_{3;2}t^3$ that of yz. The rows of the 3×8 lifting matrix come only from the monomial z:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 & | & 0 & 1 \\ \hline q_{0;1} & q_{1;1} & q_{2;1} & q_{3;1} & | & q_{0;2} & q_{1;2} & q_{2;2} & q_{3;2} & | & 0 & 0 \end{pmatrix}$$

The matrix has rank 3 if some $q_{i,j}$ does not vanish, but rank 2 if they all vanish; then the surface $\{Q = 0\}$ has a singular line.
The deformation variables ζ_1, ζ_2 vanish. We have two pure rolling factors deformations ρ_1 and ρ_2 in the second set of additional equations, and there are scrollar deformations $\xi_1, \ldots, \eta_4$. Between those exist a linear relation given by the third line of the matrix. The equations for the base can be written down independently of this linear relation, because the ζ_i vanish.
We give a specific example: $z^2 + t^4y^2 + (s^4 + t^4)x^2$ and $(s^5 + t^5)x^2 + (s^5 - t^5)y^2 + q_1(s,t)xz + q_2(s,t)yz$. We get the following nine equations:

$$-2\xi_2\xi_3 - 2\xi_1\xi_4 - 2\eta_2\eta_3 - 2\eta_1\eta_4$$
$$\xi_1^2 - \xi_3^2 - 2\xi_2\xi_4 - \eta_3^2 - 2\eta_2\eta_4$$
$$2\xi_1\xi_2 - 2\xi_3\xi_4 - 2\eta_3\eta_4$$

$$2\xi_1\xi_3 + \xi_2^2 - \xi_4^2 - \eta_4^2$$
$$2\xi_1\xi_4 + 2\xi_2\xi_3$$
$$\rho_1\xi_1 + \rho_2\eta_1 - \xi_3^2 - 2\xi_2\xi_4 + \eta_3^2 + 2\eta_2\eta_4$$
$$\rho_1\xi_2 + \rho_2\eta_2 + \xi_1^2 + \eta_1^2 - 2\xi_3\xi_4 + 2\eta_3\eta_4$$
$$\rho_1\xi_3 + \rho_2\eta_3 + 2\xi_1\xi_2 + 2\eta_1\eta_2 - \xi_4^2 + \eta_4^2$$
$$\rho_1\xi_4 + \rho_2\eta_4 + 2\xi_1\xi_3 + \xi_2^2 + 2\eta_1\eta_3 + \eta_2^2$$

Also in general we have 5 equations π_m and 4 equations $\rho_1\xi_m + \rho_2\eta_m + \chi_m$. The pure rolling factors equations are never obstructed. We have as solution to the equations therefore the (ρ_1, ρ_2)-plane with a non reduced structure. Given a general value of (ρ_1, ρ_2) we can eliminate say the η_i variables. We are then left with 5 equations π_i depending only on the x_i. Their quadratic parts satisfy a relation with constant coefficients, but even more is true: this relation can be lifted to the equations themselves. So the component has multiplicity 16. The general fibre over the reduced component has a simple elliptic singularity of degree 10.
To find the other solutions we eliminate ρ_1 and ρ_2. This gives the condition

$$\operatorname{Rank}\begin{pmatrix} \chi_1 & \chi_2 & \chi_3 & \chi_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{pmatrix} \leq 2$$

which defines a codimension 2 variety of degree 11. In general the 5 equations π_m cut out a subset of codimension 7 and degree 352. But if

$$\operatorname{Rank}\begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{pmatrix} \leq 1 \qquad (R)$$

the full equations have only solutions in the (ρ_1, ρ_2)-plane. Even if this rank condition defines a codimension 3 subspace, there are always solutions. To see this we set $\xi_i = s^{4-i}t^{i-1}\xi$ and $\eta_i = s^{4-i}t^{i-1}\eta$. The equations π_m are satisfied if $\frac{\partial}{\partial s}P_4(s,t;\xi,\eta) = 0$ and $\frac{\partial}{\partial t}P_4(s,t;\xi,\eta) = 0$. This is the intersection of two curves of type $(2,3)$ on the scroll $S_{3,3} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and there are 12 such intersection points. Those points give multiple solutions. One can compute that the rank of the Jacobi matrix of the system of equations (R) together with the π_m is five. By taking a suitable general example one finds that the multiplicity is in fact 4, and 48 is the degree of the solution of the system.

Proposition 4.17. *The general tetragonal cone with invariants* $(5,5,3;6,5)$, *which is composed with an involution of genus* 4, *has* 302 *smoothing components. The base space of a non-composed cone can be identified with a hyperplane section of the base of the corresponding composed one and only the smoothing components of lying in this hyperplane give smoothing components of the non-composed cone.*

This means that for fixed polynomials P, Q the existence of smoothings depends on an equation of degree 302 in the eight variables $q_{i;j}$. For special values the number of smoothing components may go down. This happens in the specific example given, where the condition (R) gives two-dimensional 'false' solutions. Here there are only 238 smoothing components. Besides the hyperelliptic involution the curve has another automorphism which acts on the base space. The only solutions I have found are easy to see:

$$\eta_1 = \eta_2 = \xi_3 = \xi_4 = \xi_1 + \xi_2 = \eta_3 + \eta_4 = \rho_1 + \xi_1 = \rho_2 + \eta_4 = \xi_1^2 - \eta_4^2 = 0$$

We take $\xi_1 = \eta_3 = \rho_2 = \delta$ and $\xi_2 = \eta_4 = \rho_1 = -\delta$. The total space is a surface on a scroll of type $(4,3,3,3)$ with bihomogeneous coordinates $(W,X,Y,Z;s,t)$. We set $Y_i = y_i$, $X_i = x_{i+2}$ and $Z_i = z_i$ for $i = 0,\dots,3$. The hyperplane section is $\delta = W_2 + X_0 + X_1 + Y_2 + Y_3$, so if $\delta = 0$ we have $X = t^2x$, $Y = s^2y$, $Z = z$ and $W = -(s+t)(x+y)$. The lifting equation is now $q_{0;1} - q_{1;1} + q_{2;2} - q_{3;3} = 0$. One computes that the surface is given by

$$\begin{gathered}2X^2 + Y^2 - 2(X-Y)W(s-t) + W^2(s-t)^2 + Z^2\\ 2Y^2s - XW(s^2 - 2st + 2t^2) + YW(2s^2 - 2st + t^2) + W^2(s-t)(s^2 - st + t^2)\\ - XZ(sq_{2;2} + tq_{3;2}) - YZ(sq_{0;1} + tq_{1;1}) - ZW(s^2q_{0;1} + st(q_{2;2} - q_{3;2}) + t^2q_{3;2})\end{gathered}$$

This is a $K3$ surface with an A_1-singularity.
For even more special values of the coefficients there may be higher dimensional smoothing components. This happens e.g. for $P = z^2 + t^4y^2 + s^4x^2$ and the same Q as above, where the equations π_m have the solution $\xi_1 = \xi_2 = \eta_3 = \eta_4 = 0$, giving rise to an extra component of degree 15, which is the cone over three rational normal curves of degree 5. Then all tetragonal on Y have smoothings, but depending on the position of the hyperplane the number may increase.

(4.18) *The case* $(b_1, b_2) = (8,4)$. In this case there exist five families of $K3$-surfaces, three of which have the maximal dimension 18. The general hyperplane section of the scroll $S_{8,4,2,0}$ is a scroll $S_{8,4,2}$ while for both $S_{5,4,3,2}$ and $S_{4,4,4,2}$ it is $S_{5,5,4}$. One computes that the tetragonal curves of type $(2H - 8R, 2H - 4R)$ on $S_{8,4,2}$ depend on 29 moduli and those on $S_{5,5,4}$ depend on 34 moduli.

PROPOSITION 4.19. *The general tetragonal curve of type* $(8,4,2;8,4)$ *has only pure rolling factors extensions. If the* g^1_4 *is composed with an involution of genus* 3, *then there are in general* 91 *smoothing components not of this type.*

Remark 4.20. The tetragonal curve can be a special hyperplane section of a $K3$ surface on $S_{7,4,2,1}$, $S_{6,4,2,2}$, $S_{5,4,3,2}$ or $S_{4,4,4,2}$. Therefore the genericity assumption cannot be dropped.

Proof. After a coordinate we may assume that P has the form $p_8x^2 + y^2 + p_2xz$. The g^1_4 is composed with an involution of genus 3 if and only if $p_2 \equiv 0$. In that case Q may be taken in the form $q_{12}x^2 + q_8xy + z^2$. That the curve is nonsingular implies that p_8 has no multiple roots. If the g^1_4 is not composed, the term z^2

may be absent in Q, and p_8 may have multiple roots. For the general curve this does not occur. We look therefore at curves given by

$$\begin{aligned} P&: p_8x^2 + y^2 + p_2xz \\ Q&: q_{12}x^2 + q_8xy + z^2 \ . \end{aligned}$$

The lifting matrix is a block matrix

$$\begin{pmatrix} 0 & 2I & 0 \\ \Pi & 0 & 0 \\ 0 & 0 & 2I \end{pmatrix}$$

with Π giving the equations $p_{2,0}\xi_i + p_{2,1}\xi_{i+1} + p_{2,2}\xi_{i+2} = 0$. There is one pure rolling factors deformation for the first equation, and $5+1$ for the second. The equation P leads to 7 base equations π_m in the 8 variables ρ, ξ_1, …, ξ_7. The 128 solutions are described above. The equations coming from Q are

$$\begin{aligned} \rho_1\xi_1 + \rho_2\xi_2 + \rho_3\xi_3 + \rho_4\xi_4 + \rho_5\xi_5 + \chi_1 &= 0 \\ \rho_1\xi_2 + \rho_2\xi_3 + \rho_3\xi_4 + \rho_4\xi_5 + \rho_5\xi_6 + \chi_2 &= 0 \\ \rho_1\xi_3 + \rho_2\xi_4 + \rho_3\xi_5 + \rho_4\xi_6 + \rho_5\xi_7 + \chi_3 &= 0 \end{aligned}$$

We view this as inhomogeneous linear equations for the ρ_i. The coefficient matrix

$$M = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 \\ \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ \xi_3 & \xi_4 & \xi_5 & \xi_6 & \xi_7 \end{pmatrix}$$

is the transpose of the coefficient matrix of the equations $p_{2,0}\xi_i + p_{2,1}\xi_{i+1} + p_{2,2}\xi_{i+2} = 0$, viewed as equations for the coefficients of p_2. If for a given solution of the equations π_m the matrix M has not full rank, then there exists a non-composed pencil admitting the same solution. But then also $p_{2,0}\chi_1 + p_{2,1}\chi_2 + p_{2,2}\chi_3 = 0$, an equation which in general is not satisfied. We have 8 solutions which lie on a rational normal curve and 28 solutions on the secant variety of this curve. The equations of the secant variety are the maximal minors of M. Only for 91 solutions the matrix M has full rank. □

In the general case we get components of dimension $3+1$ (the y-rolling factors deformation does not enter the equations), for solutions not on the rational curve, but on its secant variety the component has dimension 5, while we get a 6-dimensional component if p_8 and q_{12} have a common root. This does not contradict the fact that all smoothing components of Gorenstein surface singularities have the same dimension, because we here only look at the restriction to negative degree.

Proposition 4.21. *The general hyperplane section of a $K3$ surface of type $(5,4,3,2;8,4)$ or $(4,4,4,2;8,4)$ is a tetragonal curve of type $(5,5,4;8,4)$, which lies on a rational surface with two double points.*

Proof. We use coordinate transformations on the scroll to bring the hyperplane section into a normal form, while we suppose the coefficients of

the equations to be general. Let as usual $(X, Y, Z, W; s, t)$ be coordinates on the scroll. Let the hyperplane section be $a_0W_0 + a_1W_1 + a_2W_2 + \cdots = (a_0s^2 + a_1st + a_2t^2)W + \cdots$. By a transformation in (s, t) we achieve that $a_0 = a_2 = 0$, so the equation is $W_1 + \cdots$. First consider the case $(4, 4, 4, 2)$. By a suitable transformation $w \mapsto W + a_2(s,t)X + b_2(s,t)Y + c_2(s,t)Z$ we remove all terms with index 1, 2 or 3, leaving $W_1 + a_0X_0 + b_0Y_0 + c_0Z_0 + a_4X_4 + b_4Y_4 + c_4Z_4$. Taking $a_0X + b_0Y + c_0Z$ as new X and $a_4X + b_4Y + c_4Z$ as new Y brings us finally to $X_4 + W_1 + Y_0$. With coordinates $(x, y, z; s, t)$ for the scroll $S_{5,5,4}$ we get the hyperplane section by setting $Z = z$, $X = sx$, $Y = ty$ and $W = -t^3x - s^3y$. The equation P does not involve the variable W so we have quadratic singularities if $sx = ty = z = 0$, which gives the points $s = y = z = 0$ and $t = x = z = 0$.
In the case $(5, 4, 3, 2)$ we can achieve $W_1 + Z_0 + Z_3$ and we get the curve by $X = x$, $Y = z$, $Z = sty$ and $W = -(s^3 + t^3)y$. The equation $P\colon p_2X^2 + p_1XY + p_0Y^2 + XZ$ now gives $p_2x^2 + p_1xz + p_0z^2 + stxy$, which for general p_i has singular points at $x = z = st = 0$. □

To investigate the sufficiency of these conditions we look at the general cone of type $(5, 5, 4; 8, 4)$. We may suppose that P has the form $z^2 + P_2(x, y)$. The equation $P_2(x, y)$ describes a curve of type $(2, 2)$ on $S_{5,5} \cong \mathbb{P}^1 \times \mathbb{P}^1$. If this curve has a singular point, we may assume that it lies in the point $x = s = 0$. Under the assumption that the coefficient of $stxy$ does not vanish we can transform the equation into the form $(as^2 + bt^2)x^2 + 2stxy + cs^2y^2$ and unfolding the singularity we get the equation

$$P_2 = (as^2 + bt^2)x^2 + 2stxy + (cs^2 + dt^2)y^2 \ .$$

One can then write out the lifting conditions and base equations coming from the equation P. The result is that they have only trivial solutions if and only if $abcd((ad + bc - 1)^2 - 4abcd) \neq 0$, if and only if the curve P_2 is nonsingular. If a singularity is present we assume it to be in $x = s = 0$, so $d = 0$. The equation Q gives three base equations, in which 2+2 pure rolling factors variables can enter. We analyze what happens if there is a second singularity. For $b = d = 0$ the equation P_2 is divisible by s, and we do not find extensions. In case $a = d = 0$ the curve P_2 splits into two curves of type $(1, 1)$; we get two components with deformed scroll $S_{4,4,4,2}$. For $c = d = 0$ we have intersection of a line with a curve of type $(2, 1)$ and we find two components with deformed scroll $S_{5,4,3,2}$.

Remark 4.22. For the general tetragonal cone with large g we found $\dim T^1(-1) = 9$, but all deformations are obstructed. For special curves extensions may exist; also the dimension can be higher. Both conditions seem to be independent. As the number of base equations we find is always $g - 7$, having more variables increases the chances of finding solutions. In the borderline case studied above this may suffice to force the existence, but in general it does not. On the other, taking a general hyperplane section of a general tetragonal $K3$ surface will give a cone with $\dim T^1(-1) = 9$. It would be interesting to find a property of a canonical curve which gives a sufficient condition for the existence of an extension.

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Products of Harmonic Forms and Rational Curves

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Abstract. In general, the product of harmonic forms is not harmonic. We study the top exterior power of harmonic two-forms on compact Kähler manifolds. The non-harmonicity in this case is related to the geometry of the manifold and to the existence of rational curves in particular. K3 surfaces and hyperkähler manifolds are discussed in detail.

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The product of closed forms is closed again. The analogous statement for harmonic forms, however, fails. A priori, there is no reason why the product of harmonic forms should be harmonic again. This phenomenon was recently studied by Merkulov [8]. He shows that it leads to a natural A_∞-structure on a Kähler manifold. In the context of mirror symmetry Polishchuk made use of (a twisted version of) this A_∞-structure on elliptic curves to confirm Kontsevich's homological version of mirror symmetry in this case [9].

In the present paper we show that this failure of harmonicity in fact happens quite frequently. It usually is related to certain geometric properties of the manifold and to the existence of rational curves in particular. In fact, we are only interested in the product of harmonic $(1,1)$-forms, as this is the geometric relevant case. We wish to emphasize that the interplay between harmonicity and geometry is by far not completely understood. The results of this paper just seem to indicate that there is in fact a relationship. As the failure of harmonicity is related to the shape of the *'non-linear Kähler cone'* (cf. 1.2, 2.3), the results of this paper can roughly be phrased by saying that the geometry of the manifold forces the non-linear Kähler cone to be curved. For a deeper understanding of the situation one will have to study the shape of the non-linear Kähler cone or, equivalently, the non-harmonicity of products of harmonic forms.

Let us briefly indicate the main results for the special case of compact Ricci-flat Kähler manifolds. For a Kähler class $\omega \in H^2(X,\mathbb{R})$ on such a manifold there exists a unique Ricci-flat Kähler form $\tilde{\omega}$ representing it. Let $\mathcal{H}^{1,1}(\tilde{\omega})$ denote the space of $(1,1)$-forms harmonic with respect to $\tilde{\omega}$. Of course, for a different Kähler class ω' and the representing Ricci-flat Kähler form $\tilde{\omega}'$ this space might be different.

The main technical result (Prop. 2.3) says that $\mathcal{H}^{1,1}(\tilde{\omega})$ is independent of ω if and only if the top exterior power of any harmonic form $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega})$ is again harmonic. This can be used to interpret the failure of harmonicity of the top exterior power geometrically. Proposition 3.2 asserts that there always exist harmonic $(1,1)$-forms with non-harmonic top exterior power, whenever the Kähler cone (or ample cone) does not form a connected component of the (integral) cone of all classes $\alpha \in H^{1,1}(X,\mathbb{R})$ with $\int_X \alpha^N > 0$.

Note that there are many instances where the Kähler cone is strictly smaller. E.g. this is the case for any Calabi-Yau manifold that is birational to a non-isomorphic Calabi-Yau manifold (Prop. 6.1).

In Section 4 we apply the result for K3 surfaces. One finds that on any K3 surfaces containing a rational curve there exists a harmonic $(1,1)$-form α such that α^2 is not harmonic. This can be extended to arbitrary K3 surfaces by using the existence of rational curves on nearby K3 surfaces.

1 Preparations

Let X be a compact Kähler manifold. Then $\mathcal{K}_X \subset H^{1,1}(X,\mathbb{R})$ denotes the Kähler cone, i.e. the open set of all Kähler classes on X. For a class $\alpha \in H^{1,1}(X,\mathbb{R})$ we usually denote by $\tilde{\alpha} \in \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ a closed real $(1,1)$-form representing α. Let us recall the Calabi-Yau theorem [3].

Theorem 1.1 — *Let X be an N-dimensional compact Kähler manifold with a given volume form* $\mathrm{vol} \in \mathcal{A}^{N,N}(X)_{\mathbb{R}}$. *For any Kähler class $\omega \in \mathcal{K}_X$ there exists a unique Kähler form $\tilde{\omega} \in \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ representing ω, such that $\tilde{\omega}^N = c \cdot \mathrm{vol}$, with $c \in \mathbb{R}$.*

Since $\tilde{\omega}^N$ is harmonic with respect to $\tilde{\omega}$, this can be equivalently expressed by saying that any Kähler class ω can uniquely be represented by a Kähler form $\tilde{\omega}$ with respect to which the given volume form is harmonic. Note that the constant c can be computed as $c = \int_X \omega^N/\mathrm{vol}(X)$.

Definition 1.2 — *For a given volume form* $\mathrm{vol} \in \mathcal{A}^{N.N}(X)_{\mathbb{R}}$ *we let $\tilde{\mathcal{K}}_X \subset \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ be the set of Kähler forms $\tilde{\omega}$ with respect to which* vol *is harmonic.*

By the Calabi-Yau theorem the natural projection $\tilde{\mathcal{K}}_X \to \mathcal{K}_X$ is bijective. The Kähler cone $\mathcal{K}_X$ is an open subset of $H^{1,1}(X,\mathbb{R})$, whereas $\tilde{\mathcal{K}}_X$ is in general not

contained as an open subset in a linear subspace of $\mathcal{A}^{1,1}(X)$ (cf. 2.1). Thus it might be appropriate to call $\tilde{\mathcal{K}}_X$ the *non-linear Kähler cone.*
Let $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ and $c = \int_X \omega^N/\mathrm{vol}(X)$. The tangent space of $\tilde{\mathcal{K}}_X$ at $\tilde{\omega}$ can be computed as follows. Firstly, we may write $\tilde{\mathcal{K}}_X = \mathbb{R}_+ \times \tilde{\mathcal{K}}_X^c$, where $\tilde{\mathcal{K}}_X^c = \{\tilde{\omega} \in \tilde{\mathcal{K}}_X | \tilde{\omega}^N = c \cdot \mathrm{vol}\}$. Secondly, the infinitesimal deformations of $\tilde{\omega}$ in the direction of $\tilde{\mathcal{K}}_X^c$ are of the form $\tilde{\omega} + \varepsilon\tilde{v}$, where $\tilde{v}$ is a closed real $(1,1)$-form and such that $(\tilde{\omega}+\varepsilon\tilde{v})^N = \tilde{\omega}^N$. The latter condition gives $\tilde{\omega}^N + N\varepsilon\tilde{\omega}^{N-1}\tilde{v} = \tilde{\omega}^N$, i.e. $\tilde{v}$ is primitive. As any closed primitive $(1,1)$-form is harmonic, this shows that the tangent space of $\tilde{\mathcal{K}}_X^c$ at $\tilde{\omega}$ is the space $\mathcal{H}^{1,1}(\omega)_{\mathbb{R},prim}$ of real $\tilde{\omega}$-primitive $\tilde{\omega}$-harmonic $(1,1)$-forms. Thirdly, the $\mathbb{R}_+$-direction corresponds to the scaling of $\tilde{\omega}$ and this tangent direction is therefore canonically identified with $\mathbb{R}\tilde{\omega}$. Altogether, one obtains that $T_{\tilde{\omega}}\tilde{\mathcal{K}}_X = \mathcal{H}^{1,1}(\tilde{\omega})_{\mathbb{R}}$ is the space of real $\tilde{\omega}$-harmonic $(1,1)$-forms. In particular, $\tilde{\mathcal{K}}_X$ is a smooth connected submanifold of $\mathcal{A}^{1,1}(X)_{\mathbb{R}}$.
To make this approach rigorous, one first completes $\mathcal{A}^{1,1}(X)$ in the L^2_k-topology, where $k > N$. The Sobolev embedding theorem then shows that $L^2_k(\Lambda^{1,1}_{\mathbb{R}})_{cl} \to L^2_k(\Lambda^{N,N}_{\mathbb{R}})$ given by $\alpha \mapsto \alpha^N$ is a well-defined continuous multi-linear map and hence differentiable. Then, $\tilde{\mathcal{K}}_X$ is contained as an open subset in the fibre over vol. It inherits the differentiable structure and the above calculation then shows that it is smooth. Also note that the projection from the closed L^2_k forms onto cohomology is differentiable. Hence, $\tilde{\mathcal{K}}_X \to \mathcal{K}_X$ is a differentiable map. Moreover, again due to the description of the tangent space, this map is in fact a diffeomorphism. In particular, the bijection $\tilde{\mathcal{K}}_X \to \mathcal{K}_X$ yields a differentiable map $\mathcal{K}_X \to \mathcal{A}^2(X)$ (in the L^2_k-topology). This fact is used in 2.2.

Definition 1.3 — *Let X be a compact Kähler manifold with a given volume form. Then one associates to a given Kähler class $\omega \in \mathcal{K}_X$ the space $\mathcal{H}^{p,q}(\omega) := \mathcal{H}^{p,q}(\tilde{\omega})$ of (p,q)-forms that are harmonic with respect to the unique $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ representing ω.*

Note that two different Kähler forms $\tilde{\omega}_1$ and $\tilde{\omega}_2$ representing the same Kähler class $\omega_1 = \omega_2$ always have different spaces of harmonic $(1,1)$-forms. Indeed, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are $\tilde{\omega}_1$-harmonic respectively $\tilde{\omega}_2$-harmonic. Since any class, in particular $\omega_1 = \omega_2$, is represented by a unique harmonic form and $\tilde{\omega}_1 \neq \tilde{\omega}_2$, this yields $\mathcal{H}^{1,1}(\tilde{\omega}_1) \neq \mathcal{H}^{1,1}(\tilde{\omega}_2)$. But one might ask whether $\mathcal{H}^{1,1}(\tilde{\omega}_1)$ and $\mathcal{H}^{1,1}(\tilde{\omega}_2)$ can be equal for two Kähler forms $\tilde{\omega}_1, \tilde{\omega}_2$ not representing the same class, e.g. $\tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\mathcal{K}}_X$. It is quite interesting to observe that the dependence of $\mathcal{H}^{1,1}(\tilde{\omega})$ on the Kähler class ω is related to the problem discussed in the introduction. This is explained in the next section.

2 How 'harmonic' depends on the Kähler form

As before, we consider a compact Kähler manifold X with a fixed volume form and we let $\tilde{\mathcal{K}}_X$ be the associated non-linear Kähler cone. Let us begin with the following fact which relates the shape of $\tilde{\mathcal{K}}_X$ to the dependence of $\mathcal{H}^{1,1}(\omega)$ on ω.

Proposition 2.1 — *The subspace $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ is independent of ω if and only if $\tilde{\mathcal{K}}_X$ spans an $\mathbb{R}$-linear subspace of dimension $h^{1,1}(X)$.*

Proof. Let $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ be independent of $\omega \in \mathcal{K}_X$. Since for any $\omega \in \mathcal{K}_X$ the unique $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ representing it is $\tilde{\omega}$-harmonic, the assumption immediately yields $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$ for any $\omega \in \mathcal{K}_X$.
Conversely, if $\tilde{\mathcal{K}}_X$ spans an $\mathbb{R}$-linear subspace of dimension $h^{1,1}(X)$, then this subspace coincides with the tangent space of $\tilde{\mathcal{K}}_X$ at every point $\tilde{\omega} \in \tilde{\mathcal{K}}_X$. But the latter was identified with $\mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$. Hence, the linear subspace equals $\mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$ for any $\omega \in \mathcal{K}_X$ and $\mathcal{H}^{1,1}(\omega)$, therefore, does not depend on ω. □

Remark 2.2 — The assertion might be rephrased from a slightly different point of view as follows. Use the differentiable map $\mathcal{K}_X \to \mathcal{A}^2(X)$. The proposition then just says that this map is linear if and only if the Gauss map is constant. It might be instructive to rephrase some of the results later on in this spirit, e.g. Proposition 3.2.

The next proposition states that the 'global' change of $\mathcal{H}^{1,1}(\omega)$ for $\omega \in \mathcal{K}_X$ is determined by the 'harmonic' behavior with respect to a single $\omega \in \mathcal{K}_X$.

Proposition 2.3 — *Let X be a compact Kähler manifold of dimension N with a fixed Kähler form $\tilde{\omega}_0$ and volume form $\tilde{\omega}_0^N/N!$. Then the following statements are equivalent:*
i) The linear subspace $\mathcal{H}^{1,1}(\omega)_{\mathbb{R}} \subset \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ does not depend on $\omega \in \mathcal{K}_X$.
ii) For all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ one has $\alpha^N \in \mathcal{H}^{N,N}(\omega_0)$.

Proof. Let us assume *i)*. By the previous proposition the lifted Kähler cone $\tilde{\mathcal{K}}_X$ spans the $\mathbb{C}$-linear subspace $\mathcal{H}^{1,1}(\omega_0)$. Since $\tilde{\mathcal{K}}_X$ is open in $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ and all $\alpha \in \tilde{\mathcal{K}}_X$ satisfy the $\mathbb{C}$-linear equation

$$\alpha^N = (\int_X \alpha^N / \int_X \omega_0^n) \cdot \omega_0^N \tag{1}$$

which is an algebraic condition, in fact all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ satisfy (1). Hence, for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ the top exterior power α^N is harmonic, i.e. *ii)* holds true.
Let us now assume *ii)*. If $\alpha \in \mathcal{H}^{1,1}(\omega_0)$, such that its cohomology class $\omega := [\alpha]$ is a Kähler class, let $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ denote the distinguished Kähler form representing ω. If α itself is strictly positive definite, then the unicity of $\tilde{\omega}$ and *ii)* imply $\alpha = \tilde{\omega}$. Thus, the intersection of the closed subset $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ with the open cone of strictly positive definite real $(1,1)$-forms is contained in $\tilde{\mathcal{K}}_X$. This intersection is non-empty, as it contains $\tilde{\omega}_0$. Since $\tilde{\mathcal{K}}_X$ is a closed connected subset of this open cone of the same dimension as $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ this yields $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$. By Prop. 2.1 one concludes that $\mathcal{H}^{1,1}(\omega)$ does not depend on $\omega \in \mathcal{K}_X$. □

3 The positive cone

The next proposition is a first step towards a geometric understanding of the failure of harmonicity of α^N for a harmonic form α. To state it we recall the following notation.

Definition 3.1 — *For a compact Kähler manifold X the positive cone $\mathcal{C}_X \subset H^{1,1}(X,\mathbb{R})$ is the connected component of $\{\alpha \in H^{1,1}(X,\mathbb{R}) \mid \int_X \alpha^N > 0\}$ that contains the Kähler cone.*

Note that by definition $\mathcal{K}_X \subset \mathcal{C}_X$. Also note that the positive cone $\mathcal{C}_X$ might not be convex. However, for hyperkähler manifolds also $\mathcal{C}_X$ is convex, as it coincides with the cone defined by the Beauville-Bogomolov quadratic form (see Sect. 5).

Proposition 3.2 — *If X is a compact Kähler manifold such that $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$, then for any Kähler form $\tilde\omega$ there exists a $\tilde\omega$-harmonic $(1,1)$-form α such that α^N is not $\tilde\omega$-harmonic.*

Proof. Assume that there exists a Kähler form $\tilde\omega_0$ such that for all $\alpha \in \mathcal{H}^{1,1}(\tilde\omega_0)$ also α^N is $\tilde\omega_0$-harmonic. We endow X with the volume form $\tilde\omega_0^N/N!$. By Prop. 2.3 the lifted Kähler cone $\tilde{\mathcal{K}}_X$ is contained in $\mathcal{H}^{1,1}(\tilde\omega_0)$. Since $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$ there exists a sequence $\omega_t \in \mathcal{K}_X$ converging towards a $\omega \in \mathcal{C}_X \setminus \mathcal{K}_X$. As $\tilde{\mathcal{K}}_X$ is contained in the finite-dimensional space $\mathcal{H}^{1,1}(\tilde\omega_0)$ the lifted Kähler forms $\tilde\omega_t \in \tilde{\mathcal{K}}_X$ will converge towards a form (!) and just only a current $\tilde\omega \in \mathcal{H}^{1,1}(\tilde\omega_0) \setminus \tilde{\mathcal{K}}_X$. As a limit of strictly positive definite forms $\tilde\omega$ is still semi-positive definite. Moreover, $\tilde\omega$ is strictly positive definite at $x \in X$ if and only if $\tilde\omega^N$ does not vanish at x. By assumption $\tilde\omega^N = c \cdot \tilde\omega_0^N$ with $c = \int_X \omega^N / \int_X \omega_0^N$. Since $\omega \in \mathcal{C}_X$, the scalar factor c is strictly positive. Hence, $\tilde\omega^N$ is everywhere non-trivial. Thus $\tilde\omega$ is strictly positive definite. This yields the contradiction. □

The interesting thing here is that the proposition in particular can be used to determine the positivity of a class with positive top exterior power just by studying the space of harmonic forms with respect to a single given, often very special Kähler form:

Corollary 3.3 — *Let X be a compact Kähler manifold with a given Kähler form $\tilde\omega_0$. If for all $\tilde\omega_0$-harmonic $(1,1)$-forms α the top exterior power α^N is also $\tilde\omega_0$-harmonic, then any class $\omega \in \mathcal{C}_X$ is a Kähler class.* □

Here is another version of the same corollary in a more algebraic spirit.

Corollary 3.4 — *Let X be a compact Kähler manifold with a given Kähler form $\tilde\omega_0$, such that for every $\tilde\omega_0$-harmonic $(1,1)$-form α the top exterior power α^N is also $\tilde\omega_0$-harmonic. Then, a line bundle L on X is ample if and only if $c_1(L) \in \mathcal{C}_X$.* □

We conclude this section with a few examples, where the assumption of the corollary is met *a priori*. In the later sections we will discuss examples where $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$ and where Prop. 3.2 can be used to conclude the 'failure' of harmonicity.

EXAMPLES 3.5 — *i)* If X is a COMPLEX TORUS and ω is a flat Kähler form, then harmonic forms are constant forms and their products are again constant, hence harmonic. In particular, one recovers the fact that on a torus the Kähler cone and the positive cone coincide.
ii) If for two Kähler manifolds $(X, \tilde{\omega})$ and $(X', \tilde{\omega}')$ with $b_1(X) \cdot b_1(X') = 0$ the top exterior power of any harmonic $(1,1)$-forms on X or on X' is again harmonic, then the same holds for the PRODUCT $(X \times X', \tilde{\omega} \times \tilde{\omega}')$. The additional assumption on the Betti-numbers is necessary as the product of two curves of genus at least two shows. Indeed, any $\varphi \in H^{1,0}(X)$, for a curve X, is harmonic, but $\varphi \wedge \bar{\varphi}$ is not. Hence, $\alpha = \varphi \times \bar{\varphi} + \bar{\varphi} \times \varphi$ is a harmonic $(1,1)$-form on $X \times X'$ with non-harmonic α^2.
iii) Let X be a Kähler manifold and $X \to X'$ a smooth FINITE QUOTIENT. Consider the non-linear Kähler cone on X with respect to the pull-back of a volume form on X'. Then $\mathcal{H}^{1,1}(\omega)$ with $\omega \in \mathcal{K}_X$ does not depend on ω if and only if the same holds true for X'.
iv) For HERMITIAN SYMMETRIC SPACES of compact type it is known that the space of harmonic forms equals the space of forms invariant under the real form. As the latter space is invariant under products, the Kähler cone of an irreducible hermitian symmetric space coincides with the positive cone.

4 K3 SURFACES

As indicated earlier the behavior of the Kähler cone is closely related to the geometry of the manifold. We shall study this in more detail for K3 surfaces. The next proposition follows directly from the well-known description of the Kähler cone of a K3 surface.

PROPOSITION 4.1 — *Let X be a K3 surface containing a smooth rational curve. Then for any Kähler form $\tilde{\omega}$ there exists an $\tilde{\omega}$-harmonic form $(1,1)$-form α such that α^2 is not harmonic.*

Proof. If X contains a smooth rational curve, then $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$ and we apply Prop. 3.2. Indeed, a smooth rational curve $C \subset X$ determines a (-2)-class $[C]$, whose perpendicular hyperplane $[C]^\perp$ cuts $\mathcal{C}_X$ into two parts and $\mathcal{K}_X$ is contained in the part that is positive on C. □

If the harmonicity of the top exterior powers fails for a Kähler manifold with a given Kähler form $(X, \tilde{\omega})$ then it should do so for any small deformation of

$(X, \tilde{\omega})$. For a Ricci-flat Kähler structure on a K3 surface the argument can be reversed and one can use the existence of rational curves on arbitrarily near deformations to prove the assertion in the above proposition on any K3 surface with respect to a Ricci-flat Kähler form.

COROLLARY 4.2 — *Let X be an arbitrary K3 surface. If $\tilde{\omega}$ is any hyperkähler form on X, then there exists an $\tilde{\omega}$-harmonic $(1,1)$-form α such that α^2 is not $\tilde{\omega}$-harmonic.*

Proof. Let $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$. Then

$$\begin{aligned} \mathcal{H}^2(\tilde{\omega}) &= \mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathcal{H}^{2,0}(\tilde{\omega}) \oplus \mathcal{H}^{0,2}(\tilde{\omega}) \\ &= \mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathbb{C}\sigma \oplus \mathbb{C}\bar{\sigma} \end{aligned}$$

As the space of harmonic forms only depends on the underlying hyperkähler metric g, the space $\mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathbb{C}\sigma \oplus \mathbb{C}\bar{\sigma}$ contains $\mathcal{H}^{1,1}(\tilde{\omega}_{aI+bJ+cK})$ for all $(a,b,c) \in S^2$. Here, I, J, and K are the three complex structures associated with the hyperkähler metric g (cf. Sect. 5).
Assume α^2 is g-harmonic for all $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega})$. Since $\sigma = \tilde{\omega}_J + i\tilde{\omega}_K$ (up to a scalar factor) and since the product of a harmonic form with the Kähler form is again harmonic, also $\sigma\bar{\sigma}$ is harmonic. This implies that α^2 is harmonic for all $\alpha \in \mathcal{H}^2(\tilde{\omega})$, as $\sigma^2 = \bar{\sigma}^2 = \alpha\sigma = \alpha\bar{\sigma} = 0$ for $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega})$. Thus, α^2 is g-harmonic for all $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega}_{aI+bJ+cK})$ and all $(a,b,c) \in S^2$. On the other hand, it is well-known that for a non-empty (dense) subset of S^2 the K3 surface $(X, aI+bJ+cK)$ contains a smooth rational curve. Indeed, if $e \in H^2(X,\mathbb{Z})$ is any (-2)-class, then the subset of the moduli space of marked K3 surfaces for which e is of type $(1,1)$ is a hyperplane section. This hyperplane section, necessarily, cuts the complete curve given by the base $\mathbb{P}^1 = S^2$ of the twistor family. Hence, on one of the K3 surfaces $(X, aI+bJ+cK)$ the class e is a (-2)-class of type $(1,1)$ and, thus, X contains a smooth rational curve. This yields a contradiction to Prop. 4.1. $\square$

REMARK 4.3 — What are the bad harmonic $(1,1)$-forms? Certainly $\tilde{\omega}^2$ is harmonic and for any harmonic form α also $\tilde{\omega}\alpha$ is harmonic. So, if there is any bad harmonic $(1,1)$-form there must be also one that is $\tilde{\omega}$-primitive. Most likely, it is even true that the square of any primitive harmonic form is not harmonic. The proof of it should closely follow the arguments in the proof of Proposition 3.2, but there is a slight subtlety concerning the existence of sufficiently many (-2)-classes, that I do not know how to handle. We sketch the rough idea:
Assume there exists a $\tilde{\omega}$-harmonic $\tilde{\omega}$-primitive real $(1,1)$-form α such that α^2 is $\tilde{\omega}$-harmonic. As an $\tilde{\omega}$-harmonic $\tilde{\omega}$-primitive $(1,1)$-form, the form α is also of type $(1,1)$ with respect to any complex structure $\lambda = aI + bJ + cK$ induced by

the hyperkähler metric corresponding to $\tilde{\omega}$ (see Prop. 7.5 [6]). Moreover, α is also primitive with respect to all Kähler forms $\tilde{\omega}_\lambda$.
Assume that there exists a complex structure $\lambda \in S^2$, such that $\mathcal{C}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega_\lambda$ is not contained in $\mathcal{K}_X$. This condition can be easily rephrased in terms of (-2)-classes and thus becomes a question on the lattice $3U \oplus 2(-E_8)$. It looks rather harmless, but for the time being I do not know a complete proof of it. Under this assumption, we may even assume that in fact $\lambda = I$. Since α^2 is harmonic, in fact β^2 is harmonic for all $\beta \in \mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega} \subset \mathcal{H}^{1,1}(\omega)$. Going back to the proof of Prop. 3.2, we see that the second part of it can be adapted to this situation and shows that $\psi^{-1}(\mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega) \subset \mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega}$, where $\psi : \tilde{\mathcal{K}}_X \to \mathcal{K}_X$. The space $\psi^{-1}(\mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega)$ is the space of the distinguished Kähler forms whose classes are linear combinations of $[\alpha]$ and ω. Therefore, all these forms are harmonic and linear combinations of α and $\tilde{\omega}$ themselves. To conclude, we imitate the proof of Prop. 3.2 and choose a sequence $\omega_t \in \mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega$ converging towards $\omega' \in \mathcal{C}_X \setminus \mathcal{K}_X$. The corresponding sequence $\tilde{\omega}_t \in \tilde{\mathcal{K}}_X$ is contained in $\mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega}$ and converges towards a form $\tilde{\omega}'$. As in the proof of Prop. 3.2 this leads to a contradiction.

5 Hyperkähler manifolds

We will try to improve upon Proposition 3.2 in the case of hyperkähler manifolds. In particular, we will replace the question whether the top exterior power α^N of an harmonic form α is harmonic by the corresponding question for the square of α. The motivation for doing so stems from the general philosophy that hyperkähler manifolds should be treated in almost complete analogy to K3 surfaces, whereby the top intersection pairing should be replaced by the Beauville-Bogomolov form [2], which is the higher dimensional analogue of the intersection pairing on a K3 surface.
Let us begin by recalling some notations and basic facts. By a compact hyperkähler manifold X we understand a simply-connected compact Kähler manifold, such that $H^0(X, \Omega^2) = \mathbb{C}\sigma$, where σ is an everywhere non-degenerate holomorphic two-form. A Ricci-flat Kähler form $\tilde{\omega}$ turns out to be a hyperkähler form (cf. [2]), i.e. there exists a metric g and three complex structures I, J, and $K := IJ$, such that the corresponding Kähler forms $\tilde{\omega}_{aI+bJ+cK}$ are closed for all $(a,b,c) \in S^2$, such that I is the complex structure defining X, and such that $\tilde{\omega} = \tilde{\omega}_I$. One may renormalize σ, such that $\sigma = \tilde{\omega}_J + i\tilde{\omega}_K$. In particular, multiplying with σ maps harmonic forms to harmonic forms, for this holds true for the Kähler forms $\tilde{\omega}_J$ and $\tilde{\omega}_K$.
The positive cone $\mathcal{C}_X \subset H^{1,1}(X,\mathbb{R})$ is a connected component of $\{\alpha \in H^{1,1}(X,\mathbb{R}) \mid q_X(\alpha) > 0\}$, where q_X is the Beauville-Bogomolov form (cf. [2]).

Proposition 5.1 — *Let X be a $2n$-dimensional compact hyperkähler manifold with a fixed hyperkähler form $\tilde{\omega}_0$ and the unique holomorphic two-form σ. Then,*

$\alpha^2(\sigma\bar\sigma)^{n-1}$ is harmonic for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ if and only if the linear subspace $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ does not depend on $\omega \in \mathcal{K}_X$.

Proof. Assume that for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ also $\alpha^2(\sigma\bar\sigma)^{n-1}$ is harmonic. If α is in addition strictly positive definite and $\tilde\omega \in \tilde{\mathcal{K}}_X$ with $[\alpha] = \omega$, then $\alpha^2(\sigma\bar\sigma)^{n-1} = \tilde\omega^2(\sigma\bar\sigma)^{n-1}$. We adapt Calabi's classical argument to deduce that in this case $\alpha = \tilde\omega$: If $\alpha^2(\sigma\bar\sigma)^{n-1} = \tilde\omega^2(\sigma\bar\sigma)^{n-1}$, then $(\alpha-\tilde\omega)(\alpha+\tilde\omega)(\sigma\bar\sigma)^{n-1} = 0$. Since α and $\tilde\omega$ are strictly positive definite, also $(\alpha + \tilde\omega)$ is strictly positive definite. It can be shown that also $(\alpha + \tilde\omega)(\sigma\bar\sigma)^{n-1}$ is strictly positive. As $[\alpha] = \omega = [\tilde\omega]$, the difference $\alpha-\tilde\omega$ can be written as $dd^c\varphi$ for some real function φ. But by the maximum principle the equation $(\alpha+\tilde\omega)(\sigma\bar\sigma)^{n-1}dd^c\varphi = 0$ implies $\varphi \equiv const$. Hence, $\alpha = \tilde\omega$.
As in the proof of Proposition 3.2 this shows that the intersection of the closed subset $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ with the open cone of strictly positive definite forms in $\mathcal{A}^{1,1}(X)_{\mathbb{R}}$ is contained in $\tilde{\mathcal{K}}_X$ and one concludes that $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$.
Hence, $\tilde{\mathcal{K}}_X$ spans a linear subspace of the same dimension and, by Lemma 2.1 this shows that $\mathcal{H}^{1,1}(\omega)$ is independent of $\omega \in \mathcal{K}_X$.
Conversely, let $\mathcal{H}^{1,1}(\omega)$ be independent of $\omega \in \mathcal{K}_X$. Then $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$ for any $\omega \in \mathcal{K}_X$. Therefore, $\alpha^2(\sigma\bar\sigma)^{n-1} = c(\sigma\bar\sigma)^n$ with $c \in \mathbb{R}$ for α in the Zariski-dense open subset $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$. Hence, $\alpha^2(\sigma\bar\sigma)^{n-1}$ is harmonic for any $\alpha \in \mathcal{H}^{1,1}(\omega)$ (cf. the proof of Prop. 2.3). □

Similar to Proposition 3.2 one has

COROLLARY 5.2 — *Let X be a $2n$-dimensional compact hyperkähler manifold. If the positive cone $\mathcal{C}_X$ is strictly smaller than the Kähler cone, then for any hyperkähler form $\tilde\omega_0$ there exists a harmonic form $\alpha \in \mathcal{H}^{1,1}(\tilde\omega_0)$, such that $\alpha^2(\sigma\bar\sigma)^{n-1}$ is not harmonic.* □

Of course, one expects that $\mathcal{H}^{1,1}(\omega)$ does in fact depend on ω, as it is the case for K3 surfaces. This would again follow from the existence of rational curves in nearby hyperkähler manifolds in the twistor space. In fact, for the two main series of examples of higher dimensional hyperkähler manifolds, i.e. Hilbert schemes of points on K3 surfaces and generalized Kummer varieties, this trivially holds true, since in these cases $\mathcal{C}_X$ is strictly bigger that $\mathcal{K}_X$ and so Corollary 5.2 applies. But already for global deformations of these examples the situation is not clear. One might speculate that Hilbert schemes respectively Kummer varieties are dense in their deformation spaces, so that arguments similar to those in the proof of Corollary 4.2 could be applied. But an understanding of the global deformations of Hilbert schemes respectively Kummer varieties seems difficult.
Actually, it would be more interesting to reverse the argument: Assume that X is a hyperkähler manifold, such that for any small deformation X' of X the Kähler cone $\mathcal{K}_{X'}$ equals $\mathcal{C}_{X'}$. I expect that this is equivalent to saying that

$\mathcal{H}^{1,1}(\omega)$ does not depend on ω. If for some other reason than the existence of rational curves as used in the K3 surface case this can be excluded, then one could conclude that there always is a nearby deformation X' for which $\mathcal{K}_{X'}$ is strictly smaller than $\mathcal{C}_{X'}$. The latter implies the existence of rational curves on X' (cf. [6]). Along these lines one could try to attack the Kobayashi conjecture, as the existence of rational curves on nearby deformations would say that X itself cannot be hyperbolic. Unfortunately, I cannot carry this through even for K3 surface.

6 VARIOUS OTHER EXAMPLES

Here we collect a few examples where algebraic geometry predicts the failure of harmonicity of the top exterior power of harmonic $(1,1)$-forms. In all examples this is linked to the existence of rational curves.

VARIETIES OF GENERAL TYPE. Let X be a non-minimal smooth variety of general type. As I learned from Keiji Oguiso this immediately implies that the Kähler cone is strictly smaller than the positive cone. His proof goes as follows: By definition the canonical divisor K_X is big and by the Kodaira Lemma (cf. [7]) it can therefore be written as the sum $K_X = H + E$ of an ample divisor H and an effective divisor E (with rational coefficients). Consider the segment $H_t := H + tE$ with $t \in [0,1)$. If all H_t were contained in the positive cone $\mathcal{C}_X$, then K_X would be in the closure of $\mathcal{C}_X$. If the Kähler cone coincided with the positive cone $\mathcal{C}_X$, then K_X would be nef, contradicting the hypothesis that X is not minimal. Hence $t_0 := \sup\{t | H_t \in \mathcal{C}_X\} \in (0,1)$. If H_{t_0} is not nef, then $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$. Thus, it suffices to show that H_{t_0} is not nef. If H_{t_0} were nef then all expressions of the form $H_{t_0}^{N-i}.H^{i-1}.E$ would be non-negative. Then $0 = H_{t_0}^N = H_{t_0}^{N-1}(H + t_0E) = H_{t_0}^{N-1}.H + t_0H_{t_0}^{N-1}.E$, so both summands must vanish. In particular, $0 = H_{t_0}^{N-1}.H = H^2.H_{t_0}^{N-2} + t_0H.H_{t_0}^{N-2}.E$. Again this yields the vanishing of both terms and in particular $0 = H^2.H_{t_0}^{N-2}$. By induction we eventually obtain $0 = H^{N-1}.H_{t_0}$ and, furthermore, $0 = H^{N-1}.H_{t_0} = H^N + t_0H^{N-1}.E$. But this time $H^N > 0$ yields the contradiction. Therefore, for a non-minimal variety of general type one has $\mathcal{K}_X \neq \mathcal{C}_X$ and hence there exist harmonic (with respect to any Kähler metric) $(1,1)$-forms with non-harmonic top exterior power. Note that a non-minimal variety contains rational curves. As the reader will notice, the above proof goes through on any manifold X that admits a big, but not nef line bundle L (replacing the canonical divisor). Also in this case the positive cone and the Kähler cone differ.

For a CALABI-YAU MANIFOLD X the following proposition shows that if X admits a 'special' Kähler form in the sense that the top power of any harmonic $(1,1)$-form is harmonic, then X is a unique birational model.

Proposition 6.1 — *If $\varphi : X - - \to X'$ is a birational map between two Calabi-Yau manifolds, then either φ can be extended to an isomorphism or $\mathcal{K}_X$ is strictly smaller than the positive cone $\mathcal{C}_X$. In the latter case, there exists for any Kähler form $\tilde{\omega}$ a $\tilde{\omega}$-harmonic $(1,1)$-form α with α^N not harmonic.*

Proof. The arguments are very similar to the one in the previous example. Let $H^{1,1}(X,\mathbb{R}) \cong H^{1,1}(X',\mathbb{R})$ be the natural isomorphism induced by the birational map. Let $\omega' \in H^{1,1}(X,\mathbb{R})$ correspond to a Kähler class on X'. By [5] the class ω' can be represented by a closed positive current. Furthermore, the birational map φ extends to an isomorphism if and only if $\omega' \in \mathcal{K}_X$. Assume this is not the case. Then $\omega' \notin \mathcal{K}_X$. If $\omega' \in \mathcal{C}_X$ one can apply Prop. 3.2 and we are done. If $\omega' \notin \mathcal{C}_X$ we may assume that it is also not in the boundary of $\mathcal{C}_X$, as we can change ω' slightly in the open cone $\mathcal{K}_{X'}$. If ω is a small enough Kähler class on X, then the difference $\alpha := \omega' - \omega$ can still be represented by a positive current. Let $t_0 := \sup\{t | \omega_t := \omega + t\alpha \in \mathcal{C}_X\}$. Then $t_0 \in (0,1)$. If $\mathcal{K}_X = \mathcal{C}_X$ then $\omega_{t_0}^{N-i}\omega^{i-1}\alpha \geq 0$ for all i. Then the above induction argument goes through and we eventually get $\omega^{N-1}\alpha = 0$. Since α is a positive current, this is only possible for $\alpha = 0$. Hence $\mathcal{K}_X \neq \mathcal{C}_X$. Note that for hyperkähler manifolds one knows that $\mathcal{K}_{X'} \subset \mathcal{C}_X$. This simplifies the argument. $\square$

Remark 6.2 — *i)* Again, a non-trivial birational correspondence produces rational curves. We thus have another instance, where the special geometry of the variety is related to the non-harmonicity of products of harmonic forms.
ii) Most likely, finer information is encoded by Ricci-flat metrics. Those probably 'feel' contractible curves in small deformations. So, as for hyperkähler manifolds I would expect that $\mathcal{H}^{1,1}(\omega)$ depends on the Ricci-flat Kähler form representing ω.
iii) The arguments of the proof of 6.1 can be applied to the case of different birational minimal models (minimal models are not unique!). This shows that in the previous example the Kähler cone could be strictly smaller than the positive cone, even when K_X is nef or ample.

Blow-ups. This example is very much in the spirit of the previous two. Let $f : X \to Y$ be a non-trivial blow-up of a projective variety Y. Then $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$ and, therefore, for any Kähler structure on X there exist harmonic $(1,1)$-forms with non-harmonic maximal exterior power. Indeed, if L is an ample line bundle on Y then $f^*(L)$ is nef, but not ample, and it is contained in the positive cone. Hence, $f^*(L) \in \mathcal{C}_X \setminus \mathcal{K}_X$. Note that also the first example could be proved along these lines. By evoking the contraction theorem one shows that any non-minimal projective variety X admits a non-trivial contraction to a projective variety Y. The above argument then yields that $\mathcal{K}_X$ and $\mathcal{C}_X$ are different.

7 Chern forms

Let X be a compact Kähler manifold with a Ricci-flat Kähler form $\tilde{\omega}$. If F denotes the curvature of the Levi-Cevita connection ∇, then the Bianchi identity reads $\nabla F = 0$. The Kähler-Einstein condition implies $\Lambda_\omega F = 0$. The last equation can be expressed by saying that F is $\tilde{\omega}$-primitive. Analogously to the fact that any closed primitive $(1,1)$-form is in fact harmonic, one has that for F with $\nabla F = 0$ the primitivity condition $\Lambda_{\tilde{\omega}} F = 0$ is equivalent to the harmonicity condition $\nabla * F = 0$. As for untwisted harmonic $(1,1)$-forms one might ask for the harmonicity of the product F^m. Slightly less ambitious, one could ask whether the trace of this expression, an honest differential form, is harmonic. This trace is, in fact, a scalar multiple of the Chern character $ch_m(X, \tilde{\omega}) \in \mathcal{A}^{m,m}(X)_{\mathbb{R}}$.

Question. — Let $(X, \tilde{\omega})$ be a Ricci-flat Kähler manifold. Are the Chern forms $ch_m(X, \tilde{\omega})$ harmonic with respect to $\tilde{\omega}$?

By what was said about K3 surface we shall expect a negative answer to this question at least in this case:

Problem. — Let X be a K3 surface with a hyperkähler form $\tilde{\omega}$. Let $c_2 \in A^{2,2}(X)$ be the associated Chern form. Show that c_2 is not harmonic with respect to $\tilde{\omega}$!

So, this should be seen in analogy to the fact that α^2 is not harmonic for any primitive harmonic $(1,1)$-form α. Here, α is replaced by the curvature F and α^2 by trF^2. It is likely that the non-harmonicity of c_2 can be shown by standard methods in differential geometry, in particular by using the fact that c_2 is essentially $\|F\| \cdot \tilde{\omega}^2$ (see [3]), but I do not know how to do this.
Furthermore, it is not clear to me what the relation between the above question and the one treated in the previous sections is. I could imagine that the non-harmonicity of ch_m in fact implies the existence of harmonic $(1,1)$-forms with non-harmonic top exterior power.

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Extensions of Stable C^*-Algebras

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Abstract. We show that an extension of two stable C^*-algebras need not be stable. More explicitly we find an extension

$$0 \to C(Z) \otimes \mathcal{K} \to A \to \mathcal{K} \to 0$$

for some (infinite dimensional) compact Hausdorff space Z such that A is not stable. The C^*-algebra A in our example has an approximate unit consisting of projections.

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1 Introduction

It follows from BDF-theory, [1], that for every extension $0 \to \mathcal{K} \to A \to B \to 0$ of separable C^*-algebras one has A is stable if and only if B is stable. This fact prompted the question if every extension of two (separable) stable C^*-algebras is stable, a question we here answer in the negative.
In an earlier paper with J. Hjelmborg, [3], we derived a characterization of stability for C^*-algebras, actually in the hope of providing a positive answer to the extension problem. Later, in [5], the author showed that stability is not a nicely behaved property by providing an example of a (simple, separable) C^*-algebra A such that $M_2(A)$ is stable while A is non-stable. The construction of that example was inspired by ideas of Villadsen from [9]. Again using ideas of Villadsen and of results obtained in [5] and [6] the author found in [8] an example of a simple C^*-algebra that contains both a non-zero finite and an infinite projection. A key ingredient in this construction was a study of projections in the multiplier algebra of $C(Z) \otimes \mathcal{K}$, where Z is the infinite Cartesian product of 2-spheres. In particular, a recipe was derived for deciding when certain projections in this multiplier algebra, arising as infinite sums of Bott projections, are properly infinite. This recipe (restated here in Proposition 2.1) is also a crucial ingredient in the construction of the example given in this note.

2 The Construction

We review some of the notation and some of the results from [8]. Let Z denote the infinite dimensional Cartesian product space $\prod_{n=1}^{\infty} S^2$. Let $p \in M_2(C(S^2)) = C(S^2, M_2)$ be the Bott projection over S^2, so that p is a one-dimensional projection whose Euler class in $H^2(S^2, \mathbb{Z})$ is non-zero. For each non-empty finite subset $I = \{i_1, i_2, \dots, i_k\}$ of $\mathbb{N}$ and for each point $x = (x_1, x_2, x_3, \dots) \in Z$ we define the Bott projections over the copies of S^2 indexed by the set I to be

$$p_I(x_1, x_2, x_3, \dots) = p(x_{i_1}) \otimes p(x_{i_2}) \otimes \cdots \otimes p(x_{i_k}), \tag{2.1}$$

so that p_I belongs to $C(Z, M_2 \otimes \cdots \otimes M_2)$. Identifying $M_2 \otimes \cdots \otimes M_2$ with a sub-C^*-algebra of the algebra $\mathcal{K}$ of compact operators we may view p_I as an element in $C(Z, \mathcal{K}) = C(Z) \otimes \mathcal{K}$. (It is for our purposes only necessary to define p_I up to Murray–von Neumann equivalence.)
Choose a sequence $\{S_j\}_{j=1}^{\infty}$ of isometries in $\mathcal{M}(C(Z) \otimes \mathcal{K})$ with orthogonal range projections such that $\sum_{j=1}^{\infty} S_j S_j^*$ converges strictly to 1. For each sequence $\{q_j\}_{j=1}^{\infty}$ of projections in $C(Z) \otimes \mathcal{K}$ or in $\mathcal{M}(C(Z) \otimes \mathcal{K})$ define

$$\bigoplus_{j=1}^{\infty} q_j \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} S_j q_j S_j^* \in \mathcal{M}(C(Z) \otimes \mathcal{K}).$$

A projection p in a C^*-algebra A is said to be *properly infinite* if there are subprojections p_1 and p_2 of p in A satisfying $p \sim p_1 \sim p_2$ and $p_1 \perp p_2$. Equivalently, p is properly infinite if p is non-zero and

$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \precsim \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix},$$

(i.e., $p \oplus p \precsim p$.)
The following proposition was proved in [8, Proposition 4.4 (i)].

Proposition 2.1 *Let $I_1, I_2, \dots$ be a sequence of non-empty, finite subsets of $\mathbb{N}$, and suppose that $\left|\bigcup_{j \in F} I_j\right| \geq |F|$ for all finite subsets F of $\mathbb{N}$. It follows that the projection $\bigoplus_{j=1}^{\infty} p_{I_j}$ in $\mathcal{M}(C(Z) \otimes \mathcal{K})$ is not properly infinite.*

The next lemma is similar to [8, Lemma 3.1].

Lemma 2.2 *Let I be a non-empty finite subset of $\mathbb{N}$ and let e be a constant one-dimensional projection in $C(Z) \otimes \mathcal{K}$ (so that e corresponds to the trivial complex line bundle over Z). Then $e \precsim \bigoplus_{j=1}^{n} p_I$ whenever $n > |I|$.*

Proof: Write $I = \{i_1, i_2, \dots, i_k\}$, define $\rho \colon Z \to (S^2)^k$ by

$$\rho(x_1, x_2, x_3, \dots) = (x_{i_1}, x_{i_2}, \dots, x_{i_k}), \qquad (x_1, x_2, x_3, \dots) \in Z,$$

and let $\widehat{\rho}\colon C((S^2)^k) \to C(Z)$ be its induced map. Use (2.1) to see that p_I belongs to the image of $\widehat{\rho} \otimes \mathrm{id}_{\mathcal{K}}$, and hence that $\bigoplus_{j=1}^n p_I = (\widehat{\rho} \otimes \mathrm{id}_{\mathcal{K}})(q)$ for some n-dimensional projection q in $C((S^2)^k) \otimes \mathcal{K}$. The projection q corresponds to an n-dimensional complex vector bundle ξ over $(S^2)^k$. Since

$$\dim(\xi) \;=\; n \;>\; (n-1) \;\geq\; k \;\geq\; \tfrac{1}{2}\big(\dim((S^2)^k) - 1\big),$$

it follows from Husemoller, [4, 9.1.2], that ξ dominates a trivial complex line bundle. Translated into a statement about projections, this means that $f \precsim q$, where f is a constant one-dimensional projection in $C((S^2)^k) \otimes \mathcal{K}$. But then

$$e \;\sim\; (\widehat{\rho} \otimes \mathrm{id}_{\mathcal{K}})(f) \;\precsim\; (\widehat{\rho} \otimes \mathrm{id}_{\mathcal{K}})(q) \;=\; \bigoplus_{j=1}^{n} p_I.$$

$\square$

We also need the following lemma to decide that our extension is not stable. The lemma is contained in [7, Proposition 6.8] and it is a consequence of [3, Corollary 4.3]. Recall that the multiplier algebra of a stable C^*-algebra contains $B(H)$, the bounded operators on a separable Hilbert space H, as a unital sub-C^*-algebra, so the unit of the multiplier algebra of a stable C^*-algebra is a properly infinite projection.

Lemma 2.3 *Let A be a separable C^*-algebra and let I be an essential ideal in A (so that A is a sub-C^*-algebra of $\mathcal{M}(I)$). If A contains a projection Q such that $1-Q$ is not a properly infinite projection in $\mathcal{M}(I)$, then A is not stable.*

Proof: Assume to reach a contradiction that A is stable and let Q be a projection in A. It then follows from [3, Corollary 4.3] that $(1-Q)A(1-Q)$ is stable. The C^*-algebra $(1-Q)I(1-Q)$ must also be stable, being an ideal in the stable C^*-algebra $(1-Q)A(1-Q)$, and so its multiplier algebra is properly infinite. The multiplier algebra of $(1-Q)I(1-Q)$ is isomorphic to $(1-Q)\mathcal{M}(I)(1-Q)$. Therefore $1-Q$ is a properly infinite projection in $\mathcal{M}(I)$, in contradiction with the assumption in the lemma. $\square$

Our main result below shows that not all extensions of two stable C^*-algebras are stable. We keep the notation Z for the space $\prod_{j=1}^{\infty} S^2$, and $\mathcal{K}$ is the algebra of compact operators.

Theorem 2.4 *There is an extension of C^*-algebras*

$$0 \longrightarrow C(Z) \otimes \mathcal{K} \longrightarrow A \longrightarrow \mathcal{K} \longrightarrow 0 \tag{2.2}$$

such that A is non-stable and such that A contains an approximate unit consisting of projections.

Proof: Let J denote the C^*-algebra $C(Z) \otimes \mathcal{K}$. Choose a one-dimensional constant projection e in J. Choose mutually disjoint subset $I_2, I_3, \dots$ of $\mathbb{N}$ such that I_n has $n-1$ elements. Choose mutually orthogonal projections $q_{n,j}$ in $\mathcal{M}(J)$, for $n \in \mathbb{N}$ and $1 \le j \le n$, such that the sum

$$Q = \sum_{n=1}^{\infty} \sum_{j=1}^{n} q_{n,j}$$

converges strictly in $\mathcal{M}(J)$ and such that

$$q_{1,1} \sim e, \qquad q_{n,1} \sim q_{n,2} \sim \cdots \sim q_{n,n} \sim p_{I_n}, \quad n \ge 2.$$

We claim that $Q \sim 1$ in $\mathcal{M}(J)$. Indeed, observe that

$$1 \sim \bigoplus_{n=1}^{\infty} e \precsim \sum_{n=1}^{\infty} \sum_{j=1}^{n} q_{n,j} = Q \le 1,$$

where the second relation follows from Lemma 2.2. This shows that $Q \oplus Q \le 1 \oplus 1 \precsim 1 \precsim Q$ because the unit 1 is properly infinite, and hence Q is a properly infinite projection. The two projections 1 and Q define the same element of $K_0(\mathcal{M}(J))$ because this group is trivial. It therefore follows from Cuntz [2, Section 1] that $Q \sim 1$.

Choose an isometry S in $\mathcal{M}(J)$ such that $SS^* = Q$. Upon replacing Q and $q_{n,j}$ by S^*QS and $S^*q_{n,j}S$ we can assume that $Q = 1$ and hence that $\sum_{n=1}^{\infty} \sum_{j=1}^{n} q_{n,j} = 1$. Put

$$Q_j = \sum_{n=j}^{\infty} q_{n,j}, \qquad j \in \mathbb{N},$$

so that $\{Q_j\}_{j=1}^{\infty}$ is a sequence of mutually orthogonal projections in $\mathcal{M}(J)$ with $\sum_{j=1}^{\infty} Q_j = 1$. Notice that $Q_j \sim Q_{j+1} + q_{j,1}$ for all j. With $\pi \colon \mathcal{M}(J) \to \mathcal{M}(J)/J$ the quotient mapping we get $\pi(Q_1) \sim \pi(Q_2) \sim \cdots$. It follows that there is a *-homomorphism $\varphi \colon \mathcal{K} \to \mathcal{M}(J)/J$ such that $\varphi(e_{jj}) = \pi(Q_j)$ where $\{e_{ij}\}_{i,j=1}^{\infty}$ is a system of matrix units for $\mathcal{K}$. Put $A = \pi^{-1}(\varphi(\mathcal{K}))$ so that we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook] \arrow[d, equal] & A \arrow[r, "\rho"] \arrow[d, hook] & \mathcal{K} \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & J \arrow[r, hook] & \mathcal{M}(J) \arrow[r, "\pi"'] & \mathcal{M}(J)/J \arrow[r] & 0
\end{tikzcd}$$

The projection Q_1 belongs to A and

$$1 - Q_1 = \sum_{j=2}^{\infty} Q_j = \sum_{j=2}^{\infty} \sum_{n=j}^{\infty} q_{n,j} = \sum_{n=2}^{\infty} \sum_{j=2}^{n} q_{n,j} \sim \bigoplus_{n=2}^{\infty} \bigoplus_{j=2}^{n} p_{I_n}.$$

It follows from Proposition 2.1 and the choice of the sets I_n that $1 - Q_1$ is not properly infinite, and Lemma 2.3 now yields that A is not stable.
Put $P_n = Q_1 + \cdots + Q_n$. We show that $\{P_n\}_{n=1}^{\infty}$ is an approximate unit for A. Notice that $\{\rho(P_n)\}_{n=1}^{\infty}$ is an approximate unit for $\mathcal{K}$ and that $P_n \to 1$ strictly. Let a in A and $\varepsilon > 0$ be given. Then $\|\rho(a - P_m a)\| \leq \varepsilon/2$ for some m. Find x in J such that $\|\rho(a - P_m a)\| = \|a - P_m a - x\|$. Find next n such that $\|x - P_n x\| \leq \varepsilon/2$. Then $\|a - P_m a - P_n x\| \leq \varepsilon$, and therefore

$$\|(1 - P_k)a\| \leq \varepsilon + \|(1 - P_k)(P_m a + P_n x)\| = \varepsilon$$

for all $k \geq \max\{n, m\}$. □

Our example leaves open several questions regarding extensions of stable C^*-algebras (see also [7]).

Question 2.5 Let

$$0 \longrightarrow J \longrightarrow A \underset{\lambda}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

be a split exact extension with J and B stable (and separable). Does it follow that A is stable?

Question 2.6 Suppose that I and J are stable (separable) ideals of a C^*-algebra A. Does it follow that $I + J$ is stable?

Question 2.6 can equivalently be phrased as follows: Does every (separable) C^*-algebra A have a *greatest* stable ideal, i.e., a stable ideal that contains all stable ideals of A? (See [7].) It can be shown that the canonical ideal $C(Z) \otimes \mathcal{K}$ of the C^*-algebra A appearing in Theorem 2.4 is a greatest stable ideal in A. Hence even when a C^*-algebra A has a greatest stable ideal I it may be that the quotient A/I has a non-zero stable ideal.
The two questions below were suggested by Eberhard Kirchberg.

Question 2.7 Suppose that

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

is an extension of (separable) C^*-algebras, and suppose that J and B are stable and that A is of real rank zero. Does it follow that A is stable?

Question 2.8 Suppose that

$$0 \longrightarrow J \longrightarrow A \longrightarrow \mathcal{O}_2 \otimes \mathcal{K} \longrightarrow 0$$

is an extension where J is stable and separable. Does it follow that A is stable? What if we replace $\mathcal{O}_2 \otimes \mathcal{K}$ by its cone $C_0((0,1]) \otimes \mathcal{O}_2 \otimes \mathcal{K}$?

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Global L-Packets for GSp(2) and Theta Lifts

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Abstract. Let F be a totally real number field. We define global L-packets for GSp(2) over F which should correspond to the elliptic tempered admissible homomorphisms from the conjectural Langlands group of F to the L-group of GSp(2) which are reducible, or irreducible and induced from a totally real quadratic extension of F. We prove that the elements of these global L-packets occur in the space of cusp forms on GSp(2) over F as predicted by Arthur's conjecture. This can be regarded as the GSp(2) analogue of the dihedral case of the Langlands-Tunnell theorem. To obtain these results we prove a nonvanishing theorem for global theta lifts from the similitude group of a general four dimensional quadratic space over F to GSp(2) over F.

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Introduction

Let F be a number field with adeles $\mathbb{A}$ and Weil group W_F, and let $\varphi : \mathrm{W}_F \to \mathrm{GL}(n, \mathbb{C})$ be an irreducible continuous representation. There is a unique real number t such that the twist of φ by the canonical norm function on W_F raised to the t-th power has bounded image, so assume $\varphi(\mathrm{W}_F)$ is bounded; if φ factors through $\mathrm{Gal}(\overline{F}/F)$ this is automatic. For all places v of F, let π_v be the tempered irreducible admissible representation of $\mathrm{GL}(n, F_v)$ corresponding to the restriction φ_v under the local Langlands correspondence; then conjecturally $\otimes_v \pi_v$ is an irreducible unitary cuspidal automorphic representation (hereafter, cuspidal automorphic representation) of $\mathrm{GL}(n, \mathbb{A})$. This conjecture is known in

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some cases. For example, if $n = 2$ and the image of φ in $\mathrm{PGL}(2, \mathbb{C})$ is not the icosahedral group A_5, then the Langlands-Tunnell theorem asserts π is cuspidal automorphic.

Inspired by this, one can ask for a complete parameterization of the tempered cuspidal automorphic representations of $\mathrm{GL}(n)$ and of other groups ([Ko], Section 12; [LL]). Since there are tempered cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ which do not correspond to any $\varphi : \mathrm{W}_\mathbb{Q} \to \mathrm{GL}(2, \mathbb{C})$, the Weil group is inadequate. Conjecturally, there exists a locally compact group L_F, called the Langlands group of F, which is an extension of W_F by a compact group and is formally similar to the Weil group; locally, if v is an infinite place of F, then $\mathrm{L}_{F_v} = \mathrm{W}_{F_v}$ and if v is finite, then $\mathrm{L}_{F_v} = \mathrm{W}_{F_v} \times \mathrm{SU}(2, \mathbb{R})$. Moreover, the tempered cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A})$ should be in bijection with the n dimensional irreducible continuous complex representations of L_F with bounded image. For other connected reductive linear algebraic groups G over F the conjecture is more intricate, and involves L-packets attached to appropriate L-parameters $\mathrm{L}_F \to {}^L G$. In this paper we prove results about local and global theta lifts which yield parameterizations of some tempered cuspidal automorphic representations of $\mathrm{GSp}(2, \mathbb{A})$ in agreement with this conjecture.

To motivate the results we recall the conjecture, taking into account simplifications for $\mathrm{GSp}(2)$. Assume L_F exists. Then for $\mathrm{GSp}(2)$ one considers elliptic tempered admissible homomorphisms from L_F to ${}^L\,\mathrm{GSp}(2) = \widehat{\mathrm{GSp}(2)} \rtimes \mathrm{W}_F$. Concretely, since $\mathrm{GSp}(2)$ is split and one can fix an isomorphism between the dual group $\widehat{\mathrm{GSp}(2)}$ and $\mathrm{GSp}(2, \mathbb{C})$, such homomorphisms amount to continuous homomorphisms $\varphi : \mathrm{L}_F \to \mathrm{GSp}(2, \mathbb{C})$ such that $\varphi(x)$ is semi-simple for all $x \in \mathrm{L}_F$ and $\varphi(\mathrm{L}_F)$ is bounded and not contained in the Levi subgroup of a proper parabolic subgroup of $\mathrm{GSp}(2, \mathbb{C})$. Since L_F should be an extension of W_F a basic example is a continuous homomorphism $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GSp}(2, \mathbb{C})$ which is irreducible as a four dimensional complex representation. Fix such a $\varphi : \mathrm{L}_F \to \mathrm{GSp}(2, \mathbb{C})$. The conjecture first asserts that for each place v of F one can associate to the restriction $\varphi_v : \mathrm{L}_{F_v} \to \mathrm{GSp}(2, \mathbb{C})$ a finite set $\Pi(\varphi_v)$ of irreducible admissible representations of $\mathrm{GSp}(2, F_v)$, the L-packet of φ_v. These packets should have a number of properties [B], but minimally we require that $\Pi(\varphi_v)$ consists of tempered representations, and if v is finite and φ_v is unramified, then $\Pi(\varphi_v)$ consists of a single representation unramified with respect to $\mathrm{GSp}(2, \mathfrak{O}_{F_v})$ with Satake parameter $\varphi_v(\mathrm{Frob}_v)$ where Frob_v is a Frobenius element at v; also, the common central character of the elements of $\Pi(\varphi_v)$ should correspond to $\lambda \circ \varphi_v$, where $\lambda : \mathrm{GSp}(2, \mathbb{C}) \to \mathbb{C}^\times$ is the similitude quasi-character. Define

$$\begin{aligned}\Pi(\varphi) &= \{\mathit{\Pi} = \otimes_v \mathit{\Pi}_v \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GSp}(2, \mathbb{A})) : \mathit{\Pi}_v \in \Pi(\varphi_v) \text{ for all } v\} \\ &= \otimes_v \Pi(\varphi_v).\end{aligned}$$

Arthur's conjecture ([LL], [Ko], [A1], [A2]) now asserts that if $\mathit{\Pi} \in \Pi(\varphi)$ then

Π occurs with multiplicity

$$m(\Pi) = \frac{1}{|\mathbb{S}(\varphi)|} \sum_{s \in \mathbb{S}(\varphi)} \langle s, \Pi \rangle$$

in the space of cusp forms on $\mathrm{GSp}(2, \mathbb{A})$ with central character $\lambda \circ \varphi$. Here, $\mathbb{S}(\varphi)$ is the connected component group $\pi_0(S(\varphi)/\mathbb{C}^\times)$, where $S(\varphi)$ is the centralizer of the image of φ, and $\langle \cdot, \cdot \rangle : \mathbb{S}(\varphi) \times \Pi(\varphi) \to \mathbb{C}$ is defined by

$$\langle s, \Pi \rangle = \prod_v \langle s_v, \Pi_v \rangle_v,$$

where s_v is the image of s under the natural map $\mathbb{S}(\varphi) \to \mathbb{S}(\varphi_v)$ and $\Pi = \otimes_v \Pi_v$; the $\langle \cdot, \cdot \rangle_v : \mathbb{S}(\varphi_v) \times \Pi(\varphi_v) \to \mathbb{C}$ should be functions such that $\langle \cdot, \Pi_v \rangle_v$ is the character of a finite dimensional complex representation of $\mathbb{S}(\varphi_v)$ which is identically 1 if Π_v is unramified.

By looking at cases we can be more specific. Elliptic tempered admissible homomorphisms $\varphi : \mathrm{L}_F \to \mathrm{GSp}(2, \mathbb{C})$ can be divided into three types: (A) those which are irreducible and induced as a representation; (B) those which are reducible as a representation; and (C) those which are irreducible and primitive as a representation, i.e., not induced. Our result is motivated by what the conjecture predicts for φ of the first two types.

Suppose φ is of type (A). Then one can show that φ is equivalent to $\varphi(\eta, \rho)$ for some η and ρ, where $\varphi(\eta, \rho) = \mathrm{Ind}_{\mathrm{L}_E}^{\mathrm{L}_F} \rho$, E is a quadratic extension of F, $\rho : \mathrm{L}_E \to \mathrm{GL}(2, \mathbb{C})$ is an irreducible continuous representation with bounded image such that ρ is not Galois invariant but $\det \rho$ is, and $\eta : \mathrm{L}_F \to \mathbb{C}^\times$ extends $\det \rho$; the symplectic form on $\varphi(\eta, \rho)$ (regarded as $\rho \oplus \rho$) is $\langle v_1 \oplus v_2, v_1' \oplus v_2' \rangle = \eta(h)\langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle$ where $\langle \cdot, \cdot \rangle$ is any fixed nondegenerate symplectic form on $\mathbb{C}^2$ (up to multiplication by nonzero scalars there is only one) and h is a representative for the nontrivial coset of $\mathrm{L}_E \setminus \mathrm{L}_F$. Evidently,

$$\lambda \circ \varphi(\eta, \rho) = \eta, \quad \mathbb{S}(\varphi(\eta, \rho)) = 1.$$

The conjecture thus predicts that every element Π of $\Pi(\varphi) = \Pi(\varphi(\eta, \rho))$ should be cuspidal automorphic with $m(\Pi) = 1$; that is, $\Pi(\varphi)$ should be a stable global L-packet.

Type (B) parameters, however, will in general give unstable L-packets. Suppose φ is of type (B). Then $\varphi \cong \varphi(\rho_1, \rho_2)$, where $\varphi(\rho_1, \rho_2) = \rho_1 \oplus \rho_2$, $\rho_1, \rho_2 : \mathrm{L}_F \to \mathrm{GL}(2, \mathbb{C})$ are inequivalent irreducible continuous representations with bounded image and the same determinant, and the symplectic form on $\varphi(\rho_1, \rho_2)$ is $\langle v_1 \oplus v_2, v_1' \oplus v_2' \rangle = \langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle$. We see that

$$\lambda \circ \varphi(\rho_1, \rho_2) = \det \rho_1 = \det \rho_2, \quad S_\varphi = \{ \begin{bmatrix} a \cdot I_2 & 0 \\ 0 & \pm a \cdot I_2 \end{bmatrix} : a \in \mathbb{C}^\times \}.$$

Thus,

$$\mathbb{S}(\varphi(\rho_1, \rho_2)) \cong Z_2.$$

Let $s \in \mathbb{S}(\varphi(\rho_1, \rho_2))$ be nontrivial. If $\Pi \in \Pi(\varphi) = \Pi(\varphi(\rho_1, \rho_2))$, the conjecture predicts

$$m(\Pi) = \frac{1}{2}(1 + \prod_v \langle s_v, \Pi_v \rangle_v).$$

Now for each v, $\mathbb{S}(\varphi_v) = 1$ or Z_2; and if $\mathbb{S}(\varphi_v) = Z_2$, then s_v is a nontrivial element of $\mathbb{S}(\varphi_v)$. Thus, if M is the number of times $\mathbb{S}(\varphi_v) = Z_2$ and Π_v induces the nontrivial character of $\mathbb{S}(\varphi_v)$, then

$$m(\Pi) = \frac{1}{2}(1 + (-1)^M).$$

By the conjecture, Π is cuspidal automorphic if and only if M is even; if so, $m(\Pi) = 1$. The conjecture thus provides exact predictions for φ of types (A) and (B).

But as precise as they are, these predictions concern conjectural objects. Globally, the hypothetical Langlands group underlies Arthur's conjecture; locally, the existence of L-packets is required. There are at least two approaches to the avoiding L_F and testing the conjecture. One natural alternative is to consider only L-parameters that factor through the Weil group or the Galois group. Another approach is to move matters, when possible, entirely to the automorphic side of the picture and render Arthur's conjecture into a statement involving only automorphic data. Base change and automorphic induction for $\mathrm{GL}(n)$ are important examples of such a shift. There is also a translation for parameters of type (A) and (B). The reason is that for φ of type (A), η corresponds to a Hecke character χ of $\mathbb{A}^\times$ by Abelian class field theory and ρ should correspond to a non-Galois invariant tempered cuspidal automorphic representation τ of $\mathrm{GL}(2, \mathbb{A}_E)$ whose central character factors through N_F^E via χ; for φ of type (B), ρ_1 and ρ_2 should correspond to a pair of inequivalent tempered cuspidal automorphic representations τ_1 and τ_2 of $\mathrm{GL}(2, \mathbb{A})$ with the same central character χ. Our first main result proves the automorphic version of Arthur's conjecture for φ of types (A) and (B).

To explain this automorphic analogue, suppose we are given, without reference to the global Langlands group, (A) a quadratic extension E of F and a non-Galois invariant tempered cuspidal automorphic representation τ of $\mathrm{GL}(2, \mathbb{A}_E)$ whose central character factors through N_F^E via a character χ, or (B) a pair of inequivalent tempered cuspidal automorphic representations τ_1 and τ_2 of $\mathrm{GL}(2, \mathbb{A})$ with common central character χ. Then we have a corresponding conjectural ρ or ρ_1 and ρ_2, a corresponding φ of type (A) or (B), and using φ, the statement of Arthur's conjecture. However, φ can be avoided entirely in arriving at a formulation of Arthur's conjecture starting from (A) or (B). This is due to two observations: first, the local L-parameters φ_v are defined via the local Langlands correspondence for $\mathrm{GL}(2)$ independent of the existence of φ; and second, the predictions of Arthur's conjecture for parameters of type (A) and (B) only involve local data.

To be specific, let v be a place of F, $E_v = F_v \otimes_F E$, and let τ_v be the irreducible admissible representation $\otimes_{w|v} \tau_w$ of $\mathrm{GL}(2, E_v)$, where w runs over the places of E lying over v (in case (B), $E_v = F_v \times F_v$, and $\tau_v = \tau_{1,v} \otimes \tau_{2,v}$). Then, as mentioned and using no conjecture, we can associate to χ_v and τ_v a canonical local L-parameter $\varphi(\chi_v, \tau_v) : \mathrm{L}_{F_v} \to \mathrm{GSp}(2, \mathbb{C})$. The automorphic version of Arthur's conjecture now presumes that we can further associate to χ_v and τ_v a local L-packet, satisfying certain basic requirements connected with $\varphi(\chi_v, \tau_v)$, and in the unstable case (B) a local pairing. This we do in Section 8: if F' is a local field of characteristic zero, E' is a quadratic extension of F' or $E' = F' \times F'$, and τ' is an infinite dimensional irreducible admissible representation of $\mathrm{GL}(2, E')$ with central character factoring through $\mathrm{N}_{F'}^{E'}$ via a quasi-character χ' (if F' is nonarchimedean of even residual characteristic we do also assume τ' is tempered; if F' is archimedean we assume $F' = \mathbb{R}$ and $E' = \mathbb{R} \times \mathbb{R}$), then we define a finite set $\Pi(\chi', \tau')$ of irreducible admissible representations of $\mathrm{GSp}(2, F')$. We show that this local L-packet has the desired essential properties: the common central character of the elements of $\Pi(\chi', \tau')$ is χ', the character corresponding to $\lambda \circ \varphi(\chi', \tau')$; if τ' is tempered, then $\varphi(\chi', \tau')$ and the elements of $\Pi(\chi', \tau')$ are tempered; and if τ' is unitary and E'/F' and τ' are unramified, then $\Pi(\chi', \tau')$ is a singleton whose Satake parameter is $\varphi(\chi', \tau')(\mathrm{Frob}_{F'})$ (if $E' = F' \times F'$, then we say that E'/F' is unramified). We also show $|\Pi(\chi', \tau')| = 1$ or 2 and $|\mathbb{S}(\varphi(\chi', \tau'))| = |\Pi(\chi', \tau')|$ at least if F' is not of even residual characteristic and E' is a field. Additionally, when $E' = F' \times F'$ we define a function $\langle \cdot, \cdot \rangle_{F'} : \mathbb{S}(\varphi(\chi', \tau')) \times \Pi(\chi', \tau') \to \mathbb{C}$, and show that for all $\mathit{\Pi} \in \Pi(\chi', \tau')$, $\langle \cdot, \mathit{\Pi} \rangle_{F'}$ is a character of $\mathbb{S}(\varphi(\chi', \tau'))$, and if $|\mathbb{S}(\varphi(\chi', \tau'))| = |\Pi(\chi', \tau')| = 2$ then both characters of $\mathbb{S}(\varphi(\chi', \tau'))$ arise in this way. The following theorem is now the automorphic version of Arthur's conjecture for parameters of type (A) and (B).

8.6 Theorem. *Let F be a totally real number field and let E be a totally real quadratic extension of F or $E = F \times F$. Let τ be a non-Galois invariant tempered cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_E)$ whose central character factors through the norm N_F^E via a Hecke character χ of $\mathbb{A}^\times$. Thus, if $E = F \times F$, then τ is a pair τ_1, τ_2 of inequivalent tempered cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ sharing the same central character χ. Define the global L-packet:*

$$\begin{aligned}\Pi(\chi, \tau) &= \{\mathit{\Pi} = \otimes_v \mathit{\Pi}_v \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GSp}(2, \mathbb{A})) : \mathit{\Pi}_v \in \Pi(\chi_v, \tau_v) \text{ for all } v\} \\ &= \otimes_v \Pi(\chi_v, \tau_v).\end{aligned}$$

(1) *If E is a field, then every element of $\Pi(\chi, \tau)$ occurs with multiplicity one in the space of cusp forms on $\mathrm{GSp}(2, \mathbb{A})$ with central character χ.*

(2) *Suppose $E = F \times F$. Let $\mathit{\Pi} \in \Pi(\chi, \tau)$, and let $T_{\mathit{\Pi}}$ be the set of places v such that $\mathbb{S}(\varphi(\chi_v, \tau_v)) = Z_2$ and $\langle \cdot, \mathit{\Pi}_v \rangle_v$ is the nontrivial character of $\mathbb{S}(\varphi(\chi_v, \tau_v))$. If $|T_{\mathit{\Pi}}|$ is even, then $\mathit{\Pi}$ occurs with multiplicity one in the space of cusp forms on $\mathrm{GSp}(2, \mathbb{A})$ with central character χ. Conversely, if $\mathit{\Pi}$ occurs in the space of cusp forms on $\mathrm{GSp}(2, \mathbb{A})$ with central character χ, then $|T_{\mathit{\Pi}}|$ is even.*

We hope this result will be of some use to investigators of rank four motives, four dimensional symplectic Galois representations, Siegel modular forms or varieties of degree two, or Abelian surfaces. One way to think of this theorem is as an analogue for GSp(2) of the dihedral case of the Langlands-Tunnell theorem. Of course, some applications might require more information about the local L-packets of Theorem 8.6. For example, we still need to prove the sole dependence of the $\Pi(\chi_v, \tau_v)$ on the $\varphi(\chi_v, \tau_v)$, i.e., if $\varphi(\chi_v, \tau_v) \cong \varphi(\chi'_v, \tau'_v)$, then $\Pi(\chi_v, \tau_v) = \Pi(\chi'_v, \tau'_v)$. Also, detailed knowledge about the local L-packets at the ramified places and at infinity would be useful. We will return to these local concerns in a later work. The intended emphasis of this paper is, as much as possible, global.

As remarked, the proof of Theorem 8.6 uses theta lifts. Locally, χ and τ give irreducible admissible representations of $\mathrm{GO}(X, F_v)$ for various four dimensional quadratic spaces X over F_v; theta lifts of these define the local L-packets. Globally, χ and τ induce cuspidal automorphic representations of $\mathrm{GO}(X, \mathbb{A})$ for various four dimensional quadratic spaces X over F. The automorphicity asserted in Theorem 8.6 is a consequence of our second main result, which gives a fairly complete characterization of global theta lifts from $\mathrm{GO}(X, \mathbb{A})$ to $\mathrm{GSp}(2, \mathbb{A})$ for four dimensional quadratic spaces X over F. In particular, it shows that the nonvanishing of the global theta lift to $\mathrm{GSp}(2, \mathbb{A})$ of a tempered cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A})$ is equivalent to the nonvanishing of all the involved local theta lifts; in turn, these local nonvanishings are equivalent to conditions involving distinguished representations.

8.3 Theorem. *Let F be a totally real number field, and let X be a four dimensional quadratic space over F. Let $d \in F^\times/F^{\times 2}$ be the discriminant of $X(F)$, and assume that the discriminant algebra E of $X(F)$ is totally real, i.e., either $d = 1$ or $d \neq 1$ and $E = F(\sqrt{d})$ is totally real. Let $\sigma \cong \otimes_v \sigma_v$ be a tempered cuspidal automorphic representation of $\mathrm{GO}(X, \mathbb{A})$ with central character ω_σ. Let V_σ be the unique realization of σ in the space of cusp forms on $\mathrm{GO}(X, \mathbb{A})$ of central character ω_σ (Section 7). Then the following are equivalent:*

(1) *The global theta lift $\Theta_2(V_\sigma)$ of V_σ to $\mathrm{GSp}(2, \mathbb{A})$ is nonzero.*
(2) *For all places v of F, σ_v occurs in the theta correspondence with $\mathrm{GSp}(2, F_v)$.*
(3) *For all places v of F, σ_v is not of the form π_v^- for some distinguished $\pi_v \in \mathrm{Irr}(\mathrm{GSO}(X, F_v))$ (Section 3).*

Let σ lie over the cuspidal automorphic representation π of $\mathrm{GSO}(X, \mathbb{A})$ (Section 7), and let $s \in \mathrm{O}(X, F)$ be the element of determinant -1 from Lemma 6.1. If $s \cdot \pi \ncong \pi$ and one of (1), (2) or (3) holds, then $\Theta_2(V_\sigma) \neq 0$, $\Theta_2(V_\sigma)$ is an irreducible unitary cuspidal automorphic representation of $\mathrm{GSp}(2, \mathbb{A})$ with central character ω_σ, and

$$\Theta_2(V_\sigma) \cong \otimes_v \theta_2(\sigma_v^\vee) = \otimes_v \theta_2(\sigma_v)^\vee,$$

where $\theta_2(\sigma_v)$ is the local theta lift of σ_v. For all v, $\theta_2(\sigma_v)$ is tempered.

In this theorem we make no assumptions about Howe duality at the even places: we prove a version of Howe duality for the case at hand in Section 1.
As mentioned, in the proof of Theorem 8.6 we use Theorem 8.3 to show that those elements of a global L-packet which should occur in the space of cusp forms really do. Theorem 8.6 also asserts that such elements occur with multiplicity one, and in the case of an unstable global L-packet, if Π is a cuspidal automorphic element, then $|T_\Pi|$ is even. To prove these two remaining claims the key step is to show that if Π is an element of a global L-packet and V is a subspace of the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ with $V \cong \Pi$, then V has a nonzero theta lift to $\mathrm{GO}(X,\mathbb{A})$ for some four dimensional quadratic space X with $\operatorname{disc} X = d$, where $E = F(\sqrt{d})$ with $d = 1$ if $E = F \times F$. This step is a consequence of a theorem of Kudla, Rallis and Soudry, which implies that if Π_1 is a cuspidal automorphic representation of $\mathrm{Sp}(2,\mathbb{A})$, V_1 is a realization of Π_1 in the space of cusp forms, and some twisted standard partial L-function $L^S(s,\Pi_1,\chi)$ has a pole at $s = 1$, then V_1 has a nonzero theta lift to $\mathrm{O}(X,\mathbb{A})$ for some four dimensional quadratic space X with $(\cdot, \operatorname{disc} X)_F = \chi$. Using this key result, multiplicity one follows from the Rallis multiplicity preservation principle and multiplicity one for $\mathrm{GO}(X,\mathbb{A})$ for four dimensional quadratic spaces X; our understanding of the involved local theta lifts and especially the relevant theta dichotomy also plays an important role. The proof of the evenness of $|T_\Pi|$ also uses the key step, local theory, and finally the fact that a quaternion algebra over F must be ramified at an even number of places.
Theorems 8.3 and 8.6 depend on many previous works. Locally, we use the papers [R1], [R2] and [R3] which dealt with the local nonarchimedean theta correspondence for similitudes, the nonarchimedean theta correspondence between $\mathrm{GO}(X,F)$ and $\mathrm{GSp}(2,F)$ for $\dim_F X = 4$, and tempered representations and the nonarchimedean theta correspondence, respectively. Globally, the critical nonvanishing results for theta lifts of this paper depend on the main result of [R4]. In turn, the essential idea of [R4] is based on an ingenious insight of [BSP]; [R4] also uses some strong results and ideas from [KR1] and [KR2]. The multiplicity one part of Theorem 8.6 uses one of the main results of [KRS], along with the multiplicity preservation principle of [Ra]. We use nonvanishing results for L-functions at $s = 1$ from [Sh] to satisfy the hypothesis of Corollary 1.2 of [R4]. Various results and ideas from [HST] are used in this paper. We would also like to mention as inspiration the papers of H. Yoshida [Y1] and [Y2] which first looked at theta lifts of automorphic forms on $\mathrm{GSO}(X,\mathbb{A}_\mathbb{Q})$ for $\dim_\mathbb{Q} X = 4$ to $\mathrm{GSp}(2,\mathbb{A}_\mathbb{Q})$. Using results from [HPS], the paper [V] also defined local discrete series L-packets for GSp(2) using theta lifts in the case of odd residual characteristic.
This paper is organized as follows. In Section 1 we consider the local theta correspondence for similitudes. The first main goal of this section is to extend the results of [R1] to the even residual characteristic and real cases. This requires that we prove a version of Howe duality in the even residual characteristic case: we do this for tempered representations when the underlying quadratic and symplectic bilinear spaces have the same dimension. The second main goal

is to prove a case of S.S. Kudla's theta dichotomy conjecture, which is required for a complete theta lifting theory for similitudes for the relevant case. In Section 2 we review the basic theory of four dimensional quadratic spaces and their similitude groups. In particular, we define the four dimensional quadratic spaces $X_{D,d}$ of discriminant d over a field F not of characteristic two; here, D is a quaternion algebra over F and $d \in F^\times/F^{\times 2}$. Up to similitude, every four dimensional quadratic space over F is of the form $X_{D,d}$. The characterization of what irreducible representations of $\mathrm{GO}(X,F)$, $\dim_F X = 4$, occur in the theta correspondence with $\mathrm{GSp}(2,F)$ when F is a local field is given in Section 3. The case when F is nonarchimedean of odd residual characteristic was worked out in [R2] and the remaining cases are similar, but require additional argument. In Section 4 we define the local L-parameters and L-packets of Theorem 8.6; in fact, the L-parameters and L-packets are associated to irreducible admissible representations of $\mathrm{GSO}(X_{\mathrm{M}_{2\times2},d},F)$ where $X_{\mathrm{M}_{2\times2},d}$ is the four dimensional quadratic space from Section 2 over the local field F. The information is summarized in three tables which appear in the Appendix. Section 5 reviews the theory of global theta lifts for similitudes. Sections 6 and 7 explain the transition from cuspidal automorphic representations of a quaternion algebra over a quadratic extension to those of similitude groups of four dimensional quadratic spaces. Finally, in Section 8 we prove the main theorems.

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Notation. Let F be a field not of characteristic two. A quadratic space over F is a finite dimensional vector space X over F equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$. Let X be a quadratic space over F. In this and the next two paragraphs, also denote the F points of X by X; the same convention holds when we are considering quadratic spaces solely over a local field, as in Sections 1, 3 and 4. The discriminant $\operatorname{disc} X \in F^\times/F^{\times 2}$ of X is $(-1)^k \det X$ where $\dim X = 2k$ or $2k+1$. If $(X',(\cdot,\cdot)')$ is another quadratic space over F then a similitude from X to X' is an F linear map $t : X \to X'$ such that for some $\lambda \in F^\times$, $(tx,tx') = \lambda(x,x')$ for $x,x' \in X$; λ is uniquely determined, and we write $\lambda(t) = \lambda$. The group $\mathrm{GO}(X,F)$ is the set of $h \in \mathrm{GL}_F(X)$ which are similitudes from X to X. The group $\mathrm{O}(X,F)$ is the kernel of $\lambda : \mathrm{GO}(X,F) \to F^\times$, and $\mathrm{SO}(X,F)$ is the subgroup of $h \in \mathrm{O}(X,F)$ with $\det h = 1$. Assume $\dim X$ is even. Then $\mathrm{GSO}(X,F)$ is the kernel of sign : $\mathrm{GO}(X,F) \to \{\pm1\}$ defined by $h \mapsto \det(h)/\lambda(h)^{\dim X/2}$; $\mathrm{SO}(X,F) = \mathrm{GSO}(X,F) \cap \mathrm{O}(X,F)$. Let n be a positive integer. Then $\mathrm{GSp}(n,F)$ is the group of $g \in \mathrm{GL}(2n,F)$ such that for some $\lambda \in F^\times$

$$ {}^t g \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} g = \lambda \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}; $$

λ is uniquely determined, and we write $\lambda(h) = \lambda$. The group $\mathrm{Sp}(n,F)$ is the kernel of $\lambda : \mathrm{GSp}(n,F) \to F^\times$. $\mathrm{M}_{2\times2} = \mathrm{M}_{2\times2}(F)$ is the quaternion algebra of 2×2 matrices over F with canonical involution $*$.

Suppose F is a nonarchimedean field of characteristic zero with integers $\mathfrak{O}_F$, prime ideal $\mathfrak{p}_F = \pi_F \mathfrak{O}_F \subset \mathfrak{O}_F$, Hilbert symbol $(\cdot,\cdot)_F$, and valuation $|\cdot| = |\cdot|_F$ such that if μ is an additive Haar measure on F, then $\mu(xA) = |x|\mu(A)$ for $x \in F$ and $A \subset F$. Let G be a group of td-type, as in [C]. Then $\mathrm{Irr}(G)$ is the set of equivalence classes of smooth admissible irreducible representations of G. If $\pi \in \mathrm{Irr}(G)$, then $\pi^\vee \in \mathrm{Irr}(G)$ is the contragredient of π and ω_π is the central character of π. The trivial representation of G is $\mathbf{1} = \mathbf{1}_G$. If H is a closed normal subgroup of G, $\pi \in \mathrm{Irr}(H)$ and $g \in G$, then $g \cdot \pi \in \mathrm{Irr}(H)$ has the same space as π and action defined by $(g\cdot\pi)(h) = \pi(g^{-1}hg)$. If G is the F-points of a connected reductive algebraic group defined over F, then $\pi \in \mathrm{Irr}(G)$ is tempered (square integrable) if and only if ω_π is unitary and every matrix coefficient of π lies in $L^{2+\epsilon}(G/Z(G))$ for all $\epsilon > 0$ (lies in $L^2(G/Z)$). Let X be a quadratic space over F. The quadratic character $\chi_X : F^\times \to \{\pm 1\}$ associated to X is $(\cdot, \mathrm{disc}\, X)_F$. We say $\sigma \in \mathrm{Irr}(\mathrm{O}(X,F))$ ($\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$) is tempered if all the irreducible components of $\sigma|_{\mathrm{SO}(X,F)}$ ($\sigma|_{\mathrm{GSO}(X,F)}$)) are tempered. A self-dual lattice L in X is a free $\mathfrak{O}_F$ submodule of rank $\dim X$ such that $L = \{x \in X : (x,y) \in \mathfrak{O}_F \text{ for all } y \in L\}$. D_{ram} is the division quaternion algebra over F with canonical involution $*$. If E/F is a quadratic extension the quadratic character of $F^\times$ associated to E/F is $\omega_{E/F}$.

Suppose $F = \mathbb{R}$. Let $|\cdot| = |\cdot|_\mathbb{R}$ be the usual absolute value on $\mathbb{R}$, and let $(\cdot,\cdot)_\mathbb{R}$ be the Hilbert symbol of $\mathbb{R}$. If X is quadratic space over $\mathbb{R}$, then the quadratic character $\chi_X : \mathbb{R}^\times \to \{\pm 1\}$ associated to X is $(\cdot, \mathrm{disc}\, X)_\mathbb{R}$. Let G be a real reductive group as in [Wal]. Let K be a maximal compact subgroup of G, and let $\mathfrak{g}$ be the Lie algebra G. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible $(\mathfrak{g}, K)$ modules. The trivial $(\mathfrak{g},K)$ module will be denoted by $\mathbf{1} = \mathbf{1}_G$. If K_1 is a closed normal subgroup of K, π is a $(\mathfrak{g}, K_1)$ module and $s \in K$, then $s\cdot\pi$ is the $(\mathfrak{g},K_1)$ module with the same space as π and action defined by $(s\cdot\pi)(k) = \pi(s^{-1}ks)$ for $k \in K_1$ and $(s\cdot\pi)(X) = \pi(\mathrm{Ad}(s)X)$ for $X \in \mathfrak{g}$. When G satisfies $G^{\mathrm{o}} = {}^{\mathrm{o}}(G^{\mathrm{o}})$ ([Wal], p. 48-9) the concepts of tempered and square integrable $(\mathfrak{g},K)$ modules are defined in [Wal], 5.5.1; this includes $G = \mathrm{Sp}(n,\mathbb{R})$, $\mathrm{O}(p,q,\mathbb{R})$ and $\mathrm{SO}(p,q,\mathbb{R})$ for p and q not both 1. When $G^{\mathrm{o}} = {}^{\mathrm{o}}(G^{\mathrm{o}})$, then $\pi \in \mathrm{Irr}(G)$ is tempered (square integrable) if and only if π is equivalent to the underlying $(\mathfrak{g},K)$ module of an irreducible unitary representation Π of G such that $g \mapsto \langle \Pi(g)v, w\rangle$ lies in $L^{2+\epsilon}(G)$ for all $v, w \in \pi$ and $\epsilon > 0$ (lies in $L^2(G)$ for all $v,w \in \pi$). When $G = \mathrm{GSp}(n,\mathbb{R})$, $\mathrm{GO}(p,q,\mathbb{R})$ or $\mathrm{GSO}(p,q,\mathbb{R})$ with p and q not both 1, then we say that $\pi \in \mathrm{Irr}(G)$ is tempered (square integrable) if π is equivalent to the underlying $(\mathfrak{g},K)$ module of an irreducible unitary representation Π of G such that $g \mapsto \langle \Pi(g)v,w\rangle$ lies in $L^2(\mathbb{R}^\times\backslash G)$ for all $v,w \in \pi$ and $\epsilon > 0$ (lies in $L^2(\mathbb{R}^\times\backslash G)$ for all $v,w \in \pi$); this is equivalent to the irreducible constituents of $\pi|_{(\mathfrak{g}_1,K_1)}$ being tempered (square integrable), where $\mathfrak{g}_1$ is the Lie algebra and $K_1 \subset K$ is the maximal compact subgroup of $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{O}(p,q,\mathbb{R})$ or $\mathrm{SO}(p,q,\mathbb{R})$, respectively. D_{ram} is the division quaternion algebra over $\mathbb{R}$ with canonical involution $*$.

Suppose F is a number field with adeles $\mathbb{A}$ and finite adeles $\mathbb{A}_f$; set $F_\infty = F \otimes_\mathbb{Q} \mathbb{R}$. The Hilbert symbol of F is $(\cdot,\cdot)_F$. If X is quadratic space over F, then the quadratic Hecke character $\chi_X : \mathbb{A}^\times/F^\times \to \{\pm1\}$ associated to X is $(\cdot, \operatorname{disc} X(F))_F$. Let G be a reductive linear algebraic group defined over F, let $\mathfrak{g}$ be the Lie algebra of $G(F_\infty)$, and let K be a maximal compact subgroup of $G(F_\infty)$. Then $\mathrm{Irr}_{\mathrm{admiss}}(G(\mathbb{A}))$ is the set of equivalence classes of irreducible admissible $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$ modules. If $\pi \in \mathrm{Irr}_{\mathrm{admiss}}(G(\mathbb{A}))$ then the central character of π is ω_π and $\pi = \otimes_v \pi_v$ is tempered if π_v is tempered for all places v of F. A cuspidal automorphic representation of $G(\mathbb{A})$ is a $\pi \in \mathrm{Irr}_{\mathrm{admiss}}(G(\mathbb{A}))$ which is isomorphic to an irreducible submodule of the $G(\mathbb{A}_f) \times (\mathfrak{g}, K)$ module of cuspidal automorphic forms on $G(\mathbb{A})$ of central character ω_π; such a π is unitary.

1. The local theta correspondence for similitudes

In this section we recall and prove results about the local theta correspondence for similitudes. The paper [R1] dealt with the nonarchimedean odd residual characteristic case. Here we do the even residual characteristic and real cases and prove a very special, but adequate, case of S.S. Kudla's theta dichotomy conjecture. We also show that the theta correspondence for similitudes is independent of the additive character, compatible with contragredients, and respects unramified representations.

Fix the following notation. Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let n be a positive integer, and let X be a quadratic space of nonzero even dimension m over F. To simplify notation, denote the F points of X by X. Let $d = \operatorname{disc} X$. Fix a nontrivial unitary character ψ of F. The Weil representation $\omega = \omega_X = \omega_n = \omega_{X,n}$ of $\mathrm{Sp}(n, F) \times \mathrm{O}(X, F)$ defined with respect to ψ is the unitary representation on $L^2(X^n)$ given by

$$\begin{aligned}
\omega(1,h)\varphi(x) &= \varphi(h^{-1}x),\\
\omega(\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}, 1)\varphi(x) &= \chi_X(\det a)|\det a|^{m/2}\varphi(xa),\\
\omega(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, 1)\varphi(x) &= \psi(\frac{1}{2}\operatorname{tr}(bx, x))\varphi(x),\\
\omega(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 1)\varphi(x) &= \gamma\hat{\varphi}(x).
\end{aligned}$$

Here, $\hat{\varphi}$ is the Fourier transform defined by

$$\hat{\varphi}(x) = \int_{X^n} \varphi(x')\psi(\operatorname{tr}(x, x'))\, dx'$$

with dx such that $\hat{\hat{\varphi}}(x) = \varphi(-x)$ for $\varphi \in L^2(X^n)$ and $x \in X^n$, and γ is a certain fourth root of unity depending only on the anisotropic component of X, n and ψ. If $h \in \mathrm{O}(X, F)$, $a \in \mathrm{GL}(n, F)$, $b \in \mathrm{M}_n(F)$ with ${}^t b = b$ and $x = (x_1, \ldots, x_n), x' = (x'_1, \ldots, x'_n) \in X^n$, we write $h^{-1}x = (h^{-1}x_1, \ldots, h^{-1}x_n)$, $xa = (x_1, \ldots, x_n)(a_{ij})$, $(x, x') = ((x_i, x'_j))$, $bx = b^t(x_1, \ldots, x_n)$. Also, χ_X is the quadratic character of $F^\times$ defined by $\chi_X(t) = (t,d)_F$; χ_X depends only on the anisotropic component of X.

Suppose F is nonarchimedean. We will work with smooth representations of groups of td-type such as $\mathrm{Sp}(n,F)$ and $\mathrm{O}(X,F)$. We thus consider the restriction of ω to a smaller subspace. Let $\mathcal{S}(X^n)$ be the space of locally constant, compactly supported functions on X^n. Then ω preserves $\mathcal{S}(X^n)$. By ω we will usually mean ω acting on $\mathcal{S}(X^n)$; context will give the meaning. Let $\mathcal{R}_n(\mathrm{O}(X,F))$ be the set of elements of $\mathrm{Irr}(\mathrm{O}(X,F))$ which are nonzero quotients of ω, and define $\mathcal{R}_X(\mathrm{Sp}(n,F))$ similarly.

Suppose $F = \mathbb{R}$. In the analogy to the last case, we will work with Harish-Chandra modules of real reductive groups. This requires definitions. Fix $K_1 = \mathrm{Sp}(n,\mathbb{R}) \cap \mathrm{O}(2n,\mathbb{R})$ as a maximal compact subgroup of $\mathrm{Sp}(n,\mathbb{R})$. The Lie algebra of $\mathrm{Sp}(n,\mathbb{R})$ is $\mathfrak{g}_1 = \mathfrak{sp}(n,\mathbb{R})$. Let X have signature (p,q). We parameterize the maximal compact subgroups of $\mathrm{O}(X,\mathbb{R})$ as follows. Let X^+ and X^- be positive and negative definite subspaces of X, respectively, such that $X = X^+ \perp X^-$. Then the maximal compact subgroup $J_1 = J_1(X^+,X^-)$ associated to (X^+,X^-) is the set of $k \in \mathrm{O}(X,\mathbb{R})$ such that $k(X^+) = X^+$ and $k(X^-) = X^-$. Of course, $J_1 = \mathrm{O}(X^+,\mathbb{R}) \times \mathrm{O}(X^-,\mathbb{R}) \cong \mathrm{O}(p,\mathbb{R}) \times \mathrm{O}(q,\mathbb{R})$. Fix one such $J_1 = J_1(X^+,X^-)$. The Lie algebra of $\mathrm{O}(X,\mathbb{R})$ is $\mathfrak{h}_1 = \mathfrak{o}(X,\mathbb{R})$. Let $\mathcal{S}(X^n) = \mathcal{S}_\psi(X^n)$ be the subspace of $L^2(X^n)$ of functions

$$p(x)\exp[-\frac{1}{2}|c|(\mathrm{tr}(x^+,x^+) - \mathrm{tr}(x^-,x^-))].$$

Here, $p : X^n \to \mathbb{C}$ is a polynomial function on X^n, and (x^+,x^+) and (x^-,x^-) are the $n \times n$ matrices with (i,j)-th entries (x_i^+,x_j^+) and (x_i^-,x_j^-) respectively, where $x_i = x_i^+ + x_i^-$, with $x_i^+ \in X^+$ and $x_i^- \in X^-$ for $1 \le i \le n$; $c \in \mathbb{R}^\times$ is such that $\psi(t) = \exp(ict)$ for $t \in \mathbb{R}$. Then $\mathcal{S}(X^n)$ is a $(\mathfrak{g}_1 \times \mathfrak{h}_1, K_1 \times J_1)$ module under the action of ω. By ω we will usually mean the $(\mathfrak{g}_1 \times \mathfrak{h}_1, K_1 \times J_1)$ module $\mathcal{S}(X^n)$. Let $\mathcal{R}_n(\mathrm{O}(X,\mathbb{R}))$ be the set of irreducible $(\mathfrak{g}_1, J_1)$ modules which are nonzero quotients of ω, and define $\mathcal{R}_X(\mathrm{Sp}(n,\mathbb{R}))$ similarly. For uniformity, write $\mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega,\pi\otimes\sigma)$ for $\mathrm{Hom}_{(\mathfrak{g}_1\times\mathfrak{h}_1,K_1\times J_1)}(\omega,\pi\otimes\sigma)$.

We have the following foundational result on the theta correspondence for isometries.

1.1 Theorem ([H], [W1]). *Suppose F is real or nonarchimedean of odd residual characteristic. The set*

$$\{(\pi,\sigma) \in \mathcal{R}_X(\mathrm{Sp}(n,F)) \times \mathcal{R}_n(\mathrm{O}(X,F)) : \mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega,\pi\otimes\sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}_X(\mathrm{Sp}(n,F))$ and $\mathcal{R}_n(\mathrm{O}(X,F))$, and

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega,\pi\otimes\sigma) \le 1$$

for $\pi \in \mathcal{R}_X(\mathrm{Sp}(n,F))$ and $\sigma \in \mathcal{R}_n(\mathrm{O}(X,F))$.

When F is nonarchimedean of even residual characteristic partial results are known. For us the following unconditional result suffices. If F is nonarchimedean and $\sigma \in \mathrm{Irr}(\mathrm{O}(X,F))$ we say that σ is tempered if all the irreducible constituents of $\sigma|_{\mathrm{SO}(X,F)}$ are tempered.

1.2 Theorem. *Suppose F is nonarchimedean of even residual characteristic and $m = 2n$. Let $\mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$ be the subsets of $\mathcal{R}_n(\mathrm{O}(X,F))$ and $\mathcal{R}_X(\mathrm{Sp}(n,F))$ of tempered elements, respectively. Then the statement of Theorem 1.1 holds with $\mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$ replacing the sets $\mathcal{R}_X(\mathrm{Sp}(n,F))$ and $\mathcal{R}_n(\mathrm{O}(X,F))$, respectively.*

Proof. Let $\sigma \in \mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$. By 2) a), p. 69 of [MVW], there exists $\pi \in \mathcal{R}_X(\mathrm{Sp}(n,F))$ such that the homomorphism space of Theorem 1.1 is nonzero; π is tempered by (1) of Theorem 4.2 of [R3], and is unique by Theorem 4.4 of [R3]. To prove the map from $\mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$ to $\mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$ is injective and the homomorphism space has dimension at most one, let $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F))_{\mathrm{temp}}$. By putting together the proofs of Proposition II.3.1 of [Ra] and Theorem 4.4 of [R3] one can show that there is a $\mathbb{C}$ linear injection

$$\bigoplus_{\substack{\sigma\in\mathrm{Irr}(\mathrm{O}(X,F))\\ \sigma \text{ unitary}}} \mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega,\pi\otimes\sigma) \hookrightarrow \mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{Sp}(n,F)}(\mathcal{S}(\mathrm{Sp}(n,F)),\pi\otimes\pi^\vee),$$

where $\mathcal{S}(\mathrm{Sp}(n,F))$ is the space of locally constant compactly supported functions on $\mathrm{Sp}(n,F)$ and the action of $\mathrm{Sp}(n,F)\times\mathrm{Sp}(n,F)$ on $\mathcal{S}(\mathrm{Sp}(n,F))$ is defined by $((g,g')\cdot\phi)(x) = \phi(g^{-1}xg')$. The last space is one dimensional as

$$\pi\otimes\pi^\vee \cong \mathcal{S}(\mathrm{Sp}(n,F))/\bigcap_{\substack{f\in\mathrm{Hom}_{\mathrm{Sp}(n,F)}(\mathcal{S}(\mathrm{Sp}(n,F)),\pi\otimes U),\\ U \text{ a } \mathbb{C} \text{ vector space}}} \ker(f);$$

see the lemma on p. 59 of [MVW]. This proves the claims about injectivity and dimension. For surjectivity, let $\pi \in \mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$. As above, there exists $\sigma \in \mathcal{R}_n(\mathrm{O}(X,F))$ such that the homomorphism space is nonzero. An argument as in the proof of (1) of Theorem 4.2 of [R3] shows that σ must be tempered. See also [Mu]. $\square$

It is worth noting the following from the proof of Theorem 1.2: Let $m = 2n$, $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F))$ and $\sigma \in \mathrm{Irr}(\mathrm{O}(X,F))$. If $\mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{O}(X,F)}(\omega,\pi\otimes\sigma) \neq 0$, then π is tempered if and only if σ is tempered.
When F is nonarchimedean of odd residual characteristic or $F = \mathbb{R}$, then the bijection from Theorem 1.1 and its inverse are denoted by

$$\theta : \mathcal{R}_X(\mathrm{Sp}(n,F)) \xrightarrow{\sim} \mathcal{R}_n(\mathrm{O}(X,F)), \quad \theta : \mathcal{R}_n(\mathrm{O}(X,F)) \xrightarrow{\sim} \mathcal{R}_X(\mathrm{Sp}(n,F));$$

if F is nonarchimedean of even residual characteristic we use the same notation for the bijections between $\mathcal{R}_X(\mathrm{Sp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{O}(X,F))_{\mathrm{temp}}$ from Theorem 1.2.
Next we recall and prove a special case of a conjecture of S.S. Kudla on the theta correspondence for isometries. This conjecture has important implications for the theta correspondence for similitudes.

1.3 THETA DICHOTOMY CONJECTURE (S.S. KUDLA). *Assume F is nonarchimedean. Let m be a positive even integer, let $d \in F^\times/F^{\times 2}$, and let n be a positive integer such that $m \leq 2n$. There exist at most two quadratic spaces Y and Y' over F of dimension m and discriminant d; assume both exist. Then $\mathcal{R}_Y(\mathrm{Sp}(n,F)) \cap \mathcal{R}_{Y'}(\mathrm{Sp}(n,F)) = \emptyset$.*

We can prove the conjecture when m is small in comparison to $2n$:

1.4 LEMMA. *Suppose that the notation is as in Conjecture 1.3, and assume the two quadratic spaces Y and Y' exist. If $m \leq n+2$, then the theta dichotomy conjecture holds for m and n.*

Proof. Let Z be the quadratic space over F of dimension $2m$ with four dimensional anisotropic component. To prove the theta dichotomy conjecture for m and n it suffices to show that $\mathbf{1}_{\mathrm{Sp}(n)} \notin \mathcal{R}_Z(\mathrm{Sp}(n,F))$. This reduction is well known, but we recall the proof for the convenience of the reader. Assume $\mathbf{1}_{\mathrm{Sp}(n)} \notin \mathcal{R}_Z(\mathrm{Sp}(n,F))$, and suppose $\pi \in \mathcal{R}_Y(\mathrm{Sp}(n,F)) \cap \mathcal{R}_{Y'}(\mathrm{Sp}(n,F))$. To get a contradiction, let $g_0 \in \mathrm{GSp}(n,F)$ be such that $\lambda(g_0) = -1$. By the first theorem on p. 91 of [MVW], $g_0 \cdot \pi \cong \pi^\vee$. Thus, there is a nonzero $\mathrm{Sp}(n,F) \times \mathrm{Sp}(n,F)$ map from $\omega_{Y,n} \otimes g_0 \cdot \omega_{Y',n}$ to $\pi \otimes \pi^\vee$. By Lemma 1.6 below, $g_0 \cdot \omega_{Y',n} \cong \omega_{-Y',n}$, where $-Y'$ has the same space as Y' and form multiplied by -1. Also, $(\omega_{Y,n} \otimes \omega_{-Y',n})|_{\Delta\,\mathrm{Sp}(n,F)} \cong \omega_{Y\perp -Y',n}|_{\mathrm{Sp}(n,F)}$. Clearly, $Y \perp -Y' \cong Z$. Since $\mathrm{Hom}_{\mathrm{Sp}(n,F)}((\pi \otimes \pi^\vee)|_{\Delta\,\mathrm{Sp}(n,F)}, \mathbf{1}_{\mathrm{Sp}(n,F)}) \neq 0$, there now is a nonzero $\mathrm{Sp}(n,F)$ map from $\omega_{Z,n}$ to $\mathbf{1}_{\mathrm{Sp}(n,F)}$, i.e., $\mathbf{1}_{\mathrm{Sp}(n,F)} \in \mathcal{R}_Z(\mathrm{Sp}(n,F))$. We now show $\mathbf{1}_{\mathrm{Sp}(n)} \notin \mathcal{R}_Z(\mathrm{Sp}(n,F))$ for $m \leq n+2$. Let i be the Witt index of Z, i.e., $i = m-2$, and write $V_i = Z$ to indicate that Z is the orthogonal direct sum of the four dimensional anisotropic quadratic space over F with i hyperbolic planes. We must show $\mathbf{1}_{\mathrm{Sp}(n)} \notin \mathcal{R}_{V_i}(\mathrm{Sp}(n,F))$ for $0 \leq i \leq n$; we do this by induction on n. The case $n=1$ follows from Lemma 7.3 of [R2] (see its proof, which is residual characteristic independent). Let $n > 1$, and assume the claim for $n-1$. Let $i \leq n$, and assume $\mathbf{1}_{\mathrm{Sp}(n,F)} \in \mathcal{R}_{V_i}(\mathrm{Sp}(n,F))$. To find a contradiction we reduce dimensions using Jacquet functors and Kudla's filtration of the Jacquet module of the Weil representation. We use the notation of [R3]: write $\omega_{i,n} = \omega_{V_i,n}$. Since $\mathbf{1}_{\mathrm{Sp}(n,F)} \in \mathcal{R}_{V_i}(\mathrm{Sp}(n,F))$, by 2) a) of the Theorem on p. 69 of [MVW], there exists $\sigma \in \mathrm{Irr}(\mathrm{O}(V_i,F))$ and a nonzero $\mathrm{Sp}(n,F) \times \mathrm{O}(V_i,F)$ map

$$\omega_{i,n} \to \mathbf{1}_{\mathrm{Sp}(n,F)} \otimes \sigma.$$

Let N_1' be the unipotent radical of the standard maximal parabolic of $\mathrm{Sp}(n,F)$ with Levi factor isomorphic to $\mathrm{GL}(1,F) \times \mathrm{Sp}(n-1,F)$. Applying the normalized Jacquet functor with respect to N_1', which is exact, we obtain a nonzero $\mathrm{GL}(1,F) \times \mathrm{Sp}(n-1,F) \times \mathrm{O}(V_i,F)$ map

$$\mathrm{R}_{N_1'}(\omega_{i,n}) \to \mathrm{R}_{N_1'}(\mathbf{1}_{\mathrm{Sp}(n)}) \otimes \sigma = |\cdot|^{-n} \otimes \mathbf{1}_{\mathrm{Sp}(n-1,F)} \otimes \sigma.$$

Suppose first $i = 0$, so that V_0 is four dimensional and anisotropic. By Kudla's computation of the Jacquet functors of $\omega_{0,n}$, (see [R3] for a statement in our notation), $\mathrm{R}_{N_1'}(\omega_{0,n})|_{\mathrm{Sp}(n-1,F)} \cong \omega_{0,n-1}|_{\mathrm{Sp}(n-1,F)}$. Thus, $\mathbf{1}_{\mathrm{Sp}(n-1,F)} \in \mathcal{R}_{V_i}(\mathrm{Sp}(n-1,F))$. Since $i = 0 \leq n-1$, by the induction hypothesis this is a contradiction.

Suppose $i > 0$. By Kudla's filtration (two step, in this case) of $\mathrm{R}_{N_1'}(\omega_{i,n})$ either there exists a nonzero $\mathrm{GL}(1,F) \times \mathrm{Sp}(n-1,F) \times \mathrm{O}(V_i,F)$ map

$$|\cdot|^{\dim V_i/2-n} \otimes \omega_{i,n-1} \to |\cdot|^{-n} \otimes \mathbf{1}_{\mathrm{Sp}(n-1,F)} \otimes \sigma,$$

or there exists a nonzero $\mathrm{GL}(1,F) \times \mathrm{GL}(1,F) \times \mathrm{Sp}(n-1,F) \times \mathrm{O}(V_{i-1},F)$ map

$$\xi_1 \xi_1' \boldsymbol{\sigma}_1 \otimes \omega_{i-1,n-1} \to |\cdot|^{-n} \otimes \mathbf{1}_{\mathrm{Sp}(n-1,F)} \otimes \overline{\mathrm{R}}_{N_1}(\sigma);$$

here, ξ_1 and ξ_1' are quasi-characters of $\mathrm{GL}(1,F)$ and $\boldsymbol{\sigma}_1$ is a representation of $\mathrm{GL}(1,F) \times \mathrm{GL}(1,F)$ whose precise definitions we will not need, $|\cdot|^{-n}$ is regarded as a quasi-character of $\mathrm{GL}(1,F)$, N_1 is the unipotent radical of the standard parabolic of $\mathrm{O}(V_i,F)$ with Levi factor isomorphic to $\mathrm{GL}(1,F) \times \mathrm{O}(V_{i-1},F)$, and $\overline{\mathrm{R}}_{N_1}(\sigma) = \mathrm{R}_{N_1}(\sigma^\vee)^\vee$. The first case is ruled out since $|\cdot|^{\dim V_i/2-n} \neq |\cdot|^{-n}$. Since the second case must therefore hold, we get $\mathrm{Hom}_{\mathrm{Sp}(n-1,F)}(\omega_{i-1,n-1}, \mathbf{1}_{\mathrm{Sp}(n-1,F)}) \neq 0$, i.e., $\mathbf{1}_{\mathrm{Sp}(n-1,F)} \in \mathcal{R}_{V_{i-1}}(\mathrm{Sp}(n-1,F))$. This contradicts the induction hypothesis since $i-1 \leq n-1$. $\square$

The real analogue of the theta dichotomy conjecture is known. The assumption of the evenness of p and q in the following lemma is a consequence of the same assumptions in [M].

1.5 Lemma. *Suppose $F = \mathbb{R}$. Let m be a positive even integer and let n be a positive integer such that $m \leq 2n$. Then the sets $\mathcal{R}_Y(\mathrm{Sp}(n,\mathbb{R}))$ as Y runs over the isometry classes quadratic spaces over $\mathbb{R}$ of dimension m and signature of the form (p,q) with p and q even are mutually disjoint.*

Proof. We argue as in the second paragraph of the proof of Lemma 1.8 of [AB]. Suppose Y and Y' are quadratic spaces of dimension m with signatures (p,q) and (p',q') with p,q,p' and q' even. Assume $\pi \in \mathcal{R}_Y(\mathrm{Sp}(n,\mathbb{R})) \cap \mathcal{R}_{Y'}(\mathrm{Sp}(n,\mathbb{R}))$. We must show that $Y \cong Y'$, i.e., $p = p'$ and $q = q'$. We have $\mathrm{Hom}_{(\mathfrak{g}_1,K_1)}(\omega_{Y,n},\pi) \neq 0$ and $\mathrm{Hom}_{(\mathfrak{g}_1,K_1)}(\omega_{Y',n},\pi) \neq 0$; as in the proof of Lemma 1.4 this implies $\mathrm{Hom}_{(\mathfrak{g}_1,K_1)}(\omega_{Z,n},(\pi \otimes \pi^\vee)|_{\Delta(\mathfrak{g}_1,K_1)}) \neq 0$, where $Z = Y \perp -Y'$, and $-Y'$ is the quadratic space with same space as Y' but with form multiplied by -1; Z has signature $(p+q', p'+q)$. Hence, $\mathbf{1}_{\mathrm{Sp}(n,\mathbb{R})} \in \mathcal{R}_Z(\mathrm{Sp}(n,\mathbb{R}))$. We now use [M] to complete the proof. The representation $\mathbf{1}_{\mathrm{Sp}(n,\mathbb{R})}$ has only one K_1-type, namely the trivial representation of K_1. As $\mathbf{1}_{\mathrm{Sp}(n,\mathbb{R})} \in \mathcal{R}_Z(\mathrm{Sp}(n,\mathbb{R}))$, the trivial representation of K_1 appears in the joint harmonics $H(K_1, \mathrm{O}(p+q',\mathbb{R}) \times \mathrm{O}(p'+q,\mathbb{R}))$ for this theta correspondence (see I.1 and the second paragraph of II.1 of [M]). By Corollaire I.4 of [M], which computes the representations of K_1 occurring in the joint harmonics, $p+q' = p'+q$; since $p+q = p'+q'$, we have $p = p'$ and $q = q'$. $\square$

We now recall the extended Weil representation which will be used to define the theta correspondence for similitudes; see [R1] for references. Define

$$R = R_X = R_n = R_{X,n} = \{(g,h) \in \mathrm{GSp}(n,F) \times \mathrm{GO}(X,F) : \lambda(g) = \lambda(h)\}.$$

The Weil representation ω of $\mathrm{Sp}(n,F)\times \mathrm{O}(X,F)$ on $L^2(X^n)$ extends to a unitary representation of R via

$$\omega(g,h)\varphi = |\lambda(h)|^{-\frac{mn}{4}}\omega(g_1,1)(\varphi\circ h^{-1}),$$

where

$$g_1 = g\begin{bmatrix}1 & 0\\ 0 & \lambda(g)\end{bmatrix}^{-1}\in \mathrm{Sp}(n,F).$$

Evidently, the group of elements $(t,t)=(t\cdot 1,t\cdot 1)$ for $t\in F^\times$ is contained in the center of R, and we have $\omega(t,t)\varphi=\chi_X(t)^n\varphi$ for $\varphi\in L^2(X^n)$ and $t\in F^\times$. If F is nonarchimedean, then the extended Weil representation preserves $\mathcal{S}(X^n)$; when F is nonarchimedean, by ω we shall often mean ω acting on $\mathcal{S}(X^n)$.
Suppose $F=\mathbb{R}$; then ω extended to R also preserves $\mathcal{S}(X^n)$, but only at the level of Harish-Chandra modules. We need definitions. As a standard maximal compact subgroup K of $\mathrm{GSp}(n,\mathbb{R})$ take the group generated by K_1 and the order two element

$$k_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

The Lie algebra $\mathfrak{g}=\mathfrak{gsp}(n,\mathbb{R})$ of $\mathrm{GSp}(n,\mathbb{R})$ is the direct sum of its center $\mathbb{R}$ and $\mathfrak{g}_1=\mathfrak{sp}(n,\mathbb{R})$. If $p\neq q$, then any maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$ is a maximal compact subgroup of $\mathrm{O}(X,\mathbb{R})$, and we let J denote the subgroup J_1 from above. Suppose $p=q$. Then every maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$ contains a unique maximal compact subgroup of $\mathrm{O}(X,\mathbb{R})$ as a subgroup of index two, and any maximal compact subgroup of $\mathrm{O}(X,\mathbb{R})$ is contained in a unique maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$ as a subgroup of index two. As a maximal compact subgroup for $\mathrm{GO}(X,\mathbb{R})$ we take the maximal compact subgroup $J=J(X^+,X^-)$ containing $J_1=J_1(X^+,X^-)$. To get a coset representative j_0 for the nontrivial coset of J_1 in J, let $i:X^+\to X^-$ be an isomorphism of $\mathbb{R}$ vector spaces such that $(i(x^+),i(x^+))=-(x^+,x^+)$ for $x^+\in X^+$ and using $X=X^+\perp X^-$ set

$$j_0=\begin{bmatrix}0 & i^{-1}\\ i & 0\end{bmatrix}.$$

The Lie algebra $\mathfrak{h}=\mathfrak{go}(n,\mathbb{R})$ of $\mathrm{GO}(X,\mathbb{R})$ is the direct sum of its center $\mathbb{R}$ and $\mathfrak{h}_1=\mathfrak{o}(n,\mathbb{R})$. The group R is a real reductive group containing $(\mathrm{Sp}(n,\mathbb{R})\times \mathrm{O}(X,\mathbb{R}))\{(t,t):t\in\mathbb{R}^\times\}$ as an open subgroup of index one if $p\neq q$, and index two if $p=q$. As a maximal compact subgroup L of R we take $L=K_1\times J_1$ if $p\neq q$; if $p=q$, then we take L to be generated by $K_1\times J_1$ and (k_0,j_0). The Lie algebra $\mathfrak{r}$ of R is the set of pairs $(x,y)\in\mathfrak{g}\times\mathfrak{h}$ such that $x=z+x_1$ and $y=z+y_1$ for some $z\in\mathbb{R}, x_1\in\mathfrak{g}_1$ and $y_1\in\mathfrak{h}_1$. The space $\mathcal{S}(X^n)$ is evidently closed under the action of ω restricted to L and $\mathfrak{r}$. The $(\mathfrak{g}_1\times\mathfrak{h}_1,K_1\times J_1)$ module $\mathcal{S}(X^n)$ thus extends to an $(\mathfrak{r},L)$ module, which we will also denote by ω.

Before discussing the theta correspondence for similitudes it will be useful to describe the relationship between the extended Weil representations for similar quadratic spaces, and what happens to the extended Weil representation when the additive character is changed. For $\lambda \in F^\times$ and $g \in \mathrm{GSp}(n,F)$ write

$$g^{[\lambda]} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} g \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}^{-1}.$$

1.6 LEMMA. *Let X' another quadratic space over F, and suppose $t : X \to X'$ is a similitude with similitude factor λ. Let ω' be the Weil representation of $R_{X',n}$ on $L^2(X'^n)$. Then*

$$\omega(g,h)(\varphi' \circ t) = [\omega'(g^{[\lambda]}, tht^{-1})\varphi')] \circ t$$

for $(g,h) \in R_{X,n}$ and $\varphi' \in L^2(X'^n)$.

Proof. By the formulas for the Weil representation the statement holds for $g \in \mathrm{Sp}(n,F)$ and $h = 1$, up to a factor $\alpha(g)$ in the fourth roots of unity μ_4. The function $\alpha : \mathrm{Sp}(n,F) \to \mu_4$ is a character. The only normal subgroups of $\mathrm{Sp}(n,F)$ are $\{\pm 1\}$ and $\mathrm{Sp}(n,F)$; α must be trivial. It is now easy to check that the formula holds for all $(g,h) \in R$. $\square$

In the next result the dependence of ω on ψ is indicated by a subscript. Its proof is similar to that of Lemma 1.6.

1.7 LEMMA. *Let ψ' be another nontrivial unitary character of F. Let $a \in F^\times$ be such that $\psi'(t) = \psi(at)$ for $t \in F$. Then there is an isomorphism*

$$(\omega_{\psi'}, L^2(X^n)) \xrightarrow{\sim} ((\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}, 1) \cdot \omega_\psi, L^2(X^n))$$

of representations of R, where $\epsilon = a$ if F is nonarchimedean and $\epsilon = \mathrm{sign}(a)$ if $F = \mathbb{R}$. If F is nonarchimedean, the isomorphism is the identity map; if $F = \mathbb{R}$, the isomorphism sends φ' to φ, where $\varphi(x) = \varphi'(\sqrt{|a|}^{-1} x)$. This isomorphism maps $\mathcal{S}_{\psi'}(X^n)$ onto $\mathcal{S}_\psi(X^n)$ (the subscripts ψ and ψ' are relevant when $F = \mathbb{R}$).

With this preparation, we recall the theta correspondence for similitudes from [R1]. In analogy to the case of isometries, we ask when does $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ for $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F))$ and $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ define the graph of a bijection between appropriate subsets of $\mathrm{Irr}(\mathrm{GSp}(n,F))$ and $\mathrm{Irr}(\mathrm{GO}(X,F))$? In considering this, two initial observations come to mind. First, R only involves $\mathrm{GSp}(n,F)^+$, the subgroup of $\mathrm{GSp}(n,F)$ (of at most index two) of $g \in \mathrm{GSp}(n,F)^+$ with $\lambda(g) \in \lambda(\mathrm{GO}(X,F))$; thus, at first it might be better to look at representations of $\mathrm{GSp}(n,F)^+$ instead of $\mathrm{GSp}(n,F)$. Second, there should be a close relationship between $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ and $\mathrm{Hom}_{\mathrm{Sp}(n,F)\times \mathrm{O}(X,F)}(\omega, \pi_1 \otimes \sigma_1) \neq 0$ for π_1 and σ_1 irreducible constituents of

$\pi|_{\mathrm{Sp}(n,F)}$ and $\sigma|_{\mathrm{O}(X,F)}$, respectively. The basic result that builds on these remarks is Lemma 4.2 of [R1]. It asserts that if $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F)^+)$, $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ and $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$, then

$$\pi|_{\mathrm{Sp}(n,F)} = m \cdot \pi_1 \oplus \cdots \oplus m \cdot \pi_M, \quad \sigma|_{\mathrm{O}(X,F)} = m' \cdot \sigma_1 \oplus \cdots \oplus m' \cdot \sigma_M$$

with $\theta(\pi_i) = \sigma_i$ for $1 \leq i \leq M$ and $m = 1$ if and only if $m' = 1$. Here the $\pi_i \in \mathrm{Irr}(\mathrm{Sp}(n,F))$ and $\sigma_i \in \mathrm{Irr}(\mathrm{O}(X,F))$ are mutually nonisomorphic. Actually, [R1] considers the nonarchimedean case of odd residual characteristic, but the same proof works if F has even residual characteristic, $\dim X = 2n$ and π and σ are tempered, so that Theorem 1.2 applies, or if $F = \mathbb{R}$; in this case $m = m' = 1$, as $[\mathrm{GSp}(n,\mathbb{R})^+ : \mathbb{R}^\times \mathrm{Sp}(n,\mathbb{R})], [\mathrm{GO}(X,\mathbb{R}) : \mathbb{R}^\times \mathrm{O}(X,\mathbb{R})] \leq 2$ (see Table 1 in the appendix for data on $\mathrm{GSp}(n,\mathbb{R})^+$). With this in place, [R1] shows that the condition $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ defines the graph of a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$, where $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ is the set of $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F)^+)$ such that $\pi|_{\mathrm{Sp}(n,F)}$ is multiplicity free and has an irreducible constituent in $\mathcal{R}_X(\mathrm{Sp}(n,F))$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$ is similarly defined (again, this also holds if F has even residual characteristic or $F = \mathbb{R}$).
Finally, when $\mathrm{GSp}(n,F)^+$ is proper in $\mathrm{GSp}(n,F)$, [R1] shows $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ defines the graph of a bijection between suitable subsets of $\mathrm{Irr}(\mathrm{GSp}(n,F))$ and $\mathrm{Irr}(\mathrm{GO}(X,F))$ provided $m \leq 2n$ and the relevant case of Conjecture 1.3 holds. The idea is that if $[\mathrm{GSp}(n,F) : \mathrm{GSp}(n,F)^+] = 2$, then X has a certain companion nonisometric quadratic space X' with the same dimension and discriminant (this determines X' if F is nonarchimedean; if $F = \mathbb{R}$, then X' is the quadratic space of signature (q,p)). When it holds and $m \leq 2n$, Conjecture 1.3 implies that together the two theta correspondences between $\mathrm{GSp}(n,F)^+$ and $\mathrm{GO}(X,F)$ and between $\mathrm{GSp}(n,F)^+$ and $\mathrm{GO}(X',F)$ give one theta correspondence between $\mathrm{GSp}(n,F)$ and $\mathrm{GO}(X,F)$ (which is the same as that between $\mathrm{GSp}(n,F)$ and $\mathrm{GO}(X',F)$, using $\mathrm{GO}(X,F) = \mathrm{GO}(X',F)$). Since [R1] explains this in somewhat different language, we recall the argument in the proof of the summary theorem below.
For the statement of the theorem we need some notation. If F is nonarchimedean, define $\mathcal{R}_n(\mathrm{GO}(X,F))$ and $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ as above, and let $\mathcal{R}_X(\mathrm{GSp}(n,F))$ be the set of $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F))$ such that some irreducible constituent of $\pi|_{\mathrm{GSp}(n,F)^+}$ is contained in $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$. If $F = \mathbb{R}$, let $\mathcal{R}_n(\mathrm{GO}(X,\mathbb{R}))$ be the set of $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,\mathbb{R}))$ such that $\sigma|_{\mathrm{O}(X,\mathbb{R})}$ has an irreducible constituent in $\mathcal{R}_n(\mathrm{O}(X,\mathbb{R}))$, and let $\mathcal{R}_X(\mathrm{GSp}(n,\mathbb{R}))$ be the set of $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,\mathbb{R}))$ such that $\pi|_{\mathrm{Sp}(n,\mathbb{R})}$ has an irreducible constituent in $\mathcal{R}_X(\mathrm{Sp}(n,\mathbb{R}))$. Here $\sigma|_{\mathrm{O}(X,\mathbb{R})}$ and $\pi|_{\mathrm{Sp}(n,\mathbb{R})}$ mean $\sigma|_{(\mathfrak{h}_1,J_1)}$ and $\pi|_{(\mathfrak{g}_1,K_1)}$, respectively. If $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ and F is nonarchimedean we say that σ is tempered if all the irreducible constituents of $\sigma|_{\mathrm{GSO}(X,F)}$ are tempered; evidently, σ is tempered if and only if the irreducible constituents of $\sigma|_{\mathrm{O}(X,F)}$ are tempered and σ has unitary central character, and this happens if and only if the irreducible constituents of $\sigma|_{\mathrm{SO}(X,F)}$ are tempered and σ has unitary central character.

1.8 THEOREM. *Suppose first F is real or nonarchimedean of odd residual characteristic. Then*

$$\{(\pi,\sigma)\in\mathcal{R}_X(\mathrm{GSp}(n,F))\times\mathcal{R}_n(\mathrm{GO}(X,F)):\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\neq 0\}$$

is the graph of a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,F))$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$, and

$$\dim_{\mathbb{C}}\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\leq 1$$

for $\pi\in\mathcal{R}_X(\mathrm{GSp}(n,F))$ and $\sigma\in\mathcal{R}_n(\mathrm{GO}(X,F))$, in the following cases:

(1) *F is nonarchimedean and $d=1$;*
(2) *F is nonarchimedean, $d\neq 1$, and $m\leq n+2$;*
(3) *$F=\mathbb{R}$ and $p=q$;*
(4) *$F=\mathbb{R}$, $p\neq q$, p and q are even, and $p+q\leq 2n$.*

Now assume F is nonarchimedean of even residual characteristic and $m=2n$. As in Theorem 1.2, let the subscript temp denote the subset of tempered elements. Then the above statement holds with $\mathcal{R}_X(\mathrm{GSp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{GO}(X,F))_{\mathrm{temp}}$ in place of $\mathcal{R}_X(\mathrm{GSp}(n,F))$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$, respectively, in the following cases:

(5) *$d=1$ and $m=2n$;*
(6) *$d\neq 1$ and $m=2n=n+2=4$.*

Proof. (1). Since $d=1$, $\mathrm{GSp}(n,F)^+=\mathrm{GSp}(n,F)$, and the statement follows from Theorem 4.4 of [R1].
(2). This is dealt with in [R1], but we shall briefly recall the argument for the purposes of explanation. In this case we have $[\mathrm{GSp}(n,F):\mathrm{GSp}(n,F)^+]=2$. Let $g\in\mathrm{GSp}(n,F)$ be a representative for the nontrivial coset of $\mathrm{GSp}(n,F)^+$ in $\mathrm{GSp}(n,F)$. As mentioned above, by Theorem 4.4 of [R1] the condition $\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\neq 0$ defines a bijection between $\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$, and $\dim_{\mathbb{C}}\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\leq 1$ for $\pi'\in\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ and $\sigma\in\mathcal{R}_n(\mathrm{GO}(X,F))$. To prove the theorem in this case, we first claim that if $\pi'\in\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$ and $\sigma\in\mathcal{R}_n(\mathrm{GO}(X,F))$ are such that $\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\neq 0$, then $g\cdot\pi'\not\cong\pi'$ (so that $\pi=\mathrm{Ind}_{\mathrm{GSp}(n,F)^+}^{\mathrm{GSp}(n,F)}\pi'$ is irreducible), and $\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\cong\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)$; also, if $\pi\in\mathcal{R}_X(\mathrm{GSp}(n,F))$ then $\pi|_{\mathrm{GSp}(n,F)^+}$ has two irreducible components. Let X' be the other quadratic space of dimension m and discriminant d nonisometric to X. We may assume that X' is obtained from X by multiplying the form on X by $\lambda(g)$; then $\mathrm{GO}(X',F)=\mathrm{GO}(X,F)$ and $R_{X',n}=R=R_{X,n}$. Let $\omega'=\omega_{X'}$; by Lemma 1.6, $g\cdot\omega\cong\omega'$. Now since $\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\neq 0$ we have $\mathrm{Hom}_R(g\cdot\omega,g\cdot\pi'\otimes\sigma)\neq 0$, and so $\mathrm{Hom}_R(\omega',g\cdot\pi'\otimes\sigma)\neq 0$. This gives $g\cdot\pi'\in\mathcal{R}_{X'}(\mathrm{GSp}(n,F)^+)$. If now $\pi'\cong g\cdot\pi'$, then $\mathcal{R}_X(\mathrm{Sp}(n,F))\cap\mathcal{R}_{X'}(\mathrm{Sp}(n,F))\neq\emptyset$ (see Lemma 4.2 of [R1]), contradicting Lemma 1.4. Thus, $g\cdot\pi'\not\cong\pi'$. Composing with the projection $\pi\to\pi'$ gives a map $\mathrm{Hom}_R(\omega,\pi\otimes\sigma)\to\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)$; by arguments similar to those just given, this map is a $\mathbb{C}$ linear isomorphism. Let $\pi\in\mathcal{R}_X(\mathrm{GSp}(n,F))$, and suppose $\pi|_{\mathrm{GSp}(n,F)^+}=\pi'$ is irreducible. Then $\pi'\in\mathcal{R}_X(\mathrm{GSp}(n,F)^+)$, and

so $g \cdot \pi' \ncong \pi'$, a contradiction. This completes the proof of our claim. Using the claim, it is straightforward to prove the theorem in this case via analogous arguments.
(3) and (4). The arguments are similar to the nonarchimedean case in [R1]. In fact, they are easier since the indices of the various relevant subgroups are at most two. Thus, the analogues of the lemmas about induction and restriction from [GK] used in [R1] take on a simple form. For the convenience of the reader wishing to look closely at the arguments we present a table of data (See Table 1 in the Appendix). The $p = q$ and $p \neq q$ cases should be regarded as being analogous to the $d = 1$ and $d \neq 1$ nonarchimedean cases, respectively. In the table K^+ is a maximal compact subgroup of $\mathrm{GSp}(n, \mathbb{R})^+$.
(5) and (6). The arguments are similar to those for (1) and (2) as we have the inputs Theorem 1.2 and Lemma 1.4. The proofs of section 4 of [R1] are made in an abstracted context and thus residual characteristic independent; these arguments also go through with the restriction to tempered representations. The arguments in (2) for the case $[\mathrm{GSp}(n, F) : \mathrm{GSp}(n, F)^+] = 2$ also work with the restriction to tempered representations. The reader wishing to go through the details should note the remark after Theorem 1.2. $\square$

The proof of Theorem 1.8 only used Lemmas 1.4 and 1.5 when F is nonarchimedean and $d \neq 1$ and $F = \mathbb{R}$ and $p \neq q$, respectively. However, Lemmas 1.4 and 1.5 have important applications when F is nonarchimedean and $d = 1$, and $F = \mathbb{R}$ and $p = q$: see Lemma 8.4 and the proof of Proposition 4.1.
We note that if $\pi \in \mathrm{Irr}(\mathrm{GSp}(n, F))$, $\sigma \in \mathrm{Irr}(\mathrm{GO}(X, F))$ and $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ then $\chi_X^n = \omega_\pi \omega_\sigma$ where ω_π and ω_σ are the central characters of π and σ, respectively. Here, if $F = \mathbb{R}$ then the central character of $\pi \in \mathrm{Irr}(\mathrm{GSp}(n, \mathbb{R}))$ is defined by $\omega_\pi(e^z) = \exp(\pi(z))$ for $z \in \mathbb{R} \subset \mathfrak{g}$, and $\omega_\pi(-1) = \pi(-1)$, where $-1 \in K$; ω_σ is defined similarly.
The theta correspondence for similitudes from Theorem 1.8 is independent of the choice of character ψ.

1.9 Proposition. *Let ψ' be another nontrivial unitary character of F, and let $\omega_{\psi'}$ be the Weil representation of R on $\mathcal{S}_{\psi'}(X^n)$ corresponding to ψ' (the subscript ψ' in $\mathcal{S}_{\psi'}(X^n)$ is relevant when $F = \mathbb{R}$). Let $\sigma \in \mathrm{Irr}(\mathrm{GO}(X, F))$ and $\pi \in \mathrm{Irr}(\mathrm{GSp}(n, F))$. Then $\mathrm{Hom}_R(\omega_\psi, \pi \otimes \sigma) \neq 0$ if and only if $\mathrm{Hom}_R(\omega_{\psi'}, \pi \otimes \sigma) \neq 0$.*

Proof. This follows from Lemma 1.7. $\square$

Assume we are in one of the cases of Theorem 1.8. We then denote the bijection between $\mathcal{R}_X(\mathrm{GSp}(n, F))$ and $\mathcal{R}_n(\mathrm{GO}(X, F))$ by θ:

$$\theta : \mathcal{R}_X(\mathrm{GSp}(n, F)) \xrightarrow{\sim} \mathcal{R}_n(\mathrm{GO}(X, F)),$$
$$\theta : \mathcal{R}_n(\mathrm{GO}(X, F)) \xrightarrow{\sim} \mathcal{R}_X(\mathrm{GSp}(n, F)).$$

If $\pi \in \mathcal{R}_X(\mathrm{GSp}(n, F))$ and $\sigma \in \mathcal{R}_n(\mathrm{GO}(X, F))$ then $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ if and only if $\theta(\pi) = \sigma$ and $\theta(\sigma) = \pi$; if F has even residual characteristic we use $\mathcal{R}_X(\mathrm{GSp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{GO}(X,F))_{\mathrm{temp}}$. If $\sigma \in \mathcal{R}_n(\mathrm{GO}(X,F))$ we say that σ OCCURS IN THE THETA CORRESPONDENCE WITH $\mathrm{GSp}(n,F)$; similarly, if $\pi \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ we say that π occurs in the theta correspondence with $\mathrm{GO}(X,F)$. The above definition of θ is not quite compatible with the global definition; a contragredient must be introduced. If π is a cuspidal automorphic representation of $\mathrm{GSp}(X,\mathbb{A})$, the global theta lift $\Theta(\pi)$ is nonzero and cuspidal, and Theorem 1.8 applies at every place, then $\pi_v^\vee \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ for all places v of F, and $\Theta(\pi) = \otimes_v \theta(\pi_v^\vee)$ (See Section 5). However, we have the following proposition. It guarantees that if $\sigma_v = \theta(\pi_v^\vee)$ then $\theta(\sigma_v^\vee) = \pi_v$.

1.10 PROPOSITION. *Let $\pi \in \mathrm{Irr}(\mathrm{GSp}(n,F))$ and $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$ be unitary. Then $\mathrm{Hom}_R(\omega,\pi\otimes\sigma) \neq 0$ if and only if $\mathrm{Hom}_R(\omega,\pi^\vee\otimes\sigma^\vee) \neq 0$. Suppose one of (1)–(6) of Theorem 1.8 holds. Then $\pi \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ if and only if $\pi^\vee \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ and if $\pi \in \mathcal{R}_X(\mathrm{GSp}(n,F))$, then $\theta(\pi^\vee) = \theta(\pi)^\vee$. Similarly, $\sigma \in \mathcal{R}_n(\mathrm{GO}(X,F))$ if and only if $\sigma^\vee \in \mathcal{R}_n(\mathrm{GO}(X,F))$, and if $\sigma \in \mathcal{R}_n(\mathrm{GO}(X,F))$ then $\theta(\sigma^\vee) = \theta(\sigma)^\vee$. (If F has even residual characteristic, replace $\mathcal{R}_X(\mathrm{GSp}(n,F))$ and $\mathcal{R}_n(\mathrm{GO}(X,F))$ by $\mathcal{R}_X(\mathrm{GSp}(n,F))_{\mathrm{temp}}$ and $\mathcal{R}_n(\mathrm{GO}(X,F))_{\mathrm{temp}}$, respectively, in these statements.)*

Proof. Since π and σ are unitary, there exist $\mathbb{C}$ antilinear isomorphisms $\pi \xrightarrow{\sim} \pi^\vee$ and $\sigma \xrightarrow{\sim} \sigma^\vee$ intertwining the actions of $\mathrm{GSp}(n,F)$ and $\mathrm{GO}(X,F)$, respectively. It follows that there is a $\mathbb{C}$ antilinear isomorphism $\pi\otimes\sigma \xrightarrow{\sim} \pi^\vee\otimes\sigma^\vee$ intertwining the action of $\mathrm{GSp}(n,F)\times\mathrm{GO}(X,F)$. Let $\overline{\omega}$ be the representation of R on $\mathcal{S}(X^n)$ defined by $\overline{\omega}(r)\varphi = \overline{\omega(r)\overline{\varphi}}$ for $r \in R$ and $\varphi \in \mathcal{S}(X^n)$. Let $t : \omega \to \pi\otimes\sigma$ be a nonzero R map; then sending φ to $t(\overline{\varphi})$ gives a nonzero $\mathbb{C}$ antilinear R map $\overline{\omega} \to \pi\otimes\sigma$. Composing, we get a nonzero R map $\overline{\omega} \to \pi^\vee\otimes\sigma^\vee$. On the other hand, there is an R isomorphism

$$\overline{\omega} \cong (\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, 1) \cdot \omega.$$

This implies that there is a nonzero R map $\omega \to \pi^\vee \otimes \sigma^\vee$. The remaining claims of the proposition follow. $\square$

Finally, using [H], we consider how the theta correspondence for similitudes treats unramified representations. This requires some definitions. Assume F is nonarchimedean, and let $\mathbb{H}$ be the hyperbolic plane over F. Then $X \cong \mathbb{H} \perp \cdots \perp \mathbb{H} \perp X_0$, where X_0 is an anisotropic quadratic space over F of dimension $0, 2$ or 4. In particular, if $\dim_F X_0 = 2$, then $d \neq 1$ and $X_0 \cong (E, \delta\,\mathrm{N}_F^E)$, for a quadratic extension E/F, where $\delta = 1$ or is a representative for the nontrivial coset of $F^\times/\mathrm{N}_F^E(E^\times)$. We say that X is UNRAMIFIED if either $\dim X_0 = 0$ or $\dim X_0 = 2$, E/F is unramified and $\delta = 1$. If X is unramified, then there exists a lattice $L \subset X$ which is self-dual, and if L' is any other self-dual lattice in X, then there exists $h \in \mathrm{SO}(X,F)$ such that $h(L) = L'$. If $\dim_F X_0 = 0$ or $\dim_F X_0 = 2$ and E/F is unramified, we define a maximal compact subgroup J of $\mathrm{GO}(X,F)$ in the following way. First, if X is unramified, we let J be the stabilizer in $\mathrm{GO}(X,F)$ of a fixed self-dual lattice L, i.e., J is the set of $k \in$

$\mathrm{GO}(X,F)$ such that $k(L)=L$. Next assume $\dim_F X_0 = 2$, E/F is unramified but $\delta \neq 1$. Then there exists a similitude h from X to the unramified quadratic space of the same dimension and discriminant with anisotropic component (E, N_F^E); we let J be the set of $k \in \mathrm{GO}(X,F)$ of the form $h^{-1}k'h$ were k' is in the maximal compact subgroup of $\mathrm{GO}(X',F)$ we have already defined. The definition of J depends on choices, but any two subgroups defined by different choices are conjugate. Let $K = \mathrm{GSp}(n, \mathfrak{O}_F)$.

1.11 Proposition. *Suppose F is nonarchimedean of odd residual characteristic and X is such that J is defined. Let $\sigma \in \mathcal{R}_n(\mathrm{GO}(X,F))$ and $\pi \in \mathcal{R}_X(\mathrm{GSp}(n,F))$ and assume $\mathrm{Hom}_R(\omega, \pi\otimes\sigma) \neq 0$. Then π is unramified with respect to K if and only if σ is unramified with respect to J.*

Proof. By Proposition 1.9 we may assume $\psi(\mathfrak{O}_F)=1$ but $\psi(\pi_F^{-1}\mathfrak{O}_F)\neq 1$; by Lemma 1.6 we may assume X is unramified. We have $K = \mathfrak{O}_F^\times K_1 \cup k_0 \mathfrak{O}_F^\times K_1$ and $J = \mathfrak{O}_F^\times J_1 \cup j_o \mathfrak{O}_F^\times J_1$, where $K_1 = K \cap \mathrm{Sp}(n,F)$, $J_1 = J \cap \mathrm{O}(X,F)$, $\lambda(k_0)=\lambda(j_0)=\mu$, and μ is a representative for the nontrivial coset of $\mathfrak{O}_F^\times/\mathfrak{O}_F^{\times 2}$. For some irreducible component π' of $\pi|_{\mathrm{GSp}(n,F)^+}$, we have $\mathrm{Hom}_R(\omega,\pi'\otimes\sigma)\neq 0$. As $K \subset \mathrm{GSp}(n,F)^+$, it will suffice to show that σ is unramified with respect to J if and only if π' is unramified with respect to K. By the proof of Lemma 4.2 of [R1] we can write

$$\pi'|_{\mathrm{Sp}(n,F)} = \pi_1 \oplus \cdots \oplus \pi_M, \quad \sigma|_{\mathrm{O}(X,F)} = \sigma_1 \oplus \cdots \oplus \sigma_M$$

where the $\pi_i \in \mathrm{Irr}(\mathrm{Sp}(n,F))$ and the $\sigma_i \in \mathrm{Irr}(\mathrm{O}(X,F))$ are mutually nonisomorphic and $\sigma_i = \theta(\pi_i)$. Let V_i and W_i be the spaces of π_i and σ_i, respectively. Assume σ is unramified with respect to J. Let $w_1 \in \sigma$ be nonzero and fixed by J. Since $\sigma|_{\mathrm{O}(X,F)}$ has exactly one irreducible constituent unramified with respect to J_1, we may assume, say, $w_1 \in W_1$. Evidently, $\sigma(j_0)W_1 = W_1$. By (b) of Theorem 7.1 of [H], $\pi_1 = \theta(\sigma_1)$ is unramified with respect to K_1. Let $v_1 \in V_1$ be nonzero and fixed by K_1. We will show that v_1 is in fact fixed by K, i.e., $\pi'(k_0)v_1 = v_1$. As $\pi'|_{\mathrm{Sp}(n,F)}$ has exactly one irreducible constituent unramified with respect to K_1 we have $\pi'(k_0)V_1 = V_1$. Since $V_1^{K_1}$ is one dimensional, $\pi'(k_0)v_1 = \epsilon v_1$ for some $\epsilon \in \{\pm 1\}$. We must show $\epsilon = 1$. Let $T : \omega \to \pi'\otimes\sigma$ be a nonzero R map, and let $p : \pi'\otimes\sigma \to \pi_1\otimes\sigma_1$ be projection. Let $T_1 = p\circ T : \omega \to \pi_1 \otimes \sigma_1$; this is a nonzero $\mathrm{Sp}(n,F)\times \mathrm{O}(X,F)$ map. Let $\varphi \in \omega$ be such that $T_1(\varphi) = v_1 \otimes w_1$; we may assume φ is fixed by $K_1 \times J_1$. By the top of p. 107 of [MVW], there exists a locally constant compactly supported K_1 bi-invariant function $f : \mathrm{Sp}(n,F) \to \mathbb{C}$ such that

$$\varphi = \int_{\mathrm{Sp}(n,F)} f(g)\omega(g,1)\varphi_0 \, dg;$$

here $\varphi_0 \in \omega$ is a certain element fixed by $K_1 \times J_1$. One can check that φ_0 is also fixed by (k_0, j_0). We have $T_1(\omega(k_0,j_0)\varphi) = \epsilon(v_1\otimes w_1)$. However,

$$\omega(k_0,j_0)\varphi = \int_{\mathrm{Sp}(n,F)} f(k_0^{-1}gk_0)\omega(g,1)\varphi_0 \, dg.$$

Let $g \in \mathrm{Sp}(n,F)$. We claim $f(k_0^{-1}gk_0) = f(g)$. Write $g = kak'$ with $k,k' \in K_1$ and a a diagonal matrix. Since we may assume

$$k_0 = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix},$$

we have $k_0^{-1}gk_0 = k_0^{-1}kk_0ak_0^{-1}k'k_0$; hence, $f(k_0^{-1}gk_0) = f(a) = f(g)$. Thus, $\omega(k_0,j_0)\varphi = \varphi$. This implies $v_1 \otimes w_1 = T_1(\varphi) = \epsilon(v_1 \otimes w_1)$, so that $\epsilon = 1$. The implication in the other direction has a similar argument. $\square$

2. FOUR DIMENSIONAL QUADRATIC SPACES

In this section we recall background on four dimensional quadratic spaces X over a base field F and their similitude groups. We begin by characterizing the special similitude group $\mathrm{GSO}(X,F)$ of X via its even Clifford algebra. We also obtain canonical coset representatives for the nontrivial coset of $\mathrm{GSO}(X,F)$ in $\mathrm{GO}(X,F)$; these correspond to quaternion algebras over F contained in the even Clifford algebra over F, which in turn are in bijection with Galois actions on the even Clifford algebra. This leads to the concept of a quadratic quaternion algebra over F, an abstraction of the even Clifford algebra of a four dimensional quadratic space. We construct examples of four dimensional quadratic spaces from a given quadratic quaternion algebra over F and quaternion algebras over F contained in the quadratic quaternion algebra, or equivalently, Galois actions on the quadratic quaternion algebra. We prove that any four dimensional quadratic space over F is, up to similitude, one of these examples. We also describe the relationship between the examples that arise from a given quadratic quaternion algebra. To close the section, we consider four dimensional quadratic spaces over local and number fields. The material in this section is essentially well known. As some basic references we use [E], [Sch] and [Kn].

To begin, let F be a field not of characteristic two, and let $(X,(\cdot,\cdot))$ be a four dimensional quadratic space over F. For simplicity denote the F points of X by X. Set $d = \operatorname{disc} X$. Let x_1,x_2,x_3,x_4 be an orthogonal basis for X. Let C be the Clifford algebra of X, let $B = B(X)$ be the even Clifford algebra of X in C, let $E = E(X)$ be the center of B, and let $C_1 = C_1(X)$ be the subspace of C of odd elements. Then C, B, E and C_1 are 16, 8, 2 and 8 dimensional over F, respectively. The F algebra E is called the DISCRIMINANT ALGEBRA of X and is REDUCED, i.e., has no nonzero nilpotent elements. Hence, E is either a field or is isomorphic to $F \times F$; these happen when $d \neq 1$ and $d = 1$, respectively. Let $\mathrm{Gal}(E/F) = \{1,\alpha\}$. Let N_F^E and T_F^E be the norm and trace from E to F defined by $\mathrm{N}_F^E(z) = z\alpha(z)$ and $\mathrm{T}_F^E(z) = z + \alpha(z)$, respectively. Let $*$ be the involution of C which takes a product of the x_i to the product of the same x_i in the reverse order. Clearly, $*$ preserves B and C_1. If $x \in B$, then $x \in E$ if and only if $x^* = x$. For $x \in C$, define $\mathrm{N}(x) = x^*x$. Then $\mathrm{N}(x) \in E$ for $x \in B$. We may regard X as contained in C_1. Evidently, X is the set of $x \in C_1$ such

that $x^* = x$. For $x \in X$, $(x,x) = \mathrm{N}(x)$. Also, for $z \in E$ and $x \in C_1$ we have $xz = \alpha(z)x$.
To further describe the structure of B and E, suppose B is an arbitrary F algebra with center E and involution $*$ which is the identity on E. Then we say that B is a QUADRATIC QUATERNION ALGEBRA OVER F if E is two dimensional over F and reduced, and there exists a quaternion algebra D over F contained in B such that the natural map $E \otimes_F D \to B$ given by $z \otimes x \mapsto zx$ is an isomorphism of E algebras and $*$ induces the canonical involution on D. Let B be a quadratic quaternion algebra over F with center E and involution $*$. We define the norm $\mathrm{N} : B \to E$ and trace $\mathrm{T} : B \to E$ by $\mathrm{N}(x) = xx^* = x^*x$ and $\mathrm{T}(x) = x + x^*$ respectively. We also define a symmetric E-bilinear form $(\cdot,\cdot) : B \times B \to E$ by $(x,y) = \mathrm{T}(xy^*)/2$. This form is nondegenerate, i.e., if $x \in B$ is nonzero, there exists $y \in B$ such that $(x,y) \neq 0$. The definition of a quadratic quaternion algebra B includes a particular quaternion algebra over F in B, but the next straightforward result shows that all the quaternion algebras over F in B have equal status.

2.1 Proposition. *Let B be a quadratic quaternion algebra over F with center E and involution $*$. Let D be any quaternion algebra over F in B. The natural map $E \otimes_F D \to B$ is an isomorphism of E algebras, and $*$ induces the canonical involution on D.*

Given a quadratic quaternion algebra B as above, in general there may be infinitely many nonisomorphic quaternion algebras D over F in B. However, if $E \cong F \times F$, then $B \cong D \times D$, and any quaternion algebra over F in B is isomorphic to D.

2.2 Proposition. *Let X be a four dimensional quadratic space over X. The F algebra $B(X)$ is a quadratic quaternion algebra over F.*

We characterize $\mathrm{GSO}(X,F)$. Write $B = B(X)$. Define a left action of $F^\times \times B^\times$ on C_1 by $\rho(t,g)x = t^{-1}gxg^*$. This action preserves X, and a computation shows that if $x \in X$ and $(t,g) \in F^\times \times B^\times$, then $\mathrm{N}(\rho(t,g)x) = t^{-2}\,\mathrm{N}_F^E(\mathrm{N}(g))\,\mathrm{N}(x)$; thus, $\rho(t,g) \in \mathrm{GO}(X,F)$, with similitude factor $t^{-2}\,\mathrm{N}_F^E(\mathrm{N}(g))$. In fact, if $(t,g) \in F^\times \times B^\times$, then $\rho(t,g) \in \mathrm{GSO}(X,F)$. For the following see for example V (4.6.1) of [Kn], p. 273.

2.3 Theorem. *Let X be a four dimensional quadratic space over X, and write $B = B(X)$ and $E = E(X)$. Define an inclusion of $E^\times$ into $F^\times \times B^\times$ by $a \mapsto (\mathrm{N}_F^E(a), a)$. Then the following sequence is exact:*

$$1 \to E^\times \to F^\times \times B^\times \xrightarrow{\rho} \mathrm{GSO}(X,F) \to 1.$$

This theorem determines $\mathrm{GSO}(X,F)$. We also need to understand $\mathrm{GO}(X,F)$, and we now explain how to describe certain canonical coset representatives for the nontrivial coset of $\mathrm{GSO}(X,F)$ in $\mathrm{GO}(X,F)$. These coset representatives will correspond to choices of quaternion algebras over F in B. The following lemma is the key structural result for the construction of the coset representatives. It is an elaboration of a general result about Clifford algebras of even dimensional quadratic spaces (Chapter 9, Theorem 2.10 of [Sch], p. 332).

2.4 Lemma. *Let X be a four dimensional quadratic space over X, and write $B = B(X)$ and $E = E(X)$. Let D be a quaternion algebra over F contained in B. Let D' be the F algebra of elements of C which commute with all the elements of D. Then D' is a quaternion algebra over F and $X \cap D'$ is one dimensional and spanned by an anisotropic vector y, so that $D' = E + Ey$. The map $x' \otimes x \mapsto x'x$ determines an isomorphism $D' \otimes_F D \xrightarrow{\sim} C$ of F algebras. Conversely, if $y \in X$ is an anisotropic vector, then the set D of elements of B commuting with y is a quaternion algebra over F in B.*

The maps from the previous lemma are evidently inverses of each other; that is, there is a bijection

$$\text{Quaternion algebras over } F \text{ in } B \longleftrightarrow \text{Anisotropic lines in } X.$$

For the description of the nontrivial coset representatives of $\mathrm{GSO}(X, F)$ in $\mathrm{GO}(X, F)$ we also need the following. Suppose B is any quadratic quaternion algebra over F with center E with $\mathrm{Gal}(E/F) = \{1, \alpha\}$. Then a Galois action on B is an F-automorphism $a : B \to B$ such that $a^2 = 1$ and $a(zx) = \alpha(z)a(x)$ for $z \in E$ and $x \in B$. If a is a Galois action on B, then the fixed points of a are a quaternion algebra over F contained in B; conversely, if D is a quaternion algebra over F contained in B, and $a : B \to B$ is defined by $a(z \otimes x) = \alpha(z) \otimes x$, then a is a Galois action on B. These two maps are inverses of each other, and establish a bijection:

$$\text{Quaternion algebras over } F \text{ in } B \longleftrightarrow \text{Galois actions on } B.$$

Direct computation gives the following:

2.5 Proposition. *Let X be a four dimensional quadratic space over X, and write $B = B(X)$ and $E = E(X)$. Let D be a quaternion algebra over F contained in B, and let D' be as in Lemma 2.4. Let $\#$ be the involution of C obtained via the isomorphism $D' \otimes D \cong C$ from the tensor product of the canonical involutions on D' and D. Then $X^{\#} = X$; define $s : X \to X$ by $s(x) = -x^{\#}$. Then $s \in \mathrm{O}(X, F)$, $s^2 = 1$, and $\det s = -1$. Moreover, the following diagram commutes:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho} & \mathrm{GSO}(X,F) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle 1\times a} & & \Big\downarrow{\scriptstyle \text{conj. by } s} & & \\
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho} & \mathrm{GSO}(X,F) & \longrightarrow & 1
\end{array}.$$

Here, a is the Galois action on B determined by D.

When F is a local field we shall deal with representations of $\mathrm{GSO}(X, F)$ distinguished with respect to subgroups $\mathrm{SO}(Y, F)$, where Y is a three dimensional subspace of X. The above development leads to a compatible characterization of such subgroups. For the exactness of the first sequence in the next result see for example [Kn], p. 264.

2.6 Proposition. *Let X be a four dimensional quadratic space over X, and write $B = B(X)$ and $E = E(X)$. Let $y \in X$ be anisotropic, and set $Y = (F{\cdot}y)^\perp$ in X. Let D be the quaternion algebra over F in B corresponding to y. For $g \in D^\times$ and $x \in Y$, define $\rho(g)x = gxg^{-1}$. Then $\rho(g) \in \mathrm{SO}(Y,F)$ for $g \in D^\times$, the sequence*

$$1 \to F^\times \to D^\times \xrightarrow{\rho} \mathrm{SO}(Y,F) \to 1$$

is exact, there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & D^\times & \xrightarrow{\rho} & \mathrm{SO}(Y,F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho} & \mathrm{GSO}(X,F) & \longrightarrow & 1 \end{array}$$

where the inclusion of $D^\times$ in $F^\times \times B^\times$ is given by $g \mapsto (\mathrm{N}(g), g)$, and $\mathrm{SO}(Y,F)$ is included in $\mathrm{GSO}(X,F)$ by regarding $\mathrm{SO}(Y,F)$ as the stabilizer of y in $\mathrm{GSO}(X,F)$. Moreover, the element s from Proposition 2.5 corresponding to D is such that $s(y) = y$ and s is multiplication by -1 on Y, so that $s|_Y \in \mathrm{O}(Y,F)$, with $\det s|_Y = -1$.

It will be important to have some explicitly constructed four dimensional quadratic spaces, and we now reverse matters and construct such examples from a given quadratic quaternion algebra over F equipped with a Galois action. Let B be a quadratic quaternion algebra over F with center E with $\mathrm{Gal}(E/F) = \{1, \alpha\}$, involution $*$, and let $a : B \to B$ be a Galois action on B. Let D be the quaternion algebra over F in B corresponding to a, i.e., the fixed points of a. We let X_a be the set of $x \in B$ such that $a(x) = x^*$. Then X_a is a four dimensional vector space over F, and equipped with the symmetric bilinear form induced by the norm of B, X_a is a quadratic space over F. Define an explicit action of $F^\times \times B^\times$ on X_a by $\rho_a(t,g)x = t^{-1}gxa(g)^*$. Then $\rho_a(t,g) \in \mathrm{GSO}(X_a,F)$ for $(t,g) \in F^\times \times B^\times$. The relationship between the previous characterization of $\mathrm{GSO}(X_a,F)$ and the homomorphism ρ_a is given by the following proposition.

2.7 Proposition. *Let B be a quadratic quaternion algebra over F with center E with $\mathrm{Gal}(E/F) = \{1, \alpha\}$, involution $*$, and let $a : B \to B$ be a Galois action on B. Then the sequence*

$$1 \to E^\times \to F^\times \times B^\times \xrightarrow{\rho_a} \mathrm{GSO}(X_a,F) \to 1,$$

is exact, where the inclusion of $E^\times$ is defined by $z \mapsto (\mathrm{N}^E_F(z), z)$. There exists a unique F algebra isomorphism $B(X_a) \xrightarrow{\sim} B$ sending $E(X_a)$ onto E so that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E(X_a)^\times & \longrightarrow & F^\times \times B(X_a)^\times & \xrightarrow{\rho} & \mathrm{GSO}(X_a,F) & \to & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \mathrm{id}\downarrow & & \\ 1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho_a} & \mathrm{GSO}(X_a,F) & \to & 1 \end{array}$$

commutes. The map defined by $x \mapsto a(x) = x^*$ maps X_a onto X_a, and is the element $s \in \mathrm{O}(X_a, F)$ from Proposition 2.5 associated to the quaternion algebra over F in $B(X_a)$ corresponding to a.

Explicit quadratic quaternion algebras equipped with Galois actions may constructed as follows. Let E be a two dimensional reduced F algebra, so that E is either a quadratic extension of F, or $E \cong F \times F$. Let $\mathrm{Gal}(E/F) = \{1, \alpha\}$, and let D be a quaternion algebra over F with canonical involution $*$. Set $B_{D,E} = E \otimes_F D$, endow $B_{D,E}$ with the involution defined by $(z \otimes x)^* = z \otimes x^*$, and define $a = a(D,E) : B_{D,E} \to B_{D,E}$ by $a(z \otimes x) = \alpha(z) \otimes x$. Clearly $B_{D,E}$ is a quadratic quaternion algebra over F, and a is a Galois action on $B_{D,E}$; we will write $X_{D,E} = X_a$. To be even more concrete, let $d \in F^\times / F^{\times 2}$. If $d \neq 1$, let $E_d = F(\sqrt{d})$; if $d = 1$, let $E_d = F \times F$. We write $B_{D,d} = B_{D,E_d}$ and $X_{D,d} = X_{D,E_d}$. Evidently $\operatorname{disc} X_{D,d} = d$. Assume further $d = 1$. Then there is a canonical isomorphism $D \times D \xrightarrow{\sim} B_{D,1}$ of F algebras. With respect to this isomorphism, a is given by $a(x, x') = (x', x)$, and $*$ is given by $(x, x')^* = (x^*, x'^*)$. Thus, $X_{D,1}$ is the set of pairs (x, x^*) for $x \in D$, which can be identified with D. With respect to these identifications, $\rho_a(t, (g, g'))x = t^{-1} g x g'^*$ for $t \in F^\times$, $x \in D$, and $g, g' \in D^\times$.
Before turning to specific fields we address two natural questions. First, if X is an arbitrary four dimensional quadratic space over F, when can X be related to an $X_{D,E}$?

2.8 PROPOSITION. *Let X be a four dimensional quadratic space over F and write $B = B(X)$ and $E = E(X)$. There exists a quaternion algebra D over F in B and a similitude $T : X \to X_{D,E}$ so that*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho} & \mathrm{GSO}(X,F) & \longrightarrow & 1 \\
 & & {\scriptstyle \mathrm{id}}\big\downarrow & & \wr\big\downarrow & & \big\downarrow{\scriptstyle T \cdot T^{-1}} & & \\
1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B_{D,E}^\times & \xrightarrow{\rho_{a(D,E)}} & \mathrm{GSO}(X_{D,E},F) & \longrightarrow & 1
\end{array}$$

commutes, and the element $s \in \mathrm{O}(X, F)$ corresponding to D from Proposition 2.5 is mapped to the element of $\mathrm{O}(X_{D,E}, F)$ defined by $x \mapsto a(x) = x^*$, where $a = a(D,E)$. If X represents 1, then we may further choose T to be an isometry. Conversely, if X is isometric to $X_{D,E}$ for some D, then X represents 1.

Given a quadratic quaternion algebra over F, what is the relationship between the X_a for different Galois actions a on the quadratic quaternion algebra? The main ingredient for the following is the Skolem-Noether theorem.

2.9 PROPOSITION. *Let B be a quadratic quaternion algebra over F with center E, let $\mathrm{Gal}(E/F) = \{1, \alpha\}$, and let a and a' be Galois actions on B. There exists $u \in B^\times$, uniquely determined up to multiplication by elements of $E^\times$, such that $a'(x) = u^{-1} a(x) u$ for $x \in B$. We have $u a(u) = u a'(u) \in F^\times$. Let*

$\mu = ua(u) = ua'(u)$. Then u can be chosen so that $\mathrm{N}(u) = \mu$; choose such a u. The map $T : X_a \to X_{a'}$ given by $T(x) = xu$ is a well-defined similitude with similitude factor $\lambda(T) = \mu$. The diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho_a} & \mathrm{GSO}(X_a, F) & \longrightarrow & 1 \\ & & {\scriptstyle\mathrm{id}}\big\downarrow & & {\scriptstyle\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle T\cdot T^{-1}} & & \\ 1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho_{a'}} & \mathrm{GSO}(X_{a'}, F) & \longrightarrow & 1 \end{array}$$

commutes.

To close this section we consider choices of F. Suppose F is nonarchimedean of characteristic zero. Let $d \in F^\times/F^{\times 2}$. Up to isometry, there are two four dimensional quadratic spaces of discriminant d; these are distinguished by their Hasse invariant. Both spaces represent 1. One space is isometric to $X_{\mathrm{M}_{2\times2},d}$, where $\mathrm{M}_{2\times2} = \mathrm{M}_{2\times2}(F)$ is the quaternion algebra of 2×2 matrices over F; the other is isometric to $X_{D_{\mathrm{ram}},d}$, where D_{ram} is the division quaternion algebra over F. These spaces have Hasse invariant $\epsilon(d)$ and $-\epsilon(d)$, respectively, where $\epsilon(d) = (-1,-d)_F$. If $d = 1$, then $X_{\mathrm{M}_{2\times2},1}$ is isometric to $\mathrm{M}_{2\times2}(F)$ equipped with the determinant, and $X_{D_{\mathrm{ram}},1}$ is isometric to D_{ram} equipped with the norm; see the remarks before Proposition 2.8. Suppose $d \neq 1$. Then $X_{\mathrm{M}_{2\times2},d}$ and $X_{D_{\mathrm{ram}},d}$ are both isotropic. Also, $B_{\mathrm{M}_{2\times2},d}$ and $B_{D_{\mathrm{ram}},d}$ are both isomorphic to $\mathrm{M}_{2\times2}(E_d)$. Explicitly, let δ be a representative for the nontrivial coset of $F^\times/\mathrm{N}_F^{E_d}(E_d^\times)$. Then we can take

$$D_{\mathrm{ram}} = \{\begin{bmatrix} e & f\delta \\ \alpha(f) & \alpha(e)\end{bmatrix} : e, f \in E_d\} \subset \mathrm{M}_{2\times2}(E_d).$$

The Galois actions $a = a(\mathrm{M}_{2\times2}, E_d)$ and $a' = a(D_{\mathrm{ram}}, E_d)$ on $\mathrm{M}_{2\times2}(E_d)$ corresponding to $\mathrm{M}_{2\times2}(F)$ and D_{ram} are given by

$$(2.1) \quad a(\begin{bmatrix} e & f \\ g & h\end{bmatrix}) = \begin{bmatrix} \alpha(e) & \alpha(f) \\ \alpha(g) & \alpha(h)\end{bmatrix} \text{ and } a'(\begin{bmatrix} e & f \\ g & h\end{bmatrix}) = \begin{bmatrix} \alpha(h) & \delta\alpha(g) \\ \alpha(f)/\delta & \alpha(e)\end{bmatrix},$$

respectively, and $X_{\mathrm{M}_{2\times2},d}$ and $X_{D_{\mathrm{ram}},d}$ are the set of elements in $\mathrm{M}_{2\times2}(E_d)$

$$\begin{bmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \alpha(e)\end{bmatrix} \text{ and } \begin{bmatrix} f & -\delta e \\ \alpha(e) & g\end{bmatrix},$$

respectively, for $e \in E_d$ and $f, g \in F$. The element u from Proposition 2.9 can be taken to be

$$\sqrt{d}\begin{bmatrix} 0 & \delta \\ 1 & 0\end{bmatrix}.$$

Evidently, if the residual characteristic of F is odd and E_d/F is unramified, then $X_{\mathrm{M}_{2\times2},d}$ is unramified but $X_{D_{\mathrm{ram}},d}$ is not. The quadratic spaces $X_{\mathrm{M}_{2\times2},d}$ and $X_{D_{\mathrm{ram}},d}$ have isomorphic similitude groups, and from the point of view of the theta correspondence for similitudes, they are grouped together. The two quadratic spaces with discriminant 1, however, do not have isomorphic similitude groups and and are distinct from the point of view of the theta correspondence for similitudes. See the remarks before Theorem 1.8 and the proof of Theorem 1.8.

Suppose $F = \mathbb{R}$. Let $d \in \mathbb{R}^\times/\mathbb{R}^{\times 2}$. If $d = 1$, then up to isometry there are three four dimensional quadratic spaces of discriminant 1, with signatures $(4,0)$, $(2,2)$ or $(0,4)$. The quadratic space with signature $(4,0)$ is $X_{D_{\mathrm{ram}},1}$; the ramified quaternion algebra D_{ram} over $\mathbb{R}$ is the Hamilton quaternion algebra. The quadratic space with signature $(2,2)$ is $X_{\mathrm{M}_{2\times 2},1}$ where $\mathrm{M}_{2\times 2} = \mathrm{M}_{2\times 2}(\mathbb{R})$. Finally, the quadratic space with signature $(0,4)$ is not of the form $X_{D,1}$. However, as predicted by Proposition 2.8, there is an intertwining similitude with the space $X_{D_{\mathrm{ram}},1}$: the quadratic space with signature $(0,4)$ can be taken to be $X_{D_{\mathrm{ram}},1}$ with form multiplied by -1. Then the intertwining similitude is just the identity function. If $d = -1$, then up to isometry there are two quadratics spaces of discriminant -1, with signatures $(1,3)$ or $(3,1)$. The quadratic space with signature $(3,1)$ is $X_{\mathrm{M}_{2\times 2},-1}$, while the quadratic space with signature $(1,3)$ is $X_{D_{\mathrm{ram}},-1}$. From the point of view of the theta correspondence for similitudes, the spaces with signature $(4,0)$ and $(0,4)$ are grouped together, the spaces with signature $(3,1)$ and $(1,3)$ are grouped together, and the space of signature $(2,2)$ is not grouped with another four dimensional quadratic space.

When $F = \mathbb{R}$ there are further exact sequences. Let X be a four dimensional quadratic space over $\mathbb{R}$, with even Clifford algebra B; let E be the center of B. We regard $F = \mathbb{R}, E$ and B as the Lie algebras of $F^\times = \mathbb{R}^\times, E^\times$ and $B^\times$, respectively. We take the Lie algebra $\mathfrak{gso}(X,\mathbb{R})$ of $\mathrm{GSO}(X,\mathbb{R})$ to be the subalgebra of $h \in \mathrm{End}_{\mathbb{R}} X$ for which there exists a $\lambda \in \mathbb{R}$ such that $(hx, x') + (x, hx') = \lambda(x,x')$ for $x, x' \in X$; then $\lambda = \mathrm{tr}(h)/2$. Define an action of $\mathbb{R} \times B$ on X by $\rho(r,h)x = -rx + hx + xh^*$, and an inclusion of E into $\mathbb{R} \times B$ by $b \mapsto (\mathrm{T}^E_F(b), b)$. By Theorem 2.3,

$$0 \to E \to \mathbb{R} \times B \xrightarrow{\rho} \mathfrak{gso}(X,\mathbb{R}) \to 0 \tag{2.2}$$

is an exact sequence of Lie algebras. Any two maximal compact subgroups of $\mathrm{GSO}(X,\mathbb{R})$ are conjugate. Let J_0 be a maximal compact subgroup of $\mathrm{GSO}(X,\mathbb{R})$. Then there exists a unique maximal compact subgroup K_B of $B^\times$ such that $\rho(\{\pm 1\} \times K_B) = J_0$. The normalizer of J_0 is $\mathbb{R}^\times J_0$, and J_0 is contained in a unique maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$. There is an exact sequence

$$1 \to K_B \cap E^\times \to \{\pm 1\} \times K_B \xrightarrow{\rho} J_0 \to 1. \tag{2.3}$$

Suppose that $y \in X$ is anisotropic and Y and D are as in Proposition 2.6. We take the Lie algebra of $\mathfrak{so}(Y,\mathbb{R})$ of $\mathrm{SO}(Y,\mathbb{R})$ to be the subalgebra of $h \in \mathrm{End}_{\mathbb{R}} Y$ such that $(hx, x') + (x, hx') = 0$ for $x, x' \in Y$. We regard D as the Lie algebra of $D^\times$, and define an action of D on X by $\rho(h)x = hx - xh$. By Proposition

2.6 there is an exact sequence

$$0 \to \mathbb{R} \to D \to \mathfrak{so}(Y,\mathbb{R}) \to 0,$$

and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & D & \xrightarrow{\rho} & \mathfrak{so}(Y,\mathbb{R}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & \mathbb{R}\times B & \xrightarrow{\rho} & \mathfrak{gso}(X,\mathbb{R}) & \longrightarrow & 0 \end{array}$$

where D is included in $\mathbb{R}\times B$ via $h \mapsto (\mathrm{T}(h), h)$ and $\mathfrak{so}(Y,\mathbb{R})$ is included in $\mathfrak{gso}(X,\mathbb{R})$ by setting the elements of $\mathfrak{so}(Y,\mathbb{R})$ to be 0 on $\mathbb{R}\cdot y$. Any two maximal compact subgroups of $\mathrm{SO}(Y,\mathbb{R})$ are conjugate. Let J_Y be a maximal compact subgroup of $\mathrm{SO}(Y,\mathbb{R})$. Then there exists a unique maximal compact subgroup K_D of $D^\times$ such that $J_Y = \rho(K_D)$, and J_Y is contained in a unique maximal compact subgroup $J_0 = \rho(\{\pm 1\}\times K_B)$ of $\mathrm{GSO}(X,\mathbb{R})$. Also, $K_D \subset K_B$, the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm1\} & \longrightarrow & K_D & \xrightarrow{\rho} & J_Y & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_B\cap E^\times & \longrightarrow & \{\pm1\}\times K_B & \xrightarrow{\rho} & J_0 & \longrightarrow & 1 \end{array}$$

commutes, and the element $s \in \mathrm{O}(X,\mathbb{R})$ from Proposition 2.6 normalizes J_Y and J_0. Conversely, if $J_0 = \rho(\{\pm1\}\times K_B)$ is a maximal compact subgroup of $\mathrm{GSO}(X,\mathbb{R})$ there exists an anisotropic $y \in X$ and a maximal compact subgroup $J_Y = \rho(K_D) \subset \mathrm{SO}(Y,\mathbb{R})$ such that $J_Y \subset J_0$; in particular, the unique maximal compact subgroup of $\mathrm{GSO}(X,\mathbb{R})$ which contains J_0 is generated by J_0 and s. Finally, suppose F is a number field with adeles $\mathbb{A}$, X is a four dimensional quadratic space over F, B is the even Clifford algebra of X, and E is the center of B. Using Theorem 2.3 one can show that the sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho} \mathrm{GSO}(X,\mathbb{A}) \to 1$$

is exact; we identify $E^\times(\mathbb{A})$ and $\mathbb{A}_E^\times$. Similarly, if B is a quadratic quaternion algebra over F with center E, and a is a Galois action on B, then the sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho_a} \mathrm{GSO}(X_a,\mathbb{A}) \to 1$$

is exact. In addition, we have the following useful observation. Suppose D and D' are quaternion algebras over F, and E is a two dimensional reduced algebra over F. Let $S_{D,E}$ be the set of places v of F such that D_v is ramified and v splits in E; if $E \cong F\times F$, we will say that every place of F splits in E. Define $S_{D',E}$ similarly. Evidently, if $S_{D,E} = S_{D',E}$, then $B_{D,E} \cong B_{D',E}$ as E algebras. Thus, if $S_{D,E} = S_{D',E}$, then by Proposition 2.9 there exists an intertwining similitude from $X_{D,E}$ to $X_{D',E}$.

3. Local theta lifts for $\dim X = 2n = 4$

In this section we describe what irreducible representations of $\mathrm{GO}(X,F)$ occur in the theta correspondence with $\mathrm{GSp}(2,F)$ for X a four dimensional quadratic space over a local field F. This is needed to define local L-packets for $\mathrm{GSp}(2,F)$ in the next section. The description below involves distinguished representations, and was given in [R2] when F is a local field of characteristic zero with odd residual characteristic; we also do the even residual characteristic and real cases.

Fix the following notation. Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let X be a four dimensional quadratic space over F; write X for the F points of X. Let $d = \operatorname{disc} X$. As in Section 2, let B be the even Clifford algebra of X, and let E be the center of B. Let $s \in \mathrm{O}(X,F)$ be an element as in Proposition 2.5, so that $s^2 = 1$, $\det s = -1$, and s is a representative for the nontrivial coset of $\mathrm{GSO}(X,F)$ in $\mathrm{GO}(X,F)$. Suppose that $F = \mathbb{R}$. Fix a maximal compact subgroup K_B of $B^\times$, and let $J_0 = \rho(\{\pm 1\} \times K_B)$, a maximal compact subgroup of $\mathrm{GSO}(X,\mathbb{R})$. As explained in the penultimate paragraph of Section 2, we may assume that s normalizes J_0, so that the subgroup J generated by J_0 and s is a maximal compact subgroup of $\mathrm{GO}(X,\mathbb{R})$. As usual, by $\mathrm{Irr}(B^\times)$ we mean the set of equivalence classes of irreducible (B, K_B) modules, where B is regarded as the Lie algebra of $B^\times$. If $\tau \in \mathrm{Irr}(B^\times)$, the central character $\omega_\tau : E^\times \to \mathbb{C}^\times$ of τ is defined by $\omega_\tau(e^z) = \exp(\tau(z))$ for $z \in E \subset B = \mathrm{Lie}(B^\times)$, and $\omega_\tau(\epsilon) = \tau(\epsilon)$ for $\epsilon \in E^\times \cap K_B$.

Using the exact sequences of Section 2, we can describe representations of $\mathrm{GSO}(X,F)$ in terms of representations of $B^\times$. Let $\mathrm{Irr}_f(F^\times \times B^\times)$ be the set of pairs (χ, τ), where $\tau \in \mathrm{Irr}(B^\times)$ is such that ω_τ is Galois invariant, and χ is a quasi-character of $F^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}^E_F$. The exact sequences from Theorem 2.3, (2.2) and (2.3) give a bijection

$$\mathrm{Irr}_f(F^\times \times B^\times) \xrightarrow{\sim} \mathrm{Irr}(\mathrm{GSO}(X,F)), \quad (\chi,\tau) \mapsto \pi(\chi,\tau).$$

If F is nonarchimedean, $\pi(\chi,\tau)$ has the same space as τ, and is defined by $\pi(\chi,\tau)(\rho(t,g)) = \chi(t)^{-1}\tau(g)$. Suppose $F = \mathbb{R}$, and let $(\chi,\tau) \in \mathrm{Irr}_f(\mathbb{R}^\times \times B^\times)$. Since ω_τ is Galois invariant, it follows that there exists a unique $\mathbb{R}$ linear map $l_\tau : \mathbb{R} \to \mathbb{C}$ such that $\tau(z) = l_\tau(\mathrm{T}^E_{\mathbb{R}}(z))$ for $z \in E \subset \mathrm{Lie}(B^\times)$. We have $\chi(e^x) = \exp l_\tau(x)$ for $x \in \mathbb{R}$. Then $\pi(\chi,\tau)$ has the same space as τ, and $\pi(\chi,\tau)$ is defined by $\pi(\chi,\tau)(\rho(\epsilon,k)) = \chi(\epsilon)^{-1}\tau(k)$ for $\rho(\epsilon,k) \in J_0$, and by $\pi(\chi,\tau)(\rho(r,h)) = -l_\tau(r) + \tau(h)$ for $\rho(r,h) \in \mathfrak{gso}(X,\mathbb{R})$. The central character of $\pi(\chi,\tau)$ is χ.

In addition, if X is of the form X_a for some Galois action a on a quadratic quaternion algebra B (see Section 2), then it may be convenient to write $\pi = \pi(\chi,\tau)$ with respect to the first exact sequence from Proposition 2.7. By Proposition 2.7, the difference between using the exact sequences from Proposition 2.7 and Theorem 2.3 is inessential.

We describe representations of $\mathrm{GO}(X,F)$ via representations of $\mathrm{GSO}(X,F)$. Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X,F))$. If the induced representation of π to $\mathrm{GSO}(X,F)$ is irreducible, we say that π is REGULAR, and write π^+ for the induced representation. Here, if $F=\mathbb{R}$, $\mathrm{Ind}_{\mathrm{GSO}(X,\mathbb{R})}^{\mathrm{GO}(X,\mathbb{R})}\pi$ is the $(\mathfrak{go}(X,\mathbb{R}),J)=(\mathfrak{gso}(X,\mathbb{R}),J)$ module with space $\pi\oplus\pi$ and action

$$\pi^+(k)(w\oplus w')=\pi(k)w\oplus\pi(sks)w',\ k\in J_0,\quad \pi^+(s)(w\oplus w')=w'\oplus w,$$

and Lie algebra action

$$\pi^+(X)(w\oplus w')=\pi(X)w\oplus\pi(\mathrm{Ad}(s)X)w',\ X\in\mathfrak{gso}(X,\mathbb{R}).$$

If π is not regular, we say that π is INVARIANT. If π is invariant, then $s\cdot\pi\cong\pi$ and π extends to exactly two representations of $\mathrm{GO}(X,F)$; if $F=\mathbb{R}$, by $s\cdot\pi$ we mean the $(\mathfrak{gso}(X,\mathbb{R}),J_0)$ module with same space as π and action defined by $(s\cdot\pi)(k)w=\pi(sks)w$ for $k\in J_0$ and $w\in\pi$ and $(s\cdot\pi)(X)w=\pi(\mathrm{Ad}(s)X)w$ for $X\in\mathfrak{gso}(X,\mathbb{R})$ and $w\in\pi$. Before we can describe what representations of $\mathrm{GO}(X,F)$ occur in the theta correspondence with $\mathrm{GSp}(2,F)$ we must be able to adequately tell apart the two extensions of an invariant representation to $\mathrm{GO}(X,F)$. To do so we use distinguished representations.

Let $\pi\in\mathrm{Irr}(\mathrm{GSO}(X,F))$ be invariant. We say that π is DISTINGUISHED if there exists an anisotropic vector $y\in X$ such that $\mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi,\mathbf{1})\neq 0$, and if $d\neq 1$, then Y is isotropic. Here, $Y=(F\cdot y)^\perp$, as in Proposition 2.6, and $\mathbf{1}$ is the trivial representation of $\mathrm{SO}(Y,F)$, i.e., the representation with space $\mathbb{C}$ and trivial action. In the case $F=\mathbb{R}$ more comments are required. Let $y\in X$ be anisotropic, and let $Y=(F\cdot y)^\perp$. Let J_Y be a maximal compact subgroup of $\mathrm{SO}(Y,\mathbb{R})$. Then as mentioned in Section 2, J_Y is contained in a unique maximal compact subgroup J_0' of $\mathrm{GSO}(X,\mathbb{R})$. Since J_0' is conjugate to J_0, we may regard the $(\mathfrak{gso}(X,\mathbb{R}),J_0)$ module π as a $(\mathfrak{gso}(X,\mathbb{R}),J_0')$ module, and by restriction, as an $(\mathfrak{so}(Y,\mathbb{R}),J_Y)$ module. Then we say that π is distinguished if for some y, $\mathrm{Hom}_{(\mathfrak{so}(Y,\mathbb{R}),J_Y)}(\pi,\mathbf{1})\neq 0$, and if $d\neq 1$, then Y is isotropic. It is easy to verify that the nonvanishing of this homomorphism space does not depend on the choice of maximal compact subgroup of $\mathrm{SO}(Y,\mathbb{R})$ or element of $\mathrm{GSO}(X,\mathbb{R})$ used to conjugate J_0' into J_0 (use that the normalizer of J_0 is $\mathbb{R}^\times J_0$). Also, π is distinguished with respect to all anisotropic y if and only if it is distinguished with respect to one anisotropic y. If F is nonarchimedean, then this was pointed out in [R2]; if $F=\mathbb{R}$ it follows by a similar argument.

3.1 PROPOSITION. *If $F=\mathbb{R}$ assume $d=1$. Let $\pi\in\mathrm{Irr}(\mathrm{GSO}(X,F))$. Assume π is invariant. Then for all anisotropic $y\in X$ such that $Y=(F\cdot y)^\perp$ is isotropic if $d\neq 1$,* $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi,\mathbf{1})\leq 1$.

Proof. This was proven in Proposition 4.1 of [R2] if F is nonarchimedean. Suppose $F=\mathbb{R}$. Since the homomorphism spaces for different anisotropic y are all isomorphic, it suffices to show this for one y. As $d=1$, we have $B\cong D\times D$ for some quaternion algebra D over $\mathbb{R}$. Identify B with $D\times D$, and let $y \in X$ be an anisotropic vector such that the line $\mathbb{R} \cdot y$ corresponds to ΔD, where ΔD consists of the $(x, x) \in B$ with $x \in D$ (see Section 2). Let K_D be a maximal compact subgroup of $D^\times$; then ΔK_D is a maximal compact subgroup of $\Delta D^\times$, $\Delta K_D \subset K_D \times K_D$, and $J_Y = \rho(K_D) \subset \rho(\{\pm 1\} \times K_D \times K_D)$. Write $\pi = \pi(\chi, \tau)$, with $\tau \cong \tau_1 \otimes \tau_2$, $\tau_1, \tau_2 \in \mathrm{Irr}(D^\times)$, $\omega_{\tau_1} = \omega_{\tau_2} = \chi$. Since π is invariant, $\tau_1 \cong \tau_2$. We have $\mathrm{Hom}_{\mathrm{SO}(Y,\mathbb{R})}(\pi, \mathbf{1}) \cong \mathrm{Hom}_{(\Delta D, \Delta K_D)}(\tau, \chi \circ \mathrm{N}) \cong \mathrm{Hom}_{(\Delta D, \Delta K_D)}(\tau_1 \otimes \tau_1^\vee, \mathbf{1})$. This space is one dimensional. $\square$

As we shall see in the theorem below, we can now tell the two extensions of invariant representations apart to an extent sufficient for our purposes. Suppose $\pi \in \mathrm{Irr}(\mathrm{GSO}(X, F))$ is distinguished with respect to an anisotropic $y \in X$, with $d = 1$ if $F = \mathbb{R}$. Since $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi, \mathbf{1}) = 1$ by Proposition 3.1, it follows that for exactly one extension π^+ of π to $\mathrm{GO}(X, F)$ we have $\mathrm{Hom}_{\mathrm{O}(Y,F)}(\pi^+, \mathbf{1}) \neq 0$. Denote the other extension of π to $\mathrm{GO}(X, F)$ by π^-. The definitions of π^+ and π^- do not depend on the choice of y.
Before characterizing $\mathcal{R}_2(\mathrm{GO}(X, F))$ we require require two more results.

3.2 Lemma. *Let $F = \mathbb{R}$; assume $d = 1$. Then* $\mathrm{Hom}_{\mathrm{SO}(X,\mathbb{R})}(\omega, \pi_1) \neq 0$ *for* $\pi_1 \in \mathrm{Irr}(\mathrm{SO}(X, \mathbb{R}))$.

Proof. If the signature of X is $(2, 2)$, this follows from (3.6.10) of [P]. If the signature of X is $(4, 0)$ or $(0, 4)$ this follows by (6.12) of [KV]. $\square$

3.3 Proposition. *The elements of* $\mathrm{Irr}(\mathrm{GO}(X, F))$ *have multiplicity free restrictions to* $\mathrm{O}(X, F)$.

Proof. If $F = \mathbb{R}$ then the restriction of any element of $\mathrm{Irr}(\mathrm{GO}(X, \mathbb{R}))$ is multiplicity free as $[\mathrm{GO}(X, \mathbb{R}) : \mathbb{R}^\times \mathrm{O}(X, \mathbb{R})] \leq 2$. If $d = 1$ and F is nonarchimedean then this is Lemma 7.2 of [HPS]. The case $d \neq 1$ and F nonarchimedean remains. If F is of odd residual characteristic then $[\mathrm{GO}(X, F) : F^\times \mathrm{O}(X, F)] = [\mathrm{N}_F^E(E^\times) : F^{\times 2}] = 2$ so the proposition follows from Lemma 2.1 of [GK]. We now give an argument for both the even and odd residual characteristic cases. There are two four dimensional quadratic spaces over F of discriminant d. By Proposition 2.9 there is a similitude between them; thus, it suffices to prove the result for one of them. We take $X = X_{\mathrm{M}_{2\times 2}, d}$. Using Proposition 3.2 of [R2] it is easy to verify that the finite dimensional, i.e., one or two dimensional, elements of $\mathrm{Irr}(\mathrm{GO}(X, F))$ have multiplicity free restrictions to $\mathrm{O}(X, F)$. To complete the proof it will suffice to show that for infinite dimensional $\pi \in \mathrm{Irr}(\mathrm{GSO}(X, F))$, the representation $\sigma = \mathrm{Ind}_{\mathrm{GSO}(X,F)}^{\mathrm{GO}(X,F)} \pi$ (which may be reducible) has a multiplicity free restriction to $\mathrm{O}(X, F)$. Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X, F))$ and using the first exact sequence from Proposition 2.7 write $\pi = \pi(\chi, \tau)$ where $\tau \in \mathrm{Irr}(\mathrm{GL}(2, E))$ and χ is a quasi-character of $F^\times$ such that $\chi \circ \mathrm{N}_F^E = \omega_\tau$; here and below $E = E_d$. We make take s to be given by $s(x) = a(x)$, where a is the usual Galois action on $\mathrm{M}_{2\times 2}(E)$ as given in (2.1). Let V be the space of τ, i.e., the space of π. As a model for σ use $V \oplus V$ with

(3.1) $\sigma(h)(v \oplus v') = \pi(h)v \oplus \pi(shs)v'$, $h \in \mathrm{GSO}(X, F)$, $\quad \sigma(s)(v \oplus v') = v' \oplus v$.

We begin with some remarks about the restriction of π to subgroups. Let $\psi_E : E \to \mathbb{C}^\times$ be a nontrivial quasi-character of E; we may assume ψ_E is $\mathrm{Gal}(E/F)$ invariant. Let N be the subgroup of $\mathrm{GSO}(X, F)$ of elements

$$n = \rho_a(1, \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix})$$

for $x \in E$. Since the space of Whittaker functionals on τ with respect to ψ_E is one dimensional, it follows that $\dim \mathrm{Hom}_N(\pi, \psi_E) = 1$, where ψ_E is the character of N defined by $\psi_E(n) = \psi_E(x)$. This fact allows us to prove the following statements just as in the proof of Theorem 4.3 of [R2]. Let H_0 be a closed normal subgroup of $\mathrm{GSO}(X, F)$ such that $F^\times H_0$ is open, $\mathrm{GSO}(X, F)/F^\times H_0$ is finite and Abelian, and $N \subset H_0$. Then the restriction $\pi|_{H_0}$ is multiplicity free: $\pi|_{H_0} = V_1 \oplus \cdots \oplus V_M$, where V_i, $1 \le i \le M$ are mutually nonisomorphic irreducible H_0 subspaces of π (see [GK] for general results about restrictions), and, say, $\dim_{\mathbb{C}} \mathrm{Hom}_N(V_1, \psi_E) = 1$ and $\dim_{\mathbb{C}} \mathrm{Hom}_N(V_i, \psi_E) = 0$ for $2 \le i \le M$. Suppose additionally $s \cdot \pi \cong \pi$, and let $\hat{\pi}$ be an extension of π to $\mathrm{GO}(X, F)$. Then $\hat{\pi}(s)V_1 = V_1$.

Now we show $\sigma|_{\mathrm{O}(X,F)}$ is multiplicity free. Suppose first there is no quasi-character β of $E^\times$ such that $\beta|_{F^\times} = 1$ and $\beta \otimes \tau \cong \tau \circ a$. Let W be a nonzero irreducible $\mathrm{O}(X, F)$ subspace of σ. Then either there is an irreducible $\mathrm{SO}(X, F)$ subspace U of (π, V) such that $W = U \oplus U$, or there is an irreducible $\mathrm{SO}(X, F)$ subspace U of (π, V) and $i : U \to U$ such that $i^2 = 1$, $i(\pi(h)u) = \pi(shs)i(u)$ for $h \in \mathrm{SO}(X, F)$ and $W = \{u \oplus i(u) : u \in U\}$. We assert the second case is impossible; suppose it holds. Then $\pi|_{\mathrm{SO}(X,F)}$ and $(s \cdot \pi)|_{\mathrm{SO}(X,F)}$ share an irreducible component. Since $\pi|_{\mathrm{SO}(X,F)}$ is multiplicity free by the last paragraph, by Lemma 2.4 of [GK] there is a quasi-character $\gamma : \mathrm{GSO}(X, F) \to \mathbb{C}^\times$ trivial on $F^\times \mathrm{SO}(X, F)$ such that $s \cdot \pi \cong \gamma \otimes \pi$. Since

$$1 \to F^\times \mathrm{SO}(X, F) \xrightarrow{\mathrm{inc}} \mathrm{GSO}(X, F) \xrightarrow{\lambda} \mathrm{N}_F^E(E^\times)/F^{\times 2} \to 1$$

is exact, $\gamma = \eta \circ \lambda$ for some quasi-character $\eta : \mathrm{N}_F^E(E^\times) \to \mathbb{C}^\times$ with $\eta^2 = 1$. Let $T : (\eta \circ \lambda) \otimes \pi \to s \cdot \pi$ be a $\mathrm{GSO}(X, F)$ isomorphism. Then for $g \in \mathrm{GL}(2, E)$ and $v \in V$,

$$\begin{aligned} T(\eta(\lambda(\rho_a(1, g))) \cdot \pi(\rho_a(1, g)v)) &= \pi(s\rho_a(1, g)s)T(v) \\ (\eta \circ \mathrm{N}_F^E)(\det g)T(\tau(g)v) &= \tau(a(g))T(v). \end{aligned}$$

This implies $(\eta \circ \mathrm{N}_F^E) \otimes \tau \cong \tau \circ a$, contradicting our assumption; note $(\eta \circ \mathrm{N}_F^E)|_{F^\times} = 1$. Thus, $W = U \oplus U$. Let W' be another nonzero irreducible $\mathrm{O}(X, F)$ subspace of σ and assume $W' \cong W$ as $\mathrm{O}(X, F)$ representations; to show $\sigma|_{\mathrm{O}(X,F)}$ is multiplicity free it will suffice to show W and W' are identical, i.e., $W = W'$. Write $W' = U' \oplus U'$, with U' an irreducible $\mathrm{SO}(X, F)$ subspace of (π, V). Consider the composition $U \to W \xrightarrow{\sim} W' \to U'$ where the first map sends u to $u \oplus 0$, the second is our fixed isomorphism $W \cong W'$, and the last map sends $u \oplus u'$ to u. This is an $\mathrm{SO}(X,F)$ map from $(U, \pi|_{\mathrm{SO}(X,F)})$ to $(U', \pi|_{\mathrm{SO}(X,F)})$. We claim it is nonzero; suppose not. Then the same composition with the last map replaced by the map sending $u \oplus u'$ to u' gives a nonzero $\mathrm{SO}(X,F)$ map from $(U, \pi|_{\mathrm{SO}(X,F)})$ to $(U', (s\cdot\pi)|_{\mathrm{SO}(X,F)})$. However, we just saw that $\pi|_{\mathrm{SO}(X,F)}$ and $(s\cdot\pi)|_{\mathrm{SO}(X,F)}$ have no common irreducible constituents. Thus, the first composition is nonzero, and U and U' are isomorphic irreducible subspaces of $\pi|_{\mathrm{SO}(X,F)}$. Since $\pi|_{\mathrm{SO}(X,F)}$ is multiplicity free, $U = U'$ and so $W = W'$.

Now suppose there is a quasi-character β of $E^\times$ such that $\beta|_{F^\times} = 1$ and $\beta\otimes\tau \cong \tau \circ a$. Let H_0 be the subgroup of $\mathrm{SO}(X,F)$ of $\rho_a(1,g)$ for $g \in \mathrm{Sl}(2,E)$, and let $H \subset \mathrm{O}(X,F)$ be generated by H_0 and s. To prove $\sigma|_{\mathrm{O}(X,F)}$ is multiplicity free it will suffice to prove $\sigma|_H$ is multiplicity free. For this, we replace σ with a more tractable representation via twisting. Since $\beta|_{F^\times} = 1$, there is a quasi-character μ of $E^\times$ such that $\beta(x) = \mu(x/a(x))$ for $x \in E^\times$. Letting $\tau' = \mu \otimes \tau$, we have $\tau' \circ a \cong \tau'$. Since ω_τ is Galois invariant, $\beta^2 = 1$, which implies μ^2 is Galois invariant. Let ν be a quasi-character of $F^\times$ such that $\mu^2 = \nu \circ \mathrm{N}^E_F$, and set $\chi' = \nu\chi$; then $\omega_{\tau'} = \chi' \circ \mathrm{N}^E_F$. Set $\pi' = \pi(\chi', \tau')$. Since $\tau' \circ a \cong \tau'$ we have $s\cdot\pi' \cong \pi'$. Let $\sigma' = \mathrm{Ind}^{\mathrm{GO}(X,F)}_{\mathrm{GSO}(X,F)} \pi'$, and use the same model for σ' as above, so the underlying space of σ' is $V \oplus V$. Now σ' may not be isomorphic to σ, but it is easy to see that the identity map between the models for σ and σ' gives an isomorphism $\sigma|_H \cong \sigma'|_H$. We are reduced to showing $\sigma'|_H$ is multiplicity free. As $s\cdot\pi' \cong \pi'$, we have $\sigma' \cong \pi'_1 \oplus \pi'_2$, where π'_1 and π'_2 are the two extensions of π' to $\mathrm{GO}(X,F)$. Since the restrictions $\pi'_1|_{H_0} = \pi'_2|_{H_0} = \pi|_{H_0}$ are multiplicity free as $N \subset H_0$, it follows that $\pi'_1|_H$ and $\pi'_2|_H$ are multiplicity free. It will now suffice to show $\pi'_1|_H$ and $\pi'_2|_H$ do not share an irreducible component; suppose they do. By Lemma 2.4 of [GK], $\pi'_1|_H \cong \pi'_2|_H$. Let $R : \pi'_1|_H \to \pi'_2|_H$ be an H isomorphism. As indicated above, there is an irreducible H_0 subspace $V_1 \subset V$ such that $\pi'_1(s)V_1 = \pi'_2(s)V_1 = V_1$, i.e., V_1 is also an irreducible H subspace for $\pi'_1|_H$ and $\pi'_2|_H$. Since $\pi'_1|_H$ and $\pi'_2|_H$ are multiplicity free, we must have $R(V_1) = V_1$. Applying Schur's lemma to $R : V_1 \to V_1$, with V_1 regarded as an irreducible H_0 representation, there exists a nonzero scalar c such that $R(v) = cv$ for $v \in V_1$. This implies $\pi'_1(s)v = \pi'_2(s)v$ for $v \in V_1$. However, $\pi'_2(s)v = -\pi'_1(s)v$ for $v \in V$, a contradiction. $\square$

3.4 Theorem. *Let $\sigma \in \mathrm{Irr}(\mathrm{GO}(X,F))$. If F is nonarchimedean and $d \neq 1$, assume σ is infinite dimensional; if $F = \mathbb{R}$, assume $d = 1$. Then $\sigma \in \mathcal{R}_2(\mathrm{GO}(X,F))$ if and only if σ is not of the form π^- for some distinguished $\pi \in \mathrm{Irr}(\mathrm{GSO}(X,F))$.*

Proof. Suppose first F is nonarchimedean. Then this theorem was proven in [R2] in the case F has odd residual characteristic. To verify the theorem if F has even residual characteristic we proceed as follows. We note first that the background results of [R2] are valid in any residual characteristic; that is, the results of sections 2, 3 and 4 hold, and Lemma 6.1, Corollary 6,2, Lemma 6.3 and Lemma 6.4 also hold with the same proofs. We need to show that Lemmas

6.6 and 6.7 of [R2] also hold if F has even residual characteristic. Consider first the proof of Lemma 6.6. The first paragraph of the proof of Lemma 6.6 is independent of the residual characteristic. In the second paragraph, we used a result from [T] proven only in the case of odd residual characteristic; by [Sa], this also holds in the case of even residual characteristic. The third paragraph of the proof is also valid in even residual characteristic (in spite of the unnecessary mention there of odd residual characteristic). Next, we consider all remaining paragraphs but the last paragraph: these cover the case $d = 1$ and $\epsilon = -\epsilon(1)$, in the notation of [R2]. Letting D_{ram} be the ramified quaternion algebra over F, we are given $\tau, \tau' \in \mathrm{Irr}(D^\times_{\mathrm{ram}})$ with $\omega_\tau = \omega_{\tau'}$, and we must show that there exists a quadratic extension $E \subset D_{\mathrm{ram}}$ of F of such that $\mathrm{Hom}_{\mathrm{SO}(Z)}(\pi, \mathbf{1}) \neq 0$, where $\mathrm{SO}(Z)$ is the subgroup $\{\rho(x, x^{*-1}) : x \in E^\times\}$ and $\pi = \pi(\tau, \tau')$. Embed $D^\times_{\mathrm{ram}}$ into $D^\times_{\mathrm{ram}} \times D^\times_{\mathrm{ram}}$ via $x \mapsto (x, x^{*-1})$, and consider the restriction of $\tau \otimes \tau'$ to $D^\times_{\mathrm{ram}}$. Let τ'' be an irreducible component of $(\tau \otimes \tau')|_{D^\times_{\mathrm{ram}}}$; then $\omega_{\tau''} = 1$. By Proposition 18 of [W2], there exists a quadratic extension E of F in D_{ram} and a nonzero vector $v \in \tau''$ such that $\tau''(x)v = v$ for $x \in E^\times$. This implies that $\pi(h)v = v$ for $h \in \mathrm{SO}(Z)$, proving the required claim. The last paragraph of the proof of Lemma 6.6 is also valid in the case of even residual characteristic, thus completing the verification of Lemma 6.6 in this case. Next, we consider the proof of Lemma 6.7 of [R2]. To make the proof of Lemma 6.7 go through in the case of even residual characteristic it suffices to show that if K is a quadratic extension of F, $\tau \in \mathrm{Irr}(\mathrm{GL}(2, K))$ is Galois invariant with $\omega_\tau = \chi \circ \mathrm{N}^K_F$ and $\mathrm{Hom}_{\mathrm{GL}(2,F)}(\tau, \chi \circ \det) = 0$, then there exist quasi-characters ζ and ζ' of $K^\times$ extending χ such that $\epsilon(\tau \otimes \zeta^{-1}, 1/2, \psi_K) = \chi(-1)$ and $\epsilon(\tau \otimes \zeta'^{-1}, 1/2, \psi_K) = -\chi(-1)$. To show the existence of ζ, pick ζ extending χ such that ζ is very ramified (this can be done); then by 3 of Lemma 14 of [HST], $\epsilon(\tau \otimes \zeta^{-1}, 1/2, \psi_K) = \chi(-1)$. On the other hand, since $\mathrm{Hom}_{\mathrm{GL}(2,F)}(\tau, \chi \circ \det) = 0$, by the equivalence of 1 and 2 of Theorem 5.3 of [R2], there exists a quasi-character ζ' of $K^\times$ extending χ such that $\epsilon(\tau \otimes \zeta'^{-1}, 1/2, \psi_K) = -\chi(-1)$. (Note that the proof of the equivalence of 1 and 2 of Theorem 5.3 of [R2] works in any residual characteristic; the use of odd residual characteristic in the proof of Lemma 5.2 is easily seen to be unnecessary.)

Now suppose $F = \mathbb{R}$ and $d = 1$. Suppose $\sigma \in \mathcal{R}_2(\mathrm{GO}(X, \mathbb{R}))$. Then an argument as in Theorem 4.3 of [R2] shows that σ cannot be of the form π^- for some distinguished π. Conversely, suppose σ is not of the form π^- for some distinguished π. Then $\sigma \cong \pi^+$ for some regular π or distinguished π. Using Lemma 3.2, an argument as in Theorem 4.4 of [R2] shows that $\sigma \in \mathcal{R}_2(\mathrm{GO}(X, \mathbb{R}))$. $\square$

4. Definition of the local L-packets and parameters

Let F be a local field of characteristic zero, with $F = \mathbb{R}$ if F is archimedean. Let $d \in F^\times/F^{\times 2}$; assume $d = 1$ if $F = \mathbb{R}$. Let $X_{\mathrm{M}_{2\times 2},d}$ be the four dimensional quadratic space over F defined after Proposition 2.7 and discussed after Proposition 2.9. We will parameterize $\mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ as explained at the beginning of Section 3. However, since we are dealing with the concrete quadratic spaces $X_{\mathrm{M}_{2\times 2},d}$ we will use the first exact sequence from Proposition 2.7; by Proposition 2.7, the difference is trivial. We let $s \in \mathrm{O}(X_{\mathrm{M}_{2\times 2},d}, F)$, $\det s = -1$, be defined by $s(x) = a(x)$, where a is the Galois action on $\mathrm{M}_{2\times 2}(E_d)$ defining $X_{\mathrm{M}_{2\times 2},d}$; see (2.1). Using the results of the last section, we will associate to every element $[\pi]$ of $\langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ a packet $\Pi([\pi])$ of elements of $\mathrm{Irr}(\mathrm{GSp}(2,F))$ and a $\mathrm{GSp}(2)$ L-parameter $\varphi([\pi]) : \mathrm{L}_F \to \mathrm{GSp}(2,\mathbb{C})$ over F, where L_F is the Langlands group of F (i.e., $\mathrm{W}_F \times \mathrm{SU}(2,\mathbb{R})$ if F is nonarchimedean and the Weil group W_F if $F = \mathbb{R}$). We expect that $\Pi([\pi])$ is the L-packet associated to $\varphi([\pi])$ under the conjectural Langlands correspondence. Some evidence is provided by Propositions 4.1, 4.2 and 4.3 below which give some basic properties of the $\Pi([\pi])$ and $\varphi([\pi])$. More work on this issue remains to be done: for example, are the packets $\Pi([\pi])$ disjoint, and if $\varphi([\pi])$ and $\varphi([\pi'])$ are equivalent, does it follow that $\Pi([\pi]) = \Pi([\pi'])$? We will return to this topic in a subsequent work; the thrust of this paper is global results. To define the L-packets, we begin by noting that there is a surjective map

$$\mathrm{Irr}(\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)) \to \langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$$

which sends σ to the components of σ restricted to $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F)$. We will define the L-packet of elements of $\mathrm{Irr}(\mathrm{GSp}(2,F))$ associated to a point of $\langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ by considering the fiber over such a point, and applying the results of Section 3. For $\pi \in \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F)$ denote the element of $\langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ determined by π by $[\pi] = \{\pi, s\cdot\pi\}$. Let $[\pi] \in \langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$. We assume that π is infinite dimensional; if F is nonarchimedean of even residual characteristic, we assume additionally that π is tempered. Then how $[\pi]$ gives rise to irreducible representations of $\mathrm{GSp}(2,F)$ is described in Tables 2 and 3 of the Appendix. In the first step, using the results of Section 3, π gives rise to representations of various orthogonal similitude groups. This is summarized in the tables, but certain aspects deserve comment. If $d \neq 1$, then it may happen that π is invariant but not distinguished. Then the two extensions of π to $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)$ are denoted by π_1 and π_2. When $d = 1$, then π is either regular or invariant and distinguished; in the first case π induces to give π^+, and in the second case π extends to give π^+ and π^-. Additionally, if $d = 1$ and π is essentially square integrable, then π gives an element $\pi^{\mathrm{JL}} \in \mathrm{Irr}(\mathrm{GSO}(X_{D_{\mathrm{ram}},1}, F))$ via the Jacquet-Langlands correspondence, and then analogously elements of $\mathrm{Irr}(\mathrm{GO}(X_{D_{\mathrm{ram}},1}, F))$. Here, D_{ram} is the ramified quaternion algebra over F, and π is ESSENTIALLY SQUARE INTEGRABLE if and only if $\pi = (\alpha \circ \lambda) \otimes \pi'$ for some quasi-character $\alpha : F^\times \to \mathbb{C}^\times$ and square integrable $\pi' \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F))$. To apply the Jacquet-Langlands correspondence, we write as in Section 3, $\pi = \pi(\chi, \tau)$ for $\tau = \tau_1 \otimes \tau_2 \in \mathrm{Irr}(\mathrm{GL}(2,F) \times \mathrm{GL}(2,F))$; recall that the exact sequence from Proposition 2.7 is in this case

$$1 \to F^\times \times F^\times \to F^\times \times \mathrm{GL}(2,F) \times \mathrm{GL}(2,F) \to \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F) \to 1.$$

We define $\pi^{\rm JL} = \pi(\chi, \tau^{\rm JL}) \in \mathrm{Irr}(\mathrm{GSO}(X_{D_{\rm ram},1}, F))$, where $\tau^{\rm JL}$ is the irreducible representation of $D_{\rm ram}^\times \times D_{\rm ram}^\times$ corresponding to τ under the Jacquet-Langlands correspondence (π being essentially square integrable means exactly τ_1 and τ_2 are essentially square integrable); the exact sequence from Proposition 2.7 for this is

$$1 \to F^\times \times F^\times \to F^\times \times D_{\rm ram}^\times \times D_{\rm ram}^\times \to \mathrm{GSO}(X_{D_{\rm ram},1}, F) \to 1.$$

Next, using Theorem 3.4, the thus constructed representations of orthogonal similitude groups give representations of $\mathrm{GSp}(2,F)$ via theta correspondences; note that each theta correspondence used is covered by Theorem 1.8. In Tables 2 and 3 of the Appendix we indicate the appropriate theta correspondences with a subscript. We also indicate when representations do not have theta lifts. Finally, in the Table 4 of the Appendix the packets of representations associated to $[\pi]$ are defined using the representations constructed in Tables 2 and 3 of the Appendix. Note the introduction of the contragredient.
The next proposition describes a few basic properties of the L-packets $\Pi([\pi])$.

4.1 Proposition. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times2},d}, F))$. Assume π is infinite dimensional; if F is nonarchimedean of even residual characteristic, assume π is tempered. Then*

(1) *The common central character of the elements of $\Pi([\pi])$ is ω_π.*
(2) *If $d = 1$ then $|\Pi([\pi])| = 1$ unless π is essentially square integrable; in this case $|\Pi([\pi])| = 2$. If $d \neq 1$, then $|\Pi([\pi])| = 1$ unless π is invariant but not distinguished; in this case $|\Pi([\pi])| = 2$.*
(3) *If π is tempered, then all the elements of $\Pi([\pi])$ are tempered.*

Proof. (1) This follows from the remark on central characters after Theorem 1.8.
(2) Evidently, $|\Pi([\pi])| = 1$ except possibly if $d = 1$ and π is essentially square integrable, or $d \neq 1$ and π is invariant but not distinguished. If $d = 1$ and π is essentially square integrable, then $|\Pi([\pi])| = 2$ by Lemma 8.4 below. If $d \neq 1$ and π is invariant but not distinguished, then $|\Pi([\pi])| = 2$ because $\theta_{\mathrm{M}_{2\times2},d}$ is a bijection.
(3) If F is nonarchimedean, this follows from (1) of Theorem 4.2 of [R3]. If $F = \mathbb{R}$, this follows from IV.3, p. 70 and III.2, p. 49 of [M]. $\square$

Next, we associate to each $[\pi] \in \langle s\rangle\backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times2},d}, F))$ an L-parameter $\varphi([\pi]) : \mathrm{L}_F \to {}^L\mathrm{GSp}(2)$. Here L_F denotes the Langlands group of F, i.e., $\mathrm{L}_F = \mathrm{W}_F \times \mathrm{SU}(2,\mathbb{R})$ if F is nonarchimedean, and $\mathrm{L}_F = \mathrm{W}_F$ if F is archimedean ([Ko], Section 12); W_F is the Weil group of F. As is well known, the dual group $\widehat{\mathrm{GSp}(2)}$ of $\mathrm{GSp}(2,\overline{F})$ (${}^L\mathrm{GSp}(2)^0$ in the notation of [B]) is isomorphic to $\mathrm{GSp}(2,\mathbb{C})$, and we shall use such an isomorphism. But since $\mathrm{GSp}(2,\mathbb{C})$ has a non-inner automorphism, we need to be specific (the same issue arises for other groups, but for, say, GL(2) the choice is established). To do so, we will specify an isomorphism from the based root datum of ${}^L\mathrm{GSp}(2)$

to the based root datum of $\mathrm{GSp}(2,\mathbb{C})$. As a maximal split torus in $\mathrm{GSp}(2,\overline{F})$ we take the group T of elements $t = t(a,b,c) = \mathrm{diag}(a,b,a^{-1}c,b^{-1}c)$. The group X^* of characters of T is the free Abelian group with generators e_1, e_2 and e_3 defined by $e_1(t) = a, e_2(t) = b$ and $e_3(t) = c$. The group X_* of cocharacters of T is the free Abelian group with generators f_1, f_2 and f_3 defined by $f_1(x) = t(x,1,1)$, $f_2(x) = t(1,x,1)$ and $f_3(x) = t(1,1,x)$. The roots of $\mathrm{GSp}(2,\overline{F})$ with respect to T are $\{\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2 - e_3, \alpha_1 + \alpha_2, 2\alpha_1+\alpha_2, -\alpha_1, -\alpha_2, -(\alpha_1+\alpha_2), -(2\alpha_1+\alpha_2)\}$. The coroots are $\{\alpha_1^\vee = \gamma_1 = f_1 - f_2, \alpha_2^\vee = \gamma_2 = f_2, (\alpha_1+\alpha_2)^\vee = \gamma_1 + 2\alpha_2, (2\alpha_1+\alpha_2)^\vee = \gamma_1+\gamma_2, (-\alpha_1)^\vee = -\gamma_1, (-\alpha_2)^\vee = -\gamma_2, (-(\alpha_1+\alpha_2))^\vee = -(\gamma_1+2\gamma_2), (-(2\alpha_1+\alpha_2))^\vee = -(\gamma_1+\gamma_2)\}$. As simple roots we take $\Delta^* = \{\alpha_1, \alpha_2\}$; then $\Delta_* = \Delta^{*\vee} = \{\gamma_1,\gamma_2\}$. We have similar notation for $\mathrm{GSp}(2,\mathbb{C})$, which we will indicate with the addition of a prime. Let $\Psi = (X^*, \Delta^*, X_*, \Delta_*)$; the dual of Ψ is $\Psi^\vee = (X_*, \Delta_*, X^*, \Delta^*)$; let $\Psi' = (X'^*, \Delta'^*, X'_*, \Delta'_*)$. Then an isomorphism $\Psi^\vee \xrightarrow{\sim} \Psi'$ amounts to an isomorphism $f : X_* \xrightarrow{\sim} X'^*$ of Abelian groups such that $f(\Delta_*) = \Delta'^*$ and the matrix of f with respect to our bases is symmetric. One can check that there are exactly two such isomorphisms f, with matrices

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & c \end{bmatrix}, \quad c = 0 \text{ or } 1.$$

As is done implicitly in [HST], we shall fix the isomorphism corresponding to the choice $c = 1$. Our fixed isomorphism of based root data $\Psi^\vee \xrightarrow{\sim} \Psi'$ determines a T conjugacy class of isomorphisms $\widehat{\mathrm{GSp}(2)} \xrightarrow{\sim} \mathrm{GSp}(2,\mathbb{C})$ ([Sp], Theorem 9.6.2); we fix one such isomorphism in the conjugacy class. Additionally, since the action of W_F on $\widehat{\mathrm{GSp}(2)}$ is trivial, ${}^L\mathrm{GSp}(2)$ is the direct product $\widehat{\mathrm{GSp}(2)} \times \mathrm{W}_F$. Thus, in considering L-parameters we may just as well look at maps into $\widehat{\mathrm{GSp}(2)}$, which we identify with $\mathrm{GSp}(2,\mathbb{C})$ (always via our fixed isomorphism). We define a $\mathrm{GSp}(2)$ L-PARAMETER over F to be a continuous homomorphism $\varphi : \mathrm{L}_F \to \mathrm{GSp}(2,\mathbb{C})$ such that $\varphi(x)$ is semisimple for $x \in \mathrm{W}_F$, and if F is nonarchimedean then $\varphi|_{1\times\mathrm{SU}(2,\mathbb{R})}$ is a smooth representation. Let φ be a $\mathrm{GSp}(2)$ L-parameter over F. The SIMILITUDE QUASI-CHARACTER of φ is the quasi-character of L_F given by $\lambda \circ \varphi$, where $\lambda : \mathrm{GSp}(2,\mathbb{C}) \to \mathbb{C}^\times$ is the usual similitude homomorphism. If F is nonarchimedean, we say φ is UNRAMIFIED if $\varphi(\mathrm{SU}(2,\mathbb{R})) = 1$ and φ is trivial on the inertia subgroup of W_F. We say that φ is TEMPERED if $\varphi(\mathrm{L}_F)$ is bounded. If $\varphi' : \mathrm{L}_F \to \mathrm{GSp}(2,\mathbb{C})$ is another $\mathrm{GSp}(2)$ L-parameter over F we say that φ and φ' are EQUIVALENT if there exists $g \in \mathrm{GSp}(2,\mathbb{C})$ such that $g\varphi(x)g^{-1} = \varphi'(x)$ for all $x \in \mathrm{L}_F$. The CONNECTED COMPONENT GROUP of φ is the group $\mathbb{S}(\varphi) = \pi_0(S(\varphi)/\mathbb{C}^\times)$, where $S(\varphi)$ is the group of $g \in \mathrm{GSp}(2,\mathbb{C})$ such that $g\varphi(x) = \varphi(x)g$ for all $x \in \mathrm{L}_F$.

The parameter $\varphi([\pi])$ will be one of two kinds of examples of $\mathrm{GSp}(2)$ L-parameters over F. To define the first kind of example, suppose E/F is a quadratic extension and let $\rho : \mathrm{L}_E \to \mathrm{GL}(2,\mathbb{C})$ be a $\mathrm{GL}(2)$ L-parameter over E such that $\det\rho$ is Galois invariant. Let $\eta : \mathrm{L}_F \to \mathbb{C}^\times$ be a quasi-character extending $\det \rho$; there are two such quasi-characters. Let $V = \mathbb{C}^2$, and regard ρ as a representation on V. Define $\varphi(\eta, \rho) : \mathrm{L}_F \to \mathrm{GSp}(W)$ by letting the space and action of $\varphi(\eta, \rho)$ be $W = \mathrm{Ind}_{\mathrm{L}_E}^{\mathrm{L}_F} \rho$ and defining a nondegenerate symplectic form on W by

$$\langle v_1 \oplus v_2, v_1' \oplus v_2' \rangle = \eta(y)\langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle.$$

Here, y is a fixed representative for the nontrivial coset of L_E in L_F, we identify the space of $\varphi(\eta, \rho)$ with $V \oplus V$ via the map $f \mapsto f(1) \oplus f(y)$, and we have fixed a nondegenerate symplectic form on V (note that up to multiplication by elements of $\mathbb{C}^\times$, there is only one nondegenerate symplectic form on a two dimensional complex vector space). Then $\varphi(\eta, \rho)$ is a GSp(2) L-parameter over F, and the similitude quasi-character of $\varphi(\eta, \rho)$ is $\lambda \circ \varphi(\eta, \rho) = \eta$. Suppose next that $\rho_1 : \mathrm{L}_F \to \mathrm{GL}(2, \mathbb{C})$ and $\rho_2 : \mathrm{L}_F \to \mathrm{GL}(2, \mathbb{C})$ are GL(2) L-parameters over F with $\det \rho_1 = \det \rho_2$. Regard ρ_1 and ρ_2 as two dimensional representations of L_F on $V_1 = \mathbb{C}^2$ and $V_2 = \mathbb{C}^2$, respectively, and fix nondegenerate symplectic forms $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on the spaces of ρ_1 and ρ_2, respectively. Define $\varphi(\rho_1, \rho_2) : \mathrm{L}_F \to \mathrm{GSp}(W)$ by letting the space and action of $\varphi(\rho_1, \rho_2)$ be $W = \rho_1 \oplus \rho_2$ and defining a nondegenerate symplectic form on the space of $\varphi(\rho_1, \rho_2)$ by

$$\langle v_1 \oplus v_2, v_1' \oplus v_2' \rangle = \langle v_1, v_1' \rangle_1 + \langle v_2, v_2' \rangle_2.$$

Then $\varphi(\rho_1, \rho_2)$ is a GSp(2) L-parameter over F, and the similitude quasi-character of $\varphi(\rho_1, \rho_2)$ is $\lambda \circ \varphi(\rho_1, \rho_2) = \det \rho_1 = \det \rho_2$.
Now let $[\pi] \in \langle s \rangle \backslash \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2}, d}, F))$. Write $\pi = \pi(\chi, \tau)$, with $\tau \in \mathrm{Irr}(\mathrm{GL}(2, E_d))$ and χ a quasi-character of $F^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}_F^{E_d}$. Suppose first that $d \neq 1$. Let $\rho : \mathrm{L}_{E_d} \to \mathrm{GL}(2, \mathbb{C})$ be the GL(2) L-parameter over E_d corresponding to τ, and let $\eta : \mathrm{L}_F \to \mathbb{C}^\times$ be the quasi-character of L_F corresponding to χ. Then η extends $\det \rho$ and the equivalence class of $\varphi(\eta, \rho)$ depends only on $[\pi]$ and not the choice of representative π. We set $\varphi([\pi]) = \varphi(\eta, \rho)$. Suppose next $d = 1$. Then $\mathrm{GL}(2, E_d) \cong \mathrm{GL}(2, F) \times \mathrm{GL}(2, F)$. Let $\tau \cong \tau_1 \otimes \tau_2$, with $\tau_1, \tau_2 \in \mathrm{Irr}(\mathrm{GL}(2, F))$ such that $\chi = \omega_{\tau_1} = \omega_{\tau_2}$. Let $\rho_1, \rho_2 : \mathrm{L}_F \to \mathrm{GL}(2, \mathbb{C})$ be the GL(2) L-parameters over F corresponding to ρ_1 and ρ_2, respectively. Then $\det \rho_1 = \det \rho_2$ and the equivalence class of $\varphi(\rho_1, \rho_2)$ depends only on $[\pi]$ and not the choice of representative π. We set $\varphi([\pi]) = \varphi(\rho_1, \rho_2)$.
The following is an analogue of Proposition 4.1.

4.2 Proposition. *Let $\pi \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2}, d}, F))$. Assume π is infinite dimensional; if F is nonarchimedean of even residual characteristic, assume π is tempered. Then*

1. *The similitude quasi-character of $\varphi([\pi])$ corresponds to ω_π.*
2. *If $d = 1$ then $|\mathbb{S}_{\varphi([\pi])}| = 1$ unless π is a essentially square integrable; in this case $\mathbb{S}(\varphi([\pi])) = Z_2$. If $d \neq 1$ and F is not nonarchimedean of even residual characteristic, then $|\mathbb{S}(\varphi([\pi]))| = 1$ unless π is invariant but not distinguished; in this case $\mathbb{S}(\varphi([\pi])) = Z_2$.*
3. *If π is tempered, then $\varphi([\pi])$ is tempered.*

Proof. (1). This follows from the definitions and above remarks.
(2) This follows by a case by case analysis following Tables 2 and 3 of the Appendix. We note in particular that by Theorem 5.3 of [R2] if $d \neq 1$, then $\pi = \pi(\chi, \tau)$ is distinguished if and only if τ is Galois invariant and τ is the base change of a $\tau_0 \in \mathrm{Irr}(\mathrm{GL}(2, F))$ such that $\omega_{\tau_0} = \omega_{E_d/F}\chi$.
(3) Assume π is tempered. Then ρ and η in the case $d \neq 1$, and ρ_1 and ρ_2 in the case $d = 1$, have bounded image. This implies that $\varphi([\pi])$ has bounded image. □

4.3 PROPOSITION. *Suppose F is nonarchimedean, E_d/F is unramified (if $d = 1$ by convention E_d/F is unramified) and $\pi = \pi(\chi, \tau) \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F))$ is infinite dimensional with χ and τ unramified. If the residual characteristic of F is even, assume additionally π is tempered. Then*

(1) *$\varphi([\pi])$ is unramified, $|\Pi([\pi])| = 1$, and if the residual characteristic of F is odd, then the single element Π of $\Pi([\pi])$ is unramified with respect to $\mathrm{GSp}(2, \mathfrak{O}_F)$.*
(2) *([HST]) Let $\Pi([\pi]) = \{\Pi\}$. If π and Π are unitary (e.g., as in global applications, or π tempered), then Π is unramified with respect to $\mathrm{GSp}(2, \mathfrak{O}_F)$ and $\varphi([\pi])$ and Π correspond to the same conjugacy class in $\mathrm{GSp}(2, \mathbb{C})$.*

Proof. (1) Suppose $d = 1$. Evidently, $\varphi([\pi])$ is unramified. Write $\pi = \pi(\chi, \tau)$. As mentioned in Section 3 and the beginning of this section, instead of using the exact sequence of Theorem 2.3, let us use the more convenient sequence of Proposition 2.7, and let s be the representative for the nontrivial coset of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F)$ in $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},1}, F)$ from Proposition 2.7. Also, make the identification of $X_{\mathrm{M}_{2\times 2},1}$ with $\mathrm{M}_{2\times 2}(F)$ equipped with the determinant as remarked before Proposition 2.8. Then s is given by $s(x) = x^*$, with $*$ the canonical involution of matrices, and $\tau = \tau_1 \otimes \tau_2$ with $\tau_1, \tau_2 \in \mathrm{Irr}(\mathrm{GL}(2, F))$ and $\omega_{\tau_1} = \omega_{\tau_2} = \chi$. The lattice $\mathrm{M}_{2\times 2}(\mathfrak{O}_F) \subset X_{\mathrm{M}_{2\times 2},1}$ is self-dual, and the maximal compact subgroups J_0 and J of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F)$ and $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},1}, F)$ which are the stabilizers of $\mathrm{M}_{2\times 2}(\mathfrak{O}_F)$ are $\rho_a(\mathfrak{O}_F^\times \times \mathrm{GL}(2, \mathfrak{O}_F) \times \mathrm{GL}(2, \mathfrak{O}_F))$ and the subgroup generated by $\rho_a(\mathfrak{O}_F^\times \times \mathrm{GL}(2, \mathfrak{O}_F) \times \mathrm{GL}(2, \mathfrak{O}_F))$ and s, respectively. Since π is not essentially square integrable, $\Pi([\pi]) = \{\Pi = \theta_{\mathrm{M}_{2\times 2},1}(\pi^+)^\vee\}$ so that $|\Pi([\pi])| = 1$. By Proposition 1.11 to show Π is unramified it will suffice to show π^+ is unramified. Suppose $\tau_1 \not\cong \tau_2$. Then $\pi^+ = \mathrm{Ind}_{\mathrm{GSO}(X,F)}^{\mathrm{GO}(X,F)} \pi$. Using the model for π^+ as in (3.1), we see that if v is an unramified vector with respect to J_0, then $v \oplus v$ is an unramified vector for π^+. Suppose $\tau_1 \cong \tau_2$, so that π is distinguished and π^+ is the extension to $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},1}, F)$ of π defined in Section 3. Let $\tau = \tau_1$. It will suffice to show $\pi^+ = \pi(\chi, \tau \otimes \tau)^+$ is unramified. Define $T : \pi \to \pi$ by $T(v \otimes w) = w \otimes v$. To show π^+ is unramified it suffices to show $T = \pi^+(s)$. Let $Y = (F \cdot y)^\perp$, where $y \in X_{\mathrm{M}_{2\times 2},1}$ is the 2×2 identity matrix. Then $\mathrm{SO}(Y, F)$, identified as usual with the stabilizer of y in $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F)$, is the group of $\rho_a(1, g, g^{*-1})$ for $g \in \mathrm{GL}(2, F)$. To show $T = \pi^+(s)$ it will suffice to show $T\pi(shs) = \pi(h)T$ for $h \in \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},1}, F)$,

$T^2 = 1$, and $L \circ T = L$ for any nonzero element L of $\mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi, \mathbf{1})$. The first two statements follow from $s\rho_a(t,g,g')s = \rho_a(t,g',g)$ for $g, g' \in \mathrm{GL}(2,F)$. Let V be the space of τ. Fix a $\mathrm{GL}(2,F)$ isomorphism $R : (\omega_\tau^{-1} \otimes \tau, V) \to (\tau^\vee, V^\vee)$. Let $S = 1 \otimes R : V \otimes V \to V \otimes V^\vee$. We have a nonzero $\mathrm{GL}(2,F)$ invariant linear form $V \otimes V^\vee \to \mathbf{1}$ given by $v \otimes f \mapsto f(v)$. The composition of this with S gives us $L \in \mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi, \mathbf{1})$. Now $L \circ T = \epsilon L$ for some $\epsilon = \pm 1$. We must show $\epsilon = 1$. Let $v \in V$ be nonzero and fixed by $\mathrm{GL}(2, \mathfrak{O}_F)$. Then $L(v \otimes v) = L(T(v \otimes v)) = \epsilon L(v \otimes v)$. To see $L(v \otimes v) \neq 0$ and hence $\epsilon = 1$, let $V = \mathbb{C}v_0 \oplus W$ be a $\mathrm{GL}(2, \mathfrak{O}_F)$ decomposition, and define $f \in V^\vee$ by letting f be zero on W and setting $f(v) = 1$. Then $\tau^\vee(k)f = f$ for $k \in \mathrm{GL}(2, \mathfrak{O}_F)$. Evidently, $S(v \otimes v) = c(v \otimes f)$ for some $c \in \mathbb{C}^\times$ so that $L(v \otimes v) = cf(v) = c \neq 0$. Now suppose $d \neq 1$. Again, it is clear that $\varphi([\pi])$ is unramified. Write $\pi = \pi(\chi, \tau)$ again using Proposition 2.7. We will also use the notation after Proposition 2.9 regarding $X_{\mathrm{M}_{2\times 2},d}$. The representative s for the nontrivial coset of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F)$ in $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)$ is given by $s(x) = x^* = a(x)$. The lattice $X_{\mathrm{M}_{2\times 2},d} \cap \mathrm{M}_{2\times 2}(\mathfrak{O}_{E_d})$ is self dual, and the maximal compact subgroups J_0 and J of $\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F)$ and $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)$ which are the stabilizers of $X_{\mathrm{M}_{2\times 2},d} \cap \mathrm{M}_{2\times 2}(\mathfrak{O}_{E_d})$ are $\rho_a(\mathfrak{O}_F^\times \times \mathrm{GL}(2, \mathfrak{O}_{E_d}))$ and the subgroup generated by $\rho_a(\mathfrak{O}_F^\times \times \mathrm{GL}(2, \mathfrak{O}_{E_d}))$ and s, respectively. Since τ is unramified, $\tau \cong \mathrm{Ind}_P^{\mathrm{GL}(2,E_d)}(\mu_1 \otimes \mu_2)$, where B is the usual Borel subgroup of $\mathrm{GL}(2, E_d)$ and μ_1 and μ_2 are unramified and Galois invariant so that τ is Galois invariant. This implies $s \cdot \pi \cong \pi$. In the proof of Theorem 5.3 of [R2] it was shown that π is distinguished and that $L \in \mathrm{Hom}_{\mathrm{SO}(Y,F)}(\pi, \mathbf{1})$ is given by

$$L(f) = \int_{T \backslash \mathrm{GL}(2,F)} f(g_0^{-1} g) \chi(\det g)^{-1}\, dg$$

where T and g_0 are as in [R2]; here $Y = (F \cdot y)^\perp$, where y is the 2×2 identity matrix in $X_{\mathrm{M}_{2\times 2},d}$ and $\mathrm{SO}(Y,F)$ is the group of $\rho_a(\det g, g)$ for $g \in \mathrm{GL}(2,F)$. By definition, we have $\Pi([\pi]) = \{\mathit{\Pi} = \theta_{\mathrm{M}_{2\times 2},d}(\pi^+)^\vee\}$, so that $|\Pi([\pi])| = 1$. Again, by Proposition 1.11 to show that $\mathit{\Pi}$ is unramified with respect to $\mathrm{GSp}(2, \mathfrak{O}_F)$ when has F has odd residual characteristic it will suffice to show that π^+ is unramified. We proceed as in the case $d = 1$. Define $T : \pi \to \pi$ by $T(f) = f \circ a$. As in the $d = 1$ case, it will suffice to show $L \circ T = L$. For $f \in \pi$,

$$\begin{aligned}
(L \circ T)(f) &= \int_{T \backslash \mathrm{GL}(2,F)} f(a(g_0^{-1} g)) \chi(\det g)^{-1}\, dg \\
&= \int_{T \backslash \mathrm{GL}(2,F)} f(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} g_0^{-1} g) \chi(\det g)^{-1}\, dg \\
&= \mu_1(-1) \int_{T \backslash \mathrm{GL}(2,F)} f(g_0^{-1} g) \chi(\det g)^{-1}\, dg \\
&= \mu_1(-1) L(f).
\end{aligned}$$

Since $\mu_1(-1) = 1$, we have $L \circ T = L$, as desired.

(2) This follows from Lemmas 10 and 11 of [HST] after one reconciles the definitions. ([HST] for example uses Whittaker models instead of distinguished representations to define extensions to $\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F)$.) Lemma 1.6 and Proposition 2.9 are also useful for the comparison to [HST]. The reader should be aware that in Lemma 7 of [HST] the Langlands parameter should be $\mathrm{diag}(\chi_3(v), \chi_1(v)\chi_3(v), \chi_1(v)\chi_2(v)\chi_3(v), \chi_2(v)\chi_3(v))$, and in Lemma 10 of [HST] the L-parameter should be $\mathrm{diag}(\sqrt{\alpha}, -\sqrt{\alpha}, \sqrt{\beta}, -\sqrt{\beta})$. $\square$

5. Global theta lifts for similitudes

In this section we review some foundational results on global theta lifts for similitudes ([HK], [HST]). We use the following definitions. Let F be a totally real number field with ring of integers $\mathfrak{O}$, and let X be a even dimensional quadratic space defined over F of positive dimension. For each infinite place v of F fix maximal compact subgroups $J_{1,v}$ and J_v of $\mathrm{O}(X, F_v)$ and $\mathrm{GO}(X, F_v)$, and let $\mathfrak{h}_{1,v}$ and $\mathfrak{h}_v$ be the Lie algebras of $\mathrm{O}(X, F_v)$ and $\mathrm{GO}(X, F_v)$, respectively, as in Section 1. Let $J_{1,\infty}$ and J_∞ be the products of $J_{1,v}$ and J_v, respectively, over the infinite places of F, and let $\mathfrak{h}_{1,\infty}$ and $\mathfrak{h}_\infty$ be the direct sums of the $\mathfrak{h}_{1,v}$ and $\mathfrak{h}_v$, respectively, over the infinite places of v. Let n be a positive integer. For each infinite place v of F let $K_{1,v}$ and K_v be the usual maximal compact subgroups of $\mathrm{Sp}(n, F_v)$ and $\mathrm{GSp}(n, F_v)$, and let $\mathfrak{g}_{1,v}$ and $\mathfrak{g}_v$ be the Lie algebras of $\mathrm{Sp}(n, F_v)$ and $\mathrm{GSp}(n, F_v)$, respectively, as in Section 1. Define $K_{1,\infty}$, K_∞, $\mathfrak{g}_{1,\infty}$ and $\mathfrak{g}_\infty$ as in the case of $\mathrm{O}(X)$ and $\mathrm{GO}(X)$. For v a place of F, define $R(F_v) \subset \mathrm{GSp}(n, F_v) \times \mathrm{GO}(X, F_v)$ as in Section 1. Let $R(F)$ and $R(\mathbb{A})$ be the set of pairs (g, h) in $\mathrm{GSp}(n, F) \times \mathrm{GO}(X, F)$ and $\mathrm{GSp}(n, \mathbb{A}) \times \mathrm{GO}(X, \mathbb{A})$, respectively, such that $\lambda(g) = \lambda(h)$. For v an infinite place of F, let L_v be the maximal compact subgroup of $R(F_v)$ as defined in Section 1, and let $\mathfrak{r}_v$ be Lie algebra of $R(F_v)$. Let L_∞ and $\mathfrak{r}_\infty$ be defined analogously to the last two cases. To define global theta lifts we need a global version of the Weil representation. Fix a nontrivial unitary character ψ of $\mathbb{A}/F$. For v a place of F, let ω_v be the Weil representation of $R(F_v)$ on $L^2(X(F_v)^n)$ defined with respect to ψ_v as in Section 1. Again, if v is a place of F then $\mathcal{S}(X(F_v)^n) \subset L^2(X(F_v)^n)$ is an $R(F_v)$ module if v is finite and an (r_v, L_v) module if v is infinite. Let $x_1, \dots, x_m$ be a vector space basis for $X(F)$ over F. Let $(g, h) \in R(\mathbb{A})$. Then for almost all finite v, $\omega_v(g_v, h_v)$ fixes the characteristic function of $\mathfrak{O}_v x_1 + \cdots + \mathfrak{O}_v x_m$. Let $\otimes_v \mathcal{S}(X(F_v)^n)$ be the algebraic restricted direct product over all the places of F of the complex vector spaces $\mathcal{S}(X(F_v)^n)$ with respect to the characteristic function of $\mathfrak{O}_v x_1 + \cdots + \mathfrak{O}_v x_m$ for v finite. We will denote the restricted algebraic direct product $\otimes_v \mathcal{S}(X(F_v)^n)$ by $\mathcal{S}(X(\mathbb{A})^n)$; then $\mathcal{S}(X(\mathbb{A})^n)$ is an $R(\mathbb{A}_f) \times (\mathfrak{r}_\infty, L_\infty)$ module, where $R(\mathbb{A}_f)$ has the obvious meaning. Let $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$ and $(g, h) \in R(\mathbb{A})$; assume $\varphi = \otimes_v \varphi_v$. The function $\omega(g, h)\varphi : X(\mathbb{A})^n \to \mathbb{C}$ given by $(\omega(g, h)\varphi)(x) = \prod_v (\omega_v(g_v, h_v)\varphi_v)(x_v)$ is well defined (note that for infinite v, $\omega_v(g_v, h_v)\varphi_v$ is a smooth function though it may not be in $\mathcal{S}(X(F_v)^n)$, so that it can be evaluated at a point). Using the universal property of the algebraic restricted direct product, this definition extends to all $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$: if $(g, h) \in R(\mathbb{A})$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$, then $\omega(g, h)\varphi$ may be regarded as a function on $X(\mathbb{A})^n$. In particular, the elements of $\mathcal{S}(X(\mathbb{A})^n)$ may be regarded as functions on $X(\mathbb{A})^n$.

Global theta lifts are now defined as follows. For $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$ and $(g,h) \in R(\mathbb{A})$, set

$$\theta(g,h;\varphi) = \sum_{x \in X(F)^n} \omega(g,h)\varphi(x).$$

This series converges absolutely and is left $R(F)$ invariant. Fix a right $\mathrm{O}(X,\mathbb{A})$ invariant quotient measure on $\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A})$. Let f be a cusp form on $\mathrm{GO}(X,\mathbb{A})$ of central character χ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$. Let $\mathrm{GSp}(n,\mathbb{A})^+$ be the subgroup of $g \in \mathrm{GSp}(n,\mathbb{A})$ such that $\lambda(g) \in \lambda(\mathrm{GO}(X,\mathbb{A}))$. For $g \in \mathrm{GSp}(n,\mathbb{A})^+$ define

$$\theta_n(f,\varphi)(g) = \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A})} \theta(g,h_1h;\varphi)f(h_1h)\,dh_1,$$

where $h \in \mathrm{GO}(X,\mathbb{A})$ is any element such that $(g,h) \in R(\mathbb{A})$. This integral converges absolutely, does not depend on the choice of h, and the function $\theta_n(f,\varphi)$ on $\mathrm{GSp}(n,\mathbb{A})^+$ is left $\mathrm{GSp}(n,F)^+$ invariant. Moreover, $\theta_n(f,\varphi)$ extends uniquely to a $\mathrm{GSp}(n,F)$ left invariant function on $\mathrm{GSp}(n,\mathbb{A})$ with support in $\mathrm{GSp}(n,F)\,\mathrm{GSp}(n,\mathbb{A})^+$. This extended function, also denoted by $\theta_n(f,\varphi)$, is an automorphic form on $\mathrm{GSp}(n,\mathbb{A})$ of central character $\chi\chi_X^n = \chi(\cdot,\operatorname{disc} X(F))_F^n$. If V is a $\mathrm{GO}(X,\mathbb{A}_f) \times (h_\infty, J_\infty)$ subspace of the space of cusp forms on $\mathrm{GO}(X,\mathbb{A})$ of central character χ, then we denote by $\Theta_n(V)$ the $\mathrm{GSp}(n,\mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of the space of automorphic forms on $\mathrm{GSp}(n,\mathbb{A})$ of central character $\chi\chi_X^n$ generated by all the $\theta_n(f,\varphi)$ for $f \in V$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$. Similarly, fix a right $\mathrm{Sp}(n,\mathbb{A})$ invariant quotient measure on $\mathrm{Sp}(n,F)\backslash \mathrm{Sp}(n,\mathbb{A})$, let F be a cusp form on $\mathrm{GSp}(n,\mathbb{A})$ of central character χ' and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$. For $h \in \mathrm{GO}(X,\mathbb{A})$ define

$$\theta_X(F,\varphi)(h) = \int_{\mathrm{Sp}(n,F)\backslash \mathrm{Sp}(n,\mathbb{A})} \theta(g_1g,h;\varphi)F(g_1g)\,dg$$

where $g \in \mathrm{GSp}(n,\mathbb{A})$ is any element such that $(g,h) \in R(\mathbb{A})$. Again, this integral converges absolutely, does not depend on the choice of g, and the function $\theta_X(F,\varphi)$ is an automorphic form on $\mathrm{GO}(X,\mathbb{A})$ of central character $\chi'\chi_X^n$. If W is a $\mathrm{GSp}(n,\mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of the space of cusp forms on $\mathrm{GSp}(n,\mathbb{A})$ of central character χ', then we denote by $\Theta_X(W)$ the $\mathrm{GO}(X,\mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$ subspace of the space of automorphic forms on $\mathrm{GO}(X,\mathbb{A})$ of central character $\chi'\chi_X^n$ consisting of the $\theta_X(F,\varphi)$ for $F \in W$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$. We shall also occasionally consider global theta lifts of $\mathrm{O}(X,\mathbb{A}_f) \times (\mathfrak{h}_{1,\infty}, J_{1,\infty})$ subspaces of the space of cusp forms on $\mathrm{O}(X,\mathbb{A})$ and of $\mathrm{Sp}(n,\mathbb{A}_f) \times (\mathfrak{g}_{1,\infty}, K_{1,\infty})$ subspaces of the space of cusp forms on $\mathrm{Sp}(n,\mathbb{A})$. These have the obvious analogous definitions.

We will need to know how $\Theta_n(V)$ and $\Theta_X(W)$ behave if X is changed by a similitude. Let X' be another quadratic space over F, and suppose there is a similitude $t : X(F) \to X'(F)$ with similitude factor λ. Then for each place v of F, there is an isomorphism

$$\mathrm{GO}(X, F_v) \xrightarrow{\sim} \mathrm{GO}(X', F_v)$$

sending h to tht^{-1}. For each infinite v, let $J'_{1,v}$ and J'_v be the maximal compact subgroups of $\mathrm{O}(X', F_v)$ and $\mathrm{GO}(X', F_v)$ which are the images under the above isomorphism of $J_{1,v}$ and J_v, respectively. If v is infinite, then t also determines an isomorphism

$$\mathfrak{go}(X, F_v) \xrightarrow{\sim} \mathfrak{go}(X', F_v)$$

given by $h \mapsto tht^{-1}$. Via these two isomorphisms and definitions, for each v we obtain a bijection

$$\mathrm{Irr}(\mathrm{GO}(X, F_v)) \xrightarrow{\sim} \mathrm{Irr}(\mathrm{GO}(X', F_v)),$$

and thus a bijection

$$\mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GO}(X, \mathbb{A})) \xrightarrow{\sim} \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GO}(X', \mathbb{A})).$$

If f is an automorphic form on $\mathrm{GO}(X, \mathbb{A})$, then $tf : \mathrm{GO}(X', \mathbb{A}) \to \mathbb{C}$ defined by $(tf)(h) = f(t^{-1}ht)$ is an automorphic form on $\mathrm{GO}(X', \mathbb{A})$. Under this map, cusp forms are mapped to cusp forms. Let the right $\mathrm{O}(X', \mathbb{A})$ invariant quotient measure on $\mathrm{O}(X', F)\backslash \mathrm{O}(X', \mathbb{A})$ be obtained from the fixed right $\mathrm{O}(X, \mathbb{A})$ invariant quotient measure on $\mathrm{O}(X, F)\backslash \mathrm{O}(X, \mathbb{A})$ via the isomorphism $h \mapsto tht^{-1}$.

5.1 LEMMA. *Let V be a $\mathrm{GO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_\infty)$ subspace of the space of cusp forms on $\mathrm{GO}(X, \mathbb{A})$ of central character χ, and let W be a $\mathrm{GSp}(n, \mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of the space of cusp forms on $\mathrm{GSp}(n, \mathbb{A})$ of central character χ'. Then $\Theta_n(V) = \Theta_n(tV)$ and $t\Theta_X(W) = \Theta_{X'}(W)$. Moreover, $\Theta_n(V)$ and $\Theta_X(W)$ do not depend on the choice of nontrivial unitary character ψ of $\mathbb{A}/F$.*

Proof. To show $\Theta_n(V) \subset \Theta_n(tV)$ it will suffice to show that if $f \in V$ and $\varphi = \otimes_v \varphi_v$, then $\theta_n^X(f, \varphi) \in \Theta_n(tV)$; here and below the superscript X will indicate the dependence on X. By Lemma 1.6, if $(g, h) \in R_{X,n}(\mathbb{A})$ then

$$\theta_n^X(g, h; \varphi) = \theta_n^{X'}(g^{[\lambda]}, tht^{-1}; \varphi \circ t^{-1}).$$

Let $g = g_0 g_1 \in \mathrm{GSp}(n, F)\,\mathrm{GSp}(n, \mathbb{A})^+$ with $g_0 \in \mathrm{GSp}(n, F)$ and $g_1 \in \mathrm{GSp}(n, \mathbb{A})^+$. A computation shows that

$$\theta_n^X(f, \varphi)(g) = \theta_n^X(f, \varphi)(g_1) = \theta_n^{X'}(tf, \varphi \circ t^{-1})(g_1^{[\lambda]}).$$

Write

$$g_1^{[\lambda]} = g' \begin{bmatrix} 1 & 0 \\ 0 & |\lambda|_\infty^{-1} \end{bmatrix}.$$

Here $|\lambda|_\infty$ is the element of $\mathbb{A}^\times$ which is 1 at the finite places and $|\lambda|_v$ at the infinite place v. Then $g' \in \mathrm{GSp}(n,\mathbb{A})^+$. Let $h' \in \mathrm{GO}(X,\mathbb{A})$ be such that $\lambda(h') = \lambda(g')$. We have

$$\begin{aligned}
\theta_n^{X'}(tf,\varphi\circ t^{-1})(g_1^{[\lambda]}) &= \theta_n^{X'}(tf,\varphi\circ t^{-1})(g' \begin{bmatrix} 1 & 0 \\ 0 & |\lambda|_\infty^{-1} \end{bmatrix}) \\
&= \int_{\mathrm{O}(X',F)\backslash \mathrm{O}(X',\mathbb{A})} \theta_n^{X'}(g' \begin{bmatrix} 1 & 0 \\ 0 & |\lambda|_\infty^{-1} \end{bmatrix}, h_1 h' \sqrt{|\lambda|_\infty}^{-1}; \varphi\circ t^{-1}) \\
&\qquad\qquad \cdot (tf)(h_1 h' \sqrt{|\lambda|_\infty}^{-1})\, dh_1 \\
&= \chi(\sqrt{|\lambda|_\infty}^{-1})\theta_{X'}(tf,\varphi')(g'),
\end{aligned}$$

where

$$\varphi' = \prod_{v \text{ inf.}} |\lambda|_v^{n \dim X/4} \cdot (\varphi_f \circ t^{-1}) \otimes (\varphi_\infty \circ \sqrt{|\lambda|_\infty} t^{-1}).$$

Then $\varphi' \in \mathcal{S}(X'(\mathbb{A})^n)$, and

$$\begin{aligned}
\theta_n^X(f,\varphi)(g) &= \chi(\sqrt{|\lambda|_\infty}^{-1})\theta_n^{X'}(tf,\varphi')(g') \\
&= \chi(\sqrt{|\lambda|_\infty}^{-1})[\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}_f^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{sign}(\lambda)_\infty \end{bmatrix}^{-1} \theta_n^{X'}(tf,\varphi')](g).
\end{aligned}$$

Here, $\mathrm{sign}(\lambda)_\infty$ is the element of $\mathbb{A}^\times$ which is 1 at the finite places and at the infinite place v is the sign of λ in F_v. If $g \notin \mathrm{GSp}(n,F)\,\mathrm{GSp}(n,\mathbb{A})^+$, then also

$$g \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}_f^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{sign}(\lambda)_\infty \end{bmatrix}^{-1} \notin \mathrm{GSp}(n,F)\,\mathrm{GSp}(n,\mathbb{A})^+,$$

so that both sides of the last equality are by definition zero, and hence equal. Since

$$\begin{bmatrix} 1 & 0 \\ 0 & \mathrm{sign}(\lambda)_\infty \end{bmatrix}^{-1} \in K_\infty,$$

it now follows that $\theta_n^X(f,\varphi) \in \Theta_n(tV)$, so that $\Theta_n(V) \subset \Theta_n(tV)$. Similarly, $\Theta_n(tV) \subset \Theta_n(V)$. The proof of $t\Theta_X(W) = \Theta_{X'}(W)$ and the independence of ψ are analogous. □

The next two results are due to Rallis [Ra] in the case of isometries. The first describes when a theta lift is cuspidal. The second result gives the structure of a theta lift of a space of cusp forms in the case the theta lift is cuspidal. The proofs are similar to or use the proofs in [Ra]. Section 1 is also a basic input for the proof of Proposition 5.3.

5.2 Proposition (Rallis). *Let $n \geq 1$ be an integer. Let f be a cusp form on* $\mathrm{GO}(X,\mathbb{A})$. *Suppose that $\theta_k(f,\varphi)=0$ for all $0 \leq k \leq n-1$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^k)$. Then $\theta_n(f,\varphi)$ is cuspidal (though possibly zero) for all $\varphi \in \mathcal{S}(X(\mathbb{A})^n)$.*

Here, $\theta_0(f,\varphi)=0$ is taken to mean

$$0 = \int_{\mathrm{O}(X,F)\backslash \mathrm{O}(X,\mathbb{A})} f(h_1 h)\, dh_1 \tag{5.1}$$

for all $h \in \mathrm{GO}(X,\mathbb{A})$.

An analogous result holds for lifts from $\mathrm{GSp}(n)$ to $\mathrm{GO}(X)$. In this case, fix an even dimensional quadratic space X over F such that $X(F)$ is anisotropic. For an integer $k \geq 0$, let X_k be the orthogonal direct sum of X with k copies of the hyperbolic plane over F. Let f be a cusp form on $\mathrm{GSp}(n,\mathbb{A})$. Let $l \geq 0$ be an integer. If $l=0$ and $\dim X = 0$, so that $X_l = 0$, then $\theta_{X_l}(f,\varphi)$ is not defined; if $l=0$ and $\dim X > 0$ so that $X_l = X$, then $\theta_{X_l}(f,\varphi)$ is cuspidal for all $\varphi \in \mathcal{S}(X_l(\mathbb{A})^n)$ since the cuspidal condition is vacuous. Suppose $l \geq 1$. Suppose $\theta_{X_k}(f,\varphi)=0$ for all $0 \leq k \leq l-1$ and $\varphi \in \mathcal{S}(X_k(\mathbb{A})^n)$; then $\theta_{X_l}(f,\varphi)$ is cuspidal (though possibly zero) for all $\varphi \in \mathcal{S}(X_l(\mathbb{A})^n)$. Here, if $\dim X = 0$ and $k=0$ then the condition $\theta_{X_k}(f,\varphi)=0$ is taken to be empty.

5.3 Proposition (Rallis; Multiplicity preservation). *Let $2n = \dim X$. Let V be a $\mathrm{GO}(X,\mathbb{A}_f)\times(\mathfrak{h}_\infty,J_\infty)$ nonzero subspace of the space of cusp forms on $\mathrm{GO}(X,\mathbb{A})$ of central character χ. Assume that for each place v of F, $X(F_v)$ satisfies one of the conditions of (1)-(6) of Theorem 1.8. Assume that*

$$V = V_1 \oplus \cdots \oplus V_M,$$

where each V_i, $1 \leq i \leq M$, is a $\mathrm{GO}(X,\mathbb{A}_f)\times(\mathfrak{h}_\infty,J_\infty)$ subspace of V, and all the V_i are isomorphic to a single nonzero irreducible $\mathrm{GO}(X,\mathbb{A}_f)\times(\mathfrak{h}_\infty,J_\infty)$ representation σ. Let $\sigma \cong \otimes_v \sigma_v$, assume $\sigma_v|_{\mathrm{O}(X,F)}$ is multiplicity free for all v, and σ_v is tempered for $v \mid 2$. Suppose that $\Theta_n(V)$ is contained in the space of cusp forms on $\mathrm{GSp}(n,\mathbb{A})$ (necessarily of central character $\chi\chi_X^n = \chi(\cdot,\operatorname{disc} X(F))^n$), and that for any irreducible nonzero $\mathrm{GO}(X,\mathbb{A}_f)\times(\mathfrak{h}_\infty,J_\infty)$ subspace U of V we have $\Theta_n(U)\neq 0$. Then $\sigma_v \in \mathcal{R}_n(\mathrm{GO}(X,F_v))$ for all v,

$$\Theta_n(V) = \Theta_n(V_1)\oplus\cdots\oplus\Theta_n(V_M),$$

and each $\Theta_n(V_i)$, $1 \leq i \leq M$, is isomorphic to $\Pi = \otimes_v \theta(\sigma_v^\vee)$. An analogous result holds if the roles of $\mathrm{GSp}(n)$ and $\mathrm{GO}(X)$ are interchanged.

6. Tempered cuspidal automorphic representations of $B(\mathbb{A})^\times$ and $\mathrm{GSO}(X,\mathbb{A})$

Let F be a totally real number field, and let X be a four dimensional quadratic space over F. As in Section 2, let B be the even Clifford algebra of $X(F)$, and let E be the center of B. Let $d = \operatorname{disc} X(F)$. In this section we describe the relationship between tempered cuspidal automorphic representations of $B^\times(\mathbb{A})$ and $\mathrm{GSO}(X,\mathbb{A})$. From Section 2, we have exact sequences

$$1 \to E^\times \to F^\times \times B^\times(F) \xrightarrow{\rho} \mathrm{GSO}(X,F) \to 1$$

and

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho} \mathrm{GSO}(X,\mathbb{A}) \to 1.$$

6.1 Lemma. *There exist $s \in \mathrm{O}(X,F)$, and for each infinite v, a maximal compact subgroup $J_{0,v}$ of $\mathrm{GSO}(X,F_v)$, such that $\det s = -1$, $s^2 = 1$, and $sJ_{0,v}s = J_{0,v}$ for all infinite v.*

Proof. Let $y \in X(F)$ be anisotropic, and let $Y \subset X$ be the three dimensional quadratic space over F such that $Y(F) = (F\cdot y)^\perp$. Let $s \in \mathrm{O}(X,F)$ be defined with respect to y as in Propositions 2.5 and 2.6. Then $\det s = -1$ and $s^2 = 1$. For each infinite v, choose a maximal compact subgroup $J_{Y,v}$ of $\mathrm{SO}(Y,F_v)$, and let $J_{0,v}$ be the unique maximal compact subgroup of $\mathrm{GSO}(X,F_v)$ containing $J_{Y,v}$ mentioned in the penultimate paragraph of Section 2. Then s normalizes $J_{Y,v}$ and $J_{0,v}$ for each infinite v. $\square$

For the remainder of this paper we fix the following choices of compact subgroups. Let s and the maximal compact subgroups $J_{0,v}$ of $\mathrm{GSO}(X,F_v)$ be as in Lemma 6.1. For each infinite place v of F, let $K_{B,v}$ be the unique maximal compact subgroup of $B^\times(F_v)$ such that $\rho(\{\pm1\}\times K_{B,v}) = J_{0,v}$. Let $J_{0,\infty}$ be the product of the $J_{0,v}$ over the infinite places v of F, and let $\mathfrak{h}_\infty$ be the direct sum of the $\mathfrak{h}_v = \mathfrak{gso}(X,F_v) = \mathfrak{go}(X,F_v)$ over the infinite places v of F. Let $K_{B,\infty}$ be the product of the $K_{B,v}$ over the infinite places v of F and let B_∞ be the direct sum over the infinite places v of the Lie algebra $B(F_v)$ of $B^\times(F_v)$. We consider $B^\times(\mathbb{A}_f)\times(B_\infty,K_{B,\infty})$ and $\mathrm{GSO}(X,\mathbb{A}_f)\times(\mathfrak{h}_\infty,J_{0,\infty})$ modules.

We will use the following facts about the tempered cuspidal automorphic representations of $B^\times(\mathbb{A})$. Let $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ be the set of tempered cuspidal automorphic representations τ of $B^\times(\mathbb{A})$. It is well known that $B^\times(\mathbb{A})$ has the multiplicity one property, i.e., the elements of $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ of a fixed central character occur with multiplicity one in the space of cusp forms on $B^\times(\mathbb{A})$ of that central character. If $\tau \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B(\mathbb{A})^\times)$, then the unique space of cusp forms on $B^\times(\mathbb{A})$ isomorphic to τ will be denoted by V_τ. Also, $B^\times(\mathbb{A})$ has the strong multiplicity one property: if $\tau,\tau' \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ share the same central character and $\tau_v \cong \tau'_v$ for all but finitely many v, then $\tau \cong \tau'$, so that $V_\tau = V_{\tau'}$. In addition, the Jacquet-Langlands correspondence gives an injection of $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ into $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GL}(2,\mathbb{A}_E))$. This map is constructed as follows. Suppose E is a field. Since B has center E, we may regard B as an algebra over E, and by Section 2, B is a quaternion algebra over E. There is a canonical isomorphism $B^\times(\mathbb{A}) \cong B^\times(\mathbb{A}_E)$, and thus a bijection $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A})) \xrightarrow{\sim} \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}_E))$. Composing with the Jacquet-Langlands map from $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}_E))$ to $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GL}(2,\mathbb{A}_E))$, we get an injection $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A})) \hookrightarrow \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GL}(2,\mathbb{A}_E))$ which we also call the

Jacquet-Langlands correspondence, and denote by $\tau \mapsto \tau^{\mathrm{JL}}$. If $E \cong F \times F$, we get a similar injection, with $\mathrm{GL}(2, \mathbb{A}_E)$ taken to be $\mathrm{GL}(2, \mathbb{A}) \times \mathrm{GL}(2, \mathbb{A})$.
Tempered cuspidal automorphic representations of $\mathrm{GSO}(X, \mathbb{A})$ and $B^\times(\mathbb{A})$ may be related as in the local case. Let $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp,f}}(\mathbb{A}^\times \times B^\times(\mathbb{A}))$ be the set of pairs (χ, τ), where $\tau \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ and χ is a Hecke character of $\mathbb{A}^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}^E_F$. Let $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$ denote the set of tempered cuspidal automorphic representations of $\mathrm{GSO}(X, \mathbb{A})$. The above exact sequences give a bijection

$$\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp,f}}(\mathbb{A}^\times \times B^\times(\mathbb{A})) \xrightarrow{\sim} \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A})).$$

If $(\chi, \tau) \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp,f}}(\mathbb{A}^\times \times B^\times(\mathbb{A}))$, then $\pi(\chi, \tau) \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$ corresponding to (χ, τ) consists of the space of functions $F : \mathrm{GSO}(X, \mathbb{A}) \to \mathbb{C}$ for which there exists $f \in \tau$ so that $F(\rho(t, g)) = \chi(t)^{-1} f(g)$. The central character of $\pi(\chi, \tau)$ is χ. If $d = 1$, so that $E \cong F \times F$ and $B^\times(\mathbb{A}) \cong D^\times(\mathbb{A}) \times D^\times(\mathbb{A})$ (see Section 2), then every element $\tau \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ is of the form $\tau_1 \otimes \tau_2$ for some $\tau_1, \tau_2 \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(D^\times(\mathbb{A}))$, and the condition that ω_τ factors through N^E_F amounts to $\omega_{\tau_1} = \omega_{\tau_2}$. In this case ω_τ factors uniquely through N^E_F via $\chi = \omega_{\tau_1} = \omega_{\tau_2}$. Also, when dealing with a four dimensional quadratic space X_a over F defined by a Galois action a on a given quadratic quaternion algebra B over F with center E (Section 2), we will occasionally parameterize $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_a, \mathbb{A}))$ with respect to the explicit exact sequence

$$1 \to \mathbb{A}^\times_E \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho_a} \mathrm{GSO}(X_a, \mathbb{A}) \to 1$$

derived from Proposition 2.7; by that proposition, the difference between the two parameterizations is insignificant.
Tempered cuspidal automorphic representations of $\mathrm{GSO}(X, \mathbb{A})$ inherit similar properties from those of $B^\times(\mathbb{A})$. The elements of $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$ have the multiplicity one property and the strong multiplicity one property. If $\pi \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$ then the unique space of cusp forms on $\mathrm{GSO}(X, \mathbb{A})$ isomorphic to π will be denoted by V_π. If $\pi \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GSO}(X, \mathbb{A}))$, then we denote by $s \cdot \pi$ the $\mathrm{GSO}(X, \mathbb{A}_f) \times (\mathfrak{h}_\infty, J_{0,\infty})$ module with the same space as π, but with twisted action $(s \cdot \pi)(h) = \pi(shs)$ for $h \in \mathrm{GSO}(X, \mathbb{A}_f) \times J_{0,\infty}$ and $(s \cdot \pi)(x) = \pi(\mathrm{Ad}(s)x)$ for $x \in \mathfrak{h}_\infty$. Let $\pi \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X, \mathbb{A}))$. Then we denote by sV_π the space of cusp forms sf on $\mathrm{GSO}(X, \mathbb{A})$ defined by $(sf)(h) = f(shs)$ for $h \in \mathrm{GSO}(X, \mathbb{A})$ and $f \in V_\pi$. The map $f \mapsto sf$ from V_π with the twisted action $s \cdot \pi$ to sV_π with the usual action is an isomorphism; by multiplicity one, $s \cdot \pi \cong \pi$ if and only if $sV_\pi = V_\pi$.

7. From $\mathrm{GSO}(X, \mathbb{A})$ to $\mathrm{GO}(X, \mathbb{A})$

In this section F is a totally real number field and X is a four dimensional quadratic space over F. Let the notation be as in Section 6; following [HST], we explain how cuspidal automorphic representations of $\mathrm{GO}(X, \mathbb{A})$ are obtained from those of $\mathrm{GSO}(X, \mathbb{A})$. For each infinite place v of F let $J_{0,v}$ be the maximal compact subgroup of $\mathrm{GSO}(X,F_v)$ defined in Section 6, and let J_v denote the maximal compact subgroup of $\mathrm{GO}(X,F_v)$ generated by $J_{0,v}$ and s, where s is as in Lemma 6.1. Let J_∞ be the product of the J_v over the infinite places of F. We consider $\mathrm{GO}(X,\mathbb{A}_f)\times(\mathfrak{h}_\infty,J_\infty)$ modules. Let $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))$ be the set of tempered cuspidal automorphic representations of $\mathrm{GO}(X,\mathbb{A})$.

7.1 Theorem ([HST]). *The group $\mathrm{GO}(X,\mathbb{A})$ has the multiplicity one property; if $\sigma\in\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))$, denote by V_σ the unique space of cusp forms isomorphic to σ. Let $\sigma\in\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))$, and let V^0_σ be the nonzero space of cusp forms on $\mathrm{GSO}(X,\mathbb{A})$ obtained by restricting the functions in V_σ to $\mathrm{GSO}(X,\mathbb{A})$. Either V^0_σ is irreducible as a $\mathrm{GSO}(X,\mathbb{A}_f)\times(\mathfrak{h}_\infty,J_{0,\infty})$ module, or there exists $\pi\in\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X,\mathbb{A}))$ such that $s\cdot\pi\not\cong\pi$ and $V^0_\sigma=V_\pi\oplus sV_\pi$ (internal direct sum). Thus, there is a map*

$$\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))\to\langle s\rangle\backslash\,\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X,\mathbb{A})),$$

and if $\sigma\mapsto[\pi]=\{\pi,s\cdot\pi\}$, then

$$\sigma_v\hookrightarrow\mathrm{Ind}^{\mathrm{GO}(X,F_v)}_{\mathrm{GSO}(X,F_v)}\,\pi_v \tag{7.1}$$

for all v. The map $\sigma\mapsto[\pi]$ is surjective. If $[\pi]\in\langle s\rangle\backslash\,\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X,\mathbb{A}))$ and $s\cdot\pi\not\cong\pi$, then the fiber over $[\pi]$ is the set of all $\sigma\in\mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GO}(X,\mathbb{A}))$ such that (7.1) *holds for all places v of F.*

Proof. See Section 1 of [HST]. □

8. Proofs of the main theorems

Let F be a totally real number field. In this final section we prove the main results Theorems 8.3 and 8.6 presented in the Introduction. Besides the general foundational work of Sections 1, 2 and 5, the main ingredients for Theorem 8.3 are the local results of Section 3 and the general nonvanishing result for global theta lifts from [R4]. Globally, the result from [R4] requires the nonvanishing of a certain L-function at $s=1$; in the case at hand, this L-function turns out to be either a partial $\mathrm{GL}(2)\times\mathrm{GL}(2)$ L-function or a partial twisted Asai L-function, so that the nonvanishing at $s=1$ follows from [Sh]. To prove Theorem 8.6 we actually first prove a different version, Theorem 8.5. In this version, using Section 4, a global L-packet $\Pi([\pi])$ of tempered irreducible admissible representations of $\mathrm{GSp}(2,\mathbb{A})$ is assigned to every element π of $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times2},d},\mathbb{A}))$. When $s\cdot\pi\not\cong\pi$, Theorem 8.5 determines exactly what elements of $\Pi([\pi])$ are cuspidal automorphic and shows that the cuspidal automorphic elements occur with multiplicity one. In addition to an understanding of the local situation, the main tool for showing cuspidality is Theorem 8.3. For multiplicity one, we use the Rallis multiplicity preservation principle in the context of similitudes (Proposition 5.3), along with the nonvanishing result for global theta lifts from $\mathrm{Sp}(2,\mathbb{A})$ from [KRS]. This result shows that if a twisted partial standard L-function of a cuspidal automorphic representation of $\mathrm{Sp}(2,\mathbb{A})$ has a pole at $s=1$, then it has a nonzero theta lift to the isometry group of a certain four dimensional quadratic space. Theorem 8.6 follows directly from Theorem 8.5.

We begin with a lemma which computes the standard partial L-function of an $\mathrm{O}(X,\mathbb{A})$ component of a cuspidal automorphic representation of $\mathrm{GO}(X,\mathbb{A})$ for a four dimensional quadratic space X over F. In the following lemma, $L_v(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$ is the v-th Euler factor of the Asai L-function of τ^{JL} twisted by χ^{-1} ([HLR], p. 64–5); and $L_v(s,\tau_1^{\mathrm{JL}}\times\tau_2^{\mathrm{JL}\vee})$ is the v-th Euler factor of the usual Rankin–Selberg $\mathrm{GL}(2)\times\mathrm{GL}(2)$ L-function of τ_1^{JL} and $\tau_2^{\mathrm{JL}\vee}$; here, the superscript JL indicates the corresponding element under the Jacquet-Langlands correspondence (Section 6). Also, under the assumption that $X(F_v)$ is unramified (Section 1), when we say that an irreducible admissible representation of $\mathrm{GO}(X,F_v)$ (or of $\mathrm{O}(X,F_v)$ and $\mathrm{SO}(X,F_v)$) is unramified we mean with respect to the stabilizer in $\mathrm{GO}(X,F_v)$ (or in $\mathrm{O}(X,F_v)$ and $\mathrm{SO}(X,F_v)$, respectively) of a self-dual lattice in $X(F_v)$.

8.1 Lemma. *Let X be a four dimensional quadratic space over F, let B be the even Clifford algebra of $X(F)$, and let E be the center of B. Let $d=\operatorname{disc}X(F)$. Let $\sigma\in\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))$, and assume that σ lies over $[\pi=\pi(\chi,\tau)]$ (See Sections 6 and 7). Let v be a finite place of F such that $X(F_v)$ and σ_v are unramified. Let $\sigma_{1,v}$ be the unramified component of $\sigma_v|_{\mathrm{O}(X,F_v)}$. Then the standard L-function of $\sigma_{1,v}$ is*

$$L(s,\sigma_{1,v})=\begin{cases}L_v(s,\tau^{\mathrm{JL}},\chi^{-1},Asai) \text{ if } d\neq 1,\\ L_v(s,\tau_1^{\mathrm{JL}}\times\tau_2^{\mathrm{JL}\vee}) \text{ if } d=1 \text{ and } \tau\cong\tau_1\otimes\tau_2.\end{cases}$$

Proof. By definition, $L(s,\sigma_{1,v})$ (see Section 2 of [KR1]) is the standard L-function of any irreducible unramified component of $\sigma_{1,v}|_{\mathrm{SO}(X,F_v)}$. It will thus suffice to show that the standard L-function of any irreducible unramified component of $\sigma_v|_{\mathrm{SO}(X,F_v)}$ has the stated form; and since π_v is an irreducible component of $\sigma_v|_{\mathrm{GSO}(X,F_v)}$, it will be enough to show that the standard L-function of any irreducible unramified component of $\pi_v|_{\mathrm{SO}(X,F_v)}$ or $(s\cdot\pi_v)|_{\mathrm{SO}(X,F_v)}$ has the above form (s is as in Lemma 6.1). Since over a local nonarchimedean field a four dimensional quadratic space represents 1, by Proposition 2.8 there exists a quaternion algebra D over F_v contained in $B(F_v)$ and an isometry $T:X(F_v)\to X_{D,E_v}$ such that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E_v^\times & \longrightarrow & F_v^\times\times B^\times(F_v) & \xrightarrow{\rho} & \mathrm{GSO}(X,F_v) & \longrightarrow & 1\\
 & & \Big\downarrow{\scriptstyle\mathrm{id}} & & \Big\downarrow{\scriptstyle\wr} & & \Big\downarrow{\scriptstyle T\cdot T^{-1}} & & \\
1 & \longrightarrow & E_v^\times & \longrightarrow & F_v^\times\times B_{D,E_v}^\times & \xrightarrow{\rho_{a(D,E_v)}} & \mathrm{GSO}(X_{D,E_v},F_v) & \longrightarrow & 1
\end{array}$$

commutes, where $B^\times(F_v)\xrightarrow{\sim}B^\times_{D,E_v}$ is the isomorphism induced by the natural isomorphism $B(F_v)\cong E_v\otimes_{F_v}D$ of E_v algebras; $E_v=E(F_v)=F_v\otimes_F E$. Since

$X(F_v)$ is unramified, D is in particular split, i.e., there exists an isomorphism $D \xrightarrow{\sim} \mathrm{M}_{2\times 2}(F_v)$ of quaternion algebras over F_v. From this, we obtain an isomorphism $B_{D,E_v} \xrightarrow{\sim} \mathrm{M}_{2\times 2}(E_v)$ of E_v algebras and an isometry $t: X_{D,E_v} \xrightarrow{\sim} X_a$ so that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E_v^\times & \longrightarrow & F_v^\times \times B_{D,E_v}^\times & \xrightarrow{\rho_{a(D,E_v)}} & \mathrm{GSO}(X_{D,E_v}, F_v) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \wr} & & \Big\downarrow{\scriptstyle t\cdot t^{-1}} & & \\
1 & \longrightarrow & E_v^\times & \longrightarrow & F_v^\times \times \mathrm{GL}(2,E_v) & \xrightarrow{\rho_a} & \mathrm{GSO}(X_a, F_v) & \longrightarrow & 1
\end{array}$$

commutes. Here, a is the Galois action on $\mathrm{M}_{2\times 2}(E_v)$ defined by the formula (2.1). Composing, we now have an isomorphism $i: B(F_v) \xrightarrow{\sim} \mathrm{M}_{2\times 2}(E_v)$ of E_v algebras and isometry $r: X(F_v) \xrightarrow{\sim} X_a$ such that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E_v^\times & \longrightarrow & F_v^\times \times B^\times(F_v) & \xrightarrow{\rho} & \mathrm{GSO}(X, F_v) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \wr} & & \Big\downarrow{\scriptstyle r\cdot r^{-1}} & & \\
1 & \longrightarrow & E_v^\times & \longrightarrow & F_v^\times \times \mathrm{GL}(2,E_v) & \xrightarrow{\rho_a} & \mathrm{GSO}(X_a, F_v) & \longrightarrow & 1
\end{array}$$

commutes. Let π_v' be the representation of $\mathrm{GSO}(X_a, F_v)$ corresponding to π_v. By definition, $(\tau_v)^{\mathrm{JL}} = \tau_v \circ i$, and we have $\pi_v' = \pi(\chi_v, (\tau_v)^{\mathrm{JL}})$. Since the standard L-function of any unramified irreducible component of $\pi_v|_{\mathrm{SO}(X,F_v)}$ is the same as the standard L-function of any irreducible unramified component of $\pi_v'|_{\mathrm{SO}(X_a,F_v)}$, and the same holds for $(s\cdot\pi_v)|_{\mathrm{SO}(X,F_v)}$ and $(s\cdot\pi_v')|_{\mathrm{SO}(X_a,F_v)}$, it will now suffice to show that the the standard L-function of any irreducible unramified component of $\pi_v'|_{\mathrm{SO}(X_a,F_v)}$ or $(s\cdot\pi_v')|_{\mathrm{SO}(X_a,F_v)}$ has the above form. Assume first $d \neq 1$ (i.e., E is a field) and v stays prime in E; let w be the place of E lying over v. Then $E_w = E_v$. Since π_v' is unramified, so are χ_v and $(\tau^{\mathrm{JL}})_w = (\tau_v)^{\mathrm{JL}} \in \mathrm{Irr}(\mathrm{GL}(2,E_w))$. Let $(\tau^{\mathrm{JL}})_w = \mathrm{Ind}_P^{\mathrm{GL}(2,E_w)}(\mu_1\otimes\mu_2)$, where P is the usual upper triangular Borel subgroup of $\mathrm{GL}(2,E_w)$, induction is normalized, μ_1 and μ_2 are unramified quasi-characters of $E_w^\times$, and $\mu_1\otimes\mu_2$ is defined by

$$(\mu_1\otimes\mu_2)(\begin{bmatrix} a & b \\ 0 & c\end{bmatrix}) = \mu_1(a)\mu_2(c).$$

The space X_a was explicitly described in Section 2. With respect to the ordered basis

$$\begin{bmatrix} 0 & \sqrt{d} \\ 0 & 0\end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}, \begin{bmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d}\end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -(2/d)\sqrt{d} & 0\end{bmatrix}$$

the symmetric bilinear form on X_a, which is given by the determinant, has the form

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -d & 0 \\ 1 & 0 & 0 & 0\end{bmatrix}.$$

The stabilizer in $\mathrm{GSO}(X_a, F_v)$ of the isotropic subspace spanned by the first basis vector is a Borel subgroup P' of $\mathrm{GSO}(X_a, F_v)$, and $P' = \rho_\alpha(F_v^\times \times P)$. In particular, we have

$$\rho_\alpha(t, \begin{bmatrix} a & * \\ 0 & c \end{bmatrix}) = t^{-1} \begin{bmatrix} \mathrm{N}_{F_v}^{E_w}(a) & * & * \\ 0 & h & * \\ 0 & 0 & \mathrm{N}_{F_v}^{E_w}(c) \end{bmatrix},$$

with

$$h = \begin{bmatrix} a_1c_1 - a_2c_2d & (a_2c_1 - a_1c_2)d \\ a_2c_1 - a_1c_2 & a_1c_1 - a_2c_2d \end{bmatrix},$$

where $a = a_1 + a_2\sqrt{d}$ and $c = c_1 + c_2\sqrt{d}$. Here, the middle block h corresponds to multiplication by $a\alpha(c)$ on the two dimensional subspace spanned by the two middle basis vectors, using the obvious identification of this subspace with E_w. Recalling that $\chi_v \circ \mathrm{N}_{F_v}^{E_w} = \mu_1\mu_2$, a computation shows that

$$\pi'_v = \pi(\chi_v, (\tau^{\mathrm{JL}})_w) = \mathrm{Ind}_{P'}^{\mathrm{GSO}(X_a,F_v)} \mu,$$

where induction is normalized, and on the typical element of P' μ takes the value

$$\mu(\begin{bmatrix} a & * & * \\ 0 & h & * \\ 0 & 0 & \lambda a^{-1} \end{bmatrix}) = (\mu_2/\chi_v)(\lambda a^{-1})\mu_1(h),$$

where again we identify the elements of the middle block with $E_w^\times$ and $a, \lambda \in F_v^\times$. There is an $\mathrm{SO}(X_a, F_v)$ isomorphism

$$\pi'_v|_{\mathrm{SO}(X_a,F_v)} \xrightarrow{\sim} \mathrm{Ind}_{P'\cap\mathrm{SO}(X_a,F_v)}^{\mathrm{SO}(X_a,F_v)} \mu|_{P'\cap\mathrm{SO}(X_a,F_v)}$$

given by restriction of functions. We have

$$\mu|_{P'\cap\mathrm{SO}(X_a,F_v)}(\begin{bmatrix} a & * & * \\ 0 & h & * \\ 0 & 0 & a^{-1} \end{bmatrix}) = (\chi_v/\mu_2)(a)$$

since $\mathrm{N}_{F_v}^{E_w}(h) = 1$, so that $h \in \mathfrak{O}_w^\times$. By definition, the standard L-function of any irreducible unramified component of $\pi'_v|_{\mathrm{SO}(X_a,F_v)}$ is now

$$\begin{aligned} L(s, \chi_v/\mu_2)L(s, \mu_2/\chi_v)\zeta_{F_v}(2s) &= \det(1 - \chi(\pi_{F_v})^{-1}A|\pi_{F_v}|^s)^{-1} \\ &= L_v(s, \tau^{\mathrm{JL}}, \chi^{-1}, \mathrm{Asai}), \end{aligned}$$

where

$$A = \begin{bmatrix} \mu_1(\pi_{F_v}) & 0 & 0 & 0 \\ 0 & 0 & \mu_1(\pi_{F_v}) & 0 \\ 0 & \mu_2(\pi_{F_v}) & 0 & 0 \\ 0 & 0 & 0 & \mu_2(\pi_{F_v}) \end{bmatrix}.$$

For the last equality, see p. 64-65 of [HLR]. Since $s\cdot\pi'_v = \pi(\chi_v, (\tau^{\mathrm{JL}})_w \circ \alpha)$, a similar computation shows that the standard L-function of any irreducible unramified component of $(s\cdot\pi'_v)|_{\mathrm{SO}(X_a,F_v)}$ is also $L_v(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$.
Now suppose E is a field and v splits in E. Then F_v contains a square root of d; fix such a square root $\sqrt{d}$ in $F_v^\times$. Define an embedding of fields $i_1 : E \hookrightarrow F_v$ by sending a fixed square root of d in E to $\sqrt{d}$, and define another embedding $i_2 : E \hookrightarrow F_v$ by sending the fixed square root of d in E to $-\sqrt{d}$. We denote by w_1 and w_2 the places of E determined by i_1 and i_2, respectively. Then w_1 and w_2 are the two places of E lying over v, and via i_1 and i_2 we take F_v to be the completions E_{w_1} and E_{w_2} of E at w_1 and w_2, respectively. We also have an identification of $E_v = F_v\otimes_F E$ with $E_{w_1}\times E_{w_2}$ and hence with $F_v\times F_v$. Using the identification $E_v \cong F_v\times F_v$ we may identify $\mathrm{M}_{2\times 2}(E_v)$ with $\mathrm{M}_{2\times 2}(F_v)\times \mathrm{M}_{2\times 2}(F_v)$, $\mathrm{GL}(2,E_v)$ with $\mathrm{GL}(2,F_v)\times\mathrm{GL}(2,F_v)$ and a with the Galois action defined by $(x_1,x_2)\mapsto(x_2,x_1)$. Further, as explained after Proposition 2.7, we may identify X_a with $\mathrm{M}_{2\times 2}(F_v)$ and ρ_a with $\rho_a(t,(g_1,g_2))x = t^{-1}g_1xg_2^*$. Using the canonical isomorphisms $B(F_v)\cong B(E_{w_1})\times B(E_{w_2})\cong D\times D$ write $\tau_v\cong\tau_{w_1}\otimes\tau_{w_2}$ with $\tau_{w_1},\tau_{w_2}\in\mathrm{Irr}(D^\times)$; then $(\tau_v)^{\mathrm{JL}}\cong\tau_{w_1}^{\mathrm{JL}}\otimes\tau_{w_2}^{\mathrm{JL}}$. Let $\tau_{w_1}^{\mathrm{JL}} = \mathrm{Ind}_P^{\mathrm{GL}(2,F_v)}(\mu_1\otimes\mu_2)$ and $\tau_{w_2}^{\mathrm{JL}} = \mathrm{Ind}_P^{\mathrm{GL}(2,F_v)}(\mu_1'\otimes\mu_2')$, with the notation analogous to the previous case. Note that $\chi=\mu_1\mu_2=\mu_1'\mu_2'$. With respect to the ordered basis

$$\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\-2&0\end{bmatrix},\begin{bmatrix}0&0\\0&2\end{bmatrix}$$

the symmetric bilinear form on X_a has the matrix

$$\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}.$$

The stabilizer in $\mathrm{GSO}(X_a,F_v)$ of the isotropic flag

$$F_v\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix}\subset F_v\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix}+F_v\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

is a Borel subgroup P', and $P'=\rho_\alpha(F_v^\times\times P\times P)$. We have

$$\rho_\alpha(t,\begin{bmatrix}a&*\\0&c\end{bmatrix},\begin{bmatrix}a'&*\\0&c'\end{bmatrix}) = t^{-1}\begin{bmatrix}aa'&*&*&*\\0&ac'&*&*\\0&0&cc'&*\\0&0&0&a'c\end{bmatrix}.$$

Using $\mu_1\mu_2=\mu_1'\mu_2'$, a computation shows that

$$\pi_v'\cong\mathrm{Ind}_{P'}^{\mathrm{GSO}(X_a,F_v)}\mu$$

with

$$\mu(\begin{bmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & \lambda a^{-1} & * \\ 0 & 0 & 0 & \lambda b^{-1} \end{bmatrix}) = \mu_2(\lambda)(\mu_1'/\mu_2)(a)(\mu_2'/\mu_2)(b).$$

Again there is an $\mathrm{SO}(X_a, F_v)$ isomorphism

$$\pi_v'|_{\mathrm{SO}(X_a,F_v)} \xrightarrow{\sim} \mathrm{Ind}_{P'\cap \mathrm{SO}(X_a,F_v)}^{\mathrm{SO}(X_a,F_v)} \mu|_{P'\cap \mathrm{SO}(X_a,F_v)}$$

given by restriction of functions. We have

$$\mu|_{P'\cap \mathrm{SO}(X_a,F_v)}(\begin{bmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & a^{-1} & * \\ 0 & 0 & 0 & b^{-1} \end{bmatrix}) = (\mu_1'/\mu_2)(a)(\mu_2'/\mu_2)(b).$$

We now have that the standard L-function of any irreducible unramified component of $\pi_v'|_{\mathrm{SO}(X_a,F_v)}$ is

$$\begin{aligned} L(s,\mu_1'/\mu_2)L(s,\mu_2/\mu_1')L(s,\mu_2'/\mu_2)L(s,\mu_2/\mu_2') &= \det(1-\chi(\pi_{F_v})^{-1}A|\pi_{F_v}|^s)^{-1} \\ &= L_v(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai}), \end{aligned}$$

where

$$A = \begin{bmatrix} (\mu_1\mu_1')(\pi_{F_v}) & 0 & 0 & 0 \\ 0 & (\mu_2\mu_2')(\pi_{F_v}) & 0 & 0 \\ 0 & 0 & (\mu_1\mu_2')(\pi_{F_v}) & 0 \\ 0 & 0 & 0 & (\mu_2\mu_1')(\pi_{F_v}) \end{bmatrix};$$

here we have used $\chi_v = \mu_1\mu_2 = \mu_1'\mu_2'$. For the last equality, again see p. 64-65 of [HLR]. Since $s\cdot\pi_v' = \pi(\chi_v, \tau_{w_2}^{\mathrm{JL}} \otimes \tau_{w_1}^{\mathrm{JL}})$, a similar computation shows that the standard L-function of any irreducible unramified component of $(s\cdot\pi_v')|_{\mathrm{SO}(X_a,F_v)}$ is also $L_v(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$.
The argument in the case $d=1$ is similar to the last case and will be omitted. □

To prove the nonvanishing part of the main result Theorem 8.3 we will use the following theorem, which follows from Corollary 1.2 of [R4]. In the following $L^S(s,\sigma_1)$ is the standard partial L-function of σ_1 (see Section 2 of [KR1]).

8.2 THEOREM ([R4]). *Let F be a totally real number field, and let X be a four dimensional quadratic space over F. Let $d \in F^\times/F^{\times 2}$ be the discriminant of $X(F)$, and assume that the discriminant algebra E of $X(F)$ is totally real, i.e., either $d=1$ or $d\neq 1$ and $E = F(\sqrt{d})$ is totally real. Let σ_1 be a tempered cuspidal automorphic representation of $\mathrm{O}(X,\mathbb{A})$ with $\sigma_1 \cong \otimes_v \sigma_{1,v}$, and let V_{σ_1} be a realization of σ_1 in the space of cusp forms on $\mathrm{O}(X,\mathbb{A})$. Assume $\sigma_{1,v}$*

occurs in the theta correspondence for $\mathrm{O}(X, F_v)$ *and* $\mathrm{Sp}(2, F_v)$ *for all places* v. *If* $L^S(s, \sigma_1)$ *does not vanish at* $s = 1$ *then* $\Theta_2(V_{\sigma_1}) \neq 0$.

Proof. This follows from Corollary 1.2 of [R4] (see also the following remark below). Note that by the assumption on E, at each infinite place v of F we have $d = 1$ in $F_v^\times / F_v^{\times 2}$, so that the signature of $X(F_v)$ is $(4, 0), (2, 2)$ or $(0, 4)$ and the signature assumptions from Corollary 1.2 of [R4] are satisfied. $\square$

We take the opportunity here to make a correction to [R4]. Namely, in Theorem 1.1 of [R4] hypothesis (2) should be replaced with the statement: for all places v, σ_v is tempered and if σ_v first occurs in the theta correspondence with $\mathrm{Sp}(n', F_v)$ with $2n' > \dim X$, then the first occurrence of σ_v is tempered; in Corollary 1.2 of [R4] σ_v should also be assumed to be tempered for infinite v; and finally in Lemma 2.1 of [R4] the assumption on σ (in both the nonarchimedean and real cases) should be that σ is tempered, and if σ first occurs in the theta correspondence with $\mathrm{Sp}(n', F)$ with $2n' > \dim X$, then the first occurrence of σ is tempered. The corrections thus also introduce temperedness assumptions at the infinite places entirely analogous to those at the finite places (note that in the corrections to Theorem 1.1 and Lemma 2.1 we have actually weakened the nonarchimedean assumption; this was mentioned in [R4], but not explicitly stated as part of Theorem 1.1 and Lemma 2.1). The omission of these temperedness assumptions at infinity was due to a misreading of [M], Corollaire IV.5 (ii). The only place where the result from [M] is used in [R4] is in the proof of Lemma 2.1 of [R4] where it is asserted that, in the terminology of that lemma, $\theta_{k+1}(\sigma) = L(\chi_X|\cdot|^{s_X(k+1)} \otimes \delta_2 \cdots \otimes \delta_t \otimes \tau)$. The argument for this is as follows. Assume σ first occurs in the theta correspondence with $\mathrm{Sp}(n', \mathbb{R})$ with $n' \leq \dim X/2$. Then σ occurs in the theta correspondence with $\mathrm{Sp}(\dim X/2, \mathbb{R})$ (Lemme I.9, p. 14, [M]) and $\theta_{\dim X/2}(\sigma) = \Psi_{\dim X/2}(\sigma)$ (Théorème IV.3, p. 70, [M]). Since σ is tempered, by the definition of $\Psi_{\dim X/2}(\sigma)$ (III.2, p. 49, [M]), $\theta_{\dim X/2}(\sigma) = \Psi_{\dim X/2}(\sigma)$ is also tempered. The Langlands data for $\theta_{k+1}(\sigma)$ is obtained from the Langlands data of $\theta_{\dim X/2}(\sigma)$ by adjoining the quasi-characters of $\mathbb{R}^\times$: $\chi_X|\cdot|^{s_X(k+1)}, \ldots, \chi_X|\cdot|^{s_X(\dim X/2)+2}, \chi_X|\cdot|^{s_X(\dim X/2)+1}$ (Corollaire IV.5 (ii), p. 71, [M]). Since $\theta_{\dim X/2}(\sigma)$ is tempered, this implies $\theta_{k+1}(\sigma)$ has the claimed form. Next, assume σ first occurs in the theta correspondence with $\mathrm{Sp}(n', \mathbb{R})$ with $n' > \dim X/2$. Then $\theta_{n'}(\sigma)$ is tempered by assumption. Again, the Langlands data of $\theta_{k+1}(\sigma)$ is obtained from the Langlands data of $\theta_{n'}(\sigma)$ by adjoining the quasi-characters of $\mathbb{R}^\times$: $\chi_X|\cdot|^{s_X(k+1)}, \ldots, \chi_X|\cdot|^{s_X(n')+2}, \chi_X|\cdot|^{s_X(n')+1}$ (Corollaire IV.5 (ii), p. 71, [M]). Again, since $\theta_{n'}(\sigma)$ is tempered, this implies $\theta_{k+1}(\sigma)$ has the claimed form. This completes the corrected argument for the new statement of Lemma 2.1 of [R4]. The corrected statements of Theorem 1.1 and Corollary 1.2 have exactly the same proofs as in [R4].

Proof of Theorem 8.3. (1) $\Longrightarrow$ (2). Suppose $\Theta_2(V_\sigma) \neq 0$. Suppose $\Theta_2(V_\sigma)$ is contained in the space of cusp forms. Then by Proposition 5.3, (2) holds. Suppose $\Theta_2(V_\sigma)$ is not contained in the space of cusp forms. Since σ_v is infinite dimensional for at least one v (5.1) holds. By Proposition 5.2, $\Theta_1(V_\sigma)$ is contained in the space cusp forms; also, $\Theta_1(V_\sigma)$ is nonzero, for otherwise, by Proposition 5.2, $\Theta_2(V_\sigma)$ would be contained in the space of cusp forms. A standard argument as in the proof of Proposition 5.3 now shows that for all v, $\sigma_v \in \mathcal{R}_1(\mathrm{GO}(X, F_v))$. This implies (2) (Lemma 4.2 of [R1]; b), p. 67 of [MVW]; Lemme I.9 of [M]).

(2) $\iff$ (3). This is Theorem 3.4. Note that if $\operatorname{disc} X(F_v) \neq 1$, then $X(F_v)$ is isotropic, and so σ_v is infinite dimensional (as σ_v is tempered).

(2) $\implies$ (1). Suppose (2) holds. Let σ lie over $[\pi]$ (Section 7). Restrict the functions in V_σ to $\mathrm{O}(X,\mathbb{A})$ to obtain the space of functions V_σ^1. Then V_σ^1 is nonzero and contained in the space of cusp forms on $\mathrm{O}(X,\mathbb{A})$; let W be an irreducible nonzero $\mathrm{O}(X,\mathbb{A}_f) \times (\mathfrak{h}_{1,\infty}, J_{1,\infty})$ component of V_σ^1, and denote the isomorphism class of W by σ_1. To show $\Theta_2(V_\sigma) \neq 0$ it will suffice to show $\Theta_2(W) \neq 0$ (for if $\theta_2(f,\varphi) \neq 0$ for some $f \in W$ and $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$, then $\theta_2(F,\varphi)|_{\mathrm{Sp}(2,\mathbb{A})} = \theta_2(f,\varphi) \neq 0$ for any $F \in V_\sigma$ with $F|_{\mathrm{O}(X,\mathbb{A})} = f$). For this, we will use Theorem 8.2. We need to see that the hypotheses of Theorem 8.2 are satisfied. For all places v of F, $\sigma_{1,v}$ is an irreducible constituent of $\sigma_v|_{\mathrm{O}(X,F_v)}$. Since σ_v is tempered for all v, $\sigma_{1,v}$ is tempered for all v. Also, it is a basic consequence of (2) that $\sigma_{1,v} \in \mathcal{R}_2(\mathrm{O}(X,F_v))$ for all v (Lemma 4.2 of [R1]; see the discussion before Theorem 1.8). Finally, we need to see that the partial standard L-function $L^S(s,\sigma_1)$ of σ_1 does not vanish at $s=1$. Writing $\pi = \pi(\chi,\tau)$, by Lemma 8.1 we have

$$L^S(s,\sigma_1) = \begin{cases} L^S(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai}) \text{ if } d \neq 1 \\ L^S(s,\tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee}) \text{ if } d = 1 \text{ and } \tau \cong \tau_1 \otimes \tau_2. \end{cases}$$

Showing the nonvanishing of $L^S(s,\sigma_1)$ at $s=1$ is thus reduced to showing the nonvanishing of these two types of L-functions at $s=1$. For the nonvanishing of $L^S(s,\tau_1^{\mathrm{JL}} \times \tau_2^{\mathrm{JL}\vee})$ at $s=1$ see Theorem 5.2 of [Sh]. The nonvanishing of $L^S(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$ at $s=1$ also follows from [Sh]. For an explanation of this, see p. 296–7 of [F]. Note that $L^S(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$ is of the form $L^S(s,\tau',\mathrm{Asai})$: there exists a Hecke character $\hat\chi$ of $\mathbb{A}_E^\times$ extending χ, and for such a $\hat\chi$ we have $L^S(s,\tau\otimes\hat\chi^{-1},\mathrm{Asai}) = L^S(s,\tau^{\mathrm{JL}},\chi^{-1},\mathrm{Asai})$. By Theorem 8.2 we now have $\Theta_2(W) \neq 0$, and so $\Theta_2(V_\sigma) \neq 0$.

Now suppose that one of (1), (2) or (3) holds, and $s \cdot \pi \not\cong \pi$. By what we have already shown, $\Theta_2(V_\sigma) \neq 0$. We claim that $\Theta_2(V_\sigma)$ is contained in the space of cusp forms. Suppose not. Then as in the proof of (1) $\implies$ (2), $\Theta_1(V_\sigma)$ is nonzero and contained in the space of cusp forms, and in particular $\sigma_v \in \mathcal{R}_1(\mathrm{GO}(X,F_v))$ for all v. By Theorem 7.4 of [R2] this implies $s \cdot \pi_v \cong \pi_v$ at least for all finite v of odd residual characteristic. However, by strong multiplicity one for $\mathrm{GSO}(X,\mathbb{A})$ (Section 6) and $s \cdot \pi \not\cong \pi$, we have $s \cdot \pi_v \not\cong \pi_v$ for infinitely many v, a contradiction. Thus, $\Theta_2(V_\sigma)$ is contained in the space of cusp forms. By Proposition 5.3, $\Theta_2(V_\sigma)$ is a cuspidal automorphic representation of $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_σ and $\Theta_2(V_\sigma) = \otimes_v \theta_2(\sigma_v^\vee)$; by Proposition 1.10 this is also $\otimes_v \theta_2(\sigma_v)^\vee$. The proof that $\theta_2(\sigma_v)$ is tempered for all v is as in the proof of (3) of Proposition 4.1. $\square$

The next lemma was used in the proof of Proposition 4.1 to show that the two elements of an L-packet defined there are in fact distinct. It will also be used in the proof of Theorem 8.5.

8.4 Lemma. *Let v be a place of F. Let D_{ram} be the division quaternion algebra over F_v, and define the four dimensional quadratic spaces $X_{\mathrm{M}_{2\times2},1}$ and $X_{D_{\mathrm{ram}},1}$ over F_v as in Section 2. Then $\mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{GSp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v)) = \emptyset$.*

Proof. We will use the notation of Section 1. By Lemmas 1.4 and 1.5 it will suffice to show that if

$$\mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{GSp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v)) \neq \emptyset$$

then

$$\mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{Sp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{Sp}(2,F_v)) \neq \emptyset.$$

Suppose $\Pi \in \mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{GSp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v))$. Since Π is contained in $\mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{GSp}(2,F_v))$, by definition $\Pi|_{\mathrm{Sp}(2,F_v)}$ is multiplicity free; let $\Pi|_{\mathrm{Sp}(2,F_v)} = W_1 \oplus \cdots \oplus W_M$ with the W_i, $1 \leq i \leq M$, mutually non-isomorphic irreducible $\mathrm{Sp}(2,F_v)$ subspaces of Π. Also by definition, some W_i, say W_1, is in $\mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{Sp}(2,F_v))$. We assert that all the W_i are contained in $\mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{Sp}(2,F_v))$. Let $g \in \mathrm{GSp}(2,F_v)$ be such that $\pi(g)W_1 = W_i$ (if $F_v \cong \mathbb{R}$ then $M = 1$ or 2 and we may take $g = k_0$ with k_0 as in Section 1). Since $W_1 \in \mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{Sp}(2,F_v))$ there exists a nonzero $\mathrm{Sp}(2,F_v)$ map $t : \omega_{X_{\mathrm{M}_{2\times2},1}} \to W_1$. Let $h \in \mathrm{GO}(X_{\mathrm{M}_{2\times2},1},F_v)$ be such that $(g,h) \in R_{X_{\mathrm{M}_{2\times2},1}}$ (if $F_v \cong \mathbb{R}$ we take $h = j_0$ so that $(g,h) \in L$). Consider the composition

$$\omega_{X_{\mathrm{M}_{2\times2},1}} \xrightarrow{\omega(g,h)^{-1}} \omega_{X_{\mathrm{M}_{2\times2},1}} \xrightarrow{t} W_1 \xrightarrow{\pi(g)} W_i.$$

This is a nonzero $\mathrm{Sp}(2,F)$ map. Thus, $W_i \in \mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{Sp}(2,F_v))$. On the other hand, since $\Pi \in \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v))$ we have by definition that some irreducible component of $\Pi|_{\mathrm{Sp}(2,F_v)}$ is contained in $\mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{Sp}(2,F_v))$. We now have $\mathcal{R}_{X_{\mathrm{M}_{2\times2},1}}(\mathrm{Sp}(2,F_v)) \cap \mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{Sp}(2,F_v)) \neq \emptyset$ as desired. $\square$

We come now to the definition and analysis of global L-packets for GSp(2). We begin by proving Theorem 8.5, a version of the main result Theorem 8.6. In this version, global L-packets for GSp(2) are associated to elements of $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times2},d},\mathbb{A}))$; Theorem 8.6 will follow easily from Theorem 8.5. As in Section 2, let $d \in F^\times/F^{\times2}$, and let $E = E_d$ be $F(\sqrt{d})$ if $d \neq 1$ and $E = E_d = F \times F$ if $d = 1$. Assume E is totally real, i.e., in the case $d \neq 1$

assume E is totally real. Let $\pi \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, \mathbb{A}))$. The packet of irreducible admissible representations of $\mathrm{GSp}(2,\mathbb{A})$ corresponding to $[\pi]$ is defined to be

$$\Pi([\pi]) = \{\Pi = \otimes_v \Pi_v \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GSp}(2,\mathbb{A})) : \Pi_v \in \Pi([\pi_v]) \text{ for all } v\}.$$

Here, $\Pi([\pi_v])$ is defined in Section 4. By Proposition 4.3, for almost all nonarchimedean v, $\Pi([\pi_v])$ consists of a single representation unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_v)$. Thus,

$$\Pi([\pi]) = \otimes_v \Pi([\pi_v]).$$

Also, by (3) of Proposition 4.1, $\Pi([\pi_v])$ consists of tempered representations for all v. If S is any finite set of places such that for $v \notin S$, v is nonarchimedean and $\Pi([\pi_v])$ consists of a single representation unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_v)$, then the cardinality of $\Pi([\pi])$ is:

$$|\Pi([\pi])| = \prod_{v\in S} |\Pi([\pi_v])| = 2^M, \text{ where } M = \sum_{v\in S}(|\Pi([\pi_v])| - 1).$$

For $\Pi = \otimes_v \Pi_v \in \Pi([\pi])$, let T_Π be the set of places v of F such that v splits in E (as usual, if $d = 1$ so that $E = F\times F$ we say that every place of F splits in E) and Π_v is of the form $\theta_{D_{\mathrm{ram}},1}(\pi_v^{\mathrm{JL}+})^\vee$ (so necessarily π_v is square integrable); see Section 4.

8.5 Theorem. *Assume F is totally real, $d \in F^\times/F^{\times 2}$, and let $E = E_d$ be $F(\sqrt{d})$ if $d \neq 1$ and $F\times F$ if $d = 1$. Assume E is totally real, i.e., in the case $d \neq 1$ assume E is totally real. Let $\pi \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, \mathbb{A}))$ and assume $s\cdot\pi \ncong \pi$.*

(1) *If $d \neq 1$, then all the elements of $\Pi([\pi])$ occur with multiplicity one in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ with central character ω_π.*
(2) *Assume $d = 1$. Let $\Pi \in \Pi([\pi])$. If $|T_\Pi|$ is even, then Π occurs with multiplicity one in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ with central character ω_π. Conversely, if Π occurs in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ then $|T_\Pi|$ is even.*

Proof. Let $\Pi \in \Pi([\pi])$; if $d = 1$ assume $|T_\Pi|$ is even. We begin by showing that Π occurs in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_π. To prove this we will construct a four dimensional quadratic space X over F and a $\sigma \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))$ such that $\sigma_v \in \mathcal{R}_2(\mathrm{GO}(X,F_v))$ and $\theta_2(\sigma_v)^\vee = \Pi_v$ for all v; we will then apply Theorem 8.3 to show Π is cuspidal automorphic. To start, let us set up some definitions involving π. As in Section 6, write $\pi = \pi(\chi,\tau)$; however instead of the abstract exact sequence of Theorem 2.3, let us use the concrete exact sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times \mathrm{GL}(2,\mathbb{A}_E) \xrightarrow{\rho_{a(\mathrm{M}_{2\times 2},E)}} \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}) \to 1$$

of Proposition 2.7; by this proposition there is no real distinction. Thus, $\pi = \pi(\chi, \tau)$, with $\tau \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GL}(2, \mathbb{A}_E))$ and χ a Hecke character of $\mathbb{A}^\times$ such that $\omega_\tau = \chi \circ \mathrm{N}^E_F$. The X we will use will be of the form $X_{D,d}$. Specifically, let D be any quaternion algebra over F which is ramified at the places in T_Π, and which is unramified at any other of the places of F which split in E; note that if $d = 1$, we use the evenness of $|T_\Pi|$ for the existence of D (again, our convention is that if $d = 1$ so that $E = F \times F$ then every place of F is split in E). Evidently, if $d = 1$, then D is uniquely determined, but if $d \neq 1$, then there will be infinitely many choices for D. Nevertheless, if we let $B = E \otimes_F D$, then B is uniquely determined (in the case $d \neq 1$, regarded as a quaternion algebra over E, B is split at any place of E lying over a nonsplit place of F), and $X_{D,d}$ is uniquely determined up to similitudes by Proposition 2.9. By Lemma 5.1, it thus follows that our construction will realize Π as a cuspidal automorphic representation in exactly one way in spite of the ambiguity in the choice of D when $d \neq 1$. To define the σ mentioned above, note that again by Proposition 2.7 we have an exact sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}^\times \times B^\times(\mathbb{A}) \xrightarrow{\rho_{a(D,E)}} \mathrm{GSO}(X_{D,d}, \mathbb{A}) \to 1.$$

By the definition of T_Π, τ is in the image of the Jacquet-Langlands correspondence from $B^\times(\mathbb{A})$ discussed in Section 6; let $\tau^{\mathrm{JL}} \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(B^\times(\mathbb{A}))$ correspond to τ. Let $\pi' = \pi(\chi, \tau^{\mathrm{JL}})$; this is contained in $\mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GSO}(X_{D,d}, \mathbb{A}))$. We claim that for each place v there exists $\sigma_v \in \mathcal{R}_2(\mathrm{GO}(X_{D,d}, F_v))$ such that

$$\sigma_v \hookrightarrow \mathrm{Ind}^{\mathrm{GO}(X_{D,d},F_v)}_{\mathrm{GSO}(X_{D,d},F_v)} \pi'_v$$

and $\theta(\sigma_v)^\vee = \Pi_v$. This is clear from the definition of $\Pi([\pi_v])$ and D if v is not a nonsplit place with $D(F_v)$ ramified; assume we are in this last case. Let w be the place of E lying over v. By Proposition 2.9 and the consideration of examples after this proposition, there exists an isomorphism $i : B(F_v) \xrightarrow{\sim} \mathrm{M}_{2\times 2}(E_w)$ of E_w algebras and a similitude $T : X_{D,d}(F_v) \to X_{\mathrm{M}_{2\times 2},d}(F_v)$ such that

$$\begin{array}{ccccccccc}
1 & \longrightarrow & E_w^\times & \longrightarrow & F_v^\times \times B^\times(F_v) & \longrightarrow & \mathrm{GSO}(X_{D,d}, F_v) & \longrightarrow & 1 \\
 & & \big\downarrow \mathrm{id} & & \big\downarrow \mathrm{id}\times i & & \big\downarrow T\cdot T^{-1}=j & & \\
1 & \longrightarrow & E_w^\times & \longrightarrow & F_v^\times \times \mathrm{GL}(2, E_w) & \longrightarrow & \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d}, F_v) & \longrightarrow & 1
\end{array}$$

commutes. By the definition of $\Pi([\pi_v])$, there exists $\hat{\pi}_v \in \mathrm{Irr}(\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F_v))$ such that $\pi_v \hookrightarrow \hat{\pi}_v|_{\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},F_v)}$ and $\theta(\hat{\pi}_v) = \Pi_v^\vee$, i.e.,

$$\mathrm{Hom}_{R_{X_{\mathrm{M}_{2\times 2},d}}(F_v)}(\omega_{X_{\mathrm{M}_{2\times 2},d}(F_v)}, \Pi_v^\vee \otimes \hat{\pi}_v) \neq 0.$$

By Lemma 1.6, we obtain

$$\mathrm{Hom}_{R_{X_{D,d}}(F_v)}(\omega_{X_{D,d}(F_v)}, \Pi_v^\vee \otimes \sigma_v) \neq 0,$$

where $\sigma_v = \hat{\pi}_v \circ j$, so that $\theta(\sigma_v) = \Pi_v^\vee$. Since $\pi_v \circ j \hookrightarrow \sigma_v|_{\mathrm{GSO}(X_{D,d},F_v)}$ and $\pi_v \circ j = \pi'_v$ by the commutativity of the diagram, we get $\pi'_v \hookrightarrow \sigma_v|_{\mathrm{GSO}(X_{D,d},F_v)}$ as desired. Now Π_v is unramified for almost all finite v, and so by Proposition 1.11 σ_v is unramified for almost all finite v. We may form the restricted direct product $\sigma = \otimes_v \sigma_v \in \mathrm{Irr}_{\mathrm{admiss}}(\mathrm{GO}(X,\mathbb{A}))$. Since $s \cdot \pi \not\cong \pi$ we have $s \cdot \pi' \not\cong \pi'$. By Theorem 7.1 it follows that $\sigma \in \mathrm{Irr}^{\mathrm{temp}}_{\mathrm{cusp}}(\mathrm{GO}(X,\mathbb{A}))$, and σ lies over $[\pi']$. By Theorem 8.3, $\Theta_2(V_\sigma)$ is cuspidal and $\Theta_2(V_\sigma) \cong \Pi$.

Having shown that Π occurs in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_π, we will now show that the multiplicity with which Π occurs is one. Our strategy will be to use the multiplicity preservation principle of Rallis (Proposition 5.3) along with the fact that for a four dimensional quadratic space X over F, $\mathrm{GO}(X,\mathbb{A})$ has the (weak) multiplicity one property (Theorem 7.1). Let W be the $\mathrm{GSp}(2,\mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_π generated by the subspaces isomorphic to Π. Let U be an irreducible nonzero $\mathrm{GSp}(2,\mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$ subspace of W. Then $U \cong \Pi$. To be in a position to apply Proposition 5.3 we must show that $\Theta_{X_{D,d}}(U)$ is nonzero and contained in the space of cusp forms on $\mathrm{GO}(X,\mathbb{A})$ of central character ω_π.

As a first step, we will prove that $\Theta_{X_{D',d}}(U)$ is nonzero and cuspidal for some quaternion algebra D' over F. In the following argument showing that $\Theta_{X_{D',d}}(U)$ is nonzero and cuspidal for some D' we ask the reader to take note that we only use that $\Pi \in \Pi([\pi])$; this will be germane in a subsequent part of the proof. We begin with a reduction to isometries. Restrict the functions in U to $\mathrm{Sp}(2,\mathbb{A})$. This space of restricted functions is nonzero and is an $\mathrm{Sp}(2,\mathbb{A}_f) \times (\mathfrak{g}_{1,\infty}, K_{1,\infty})$ subspace of the space of cusp forms on $\mathrm{Sp}(2,\mathbb{A})$; let U_1 be a nonzero $\mathrm{Sp}(2,\mathbb{A}_f) \times (\mathfrak{g}_{1,\infty}, K_{1,\infty})$ irreducible subspace of this space, and let Π_1 be the isomorphism class of U_1. As in the proof of (2) $\Longrightarrow$ (1) of Theorem 8.3, to show $\Theta_{X_{D',d}}(U) \neq 0$ for some D' it will suffice to show $\Theta_{X_{D',d}}(U_1) \neq 0$ for some D'. To prove this, we will use Theorem 7.1 of [KRS]. This application requires an understanding the behavior of the partial twisted standard L-function $L^S(s, \Pi_1, \chi_{X_{D,d}})$ at $s = 1$; we now compute this L-function. As $U \cong \Pi$, $\Pi_{1,v}$ is an irreducible component of $\Pi_v|_{\mathrm{Sp}(n,F_v)}$ for all v. Let S be a finite set of places of F such that for $v \notin S$, v is finite, $X_{\mathrm{M}_{2\times 2},d}(F_v)$ is unramified (i.e., v is odd and v is unramified in E_d) and χ_v and τ_w for $w|v$ are unramified. For $v \notin S$, by Proposition 4.3 and its proof, $|\Pi([\pi_v])| = 1$, Π_v is the single element of $\Pi([\pi_v])$, Π_v is unramified and $\Pi_v = \theta_{\mathrm{M}_{2\times 2},d}(\sigma'_v)^\vee = \theta_{\mathrm{M}_{2\times 2},d}({\sigma'_v}^\vee)$, with $\sigma'_v = \pi_v^+ \in \mathrm{Irr}(\mathrm{GO}(X_{\mathrm{M}_{2\times 2},d}, F_v))$ unramified. Let $v \notin S$; we assert that there exists an unramified component $\sigma'_{1,v}$ of $\sigma'_v|_{\mathrm{O}(X_{\mathrm{M}_{2\times 2},d},F_v)}$ such that $\Pi_{1,v} = \theta({\sigma'_{1,v}}^\vee)$. To see this let, as in Section 1, $\mathrm{GSp}(2,F_v)^+$ be the subgroup of $g \in \mathrm{GSp}(2,F_v)$ such that $\lambda(g) \in \lambda(\mathrm{GO}(X_{D,d},F_v))$; again, $\mathrm{GSp}(2,F_v)^+$ has index one or two in $\mathrm{GSp}(2,F_v)$. Let $\Pi_v|_{\mathrm{GSp}(2,F_v)^+} = \Pi_v^1 \oplus \cdots \oplus \Pi_v^M$, where the $\Pi_v^i \in \mathrm{Irr}(\mathrm{GSp}(2,F_v)^+)$ are mutually nonisomorphic and $M = 1$ or 2. We have by construction

$$\mathrm{Hom}_{R_{X_{\mathrm{M}_{2\times 2},d}(F_v)}}(\omega_{X_{\mathrm{M}_{2\times 2},d}(F_v)}, \Pi_v \otimes {\sigma'_v}^\vee) \neq 0.$$

This implies that for some i,

$$\mathrm{Hom}_{R_{X_{\mathrm{M}_{2\times 2},d}(F_v)}}(\omega_{X_{\mathrm{M}_{2\times 2},d}(F_v)}, \Pi_v^i \otimes {\sigma_v'}^\vee) \neq 0.$$

By the proof of Proposition 1.11, Π_v^i is unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_v)$ (which is contained in $\mathrm{GSp}(2,F_v)^+$). As $\Pi_v|_{\mathrm{Sp}(2,F_v)}$ has only one irreducible component unramified with respect to $\mathrm{Sp}(2,\mathfrak{O}_v)$, namely $\Pi_{1,v}$, it follows that $\Pi_{1,v}$ is an irreducible component of $\Pi_v^i|_{\mathrm{Sp}(2,F_v)}$. By Lemma 4.2 of [R1], there exists an irreducible component $\sigma_{1,v}'$ of σ_v' such that

$$\mathrm{Hom}_{\mathrm{Sp}(2,F_v)\times \mathrm{O}(X_{\mathrm{M}_{2\times 2},d},F_v)}(\omega_{X_{\mathrm{M}_{2\times 2},d}(F_v)}, \Pi_{1,v} \otimes {\sigma_{1,v}'}^\vee) \neq 0.$$

By (b) of Theorem 7.1 of [H], $\sigma_{1,v}'$ is unramified. This proves our assertion. By Section 7 of [KR2] and Lemma 8.1 (or rather its proof), the twisted partial standard L-function of Π_1 now is

$$\begin{aligned} L^S(s,\Pi_1,\chi_{X_{D,d}}) &= \zeta_F^S(s)\prod_{v\notin S} L(s,\sigma_{1,v}') \\ &= \begin{cases} \zeta_F^S(s) L^S(s,\tau,\chi^{-1},\mathrm{Asai}) \text{ if } d\neq 1 \\ \zeta_F^S(s) L^S(s,\tau_1\times {\tau_2}^\vee) \text{ if } d=1 \text{ and } \tau\cong\tau_1\otimes\tau_2, \end{cases} \end{aligned}$$

where $\zeta_F^S(s)$ is the partial zeta function of F. We noted in the proof of Theorem 8.3 that L-functions of the type $L^S(s,\tau,\chi^{-1},\mathrm{Asai})$ or $L^S(s,\tau_1\times\tau_2^\vee)$ do not vanish at $s=1$; hence, $L^S(s,\Pi_1,\chi_{X_{D,d}})$ has a pole at $s=1$ (in fact, by Corollary 7.2.3 of [KR2] the pole must be simple).

Now we apply [KRS]. By Lemma 1.1 of [L], for some $f\in U_1$, f has a nonzero T-th Fourier coefficient with $\det T\neq 0$. Here, $T\in \mathrm{M}_2(F)$ is a symmetric matrix. Define a quadratic Hecke character χ of $\mathbb{A}^\times$ by $\chi_{X_{D,d}} = \chi_T\chi$, where we also write T for the two dimensional quadratic space defined by T. Since $L^S(s,\Pi_1,\chi_T\chi) = L^S(s,\Pi_1,\chi_{X_{D,d}})$ has a pole at $s=1$, by (i) and (ii) of Theorem 7.1 of [KRS], $\Theta_{X'}(U_1)\neq 0$, where $X' = X_T \perp X''$, with X'' some two dimensional quadratic space over F such that $\chi_{X''}=\chi$. We have

$$\chi_{X'} = \chi_{X_T}\cdot\chi_{X''} = \chi_{X_T}\cdot\chi = \chi_{X_T}^2\cdot\chi_{X_{D,d}} = \chi_{X_{D,d}}$$

which implies $\operatorname{disc} X'(F) = \operatorname{disc} X_{D,d}(F) = d$. By Proposition 2.8 and Lemma 5.1, we now know that $\Theta_{X_{D',d}}(U_1)\neq 0$ for some quaternion algebra D' over F. As mentioned, this implies $\Theta_{X_{D',d}}(U)\neq 0$.

Next, we claim $\Theta_{X_{D',d}}(U)$ is contained in the space of cusp forms on $\mathrm{GO}(X_{D',d},\mathbb{A})$ of central character ω_π; suppose not. Then by the remark after Proposition 5.2 there exists a two dimensional quadratic space X_0 over F such that $\Theta_{X_0}(U)$ is nonzero and is contained in the space of cusp forms of central character ω_π on $\mathrm{GO}(X_0,\mathbb{A})$. By a standard argument as in the proof of Proposition 5.3, for all but finitely many places v of F, $X_0(F_v)$ is unramified, Π_v is unramified with respect to $\mathrm{GSp}(2,\mathfrak{O}_v)$, there exists a unitary $\rho_v \in \mathrm{Irr}(\mathrm{GO}(X_0,F_v))$ which is unramified with respect to the stabilizer in $\mathrm{GO}(X_0,F_v)$ of a self-dual lattice and

$$\mathrm{Hom}_{R_{X_0}(F_v)}(\omega_{X_0(F_v)},\Pi_v^\vee\otimes\rho_v)\neq 0.$$

Let v be one such place. Let ρ_0 be an irreducible unramified component of $\rho_v|_{\mathrm{O}(X_0,F_v)}$. By Lemma 4.2 of [R1], there exists an irreducible component Π_0 of $\Pi_v^\vee|_{\mathrm{Sp}(2,F_v)}$ such that

$$\mathrm{Hom}_{\mathrm{Sp}(2,F_v)\times\mathrm{O}(X_0,F_v)}(\omega_{X_0(F_v)},\Pi_0\otimes\rho_0)\neq 0.$$

Now $\mathrm{SO}(X_0,F_v)$ is Abelian as $\dim X_0=2$; since ρ_0 is unitary, ρ_0 is therefore tempered. (Recall the definition of a tempered representation of $\mathrm{O}(X_0,F_v)$ preceding Theorem 1.2). Also, it is not difficult to show that $\rho_0\in\mathcal{R}_1(\mathrm{O}(X_0,F_v))$ (in fact, the only element of $\mathrm{Irr}(\mathrm{O}(X_0,F_v))$ not contained in $\mathcal{R}_1(\mathrm{O}(X_0,F_v))$ is sign). Applying now Theorem 4.4 of [R3], we conclude that Π_0 is not tempered, contradicting the temperedness of Π_v (see (3) of Proposition 4.1). We have shown $\Theta_{X_{D',d}}(U)\neq 0$ is nonzero and cuspidal for some D'; as promised, the argument used only that the cuspidal automorphic representation Π is contained in $\Pi([\pi])$.

Now we will show that $\Theta_{X_{D,d}}(U)$ is nonzero and contained in the space of cusp forms of central character ω_π. By Lemma 5.1, it will suffice to show that there is a similitude between $X_{D,d}(F)$ and $X_{D',d}(F)$. Let $B'=E\otimes_F D'$. We assert that $B\cong B'$ as E algebras. As in the last paragraph of Section 2, let $S_{D,E}$ be the set of places v of F such that v splits in E and $D(F_v)$ is ramified; define $S_{D',E}$ similarly. As observed in Section 2, it will suffice to show that $S_{D,E}=S_{D',E}$. Let v be a place of F that splits in E. As v splits in E, $d=1$ in $F_v^\times/F_v^{\times 2}$. By Proposition 5.3, since $\Theta_{X_{D',d}}(U)$ is nonzero and cuspidal, $\Pi_v\in\mathcal{R}_{X_{D',d}(F_v)}(\mathrm{GSp}(2,F_v))$; by construction, $\Pi_v\in\mathcal{R}_{X_{D,d}(F_v)}(\mathrm{GSp}(2,F_v))$. By Lemma 8.4 we must have $X_{D',d}(F_v)\cong X_{D,d}(F_v)$. This implies $D'(F_v)\cong D(F_v)$ so that D is ramified at v if and only if D' is ramified at v. This proves $S_{D,E}=S_{D',E}$. Since $B\cong B'$ as E algebras, by Proposition 2.9 there exists a similitude between $X_{D,d}(F)$ and $X_{D',d}(F)$.

We now apply Proposition 5.3 to conclude that the multiplicity of Π in W is the same as the multiplicity of $\Theta_{X_{D,d}}(U)$ in the space of cusp forms on $\mathrm{GO}(X_{D,d},\mathbb{A})$ of central character ω_π. By part of Theorem 7.1, this multiplicity is one.

To complete the proof we still must show that if $d=1$, $\Pi\in\Pi([\pi])$ and Π occurs in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$, then $|T_\Pi|$ is even. Let U be a realization of Π in the space of cusp forms on $\mathrm{GSp}(2,\mathbb{A})$ of central character ω_π. An argument just as above (which just used $\Pi\in\Pi([\pi])$ and nothing about the parity of $|T_\Pi|$) shows that $\Theta_{X_{D',1}}(U)$ is nonzero and cuspidal for some quaternion algebra D' over F. We claim that T_Π is exactly the set of places where D' is ramified; this will show that $|T_\Pi|$ is even. Let $v\in T_\Pi$. Then by the definition of T_Π, $\Pi_v\in\mathcal{R}_{X_{D_{\mathrm{ram}},1}}(\mathrm{GSp}(2,F_v))$. On the other hand, since $\Theta_{X_{D',1}}(U)$

is nonzero and cuspidal, $\Pi_v \in \mathcal{R}_{X_{D'(F_v),1}(F_v)}(\mathrm{GSp}(2,F_v))$ (Proposition 5.3). By Lemma 8.4, $X_{D_{\mathrm{ram}},1}(F_v) \cong X_{D'(F_v),1}(F_v)$, which implies $D'(F_v)$ is ramified. Suppose next $D'(F_v)$ is ramified. Again, $\Pi_v \in \mathcal{R}_{X_{D'(F_v),1}(F_v)}(\mathrm{GSp}(2,F_v))$. By Lemma 8.4 and the definition of $\Pi([\pi_v])$, we must have $v \in T_\Pi$. □

Finally, we prove Theorem 8.6. This result is essentially a restatement of Theorem 8.5, and will follow immediately from that theorem after we make some definitions.

First we make the definitions mentioned preceding the statement of Theorem 8.6 in the Introduction. Let F' be a local field of characteristic zero, and let E' be a quadratic extension of F' or $E' = F' \times F'$; if F' is archimedean, assume $F' = \mathbb{R}$ and $E' = \mathbb{R}\times\mathbb{R}$. If E' is a field, write $E' = F'(\sqrt{d})$; otherwise, let $d = 1$. Let $\tau' \in \mathrm{Irr}(\mathrm{GL}(2,E'))$ be infinite dimensional and assume the central character of τ' factors through $\mathrm{N}_{F'}^{E'}$ via χ'; if F' has even residual characteristic, assume additionally that τ' is tempered. By Proposition 2.7 the following sequence is exact:

$$1 \to E'^{\times} \to F'^{\times} \times \mathrm{GL}(2,E') \xrightarrow{\rho_{a(\mathrm{M}_{2\times 2},E')}} \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},F') \to 1.$$

Using this exact sequence, define $\pi' = \pi(\chi',\tau') \in \mathrm{Irr}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},F'))$ as in Section 3. Define $\varphi(\chi',\tau') = \varphi([\pi'])$ and $\Pi(\chi',\tau') = \Pi([\pi'])$, where $\varphi([\pi'])$ and $\Pi([\pi'])$ are defined as in Section 4. If $E' = F' \times F'$, define

$$\langle\cdot,\cdot\rangle_{F'} : \mathbb{S}(\varphi(\chi',\tau')) \times \Pi(\chi',\tau') \to \mathbb{C}$$

as follows. If $|\mathbb{S}(\varphi(\chi',\tau'))| = |\Pi(\chi',\tau')| = 1$ set $\langle\cdot,\cdot\rangle_{F'}$ to be identically 1; if $|\mathbb{S}(\varphi(\chi',\tau'))| = |\Pi(\chi',\tau')| = 2$ (see Propositions 4.1 and 4.2) then define $\langle\cdot,\theta_{\mathrm{M}_{2\times 2},1}(\pi'^{+})^{\vee}\rangle_{F'} = 1$ and let $\langle\cdot,\theta_{D_{\mathrm{ram}},1}(\pi'^{\mathrm{JL}+})^{\vee}\rangle_{F'}$ to be the nontrivial character of $\mathbb{S}(\varphi(\chi',\tau')) = Z_2$ (see Table 4). The claims from the Introduction concerning these definitions follow from Propositions 4.1, 4.2 and 4.3.

Next, let E, τ and χ be as in the statement of Theorem 8.6. If E is a field, write $E = F(\sqrt{d})$; otherwise, let $d = 1$. By Proposition 2.7 the following sequence is exact:

$$1 \to \mathbb{A}_E^{\times} \to \mathbb{A}^{\times} \times \mathrm{GL}(2,\mathbb{A}_E) \xrightarrow{\rho_{a(\mathrm{M}_{2\times 2},E)}} \mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}) \to 1.$$

Using this exact sequence, define $\pi = \pi(\chi,\tau) \in \mathrm{Irr}_{\mathrm{cusp}}^{\mathrm{temp}}(\mathrm{GSO}(X_{\mathrm{M}_{2\times 2},d},\mathbb{A}))$ as in Section 6.

Proof of Theorem 8.6. This follows from the definitions involved and Theorem 8.5. □

Appendix

	$p = q$	$p \neq q$
$\lambda(\mathrm{GO}(X,\mathbb{R}))$	$\mathbb{R}^\times$	$\mathbb{R}^\times_{>0}$
$[\mathrm{GSp}(n,\mathbb{R}) : \mathrm{GSp}(n,\mathbb{R})^+]$	1	2
$\mathrm{GSp}(n,\mathbb{R})^+$	$\mathrm{GSp}(n,\mathbb{R}) =$ $\mathbb{R}^\times(\mathrm{Sp}(n,\mathbb{R}) \cup \mathrm{Sp}(n,\mathbb{R})k_0)$	$\mathrm{Sp}(n,\mathbb{R})\mathbb{R}^\times$
K^+	$K = K_1 \cup K_1 k_0$	K_1
$\mathrm{GO}(X,\mathbb{R})$	$\mathbb{R}^\times(\mathrm{O}(X,\mathbb{R}) \cup \mathrm{O}(X,\mathbb{R})j_0)$	$\mathrm{O}(X,\mathbb{R})\mathbb{R}^\times$
J	$J_1 \cup J_1 j_0$	J_1
L	$(K_1 \times J_1) \cup (K_1 \times J_1)(k_0, j_0)$	$K_1 \times J_1$

TABLE 1

<table>
<tr><td colspan="2">$d \neq 1$</td></tr>
<tr><td rowspan="2">π regular or distinguished</td><td>π regular: $\pi \to \pi^+ \to \theta_{\mathrm{M}_{2\times 2},d}(\pi^+)$</td></tr>
<tr><td>π distinguished: $\pi \begin{array}{l} \nearrow \pi^+ \to \theta_{\mathrm{M}_{2\times 2},d}(\pi^+) \\ \searrow \pi^- \text{ does not lift to } \mathrm{GSp}(2,F) \end{array}$</td></tr>
<tr><td>π invariant but not distinguished</td><td>$\pi \begin{array}{l} \nearrow \pi_1 \to \theta_{\mathrm{M}_{2\times 2},d}(\pi_1) \\ \searrow \pi_2 \to \theta_{\mathrm{M}_{2\times 2},d}(\pi_2) \end{array}$</td></tr>
</table>

TABLE 2

<table>
<tr><td colspan="2">d=1</td></tr>
<tr><td rowspan="2">π not essentially square integrable</td><td>π regular: $\pi \to \pi^+ \to \theta_{M_{2\times 2},1}(\pi^+)$</td></tr>
<tr><td>π invariant and hence distinguished: $$\begin{tikzcd} & \pi^+ \arrow[r] & \theta_{M_{2\times 2},1}(\pi^+) \\ \pi \arrow[ur] \arrow[dr] & & \\ & \pi^- & \text{does not lift to GSp}(2,F) \end{tikzcd}$$</td></tr>
<tr><td rowspan="2">π essentially square integrable</td><td>π regular: $$\begin{tikzcd} \pi \arrow[r] \arrow[d] & \pi^+ \arrow[r] & \theta_{M_{2\times 2},1}(\pi^+) \\ \pi^{JL} \arrow[r] & \pi^{JL+} \arrow[r] & \theta_{D_{ram},1}(\pi^{JL+}) \end{tikzcd}$$</td></tr>
<tr><td>π invariant and hence distinguished: $$\begin{tikzcd} & \pi^+ \arrow[r] & \theta_{M_{2\times 2},1}(\pi^+) \\ \pi \arrow[ur] \arrow[r] \arrow[d] & \pi^- & \text{does not lift to GSp}(2,F) \\ \pi^{JL} \arrow[r] \arrow[dr] & \pi^{JL+} \arrow[r] & \theta_{D_{ram},1}(\pi^{JL+}) \\ & \pi^{JL-} & \text{does not lift to GSp}(2,F) \end{tikzcd}$$</td></tr>
</table>

TABLE 3

d	$[\pi]$	$\Pi([\pi])$
1	π not essentially square integrable	$\{\theta_{M_{2\times 2},1}(\pi^+)^\vee\}$
1	π essentially square integrable	$\{\theta_{M_{2\times 2},1}(\pi^+)^\vee, \theta_{D_{ram},1}(\pi^{JL+})^\vee\}$
$\neq$1	π regular or invariant and distinguished	$\{\theta_{M_{2\times 2},d}(\pi^+)^\vee\}$
$\neq$1	π invariant but not distinguished	$\{\theta_{M_{2\times 2},d}(\pi_1)^\vee, \theta_{M_{2\times 2},d}(\pi_2)^\vee\}$

TABLE 4

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Compactification of the Bruhat-Tits Building of PGL by Lattices of Smaller Rank

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Abstract. In this paper we construct a compactification of the Bruhat-Tits building associated to the group $PGL(V)$ by attaching all the Bruhat-Tits buildings of $PGL(W)$ for non-trivial subspaces W of V as a boundary.

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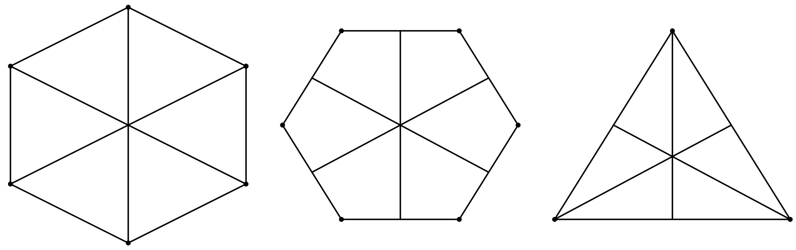
1 Introduction

Let K be a non-archimedean local field and V a finite-dimensional vector space over K. In this paper we construct a compactification of the Bruhat-Tits building X associated to the group $PGL(V)$ by attaching all the Bruhat-Tits buildings of $PGL(W)$ for non-trivial subspaces W of V as a boundary. Since the vertices of such a building correspond to homothety classes of lattices of full rank in W, we can also view this process as attaching to X (whose underlying simplicial complex is defined by lattices of full rank in V) all the lattices in V of smaller rank.

This compactification differs from both the Borel-Serre compactification and Landvogt's polyhedral compactification of X. The different features of these three constructions can be illustrated in the case of a three-dimensional V by looking at the compactification of one apartment:



Borel-Serre Landvogt Our compactification

In the Borel-Serre compactification, the points on the boundary correspond to the rays, i.e. to the half-lines emanating from the origin, and parallel lines have the same limits. In Landvogt's compactification, parallel lines have different limit points, whereas the rays in one segment (chamber) all converge to the corner vertex. In our compactification all rays contained in the two chambers around a boundary vertex converge to this vertex, and so do lines which are parallel to the middle axis. The two rays at the boundary of this double chamber (they look shorter in our picture) converge to points on the boundary lines, and their parallels converge to different points on these lines.
The idea to attach lattices of smaller rank to compactify X already appeared in Mustafin's paper [Mu]. The goal of this paper is a generalization of Mumford's p-adic Schottky uniformization to higher dimensions. Mustafin's construction and investigation of the compactified building take up about one page. He works with lattices and defines the compactification as the union of X and all the lattices in V of smaller rank, i.e. he only uses a set of vertices as the boundary. His construction remains rather obscure (at least to the author), and does not include proofs.
The construction of our compactification is based on the same idea of attaching lattices of smaller rank, but is entirely different. First we compactify one apartment Λ in X (corresponding to a maximal torus T in $PGL(V)$) by attaching some apartments of lower dimension corresponding to certain tori which are quotients of T. We define a continuous action of the normalizer N of T on this compactification $\overline{\Lambda}$. Then we glue all the compactified apartments corresponding to maximal tori in $PGL(V)$ together. To be precise, we take a certain compact subgroup $U_0^\wedge$ in G, and we define for each $x \in \overline{\Lambda}$ a subgroup P_x of G, which turns out later to be the stabilizer of x. Our compactification $\overline{X}$ is defined as the quotient of $U_0^\wedge \times \overline{\Lambda}$ by the following equivalence relation: $(g,x) \sim (h,y)$ iff there exists some n in N such that $nx = y$ and $g^{-1}hn$ lies in P_x. This is similar to the construction of the building X.
Then we prove the following results: X is an open, dense subset of $\overline{X}$, and $\overline{X}$ carries a G-action compatible with the one on X. Besides, $\overline{X}$ is compact and contractible and can be identified with the union of all Bruhat-Tits buildings corresponding to non-zero subspaces W of V.
In order to prove these results, we have to investigate in detail the structure of our stabilizer groups P_x. In particular, we show a mixed Bruhat decomposition theorem for them.

It is of course a natural question whether a similar compactification also exists for other groups. At present, I can see no generalization of this approach to arbitrary reductive groups, but probably some other cases can be treated individually. In order to facilitate such generalizations we work in a group theoretic set up where possible, and do not use the realization of X as the space of norms up to similarity.
Moreover, it would be interesting to see if there exists an analogue of our construction in the archimedean world of symmetric spaces.
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2 The Bruhat-Tits building for PGL

Throughout this paper we denote by K a non-archimedean local field, by R its valuation ring and by k the residue class field. Besides, v is the valuation map, normalized so that it maps a prime element to 1.
We adopt the convention that "$\subset$" always means strict subset, whereas we write "$\subseteq$", if equality is permitted.
Let V be an n-dimensional vector space over K. Let us recall the definition of the Bruhat-Tits building for the group $\mathbf{G} = PGL(V)$ (see [Br-Ti] and [La]).
We fix a maximal K-split torus $\mathbf{T}$ and let $\mathbf{N} = N_G \mathbf{T}$ be its normalizer. Note that $\mathbf{T}$ is equal to its centralizer in $\mathbf{G}$. We write $G = \mathbf{G}(K)$, $T = \mathbf{T}(K)$ and $N = \mathbf{N}(K)$ for the groups of rational points. By $X_*(\mathbf{T})$ respectively $X^*(\mathbf{T})$ we denote the cocharacter respectively the character group of $\mathbf{T}$. We have a natural perfect pairing

$$\begin{array}{rll} <,>: & X_*(\mathbf{T}) \times X^*(\mathbf{T}) & \longrightarrow \mathbb{Z} \\ & (\lambda, \chi) & \longmapsto < \lambda, \chi >, \end{array}$$

where $< \lambda, \chi >$ is the integer such that $\chi \circ \lambda(t) = t^{<\lambda,\chi>}$ for all $t \in \mathbb{G}_m$. Let Λ be the $\mathbb{R}$-vector space $\Lambda = X_*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. We can identify the dual space Λ^* with $X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, and extend $<,>$ to a pairing

$$<,>: \Lambda \times \Lambda^* \longrightarrow \mathbb{R}.$$

Since $<,>$ is perfect, there exists a unique homomorphism $\nu : T \to \Lambda$ such that

$$< \nu(z), \chi >= -v(\chi(z))$$

for all $z \in T$ and $\chi \in X^*(\mathbf{T})$ (compare [La], Lemma 1.1). Besides, by [La], Proposition 1.8, there exists an affine Λ-space A together with a homomorphism $\nu : N \to \mathrm{Aff}(A)$ extending $\nu : T \to \Lambda$. Here $\mathrm{Aff}(A)$ denotes the space of affine bijections $A \to A$. The pair (A, ν) is unique up to unique isomorphism. It is called the empty appartment defined by $\mathbf{T}$.
Let g be the Lie algebra of $\mathbf{G}$. We have the root decomposition

$$g = g^T \oplus \bigoplus_{a \in \Phi} g_a,$$

where $\Phi = \Phi(\mathbf{T}, \mathbf{G})$ is the set of roots and where $g^T = \{X \in g : \mathrm{Ad}(t)X = X \text{ for all } t \in T\}$ and $g_a = \{X \in g : \mathrm{Ad}(t)X = a(t)X \text{ for all } t \in T\}$ for all $a \in \Phi$ (see [Bo], 8.17 and 21.1). By [Bo], 21.6, Φ is a root system in Λ^* with Weyl group $W = N/T$. For all $a \in \Phi$ there exists a unique closed, connected, unipotent subgroup $\mathbf{U}_a$ of $\mathbf{G}$ which is normalized by $\mathbf{T}$ and has Lie algebra g_a (see [Bo], 21.9). We denote the K-rational points of $\mathbf{U}_a$ by U_a.
In our case $\mathbf{G} = PGL(V)$ we can describe these data explicitly. Our torus $\mathbf{T}$ is the image of a maximal split torus $\mathbf{T}^\sim$ in $GL(V)$. Hence there exists a basis $v_1, \ldots, v_n$ of V such that $\mathbf{T}^\sim$ is the group of diagonal matrices in $GL(V)$ with respect to $v_1, \ldots, v_n$. From now on we will fix such a basis. Let $\mathbf{N}^\sim$ be the normalizer of $\mathbf{T}^\sim$ in $GL(V)$. Then $\mathbf{N}$ is the image of $\mathbf{N}^\sim$ in $PGL(V)$. Hence N is the semidirect product of T and the group of permutation matrices, which is isomorphic to $W = N/T$.
Since W is the Weyl group corresponding to Φ, it acts as a group of reflections on Λ, and we have a natural homomorphism

$$W \longrightarrow GL(\Lambda).$$

Since $\mathrm{Aff}(\Lambda) = \Lambda \rtimes GL(\Lambda)$, we can use this map together with $\nu : T \to \Lambda$ to define

$$\nu : N = T \rtimes W \longrightarrow \Lambda \rtimes GL(\Lambda) = \mathrm{Aff}(\Lambda).$$

Hence (Λ, ν) is an empty apartment, and we write from now on $A = \Lambda$.
Denote by χ_i the character

$$\begin{array}{rcl} \chi_i : & \mathbf{T}^\sim & \longrightarrow \mathbb{G}_m \\ & \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} & \longmapsto t_i. \end{array}$$

Then for all i and j we have characters $a_{ij} := \chi_i - \chi_j$, and

$$\Phi = \{a_{ij} : i \neq j\}.$$

For $a = a_{ij}$ we define now $\mathbf{U}_a^\sim$ as the subgroup of $GL(V)$ such that $\mathbf{U}_a^\sim(\overline{K})$ is the group of matrices $U = (u_{kl})_{k,l}$ such that the diagonal elements u_{kk} are equal to one, u_{ij} is an element in $\overline{K}$ and the rest of the entries u_{kl} is zero. Its image in $PGL(V)$ is isomorphic to $\mathbf{U}_a^\sim$ and coincides with the group $\mathbf{U}_a$. Define

$$\psi_a : U_a \longrightarrow \mathbb{Z} \cup \{\infty\}$$

by mapping the matrix $U = (u_{kl})_{k,l}$ to $v(u_{ij})$. Then we put for all $l \in \mathbb{Z}$

$$U_{a,l} = \{u \in U_a : \psi_a(u) \geq l\}.$$

We also define $U_{a,\infty} = \{1\}$, and $U_{a,-\infty} = U_a$. An affine function $\theta : \Lambda \to \mathbb{R}$ of the form $\theta(x) = a(x) + l$ for some $a \in \Phi$ and some $l \in \mathbb{Z}$ is called an affine root. We can define an equivalence relation $\sim$ on Λ as follows:

$$x \sim y \quad \text{iff} \quad \begin{array}{l} \theta(x) \text{ and } \theta(y) \text{ have the same sign} \\ \text{or are both equal to } 0 \text{ for all affine roots } \theta. \end{array}$$

The equivalence classes with respect to this relation are called the faces of Λ. These faces are simplices which partition Λ (see [Bou], V, 3.9). There exists a W-invariant scalar product on Λ (uniquely determined up to scalar factor), see [Bou], VI, 1.1 and 1.2, and all the reflections at affine hyperplanes are contained in $\nu(N)$.

For all $x \in \Lambda$ let U_x be the group generated by $U_{a,-a(x)} = \{u \in U_a : \psi_a(u) \geq -a(x)\}$ for all $a \in \Phi$. Besides, put $N_x = \{n \in N : \nu(n)x = x\}$, and

$$P_x = U_x N_x = N_x U_x.$$

Now we are ready to define the building $X = X(PGL(V))$ as

$$X = G \times \Lambda / \sim,$$

where the equivalence relation $\sim$ is defined as follows (see [La], 13.1):

$$(g,x) \sim (h,y) \qquad \begin{array}{l} \text{iff there exists an element } n \in N \\ \text{such that } \nu(n)x = y \text{ and } g^{-1}hn \in P_x. \end{array}$$

We have a natural action of G on X via left multiplication on the first factor. The G-action on X continues the N-action on Λ, so that we will write nx for our old $\nu(n)x$ if $x \in \Lambda$ and $n \in N$. Besides, we can embed the apartment Λ in X, mapping $a \in \Lambda$ to the class of $(1,a)$. (This is injective, see [La], Lemma 13.2.) For $x \in \Lambda$ the group P_x is the stabilizer of x. A subset of X of the form $g\Lambda$ for some $g \in G$ is called apartment in X. Similarly, we define the faces in $g\Lambda$ as the subsets gF, where F is a face in Λ. Then two points (and even two faces) in X are always contained in a common apartment ([La], Proposition 13.12 and [Br-Ti], 7.4.18). Any apartment which contains a point of a face contains the whole face, and even its closure (see [La], 13.10, 13.11, and [Br-Ti], 7.4.13, 7.4.14). We fix once and for all a W-invariant scalar product on Λ, which induces a metric on Λ. Using the G-action it can be continued to a metric d on the whole of X (see [La], 13.14 and [Br-Ti], 7.4.20).

Note that if $n = 2$, then X is an infinite regular tree, with $q+1$ edges meeting in every vertex, where q is the cardinality of the residue field.

We denote by X^0 the set of vertices (i.e. 0-dimensional faces) in X. We define a simplex in X^0 to be a subset $\{x_1, \ldots, x_k\}$ of X^0 such that $x_1, \ldots, x_k$ are the vertices of a face in X.

Let $\eta_i : \mathbb{G}_m \to \mathbf{T}$ be the cocharacter induced by mapping x to the diagonal matrix with diagonal entries $d_1, \ldots, d_n$ such that $d_k = 1$ for $k \neq i$ and $d_i = x$. Then $\eta_1, \ldots, \eta_{n-1}$ is an $\mathbb{R}$-basis of Λ, and the set of vertices in Λ is equal to $\bigoplus_{i=1}^{n-1} \mathbb{Z}\eta_i$.

Let $\mathcal{L}$ be the set of all homothety classes of R-lattices of full rank in V. We write $\{M\}$ for the class of a lattice M. Two different lattice classes $\{M'\}$ and $\{N'\}$ are called adjacent, if there are representatives M and N of $\{M'\}$ and $\{N'\}$ such that

$$\pi N \subset M \subset N.$$

This relation defines a flag complex, namely the simplicial complex with vertex set $\mathcal{L}$ such that the simplices are the sets of pairwise adjacent lattice classes. We have a natural G-action on $\mathcal{L}$ preserving the simplicial structure.
Moreover, there is a G-equivariant bijection

$$\varphi : \mathcal{L} \longrightarrow X^0$$

preserving the simplicial structures. If $\{N\} \in \mathcal{L}$ can be written as $\{N\} = g\{M\}$ for some $g \in G$ and $M = \pi^{k_1} R v_1 + \ldots + \pi^{k_n} R v_n$, then $\varphi(\{N\})$ is given by the pair $(g, \varphi\{M\}) \in G \times \Lambda$, where

$$\varphi(\{M\}) = \sum_{i=1}^{n-1} (k_n - k_i) \eta_i$$

is a vertex in Λ.

3 Compactification of one apartment

We write $\underline{n}$ for the set $\{1, \ldots, n\}$. We continue to fix the basis $v_1, \ldots, v_n$ of V and the maximally split torus $\mathbf{T}$ from section 2. Recall that $\Lambda = \bigoplus_{i=1}^{n-1} \mathbb{R}\eta_i$, and that η_n satisfies the relation $\eta_1 + \ldots + \eta_n = 0$. We will often write $\Lambda = \sum_{i=1}^n \mathbb{R}\eta_i$, bearing this relation in mind.
Let now I be a non-empty subset of $\underline{n}$, and let V_I be the subspace of V generated by the v_i for $i \in I$. We write $\mathbf{G}^{V_I}$ for the subgroup of $\mathbf{G} = PGL(V)$ consisting of the elements fixing the subspace V_I, and $\mathbf{G}_I$ for the group $PGL(V_I)$. Then we have a natural restriction map

$$\rho_I : \mathbf{G}^{V_I} \longrightarrow \mathbf{G}_I.$$

The torus $\mathbf{T}$ is contained in $\mathbf{G}^{V_I}$, and its image under ρ_I is a maximal K-split torus $\mathbf{T}_I$ in $\mathbf{G}_I$, namely the torus induced by the diagonal matrices with respect to the base $\{v_i : i \in I\}$ of V_I. As usual, we write T_I, G_I and G^{V_I} for the groups of K-rational points.
We put $\Lambda_I = X_*(\mathbf{T}_I) \otimes_{\mathbb{Z}} \mathbb{R}$. Then ρ_I induces a surjective homomorphism $\rho_{I*} : X_*(\mathbf{T}) \to X_*(\mathbf{T}_I)$, hence a surjective homomorphism of $\mathbb{R}$-vector spaces

$$r_I : \Lambda \longrightarrow \Lambda_I.$$

For all $i \in I$ we write η_i^I for the cocharacter of $\mathbf{T}_I$ induced by mapping x to the diagonal matrix with entry x at the i-th place, i.e. $\eta_i^I = \rho_{I*}\eta_i$.

Then $\Lambda_I = \sum_{i\in I} \mathbb{R}\eta_i^I$, subject to the relation $\sum_{i\in I}\eta_i^I = 0$. In particular, $\Lambda_{\{i\}} = 0$. Note that $r_I(\sum_{i=1}^n x_i\eta_i) = \sum_{i\in I} x_i\eta_i^I$.
Let $\nu_I : T_I \to \Lambda_I$ be the unique homomorphism satisfying $< \nu_I(z), \chi >= -v(\chi(z))$ for all $\chi \in X^*(\mathbf{T}_I)$. It is compatible with ν, i.e. the following diagram is commutative:

$$\begin{array}{ccc} T & \xrightarrow{\nu} & \Lambda \\ {\scriptstyle\rho_I}\downarrow & & \downarrow{\scriptstyle r_I} \\ T_I & \xrightarrow{\nu_I} & \Lambda_I \end{array}$$

Now we define

$$\overline{\Lambda} = \Lambda \cup \bigcup_{\emptyset \neq I \subset \underline{n}} \Lambda_I = \bigcup_{\emptyset\neq I\subseteq \underline{n}} \Lambda_I.$$

Recall that we write "$\subset$" for a strict subset, and "$\subseteq$" if equality is permitted. Here $\Lambda_{\underline{n}} = \Lambda$ and $r_{\underline{n}}$ is the identity.
Let us now define a topology on $\overline{\Lambda}$. For all $I \subset \underline{n}$ we put

$$D_I = \sum_{i\notin I} \mathbb{R}_{\geq 0}(-\eta_i).$$

We think of D_I as a "corner" around Λ_I. For all open and bounded subsets $U \subset \Lambda$ we define

$$C_U^I = (U + D_I) \cup \bigcup_{I\subseteq J\subset \underline{n}} r_J(U + D_I).$$

We take as a base of our topology on $\overline{\Lambda}$ the open subsets of Λ together with these sets C_U^I for all non-empty $I \subset \underline{n}$ and all open bounded subsets U of Λ.
Note that every point $x \in \overline{\Lambda}$ has a countable fundamental system of neighborhoods. This is clear for $x \in \Lambda$. If x is in Λ_I for some $I \subset \underline{n}$, then choose some $z \in \Lambda$ with $r_I z = x$, and choose a countable decreasing fundamental system of bounded open neighborhoods $(V_k)_{k\geq 1}$ of z in Λ. Put $U_k = V_k + \sum_{i\notin I} k(-\eta_i)$. This is an open neighborhood of $z + k\sum_{i\notin I}(-\eta_i)$. Then $(C^I_{U_k})_{k\geq 1}$ is a fundamental system of open neighborhoods of x.
Hopefully the next result will shed some light on the definition of the topological space $\overline{\Lambda}$.
Recall from section 2, that we have a G-equivariant bijection φ between equivalence classes of lattices of full rank in V and vertices in the building X. If we restrict φ to lattices which can be diagonalized with respect to $v_1, \dots, v_n$, i.e. which have an R-basis consisting of multiples of these elements, then we get a bijection between these diagonal lattices and vertices in Λ. Applying this to the group $\mathbf{G}_I = PGL(V_I)$, we get a bijection φ_I between classes of diagonal lattices in V_I with respect to the v_i for $i \in I$, and vertices in Λ_I.

Proposition 3.1 *Let $(M_k)_{k\geq 1}$ be a sequence of diagonal lattices in V and let N be a diagonal lattice in V_I. The sequence of vertices $\varphi(\{M_k\})$ converges to the* *vertex $\varphi_I(\{N\}) \in \Lambda_I$ in our topology on $\overline{\Lambda}$ iff after passing to a subsequence there are lattices M'_k equivalent to M_k such that $M'_{k+1} \subseteq M'_k$ and such that $\bigcap_k M'_k$ is equivalent to N.*

PROOF: We can write $M'_k = \bigoplus_{i=1}^n \pi^{a_{i,k}} R v_i$ for some integers $a_{i,k}$. Since $M'_{k+1} \subset M'_k$, we have $a_{i,k+1} \geq a_{i,k}$. Therefore for all i the sequence $a_{i,k}$ becomes stationary or goes to infinity, so that $\bigcap_k M'_k = \bigoplus_{i \in I'} \pi^{a_i} R v_i$, where I' is the set of all i, such that $a_{i,k}$ becomes stationary, i.e. $a_{i,k} = a_i$ for all k big enough. Let us call this intersection module N'. It is a lattice in $V_{I'}$, and by assumption equivalent to N, so that $I = I'$.
Besides, we have $\varphi(\{M'_k\}) = \sum_{i=1}^n (-a_{i,k}) \eta_i$, and $\varphi_I(\{N'\}) = \sum_{i \in I} (-a_i) \eta_i^I$. If k is big enough, we have $\varphi(\{M'_k\}) = \sum_{i \in I} (-a_i) \eta_i + \sum_{i \notin I} (-a_{i,k}) \eta_i$, with $a_{i,k}$ arbitrarily large. If we take one of the systems of fundamental neighborhoods of $\varphi_I(\{N\}) = \varphi_I(\{N'\})$ constructed previously, we find that every one of them must contain a point $\varphi(\{M'_k\})$, so that $\varphi(\{M_k\})$ converges indeed to $\varphi_I(\{N\})$.
To prove the other direction, assume that $\varphi(\{M_k\})$ converges to $\varphi_I(\{N\})$ for $M_k = \bigoplus_{i=1}^n \pi^{a_{i,k}} R v_i$ and $N = \bigoplus_{i \in I} \pi^{b_i} R v_i$. Looking at the fundamental neighborhoods as above, we find that for any fixed $i_0 \in I$ the sequence $a_{i,k} - a_{i_0,k}$ is unbounded for $i \notin I$, and goes to $b_i - b_{i_0}$ for $i \in I$. This implies our claim. □

We will now show that the space $\overline{\Lambda}$ is compact.
Fix some $i \in \underline{n}$. We write D_i for $D_{\{i\}} = \sum_{j \neq i} \mathbb{R}_{\geq 0}(-\eta_j)$, the "corner in Λ around the point $\Lambda_{\{i\}}$". Besides, let

$$E_i = D_i \cup \bigcup_{i \in J \subset \underline{n}} r_J(D_i) \subset \overline{\Lambda},$$

the "closed corner in $\overline{\Lambda}$ around the point $\Lambda_{\{i\}}$".

LEMMA 3.2 *i) Each point in $\overline{\Lambda}$ lies in one of the E_i.*
ii) Each E_i is closed in $\overline{\Lambda}$.

PROOF: i) Let $x = \sum_{j=1}^n x_j(-\eta_j)$ be a point in Λ. Note that the relation $\sum_{j=1}^n \eta_j = 0$ implies that we can write $x = \sum_{j \neq i} (x_j - x_i)(-\eta_j)$ for all i. Now $E_i \cap \Lambda = D_i$ is the set of all x such that all $x_j - x_i$ are non-negative. In other words, a point $x = \sum_{j=1}^n x_j(-\eta_j)$ is in D_i iff x_i is the minimum of all the coefficients x_j. This implies that for given x we always find some E_i containing it. As similar argument holds if x is contained in a boundary piece Λ_J, since if J contains i, the boundary piece $E_i \cap \Lambda_J$ is the set of all $x = \sum_{j \in J} x_j(-\eta_j^J)$ such that x_i is the minimum of all the x_j for $j \in J$. (If i is not contained in J, then of course $E_i \cap \Lambda_J$ is empty.)
ii) Take some $x \in \overline{\Lambda}$ not contained in E_i. Then x is in some Λ_J for $J \subseteq \underline{n}$ (possibly $\underline{n}$). Since the point $\Lambda_{\{i\}}$ is contained in E_i, we know that $J \neq \{i\}$. Let us first assume that i is contained in J. We write $x = \sum_{j \in J, j \neq i} x_j(-\eta_j^J)$. Since x is not in E_i, our considerations in part i) imply that one of the x_j for $j \in J$, say x_{j_0}, must be negative. The point $z = \sum_{j \in J, j \neq i} x_j(-\eta_j)$ in Λ

projects to x, i.e. $r_J(z) = x$. This point must also be in the complement of E_i. Since $E_i \cap \Lambda = D_i$ is closed, we find a bounded open neighborhood U of z in Λ which is disjoint from E_i, and which contains only points $y = \sum_{j \neq i} y_j(-\eta_j)$ with $y_{j_0} < 0$. Then the open neighborhood C_U^J of x is also disjoint from E_i, which proves our claim.

If $i \notin J$ and $x = \sum_{j \in J} x_j(-\eta_j^J)$, we choose some $j_0 \in J$ and some $x_i > x_{j_0}$. The point $z = \sum_{j \in J} x_j(-\eta_j) + x_i(-\eta_i) \in \Lambda$ is not contained in E_i. Hence we can again find a bounded open neighborhood U of z in Λ which is disjoint from E_i, and which contains only points $y = \sum_{j \neq i} y_j(-\eta_j)$ with $y_{j_0} < 0$. Then C_U^J is an open neighborhood of x contained in the complement of E_i. $\square$

Let $R_{\geq 0, \infty}$ be the compactified half-line $\mathbb{R}_{\geq 0} \cup \{\infty\}$ with the topology generated by all intervals $[0, a[$, $]b, c[$ and $]b, \infty]$ for $a > 0$ and $b, c \geq 0$. The space $\mathbb{R}_{\geq 0, \infty}$ is compact and contractible. A contraction map $r : \mathbb{R}_{\geq 0, \infty} \times [0, 1] \to \mathbb{R}_{\geq 0, \infty}$ is given by

$$r(x,t) = \frac{(1-t)x}{1+tx} \text{ for } x \in \mathbb{R}, \quad \text{and} \quad r(\infty, t) = \begin{cases} \infty, & \text{if } t = 0 \\ \frac{1-t}{t}, & \text{if } t \neq 0 \end{cases},$$

see [La], 2.1. Let us fix some $i \in \underline{n}$. We will now compare E_i to $\mathbb{R}^{n-1}_{\geq 0, \infty}$, which we write as $\bigoplus_{j \neq i} \mathbb{R}_{\geq 0, \infty} e_j$ for a basis e_j. Recall from the proof of Lemma 3.2 that in the case $i \in I$ we can describe $E_i \cap \Lambda_I$ as the set of all $x = \sum_{j \in I, j \neq i} x_j(-\eta_j^I)$ with non-negative x_j. Hence the following map is a bijection

$$\begin{array}{llll} \alpha_I : & E_i \cap \Lambda_I & \to & \{\sum\limits_{j \neq i} x_j e_j \in \bigoplus\limits_{j \neq i} \mathbb{R}_{\geq 0, \infty} e_j : x_j = \infty \text{ iff } j \notin I\} \\ & \sum_{j \in I, j \neq i} x_j(-\eta_j^I) & \mapsto & \sum\limits_{j \in I, j \neq i} x_j e_j + \sum\limits_{j \notin I} \infty\, e_j. \end{array}$$

For $I = \underline{n}$ the map $\alpha_{\underline{n}} : E_i \cap \Lambda = D_i \to \bigoplus_{j \neq i} \mathbb{R}_{\geq 0} e_j$ can be continued to a homomorphism of $\mathbb{R}$-vector spaces

$$\alpha_\Lambda : \Lambda \longrightarrow \bigoplus_{j \neq i} \mathbb{R} e_j,$$

which is a homeomorphism. Putting all the maps α_I together, we get a bijection

$$\alpha : E_i \longrightarrow \bigoplus_{j \neq i} \mathbb{R}_{\geq 0, \infty} e_j,$$

whose restriction to $E_i \cap \Lambda$ is a homeomorphism. We even have the following fact:

Lemma 3.3 *With respect to the topology on E_i induced by $\overline{\Lambda}$, the map α is a homeomorphism on the whole of E_i.*

Proof: For all $j \neq i$ choose an open interval A_j in $\mathbb{R}_{\geq 0, \infty}$, which is either of the form $A_j = [0, a_j[$ or $A_j =]b_j, c_j[$ or of the form $A_j =]b_j, \infty]$. We claim that the preimage of $A = \sum_{j \neq i} A_j e_j$ is open in E_i.

We put $A'_j =]-1, a_j[$, if $A_j = [0, a_j[$. In all the other cases we put $A'_j = A_j$. Let $A' = \sum_{j \neq i} A'_j e_j$ and put

$$W = \alpha_\Lambda^{-1}(A' \cap \bigoplus_{j \neq i} \mathbb{R} e_j).$$

Since α_Λ is a homeomorphism, W is open in Λ. Obviously, we have $W \cap E_i = \alpha^{-1}(A) \cap \Lambda$. Now put

$$I = \{j \in \underline{n} : \infty \notin A_j\} \cup \{i\}.$$

We can assume that $I \neq \underline{n}$. Choose some positive real number b strictly bigger than all the b_j for $j \notin I$. Then $U = W \cap \{x = \sum_{j \neq i} x_j(-\eta_j) : x_j < b \text{ for } j \notin I\}$ is an open bounded subset of Λ. Note that $U + D_I = W$. We claim that $\alpha^{-1}(A) = C_U^I \cap E_i$.

Indeed, every element u in $W = U + D_I$ can be written as $u = \sum_{j \neq i} x_j(-\eta_j)$ with $x_j \in A'_j$. Let J be a subset of $\underline{n}$ containing I. If $r_J(u) = \sum_{j \in J, j \neq i} x_j(-\eta_j^J)$ is in E_i, we have $x_j \in A_j$ for all $j \in J$ not equal to i. This implies that $\alpha(r_J(u))$ is contained in A. On the other hand, let $y = \sum_{j \neq i} y_j e_j$ be an element in A, i.e. $y_j \in A_j$. Put $J = \{j \neq i : y_j \neq \infty\} \cup \{i\}$. Then J contains I. We put $x_j = y_j$, if $j \neq i$ is in J. If $j \notin J$, we choose an arbitrary element in $A_j \cap \mathbb{R}$ and call it x_j. Then $x = \sum_{j \neq i} x_j(-\eta_j)$ is contained in $W \cap D_i$, so that $r_J(x)$ is an element in $C_U^I \cap E_i$ which satisfies $\alpha(r_J(x)) = y$. Hence we also find $\alpha^{-1}(A) \subset C_U^I \cap E_i$.

Therefore α is continuous. It remains to show that α is open. Let U be an open, bounded subset of Λ and $I \subset \underline{n}$ non-empty. We will show that $\alpha(C_U^I \cap E_i)$ is open. Let x be a point in $C_U^I \cap E_i$ lying in Λ_J for some J containing I and i. Hence $x = \sum_{j \in J, j \neq i} x_j(-\eta_j^J)$ with non-negative x_j. Since x is contained in C_U^I, we can find some $z = \sum_{j \neq i} z_j(-\eta_j)$ in $U + D_I$ such that $r_J(z) = x$ (hence $z_j \geq 0$ for $j \in J$) and $z_j > 0$ for $j \notin J$. Then $z \in (U + D_I) \cap E_i$.

Since the restriction of α to $E_i \cap \Lambda$ is open, we find open intervals A_j (of the form $[0, a_j[$ or $]b_j, c_j[$) in $\mathbb{R}_{\geq 0}$ such that $A = \sum_{j \neq i} A_j e_j$ is an open neighborhood of $\alpha(z)$ in $\bigoplus_{j \neq i} \mathbb{R}_{\geq 0} e_j$ which is contained in $\alpha((U + D_I) \cap E_i)$. We can also assume that for $j \notin J$ the interval A_j does not contain 0.

Now put $A'_j = A_j$ if $j \neq i$ is contained in J, and put $A'_j =]b_j, \infty]$ if j is not contained in J, and $A_j =]b_j, c_j[$. (The interval A_j looks indeed like this since we took care to stay away from zero.)

It is easy to see that $A' = \sum_{j \neq i} A'_j e_j$ is contained in $\alpha(C_U^I \cap E_i)$. Hence we found an open neighborhood A' around $\alpha(x)$ in $\alpha(C_U^I \cap E_i)$. $\square$

Theorem 3.4 *The topological space $\overline{\Lambda}$ is compact and contractible, and Λ is an open, dense subset of $\overline{\Lambda}$.*

Proof: By the previous result, all E_i are compact and contractible. Since $\overline{\Lambda}$ is the union of the E_i, it is also compact. A straightforward calculation shows that the contraction maps are compatible, so that $\overline{\Lambda}$ is contractible. It is clear that Λ is open and dense in $\overline{\Lambda}$. □

Our next goal is to extend the action of N on Λ to a continuous action on the compactification $\overline{\Lambda}$. Recall that we identified W with the group of permutation matrices in N, so that $N = T \rtimes W$. For $w \in W$ we denote the induced permutation of the set $\underline{n}$ also by w, i.e. we abuse notation so that $w(v_i) = v_{w(i)}$. Let I be a non-empty subset of $\underline{n}$. We define a map

$$w : \Lambda_I \longrightarrow \Lambda_{w(I)}$$

by sending η_i^I to $\eta_{w(i)}^{w(I)}$. This gives an action of W on $\overline{\Lambda}$. Note that it is compatible with r_J, i.e. we have

$$w \circ r_J = r_{w(J)} \circ w$$

on Λ. Besides, we can combine the maps $\nu_I : T_I \to \Lambda_I$ with the restriction map $\rho_I : \mathbf{T} \to \mathbf{T}_I$ to define a map $\nu_I \circ \rho_I : T \to \Lambda_I$, so that T acts by affine transformations on Λ_I. Recall that $r_I(\nu(t)) = \nu_I(\rho_I(t))$ for all $t \in T$.

It is easy to check that these two actions give rise to an action of $N = T \rtimes W$ on $\overline{\Lambda}$, which we denote by ν.

Lemma 3.5 *The action $\nu : N \times \overline{\Lambda} \longrightarrow \overline{\Lambda}$ is continuous and extends the action of N on Λ.*

Proof: Let first w be an element of W, and let C_U^I one of our open basis sets. Then $\nu(w)(U + D_I) = \nu(w)(U) + D_{w(I)}$, since $\nu(w)$ is a linear map on Λ. Besides, we have $\nu(w)(r_J(U + D_I)) = r_{w(J)}(\nu(w)(U) + D_{w(I)})$, so that $\nu(w)(C_U^I) = C_{\nu(w)(U)}^{w(I)}$.

Now take some element $t \in T$. Then $\nu(t)(U + D_I) = \nu(t)(U) + D_I$, since $\nu(t)$ acts by translation. Besides, we have $\nu(t)(r_J(U + D_I)) = r_J(\nu(t)(U + D_I))$, so that $\nu(t)(C_U^I) = C_{\nu(t)(U)}^I$.

Hence for all $n \in N$ the action $\nu(n)$ on $\overline{\Lambda}$ is continuous. Since the kernel of the map $\nu : T \to \Lambda$ is an open subgroup of N (see [La], Prop. 1.2), which obviously acts trivially on $\overline{\Lambda}$, we find that the action is indeed continuous. □

4 Compactification of the building

We can now define the compactification of the building X. For all non-empty subsets Ω of $\overline{\Lambda}$ and all roots $a \in \Phi$ we put

$$\begin{aligned} f_\Omega(a) &= \inf\{t : \Omega \subseteq \overline{\{z \in \Lambda : a(z) \geq -t\}}\} \\ &= -\sup\{t : \Omega \subseteq \overline{\{z \in \Lambda : a(z) \geq t\}}\} \end{aligned}$$

Here we put $\inf \emptyset = \sup \mathbb{R} = \infty$ and $\inf \mathbb{R} = \sup \emptyset = -\infty$. Moreover, if $\Omega = \{x\}$ consists of one point only, then we write $f_x(a) = f_{\{x\}}(a)$. Note that

$$f_x(a) = -a(x) \quad \text{for all } x \in \Lambda,$$

and that

$$f_{\Omega_1}(a) \leq f_{\Omega_2}(a), \quad \text{if } \Omega_1 \subseteq \Omega_2.$$

Recall our generalized valuation map $\psi_a : U_a \to \mathbb{Z} \cup \{\infty\}$ from section 2. We can now define a subgroup

$$U_{a,\Omega} = U_{a,f_\Omega(a)} = \{u \in U_a : \psi_a(u) \geq f_\Omega(a)\}$$

of U_a, where $U_{a,\infty} = 1$ and $U_{a,-\infty} = U_a$. By U_Ω we denote the subgroup of G generated by all the $U_{a,\Omega}$ for roots $a \in \Phi$. Note that if $\Omega = \{x\}$ for some point $x \in \Lambda$, then this coincides with our previous definition of U_x. We will now investigate these groups U_x for boundary points of $\overline{\Lambda}$.

Proposition 4.1 *Recall that we denote by a_{ij} the root of T induced by the character $\chi_i - \chi_j$. Put $a = a_{ij}$, and let x be a point in Λ_I for some $I \subseteq \underline{n}$.*
i) If $j \notin I$, we have $f_x(a) = -\infty$, so that $U_{a,x}$ is the whole group U_a.
ii) If $j \in I$ and $i \notin I$, then we have $f_x(a) = \infty$, so that $U_{a,x} = 1$.
iii) If i and j are contained in I, then a is equal to $\rho_I^*(b)$ for some root b of the torus T_I in G_I. In this case we have $f_x(a) = -b(x)$. For any $z \in \Lambda$ with $r_I(z) = x$ we also have $f_x(a) = -a(z)$.

Proof: i) Choose some $z \in \Lambda$ such that $r_I(z) = x$. If i is contained in I, put $z_k = z + \sum_{l \notin I} k(-\eta_l)$. If i is not in I, then we define $z_k = z + \sum_{l \notin I, l \neq j} k(-\eta_l) - 2k\eta_j$. In both cases we find that $a(z_k)$ equals $a(z) + k$, hence it goes to infinity. Since the z_k converge to x, we find that x lies indeed in the closure of any set of the form $\{a \geq s\}$, which implies our claim.
ii) We choose again some $z \in \Lambda$ with $r_I(z) = x$. Let V_k be a countable decreasing fundamental system of bounded open neighborhoods of z. This defines a fundamental system of open neighborhoods $C^I_{U_k}$ around x, where $U_k = V_k + \sum_{i \notin I} k(-\eta_i)$. Now suppose that x is contained in the closure of the set $\{z \in \Lambda : a(z) \geq s\}$. Then we find for all k some y_k in $C^I_{U_k} \cap \Lambda$ satisfying $a(y_k) \geq s$. We can write $y_k = z_k + \lambda_k$ for some $z_k \in V_k$ and $\lambda_k = \sum_{l \notin I} \lambda_{k,l}(-\eta_l)$ with $\lambda_{k,l} \geq k$. Now $a(z_k)$ is bounded, but $a(\lambda_k) = -\lambda_{k,i}$, so that $a(y_k)$ cannot be bounded from below. Hence we find indeed that $f_x(a)$ must be ∞.
iii) Recall that $\mathbf{G}_I$ is the group $PGL(V_I)$, and $\mathbf{T}_I$ is the maximal K-split torus induced by the diagonal matrices with respect to the v_i for $i \in I$. Then the root system corresponding to $\mathbf{T}_I$ and $\mathbf{G}_I$ is

$$\Phi_I = \{b_{ij} : i \neq j \text{ in } I\}$$

where b_{ij} is the character mapping a diagonal matrix with entries t_i for $i \in I$ to t_i/t_j. Hence it is clear that in our case $i, j \in I$ the root $a = a_{i,j}$ of T is induced by the root $b = b_{ij}$ of T_I. Note that this implies that for all $z \in \Lambda$ we have $a(z) = b(r_I(z))$.
It suffices to show that

$$\overline{\{z \in \Lambda : a(z) \geq s\}} \cap \Lambda_I = \{x \in \Lambda_I : b(x) \geq s\}.$$

Take some x contained in the left hand side, and choose some $z \in \Lambda$ with $r_I(z) = x$. Besides, we take again fundamental neighborhoods V_k around z and use them to construct the open neighborhoods $C^I_{U_k}$ around x. Each $C^I_{U_k}$ must contain some $y_k \in \Lambda$ satisfying $a(y_k) \geq s$. Note that we can write $y_k = z_k + \lambda_k$, where z_k is in V_k and λ_k is a linear combination of η_i for $i \notin I$. Besides $a(y_k) = a(z_k)$, so that the sequence of $a(y_k)$ converges to $a(z)$. Since all $a(y_k)$ are $\geq s$, we find $b(x) = b(r_I z) = a(z) \geq s$.
On the other hand, suppose that x is a point in Λ_I satisfying $b(x) \geq s$. Again, we choose some $z \in \Lambda$ with $r_I(z) = x$, and neighborhoods V_k around z. For any k the point $z_k = z + k\sum_{l \notin I}(-\eta_l)$ lies in $C^I_{U_k}$. Besides, $a(z_k) = a(z) = b(r_I(z)) = b(x)$ is bounded below by s. This implies that x lies indeed in the closure of $\{z \in \Lambda : a(z) \geq s\}$. □

Proposition 4.2 *Let x be in Λ_I and let $a = a_{ij} \in \Phi$ be a root.*
i) Each $U_{a,x}$ (and hence also U_x) leaves the vector space V_I invariant. Hence $U_{a,x}$ is contained in G^{V_I}.
ii) If i and j are not both in I, we have $\rho_I(U_{a,x}) = 1$. If i and j are both in I, and the root a is induced by the root b of T_I, then ρ_I induces an isomorphism $U_{a,x} \to U^I_{b,x}$, where $U^I_{b,x}$ is defined with the root group U^I_b in $\mathbf{G}_I$ as described in section 2.

Proof: Recall that $u \in U_a$ maps v_l to itself, if l is not equal to j, and it maps v_j to $v_j + \omega v_i$, where $\psi_a(u) = v(\omega)$. Hence our claim in i) is clear if j is not contained in I or if both i and j are contained in I. In the remaining case we saw in 4.1 that $U_{a,x}$ is trivial, so that i) holds in any case.
Let us now prove ii). If j is not contained in I, then each $u \in U_a$ induces the identity map on V_I. If j is in I, but i is not, then $U_{a,x}$ is trivial. Hence in both cases we find that $\rho_I(U_{a,x}) = 1$. Let us assume that both i and j are contained in I, and let u be an element of $U_{a,x}$. Then $\rho_I(u) \in PGL(V_I)$ is induced by the matrix mapping v_l to v_l for all $l \neq j$ in I, and v_j to $v_j + \omega v_i$ with some ω having valuation $\geq f_x(a)$. By our description of the groups U^I_b in section 2 we find that $\rho_I(u)$ is contained in U^I_b and has valuation $\psi_b(\rho_I x) = v(\omega) \geq f_x(a)$. By 4.1, $f_x(a) = -b(x)$, so that $\rho_I(u)$ lies indeed in $U^I_{b,x}$. The homomorphism $\rho_I : U_{a,x} \to U^I_{b,x}$ is obviously bijective. □
Note that the map $x \mapsto f_x(a)$ is in general not continuous on $\overline{\Lambda}$. Take some $z \in \Lambda$ and define a sequence $x_k = z + \sum_{i \notin I} k(-\eta_i)$ for some non-empty I such that the complement $\underline{n}\backslash I$ contains at least two elements i and j. Then $a = a_{ij}$ has the property that $a(x_k) = a(z)$, so that $f_{x_k}(a)$ is constant. But the sequence x_k converges to the point $r_I(z)$ in Λ_I, for which $f_{r_I(z)}(a) = -\infty$ holds.
Nevertheless, we have the following result:

Lemma 4.3 *Let x_k be a sequence of points in $\overline{\Lambda}$, which converges to $x \in \overline{\Lambda}$. Let $u_k \in U_{a,x_k}$ be a sequence of elements, converging to some u in the big group U_a. Then u lies in fact in $U_{a,x}$.*

Proof: Note first of all, that the statement is clear if $f_x(a) = -\infty$, since then $U_{a,x} = U_a$. It is also clear if $f_{x_k}(a)$ converges to $f_x(a)$, since the map $\psi_a : U_a \to \mathbb{Z} \cup \{\infty\}$ is continuous. Assume that $f_x(a) = \infty$. Then any set $\overline{\{z : a(z) \geq s\}}$ contains only finitely many elements x_k. This implies that the sequence $f_{x_k}(a)$ goes to $\infty = f_x(a)$, so that in this case our claim holds by continuity.
The only case which is left is that $f_x(a)$ is real. Assume that $x \in \Lambda_I$. By 4.1, we must have $a = a_{ij}$ with i and j in I. Choose some $z \in \Lambda$ with $r_I(z) = x$, and a decreasing fundamental system of open neighborhoods V_k around z. As before, we use them to define neighborhoods $C^I_{U_k}$ around x. Since x_k converges to x, we can assume that x_k is contained in $C^I_{U_k}$. Then $x_k \in \Lambda_{J_k}$ for some J_k containing I. By definition of $C^I_{U_k}$ we find some $y_k \in V_k$ and some coefficients α_l such that $z_k = y_k + \sum_{l \notin I} \alpha_l(-\eta_l)$ satisfies $r_{J_k}(z_k) = x_k$. By 4.1 we have $f_{x_k}(a) = -a(z_k) = -a(y_k)$ and $f_x(a) = -a(z)$. Hence $f_{x_k}(a)$ converges to $f_x(a)$, and our claim follows again by continuity. □

Proposition 4.4 *For $x \in \overline{\Lambda}, n \in N$ and $a \in \Phi$ we have*

$$nU_{a,x}n^{-1} = U_{\overline{n}(a),\nu(n)(x)},$$

where ν denotes the action of N on $\overline{\Lambda}$, and $n \mapsto \overline{n}$ denotes the quotient map from N to the Weyl group W (which acts on the roots). In particular, we have $nU_xn^{-1} = U_{\nu(n)(x)}$.

Proof: Fix some $n \in N$ and denote by p the permutation matrix mapping to $\overline{n}$, i.e. $n = tp$ for some $t \in T$. We denote by p also the corresponding permutation of $\underline{n}$. If $a = a_{ij}$, then $\overline{n}(a) = a_{p(i)p(j)}$. By 4.1 we find that if $x \in \Lambda_I$ and $j \notin I$ both $f_x(a)$ and $f_{\nu(n)(x)}(\overline{n}a)$ are equal to $-\infty$. Since $nU_an^{-1} = U_{\overline{n}a}$, our claim holds in this case. Similarly, if $j \in I$ and $i \notin I$ both $f_x(a)$ and $f_{\nu(n)(x)}(\overline{n}a)$ are equal to ∞, so that both $U_{a,x}$ and $U_{\overline{n}a,\nu(n)(x)}$ are trivial.
We can therefore assume that x is in some Λ_I such that both i and j are contained in I. Recall that we composed the action of N on Λ from the natural action of the Weyl group on Λ and translation by $\nu(t)$ for elements in the torus T. Hence for all $z \in \Lambda$ we have $\nu(n)(z) = \nu(n)(0) + \overline{n}(z)$. We can now calculate

$$\begin{aligned} a(\nu(n)(z)) &= a(\nu(n)(0)) + a(\overline{n}(z)) \\ &= a(\nu(n)(0)) + (\overline{n}^{-1}a)(z), \end{aligned}$$

so that $\nu(n^{-1})\{z \in \Lambda : (\overline{n}a)(z) \geq s\} = \{z \in \Lambda : a(z) \geq a(\nu(n^{-1})(0)) + s\}$. Since $\nu(n^{-1})$ is a homeomorphism, we also have

$$\nu(n^{-1})\overline{\{z : \overline{n}a(z) \geq s\}} = \overline{\{z : a(z) \geq a(\nu(n^{-1})(0)) + s\}}.$$

Hence

$$\begin{aligned} f_{\nu(n)(x)}(\overline{n}(a)) &= \inf\{t : \nu(n)(x) \in \overline{\{\overline{n}(a) \geq -t\}}\} \\ &= \inf\{t : x \in \overline{\{z : a(z) \geq a(\nu(n^{-1})(0)) - t\}}\} \\ &= f_x(a) + a(\nu(n^{-1})(0)). \end{aligned}$$

Since we have $nU_{a,s}n^{-1} = U_{\overline{n}(a),s+a(\nu(n^{-1})(0))}$ for all real numbers s (see [La], 11.6), we find that indeed

$$nU_{a,x}n^{-1} = U_{\overline{n}(a),f_x(a)+a(\nu(n^{-1})(0))} = U_{\overline{n}(a),f_{\nu(n)(x)}(\overline{n}(a))} = U_{\overline{n}(a),\nu(n)(x)},$$

whence our claim. □

Recall that - forgetting about the special nature of our ground field - our root system Φ in Λ^* defines a finite set of hyperplanes in Λ^* and therefore a decomposition of Λ^* into faces (see [Bou],V,1). The maximal faces are called (spherical) chambers. Any chamber defines an order on Λ^* ([Bou], VI, 1.6). We denote the positive roots with respect to this order by $\Phi^+ = \Phi^+(C)$, and the negative roots by $\Phi^- = \Phi^-(C)$. In fact, for any subset Ψ of Φ such that Ψ is additively closed and Φ is the disjoint union of Ψ and $-\Psi$, there exists a chamber C such that $\Psi = \Phi^+(C)$ (see [Bou], VI, 1.7). In particular, Φ^- is the set of positive roots for a suitable chamber.

The following lemma will be useful to reduce claims about the groups $U_{a,\Omega}$ for $\Omega \subset \overline{\Lambda}$ to claims about subsets of Λ, where we can apply the "usual" theory of the Bruhat-Tits building.

Lemma 4.5 *Let $\Omega \subset \overline{\Lambda}$ be a non-empty set, and fix some chamber C. Put $\Phi^+ = \Phi^+(C)$. Assume that for every $a \in \Phi^+$ we have m elements*

$$u_{a,1}, \dots, u_{a,m} \in U_{a,\Omega},$$

such that at least one of all these $u_{a,i}$'s is non-trivial. Then there exists a non-empty subset Ω' of Λ such that $u_{a,i} \in U_{a,\Omega'}$ for all $i = 1, \dots, m$ and such that $\Omega \subset \overline{\Omega}'$. In particular, we have $U_{a,\Omega'} \subset U_{a,\Omega}$ for all roots $a \in \Phi$.

Proof: We denote by l_a the infimum of all $\psi_a(u_{a,i})$ for $i = 1, \dots, m$. If all $u_{a,i}$ are trivial, then $l_a = \infty$. This cannot happen for all $a \in \Phi^+$. Then we put

$$\Omega'_a = \overline{\{z \in \Lambda : a(z) \geq -l_a\}}.$$

(If $l_a = \infty$, then $\Omega'_a = \overline{\Lambda}$.) Besides, put $\Omega' = \Lambda \cap \bigcap_{a\in\Phi^+} \Omega'_a$. Note that Ω' contains the intersection of all sets $\{z \in \Lambda : a(z) \geq -l_a\}$ for $a \in \Phi^+$. If l is the minimum of all the l_a, then this set contains all $z \in \Lambda$ satisfying $a(z) \geq -l$ for all $a \in \Phi^+$. By looking at a base of Φ corresponding to Φ^+, we see that such z's exist. Hence Ω' is non-empty.

By construction, $f_{\Omega'_a}(a) = l_a$, and the inclusion $\Omega' \subset \Omega'_a$ gives us $f_{\Omega'}(a) \leq f_{\Omega'_a}(a)$. Therefore $\psi_a(u_{a,i}) \geq l_a \geq f_{\Omega'}(a)$, which implies that $u_{a,i}$ is indeed contained in $U_{a,\Omega'}$ for all $i = 1, \dots, m$. It remains to show that $\Omega \subset \overline{\Omega}'$. Since $f_{\Omega'}(a) = f_{\overline{\Omega}'}(a)$ for all roots a, this implies that $f_\Omega(a) \leq f_{\Omega'}(a)$, hence $U_{a,\Omega'} \subset U_{a,\Omega}$ for all $a \in \Phi$.

We are done if we prove the following claim:

(∗) For any $\Psi \subseteq \Phi^+$ and real numbers $s_a$we have $\displaystyle\bigcap_{a\in\Psi} \{\overline{a \geq s_a}\} = \overline{\bigcap_{a\in\Psi} \{a \geq s_a\}}$.

It is clear that the right hand side is contained in the left hand side. So suppose x is an element in $\bigcap_{a\in\Psi}\{\overline{a \geq s_a}\}$. Let I be the subset of $\underline{n}$ such that $x \in \Lambda_I$. We choose a system of open neighborhoods V_k of some point in Λ projecting to x and construct $C^I_{U_k}$. We are done if we can show that any $C^I_{U_k}$ intersects $\bigcap_{a\in\Psi}\{a \geq s_a\}$ non-trivially. Let z_k be a point in $U_k = V_k + k\sum_{l\notin I}(-\eta_l)$ with $r_I(z_k) = x$. Besides, let s_k be the maximum of 0 and all the numbers $s_a - a(z_k)$ for all $a \in \Psi$.
Note that Φ^+ defines a linear ordering of the set $\underline{n} = \{1, \dots, n\}$, namely $i \prec j$, iff $a_{ij} \in \Phi^+$. Hence there is a permutation π of $\underline{n}$ satisfying $\pi(1) \prec \pi(2) \prec \dots \prec \pi(n)$. Put $z = z_k - \sum_{l\notin I}(k + \pi^{-1}(l)s_k)\eta_l$. This is an element of $C^I_{U_k}$. It remains to show that indeed $a(z) \geq s_a$ for all $a \in \Psi$.
Let $a = a_{ij}$ be a root in Ψ. If both i and j are in I, we can apply 4.1 to deduce $a(z) = a(z_k) = -f_x(a) \geq s_a$. Since x is contained in $\{\overline{a \geq s_a}\}$, it cannot happen that j is in I, but i is not. If j is not in I, we find that

$$a(z) = \begin{cases} a(z_k) + k + \pi^{-1}(j)s_k & \geq s_a \quad ,\text{if } i \in I \\ a(z_k) + (\pi^{-1}(j) - \pi^{-1}(i))s_k & \geq s_a \quad ,\text{if } i \notin I \end{cases},$$

since $a \in \Phi^+$ implies that $\pi^{-1}(i) < \pi^{-1}(j)$. Hence we get $a(z) \geq s_a$ for all $a \in \Psi$, which proves $(*)$. □

Corollary 4.6 *Assume that a and b are roots in Φ which are not linear equivalent (i.e. $a \neq \pm b$), and so that $a+b$ is in Φ. If both $f_\Omega(a)$ and $f_\Omega(b)$ are real numbers, then*

$$f_\Omega(a+b) \leq f_\Omega(a) + f_\Omega(b).$$

If $f_\Omega(a) = -\infty$ and $f_\Omega(b) \neq \infty$, then $f_\Omega(a+b) = -\infty$.

Proof: By $(*)$ in the proof of the preceding lemma, we have

$$\{\overline{a \geq s}\} \cap \{\overline{b \geq r}\} = \{\overline{a \geq s, b \geq r}\} \subset \{\overline{a+b \geq s+r}\},$$

which implies our claim. □
Recall that U_Ω is the subgroup of G generated by all the $U_{a,\Omega}$. Now we prove a statement about the structure of these groups U_Ω which will we crucial for our later results.
For $\Phi^+ = \Phi^+(C)$ we denote by $\mathbf{U}_{\Phi^+}$ the corresponding subgroup of $\mathbf{G}$ (see [Bo], 21.9), and by U_{Φ^+} the set of K-rational points. Similarly, we have $\mathbf{U}_{\Phi^-}$ and U_{Φ^-}. For any non-empty subset Ω of $\overline{\Lambda}$ define

$$U^+_\Omega = U_{\Phi^+} \cap U_\Omega \quad \text{and} \quad U^-_\Omega = U_{\Phi^-} \cap U_\Omega.$$

Of course, these groups depend on the choice of some chamber C. We can use them to get some information about U_Ω.

Theorem 4.7 *i) The multiplication map induces a bijection*

$$\prod_{a\in\Phi^\pm} U_{a,\Omega} \longrightarrow U^\pm_\Omega,$$

where the product on the left hand side may be taken in arbitrary order.
ii) $U_a \cap U_\Omega = U_{a,\Omega}$ *for all* $a \in \Phi$.
iii) $U_\Omega = U_\Omega^- U_\Omega^+ (N \cap U_\Omega)$.

Proof: Note that by [La], 12.5, our claim holds for all non-empty subsets $\Omega \subset \Lambda$. Now take $\Omega \subseteq \overline{\Lambda}$, and denote by L_a the group generated by $U_{a,\Omega}$ and $U_{-a,\Omega}$, and by Y the subgroup of N generated by all $N \cap L_a$ for $a \in \Phi$. For all $a \in \Phi^+$ choose an element $u_a \in U_{a,\Omega}$. By 4.5 we find a subset Ω' of Λ such that $u_a \in U_{a,\Omega'}$. Hence by [La], 12.5, the product of the u_a in arbitrary order lies in $U_{\Omega'}^+ \subset U_\Omega^+$.
A similar argument using 4.5 shows that the image of $\prod_{a\in\Phi^+} U_{a,\Omega}$ under the multiplication map is indeed a subgroup of U_Ω^+, which is independent of the ordering of the factors. We denote it by H^+. Similarly, we define the subgroup H^- of U_Ω^- as the image of $\prod_{a\in\Phi^-} U_{a,\Omega}$ under the multiplication map. Now we can imitate the argument in [La], Proposition 8.9, to prove that the set H^-H^+Y does not depend on the choice of the chamber defining Φ^+ and is invariant under multiplication from the left by Y and $U_{a,\Omega}$ for arbitrary roots $a \in \Phi$. Hence $H^-H^+Y = U_\Omega$.
Since $U_{\Phi^+} \cap U_{\Phi^-} = \{1\}$ and $N \cap U_{\Phi^+}U_{\Phi^-} = \{1\}$ (by [BoTi], 5.15), we find $U_\Omega^- = (H^-H^+Y) \cap U_{\Phi^-} = H^-$ and $U_\Omega^+ = H^+$, which proves i) and ii). Similarly, $N \cap U_\Omega = Y$, whence iii). □
For any subset Ω of $\overline{\Lambda}$ we write $N_\Omega = \{n \in N : \nu(n)x = x \text{ for all } x \in \Omega\}$. Besides, put

$$P_\Omega = U_\Omega N_\Omega = N_\Omega U_\Omega,$$

which is a group since as in 4.4 one can show that N_Ω normalizes U_Ω. If $\Omega = \{x\}$ we write $P_\Omega = P_x$.
We can now also describe the groups P_Ω for any non-empty subset Ω of $\overline{\Lambda}$:

Corollary 4.8 *Fix some* $\Phi^+ = \Phi^+(C)$ *as above, and let* Ω *be a non-empty subset of* $\overline{\Lambda}$.
i) $P_\Omega = U_\Omega^- U_\Omega^+ N_\Omega = N_\Omega U_\Omega^+ U_\Omega^-$.
ii) $P_\Omega \cap U_{\Phi^\pm} = U_\Omega^\pm$ *and* $P_\Omega \cap N = N_\Omega$.

Proof: i) The first equality follows from part iii) of the Theorem, if we show that $N \cap U_\Omega \subset N_\Omega$. It suffices to show for each root a that $N \cap L_a \subset N_\Omega$. If both $f_\Omega(a)$ and $f_\Omega(-a)$ are real numbers, this follows from [La], 12.1. If $f_\Omega(a) = \infty$ of if $f_\Omega(-a) = \infty$, then our claim is trivial. Note that if $f_\Omega(a) = -\infty$, then $f_\Omega(-a)$ is either ∞ (then we are done) or $-\infty$. Hence the only remaining case is $f_\Omega(a) = f_\Omega(-a) = -\infty$. If $a = a_{ij}$, 4.1 implies that $\Omega \cap \Lambda_J$ can then only be non-empty if i and j are not contained in J. Now 4.2 implies our claim.
ii) Let us show first that $P_\Omega \cap U_{\Phi^-} = U_\Omega^-$. Obviously, the right hand side is contained in the left hand side. Take some $u \in P_\Omega \cap U_{\Phi^-}$. Using i), we can write it as u^-u^+n for $u^\pm \in U_\Omega^\pm$ and $n \in N_\Omega$. Then n must be in $U_{\Phi^+}U_{\Phi^-} \cap N$, which is trivial by [BoTi], 5.15. Hence u^+ is contained in $U_{\Phi^-} \cap U_{\Phi^+}$, which is also trivial. Therefore $u = u^- \in U_\Omega^-$. The corresponding statement for the

+-groups follows by taking Φ^- as the set of positive roots. It remains to show $P_\Omega \cap N = N_\Omega$. Take $u \in P_\Omega \cap N$. Then we write it again as $u = u^- u^+ n$. Hence $u^- u^+$ is contained in $U_{\Phi^-} U_{\Phi^+} \cap N$, which is trivial, so that $u = n \in N_\Omega$, as claimed. $\square$

Now we can show a weak version of the mixed Bruhat decomposition for our groups P_x. (The weakness lies in the fact that we can not take two arbitrary points in $\overline{\Lambda}$ in the next statement.)

Theorem 4.9 *Let $x \in \overline{\Lambda}$ and $y \in \Lambda$. Then we have $G = P_x N P_y$.*

Proof: Let Λ_I be the component of $\overline{\Lambda}$ containing x. Then we can write $x = \sum_{i \in I} x_i \eta_i^I$ with some real coefficients x_i. We define a sequence of points in Λ by

$$z_k = \sum_{i \in I} x_i \eta_i - k \sum_{i \notin I} \eta_i.$$

Obviously, z_k converges towards x. Now we choose a linear ordering $\prec$ on the set $\underline{n}$ in such a way that $i \in I$ and $j \notin I$ implies $i \prec j$. The set $\Phi^+ = \{a_{ij} \in \Phi : i \prec j\}$ defines an order corresponding to some chamber. Note that for any root a_{ij} in Φ^- we have

$$a_{ij}(z_k) = \begin{cases} x_i - x_j & \text{if } i, j \in I \\ -k - x_j & \text{if } i \notin I, j \in I \\ 0 & \text{if } i, j \notin I. \end{cases}$$

Hence $a_{ij}(z_k)$ is bounded from above by a constant c independent of k and of the root $a_{ij} \in \Phi^-$. Therefore U_{a_{ij}, z_k} is contained in $U_{a_{ij}, -c} = \{u \in U_{a_{ij}} : \psi_a(u) \geq -c\}$, which is a compact subgroup of $U_{a_{ij}}$. By 4.7, we find that all $U_{z_k}^-$ are contained in a compact subset of U_{Φ^-}.

We have the "usual" mixed Bruhat decomposition for two points in Λ (see [La], 12.10), hence $G = P_{z_k} N P_y$ for all k. Using 4.8, we can write an element $g \in G$ as

$$g = u_k^- u_k^+ n_k v_k$$

with $u_k^\pm \in U_{z_k}^\pm$, $n_k \in N$ and $v_k \in U_y$. Let us denote the kernel of the map $\nu : T \to \Lambda$ by $Z \subset T$. Then the group $U_y^\wedge = U_y Z$ is compact and open in G by [La], 12.12. Since all v_k lie in this compact subset, by switching to a subsequence we can assume that v_k converges to some element $v \in U_y^\wedge$. Hence the sequence $u_k^- u_k^+ n_k$ is also convergent in G. Since the Weyl group $W = N/T$ is finite, by passing to a subsequence we can assume that $n_k = t_k n$ for some $t_k \in T$ and some fixed $n \in N$. Besides, we can assume that u_k^- converges to some $u^- \in U_{\Phi^-}$, since the u_k^- are contained in a compact subset of U_{Φ^-}. Hence the sequence $u_k^+ t_k$ converges in G. Its limit must be contained in the Borel group $U_{\Phi^+} T$. Hence u_k^+ converges towards some $u^+ \in U_{\Phi^+}$, and t_k converges towards some $t \in T$.

Using 4.7, u_k^+ is a product of $u_{a,k} \in U_{a, z_k}$ for all $a \in \Phi^+$, and, applying 4.3, we deduce that the $u_{a,k}$ converge towards some element $u_a \in U_{a,x}$. Hence we see that u^+ is contained in $U_{\Phi^+} \cap U_x = U_x^+$. Similarly, u^- lies in U_x^-. Therefore $g = u_k^- u_k^+ t_k n v_k$ converges towards $u^- u^+ t n v$, which is contained in $U_x^- U_x^+ N U_y \subseteq P_x N P_y$. Hence g lies indeed in $P_x N P_y$. $\square$

Recall that $Z \subset T$ denotes the kernel of the map $\nu : T \to \Lambda$, and that the group $U_0^\wedge = U_0 Z$ is compact. We define our compactification $\overline{X}$ of the building X as

$$\overline{X} = U_0^\wedge \times \overline{\Lambda} / \sim,$$

where the equivalence relation $\sim$ is defined as follows:

$$(g,x) \sim (h,y) \qquad \text{iff there exists an element } n \in N$$
$$\text{such that } \nu(n)x = y \text{ and } g^{-1}hn \in P_x.$$

(Using 4.4, it is easy to check that $\sim$ is indeed an equivalence relation.) We equip $\overline{X}$ with the quotient topology. The inclusion $U_0^\wedge \times \Lambda \hookrightarrow G \times \Lambda$ induces a bijection $(U_0^\wedge \times \Lambda)/\sim \to X$, which is a homeomorphism if we endow the left hand side with the quotient topology (see [Bo-Se], p.221). Hence X is open and dense in $\overline{X}$.

We have a natural action of $U_0^\wedge$ on $\overline{X}$ via left multiplication on the first factor, which can be continued to an action of G in the following way: If $g \in G$ and $(v,x) \in U_0^\wedge \times \overline{\Lambda}$, we can use the mixed Bruhat decomposition to write $gv = unh$ for some $u \in U_0^\wedge$, $n \in N$ and $h \in P_x$. Then we define $g(v,x) = (u, \nu(n)x)$. Using 4.4, one can show that this induces a well-defined action on $\overline{X}$.

Mapping x to the class of $(1,x)$ defines a map $\overline{\Lambda} \to \overline{X}$. This is injective, since by 4.8 we have $P_x \cap N = N_x$. The G-action on $\overline{X}$ continues the N-action on $\overline{\Lambda}$, so that we will write nx instead of $\nu(n)x$ for $x \in X$.

The following important fact follows immediately from the definition of $\overline{X}$:

Lemma 4.10 *For all $x \in \overline{\Lambda}$ the group P_x is the stabilizer of x in G.*

We can use the mixed Bruhat decomposition to prove the following important fact:

Proposition 4.11 *For any two points $x \in \overline{X}$ and $y \in X$ there exists a compactified apartment containing x and y, i.e. there exists some $g \in G$ such that x and y both lie in $g\overline{\Lambda}$.*

Proof: We can assume that y lies in Λ. The point x lies in $h\overline{\Lambda}$ for some $h \in G$, so $x = hx'$ for some $x' \in \overline{\Lambda}$. By our mixed Bruhat decomposition 4.9 we can write $h = qnp$ for $q \in P_y$, $n \in N$ and $p \in P_{x'}$. Therefore $x = hx' = qnpx' = qnx' \in q\overline{\Lambda}$, and $y = qy$ lies also in $q\overline{\Lambda}$, whence our claim. $\square$

5 Properties of $\overline{X}$

In this section we want to check that $\overline{X}$ is compact and we want to identify it with the set $\bigcup_{W \subseteq V} X(PGL(W))$.

Hence we see that we can compactify the Bruhat-Tits building for $PGL(V)$ by attaching all the Bruhat-Tits buildings for PGL of the smaller subspaces at infinity.
The following lemma is similar to [La], 8.11.

Lemma 5.1 *Let z be a point in Λ_I and y be a point in Λ_J for some $I \subseteq J \subseteq \underline{n}$. Then we find a chamber such that the corresponding set of positive roots Φ^+ satisfies $U_y^+ \subseteq U_z^+$.*

Note that the assumptions are fulfilled if $J = \underline{n}$, i.e. if y lies in Λ.
Proof: Since $I \subseteq J$, we can define a projection map $\Lambda_J \to \Lambda_I$, which we also denote by r_I. To be precise, r_I maps a point $\sum_{i \in J} x_i \eta_i^J \in \Lambda_J$ to $\sum_{i \in I} x_i \eta_i^I \in \Lambda_I$. Put $y^* = r_I(y)$.
Recall that we denote by Φ_I the set of roots of $\mathbf{T}_I$ in $\mathbf{G}_I = PGL(V_I)$. There exists a chamber in Λ_I^* with respect to Φ_I such that the corresponding subset Φ_I^+ of positive roots satisfies $U_{y^*}^{I+} \subseteq U_z^{I+}$. Here U_z^{I+} is defined exactly as the groups U_x^+ for $x \in \Lambda$ in section 4, just replacing Λ by Λ_I.
Now $\Phi_I = \{b_{ij} : i, j \in I\}$, where b_{ij} is the character mapping a diagonal matrix with entries t_l (for $l \in I$) to t_i/t_j. We define a linear ordering on I by $i \prec j$ iff $b_{ij} \in \Phi_I^+$. This can be continued to a linear ordering on $\underline{n}$ in such a way that $i \prec j$ whenever $i \in I$ and $j \notin I$. Let us put $\Phi^+ = \{a_{ij} : i \prec j\}$. We claim that this satisfies our claim. In fact, take $a = a_{ij} \in \Phi^+$. If i and j are contained in I, we use 4.2 and the construction of Φ_I^+ to deduce that $U_{a,y} \subseteq U_{a,z}$. By definition of Φ^+ it cannot happen that j is in I, but i is not. Hence the only remaining case is that $j \notin I$. Then $f_z(a) = -\infty$ by 4.1, so that trivially $U_{a,y} \subseteq U_{a,z}$. $\square$
Recall that $P_\Omega = N_\Omega U_\Omega$ with $N_\Omega = \{n \in N : nx = x \text{ for all } x \in \Omega\}$.

Theorem 5.2 *Fix a nonempty $\Omega \subset \overline{\Lambda}$. We denote by $(*)$ the following condition:*

$(*)$ *The set $\{J \subseteq \underline{n} : \Omega \cap \Lambda_J \neq \emptyset\}$ contains a maximal element with respect to inclusion.*

If $(*)$ if satisfied, then $P_\Omega = \cap_{x \in \Omega} P_x$. In particular, P_Ω is the stabilizer of Ω.

Note that $(*)$ is satisfied if $\Omega \cap \Lambda$ is not empty.
Proof: To begin with, the inclusion $P_\Omega \subseteq P_x$ for all $x \in \Omega$ is trivial. Let J be the maximal subset of $\underline{n}$ satisfying $\Omega \cap \Lambda_J \neq \emptyset$, and choose some $x_0 \in \Omega \cap \Lambda_J$. We will first prove that $P_{\Omega^\sim} \cap P_x = P_{\Omega^\sim \cup \{x\}}$ for all $x \in \Omega$ and all subsets $\Omega^\sim$ of Ω containing x_0. Assume that x lies in in Λ_I for some $I \subseteq J$. By 5.1, we find some set of positive roots Φ^+ such that $U_{x_0}^+ \subseteq U_x^+$, hence also $U_{\Omega^\sim}^+ \subseteq U_x^+$. Let now g be an element in $P_{\Omega^\sim} \cap P_x$, and write $g = nu^-u^+$ for some $n \in N_{\Omega^\sim}$ and some $u^\pm \in U_{\Omega^\sim}^\pm$ (using 4.8). Since g and u^+ are contained in P_x, this also holds for nu^-, so that we can write $nu^- = mv^+v^-$ for some $m \in N_x$ and $v^\pm \in U_x^\pm$. So $m^{-1}n$ is contained in $N \cap U_{\Phi^+} U_{\Phi^-}$, hence trivial by [BoTi], 5.15. We find that $m = n$ and $u^- = v^+v^-$, which implies $v^+ = 1$. Hence n is

contained in $N_x \cap N_{\Omega^\sim} = N_{\Omega^\sim \cup \{x\}}$, and u^- is contained in $U_x^- \cap U_{\Omega^\sim}^-$. Note that for all $a \in \Phi$ we have the inclusion $U_{a,\Omega^\sim} \cap U_{a,x} \subseteq U_{a,\Omega^\sim \cup \{x\}}$, so that we can use 4.7 to deduce $u^- \in U^-_{\Omega^\sim \cup \{x\}}$. A similar argument as above gives $u^+ \in U^+_{\Omega^\sim} \cap U_x^+ \subset U^+_{\Omega^\sim \cup \{x\}}$. Hence our claim is proven.
Therefore any finite subset $\Omega^\sim \subset \Omega$ containing x_0 satisfies our claim, i.e. $P_{\Omega^\sim} = \cap_{x \in \Omega^\sim} P_x$.
We can write $\Omega = \bigcup_{\sigma \in \Sigma} \Omega_\sigma$, where Ω_σ for $\sigma \in \Sigma$ runs over all finite subsets of Ω containing x_0. Let us consider some $g \in \bigcap_{x \in \Omega} P_x = \bigcap_{\sigma \in \Sigma} P_{\Omega_\sigma}$. We fix some set of positive roots Φ^+ and write $g = n_\sigma u_\sigma^+ u_\sigma^-$ for $n_\sigma \in N_{\Omega_\sigma}$ and $u_\sigma^\pm \in U^\pm_{\Omega_\sigma}$ by 4.8. Put $T_{x_0} = T \cap N_{x_0}$. Then N_{x_0}/T_{x_0} is finite. By the pigeon hole principle, there must be one class m in N_{x_0}/T_{x_0} such that the set Σ' of all the $\sigma \in \Sigma$ so that n_σ is equal to m modulo T_{x_0} still has the property that $\bigcup_{\sigma \in \Sigma'} \Omega_\sigma = \Omega$. (If not, we could find for any class in N_{x_0}/T_{x_0} an element in Ω not contained in any Ω_σ with the property that n_σ lies in our class. Collecting these elements together with x_0 in some finite set gives a contradiction.)
Hence for $\sigma \in \Sigma'$ we can write $n_\sigma = m t_\sigma$ for some fixed $m \in N_{x_0}$ and some $t_\sigma \in T_{x_0}$. For σ and τ in Σ' we get $t_\sigma u_\sigma^+ u_\sigma^- = t_\tau u_\tau^+ u_\tau^-$. Using the fact that T normalizes U_{Φ^+} and U_{Φ^-} and that $N \cap U_{\Phi^+} U_{\Phi^-}$ is trivial, we find $u_\sigma^- = u_\tau^-$, $u_\sigma^+ = u_\tau^+$ and $t_\sigma = t_\tau$. Therefore the elements $t = t_\sigma$, $u^\pm = u_\sigma^\pm$ are independent of the choice of $\sigma \in \Sigma'$. Note that by definition of the T-action on $\overline{\Lambda}$ the element t in T_{x_0} stabilizes not only x_0, but also every point in the components Λ_I for $I \subseteq J$. Thus $t \in N_\Omega$. Besides, by 4.7 we deduce $u^+ \in \bigcap_{\sigma \in \Sigma'} U^+_{\Omega_\sigma} = U^+_\Omega$. Similarly, $u^- \in U^-_\Omega$. Hence we find indeed that $g = m t u^+ u^-$ is contained in P_Ω. $\square$

Corollary 5.3 *Let Ω and Ω' be two non-empty subsets of $\overline{\Lambda}$ such that Ω, Ω' and $\Omega \cup \Omega'$ satisfy condition $(*)$ in the Theorem. Then $P_\Omega \cap P_{\Omega'} = P_{\Omega \cup \Omega'}$.*

Proof: This is an immediate consequence from 5.2. $\square$
The following result is similar to [La], 9.6.

Proposition 5.4 *Let $g \in G$ and let J be a subset of $\underline{n}$ so that $\Lambda_J \cap g^{-1}\overline{\Lambda}$ is not empty. Then there exists some element $n \in N$ such that*

$$gx = nx \text{ for all } x \in g^{-1}\overline{\Lambda} \cap (\bigcup_{I \subseteq J} \Lambda_I).$$

Proof: Note that the set $\Omega = g^{-1}\overline{\Lambda} \cap (\bigcup_{I \subseteq J} \Lambda_I)$ satisfies condition $(*)$ from 5.2. Fix some $x_0 \in \Lambda_J \cap g^{-1}\overline{\Lambda}$. For all $x \in \Omega$ we have $g^{-1}N \cap P_x \neq \emptyset$, since $x = g^{-1}y$ for some $y \in \overline{\Lambda}$.
We will now show that for all finite subsets Δ of Ω containing x_0 we have $g^{-1}N \cap P_\Delta \neq \emptyset$. Let us suppose that this claim holds for some Δ and let us show it for $\Delta \cup \{x\}$, where x is some point in Ω. So there is some $n_\Delta \in N$ with $g^{-1}n_\Delta \in P_\Delta$. We also find some $n_x \in N$ satisfying $g^{-1}n_x \in P_x$. By 5.1,

we find a set of positive roots Φ^+ such that $U^+_{x_0} \subseteq U^+_x$, so that also $U^+_\Delta \subseteq U^+_x$. Hence we apply 4.8 to deduce

$$n_\Delta^{-1} n_x \in P_\Delta P_x = N_\Delta U^-_\Delta U^+_\Delta U^+_x U^-_x N_x = N_\Delta U^-_\Delta U^-_x U^+_x N_x \subseteq N_\Delta U_{\Phi^-} U_{\Phi^+} N_x.$$

Since $N \cap U_{\Phi^-} U_{\Phi^+}$ is trivial, we find $n'_\Delta \in N_\Delta$ and $n'_x \in N_x$ such that $n = n_x n'_x = n_\Delta n'_\Delta$ satisfies $g^{-1} n \in P_\Delta \cap P_x$, which is equal to $P_{\Delta \cup \{x\}}$ by 5.3. This proves our claim.

Now we write as in Theorem 5.2 $\Omega = \bigcup_{\sigma \in \Sigma} \Omega_\sigma$, where Ω_σ runs over all finite subsets of Ω containing x_0. For all σ we choose some $n_\sigma \in N$ such that $g^{-1} n_\sigma \in P_{\Omega_\sigma}$. Put $n_\sigma = n_0$ if Ω_σ is the set $\{x_0\}$. The same argument as in 5.2 shows that we can find a subset $\Sigma' \subseteq \Sigma$ such that $\Omega = \bigcup_{\sigma \in \Sigma'} \Omega_\sigma$ and such that $n_\sigma^{-1} n_0$ is equal to some fixed $m \in N_{x_0}$ modulo T_{x_0} for all $\sigma \in \Sigma'$. Since any element in T_{x_0} leaves the components Λ_I for $I \subseteq J$ pointwise invariant, it also stabilizes Ω. Therefore $n_\sigma^{-1} n_0 m^{-1}$ lies in $N_\Omega \subset P_\Omega$. Since $g^{-1} n_\sigma$ is contained in P_{Ω_σ}, the same holds for $g^{-1} n_0 m^{-1}$, so that $g^{-1} n_0 m^{-1}$ lies in $\bigcap_{\sigma \in \Sigma'} P_{\Omega_\sigma}$, which is equal to P_Ω by 5.3. Hence $n = n_0 m^{-1}$ satisfies $nx = gx$ for all $x \in \Omega$, as desired. □

Now we can prove

Theorem 5.5 *$\overline{X}$ is compact.*

By [Bou-T], I.10.4, Proposition 8, we know that since $U_0^\wedge \times \overline{\Lambda}$ is compact, the quotient after $\sim$ is Hausdorff (hence compact), iff the relation $\sim$ is closed in $(U_0^\wedge \times \overline{\Lambda}) \times (U_0^\wedge \times \overline{\Lambda})$.

Let $(u_k)_k$ and $(v_k)_k$ be sequences in $U_0^\wedge$ converging to u respectively v, and let x_k and y_k be sequences in $\overline{\Lambda}$ converging to x respectively y, so that $(u_k, x_k) \sim (v_k, y_k)$. We have to show that $(u, x) \sim (v, y)$. By definition of $\sim$ we have $x_k = u_k^{-1} v_k y_k$, so that both 0 and x_k lie in $\overline{\Lambda}$ and in $u_k^{-1} v_k \overline{\Lambda}$. Using 5.4, we find some $n_k \in N$ such that $n_k z = v_k^{-1} u_k z$ for all $z \in \overline{\Lambda} \cap u_k^{-1} v_k \overline{\Lambda}$. In particular, n_k lies in N_0 and $n_k x_k = v_k^{-1} u_k x_k = y_k$.

Hence $g_k = u_k^{-1} v_k n_k$ lies in $P_{x_k} \cap P_0 = P_{\{0, x_k\}}$. Since N_0 is compact, we can pass to a subsequence and assume that n_k converges towards some $n \in N_0$, so that g_k converges towards some $u^{-1} v n$.

By 4.8, we can write $g_k = w_k^- w_k^+ m_k$ for some $w_k^\pm \in U^\pm_{\{0, x_k\}}$ and $m_k \in N_{\{0, x_k\}}$. We can again assume that m_k converges towards some $m \in N_0$. Since N acts continuously on $\overline{\Lambda}$, m lies also in N_x. Besides, U_0^- is compact, so that we can assume that w_k^- converges towards some $w^- \in U_0^-$. Using 4.7 and 4.3 we find that w^- lies in fact in U_x^-. Now w_k^+ also converges towards some w^+ which lies in U_x^+ by the same argument. Therefore $u^{-1} v n$ lies in P_x. Since N acts continuously on $\overline{\Lambda}$, we have $nx = y$, so that indeed $(u, x) \sim (v, y)$. □

Theorem 5.6 *The space $\overline{X}$ is contractible.*

Proof: Recall from 3.4 that $\overline{\Lambda}$ is contractible. If $x = \sum_{i \neq j \in J} x_j (-\eta_j^J)$ is a point in $E_i \cap \Lambda_J$, where J contains i, then the contraction map is given by

$$r(x,t) = \begin{cases} x, & \text{if } t = 0 \\ \sum_{i \neq j \in J} \frac{(1-t) x_j}{1 + t x_j} (-\eta_j) + \sum_{j \notin I} \frac{1-t}{t} (-\eta_j), & \text{if } t \neq 0. \end{cases}$$

Now we define

$$\begin{array}{rll} R: & U_0^\wedge \times \overline{\Lambda} \times [0,1] & \longrightarrow U_0^\wedge \times \overline{\Lambda} \\ & ((g,x),t) & \longmapsto (g, r(x,t)). \end{array}$$

Obviously, R is continuous. In order to show that R is a contraction map for $\overline{X}$, it suffices to prove that it is compatible with our equivalence relation.
Let us first check that $nr(x,t) = r(nx,t)$ for all $n \in N_0$ and $x \in \overline{\Lambda}$. Write $n = tp$ for $t \in T$ and a permutation matrix p. Then t lies in N_0, so that it acts trivially on all points in $\overline{\Lambda}$. Besides, a straightforward calculation shows $pr(x,t) = r(px,t)$.
Now assume that (g,x) and (h,y) in $U_0^\wedge \times \overline{\Lambda}$ are equivalent. Hence there is some $n \in N$ with $nx = y$ and $g^{-1}hn \in P_x$. Using 5.4, we can assume that n lies in fact in N_0.
Now fix some $t \in [0,1]$. We already know that $r(y,t) = r(nx,t) = nr(x,t)$, so that our claim, namely $(g, r(x,t)) \sim (h, r(y,t))$ is proven, if we show that $g^{-1}hn \in P_{r(x,t)}$. We already know that $g^{-1}hn$ lies in $P_0 \cap P_x$, which is equal to $P_{\{0,x\}}$ by 5.3. Let us put $x_t = r(x,t)$. We are done if we show that $P_{\{0,x\}} \subset P_{x_t}$. We can assume that $t > 0$. The point x is contained in some $E_i \cap \Lambda_J$. Recall from 4.8 that $P_{\{0,x\}} = U^-_{\{0,x\}} U^+_{\{0,x\}} N_{\{0,x\}}$ and $P_{x_t} = U^-_{x_t} U^+_{x_t} N_{x_t}$ for some fixed Φ^+. A straightforward calculation yields

$$\begin{array}{lcl} f_{x_t}(a_{kl}) \leq & f_x(a_{kl}), & \text{if } k \neq i \text{ and } l \text{ are in } J \text{ and } f_x(a_{kl}) \geq 0, \\ & & \text{or if } k \notin J \text{ and } l \in J \\ f_{x_t}(a_{kl}) \leq & 0, & \text{if } k \neq i \text{ and } l \text{ are in } J \text{ and } f_x(a_{kl}) < 0, \\ & & \text{or if } k = i \text{ and } l \in J, \text{ or if } l \notin J \end{array}$$

Hence for all $a \in \Phi$ we have $U_{a,0} \subset U_{a,x_t}$ or $U_{a,x} \subset U_{a,x_t}$, which implies that $U^+_{\{0,x\}} \subset U^+_{x_t}$ and $U^-_{\{0,x\}} \subset U^-_{x_t}$. Besides, we have $N_{\{0,x\}} \subset N_{x_t}$, since r is compatible with the action of N_0, so that our claim follows. □
The following result shows that the boundary of our compactification $\overline{X}$ consists of the Bruhat-Tits buildings of all groups $PGL(W)$, where W is a non-trivial subspace of V:

Theorem 5.7 *There is a bijection between $\overline{X}$ and the set*

$$\bigcup_{0 \neq W \subseteq V} PGL(W) = X \cup \bigcup_{0 \neq W \subset V} PGL(W).$$

Proof: Let us first fix some non-empty $I \subset \underline{n}$. We will start by embedding the building corresponding to $\mathbf{G}_I = PGL(V_I)$ in $\overline{X}$. Recall that we write $\mathbf{G}^{V_I}$ for the subgroup of $\mathbf{G}$ leaving V_I invariant, and ρ_I for the map $\mathbf{G}^{V_I} \to \mathbf{G}_I$. Then the building for $\mathbf{G}_I$ is defined as $G_I \times \Lambda_I / \sim$, where $\sim$ is the equivalence relation from section 2 (replacing V by V_I everywhere). Let (h,x) be an element of $G_I \times \Lambda_I$. We choose an arbitrary lift $h^{\uparrow}$ of h in G^{V_I}, and map (h,x) to the point $h^{\uparrow}x$ in $\overline{X}$. This induces a map

$$j_I : G_I \times \Lambda_I \longrightarrow \overline{X}.$$

We claim that this is independent of the choice of a lift. We have to show that any element g in the kernel of ρ_I stabilizes each $x \in \Lambda_I$. Note that G^{V_I} is a parabolic subgroup of G. Let us fix a Borel group $B \supset T$ contained in G^{V_I}. Then there is a set of positive roots Φ^+ in Φ such that $B = U_{\Phi^+}T$. Besides, we can write $G^{V_I} = BW'B$ for some subgroup W' of W. Hence we find that $G^{V_I} = U_{\Phi^+}N^{V_I}U_{\Phi^+}$, where $N^{V_I} = N \cap G^{V_I}$. Note that since U_{Φ^+} is contained in G^{V_I}, no root of the form $a = a_{ij}$ such that $j \in I$, but $i \notin I$ can be contained in Φ^+. We write $g = u^+nv^+$ for some n in N^{V_I} and u^+, v^+ in U_{Φ^+}. Using $\rho_I(g) = 1$, we find that $\rho_I(n) = 1$ and $\rho_I(u^+v^+) = 1$. Hence n stabilizes each $x \in \Lambda_I$, so $n \in P_x \cap N = N_x$.

Let us write $[I]$ for the subset of all roots that are linear combinations of roots a_{ij} with i and j in I. We write $u^+ = u_1^+u_2^+$ and $v^+ = v_2^+v_1^+$ for some $u_1^+, v_1^+ \in m(\prod_{a \in \Phi^+ \backslash [I]} U_a)$ and $u_2^+, v_2^+ \in m(\prod_{a \in \Phi^+ \cap [I]} U_a)$, where m is the multiplication map. As in 4.2, we find that $\rho_I(u_1^+)$ and $\rho_I(v_1^+)$ are trivial, and that ρ_I is injective on all U_a for $a \in \Phi^+ \cap [I]$. Hence we deduce that $u_2^+v_2^+$ is trivial. Besides, u_2^+ commutes with n, so that $g = u_1^+u_2^+nv_2^+v_1^+ = u_1^+nv_1^+$. Recall that the roots a_{ij} for $i \notin I$ and $j \in I$ do not lie in Φ^+, so that by 4.1 we have $U_a = U_{a,x}$ for all $a \in \Phi^+ \backslash [I]$ and all $x \in \Lambda_I$. Hence g is indeed contained in P_x for all these x.

Now we claim that j_I induces an injection

$$j_I : X(G_I) = G_I \times \Lambda_I / \sim \; \hookrightarrow \overline{X}.$$

It suffices to show that for all $x \in \Lambda_I$ the map ρ_I induces a surjection

$$\rho_I : P_x \longrightarrow P_x^I,$$

where P_x^I is defined in the same way as P_x (see section 2), just replacing V by V_I. Note that this also implies $P_x = \rho_I^{-1}(P_x^I)$.

First of all, let us show that P_x is contained in G^{V_I}, so that our statement makes sense. By 4.2, we find that U_x is contained in G^{V_I}. Besides, if n is an element of N_x, we find that $n = tp$ for a torus element t and a permutation matrix p. The corresponding permutation must leave I intact, so that indeed $n \in N^{V_I}$.

Besides, by 4.2 we know that ρ_I maps U_x to U_x^I, and it is easy to see that ρ_I also maps N_x to N_x^I, so that we get a homomorphism $\rho_I : P_x \to P_x^I$. Now take some $h = nu$ in $P_x^I = N_x^I U_x^I$, and choose a lift $n^{\uparrow}$ of n in N^{V_I}. Since n stabilizes x (viewed in the apartment Λ_I), the lift $n^{\uparrow}$ stabilizes the point $x \in \overline{\Lambda}$. Hence $n^{\uparrow}$ lies in P_x. By 4.2, the element u has a lift $u^{\uparrow}$ in U_x, hence $h^{\uparrow} = n^{\uparrow}u^{\uparrow}$ is an element in P_x projecting to h via ρ_I. This proves surjectivity.

Let now W be an arbitrary non-trivial subspace of V. Then there is a linear isomorphism
$$f : V_I \longrightarrow W$$
for some $I \subset \underline{n}$. Let $\mathbf{S}$ be the maximal torus in $PGL(W)$ induced by the diagonal matrices with respect to the basis $f(v_i)$ (for all $i \in I$). Conjugation by f induces an isomorphism $PGL(W) \to PGL(V_I) = \mathbf{G}_I$, which maps $\mathbf{S}$ to $\mathbf{T}_I$, and the normalizer $N(\mathbf{S})$ of $\mathbf{S}$ to $N(\mathbf{T}_I)$, the normalizer of $\mathbf{T}_I$ in G_I. Hence we get an $\mathbb{R}$-linear isomorphism
$$\tau : X_*(\mathbf{S})_{\mathbb{R}} \xrightarrow{\sim} X_*(\mathbf{T}_I)_{\mathbb{R}} = \Lambda_I.$$
One can check that for all $x \in X_*(\mathbf{S})_{\mathbb{R}}$ and $n \in N(\mathbf{S})$ we have $\tau(nx) = (f^{-1}nf)\tau(x)$.
Choose some $f^{\uparrow} \in G$ whose restriction to V_I is given by f. Then we define a map
$$j_W : PGL(W) \times X_*(\mathbf{S})_{\mathbb{R}} \longrightarrow G_I \times \Lambda_I \xrightarrow{j_I} \overline{X} \xrightarrow{f^{\uparrow}} \overline{X},$$
where the first map is given by
$$(g, x) \longmapsto (f^{-1}gf, \tau(x)).$$
Since this maps the equivalence relation on $PGL(W) \times X_*(\mathbf{S})_{\mathbb{R}}$ defining the building $X(PGL(W))$ to the equivalence relation on $G_I \times \Lambda_I$ defining the building $X(G_I)$, we have an injection
$$j_W : X(PGL(W)) \xrightarrow{\sim} X(G_I) \xrightarrow{j_I} \overline{X} \xrightarrow{f^{\uparrow}} \overline{X}.$$
Of course, we have to check that this is well-defined. First of all, if f is fixed, then j_W does not depend on the choice of a lifting $f^{\uparrow}$, since two such liftings differ by something in the kernel of ρ_I, and this acts trivially on the image of j_I, as we have seen above.
What happens if we choose another isomorphism $g : V_J \to W$ for some $J \subset \underline{n}$? First let us consider the case that we construct j_W using $f' = s \circ f$ for some isomorphism $s : W \to W$. Then we also use a different construction of the building $X(PGL(W))$, since we use another torus. Following [La], 13.18, we find that there exists a unique $PGL(W)$–equivariant isometry between these two constructions (which we tacitly use to identify them). A straightforward calculation shows that our map j_W is compatible with this isometry.
Now assume that we use an isomorphism $g : V_J \to W$ to construct j_W. Then there exists some isomorphism $r : V_I \to V_J$ mapping the basis v_i for $i \in I$ to the basis v_j for $j \in J$ in some way. The map r can be lifted to a permutation matrix $n \in N$. It is easy to see that this implies that our construction of j_W does indeed not change if we choose $g \circ r$ instead of g. Hence j_W is well-defined.
We put all these maps j_W together and get a map
$$j : X \cup \bigcup_{0 \neq W \subset V} X(PGL(W)) \longrightarrow \overline{X},$$

which is obviously surjective. It remains to show that j is injective. Let us first assume that we have some $(g,x) \in PGL(W) \times X_*(\mathbf{S})_{\mathbb{R}}$ and $(h,y) \in PGL(W') \times X_*(\mathbf{S}')_{\mathbb{R}}$ such that $j_W(g,x) = j_{W'}(h,y)$. Hence there are isomorphisms $f_I : V_I \to W$ and $f_J : V_J \to W'$ and points $x_0 \in \Lambda_I$, $y_0 \in \Lambda_J$ such that $g^{\uparrow} f_I^{\uparrow} x_0 = h^{\uparrow} f_J^{\uparrow} y_0$. In particular, there is some $n \in N$ mapping x_0 to y_0. Hence n maps Λ_I to Λ_J, which implies $nV_I = V_J$. We have already seen that $P_{x_0} \subset G^{V_I}$, so that $h^{\uparrow} f_J^{\uparrow} n V_I = g^{\uparrow} f_I^{\uparrow} V_I$, which implies $W = f_I(V_I) = f_J(V_J) = W'$. Since we already know that j_W is injective, our claim follows. □

To conclude this paper, let us show that we can identify the vertices in $\overline{X}$ with the equivalence classes $\{N\}$ of R-modules of arbitrary rank in V. Together with Proposition 3.1, this is the link to Mustafin's paper [Mu].

We call a point x in $\overline{\Lambda}$, say $x \in \Lambda_I$, a vertex in $\overline{\Lambda}$, if it is a vertex in the apartment Λ_I. By section 2 this means, that $x = \sum_{i \in I} x_i \eta_i^I$ with integer coefficients x_i. We call a point y in $\overline{X}$ a vertex if $y = gx$ for some $g \in G$ and some vertex $x \in \overline{\Lambda}$, and we denote the set of vertices in $\overline{X}$ by $\overline{X}^0$.

We call two R-lattices in V equivalent, if they differ by a factor in $K^{\times}$. Let us denote by $\overline{\mathcal{L}}$ the set of all equivalence classes of R-lattices in V of *arbitrary* positive rank. We write $\{M\}$ for the class of such a lattice.

Our last result shows that the vertices in $\overline{X}$ correspond to elements of $\overline{\mathcal{L}}$, i.e. lattice classes of arbitrary rank, which explains the title of this paper.

Lemma 5.8 *The G-equivariant bijection $\varphi : \mathcal{L} \longrightarrow X^0$ can be continued to a G-equivariant bijection*

$$\varphi : \overline{\mathcal{L}} \longrightarrow \overline{X}^0$$

in the following way: We write $\{M\} \in \overline{\mathcal{L}}$ as $\{M\} = g\{L\}$, where $L = \sum_{i \in I} \pi^{k_i} R v_i$ for some non-empty $I \subseteq \underline{n}$. Then

$$\varphi(\{M\}) = g(\sum_{i \in I} k_i(-\eta_i^I)).$$

Proof: We only need to check that φ is injective and well-defined, which amounts to the following claim: For the vertex $x = \sum_{i \in I} k_i(-\eta_i^I) \in \Lambda_I$ let $L_x = \sum_{i \in I} \pi^{k_i} R v_i$. Then P_x is the stabilizer of $\{L_x\}$. Using 4.1, it is easy to see that all $U_{a,x}$ leave $\{L_x\}$ invariant. Besides, N_x leaves $\{L_x\}$ invariant, so that P_x is contained in $S_{\{L_x\}}$, the stabilizer of $\{L_x\}$.

Let now g be an element in $S_{\{L_x\}}$. Note that L_x is a lattice of full rank in V_I, so that g is contained in G^{V_I}, and $\rho_I(g) \in PGL(V_I) = G_I$ stabilizes $\{L_x\}$. In G_I we have the (usual) Bruhat decomposition $\rho_I(g) = pnq$ with p and q in P_x^I and $n \in N(T_I)$ (see [La], 12.10). Here P_x^I is the stabilizer of x in the building for G_I, so that p and q leave the class $\{L_x\}$ in V_I invariant. Hence $n \in G_I$ also stabilizes $\{L_x\}$. A straightforward calculation now shows that n fixes $x \in \Lambda_I$. Therefore $\rho_I(g)$ lies in P_x^I, so that g is indeed contained in P_x, as we have seen in the proof of 5.7. □

Of course, one could also set up a bijection between $\overline{\mathcal{L}}$ and $\bigcup_{0 \neq W \subseteq V} X(PGL(W))^0$ by using analogues of the map φ on every single building $X(PGL(W))$. It is easy to see that this is compatible with the map we just described, and the identification of $\overline{X}$ and $\bigcup_{0 \neq W \subseteq V} X(PGL(W))$ given in 5.7.

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Partition-Dependent Stochastic Measures and q-Deformed Cumulants

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Abstract. On a q-deformed Fock space, we define multiple q-Lévy processes. Using the partition-dependent stochastic measures derived from such processes, we define partition-dependent cumulants for their joint distributions, and express these in terms of the cumulant functional using the number of restricted crossings of P. Biane. In the single variable case, this allows us to define a q-convolution for a large class of probability measures. We make some comments on the Itô table in this context, and investigate the q-Brownian motion and the q-Poisson process in more detail.

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1. Introduction

In [RW97], Rota and Wallstrom introduced, in the context of usual probability theory, the notion of partition-dependent stochastic measures. These objects give precise meaning to the following heuristic expressions. Start with a Lévy process $X(t)$. For a set partition $\pi = (B_1, B_2, \ldots, B_k)$, temporarily denote by $c(i)$ the number of the class $B_{c(i)}$ to which i belongs. Then, heuristically,

$$\mathrm{St}_\pi(t) = \int_{\substack{[0,t)^k \\ \text{all } s_i\text{'s distinct}}} dX(s_{c(1)})dX(s_{c(2)}) \cdots dX(s_{c(n)}).$$

In particular, denote by Δ_n the higher diagonal measures of the process defined by

$$\Delta_n(t) = \int_{[0,t)} (dX(s))^n.$$

These objects were used to define the Itô multi-dimensional stochastic integrals through the usual product measures, by employing the Möbius inversion on the lattice of all partitions. In particular this approach unifies a number of combinatorial results in probability theory.

The formulation of the algebraic (noncommutative, quantum) probability goes back to the beginnings of quantum mechanics and operator algebras. While a number of results have been obtained in a general context, in many cases the lack of tight hypotheses guaranteed that the conclusions of the theory would be somewhat loose. In the last twenty years of the twentieth century a particular noncommutative probability theory, the free probability theory [VDN92, Voi00], appeared, whose wealth of results approaches that of the classical one. This theory is based on a new notion of independence, the so-called *free independence*. In particular, one defines the (additive) free convolution, a new binary operation on probability measures: $\mu \boxplus \nu$ is the distribution of the sum of freely independent operators with distributions μ, ν. Note that this is precisely the relation between independence and the usual convolution. Many limit theorems for independent random variables carry over to free probability [BP99] by adapting the method of characteristic functions, using the R-transform of Voiculescu in place of the Fourier transform. Applications of the theory range from von Neumann algebras to random matrix theory and asymptotic representations of the symmetric group.

In [Ans00, Ans01a] (see also [Ans01b]) we investigated the analogs of the multiple stochastic measures of Rota and Wallstrom in the context of free probability theory. In this analysis, the starting object $X(t)$ is a stationary process with *freely* independent *bounded* increments. One important fact observed was that in the classical case the expectation of $\mathrm{St}_\pi(t)$ is the combinatorial cumulant of the distribution of $X(t)$. This means that the expectation of $\mathrm{St}_\pi(t)$ is equal to $\prod_{j=1}^{k} r_{|B_j|}$, where r_i is the i-th coefficient in the Taylor series expansion of the logarithm of the Fourier transform of the distribution of $X(t)$. See [Shi96, Nic95] or Section 6.1. The importance of cumulants lies in their relation to independence: since independence corresponds to a factorization property of the joint Fourier transforms, it can also be expressed as a certain additivity property of cumulants, the "mixed cumulants of independent quantities equal 0" condition. See Section 4.1.

It was observed by Speicher that in free probability, the condition of free independence can also be expressed in terms of a certain different family of cumulants, the so-called free cumulants; see [Spe97a] for a review. One also naturally obtains partition-dependent free cumulants, but only for partitions that are *noncrossing*. In [Ans00], we showed that in the free case $\mathrm{St}_\pi(t) = 0$ if the partition π is crossing, and for a noncrossing partition π the expectation of $\mathrm{St}_\pi(t)$ is the corresponding free cumulant of the distribution of $X(t)$. Thus both in the classical and in the free case $\mathrm{St}_\pi(t)$ can be considered as an operator-valued version of combinatorial cumulants (no relation to [Spe98]).

We try out this idea on the q-deformed probability theory. This is a noncommutative probability theory in an algebra on the q-deformed Fock space, developed by a number of authors, see the References. Free probability corresponds to $q = 0$, while classical (bosonic) and anti-symmetric (fermionic) theories correspond to $q = 1, -1$ (in these cases q-Fock spaces degenerate to the symmetric and anti-symmetric Fock spaces, respectively). For the intermediate values of q, it is known that the theory cannot be as good as the classical and the free ones, since one cannot define a notion of q-independence satisfying all the desired properties [vLM96, Spe97b].

In this paper we try a different approach. As mentioned above, whenever we have a family of operators which corresponds to a family of measures that is in some sense infinitely divisible, we should be able to define the partition-dependent stochastic measures, and then *define* the combinatorial cumulants as their expectations. One definition of cumulants appropriate for the q-deformed probability theory has already been given in [Nic95], based on an analog of the canonical form introduced by Voiculescu in the context of free probability. The advantage of the approach of that paper is that Nica's cumulants are defined for any probability distribution all of whose moments are finite. However, the canonical form of that paper is not self-adjoint, and it is also not appropriate for our approach since it does not provide us with a natural additive process. Instead, as a canonical form we choose the q-analog of the families of [HP84, AH84, Sch91, GSS92], which in the classical and the free case provide representations for all (classically, resp. freely) infinitely divisible distributions all of whose moments are finite. We provide an explicit formula for the resulting combinatorial cumulants, involving as expected a notion of the number of crossings of a partition. The appropriate one for our context happens to be the number of "restricted crossings" of [Bia97]; in particular the resulting cumulants are different from those of [Nic95]. Our approach makes sense only for distributions corresponding to q-infinitely divisible families (although strictly speaking, one can use our definition in general). However, our canonical form of an operator is self-adjoint, and this in turn leads to a notion of q-convolution on a large class of probability measures. The fact that this convolution is not defined on all probability measures is actually to be expected, see Section 6.1.

After finishing this article, we learned about a physics paper [NS94] which seems to have been overlooked by the authors of both [Bia97] and [SY00b]. The goals of that paper are different from ours, but in particular it defines and investigates the same q-Poisson process as we do (Section 6.3) and, in that case, points out the relation between the moments of the process and the number of restricted crossings of corresponding partitions. It would be interesting to see if the results of that paper can be extended to our context, and how our results fit together with the "partial cumulants" approach.

The paper is organized as follows. Following the Introduction, in Section 2 we provide some background on the combinatorics of partitions and on q-Fock spaces, and define the q-Lévy processes. In Section 3, we define the joint moments and q-cumulants, and express partition-dependent q-cumulants in terms of the q-cumulant functional. In Section 4 we show that the cumulant functional is the generator, in the sense defined in that section, of the family of time-dependent moment functionals, and characterize all such generators. We also discuss the notion of a product state that arises from this construction. In Section 5, we provide some information about the Itô product formula in this context, by calculating the quadratic co-variation of two q-Lévy processes. In a long Section 6 we define the q-convolution, describe how our construction relates to the Bercovici-Pata bijection, and investigate the q-Brownian motion and the q-Poisson process in more detail. Finally, in the last section we make a few preliminary comments on the von Neumann algebras generated by the processes.

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2. PRELIMINARIES

2.1. NOTATION. Fix a parameter $q \in (-1, 1)$; we will usually omit the dependence on q in the notation. The analogs of the results of this paper for $q = \pm 1$ are in most cases well-known; we will comment on them throughout the paper. For n a non-negative integer, denote by $[n]_q$ the corresponding q-integer, $[0]_q = 0$, $[n]_q = \sum_{i=0}^{n-1} q^i$.

For a collection $\left\{ y_j^{(i)} \right\}$ of numbers and two multi-indices $\vec{v} = (v(1), \ldots, v(k))$ and $\vec{u} = (u(1), \ldots, u(k))$, we will throughout the paper use the notation $\mathbf{y}_{\vec{v}}^{(\vec{u})}$ to denote $\prod_{j=1}^{k} y_{v(j)}^{(u(j))}$.

Denote by $[k \ldots n]$ the ordered set of integers in the interval $[k, n]$.

For a family of functions $\{F_j\}$, where F_j is a function of j arguments, $\vec{v}$ a vector with k components, and $B \subset [1 \ldots k]$, denote $F(\vec{v}) = F_k(\vec{v})$ and

$$F(B : \vec{v}) = F_{|B|}(v(i(1)), v(i(2)), \ldots, v(i(|B|))),$$

where $B = (i(1), i(2), \ldots, i(|B|))$. In particular, we use this notation for joint moments and cumulants (see below).

2.2. PARTITIONS. For an ordered set S, denote by $\mathcal{P}(S)$ the lattice of set partitions of that set. Denote by $\mathcal{P}(n)$ the lattice of set partitions of the set $[1 \ldots n]$, and by $\mathcal{P}_2(n)$ the collection of its pair partitions, i.e. of partitions into 2-element classes. Denote by $\leq$ the lattice order, and by $\hat{1}_n = ((1, 2, \ldots, n))$ the largest and by $\hat{0}_n = ((1)(2) \ldots (n))$ the smallest partition in this order.

Fix a partition $\pi \in \mathcal{P}(n)$, with classes $\{B_1, B_2, \ldots, B_l\}$. We write $B \in \pi$ if B is a class of π. Call a class of π a *singleton* if it consists of one element. For a class B, denote by $a(B)$ its first element, and by $b(B)$ its last element. Order the classes according to the order of their last elements, i.e. $b(B_1) < b(B_2) < \ldots < b(B_l)$. Call a class $B \in \pi$ an interval if $B = [a(B) \ldots b(B)]$. Call π an *interval partition* if all the classes of π are intervals.

Following [Bia97], we define the number of *restricted crossings* of a partition π as follows. For B a class of π and $i \in B, i \neq a(B)$, denote

$p(i) = \max\{j \in B, j < i\}$. For two classes $B, C \in \pi$, a restricted crossing is a quadruple $(p(i) < p(j) < i < j)$ with $i \in B, j \in C$. The number of restricted crossings of B, C is

$$\begin{aligned}\operatorname{rc}(B,C) = &\left|\{i \in B, j \in C : p(i) < p(j) < i < j\}\right| \\ &+ \left|\{i \in B, j \in C : p(j) < p(i) < j < i\}\right|,\end{aligned}$$

and the number of restricted crossings of π is $\operatorname{rc}(\pi) = \sum_{i<j} \operatorname{rc}(B_i, B_j)$. It has the following graphical representation. Draw the points $[1 \ldots n]$ in a sequence on the x-axis, and to represent the partition π connect each i with $p(i)$ (if it is well-defined) by a semicircle above the x axis. Then the number of intersections of the resulting semicircles is precisely $\operatorname{rc}(\pi)$. See Figure 1 for an example. We say that a partition π is *noncrossing* if $\operatorname{rc}(\pi) = 0$. Denote by $NC(n) \subset \mathcal{P}(n)$ the collection of all noncrossing partitions, which in fact form a sub-lattice of $\mathcal{P}(n)$.

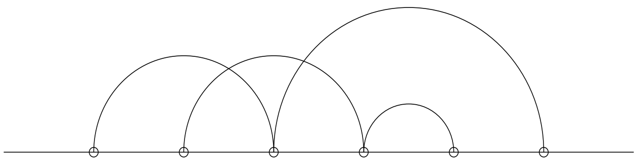

Figure 1. A partition of 6 elements with 2 restricted crossings.

We need some auxiliary notation. For $\sigma, \pi \in \mathcal{P}(n)$, we define $\pi \wedge \sigma \in \mathcal{P}(n)$ to be the meet of π and σ in the lattice, i.e.

$$i \overset{\pi\wedge\sigma}{\sim} j \ \Leftrightarrow\ i \overset{\pi}{\sim} j \text{ and } i \overset{\sigma}{\sim} j.$$

For $\pi \in \mathcal{P}(n)$, we define $\pi^{op} \in \mathcal{P}(n)$ to be π taken in the opposite order, i.e.

$$i \overset{\pi^{op}}{\sim} j \ \Leftrightarrow\ (n-i+1) \overset{\pi}{\sim} (n-j+1).$$

For $\pi \in \mathcal{P}(n), \sigma \in \mathcal{P}(k)$, we define $\pi + \sigma \in \mathcal{P}(n+k)$ by

$$i \overset{\pi+\sigma}{\sim} j \Leftrightarrow ((i,j \leq n, i \overset{\pi}{\sim} j) \text{ or } (i,j > n, (i-n) \overset{\sigma}{\sim} (j-n))).$$

We'll denote $m\pi = \pi + \pi + \ldots + \pi$ m times.
Finally, using the above notation, for a subset $B \subset [1 \ldots n]$ and $\pi \in \mathcal{P}(n)$, $(B : \pi)$ is the restriction of π to B.

2.3. The q-Fock space. Let H be a (complex) Hilbert space. Let $\mathcal{F}_{\mathrm{alg}}(H)$ be its algebraic full Fock space, $\mathcal{F}_{\mathrm{alg}}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}$, where $H^{\otimes 0} = \mathbb{C}\Omega$ and Ω is the vacuum vector. For each $n \geq 0$, define the operator P_n on $H^{\otimes n}$ by

$$\begin{aligned}&P_0(\Omega) = \Omega,\\ &P_n(\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n) = \sum_{\alpha \in \mathrm{Sym}(n)} q^{i(\alpha)} \eta_{\alpha(1)} \otimes \eta_{\alpha(2)} \otimes \ldots \otimes \eta_{\alpha(n)},\end{aligned}$$

where $\mathrm{Sym}(n)$ is the group of permutations of n elements, and $i(\alpha)$ is the number of inversions of the permutation α. For $q = 0$, each $P_n = \mathrm{Id}$. For $q = 1$, $P_n = n! \times$ the projection onto the subspace of symmetric tensors. For $q = -1$, $P_n = n! \times$ the projection onto the subspace of anti-symmetric tensors. Define the q-deformed inner product on $\mathcal{F}_{\mathrm{alg}}(H)$ by the rule that for $\zeta \in H^{\otimes k}$, $\eta \in H^{\otimes n}$,

$$\langle \zeta, \eta \rangle_q = \delta_{nk} \langle \zeta, P_n \eta \rangle ,$$

where the inner product on the right-hand-side is the usual inner product induced on $H^{\otimes n}$ from H. All inner products are linear in the second variable. It is a result of [BS91] that the inner product $\langle \cdot, \cdot \rangle_q$ is positive definite for $q \in (-1, 1)$, while for $q = -1, 1$ it is positive semi-definite. Let $\mathcal{F}_q(H)$ be the completion of $\mathcal{F}_{\mathrm{alg}}(H)$ with respect to the norm corresponding to $\langle \cdot, \cdot \rangle_q$. For $q = -1, 1$ one first needs to quotient out by the vectors of norm 0 and then complete; the result is the anti-symmetric, respectively, symmetric Fock space, with the inner product multiplied by $n!$ on the n-particle space.
For ξ in H, define the (left) creation and annihilation operators on $\mathcal{F}_{\mathrm{alg}}(H)$ by, respectively,

$$\begin{aligned} &a^*(\xi)\Omega = \xi, \\ &a^*(\xi)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \xi \otimes \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n, \end{aligned}$$

and

$$\begin{aligned} &a(\xi)\Omega = 0, \\ &a(\xi)\eta = \langle \xi, \eta \rangle \Omega, \\ &a(\xi)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \sum_{i=1}^n q^{i-1} \langle \xi, \eta_i \rangle \eta_1 \otimes \ldots \otimes \hat{\eta}_i \otimes \ldots \otimes \eta_n, \end{aligned}$$

where as usual $\hat{\eta}_i$ means omit the i-th term. For $q \in (-1, 1)$, both operators can be extended to bounded operators on $\mathcal{F}_q(H)$, on which they are adjoints of each other [BS91]. They satisfy the commutation relations $a(\xi)a^*(\eta) - qa^*(\eta)a(\xi) = \langle \xi, \eta \rangle \mathrm{Id}$. For $q = \pm 1$, we first need to compress the operators by the projection onto the symmetric / anti-symmetric Fock space, respectively, and the resulting operators differ from the usual ones by $\sqrt{n}$, but satisfy the usual commutation relations (thanks to a different inner product). For $q = 1$ the resulting operators are unbounded, but still adjoints of each other [RS75].
Denote by φ the vacuum vector state $\varphi[X] = \langle \Omega, X\Omega \rangle_q$.

2.4. Gauge operators. We now need to define differential second quantization. Consider the number operator, the differential second quantization of the identity operator. One choice, made in [Møl93], is to define it as the operator that has $H^{\otimes n}$ as an eigenspace with eigenvalue n. For a general self-adjoint operator T, this gives the true differential second quantization derived from the q-second quantization functor of [BKS97]. The resulting operators are self-adjoint, but do not satisfy nice commutation relations.

Another choice for the number operator is the operator that has $H^{\otimes n}$ as an eigenspace with eigenvalue $[n]_q$. For a general (bounded) operator T, the corresponding construction is

$$p(T)\Omega = 0,$$
$$p(T)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \sum_{i=1}^n q^{i-1} \eta_1 \otimes \ldots \otimes (T\eta_i) \otimes \ldots \otimes \eta_n.$$

Similar operators were used in [Śni00], where stochastic calculus with respect to the corresponding processes was developed. They do have nice commutation properties, but are in general not symmetric.

Finally, another natural choice for the number operator is $\sum_i a^*(e_i)a(e_i)$, where $\{e_i\}$ is an orthonormal basis for H; the resulting operator is then independent of the choice of the basis. For a general bounded operator T, the corresponding construction is $\sum_i a^*(Te_i)a(e_i)$. It is easy to see that this sum converges strongly, to the following operator.

Definition 2.1. Let T be an operator on H with dense domain $\mathcal{D}$. The corresponding *gauge operator* $p(T)$ is an operator on $\mathcal{F}_q(H)$ with dense domain $\mathcal{F}_{\mathrm{alg}}(\mathcal{D})$ defined by

$$p(T)\Omega = 0,$$
$$p(T)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \sum_{i=1}^n q^{i-1} (T\eta_i) \otimes \eta_1 \otimes \ldots \otimes \hat{\eta}_i \otimes \ldots \otimes \eta_n,$$

for $\eta_1, \eta_2, \ldots, \eta_n \in \mathcal{D}$.

Proposition 2.2. *If T is essentially self-adjoint on a dense domain $\mathcal{D}$ and $T(\mathcal{D}) \subset \mathcal{D}$, then $p(T)$ is essentially self-adjoint on a dense domain $\mathcal{F}_{\mathrm{alg}}(\mathcal{D})$.*

Proof. We first show that $p(T)$ is symmetric on $\mathcal{F}_{\mathrm{alg}}(\mathcal{D})$. Fix n, and denote by β_j the cycle in $\mathrm{Sym}(n)$ given by $\beta_j = (12\ldots j)$. For a permutation $\alpha \in \mathrm{Sym}(n)$, write $\alpha(\eta_1 \otimes \ldots \otimes \eta_n) = \eta_{\alpha(1)} \otimes \ldots \otimes \eta_{\alpha(n)}$. For $\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n \in \mathcal{D}$,

$$\begin{aligned}
&\langle p(T)\eta_1 \otimes \ldots \otimes \eta_n, \xi_1 \otimes \ldots \otimes \xi_n \rangle_q \\
&= \sum_{j=1}^n q^{j-1} \left\langle \beta_j^{-1}(\eta_1 \otimes \ldots \otimes (T\eta_j) \otimes \ldots \otimes \eta_n), \xi_1 \otimes \ldots \otimes \xi_n \right\rangle_q \\
&= \sum_{j=1}^n \sum_{\alpha \in \mathrm{Sym}(n)} q^{j-1} q^{i(\alpha)} \left\langle \beta_j^{-1}(\eta_1 \otimes \ldots \otimes (T\eta_j) \otimes \ldots \otimes \eta_n), \alpha(\xi_1 \otimes \ldots \otimes \xi_n) \right\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^n \sum_{\substack{\alpha \in \mathrm{Sym}(n) \\ \alpha(1)=k}} q^{j-1} q^{i(\alpha)} \left\langle \beta_j^{-1}(\eta_1 \otimes \ldots \otimes \eta_n), \alpha(\xi_1 \otimes \ldots \otimes (T^*\xi_k) \otimes \ldots \otimes \xi_n) \right\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^n \sum_{\substack{\alpha \in \mathrm{Sym}(n) \\ \alpha(1)=k}} q^{j-1} q^{i(\alpha)} \left\langle \eta_1 \otimes \ldots \otimes \eta_n, (\beta_j \alpha \beta_k) \beta_k^{-1}(\xi_1 \otimes \ldots \otimes (T^*\xi_k) \otimes \ldots \otimes \xi_n) \right\rangle
\end{aligned}$$

Using the combinatorial lemma immediately following this proof, this expression is equal to

$$= \sum_{k=1}^{n} \sum_{\gamma \in \mathrm{Sym}(n)} q^{k-1} q^{i(\gamma)} \left\langle \eta_1 \otimes \ldots \otimes \eta_n, \gamma(\beta_k^{-1}(\xi_1 \otimes \ldots \otimes (T^* \xi_k) \otimes \ldots \otimes \xi_n)) \right\rangle$$
$$= \langle \eta_1 \otimes \ldots \otimes \eta_n, p(T^*) \xi_1 \otimes \ldots \otimes \xi_n \rangle_q .$$

Now we show that the operator $p(T)$ is essentially self-adjoint on $\mathcal{F}_{\mathrm{alg}}(\mathcal{D})$. For $q = 1$, the proof is contained in [RS75, X.6, Example 3]. For $q \in (-1, 1)$ we proceed similarly. Let $\mathcal{D}_n = \mathcal{D}^{\otimes n}$. Let $E.$ be the spectral measure of the closure $\bar{T}$ of T, and $C \in \mathbb{R}_+$. Let $\{\eta_i\}_{i=1}^n \subset (E_{[-C.C]} H) \cap \mathcal{D}$; then $\|T \eta_i\| \leq C \|\eta_i\|$. Let $\vec{\eta} = \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n$. Then

$$\left\| p(T)^k \vec{\eta} \right\|_q^2 = \left\langle p(T)^k \vec{\eta}, P_n p(T)^k \vec{\eta} \right\rangle \leq \|P_n\| \, (n^k C^k \|\vec{\eta}\|)^2 .$$

It was shown in [BS91] that $\|P_n\| \leq [n]_{|q|}! \leq n!$. We conclude that $\left\| p(T)^k \vec{\eta} \right\|_q \leq \sqrt{n!} n^k C^k \|\vec{\eta}\|$ and so

$$\limsup_{k \to \infty} \frac{1}{k} \left\| p(T)^k \vec{\eta} \right\|_q^{1/k} = 0.$$

Therefore $\vec{\eta}$ is an analytic vector for $p(T)$. The linear span of such vectors is invariant under $p(T)$ and is a dense subset of $\mathcal{D}_n$. Therefore by Nelson's analytic vector theorem, $p(T)$ is essentially self-adjoint on $\mathcal{D}_n$.
The rest of the argument proceeds as in [RS72, VIII.10, Example 2]. An operator A is essentially self-adjoint iff the range of $A \pm i$ is dense. Since $p(T)$ restricted to $H^{\otimes n}$ is essentially self-adjoint, this property holds for each such restriction, and then for the operator $p(T)$ itself, which therefore has to be essentially self-adjoint. ☐

Lemma 2.3. *For a fixed k, every permutation $\gamma \in \mathrm{Sym}(n)$ appears in the collection*

$$\{\beta_j \alpha \beta_k : 1 \leq j \leq n, \alpha(1) = k\}$$

exactly once. Moreover, for such α, $i(\beta_j \alpha \beta_k) = i(\alpha) + j - k$.

Proof. It suffices to show the first property for the collection $\{\beta_j \alpha\}$. This collection contains at most $n!$ distinct elements. On the other hand, for $\gamma \in \mathrm{Sym}(n)$, let $j = \gamma^{-1}(k)$, and $\alpha = \beta_j^{-1} \gamma$; then j, α satisfy the conditions and $\beta_j \alpha = \gamma$.
For the second property, first take $\gamma \in \mathrm{Sym}(n)$ such that $\gamma(1) = 1$ and show that $i(\beta_j \gamma) = i(\gamma) + (j-1)$. Indeed, β_j only reverses the order of $(j-1)$ pairs (a, j) with $a < j$. β_j sends such a pair to $((a+1), 1)$, and since $\gamma(1) = 1$, γ preserves the order of such a pair.
We conclude that $i(\beta_j \alpha \beta_k) = i(\alpha \beta_k) + (j-1)$. Now we show that $i(\alpha \beta_k) = i(\alpha) - (k-1)$. Indeed, β_k only reverses the order of $(k-1)$ pairs (a, k) for $a < k$. The pre-image of such a pair under α is $(\alpha^{-1}(a), 1)$, and so α reverses its order. ☐

These gauge operators themselves do not satisfy nice commutation relations. Nevertheless, we can still calculate their combinatorial cumulants. Another advantage of this definition is that it naturally generalizes to the "Yang-Baxter" commutation relations of [BS94]. However, in this more general context partition-dependent cumulants are not expressed in terms of the cumulant functional, so we do not pursue this direction in more detail.

For $q = 0$, $p(T)$ are precisely the gauge operators on the full Fock space as defined in [GSS92]. For $q = 1$, again we first need to compress $p(T)$ by the projection onto the symmetric Fock space, and the result is the usual differential second quantization. For $q = -1$, we first need to compress $p(T)$ by the projection onto the anti-symmetric Fock space, and the result is the anti-symmetric differential second quantization.

2.5. The processes. Let V be a Hilbert space, and let H be the Hilbert space $L^2(\mathbb{R}_+, dx) \otimes V$. Let $\xi \in V$, and let T be an essentially self-adjoint operator on a dense domain $\mathcal{D} \subset V$ so that $\mathcal{D}$ is equal to the linear span of $\{T^n \xi\}_{n=0}^{\infty}$ and moreover ξ is an analytic vector for T. Given a half-open interval $I \subset \mathbb{R}_+$, define $a_I(\xi) = a(\mathbf{1}_I \otimes \xi)$, $a_I^*(\xi) = a^*(\mathbf{1}_I \otimes \xi)$, $p_I(T) = p(\mathbf{1}_I \otimes T)$. Here $\mathbf{1}_I$ is the indicator function of the set I, considered both as a vector in $L^2(\mathbb{R}_+)$ and a multiplication operator on it. For $\lambda \in \mathbb{R}$, denote $p_I(\xi, T, \lambda) = a_I(\xi) + a_I^*(\xi) + p_I(T) + |I| \lambda$. Denote by a_t, a_t^*, p_t the appropriate objects corresponding to the interval $[0, t)$. We will call a process of the form $I \mapsto p_I(\xi, T, \lambda)$ a *q-Lévy process*. For $q = 1$ this is indeed a Lévy process.

Now fix a k-tuple $\{T_j\}_{j=1}^k$ of essentially self-adjoint operators on a common dense domain $\mathcal{D} \subset V$, $T_j(\mathcal{D}) \subset \mathcal{D}$, a k-tuple $\{\xi_j\}_{j=1}^k \subset \mathcal{D}$ of vectors, and $\{\lambda_j\}_{j=1}^k \subset \mathbb{R}$. We will make an extra assumption that

$$
\begin{aligned}
&\forall i, j \in [1 \ldots k], l \in \mathbb{N}, \vec{u} \in [1 \ldots k]^l, \\
&\mathbf{T}_{\vec{u}} \xi_i = T_{u(1)} T_{u(2)} \ldots T_{u(l)} \xi_i \text{ is an analytic vector for } T_j, \\
&\text{ and } \mathcal{D} = \operatorname{span}\left(\left\{ \mathbf{T}_{\vec{u}} \xi_i : i \in [1 \ldots k], l \in \mathbb{N}, \vec{u} \in [1 \ldots k]^l \right\} \right).
\end{aligned} \tag{1}
$$

Denote by $\mathbf{X}$ the k-tuple of processes $(X^{(1)}, \ldots, X^{(k)})$, where $X^{(j)}(I) = p_I(\xi_j, T_j, \lambda_j)$. In particular $\mathbf{X}(t) = \mathbf{X}([0, t))$. We call such a k-tuple a *multiple q-Lévy process.*

Remark 2.4. The assumption (1) is not essential for most of the paper. Most of the analysis could be done purely algebraically: see Remark 5.1. We will make this assumption to guarantee that we have a correspondence between self-adjoint processes and semigroups of measures, rather than between symmetric processes and semigroups of moment sequences.

3. Cumulants

3.1. Joint distribution. Since the processes in $\mathbf{X}$ do not necessarily commute, by their joint distribution we will mean the collection of their joint moments. We organize this information as follows.

Denote by $\mathbb{C}\langle \mathbf{x} \rangle = \mathbb{C}\langle x_1, x_2, \ldots, x_k \rangle$ the algebra of polynomials in k formal noncommuting indeterminates with complex coefficients. Note that in a more abstract language, this is just the tensor algebra of the complex vector space V_0 with a distinguished basis $\{x_i\}_{i=1}^k$. While we take V_0 to be k-dimensional, the same arguments will work for an arbitrary V_0, as long as we use a more functorial definition of a process, namely for $f = \sum a_i x_i \in V_0$, we would define $T(f) = \sum a_i T_i, \xi(f) = \sum a_i \xi_i, \lambda(f) = \sum a_i \lambda_i$. See [Sch91] for a more detailed description of this approach.

Define a functional M on $\mathbb{C}\langle \mathbf{x} \rangle$ by the following action on monomials: $M(1, t; \mathbf{X}) = 1$, for a multi-index $\vec{u}$,

$$M(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \varphi \left[\mathbf{X}^{(\vec{u})}(t) \right],$$

and extend linearly. We will call $M(\cdot, t; \mathbf{X})$ the *moment functional* of the process $\mathbf{X}$ at time t.

If we equip $\mathbb{C}\langle \mathbf{x} \rangle$ with a conjugation $*$ extending the conjugation on $\mathbb{C}$ so that each $x_i^* = x_i$, it is clear that M is a positive functional, i.e. $M(ff^*, t; \mathbf{X}) \geq 0$ for all $f \in \mathbb{C}\langle \mathbf{x} \rangle$.

For a partition $\pi \in \mathcal{P}(n)$ and a monomial $\mathbf{x}_{\vec{u}}$ of degree n, denote $M_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \prod_{B \in \pi} M(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X})$. These are the *combinatorial moments* of $\mathbf{X}$ at time t.

For a one-dimensional process, the functional $M(\cdot, t; X)$ can be extended to a probability measure μ_t such that $\mu_t(x^n) = M(x^n, t; X)$. Specifically, $\mu_t(S) = \varphi[E_S]$, where $E.$ is the spectral measure of $X(t)$.

3.2. Multiple stochastic measures and cumulants. For a set S and a partition $\pi \in \mathcal{P}(n)$, denote

$$S_\pi^n = \left\{ \vec{v} \in S^n : v(i) = v(j) \Leftrightarrow i \overset{\pi}{\sim} j \right\}$$

and

$$S_{\leq \pi}^n = \left\{ \vec{v} \in S^n : v(i) = v(j) \Rightarrow i \overset{\pi}{\sim} j \right\}.$$

Fix t. For $N \in \mathbb{N}$ and a subdivision of $[0, t)$ into N disjoint ordered half-open intervals $\mathcal{I} = \{I_1, I_2, \ldots, I_N\}$, let $\delta(\mathcal{I}) = \max_{1 \leq i \leq N} |I_i|$. Denote X_i, a_i, a_i^*, p_i the appropriate objects for the interval I_i. Fix a monomial $\mathbf{x}_{\vec{u}} \in \mathbb{C}\langle x_1, x_2, \ldots, x_k \rangle$ of degree n.

Definition 3.1. The *stochastic measure* corresponding to the partition π, monomial $\mathbf{x}_{\vec{u}}$, and subdivision $\mathcal{I}$ is

$$\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I}) = \sum_{\vec{v} \in [1 \ldots N]_\pi^n} \mathbf{X}_{\vec{v}}^{(\vec{u})}.$$

The stochastic measure corresponding to the partition π and the monomial $\mathbf{x}_{\vec{u}}$ is

$$\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \lim_{\delta(\mathcal{I}) \to 0} \mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I})$$

if the limit, along the net of subdivisions of the interval $[0,t)$, exists. In particular, denote by $\Delta_n(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I}) = \mathrm{St}_{\hat{1}}(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I})$ and

$$\Delta_n(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \mathrm{St}_{\hat{1}}(\mathbf{x}_{\vec{u}}, t; \mathbf{X})$$

the n-dimensional *diagonal measure.*

Definition 3.2. The *combinatorial cumulant* corresponding to the partition π and the monomial $\mathbf{x}_{\vec{u}}$ is

$$R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \lim_{\delta(\mathcal{I})\to 0} \varphi\left[\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I})\right]$$

if the limit exists. In particular, denote by

$$R(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = R_{\hat{1}}(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \lim_{\delta(\mathcal{I})\to 0} \varphi\left[\Delta_n(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I})\right]$$

the n-th joint cumulant of $\mathbf{X}$ at time t. Note that the functional $R(\cdot, t; \mathbf{X})$ can be linearly extended to all of $\mathbb{C}\langle \mathbf{x}\rangle$. We call this functional the *cumulant functional* of the process $\mathbf{X}$ at time t. For $t = 1$ we call the corresponding functional the cumulant functional of the process $\mathbf{X}$.

We will omit the dependence on $\mathbf{X}$ in the notation if it is clear from the context. Clearly if $\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t)$ is well-defined, its expectation is equal to $R_\pi(\mathbf{x}_{\vec{u}}, t)$.
By definition of S^n_π, for any $\mathcal{I}$

$$\mathbf{X}^{(\vec{u})}(t) = \sum_{\pi \in \mathcal{P}(n)} \mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I}). \tag{2}$$

If $\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t)$ are well-defined, then

$$\mathbf{X}^{(\vec{u})}(t) = \sum_{\pi \in \mathcal{P}(n)} \mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}),$$

and so

$$M(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\pi \in \mathcal{P}(n)} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}); \tag{3}$$

in fact for this last property to hold it suffices that the combinatorial cumulants exist.
The following general algebraic notion of independence is due to Kümmerer.

Lemma 3.3. *A multiple q-Lévy process $\mathbf{X}(t)$ has* pyramidally independent *increments. That is, for a family of intervals $\left\{\{I_i\}_{i=1}^{n_1+n_3}, \{J_j\}_{j=1}^{n_2}\right\}$ in $\mathbb{R}_+$ such that for all i,j, $I_i \cap J_j = \emptyset$,*

$$\begin{aligned}\varphi\left[\left(\prod_{i=1}^{n_1} X^{(u(i))}(I_i)\right)\left(\prod_{j=1}^{n_2} X^{(v(j))}(J_j)\right)\left(\prod_{i=n_1+1}^{n_1+n_3} X^{(u(i))}(I_i)\right)\right] \\ = \varphi\left[\prod_{i=1}^{n_1+n_3} X^{(u(i))}(I_i)\right]\varphi\left[\prod_{j=1}^{n_2} X^{(v(j))}(J_j)\right].\end{aligned}$$

We record the following facts we will use in the proof. Their own proof is immediate.

Lemma 3.4. *Choose two families of intervals* $\left\{\{I_i\}_{i=1}^{n_1}, \{J_j\}_{j=1}^{n_2}\right\}$ *such that* $(\bigcup I_i) \cap (\bigcup J_j) = \emptyset$. *Let* $\mathbf{y} = \prod_{i=1}^{n_1} y_{I_i}^{(u(i))}$, *where each* $y_I^{(s)}$ *is one of* $a_I(\xi_s), a_I^*(\xi_s), p_I(T_s), |I|\,\lambda_s$. *Also let* $\vec{\eta}_1 \in \bigoplus_{j=0}^{n_1}(L^2(\bigcup I_i) \otimes V)^{\otimes j}$ *and* $\vec{\eta}_2 \in \bigoplus_{i=0}^{n_2}(L^2(\bigcup J_j) \otimes V)^{\otimes i}$, *where these two spaces are naturally embedded in* $\mathcal{F}_{\mathrm{alg}}(L^2(\mathbb{R}_+) \otimes V)$. *Then*

$$\mathbf{y}\vec{\eta}_2 = ((\mathbf{y} - \langle \Omega, \mathbf{y}\Omega\rangle_q)\Omega) \otimes \vec{\eta}_2 + \langle \Omega, \mathbf{y}\Omega\rangle_q \vec{\eta}_2$$

and

$$\langle \vec{\eta}_1, \vec{\eta}_2\rangle_q = \langle \vec{\eta}_1, \Omega\rangle_q \langle \Omega, \vec{\eta}_2\rangle_q .$$

Proof of Lemma 3.3. Fix a family of intervals $\left\{\{I_i\}_{i=1}^{n_1+n_3}, \{J_j\}_{j=1}^{n_2}\right\}$ such that for all i, j, $I_i \cap J_j = \emptyset$. Denote

$$\begin{aligned}
\vec{\eta}_1 &= \left(\prod_{i=n_1}^{1} p_{I_i}(\xi_{u(i)}, T_{u(i)}, \lambda_{u(i)})\right)\Omega && \in \bigoplus_{j=0}^{n_1}(L^2(\bigcup I_i) \otimes V)^{\otimes j},\\
\vec{\eta}_2 &= \left(\prod_{i=1}^{n_2} p_{J_i}(\xi_{v(i)}, T_{v(i)}, \lambda_{v(i)})\right)\Omega && \in \bigoplus_{j=0}^{n_2}(L^2(\bigcup J_i) \otimes V)^{\otimes j},\\
\vec{\eta}_3 &= \left(\prod_{i=n_1+1}^{n_1+n_3} p_{I_i}(\xi_{u(i)}, T_{u(i)}, \lambda_{u(i)})\right)\Omega && \in \bigoplus_{j=0}^{n_3}(L^2(\bigcup I_i) \otimes V)^{\otimes j}.
\end{aligned}$$

Then

$$\begin{aligned}
&\varphi\left[\left(\prod_{i=1}^{n_1} X^{(u(i))}(I_i)\right)\left(\prod_{j=1}^{n_2} X^{(v(j))}(J_j)\right)\left(\prod_{i=n_1+1}^{n_1+n_3} X^{(u(i))}(I_i)\right)\right]\\
&= \Bigg\langle \Big(\prod_{i=n_1}^{1} p_{I_i}(\xi_{u(i)}, T_{u(i)}, \lambda_{u(i)})\Big)\Omega,\\
&\qquad \Big(\prod_{j=1}^{n_2} p_{J_j}(\xi_{v(j)}, T_{v(j)}, \lambda_{v(j)}) \prod_{i=n_1+1}^{n_1+n_3} p_{I_i}(\xi_{u(i)}, T_{u(i)}, \lambda_{u(i)})\Big)\Omega \Bigg\rangle_q\\
&= \left\langle \vec{\eta}_1, \left(\prod_{j=1}^{n_2} p_{J_j}(\xi_{v(j)}, T_{v(j)}, \lambda_{v(j)})\right)\vec{\eta}_3 \right\rangle_q\\
&= \left\langle \vec{\eta}_1, (\vec{\eta}_2 - \langle \Omega, \vec{\eta}_2\rangle_q \Omega) \otimes \vec{\eta}_3 + \langle \Omega, \vec{\eta}_2\rangle_q \vec{\eta}_3 \right\rangle_q\\
&= \langle \Omega, \vec{\eta}_2\rangle_q \langle \vec{\eta}_1, \vec{\eta}_3\rangle_q\\
&= \varphi\left[\prod_{j=1}^{n_2} X^{(v(j))}(J_j)\right] \varphi\left[\prod_{i=1}^{n_1+n_3} X^{(u(i))}(I_i)\right].
\end{aligned}$$

PROPOSITION 3.5. *For a noncrossing partition σ,*

$$M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X})$$

if the combinatorial cumulants are well-defined.

Proof. A noncrossing partition is determined by the property that it contains a class that is an interval and the restriction of the partition to the complement of that class is still noncrossing. Using this fact and Lemma 3.3, we can conclude that for $\pi \in \mathcal{P}(n), \pi \leq \sigma$ and $\vec{v} \in [1 \dots N]^n_\pi$,

$$\varphi\left[\mathbf{X}^{(\vec{u})}_{\vec{v}}\right] = \prod_{B \in \sigma} \varphi\left[\mathbf{X}^{(B:\vec{u})}_{(B:\vec{v})}\right].$$

Therefore

$$\varphi\left[\mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I})\right] = \prod_{B \in \sigma} \varphi\left[\mathrm{St}_{(B:\pi)}(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X}, \mathcal{I})\right].$$

Thus if the combinatorial cumulants are well-defined,

$$R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \prod_{B \in \sigma} R_{(B:\pi)}(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X}),$$

and so

$$\sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} \prod_{B \in \sigma} R_{(B:\pi)}(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X}) = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}).$$

If $\sigma = (B_1, B_2, \dots, B_l)$, the left-hand-side of this equation is equal to

$$\prod_{i=1}^{l} \sum_{\pi_i \in \mathcal{P}(B_i)} R_{\pi_i}(\mathbf{x}_{(B_i:\vec{u})}, t; \mathbf{X}).$$

Combining this equation with equation (3), we obtain

$$M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}). \qquad \square$$

We emphasize that while σ is noncrossing, π need not be. Note that on the operator level we have for any $\sigma \in \mathcal{P}(n)$,

$$\sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \leq \sigma}} \mathrm{St}_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}, \mathcal{I}) = \sum_{\vec{v} \in [1 \dots N]^n_{\leq \sigma}} \mathbf{X}^{(\vec{u})}_{\vec{v}}.$$

PROPOSITION 3.6. *For the monomial $\mathbf{x}_{\vec{u}}$ of degree n, the cumulant functional of the multiple q-Lévy process $\mathbf{X}$ is given by*

$$R(\mathbf{x}_{\vec{u}}, t) = \begin{cases} t\lambda_{u(1)} & \text{if } n = 1, \\ t\left\langle \xi_{u(1)}, \prod_{j=2}^{n-1} T_{u(j)} \xi_{u(n)} \right\rangle & \text{if } n \geq 2. \end{cases}$$

Proof. By definition,

$$R(\mathbf{x}_{\vec{u}}, t) = \lim_{\delta(\mathcal{I}) \to 0} \varphi \left[\sum_{i=1}^{N} \prod_{j=1}^{n} p_i(\xi_{u(j)}, T_{u(j)}, \lambda_{u(j)}) \right].$$

For $n = 1$,

$$\langle \Omega, p_i(\xi, T, \lambda)\Omega \rangle_q = |I_i| \lambda,$$

and so $R(x, t) = t\lambda$.

Now let $n \geq 2$. Decomposing each $p_i(\xi, T, \lambda)$ into the four defining summands, we see that

$$\varphi \left[\sum_{i=1}^{N} \prod_{j=1}^{n} p_i(\xi_{u(j)}, T_{u(j)}, \lambda_{u(j)}) \right] = \sum_{S_1, S_2, S_3, S_4} \sum_{i=1}^{N} \left\langle \Omega, y_i^{(1)} y_i^{(2)} \ldots y_i^{(n)} \Omega \right\rangle_q . \tag{4}$$

Here the sum is taken over all decompositions of $[1 \ldots n]$ into four disjoint subsets S_1, S_2, S_3, S_4, and for each choice of these subsets

$$y_i^{(j)} = \begin{cases} a_i(\xi_{u(j)}) & \text{if } j \in S_1, \\ a_i^*(\xi_{u(j)}) & \text{if } j \in S_2, \\ p_i(T_{u(j)}) & \text{if } j \in S_3, \\ |I_i| \lambda_{u(j)} & \text{if } j \in S_4. \end{cases}$$

The term corresponding to $S_1 = \{1\}, S_2 = \{n\}, S_3 = [2 \ldots (n-1)], S_4 = \emptyset$ is equal to

$$\left\langle \mathbf{1}_{[0,t)} \otimes \xi_{u(1)}, (\mathbf{1}_{[0,t)} \otimes \prod_{j=2}^{n-1} T_{u(j)})(\mathbf{1}_{[0,t)} \otimes \xi_{u(n)}) \right\rangle = t \left\langle \xi_{u(1)}, \prod_{j=2}^{n-1} T_{u(j)} \xi_{u(n)} \right\rangle.$$

We show that the limit of each of the remaining terms is 0. Indeed, $y_i^{(1)} y_i^{(2)} \ldots y_i^{(n)} \Omega \in H^{\otimes(|S_2| - |S_1|)}$, so if $|S_1| \neq |S_2|$ the corresponding term in (4) is 0 even for finite N. Otherwise denote by $b(S_1, S_2)$ the set of all bijections $S_1 \to S_2$. All the terms that are not 0 are of the form

$$\sum_{i=1}^{N} \left(\left(\prod_{j_4 \in S_4} \lambda_{u(j_4)} \right) |I_i|^{|S_4|} \sum_{g \in b(S_1, S_2)} \sum_{\substack{S'_{j_1} \subset S_3 : j_1 \in S_1, \\ \bigcup_{j_1 \in S_1} S'_{j_1} = S_3}} Q_{g, \{S_{j_1}, j_1 \in S_1\}}(q) \, |I_i|^{|S_1|} \right.$$
$$\left. \times \prod_{j_1 \in S_1} \left\langle \xi_{u(j_1)}, \prod_{j_3 \in S'_{j_1}} T_{u(j_3)} \xi_{u(g(j_1))} \right\rangle \right),$$

where each $Q_{g, \{S'_{j_1} : j_1 \in S_1\}}(q)$ is a polynomial independent of i, and $|S_1| \geq 2$ or $|S_4| \geq 1, |S_1| \geq 1$; in both cases $|S_4| + |S_1| \geq 2$. Thus each of these terms is bounded by

$$C \sum_{i=1}^{N} |I_i|^{|S_1| + |S_4|} \leq C \delta(\mathcal{I}) t^{|S_1| + |S_4| - 1},$$

where C is a constant independent of the subdivision $\mathcal{I}$. Therefore such a term converges to 0 as $\delta(\mathcal{I}) \to 0$. $\square$

CONSTRUCTION 3.7 (An un-crossing map). Fix a partition π with l classes $B_1, \dots, B_l$. In preparation for the next theorem, we need the following combinatorial construction. Define the map $F : \mathcal{P}(n) \to \mathcal{P}(n)$ as follows. If π is an interval partition, $F(\pi) = \pi$. Otherwise, let i be the largest index of a non-interval class B_i of π. Let $j_2 = \max\{s \in B_i : (s-1) \notin B_i\}$ and $j_1 = p(j_2)$. Let α be the power of a cycle permutation

$$((j_1+1)(j_1+2)\dots b(B_i))^{b(B_i)-j_2+1}.$$

Then $F(\pi) = \alpha \circ \pi$, by which we mean $i \overset{\pi}{\sim} j \Leftrightarrow \alpha(i) \overset{F(\pi)}{\sim} \alpha(j)$. Also define $c_b(\pi) = |\{s : j_1 < b(B_s) < b(B_i)\}| - |\{s : j_1 < a(B_s) < b(B_i)\}|$. Then $\mathrm{rc}\,(\pi) = \mathrm{rc}\,(F(\pi)) + c_b(\pi)$. Indeed, for $B, C \in \pi, B, C \neq B_i$, $\mathrm{rc}\,(B,C) = \mathrm{rc}\,(\alpha(B), \alpha(C))$. The number of restricted crossings of B_i, B_j with $b_i \in B_i, b_j \in B_j$ and $p(b_i) < p(b_j) < b_i < b_j \leq j_1$ or $p(b_j) < p(b_i) < b_j < b_i \leq j_1$ is equal to the corresponding number for $\alpha(B_i), \alpha(B_j)$, while there are no restricted crossings for $b_i > j_2$ for B_i and $b_i > j_1$ for $\alpha(B_i)$. Finally, there are $c_b(\pi)$ restricted crossings of the form $p(j) < j_1 < j < j_2$ in π. See Figure 2 for an example.

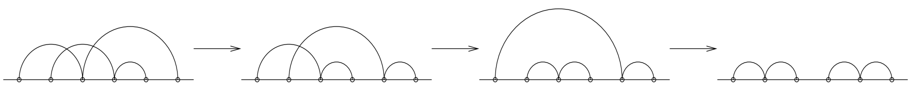

FIGURE 2. Iteration of F on a partition of 6 elements.

Clearly $F^n(\pi)$ is an interval partition. Therefore $\sum_{s=0}^n c_b(F^s\pi) = \mathrm{rc}\,(\pi)$.

THEOREM 3.8. *The combinatorial cumulants can be expressed in terms of the cumulant functional: for $\pi \in \mathcal{P}(n)$ and $\mathbf{x}_{\vec{u}}$ a monomial of degree n,*

$$R_\pi(\mathbf{x}_{\vec{u}}, t) = q^{\mathrm{rc}(\pi)} \prod_{i=1}^{l} R(\mathbf{x}_{(B_i : \vec{u})}, t).$$

Proof. The same argument as in the previous proposition shows that

$$R_\pi(\mathbf{x}_{\vec{u}}, t) = \lim_{\delta(\mathcal{I}) \to 0} \varphi\left[\sum_{\vec{v} \in [1\dots N]^n_\pi} y^{(1)}_{v(1)} y^{(2)}_{v(2)} \dots y^{(n)}_{v(n)}\right],$$

with

$$y^{(j)}_i = \begin{cases} |I_i|\,\lambda_{u(j)} & \text{if } (j) \text{ is a singleton in } \pi, \\ a_i(\xi_{u(j)}) & \text{if } j \text{ is the first element of its class in } \pi, \\ a^*_i(\xi_{u(j)}) & \text{if } j \text{ is the last element of its class in } \pi, \\ p_i(T_{u(j)}) & \text{otherwise.} \end{cases} \tag{5}$$

Fix $\vec{v}$. Let B be the class of π containing n. If B is an interval, then by Lemma 3.4

$$\left\langle \Omega, \left(\prod_{j=1}^{n} y_{v(j)}^{(j)} \right) \Omega \right\rangle_q = \left\langle \Omega, \left(\prod_{j=1}^{a(B)-1} y_{v(j)}^{(j)} \right) \Omega \right\rangle_q \left\langle \Omega, \left(\prod_{j=a(B)}^{n} y_{v(j)}^{(j)} \right) \Omega \right\rangle_q .$$

Therefore

$$R_\pi(\mathbf{x}_{\vec{u}}, t) = R_{(B_1, \ldots, B_{l-1})}(\mathbf{x}_{([1 \ldots n] \setminus B : \vec{u})}, t) R(\mathbf{x}_{(B:\vec{u})}, t).$$

Now suppose B is not an interval. Use the notation α, j_1, j_2, c_b of Construction 3.7. Denote

$$\eta(j_2) = \left(\prod_{i=j_2}^{n} y_{v(i)}^{(i)} \right) \Omega \in H$$

and

$$\vec{\eta}(j_1) = \left(\prod_{i=j_1+1}^{j_2-1} y_{v(i)}^{(i)} \right) \Omega \in H^{\otimes (c_b(\pi))}.$$

Note that $y_{v(j_1)}^{(j_1)}$ is either $a_{v(j_1)}(\xi_{u(j_1)})$ or $p_{v(j_1)}(T_{u(j_1)})$. Then

$$\begin{aligned} \left(\prod_{i=j_1}^{n} y_{v(i)}^{(i)} \right) \Omega = \left(\prod_{i=j_1}^{j_2-1} y_{v(i)}^{(i)} \right) \eta(j_2) &= y_{v(j_1)}^{(j_1)} \left(\left(\Big(\prod_{i=j_1+1}^{j_2-1} y_{v(i)}^{(i)} \Big) \Omega \right) \otimes \eta(j_2) \right) \\ &= y_{v(j_1)}^{(j_1)} (\vec{\eta}(j_1) \otimes \eta(j_2)) = q^{c_b(\pi)} (y_{v(j_1)}^{(j_1)} \eta(j_2)) \otimes \vec{\eta}(j_1) \end{aligned}$$

and

$$\begin{aligned} y_{v(j_1)}^{(j_1)} \left(\prod_{i=j_2}^{n} y_{v(i)}^{(i)} \prod_{i=j_1+1}^{j_2-1} y_{v(i)}^{(i)} \right) \Omega &= y_{v(j_1)}^{(j_1)} \left(\prod_{i=j_2}^{n} y_{v(i)}^{(i)} \right) \vec{\eta}(j_1) \\ &= y_{v(j_1)}^{(j_1)} \left(\left(\Big(\prod_{i=j_2}^{n} y_{v(i)}^{(i)} \Big) \Omega \right) \otimes \vec{\eta}(j_1) \right) \\ &= y_{v(j_1)}^{(j_1)} (\eta(j_2) \otimes \vec{\eta}(j_1)) = (y_{v(j_1)}^{(j_1)} \eta(j_2)) \otimes \vec{\eta}(j_1). \end{aligned}$$

Therefore

$$\left\langle \Omega, \left(\prod_{j=1}^{n} y_{v(j)}^{(j)} \right) \Omega \right\rangle_q = q^{c_b(\pi)} \left\langle \Omega, \left(\prod_{j=1}^{n} y_{v(\alpha(j))}^{(\alpha(j))} \right) \Omega \right\rangle_q .$$

The right-hand-side contains precisely the product of y's corresponding to the partition $F(\pi)$. The result follows by iterating these two steps. ∎

Remark 3.9 (Comments on Proposition 3.5). For $q = 0$, $R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = 0$ unless π is noncrossing. Then for $\sigma \in NC(n)$,

$$M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\pi \in NC(n)\\ \pi \leq \sigma}} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}).$$

Therefore for $\pi \in NC(n)$,

$$R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\sigma \in NC(n)\\ \sigma \leq \pi}} \text{Möb}_{NC}(\sigma, \pi) M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}),$$

where Möb_{NC} is the Möbius function on the lattice of noncrossing partitions. For $q = 1$, if $\sigma \in \mathcal{P}(n)$, $\sigma = (B_1, B_2, \ldots, B_l)$, then

$$\begin{aligned} M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) &= \prod_{i=1}^{l} M(\mathbf{x}_{(B_i : \vec{u})}, t; \mathbf{X}) \\ &= \prod_{i=1}^{l} \sum_{\pi_i \in \mathcal{P}(B_i)} R_{\pi_i}(\mathbf{x}_{(B_i : \vec{u})}, t; \mathbf{X}) \\ &= \sum_{\pi \leq \sigma} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}). \end{aligned}$$

Therefore for $\pi \in \mathcal{P}(n)$,

$$R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = \sum_{\substack{\sigma \in \mathcal{P}(n)\\ \sigma \leq \pi}} \text{Möb}_{\mathcal{P}}(\sigma, \pi) M_\sigma(\mathbf{x}_{\vec{u}}, t; \mathbf{X}),$$

where $\text{Möb}_{\mathcal{P}}$ is the Möbius function on the lattice of all partitions. Note that $\mathbf{X}(I)$ commute with $\mathbf{X}(J)$ for $I \cap J = \emptyset$ on the symmetric Fock space.
Thus for $q = 0, 1$, the cumulant functional at time 1 can be expressed through the moment functional at time 1. We will show how to do this for arbitrary q in the next section.

4. Characterization of generators

Denote by $R(f; \mathbf{X}) = R(f, 1; \mathbf{X})$ the cumulant functional.

Lemma 4.1. *The family of the moment functionals of a multiple q-Lévy process is determined by its cumulant functional. The functional $R(\cdot; \mathbf{X})$ on $\mathbb{C}\langle \mathbf{x} \rangle$ is the* generator *of the family of functionals $M(\cdot, t; \mathbf{X})$, that is,*

$$\frac{d}{dt}\Big|_{t=0} M(f, t; \mathbf{X}) = R(f; \mathbf{X}).$$

Proof. It suffices to prove these statements for a monomial $\mathbf{x}_{\vec{u}}$ of degree n. By equation (3), Theorem 3.8 and Proposition 3.6,

$$\begin{aligned} M(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) &= \sum_{\pi \in \mathcal{P}(n)} R_\pi(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) \\ &= \sum_{\pi \in \mathcal{P}(n)} q^{\mathrm{rc}(\pi)} \prod_{B \in \pi} R(\mathbf{x}_{(B:\vec{u})}, t; \mathbf{X}) \\ &= \sum_{\pi \in \mathcal{P}(n)} q^{\mathrm{rc}(\pi)} t^{|\pi|} \prod_{B \in \pi} R(\mathbf{x}_{(B:\vec{u})}, 1; \mathbf{X}), \end{aligned}$$

which implies the first statement. By differentiating this equality, we obtain

$$\frac{d}{dt}\Big|_{t=0} M(\mathbf{x}_{\vec{u}}, t; \mathbf{X}) = R_{\hat{1}}(\mathbf{x}_{\vec{u}}, 1; \mathbf{X}) = R(\mathbf{x}_{\vec{u}}; \mathbf{X}). \qquad \square$$

Definition 4.2. A functional ψ on $\mathbb{C}\langle \mathbf{x} \rangle$ is *conditionally positive* if its restriction to the subspace of polynomials with zero constant term is positive semi-definite.

We say that the functional ψ is *analytic* if for any i and any multi-index $\vec{u}$,

$$\limsup_{n \to \infty} \frac{1}{n} \psi[(\mathbf{x}_{\vec{u}})^* x_i^{2n} \mathbf{x}_{\vec{u}}]^{1/2n} < \infty.$$

The following proposition is an analog of the Schoenberg correspondence for our context. Note that the formulation of the result does not involve q: the dependence on q is hidden in Theorem 3.8.

Proposition 4.3. *A functional ψ is analytic and conditionally positive if and only if it is the generator of the family of the moment functionals for some multiple q-Lévy process.*

Proof. The proof is practically identical to that of [GSS92], or indeed of [Sch91]. We provide an outline for the reader's convenience.

Suppose ψ is the generator of the family of moment functionals $M(\cdot, t; \mathbf{X})$ for a multiple q-Lévy process $\mathbf{X}(t) = p_t(\xi, \mathbf{T}, \lambda)$. From the fact that each of the moment functionals is positive and equals 1 on the constant 1 it follows by differentiating that the cumulant functional is conditionally positive. Since $\psi = R(\cdot; \mathbf{X})$, for $\mathbf{x}_{\vec{u}}$ of degree l

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n} \psi[(\mathbf{x}_{\vec{u}})^* x_i^{2n} \mathbf{x}_{\vec{u}}]^{1/2n} &= \limsup_{n \to \infty} \frac{1}{n} R((\mathbf{x}_{\vec{u}})^* x_i^{2n} \mathbf{x}_{\vec{u}}, t; \mathbf{X})^{1/2n} \\ &= \limsup_{n \to \infty} \frac{1}{n} \left\langle \xi_{u(l)}, \prod_{j=l-1}^{1} T_{u(j)} T_i^{2n} \prod_{j=1}^{l-1} T_{u(j)} \xi_{u(l)} \right\rangle^{1/2n} \\ &= \limsup_{n \to \infty} \frac{1}{n} \left\| T_i^n \prod_{j=1}^{l-1} T_{u(j)} \xi_{u(l)} \right\|^{1/n} < \infty \end{aligned}$$

since the vector $\prod_{j=1}^{l-1} T_{u(j)} \xi_{u(l)}$ is analytic for T_i.

Now suppose ψ is conditionally positive and analytic. Then it gives rise to a multiple q-Lévy process, as follows. Denote by $\delta_0(f)$ the constant term of $f \in \mathbb{C}\langle \mathbf{x} \rangle$. ψ induces a positive semi-definite inner product on the space $\mathbb{C}\langle \mathbf{x} \rangle$ by $\langle f, g \rangle_\psi = \psi[(f - \delta_0(f))^*(g - \delta_0(g))]$. Let $\mathcal{N}_\psi$ be the subspace of vectors of length 0 with respect to this inner product. Let V be the Hilbert space obtained by completing the quotient $(\mathbb{C}\langle \mathbf{x} \rangle)/\mathcal{N}_\psi$ with respect to this inner product, with the induced inner product. Denote by ρ the canonical mapping $\mathbb{C}\langle \mathbf{x} \rangle \to V$, let $\mathcal{D}$ be its image, and for $f, g \in \mathbb{C}\langle \mathbf{x} \rangle$ define the operator $\Gamma(f) : \mathcal{D} \to \mathcal{D}$ by

$$\Gamma(f)\rho(g) = \rho(fg) - \rho(f)\delta_0(g).$$

The operator Γ is well defined since, by the Cauchy-Schwartz inequality,

$$\|\Gamma(f)\rho(g)\|_\psi = \psi[(g - \delta_0(g))^* f^* f (g - \delta_0(g))] \leq \|\rho(g)\|_\psi \, \|f^* f(g - \delta_0(g))\|_\psi \, .$$

Clearly $\mathcal{D}$ is dense in V, invariant under $\Gamma(f)$, and $\Gamma(f)$ is symmetric on it if f is symmetric.

Put, for $i \in [1 \ldots k]$, $\lambda_i = \psi[x_i], \xi_i = \rho(x_i), T_i = \Gamma(x_i)$. Each T_i takes $\mathcal{D}$ to itself. By construction, $\Gamma(x_i)\rho(\mathbf{x}_{\vec{u}}) = \rho(x_i \mathbf{x}_{\vec{u}})$, and so

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \|T_i^n \rho(\mathbf{x}_{\vec{u}})\|_\psi^{1/n} &= \limsup_{n\to\infty} \frac{1}{n} \|x_i^n \mathbf{x}_{\vec{u}}\|_\psi^{1/n} \\ &= \limsup_{n\to\infty} \frac{1}{n} \psi[(\mathbf{x}_{\vec{u}})^* x_i^{2n} \mathbf{x}_{\vec{u}}]^{1/2n} < \infty \end{aligned}$$

since the functional ψ is analytic. Therefore each of the vectors $\rho(\mathbf{x}_{\vec{u}})$ is analytic for T_i, and the linear span of these vectors is $\mathcal{D}$. In particular, T_i is essentially self-adjoint on $\mathcal{D}$.

Define the multiple q-Lévy process $\mathbf{X}$ by $X^{(i)}(t) = p_t(\xi_i, T_i, \lambda_i)$. Then

$$R(\mathbf{x}_{\vec{u}}; \mathbf{X}) = \psi[\mathbf{x}_{\vec{u}}].$$

Indeed, for $n = 1$

$$R(x_i; \mathbf{X}) = \lambda_i = \psi[x_i].$$

For $n \geq 2$,

$$\begin{aligned} R(\mathbf{x}_{\vec{u}}; \mathbf{X}) &= \left\langle \xi_{u(1)}, \prod_{j=2}^{n-1} T_{(u(j))} \xi_{u(n)} \right\rangle = \left\langle \rho(x_{u(1)}), \prod_{j=2}^{n-1} \Gamma(x_{u(j)}) \rho(x_{u(n)}) \right\rangle_\psi \\ &= \left\langle \rho(x_{u(1)}), \rho(\prod_{j=2}^{n} x_{u(j)}) \right\rangle_\psi = \psi[\prod_{j=1}^{n} x_{u(j)}] \\ &= \psi[\mathbf{x}_{\vec{u}}]. \end{aligned}$$

Therefore ψ is the generator of the moment functional family of $\mathbf{X}$. $\square$

4.1. Product states. For arbitrary q, the relation in the proof of Lemma 4.1 can be inverted.

Definition 4.4. Let Φ be any functional on $\mathbb{C}\langle \mathbf{x} \rangle$. Define the functional $\Psi = \log_q(\Phi)$ on monomials recursively by

$$\Psi(\mathbf{x}_{\vec{u}}) = \Phi(\mathbf{x}_{\vec{u}}) - \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \neq \hat{1}}} q^{\mathrm{rc}(\pi)} \prod_{B \in \pi} \Psi(\mathbf{x}_{(B:\vec{u})})$$

and extend linearly.

The definition has the form

$$\Psi(\mathbf{x}_{\vec{u}}) = \sum_{\sigma \in \mathcal{P}(n)} c(\sigma) \prod_{B \in \sigma} \Phi(\mathbf{x}_{(B:\vec{u})})$$

for some coefficient family $\{c(\sigma) : \sigma \in \mathcal{P}(k)\}$. For $q = 1$, Φ is the convolution exponential of Ψ [Sch91]. Lemma 4.1 and the discussion in Section 6 justify the notations $\Psi = \log_q(\Phi)$, $\Phi = \exp_q(\Psi)$. Note that this operation on functionals appears to bear no relation to the q-exponential power series.
It is clear that for any q-Lévy process, $R(\cdot, t; \mathbf{X}) = \log_q M(\cdot, t; \mathbf{X})$ and, moreover, that $M(\cdot, t; \mathbf{X}) = \exp_q(tR(\cdot; \mathbf{X}))$.

Definition 4.5. Let Φ_1 be a functional on $\mathbb{C}\langle x_1, x_2, \ldots, x_{k_1} \rangle$, Φ_2 a functional on $\mathbb{C}\langle x_1, x_2, \ldots, x_{k_2} \rangle$. Define their *product functional* $\Phi_1 \times_q \Phi_2$ on $\mathbb{C}\langle x_1, x_2, \ldots, x_{k_1+k_2} \rangle$ by the "mixed cumulants are 0" rule:

$$\log_q(\Phi_1 \times_q \Phi_2)(\mathbf{x}_{\vec{u}}) = \begin{cases} \log_q(\Phi_1)(\mathbf{x}_{\vec{u}}) & \text{if } \forall i, u(i) \leq k_1, \\ \log_q(\Phi_2)(\mathbf{x}_{\vec{u}}) & \text{if } \forall i, u(i) > k_1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that it is more natural to think of this construction as taking the product of two one-parameter families of functionals,

$$\exp_q(t \log_q(\Phi_1)) \times_q \exp_q(t \log_q(\Phi_2)) = \exp_q(t \log_q(\Phi_1 \times_q \Phi_2)).$$

Denote

$$\begin{aligned} \mathcal{ID}_c(q,k) &= \{\Phi : \Phi = M(\cdot, 1; \mathbf{X}) \text{ for some } k\text{-dimensional } q\text{-Lévy process } \mathbf{X}\} \\ &= \left\{\Phi : \log_q(\Phi) \text{ is conditionally positive and analytic}\right\}. \end{aligned}$$

The notation stands for "combinatorially infinitely divisible".

Lemma 4.6. *For $\Phi_1 \in \mathcal{ID}_c(q,k_1), \Phi_2 \in \mathcal{ID}_c(q,k_2)$, their product functional is a state, that is, a positive functional that equals* 1 *on the identity element.*

Proof. It suffices to show that $\Phi_1 \times_q \Phi_2 \in \mathcal{ID}_c(q, k_1 + k_2)$. Let $\mathbf{X}_1, \mathbf{X}_2$ be the q-Lévy processes whose distributions at time 1 are Φ_1, Φ_2, respectively. Let $X^{(i,1)}(t) = p_t(\xi_{i,1}, T_{i,1}, \lambda_{i,1})$, $X^{(i,2)}(t) = p_t(\xi_{i,2}, T_{i,2}, \lambda_{i,2})$. Here $\xi_{i,1} \in V_1$, $T_{i,1}$ is an operator on V_1 with domain $\mathcal{D}_1$, $\xi_{i,2} \in V_2$, $T_{i,2}$ is an operator on V_2 with domain $\mathcal{D}_2$. Let $V = V_1 \oplus V_2$. Identify $\xi_{i,1}$ with $\xi_{i,1} \oplus 0$, $\xi_{i,2}$ with $0 \oplus \xi_{i,2}$, $T_{i,1}$ with $\left(\begin{smallmatrix} T_{i,1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $T_{i,2}$ with $\left(\begin{smallmatrix} 0 & 0 \\ 0 & T_{i,2} \end{smallmatrix}\right)$. It is easy to see that this identification does not change the cumulants or the moments of the processes $\mathbf{X}_1, \mathbf{X}_2$, and that condition (1) holds for the (k_1+k_2)-dimensional process $\mathbf{X} = (X^{(1,1)}, \dots, X^{(k_1,1)}, X^{(1,2)}, \dots, X^{(k_2,2)})$. Then $\Phi_1 \times_q \Phi_2$ is equal to $M(\cdot, 1; \mathbf{X})$. □

For $q = 1$, the product state is the usual (tensor) product state, while for $q = 0$ it is the (reduced) free product state. Already for $q = -1$, the situation is unclear. The parity of $\mathrm{rc}\,(\pi)$ can differ from the parity of the number of left-reduced crossings of [Nic95] even for partitions all of whose classes have even order. Therefore even for $q = -1$, our cumulants are different from the q-cumulants of that paper. In particular, the results of [MN97] about graded independence do not apply. Note also that our product state construction is defined only on the polynomial algebras $\mathbb{C}\langle\mathbf{x}\rangle$, not on general algebras. So we do not obtain a universal product in the sense of [Spe97b].
A state Φ is tracial if for all a, b, $\Phi(ab) = \Phi(ba)$. For $q = 0, 1$, the product state of two tracial states is tracial [VDN92]. This property remains true for the q-Brownian motion (see below). However, the number of the restricted crossings of a partition is not invariant under cyclic permutations of the underlying set. For example, $\mathrm{rc}\,(((1,3,5)(2,4))) = 2$ while $\mathrm{rc}\,(((1,3)(2,4,5))) = 1$. So for general q, the product state of two tracial states need not be tracial.

5. The Itô table

In general we do not know how to calculate the partition-dependent stochastic measures $\mathrm{St}_\pi(\mathbf{X})$; indeed we don't expect a nice answer for a general process. In particular we don't expect that a functional Itô formula exists for q-Lévy processes. However, one ingredient of it is present, namely, we can calculate all the higher diagonal measures. These are higher variations of the processes, and appear in the functional Itô formula for the free Lévy processes [Ans01b].

Remark 5.1 (Algebraic approach). Unless we are considering higher diagonal measures of a single one-dimensional process, for this section we also need a more general setup than the one we had before. First, we need to consider multiple processes whose components are of the form $X(t) = p_t(\xi, \eta, T, \lambda) = a_t(\xi) + a_t^*(\eta) + p_t(T) + t\lambda$. Second, we no longer can require T to be symmetric and λ to be real. The solution in [Sch91] is to require that T be a linear operator with domain $\mathcal{D}$, not necessarily dense, so that the restriction of T^* to $\mathcal{D}$ is a well-defined linear operator.
We describe briefly how to modify this paper for the algebraic context. The gauge operators are defined in the same way, and the multiple q-Lévy process are modified as in the previous paragraph, except that we drop the assumption (1). The moments and cumulants can be modified to include $*$-quantities, i.e. use words in both $\mathbf{X}$ and $\mathbf{X}^*$ in the definitions, and consider them as functionals on $\mathbb{C}\langle x_1, x_2, \dots, x_k, x_1^*, x_2^*, \dots, x_k^*\rangle$ with the obvious conjugation. All the relations between moments and cumulants, and between partition-dependent cumulants and the cumulant functional, remain the same, and it is clear how to

modify the formula for the cumulant functional in terms of ξ, η, T, λ. In the algebraic context, generators of the families of moment functionals for symmetric processes are precisely all the conditionally positive functionals.

For the Itô table, we first need a technical lemma.

LEMMA 5.2. *For $f, g \in L^2(\mathbb{R}_+)$,*

$$\lim_{\delta(\mathcal{I})\to 0} \left| \sum_{i=1}^N \Big(\int_{I_i} f(x)dx \Big) \Big(\int_{I_i} g(y)dy \Big) \right| = 0, \tag{6a}$$

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \sum_{i=1}^N \Big(\mathbf{1}_{I_i}(x) f(x) \Big) \Big(\int_{I_i} g(y)dy \Big) \right\|_2 = 0, \tag{6b}$$

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \sum_{i=1}^N \Big(\mathbf{1}_{I_i}(x) f(x) \Big) \Big(\mathbf{1}_{I_i}(y) g(y) \Big) \right\|_2 = 0. \tag{6c}$$

Proof. We repeatedly use the Cauchy-Schwartz inequality for sequences and functions:

$$\begin{aligned}
\left| \sum_{i=1}^N \Big(\int_{I_i} f(x)dx \Big) \Big(\int_{I_i} g(y)dy \Big) \right| &\leq \sqrt{ \sum_{i=1}^N \Big(\int_{I_i} f(x)dx \Big)^2 \sum_{j=1}^N \Big(\int_{I_j} g(y)dy \Big)^2 } \\
&\leq \sqrt{ \sum_{i=1}^N |I_i| \int_{I_i} f^2(x)dx \sum_{j=1}^N |I_j| \int_{I_j} g^2(y)dy } \\
&\leq \delta(\mathcal{I}) \sqrt{ \sum_{i=1}^N \int_{I_i} f^2(x)dx \sum_{j=1}^N \int_{I_j} g^2(y)dy } \\
&\leq \delta(\mathcal{I}) \left\| f \right\|_2 \left\| g \right\|_2 .
\end{aligned}$$

$$\begin{aligned}
\left\| \sum_{i=1}^N \Big(\mathbf{1}_{I_i}(x) f(x) \Big) \Big(\int_{I_i} g(y)dy \Big) \right\|_2 &= \sqrt{ \sum_{i=1}^N \Big(\int_{I_i} f^2(x)dx \Big) \Big(\int_{I_i} g(y)dy \Big)^2 } \\
&\leq \sqrt{ \sum_{i=1}^N \int_{I_i} f^2(x)dx \, |I_i| \int_{I_i} g^2(y)dy } \\
&\leq \sqrt{\delta(\mathcal{I})} \left\| f \right\|_2 \left\| g \right\|_2 .
\end{aligned}$$

The last property requires a bit more work, since uniform estimates do not hold in this case. By the Cauchy-Schwartz inequality as above, we may assume that $f = g$; also without loss of generality we assume that $\|f\|_2 = 1$. Let $\mathcal{I} = (I_1, I_2, \ldots, I_M)$ be a subdivision of $[0, t)$, and $\varepsilon > 0$. For $N > \max(M, \frac{8}{\varepsilon^2})$ large enough, we can choose a subdivision $\mathcal{J}' = (J'_1, J'_2, \ldots, J'_N)$ so that all $\int_{J'_j} f^2(x)dx < \frac{2}{N}$ and no I_i is a subset of any J'_j. Let $\mathcal{J}$ be the smallest common refinement of $\mathcal{I}, \mathcal{J}'$. Then $\mathcal{J}$ consists of at most $M+N$ intervals J_j, and for each of them $\int_{J_j} f^2(x)dx < \frac{2}{N}$. Therefore

$$\begin{aligned}\left\| \sum_j \Big(\mathbf{1}_{J_j}(x)f(x)\Big)\Big(\mathbf{1}_{J_j}(y)f(y)\Big) \right\|_2 &= \sqrt{\sum_j \Big(\int_{J_j} f^2(x)dx\Big)\Big(\int_{J_j} f^2(y)dy\Big)} \\ &\leq \sqrt{\frac{4(M+N)}{N^2}} \\ &\leq \varepsilon.\end{aligned}$$

We conclude that $\left\| \sum_{i=1}^N \big(\mathbf{1}_{I_i}(x)f(x)\big)\big(\mathbf{1}_{I_i}(y)g(y)\big) \right\|_2$ converges to 0 along the net of subdivisions $\mathcal{I}$ as $\delta(\mathcal{I}) \to 0$. $\square$

Proposition 5.3. *The Itô table for q-Lévy processes* $X^{(i)}(t) = a_t(\xi_i) + a_t^*(\eta_i) + p_t(T_i) + t\lambda_i$ *is*

$dX^{(1)}dX^{(2)}$	$da(\xi_2)$	$da^*(\eta_2)$	$dp(T_2)$	$\lambda_2 dt$
$da(\xi_1)$	0	$\langle \xi_1, \eta_2 \rangle\, dt$	$da(T_2^*\xi_1)$	0
$da^*(\eta_1)$	0	0	0	0
$dp(T_1)$	0	$da^*(T_1\eta_2)$	$dp(T_1T_2)$	0
$\lambda_1 dt$	0	0	0	0

More precisely, the quadratic co-variation of these processes is

$$\Delta_2(x_1x_2, t; (X^{(1)}, X^{(2)})) = [X^{(1)}, X^{(2)}](t) = p_t(T_2^*\xi_1, T_1\eta_2, T_1T_2, \langle \xi_1, \eta_2 \rangle).$$

Here the convergence in the definition of Δ *is the pointwise convergence on the dense set* $\mathcal{F}_{\mathrm{alg}}(L^2(\mathbb{R}_+) \otimes \mathcal{D})$.

Proof. We need to show that for $\vec{\zeta} \in \mathcal{F}_{\mathrm{alg}}(L^2(\mathbb{R}_+) \otimes \mathcal{D})$,

$$\lim_{\delta(\mathcal{I}) \to 0} \left\| \left(\sum_{u=1}^N (y_u^{((1),i)} y_u^{((2),j)}) - y^{(i,j)} \right) \vec{\zeta} \right\|_q = 0,$$

where $y^{((1),i)}, y^{((2),j)}$ are labels for rows, respectively, columns of the Itô table, and $y^{(i,j)}$ is the corresponding entry of the table. All of these are obtained by applying Lemma 5.2, possibly with one or both of f, g equal to $\mathbf{1}_{[0,t)}$. More precisely, we use equation (6a) for the product $da(\xi_1)da(\xi_2)$, equation (6b) for the products $da^*(\eta_1)da(\xi_2), dp(T_1)da(\xi_2), da(\xi_1)dp(T_2)$ and equation (6c) for the products $da^*(\eta_1)da^*(\eta_2), dp(T_1)da^*(\eta_2), da^*(\eta_1)dp(T_2), dp(T_1)dp(T_2)$.
We do the case $dp(T_1)da^*(\eta_2)$ as an example. The linear span of the vectors of the form $\vec{\zeta} = (f_1 \otimes \zeta_1) \otimes (f_2 \otimes \zeta_2) \otimes \ldots \otimes (f_n \otimes \zeta_n)$, for $f_1, f_2, \ldots, f_n \in$

$L^2(\mathbb{R}_+), \zeta_1, \zeta_2, \ldots, \zeta_n \in \mathcal{D}$, is dense in $\mathcal{F}_{\text{alg}}(L^2(\mathbb{R}_+) \otimes \mathcal{D})$. For such a vector,

$$\begin{aligned}
\sum_{i=1}^{N} p_i(T_1) a_i^*(\eta_2)\vec{\zeta} &= \sum_{i=1}^{N} p_i(T_1)(\mathbf{1}_{I_i} \otimes \eta_2) \otimes \vec{\zeta} \\
&= \sum_{i=1}^{N} \Big((\mathbf{1}_{I_i}\mathbf{1}_{I_i} \otimes T_1\eta_2) \otimes \vec{\zeta} \\
&\quad + \sum_{k=1}^{n} q^k (\mathbf{1}_{I_i} f_k \otimes T_1 \zeta_k) \otimes (\mathbf{1}_{I_i} \otimes \eta_2) \otimes (f_1 \otimes \zeta_1) \otimes \ldots \otimes (f_n \otimes \zeta_n) \Big) \\
&= \sum_{i=1}^{N} a_i^*(T_1\eta_2)\vec{\zeta} \\
&\quad + \sum_{i=1}^{N} \sum_{k=1}^{n} q^k (\mathbf{1}_{I_i} f_k \otimes T_1 \zeta_k) \otimes (\mathbf{1}_{I_i} \otimes \eta_2) \otimes (f_1 \otimes \zeta_1) \otimes \ldots \otimes (f_n \otimes \zeta_n).
\end{aligned}$$

The first term is equal to $a_t^*(T_1\eta_2)\vec{\zeta}$; we need to show that the second term tends to 0 as $\delta(\mathcal{I}) \to 0$. It suffices to do so for each fixed k. The operator P_n is bounded, so it suffices to show that

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \sum_{i=1}^{N} (\mathbf{1}_{I_i} f_k \otimes T_1\zeta_k) \otimes (\mathbf{1}_{I_i} \otimes \eta_2) \otimes (f_1 \otimes \zeta_1) \otimes \ldots \otimes (f_n \otimes \zeta_n) \right\| = 0,$$

where we are using the usual norm on $(L^2(\mathbb{R}_+) \otimes V)^{\otimes n}$. But for this it suffices to show that

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \sum_{i=1}^{N} (\mathbf{1}_{I_i} f_k \otimes T_1\zeta_k) \otimes (\mathbf{1}_{I_i} \otimes \eta_2) \right\| = 0,$$

and in fact only that $\lim_{\delta(\mathcal{I})\to 0} \left\| \sum_{i=1}^{N} (\mathbf{1}_{I_i} f_k) \otimes \mathbf{1}_{I_i} \right\| = 0$. Now apply the lemma. ◻

Remark 5.4. Note that the Itô table does not depend on q. The Itô table was known for $q = 1$ [HP84] (with a somewhat different set of convergence), $q = -1$ [AH84] and $q = 0$ [Spe91]; for the q-Brownian motion ($T = 0$) it was known for all q [Śni00]. In all of these cases it is only a facet of a well-defined theory of stochastic integration.

Corollary 5.5. *For a one-dimensional self-adjoint process $X(t) = p_t(\xi, T, \lambda)$ and $k \geq 2$,*

$$\Delta_k(t; X) = p_t(T^{k-1}\xi, T^k, \langle \xi, T^{k-2}\xi \rangle).$$

6. Single-variable analysis

Denote by $\mathcal{M}_c$ (for "combinatorial") the space of finite positive Borel measures on $\mathbb{R}$ all of whose moments are finite, and by $\mathcal{M}_c^1 \subset \mathcal{M}_c$ the subset of probability measures. For $\mu \in \mathcal{M}_c$ considered as a functional on $\mathbb{C}[x]$, denote its moments $\mu(x^n)$ by $m_n(\mu)$. For $\mu \in \mathcal{M}_c^1$ and $n \geq 1$, the q-cumulants $r_n(\mu) = (\log_q \mu)(x^n)$ are determined by

$$r_n(\mu) = m_n(\mu) - \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \neq \hat{1}}} q^{\mathrm{rc}(\pi)} \prod_{B \in \pi} r_{|B|}(\mu). \tag{7}$$

The expressions for the first few cumulants in terms of the moments and q are

$$\begin{aligned}
r_1 &= m_1, \\
r_2 &= m_2 - m_1^2, \\
r_3 &= m_3 - 3m_2 m_1 + 2m_1^3, \\
r_4 &= m_4 - 4m_3 m_1 - (2+q)m_2^2 + (10+2q)m_2 m_1^2 - (5+q)m_1^4, \\
r_5 &= m_5 - 5m_4 m_1 - (5+4q+q^2)m_3 m_2 + (15+4q+q^2)m_3 m_1^2 \\
&\quad + (15+12q+3q^2)m_2^2 m_1 - (35+20q+5q^2)m_2 m_1^3 + (14+8q+2q^2)m_1^5.
\end{aligned}$$

While these cumulants are well-defined for arbitrary $\mu \in \mathcal{M}_c^1$, our results apply only to a special class of them. For a sequence $\mathbf{r} = (r_0 = 0, r_1, r_2, \dots)$ in $\mathbb{R}$, let $\psi_{\mathbf{r}}$ be the functional on $\mathbb{C}[x]$ defined by $\psi_{\mathbf{r}}(\sum_{i=0}^n a_i x^i) = \sum_{i=0}^n a_i r_i$. The functional $\psi_{\mathbf{r}}$ is analytic iff $\limsup_{n\to\infty} \frac{1}{n} r_{2(n+2)}^{1/2n} < \infty$. It is conditionally positive iff the functional $\psi_{(r_2, r_3, \dots)}$ is positive semi-definite. These conditions imply [Shi96] that for $n \geq 0$, $r_{n+2} = m_n(\tau)$ for some $\tau \in \mathcal{M}_c$ that is uniquely determined by its moments. Denote by $\mathcal{M}_u$ (for "unique") the subspace of finite positive Borel measures in $\mathcal{M}_c$ that are of this form, i.e. for which $\limsup_{n\to\infty} \frac{1}{n} m_{2n}(\tau)^{1/2n} < \infty$. Equivalently, $\tau \in \mathcal{M}_u$ if its exponential moment-generating function $\int_{\mathbb{R}} \exp(\theta x) d\tau(x)$ is defined for θ in a neighborhood of 0.

Definition 6.1. Let $\tau \in \mathcal{M}_u$, and $\lambda \in \mathbb{R}$. Define $\mathrm{LH}_q^{-1}(\lambda, \tau)$ to be the probability measure in $\mathcal{M}_c^1$ determined by the cumulant sequence $r_1 = \lambda, r_n = m_{n-2}(\tau)$ for $n \geq 2$. Equivalently, $\mathrm{LH}_q^{-1}(\lambda, \tau)$ is the distribution at time 1 of the q-Lévy process $p_t(\xi, T, \lambda)$ such that the operator T has distribution τ with respect to the vector functional $\langle \xi, \cdot \xi \rangle$. Note that $\mathrm{LH}_q^{-1}(\lambda, \tau)$ is in fact in $\mathcal{M}_u^1$. Denote by $\mathcal{ID}_c(q)$ the image of the map LH_q^{-1}; clearly $\mathcal{ID}_c(q) = \mathcal{ID}_c(q, 1)$. Call a measure in $\mathcal{ID}_c(q)$ *q-infinitely divisible*.

It is clear that LH_q^{-1} is injective. We define $\mathrm{LH}_q : \mathcal{ID}_c(q) \to \mathbb{R} \times \mathcal{M}_u$ to be the inverse of LH_q^{-1}. This is an analog of the Lévy-Hinchin representation, or more precisely of the canonical representation; see Section 6.1.
Note that for the process $p_t(\xi, T, \lambda)$ in the definition above, we can identify the Hilbert space V with $L^2(\mathbb{R}, \tau)$, so that ξ corresponds to the constant function

1, and T corresponds to the operator of multiplication by the variable x. The Hilbert space H is then equal to $L^2(\mathbb{R}_+ \times \mathbb{R}, dx \otimes \tau)$.

Definition 6.2. For $\mu, \nu \in \mathcal{ID}_c(q)$, define their q-convolution $\mu *_q \nu$ by the rule that $\mathrm{LH}_q(\mu *_q \nu) = \mathrm{LH}_q(\mu) + \mathrm{LH}_q(\nu)$.

Lemma 6.3. *$(\mathcal{ID}_c(q), *_q)$ is an Abelian semigroup. In particular, the q-convolution of two positive measures is positive.*

Proof. The sum of two measures in $\mathcal{M}_u$ is in $\mathcal{M}_u$. □

Lemma 6.4 (Relation to product states). *For $\mu_1, \mu_2 \in \mathcal{ID}_c(q)$,*

$$(\mu_1 *_q \mu_2)(x^n) = (\mu_1 \times_q \mu_2)((x_1 + x_2)^n).$$

Proof. Using the representation from the proof of Lemma 4.6, let $\xi = \xi_1 \oplus \xi_2 \in V$, $T = \left(\begin{smallmatrix} T_1 & 0 \\ 0 & T_2 \end{smallmatrix}\right)$ an operator on V with domain $\mathcal{D}_1 \oplus \mathcal{D}_2$, $\lambda = \lambda_1 + \lambda_2$. Let V' be the closure of the span $\left(\left\{T^j \xi\right\}_{j=0}^{\infty}\right)$. Then $T\left(\operatorname{span}\left(\left\{T^j \xi\right\}_{j=0}^{\infty}\right)\right) \subset V'$. Define T' to be the restriction $T \restriction V'$. Then $X(t) = p_t(\xi, T', \lambda)$ is a q-Lévy process. Its distribution is equal to $\mu_1 *_q \mu_2$. Indeed, if we denote this distribution by μ, then

$$r_1(\mu) = \lambda = \lambda_1 + \lambda_2 = r_1(\mu_1) + r_1(\mu_2),$$

and for $n \geq 2$,

$$r_n(\mu) = \left\langle \xi, (T')^{n-2} \xi \right\rangle = \left\langle \xi_1, T_1^{n-2} \xi_1 \right\rangle + \left\langle \xi_2, T_2^{n-2} \xi_2 \right\rangle = r_n(\mu_1) + r_n(\mu_2).$$

But $\mu_1 \times_q \mu_2 = M(\cdot, 1; (X^{(1)}, X^{(2)}))$, and it is clear that

$$M(x^n, 1; X) = M((x_1 + x_2)^n, 1; (X^{(1)}, X^{(2)})).$$ □

6.1. The Bercovici-Pata bijection. One would not expect the q-cumulants to be defined precisely for all probability measures in $\mathcal{M}_c^1$, rather than for more general moment sequences. Indeed, such a construction would provide a continuous bijection Λ on $\mathcal{M}_c^1$ with the property that $r_n(q = 1, \mu) = r_n(q = 0, \Lambda(\mu))$. In particular, this would imply that $\Lambda(\mu * \nu) = \Lambda(\mu) \boxplus \Lambda(\nu)$, where $*$ is the usual convolution while $\boxplus$ is the additive free convolution. Such a map is not known, and indeed for the space of all probability measures it is known not to exist, since the analog of the Cramér theorem does not hold in free probability [BV95]. However, there is a remarkable bijection [BP99] between the usual and the free infinitely divisible measures. We now show that as long as we restrict ourselves to infinitely divisible measures in $\mathcal{M}_c^1$, this is precisely the map obtained by identifying the cumulants as above, and in particular our spaces $\mathcal{ID}_c(q)$ provide an interpolation between the usual and the free infinitely divisible measures in cases $q = 0$ and $q = 1$.
The bijection is defined as follows. Let σ be a finite positive Borel measure on $\mathbb{R}$ and $\gamma \in \mathbb{R}$. Denoting by $\mathcal{F}$ the Fourier transform, define $\mu_*^{\gamma,\sigma}$ to be the

probability measure with the Lévy-Hinchin representation

$$\log \mathcal{F}_{\mu_*^{\gamma,\sigma}}(\theta) = i\gamma\theta + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2} \right) \frac{1+x^2}{x^2} d\sigma(x).$$

Denoting by $\mathcal{R}$ the R-transform [VDN92, Voi00], define $\mu_{\boxplus}^{\gamma,\sigma}$ to be the probability measure with the free Lévy-Hinchin representation

$$\mathcal{R}_{\mu_{\boxplus}^{\gamma,\sigma}}(z) = \gamma + \int_{\mathbb{R}} \frac{z+x}{1-zx} d\sigma(x).$$

Then $\Lambda(\mu_*^{\gamma,\sigma}) = \mu_{\boxplus}^{\gamma,\sigma}$.

Lemma 6.5. *Let $\lambda \in \mathbb{R}, \tau \in \mathcal{M}_u$. For $d\sigma(x) = \frac{1}{1+x^2} d\tau(x)$ and $\gamma = \lambda - m_1(\sigma)$, $\mu_*^{\gamma,\sigma} = \mathrm{LH}_1^{-1}(\lambda, \tau)$ and $\mu_{\boxplus}^{\gamma,\sigma} = \mathrm{LH}_0^{-1}(\lambda,\tau)$.*

Proof. Since $\sigma \in \mathcal{M}_u$, $\mu_*^{\gamma,\sigma}$ has finite variance. Then

$$\begin{aligned} \log \mathcal{F}_{\mu_*^{\gamma,\sigma}}(\theta) &= i\gamma\theta + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x + \frac{i\theta x^3}{1+x^2} \right) \frac{1}{x^2} d\tau(x) \\ &= i\lambda\theta + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \right) \frac{1}{x^2} d\tau(x) \end{aligned}$$

is the canonical representation of $\log \mathcal{F}_{\mu_*^{\gamma,\sigma}}$. It has a convergent power series expansion

$$i\lambda\theta + \sum_{n=2}^{\infty} \frac{1}{n!} (i\theta)^n m_{n-2}(\tau).$$

It is well-known [Shi96] that the classical ($q=1$)-cumulants of μ are the coefficients in such a power series expansion of $\log \mathcal{F}_\mu$. Similarly,

$$\begin{aligned} \mathcal{R}_{\mu_{\boxplus}^{\gamma,\sigma}}(z) &= \gamma + \int_{\mathbb{R}} \left(\frac{z}{1-zx} + \frac{x}{x^2+1} \right) d\tau(x) \\ &= \lambda + \int_{\mathbb{R}} \frac{z}{1-zx} d\tau(x), \end{aligned}$$

and for $z = i\theta$, it has an expansion

$$\lambda + \sum_{n=2}^{\infty} (i\theta)^{n-1} m_{n-2}(\tau).$$

Here the sum in the last expression need not converge, so what we mean by it is that for $k \geq 2$,

$$\lim_{\theta \to 0} \frac{1}{(i\theta)^k} \left(\mathcal{R}_{\mu_{\boxplus}^{\gamma,\sigma}}(i\theta) - \lambda - \sum_{n=2}^{k} (i\theta)^{n-1} m_{n-2}(\tau) \right) = m_{k-1}(\tau).$$

Again, it is well-known [Spe97a] that the free ($q=0$)-cumulants of μ are the coefficients in such an expansion of $\mathcal{R}_\mu$. □

Lemma 6.6. *The mapping $(q, \lambda, \tau) \mapsto \mathrm{LH}_q^{-1}(\lambda,\tau)$ has the following properties.*

a. $\mathrm{LH}_q^{-1}(\lambda_1, \tau_1) *_q \mathrm{LH}_q^{-1}(\lambda_2, \tau_2) = \mathrm{LH}_q^{-1}(\lambda_1 + \lambda_2, \tau_1 + \tau_2)$.

b. *Denoting by D_c the dilation operator, $D_c(\mu)(S) = \mu(c^{-1}S)$,*

$$D_c(\mathrm{LH}_q^{-1}(\lambda, \tau)) = \mathrm{LH}_q^{-1}(c\lambda, c^2 D_c(\tau)).$$

c. *For any q, $\mathrm{LH}_q^{-1}(\lambda, 0) = \delta_\lambda$, and for any $\mu \in \mathcal{ID}_c(q)$, $\mu *_q \delta_\lambda = \mu * \delta_\lambda$.*

d. *For $q \in [-1, 1]$ and fixed λ, τ, the mapping $q \mapsto \mathrm{LH}_q^{-1}(\lambda, \tau)$ is weakly continuous.*

e. *For a fixed $q \in [-1, 1]$, the mapping $\mathrm{LH}_q^{-1} : \mathbb{R} \times \mathcal{M}_u \to \mathcal{ID}_c(q)$ is a homeomorphism in the weak topology.*

Proof. The first and the third properties are immediate. For the second one, we observe that $m_k(D_c(\mu)) = c^k m_k(\mu)$ and so $r_k(D_c(\mu)) = c^k r_k(D_c(\mu))$. The last two follow from the following fact [Dur91]. Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of finite measures in $\mathcal{M}_c$ that converges weakly to a finite measure $\mu \in \mathcal{M}_c$. Then for all k, $m_k(\mu_n) \to m_k(\mu)$. Conversely, let $\{\mu_n\}_{n=1}^\infty$ be a sequence of finite measures in $\mathcal{M}_c$ such that for any k, $m_k(\mu_n) \to m_k$. If the family $\{m_k\}_{k=0}^\infty$ are the moments of a unique finite positive measure μ, then $\mu_n \to \mu$ weakly. $\square$

For $q = 0, 1$, it is known [BNTr00] that the map $(\gamma, \sigma) \mapsto \mathrm{LH}_q^{-1}(\gamma + m_1(\sigma), \frac{1}{1+x^2}\sigma)$ can be extended to a weak homeomorphism between the weak closures of $\mathbb{R} \times \mathcal{M}_u$ and $\mathcal{ID}_c(q)$.

Corollary 6.7. *Let $\tau \in \mathcal{M}_u, \lambda \in \mathbb{R}$. Fix three sequences $\{A(n)\}_{n=1}^\infty$, $\{B(n)\}_{n=1}^\infty \subset \mathbb{R}$, $\{N(1) < N(2) < \ldots\} \subset \mathbb{N}$. By limits of sequences of measures we will always mean weak limits.*

a. *Every measure in $\mathcal{ID}_c(q)$ arises as a limit*

$$\lim_{n\to\infty} (\underbrace{\mu_n *_q \mu_n *_q \ldots *_q \mu_n}_{N(n)\ times}) = \mathrm{LH}_q^{-1}(\lambda, \tau) \tag{8}$$

for some $\{\mu_n\}_{n=1}^\infty \subset \mathcal{ID}_c(q)$. The statement (8) is equivalent to

$$\lim_{n\to\infty} (N(n) m_1(\mu_n)) = \lambda, \qquad \lim_{n\to\infty} (N(n) x^2 \mu_n) = \tau.$$

b. *Let $\mu \in \mathcal{ID}_c(q)$. The statement*

$$\lim_{n\to\infty} (D_{B(n)^{-1}} (\underbrace{\mu *_q \ldots *_q \mu}_{N(n)\ times}) *_q \delta_{-A(n)}) = \mathrm{LH}_q^{-1}(\lambda, \tau)$$

is equivalent to

$$\lim_{n\to\infty} (\frac{N(n)}{B(n)} m_1(\mu) - A(n)) = \lambda, \qquad \lim_{n\to\infty} \frac{N(n)}{B(n)^2} = t, \qquad \tau = t\delta_0.$$

Hence only $\mathrm{LH}_q^{-1}(\lambda, t\delta_0)$ arise as such limits.

Proof. Denote by (λ_n, τ_n) the components of $\mathrm{LH}_q(\mu_n)$. From the preceding Lemma it follows that the statement (8) is equivalent to

$$\lim_{n\to\infty} (N(n)\lambda_n) = \lambda, \qquad \lim_{n\to\infty} (N(n)\tau_n) = \tau.$$

So to fulfill (8), it suffices to take $\mu_n = \mathrm{LH}_q^{-1}(\frac{1}{N(n)}\lambda, \frac{1}{N(n)}\tau)$.

Now we prove the equivalence. It is clear that $\lambda_n = r_1(\mu_n) = m_1(\mu_n)$. The family $\{\mu_n\}$ satisfies (8) iff, in addition, for all $k > 1$,

$$m_k(N(n)\tau_n) = N(n)m_k(\tau_n) = N(n)r_{k+2}(\mu_n) \overset{n\to\infty}{\longrightarrow} m_k(\tau).$$

This is equivalent to $r_{k+2}(\mu_n) = \frac{1}{N(n)}m_k(\tau) + o(\frac{1}{N(n)})$. By induction on k and using (7), this is equivalent to

$$m_k(x^2\mu_n) = m_{k+2}(\mu_n) = \frac{1}{N(n)}m_k(\tau) + o(\frac{1}{N(n)}),$$

i.e.

$$m_k(N(n)x^2\mu_n) \overset{n\to\infty}{\longrightarrow} m_k(\tau)$$

and

$$(N(n)x^2\mu_n) \overset{n\to\infty}{\longrightarrow} \tau.$$

The second statement follows from the first one with $\mu_n = D_{B(n)^{-1}}(\mu) *_q \delta_{-\frac{A(n)}{N(n)}}$. For $k \geq 2$,

$$m_k(\mu_n) = \frac{N(n)}{B(n)^k}m_k(\mu) \overset{n\to\infty}{\longrightarrow} m_{k-2}(\tau).$$

So $\lim_{n\to\infty} \frac{N(n)}{B(n)^2} = t$ for some t, and $m_k(\tau) = 0$ for $k \geq 0$, i.e. $\tau = t\delta_0$. So only shifted q-Gaussian distributions (see below) can arise as such a limit among the measures in $\mathcal{ID}_c(q)$. This means that the combinatorial framework is, in general, not adequate for identifying the domains of partial attraction. $\square$

Remark 6.8. While the results of this section are of most interest in the one-dimensional case, there is no difficulty with the extension to k dimensions. That is, to every functional in $\mathcal{ID}_c(q,k)$ there corresponds a unique conditionally positive analytic functional, which can be identified with a pair of $\vec{\lambda} \in \mathbb{R}^k$ and a positive analytic functional on $\mathbb{C}\langle x_1, x_2, \ldots, x_k\rangle$. Using this bijection, we can define a convolution on $\mathcal{ID}_c(q,k)$, as well as a multi-dimensional extension of the bijection Λ.

Now we consider the q-Lévy processes in the simplest case of one-dimensional V. There are essentially two distinct situations, $T = 0$ and $T = 1$.

6.2. The q-Brownian motion.

Denote $\omega(\xi) = a(\xi) + a^*(\xi)$.

Definition 6.9. Let $V = \mathbb{C}$, $\xi = 1 \in V, T = 0, \lambda = 0$ and $\xi_t = \mathbf{1}_{[0,t)}$. Then the *$q$-Brownian motion* is the process $X(t) = p(\xi_t, 0, 0) = \omega(\xi_t)$. The distribution of $X(t)$ is the q-Gaussian distribution with parameter t, given by $\mathrm{LH}_q^{-1}(0, t\delta_0)$.

See, for example, [BKS97] for an explicit form of the q-Gaussian distribution.

Definition 6.10. q-Hermite polynomials are defined by the recursion relation

$$xH_{q,n}(x,t) = H_{q,n+1}(x,t) + [n]_q t H_{q,n-1}(x,t)$$

with initial conditions $H_{q,0}(x,t) = 1$, $H_{q,1}(x,t) = x$.

Lemma 6.11. *The following chaos representation holds:*

$$H_{q,n}(X(t),t)\Omega = \xi_t^{\otimes n}.$$

Therefore the q-Gaussian distribution with parameter t is the orthogonalization measure of the q-Hermite polynomials with parameter t.

Proof. For $n = 0$, $1\Omega = \Omega$. For $n = 1$, $X(t)\Omega = \xi_t$. For $n \geq 2$ by induction

$$\begin{aligned} H_{q,n+1}(X(t),t)\Omega &= X(t)\xi_t^{\otimes n} - [n]_q t \xi_t^{\otimes(n-1)} \\ &= \xi_t^{\otimes(n+1)} + [n]_q t \xi_t^{\otimes(n-1)} - [n]_q t \xi_t^{\otimes(n-1)} \\ &= \xi_t^{\otimes n}. \end{aligned}$$

Since $\xi_t^{\otimes n}$ are orthogonal in $\mathcal{F}_q(H)$ for different n, the polynomials $H_{q,n}$ are orthogonal for different n with respect to the distribution of $X(t)$. □

For the q-Brownian motion, $\Delta_2(t) = t$ and $\Delta_k(t) = 0$ for $k > 2$. But in this case, we can in fact calculate all the partition-dependent stochastic measures. Temporarily denote by s_1, s_2 the numbers of singleton and 2-element classes of π, respectively. For a singleton (i), define its depth as

$$d(i) = \left|\{j | \exists a, b \in B_j : a < i < b\}\right|.$$

Define the singleton depth $\mathrm{sd}\,(\pi)$ to be the sum of depths of all the singletons of π. In the single-variable case, we will omit the polynomial from the notation for stochastic measures.

Proposition 6.12. *The partition-dependent stochastic measures of the q-Brownian motion are*

$$\mathrm{St}_\pi(t;X) = \begin{cases} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2} H_{q,s_1}(X(t),t) & \text{if all the classes of } \pi \text{ contain} \\ & \quad \text{at most 2 elements,} \\ 0 & \text{otherwise,} \end{cases}$$

where the defining limits are taken in the $L^p(\varphi)$ norm, for any $p \geq 1$ (where $\|X\|_p = \varphi\left[|X|^p\right]^{1/p}$).

The result is known for $q = 1$ [RW97] when a different mode of convergence is used, and for $q = 0$ [Ans00] when the limit is taken in the operator norm. The preceding proposition probably holds with the operator norm convergence as well.
Throughout, we will use the following explicit formula for the moments of the q-Brownian motion, implicitly contained already in [BS91]:

$$\varphi\left[\omega(\eta_1)\omega(\eta_2)\ldots\omega(\eta_{2n})\right] = \sum_{\pi \in \mathcal{P}_2(2n)} q^{\mathrm{rc}(\pi)} \prod_{i=1}^n \left\langle \eta_{a(B_i)}, \eta_{b(B_i)} \right\rangle.$$

Lemma 6.13. *If π has a class of at least three elements, then* $\mathrm{St}_\pi(t;X) = 0$, *where the limit is taken in the operator norm.*

Proof. For $\vec{v} \in [1 \ldots N]^n_\pi$ and $B \in \pi$, denote by $v(B)$ the value of v on any element of B. Denote by $\pi(\vec{v})$ the partition induced by $\vec{v}$, given by

$$i \overset{\pi(\vec{v})}{\sim} j \Leftrightarrow v(i) = v(j).$$

Assume $t > 1$ to simplify notation.

$$\|\mathrm{St}_\pi(t; X, \mathcal{I})\|_{2k}^{2k} = \left\| \sum_{\vec{v} \in [1 \ldots N]^n_\pi} \mathbf{X}_{\vec{v}}(t) \right\|_{2k}^{2k},$$

which equals to

$$\begin{aligned}
&\varphi\left[\left(\left(\sum_{\vec{v} \in [1 \ldots N]^n_\pi} \mathbf{X}_{\vec{v}}(t)\right)\left(\sum_{\vec{v} \in [1 \ldots N]^n_\pi} \mathbf{X}_{\vec{v}}(t)\right)^*\right)^k\right] \\
&= \varphi\left[\sum_{\substack{\vec{v} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{2k}) \\ \vec{v}_{2i+1} \in [1 \ldots N]^n_\pi, \vec{v}_{2i} \in [1 \ldots N]^n_{\pi^{op}}}} \mathbf{X}_{\vec{v}}(t)\right] \\
&= \sum_{\substack{\vec{v} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{2k}) \\ \vec{v}_{2i+1} \in [1 \ldots N]^n_\pi, \vec{v}_{2i} \in [1 \ldots N]^n_{\pi^{op}}}} \sum_{\substack{\tau \in \mathcal{P}_2(2nk) \\ \tau \leq \pi(\vec{v})}} q^{\mathrm{rc}(\tau)} \prod_{B \in \tau} \left|I_{v(B)}\right| \\
&\leq \sum_{\tau \in \mathcal{P}_2(2nk)} q^{\mathrm{rc}(\tau)} \delta(\mathcal{I})^{k(n-2|\pi|)} t^{2k|\pi|} \\
&\leq Q_{2nk}(q) \delta(\mathcal{I})^{k(n-2|\pi|)} t^{2k|\pi|},
\end{aligned}$$

where $Q_{2n}(q) = \sum_{\tau \in \mathcal{P}_2(2n)} q^{\mathrm{rc}(\tau)}$. Therefore

$$\|\mathrm{St}_\pi(t; X, \mathcal{I})\|_{2k} \leq Q_{2nk}(q)^{1/2k} t^{|\pi|} \delta(\mathcal{I})^{(n-2|\pi|)/2}.$$

$Q_{2n}(q)$ is the $2n$-th moment of the q-Gaussian distribution. By [AB98], it is equal to $\sum_{\tau \in NC_2(2n)} \prod_{B \in \tau} [d(B)]_q$. Here $NC_2(2n)$ is the collection of noncrossing pair partitions on the set of $2n$ elements, and for $\tau \in NC(n)$ and an arbitrary class $B \in \tau$, we can define its depth in τ by $d(B) = |\{i : a(B_i) \leq B \leq b(B_i)\}|$; note that this differs by 1 from our definition of singleton depth above. For $q \in [-1, 1)$, $[k]_q \leq \frac{2}{1-q}$, and so the sum is bounded by $c_n \left(\frac{2}{1-q}\right)^n$, where c_n is the n-th Catalan number. Therefore $Q_{2nk}(q)^{1/2k} \leq 2^n \left(\frac{2}{1-q}\right)^{n/2}$. We conclude that

$$\|\mathrm{St}_\pi(t; X, \mathcal{I})\|_{2k} \leq 2^n \left(\frac{2}{1-q}\right)^{n/2} t^{|\pi|} \delta(\mathcal{I})^{(n-2|\pi|)/2}. \tag{9}$$

All the vectors ξ_t lie in the real subspace $L^2(\mathbb{R}_+, \mathbb{R})$ of $L^2(\mathbb{R}_+, \mathbb{C})$. The state φ is faithful on the algebra generated by $\left\{\omega(\xi) : \xi \in L^2(\mathbb{R}_+, \mathbb{R})\right\}$, in fact Ω is separating for this algebra [BS94]. Therefore the estimate (9) holds for the operator norm of $\mathrm{St}_\pi(t; X, \mathcal{I})$. So this norm converges to 0 as $\delta(\mathcal{I}) \to 0$. □

LEMMA 6.14. *Let π contain only classes of at most 2 elements. Suppose that one of the following conditions holds:*

a. *$B, C \in \pi$ are 2-element classes with $a(B) < a(C) = b(B) - 1 < b(C)$. Let α be the transposition $(a(C) b(B))$.*
b. *$B \in \pi$ is a 2-element class and $(j) \in \pi$ is a singleton with $a(B) < j = b(B) - 1$. Let α be the transposition $(j\ b(B))$.*

Then $\mathrm{St}_\pi = q\mathrm{St}_{\alpha\circ\pi}$, meaning

$$\lim_{\delta(\mathcal{I})\to 0} \left\| \mathrm{St}_\pi(t; X, \mathcal{I}) - q\mathrm{St}_{\alpha\circ\pi}(t; X, \mathcal{I}) \right\|_p = 0,$$

for any $p \geq 1$.

Proof. We prove only the first case, the proof of the second case is similar. For a multi-index $\vec{v}$, denote $\alpha(\vec{v}) = (v(\alpha(1)), v(\alpha(2)), \ldots, v(\alpha(n)))$.

$$\varphi\left[\left(\left(\sum_{\vec{v}\in[1\ldots N]^n_\pi} \mathbf{X}_{\vec{v}}(t) - q\mathbf{X}_{\alpha(\vec{v})}(t)\right)\left(\sum_{\vec{v}\in[1\ldots N]^n_\pi} \mathbf{X}_{\vec{v}}(t) - q\mathbf{X}_{\alpha(\vec{v})}(t)\right)^*\right)^k\right]$$
$$= \varphi\left[\sum_{S\subset[1\ldots 2nk]} \sum_{\substack{\sigma\in\mathcal{P}(2nk)\\ \sigma\wedge 2k\hat{1}_n = \sum_{j=1}^{2nk}\pi_j(S)}} \sum_{\vec{v}\in[1\ldots N]^{2nk}_\sigma} (-q)^{|S|}\mathbf{X}_{\vec{v}}(t)\right],$$

where

$$\pi_j(S) = \begin{cases} \alpha\circ\pi & \text{if } j \in S, j \text{ odd},\\ (\alpha\circ\pi)^{op} & \text{if } j\in S, j \text{ even},\\ \pi & \text{if } j\notin S, j \text{ odd},\\ \pi^{op} & \text{if } j \notin S, j \text{ even}.\end{cases}$$

First consider all the terms with $\sigma \in \mathcal{P}_2(2nk)$.

$$\varphi\left[\sum_{S\subset[1\ldots 2nk]} \sum_{\substack{\sigma\in\mathcal{P}_2(2nk)\\ \sigma\wedge 2k\hat{1}_n = \sum_{j=1}^{2nk}\pi_j(S)}} \sum_{\vec{v}\in[1\ldots N]^{2nk}_\sigma} (-q)^{|S|}\mathbf{X}_{\vec{v}}(t)\right] \tag{10}$$
$$= \sum_{S\subset[1\ldots 2nk]} \sum_{\substack{\sigma\in\mathcal{P}_2(2nk)\\ \sigma\wedge 2k\hat{1}_n = \sum_{j=1}^{2nk}\pi_j(S)}} \sum_{\vec{v}\in[1\ldots N]^{2nk}_\sigma} (-q)^{|S|} q^{\mathrm{rc}(\sigma)} \prod_{B\in\sigma}\left|I_{v(B)}\right|.$$

Since $\sigma \in \mathcal{P}_2(2nk)$, it is completely determined by the collection $\{\pi_j(S)\}$ and the partition σ_s induced by σ on the singleton classes of $\sum_{j=1}^{2nk}\pi_j(S)$. Note that there is a natural (order-preserving) identification of the singleton classes of π

and $\alpha \circ \pi$, so we can consider σ_s as a partition on the singletons of $k(\pi + \pi^{op})$. Denote by σ' the partition obtained from $k(\pi+\pi^{op})$ by identifying its singleton classes using σ_s.

It is easy to see that

$$\mathrm{rc}\,(\sigma) = \mathrm{rc}\,(\sigma_s) + \mathrm{rc}\left(\sum_{j=1}^{2nk} \pi_j(S)\right) + (2nk)\mathrm{sd}\,(\pi)\,.$$

In its turn, $\mathrm{rc}\left(\sum_{j=1}^{2nk} \pi_j(S)\right) = (2nk)\mathrm{rc}\,(\pi) - |S|$. Therefore, continuing expression (10),

$$= q^{(2nk)(\mathrm{rc}(\pi)+\mathrm{sd}(\pi))} \sum_{S \subset [1\ldots 2nk]} \sum_{\substack{\sigma \in \mathcal{P}_2(2nk) \\ \sigma \wedge 2k\hat{1}_n = \sum_{j=1}^{2nk} \pi_j(S)}} \sum_{\vec{v} \in [1\ldots N]^{2nk}_{\sigma}} (-1)^{|S|} q^{\mathrm{rc}(\sigma_s)} \prod_{B \in \sigma} \left|I_{v(B)}\right|$$

$$= q^{(2nk)(\mathrm{rc}(\pi)+\mathrm{sd}(\pi))} \sum_{\sigma_s} \sum_{\vec{v} \in [1\ldots N]^{2nk}_{\sigma'}} q^{\mathrm{rc}(\sigma_s)} \prod_{B \in \sigma'} \left|I_{v(B)}\right| \sum_{S \subset [1\ldots 2nk]} (-1)^{|S|}.$$

In this expression, the only dependence on S is in $(-1)^{|S|}$, and the sum $\sum_{S \subset [1\ldots 2nk]} (-1)^{|S|} = 0$.

Therefore the non-zero contributions come only from the terms with $\sigma \notin \mathcal{P}_2(2nk)$. The rest of the argument proceeds as in the previous lemma, and shows that $\left\|\mathrm{St}_\pi - q\mathrm{St}_{\alpha\circ\pi}\right\|_p = 0$. □

Proof of Proposition 6.12. Using the lemmas, it suffices to prove the proposition for an interval partition π whose classes have at most 2 elements. Moreover, by the same arguments as in the preceding lemmas it is easy to see that each 2-element class contributes a factor of t. It remains to show that

$$\mathrm{St}_{\hat{0}_n}(t; X) = H_{q,n}(X(t), t).$$

For $n = 1$, $\sum_{i=1}^{N} X_i(t) = X(t)$. For $n = 2$,

$$\sum_{i \neq j}^{N} X_i(t) X_j(t) = \Big(\sum_{i=1}^{N} X_i(t)\Big)^2 - \sum_{i=1}^{N} X_i^2(t) = X^2(t) - t = H_{q,2}(X(t), t).$$

For $n > 2$, it suffices to show that $\mathrm{St}_{\hat{0}_n}(t; X)$ satisfy the same recursion relations as the q-Hermite polynomials. Indeed,

$$X(t)\mathrm{St}_{\hat{0}_n}(t; X, \mathcal{I}) = \mathrm{St}_{\hat{0}_{(n+1)}}(t; X, \mathcal{I}) + \sum_{i=2}^{n+1} \mathrm{St}_{\pi_i}(t; X, \mathcal{I}),$$

where $\pi_i = ((1,i)(2)\ldots(\hat{\imath})\ldots(n{+}1)) \in \mathcal{P}(n{+}1)$. By the second case of Lemma 6.14 and using induction on n,

$$\mathrm{St}_{\pi_i}(t; X) = tq^{i-2}\mathrm{St}_{\hat{0}_{n-1}}(t; X).$$

Therefore

$$\mathrm{St}_{\hat{0}_{n+1}}(t; X) = X(t)\mathrm{St}_{\hat{0}_n}(t; X) - \sum_{i=2}^{n+1} tq^{i-2}\mathrm{St}_{\hat{0}_{n-1}}(t; X).$$

This implies by induction that $\mathrm{St}_{\hat{0}_{n+1}}(t;X)$ is well-defined, and

$$
\begin{aligned}
X(t)\mathrm{St}_{\hat{0}_n}(t;X) &= \mathrm{St}_{\hat{0}_{n+1}}(t;X) + \sum_{i=2}^{n+1} tq^{i-2}\mathrm{St}_{\hat{0}_{n-1}}(t;X) \\
&= \mathrm{St}_{\hat{0}_{n+1}}(t;X) + t[n]_q\mathrm{St}_{\hat{0}_{n-1}}(t;X).
\end{aligned}
\tag{11}
$$

□

REMARK 6.15 (A combinatorial corollary). Denote by $\mathcal{P}_{1,2}(n)$ the collection of all partitions in $\mathcal{P}(n)$ that have classes of only 1 or 2 elements, and by $s_1(\pi), s_2(\pi)$ the number of 1- and 2-elements classes, respectively. Then using equation (2), we have a combinatorial corollary of the preceding proposition:

$$
x^n = \sum_{\pi \in \mathcal{P}_{1,2}(n)} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2(\pi)} H_{q,s_1(\pi)}(x,t).
$$

Using the Möbius function on $\mathcal{P}(n)$, this relation can be inverted, to obtain

$$
H_{q,n}(x,t) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^{s_2(\pi)} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2(\pi)} x^{s_1(\pi)},
$$

which is a well-known expansion for q-Hermite polynomials. In particular,

$$
\begin{aligned}
X(t)^n\Omega &= \sum_{\pi \in \mathcal{P}_{1,2}(n)} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2(\pi)} H_{q,s_1(\pi)}(X(t),t)\Omega \\
&= \sum_{\pi \in \mathcal{P}_{1,2}(n)} q^{\mathrm{rc}(\pi)+\mathrm{sd}(\pi)} t^{s_2(\pi)} \xi_t^{\otimes s_1(\pi)} \\
&= H_{q,n}(\xi_t, -t),
\end{aligned}
$$

where ξ_t is considered as an element of the tensor algebra, with the tensor multiplication.

6.3. THE q-POISSON PROCESS. The following representation is similar to but different from that of [SY00b].

DEFINITION 6.16. Let $V = \mathbb{C}$, $\xi = 1 \in V, T = \mathrm{Id}, \lambda = 1$ and $\xi_t = \mathbf{1}_{[0,t)}, T_t = \mathbf{1}_{[0,t)}$. The *$q$-Poisson process* is the process $X(t) = p(\xi_t, T_t, t)$. The distribution of $X(t)$ is the q-Poisson distribution with parameter t, given by $\mathrm{LH}_q^{-1}(t, t\delta_1)$.

We use the definitions of the q-Poisson distribution and the q-Poisson-Charlier polynomials that were introduced in [SY00a]. See that paper for an explicit formula for the q-Poisson distribution.

DEFINITION 6.17. q-Poisson-Charlier polynomials are defined by the recursion relations

$$
xC_{q,n}(x,t) = C_{q,n+1}(x,t) + ([n]_q + t)C_{q,n}(x,t) + [n]_q tC_{q,n-1}(x,t) \tag{12}
$$

with initial conditions $C_{q,0}(x,t) = 1$, $C_{q,1}(x,t) = x - t$.

Remark 6.18. Let $S_{k,n;q} = \sum_{\pi \in \Pi(n,k)} q^{\mathrm{rc}(\pi)}$, where $\Pi(n,k)$ is the set of partitions in $\mathcal{P}(n)$ with k classes. It is appropriate to call these q-Stirling numbers: they interpolate between the usual Stirling numbers for $q = 1$ and $\frac{1}{n-k+1}\binom{n}{k}\binom{n-1}{k-1}$ for $q = 0$. Then according to [Bia97] (cf. [NS94]), the generating function

$$\sum_{k,n \geq 0} S_{k,n;q} t^k z^n$$

has the continued fraction expansion

$$\cfrac{1}{1 - ([0]_q + t)z - \cfrac{[1]_q t z^2}{1 - ([1]_q + t)z - \cfrac{[2]_q t z^2}{\cdots}}}.$$

It is also the moment-generating function (in z) of the probability measure with q-cumulants $r_n = t$ for $n \geq 1$. The formula says precisely that the orthogonal polynomials with respect to that measure satisfy the 3-term recursion relation (12). These are then the orthogonal polynomials with respect to the q-Poisson distribution with parameter t. A more direct proof follows from the following lemma, which is almost verbatim from [SY00b].

Lemma 6.19. *The following chaos representation holds:*

$$C_{q,n}(X(t), t)\Omega = \xi_t^{\otimes n}.$$

Therefore the distribution of $X(t)$ is the orthogonalization measure of the q-Poisson-Charlier polynomials.

For the q-Poisson process, for $k > 0$, $\Delta_k(t) = X(t)$ independently of k. The situation with the more general stochastic measures is more complicated. In particular, it is *not* true that $\mathrm{St}_{\hat{0}_n}(t; X) = C_{q,n}(X(t), t)$, unlike in the classical and the free case [RW97, Ans00]. Nevertheless, the analog of equation (11), which is a form of q-Kailath-Segall formula for centered processes, does hold, as follows:

Lemma 6.20. *For $n \geq 0$,*

$$\begin{aligned} C_{q,n+1}(X(t), t) &= (X(t) - t) C_{q,n}(X(t), t) \\ &\quad + \sum_{j=1}^{n} (-1)^j [n]_q [n-1]_q \ldots [n-j+1]_q \Delta_{j+1}(t; X) C_{q,n-j}(X(t), t). \end{aligned}$$

Proof. We need to show that

(13)

$$C_{q,n+1}(x, t) = (x - t) C_{q,n}(x, t) + \sum_{j=1}^{n} (-1)^j [n]_q [n-1]_q \ldots [n-j+1]_q x C_{q,n-j}(x, t).$$

We will prove this by induction. The formula holds for $n = 0$. Suppose the formula true for $n-1$, i.e.

$$C_{q,n}(x,t) = (x-t)C_{q,n-1}(x,t) + \sum_{j=1}^{n-1} (-1)^j [n-1]_q [n-2]_q \ldots [n-j]_q x C_{q,n-j-1}(x,t).$$

Then

$$\begin{aligned} &- [n]_q C_{q,n}(x,t) \\ &\quad = -[n]_q (x-t) C_{q,n-1}(x,t) + \sum_{j=1}^{n-1} (-1)^{j+1} [n]_q [n-1]_q \ldots [n-j]_q x C_{q,n-j-1}(x,t) \\ &\quad = [n]_q t C_{q,n-1}(x,t) + \sum_{j=0}^{n-1} (-1)^{j+1} [n]_q [n-1]_q \ldots [n-j]_q x C_{q,n-j-1}(x,t) \\ &\quad = [n]_q t C_{q,n-1}(x,t) + \sum_{j=1}^{n} (-1)^{j} [n]_q [n-1]_q \ldots [n-j+1]_q x C_{q,n-j}(x,t). \end{aligned}$$

Add to it the recursion relation (12)

$$[n]_q C_{q,n}(x,t) + C_{q,n+1}(x,t) = (x-t)C_{q,n}(x,t) - [n]_q t C_{q,n-1}(x,t)$$

to obtain (13). □

7. von Neumann algebras

In this section we list some preliminary results on the algebras generated by the q-Lévy processes. Throughout the section we consider only $q \in (-1, 1)$.

Let $\mathbf{X}$ be a centered q-Lévy process with $X^{(i)} = p(\xi_i, T_i, 0)$, $i \in [1 \ldots k]$. We further assume that the Hilbert space V has a real Hilbert subspace $V_{\mathbb{R}}$ so that V is the complexification of $V_{\mathbb{R}}$. Then the Hilbert space H is the complexification of its real subspace $L^2(\mathbb{R}_+, \mathbb{R}, dx) \otimes V_{\mathbb{R}}$. So H has a natural conjugation $\bar{}$ defined on it. Assume that $\{\xi_i\}_{i=1}^k \subset V_{\mathbb{R}}$, and that for each i, $T_i(V_{\mathbb{R}}) \subset V_{\mathbb{R}}$ and T_i is the complexification of its restriction to $V_{\mathbb{R}}$. Denote by $\mathcal{B}(\mathcal{F}_q(H))$ the algebra of all bounded linear operators on $\mathcal{F}_q(H)$, and by $\mathcal{A}_{\mathbf{X}}$ its von Neumann subalgebra generated by $\left\{X^{(i)}(t) : i \in [1 \ldots k], t \in [0, \infty)\right\}$. As usual, if the operators comprising $\mathbf{X}$ are not bounded, we mean the algebra generated by their spectral projections.

First consider the multi-dimensional q-Brownian motion. Let $\{\xi_i\}_{i=1}^k$ be an orthonormal basis for V, let $V_{\mathbb{R}}$ be the real linear span of $\{\xi_i\}_{i=1}^k$, and all $T_i = 0$. Since the space of simple functions is dense in $L^2(\mathbb{R}_+)$, the resulting algebra is the same as the one obtained from the q-Gaussian functor. The algebra $\mathcal{A}$ is known to have the following properties [BS94, BKS97].

a. The vacuum vector Ω is a cyclic vector for $\mathcal{A}$.
b. The vacuum expectation φ is a trace on $\mathcal{A}$.
c. The vacuum vector Ω is a cyclic vector for the commutant $\mathcal{A}'$ of $\mathcal{A}$. Therefore it is a separating vector for $\mathcal{A}$, and the vacuum expectation φ is faithful on $\mathcal{A}$.

d. Define an anti-linear involution J on $\mathcal{F}_q(H)$ by

$$J(\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n) = \bar{\eta}_n \otimes \ldots \otimes \bar{\eta}_2 \otimes \bar{\eta}_1.$$

Then $\mathcal{A}' = J\mathcal{A}J$.

e. $\mathcal{A}$ is a factor. Therefore $\mathcal{A}$ is a II_1 factor in standard form.

We now investigate these properties for more general processes.

Lemma 7.1. *If* $\operatorname{span}(\{\xi_i : i \in [1 \ldots k]\})$ *is dense in* V*, the vacuum vector* Ω *is a cyclic vector for* $\mathcal{A}_{\mathbf{X}}$.

Proof. For a multi-index $\vec{u}$ of length n and a family of intervals $\{I_i\}$,

$$\prod_{i=1}^{n} X^{(u(i))}(I_i)\Omega = (\mathbf{1}_{I_1} \otimes \xi_{u(1)}) \otimes (\mathbf{1}_{I_2} \otimes \xi_{u(2)}) \otimes \ldots \otimes (\mathbf{1}_{I_n} \otimes \xi_{u(n)}) + \vec{\eta},$$

with $\vec{\eta} \in \bigoplus_{j=0}^{n-1} (L^2(\mathbb{R}_+) \otimes V)^{\otimes j}$. So if $\operatorname{span}(\{\xi_i : i \in [1 \ldots k]\})$ is dense in V, by induction on n we see that Ω is a cyclic vector for $\mathcal{A}_{\mathbf{X}}$. $\square$

Remark 7.2. We could also consider the algebra generated by the process and its higher diagonal measures determined in Section 5. We describe the construction in the one-dimensional case. Let $X = p(\xi, T, 0)$, and define

$$\Delta_n = p(T^{n-1}\xi, T^n, \langle \xi, T^{n-2}\xi \rangle).$$

Let $\mathcal{A}_{\mathbf{X},\Delta}$ be the von Neumann algebra generated by all the processes $\Delta_n(t)$ for $n \geq 1$. Then Ω is a cyclic vector for $\mathcal{A}_{\mathbf{X},\Delta}$. We may describe this construction in more detail elsewhere.

Lemma 7.3. *Let* $q = 0$*. If the cumulant functional* $R(\cdot; \mathbf{X})$ *is a trace on* $\mathbb{C}\langle \mathbf{x} \rangle$*, then* φ *is a trace on* $\mathcal{A}_{\mathbf{X}}$.

Proof. Let $\{I_i\}_{i=1}^{l}$ be a family of disjoint intervals. It suffices to show the trace property for the family of operators $\left\{X^{(u(i))}(I_{v(i)})\right\}_{i=1}^{n}$ for arbitrary multi-indices $\vec{u}, \vec{v}$. However, it is easy to see that

$$\begin{aligned}
&\varphi\left[\prod_{i=1}^{n} X^{(u(i))}(I_{v(i)})\right] \\
&\quad = \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi(\vec{v})}} \prod_{\substack{B \in \sigma \\ B=(j(1),j(2),\ldots,j(l))}} \left| \bigcap_{j \in B} I_{v(j)} \right| \left\langle \xi_{j(1)}, T_{j(2)} \ldots T_{j(l-1)} \xi_{j(l)} \right\rangle \\
&\quad = \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi(\vec{v})}} R_\sigma(\mathbf{x}_{\vec{u}}; \mathbf{X}) \prod_{B \in \sigma} \left| \bigcap_{j \in B} I_{v(j)} \right|.
\end{aligned}$$

If $R(\cdot; \mathrm{X})$ is a trace, this expression is symmetric under simultaneous cyclic permutations of the components of $\vec{u}$ and $\vec{v}$. $\square$

The hypothesis of Lemma 7.1 is rarely satisfied. It does hold for the q-Brownian motion, and it also holds for the q-Poisson process. For the remained of the section we investigate the latter.
Let $\{\xi_i\}_{i=1}^k$ be an orthonormal basis for V, with $V_{\mathbb{R}}$ the real linear span of $\{\xi_i\}_{i=1}^k$. Let T_i be the orthogonal projection on ξ_i. The process $\mathbf{X}$ with $X^{(i)} = p(\xi_i, T_i, 0)$ is the centered k-dimensional q-Poisson process. By Lemma 7.1, Ω is a cyclic vector for $\mathcal{A}_{\mathbf{X}}$; a related statement is contained in Lemma 6.19.
First let $q = 0$. Then by Lemma 7.3, φ is a trace on $\mathcal{A}_{\mathbf{X}}$. By the same arguments used in [BS94] for the q-Brownian motion, it is easy to see that Ω is separating for $\mathcal{A}_{\mathbf{X}}$, and $\mathcal{A}'_{\mathbf{X}} = J\mathcal{A}_{\mathbf{X}}J$. In fact, using a different representation of the process [NS96] it follows that $\mathcal{A}_{\mathbf{X}}$ is the reduced von Neumann algebra of the free group on infinitely many generators. The preceding discussion shows that it is given in standard form.
For $q \neq 0$, for simplicity we consider the 1-dimensional process. Then $H = L^2(\mathbb{R}_+)$. We extend the mapping $I \mapsto X(I)$ to the map on all of $H_{\mathbb{R}}$, namely for $f \in L^2(\mathbb{R}_+, \mathbb{R}, dx)$, $X(f) = a(f) + a^*(f) + p(M_f)$, where M_f is the (possibly unbounded) operator of multiplication by f. Then $\mathcal{A}_X$ is the von Neumann algebra generated by $\{X(f) : f \in H_{\mathbb{R}}\}$.

Proposition 7.4. *For the q-Poisson process X, Ω is a separating vector for $\mathcal{A}_X$.*

Proof. Define the *Wick map* $W : \mathcal{F}_{\mathrm{alg}}(H_{\mathbb{R}}) \to \mathcal{A}_X$ as follows. For $f, f_1, f_2, \ldots \in H_{\mathbb{R}}$, let $W(\Omega) = \mathrm{Id}$, $W(f) = X(f)$, inductively

$$\begin{aligned} W(f \otimes f_1 \otimes \ldots \otimes f_n) = X(f)&W(f_1 \otimes \ldots \otimes f_n) \\ &- \sum_{i=1}^n q^{i-1} \langle f, f_i \rangle W(f_1 \otimes \ldots \otimes \hat{f}_i \otimes \ldots \otimes f_n) \\ &- \sum_{i=1}^n q^{i-1} W(f f_i \otimes f_1 \otimes \ldots \otimes \hat{f}_i \otimes \ldots \otimes f_n), \end{aligned}$$

and extend $\mathbb{R}$-linearly. Clearly

$$W(f_1 \otimes \ldots \otimes f_n)\Omega = f_1 \otimes \ldots \otimes f_n. \tag{14}$$

For $f \in H_{\mathbb{R}}$, define the operator $X_r(f)$ with dense domain $\mathcal{F}_{\mathrm{alg}}(H_{\mathbb{R}})$ by

$$X_r(f) f_1 \otimes \ldots \otimes f_n = W(f_1 \otimes \ldots \otimes f_n) X(f)\Omega = W(f_1 \otimes \ldots \otimes f_n) f.$$

$X_r(f)$ commutes with $\mathcal{A}_X$ on its domain of definition. Indeed,

$$X(g) X_r(f)\Omega = X(g) f = W(g) f = X_r(f) g = X_r(f) X(g)\Omega.$$

Also,

$$X(g) X_r(f) f_1 \otimes \ldots \otimes f_n = X(g) W(f_1 \otimes \ldots \otimes f_n) f$$

and

$$\begin{aligned}
&X_r(f)X(g)f_1 \otimes \ldots \otimes f_n \\
&\quad = X_r(f)X(g)W(f_1 \otimes \ldots \otimes f_n)\Omega \\
&\quad = X_r(f)\Big[W(g \otimes f_1 \otimes \ldots \otimes f_n) + \sum_{i=1}^{n} q^{i-1} \langle g, f_i \rangle W(f_1 \otimes \ldots \otimes \hat{f}_i \otimes \ldots \otimes f_n) \\
&\qquad + \sum_{i=1}^{n} q^{i-1} W(g f_i \otimes f_1 \otimes \ldots \otimes \hat{f}_i \otimes \ldots \otimes f_n)\Big]\Omega \\
&\quad = X(g)W(f_1 \otimes \ldots \otimes f_n)f.
\end{aligned}$$

Next,

$$\begin{aligned}
X_r(f_n)X_r(f_{n-1})\ldots X_r(f_2)X_r(f_1)\Omega &= W(\ldots W(W(f_1)f_2)\ldots f_{n-1})f_n \\
&= f_1 \otimes f_2 \otimes \ldots \otimes f_n + \vec{\eta},
\end{aligned}$$

with $\vec{\eta} \in \bigoplus_{i=0}^{n-1} H^{\otimes i}$. Therefore Ω is separating for $\mathcal{A}_X$. □

As a consequence, the map W is in fact determined by the condition (14).

Lemma 7.5. *Assume $q \neq 0$. Then for the q-Poisson process X,*

a. *φ is not a trace on $\mathcal{A}_X$.*
b. *$\mathcal{A}_X$ and $J\mathcal{A}_X J$ do not commute.*

Proof. Let I_1, I_2 be two disjoint intervals. It is easy to see that

$$\varphi\left[X(I_1)X(I_2)X(I_1)X(I_2)X(I_1)\right] = q^2 \left|I_1\right| \left|I_2\right|,$$

while

$$\varphi\left[X(I_1)X(I_1)X(I_2)X(I_1)X(I_2)\right] = q \left|I_1\right| \left|I_2\right|.$$

Therefore φ is not a trace on $\mathcal{A}_X$.
Moreover, for an interval I, $(X(I)JX(I)J)(\eta_1 \otimes \eta_2 \otimes \eta_3)$ contains the term $\mathbf{1}_I \otimes \eta_1 \otimes \eta_2 \otimes \eta_3$ with coefficient q^3, while $(JX(I)JX(I))(\eta_1 \otimes \eta_2 \otimes \eta_3)$ contains no such term. So already on $H^{\otimes 3}$, $\mathcal{A}_X$ and $J\mathcal{A}_X J$ do not commute. □

We conclude that even for the q-Poisson process, the Fock representation of the corresponding algebra provides little immediate information about the algebra. The subject certainly deserves further investigation.

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Excellent Special Orthogonal Groups

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Abstract. In this paper we give a complete classification of excellent special orthogonal groups SO(q) where q is a regular quadratic form over a field of characteristic 0.

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Contents

0. Introduction

Let F be a field of characteristic $\neq 2$ and let φ be a regular quadratic form over F. Then φ is said to be *excellent* if, for any field extension E/F, the anisotropic part of $\varphi_E := \varphi \otimes_F E$ is defined over F. This notion was introduced by M. Knebusch in [Kn1, Kn2]. In [KR], a similar notion for semisimple algebraic groups was introduced and studied for special linear and special orthogonal groups. Let us recall that the main result of [KR] says that the following conditions are equivalent.

(i) *The special orthogonal group* $\mathbb{SO}(\varphi)$ *is excellent.*
(ii) *For every field extension* E/F *there is an element* $a \in E^*$ *and a form* ψ *over* F *such that the anisotropic part of* φ_E *is isomorphic to* $a\psi_E$.

In general, if φ is excellent $\mathbb{SO}(\varphi)$ is also excellent. The converse holds for odd-dimensional forms (see [KR]). For even-dimensional forms there are examples of non-excellent forms φ such that the group $\mathbb{SO}(\varphi)$ is excellent.

We say that the form φ is *quasi-excellent* if the group $\mathbb{SO}(\varphi)$ is excellent. Taking into account the criterion mentioned above, we can rewrite the definition as follows: φ *is quasi-excellent if for any field extension* E/F *there exists a form* ψ *over* F *such that* $(\varphi_E)_{\mathrm{an}}$ *is similar to* ψ_E . In this case we write $(\varphi_E)_{\mathrm{an}} \sim \psi_E$.

To study even-dimensional quasi-excellent forms, it is very convenient to give another definition.

Definition 0.1. We say that a sequence of quadratic forms $\varphi_0, \varphi_1, \dots, \varphi_h$ over F is *quasi-excellent* if the following conditions hold:

- the forms $\varphi_0, \dots, \varphi_{h-1}$ are regular and of dimension > 0;
- the form φ_0 is anisotropic and the form φ_h is zero;
- for $i = 1, \dots, h$, we have $((\varphi_0)_{F_i})_{\mathrm{an}} \sim (\varphi_i)_{F_i}$ where $F_i = F(\varphi_0, \dots, \varphi_{i-1})$.

Then the number h is called the *height of the sequence.* (It coincides with the height of φ_0 defined by Knebusch in [Kn1], 5.4.)

It is not difficult to show that we have a surjective map (see Lemma 2.2 and Corollary 2.4 below):

$$\begin{aligned}\{\text{quasi-excellent sequences}\} &\to \{\text{even-dim. quasi-excellent anisotropic forms}\}\\ (\varphi_0, \varphi_1, \dots, \varphi_h) &\mapsto \varphi_0\end{aligned}$$

Any regular quadratic form of dimension $n > 0$ over F is isomorphic to a diagonal form $\langle a_1, \dots, a_n\rangle := a_1X_1^2 + \dots + a_nX_n^2$ with $a_1, \dots, a_n \in F^*$ and variables $X_1, \dots, X_n$. A d-fold *Pfister form* is a form of the type

$$\langle\langle a_1, \dots, a_d\rangle\rangle := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_d\rangle \ .$$

Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence. We prove in Lemma 2.5 that φ_{h-1} is similar to some d-fold Pfister form, and then say that d is the *degree of the sequence.*

Example 0.2. Let $a_1, a_2, \dots, a_d, k_0, k_1, k_2, u, v, c \in F^*$. Set

$$\begin{aligned}\phi_0 &= k_0\,\langle\langle a_1, a_2, \dots, a_{d-1}\rangle\rangle \otimes (\langle\langle u, v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle)\,,\\ \phi_1 &= k_1\,\langle\langle a_1, a_2, \dots, a_{d-1}\rangle\rangle \otimes \langle -u, -v, uv, a_d\rangle\,,\\ \phi_2 &= k_2\,\langle\langle a_1, a_2, \dots, a_d\rangle\rangle\,,\\ \phi_3 &= 0\,.\end{aligned}$$

Suppose that ϕ_0 , ϕ_1 and ϕ_2 are anisotropic. Then the sequence $(\phi_0, \phi_1, \phi_2, \phi_3)$ is quasi-excellent of degree d (see Lemma 9.1). We notice that $\dim \phi_{h-1} = 2^d$, $\dim \phi_{h-2} = 2^{d+1}$ and $\dim \phi_{h-3} = 3 \cdot 2^d$.

Clearly, the sequences (ϕ_1, ϕ_2, ϕ_3) and (ϕ_2, ϕ_3) are also quasi-excellent. In particular, the forms ϕ_0, ϕ_1, ϕ_2, and ϕ_3 are quasi-excellent.

Definition 0.3. Let $(\varphi_0, \ldots, \varphi_h)$ be a quasi-excellent sequence of degree d. We say that the sequence is of the

- "first type" if $\dim \varphi_{h-2} \neq 2^{d+1}$ or $h = 1$
- "second type" if $\dim \varphi_{h-2} = 2^{d+1}$ and, if $h \geq 3$, $\dim \varphi_{h-3} \neq 3 \cdot 2^d$
- "third type" if $\dim \varphi_{h-2} = 2^{d+1}$ and $\dim \varphi_{h-3} = 3 \cdot 2^d$, (here $h \geq 3$).

Example 0.4. Let $(\phi_0, \phi_1, \phi_2, \phi_3)$ be the sequence constructed in Example 0.2. Assume that ϕ_0, ϕ_1 and ϕ_2 are anisotropic.

- The sequence (ϕ_2, ϕ_3) is of the first type,
- The sequence (ϕ_1, ϕ_2, ϕ_3) is of the second type,
- The sequence $(\phi_0, \phi_1, \phi_2, \phi_3)$ is of the third type.

According to Knebusch [Kn2], 7.4, a regular quadratic form ψ is called a *Pfister neighbor*, if there exist a Pfister form π, some $a \in F^*$, and a form η with $\dim \eta < \dim \psi$, such that $\psi \perp \eta \simeq a\pi$. The form η is called the *complementary* form of the Pfister neighbor ψ.

Example 0.5. Let $(\varphi_1, \ldots, \varphi_h)$ be a quasi-excellent sequence. Let φ_0 be an anisotropic Pfister neighbor whose complementary form is similar to φ_1. Then the sequence $(\varphi_0, \varphi_1, \ldots, \varphi_h)$ is quasi-excellent. Moreover, this sequence is of the same type as the sequence $(\varphi_1, \ldots, \varphi_h)$. (Note that $((\varphi_0)_{F_1})_{\mathrm{an}} \sim (\varphi_1)_{F_1}$ by [Kn2], p. 3.)

Clearly, Examples 0.4 and 0.5 give rise to the construction of many examples of quasi-excellent sequences of prescribed type: We start with a quasi-excellent sequence given in Example 0.4. We can then apply the construction presented in Example 0.5 to obtain a new quasi-excellent sequence. Since we can apply the construction in Example 0.5 many times, we get quasi excellent sequences of arbitrary height.

The main goal of this paper is to prove (under certain assumptions) that all quasi-excellent sequences can be constructed by using this recursive procedure. To be more accurate, for sequences of the first type, we prove the following classification result:

Theorem 0.6. *Let $(\varphi_0, \ldots, \varphi_h)$ be a quasi-excellent sequence of the first type. Then for any $i < h$ the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1}.*

For sequences of the second and the third type, we state our classification results as conjectures which we will prove for sequences of degree 1. For sequences of arbitrary degree we will deduce our conjectures from some classical conjectures which now seem to be settled for all fields of characteristic 0, cf. [Vo, OVV].

Conjecture 0.7. *Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence of the second type. Then for any $i < h-2$ the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1}. Besides, the forms φ_{h-2} and φ_{h-1} look as follows:*

$$\varphi_{h-2} \sim \langle\langle a_1, \dots, a_{d-1}\rangle\rangle \otimes \langle -u, -v, uv, a_d\rangle\,,$$
$$\varphi_{h-1} \sim \langle\langle a_1, \dots, a_{d-1}, a_d\rangle\rangle\,.$$

(For $d = 1$ we put $\langle\langle a_1, \dots, a_{d-1}\rangle\rangle = \langle 1\rangle$.)

Conjecture 0.8. *Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence of the third type. Then for any $i < h-3$ the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1}. Besides, the forms φ_{h-3} , φ_{h-2} and φ_{h-1} look as follows:*

$$\varphi_{h-3} \sim \langle\langle a_1, \dots, a_{d-1}\rangle\rangle \otimes (\langle\langle u, v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle),$$
$$\varphi_{h-2} \sim \langle\langle a_1, \dots, a_{d-1}\rangle\rangle \otimes \langle -u, -v, uv, a_d\rangle\,,$$
$$\varphi_{h-1} \sim \langle\langle a_1, \dots, a_{d-1}, a_d\rangle\rangle\,.$$

The main results of this paper are Theorem 0.6 and the following two theorems.

Theorem 0.9. *Conjectures 0.7 and 0.8 are true for quasi-excellent sequences of degree* 1. *The well-known so far unpublished result by Rost, that the Milnor invariant e^4 is bijective, implies that 0.7 and 0.8 are also true for sequences of degree* 2 .

Theorem 0.10. *Modulo results proved in* [Vo, OVV] *both Conjectures 0.7 and 0.8 are true over any field of characteristic* 0.

All results of this paper are due to the first-named author Oleg Izhboldin. The second-named author is responsible for a final version of Oleg's beautiful draft which he could not complete because of his sudden death on April 17, 2000.

1. Notation and background material

We fix a ground field F of characteristic different from 2 and set $F^* = F\backslash\{0\}$. If two quadratic forms φ and ψ are isomorphic we write $\varphi \simeq \psi$. We say that φ and ψ are *similar* if $\varphi \simeq a\psi$ for some $a \in F^*$, and write $\varphi \sim \psi$. A regular quadratic form φ of dimension $\dim \varphi > 0$ is said to be *isotropic* if there is a non-zero vector v in the underlying vector space of φ such that $\varphi(v) = 0$, and *anisotropic* otherwise. The zero form 0 is assumed to be anisotropic. As has been shown by Witt [W], any regular quadratic form φ has a decomposition

$$\varphi \simeq i \times \langle 1, -1\rangle \perp \varphi_{\mathrm{an}}$$

where φ_{an} is anisotropic and $i \geq 0$. Moreover, the number $i =: i(\varphi)$ and, up to isomorphism, the form φ_{an} are uniquely determined by φ . We call $i(\varphi)$ the

Witt index of φ. If $i(\varphi) > 0$ then φ is isotropic. A form $\varphi \neq 0$ is said to be *hyperbolic* if $\varphi_{\mathrm{an}} = 0$. We have a *Witt equivalence* relation $\varphi \sim_{\mathrm{w}} \psi$ defined by

$$\varphi \sim_{\mathrm{w}} \psi \iff \varphi_{\mathrm{an}} \simeq \psi_{\mathrm{an}}$$

The Witt equivalence classes $[\varphi]$ of regular quadratic forms φ over F form a commutative ring $W(F)$ with zero element $[0]$ and unit element $[\langle 1 \rangle]$. The operations in this *Witt ring* $W(F)$ are induced by:

$\langle a_1, \ldots, a_m \rangle \perp \langle b_1, \ldots, b_n \rangle = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \rangle$

$\langle a_1, \ldots, a_m \rangle \otimes \langle b_1, \ldots, b_n \rangle = \langle a_1 b_1, \ldots, a_1 b_n, \ \ldots, a_m b_1, \ldots, a_m b_n \rangle$.

In particular, there is a surjective ring homomorphism

$$e^0 : W(F) \to \mathbb{Z}/2\mathbb{Z}, \ [\varphi] \mapsto (\dim \varphi) \bmod 2\mathbb{Z}.$$

Its kernel $I(F) := \ker(e^0)$ is called the *fundamental ideal* of $W(F)$. Since $\langle a, b \rangle \sim_{\mathrm{w}} \langle\langle -a \rangle\rangle \perp - \langle\langle b \rangle\rangle$ the ideal $I(F)$ is generated by the classes of the 1-fold Pfister forms $\langle\langle a \rangle\rangle = \langle 1, -a \rangle$ with $a \in F^*$. Consequently, the nth power ideal $I^n(F)$ of $I(F)$ is generated by the classes of n-fold Pfister forms $\langle\langle a_1, \ldots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$. We will use the *Arason-Pfister Hauptsatz* [AP]:

Theorem 1.1. (Arason-Pfister) *If* $[\varphi] \in I^n(F)$ *and* $\dim \varphi_{\mathrm{an}} < 2^n$ *then* $\varphi \sim_{\mathrm{w}} 0$.

Put $F^{*2} = \{x^2 \in F^* \mid x \in F^*\}$ and $d(\varphi) := (-1)^{\binom{m}{2}} \det(\varphi)$ with $m = \dim \varphi$. Then there is a surjective group homomorphism

$$e^1 : I(F) \to F^*/F^{*2}, \ [\varphi] \mapsto d(\varphi)\, F^{*2},$$

satisfying $\ker(e^1) = I^2(F)$, see [Pf1], 2.3.6. For any ideal I in $W(F)$ we write $\varphi \equiv \psi \bmod I$ when $[\varphi \perp - \psi] \in I$. Let $\mu = \langle a_1, \ldots, a_m \rangle$ with $a_1, \ldots, a_m \in F^*$. If m is odd then $d\big(\langle a_1, \ldots, a_m, -d(\mu) \rangle\big) = \prod_{i=1}^m a_i^2 \in F^{*2}$, and we obtain the following remark which will be used for the classification of quasi-excellent sequences of the first type.

Remark 1.2. *If* $\dim \mu$ *is odd then* $\mu \equiv \langle\, d(\mu) \,\rangle \bmod I^2(F)$.

Of special interest for us is the *function field* $F(\varphi)$ of a regular quadratic form φ. Assuming that $\dim \varphi \geq 2$ and $\varphi \not\simeq \langle 1, -1 \rangle$ we let $F(\varphi)$ be the function field of the projective variety defined by φ. Its transcendence degree is $(\dim \varphi) - 2$ and $\varphi_{F(\varphi)}$ is isotropic. Moreover, $F(\varphi)$ is purely transcendental over F iff φ is isotropic (cf. [Kn1], 3.8). We denote by $F(\varphi, \psi)$ the function field of the product of the varieties defined by the forms φ and ψ.

We say that φ is a *subform* of ψ, and write $\varphi \subset \psi$, if φ is isomorphic to an orthogonal summand of ψ. We will use the following two consequences of the *Cassels-Pfister Subform Theorem* [Pf1], 1.3.4:

Theorem 1.3. ([Kn1], 4.4, and [S], 4.5.4 (ii)) *Let* λ *be an anisotropic form and* ρ *be a Pfister form. Then the following conditions are equivalent:*

- *there exists a form* μ *such that* $\lambda \simeq \rho \otimes \mu$,
- *there exists a form* ν *such that* $\lambda \sim_{\mathrm{w}} \rho \otimes \nu$,
- $\lambda_{F(\rho)}$ *is hyperbolic.*

Moreover, in these cases $k\rho \subset \lambda$ *for any* $k \in D(\lambda) := \{a \in F^* \mid \langle a \rangle \subset \varphi\}$.

Theorem 1.4. ([Kn1], 4.5) *Let* φ *and* ψ *be forms of dimension* ≥ 2 *satisfying* $\varphi \not\simeq \langle 1, -1\rangle$ *and* $\psi \not\sim_{\mathrm{w}} 0$. *If* $\psi_{F(\varphi)} \sim_{\mathrm{w}} 0$ *then* φ *is similar to a subform of* ψ, *hence* $\dim\varphi \leq \dim\psi$.
Consequently, if $\dim\varphi > \dim\psi$ *then* $\psi_{F(\varphi)}$ *is not hyperbolic.*

If, in addition, φ and ψ are anisotropic and the dimensions of φ and ψ are separated by a 2-power, then $\psi_{F(\varphi)}$ is *not isotropic* as Hoffmann has shown.

Theorem 1.5. (Hoffmann ([H1], Theorem 1) *Let* φ *and* ψ *be anisotropic forms with* $\dim\psi \leq 2^n < \dim\varphi$ *for some* $n > 0$. *Then* $\psi_{F(\varphi)}$ *is anisotropic.*

In accordance with the definition given in the introduction, a form φ is a *Pfister neighbor* of a d-fold Pfister form π if $\dim\varphi > 2^{d-1}$ and φ is similar to a subform of π.

Theorem 1.6. (Hoffmann [H1], Corollaries 1, 2) *Let* φ *be an anisotropic form of dimension* $2^n + m$ *with* $0 < m \leq 2^n$. *Then* $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} \geq 2^n - m$.
If, in addition, φ *is a Pfister neighbor then* $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} = 2^n - m$.

The following theorem is a result by Izhboldin on "virtual Pfister neighbors", cf. [Izh], Theorem 3.5.

Theorem 1.7. (Izhboldin) *Let* φ *be an anisotropic form of dimension* $2^n + m$ *with* $0 < m \leq 2^n$. *Assume that there is a field extension* E/F *such that* φ_E *is an anisotropic Pfister neighbor. Then either* $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} \geq 2^n$ *or* $\dim(\varphi_{F(\varphi)})_{\mathrm{an}} = 2^n - m$.

Theorem 1.8. (Knebusch [Kn2], 7.13) *Let* φ *and* ψ *be anisotropic forms such that* $(\varphi_{F(\varphi)})_{\mathrm{an}} \simeq \psi_{F(\varphi)}$. *Then* φ *is a Pfister neighbor and* $-\psi$ *is the complementary form of* φ.

The following theorem is a special case of the *Knebusch-Wadsworth Theorem* [Kn1], 5.8. It will be used in Lemma 2.5 below.

Theorem 1.9. (Knebusch-Wadsworth) *Let* φ *be an anisotropic form such that* $\varphi_{F(\varphi)}$ *is hyperbolic. Then* φ *is similar to a Pfister form.*

Knebusch introduced in [Kn1] a *generic splitting tower* $K_0 \subset K_1 \subset \cdots \subset K_h$ of a form $\psi \not\sim_{\mathrm{w}} 0$ which is easily described as follows. Let $K_0 = F$ and $\psi_0 \simeq \psi_{\mathrm{an}}$ and proceed inductively by letting $K_i = K_{i-1}(\psi_{i-1})$ and $\psi_i \simeq ((\psi_{i-1})_{K_i})_{\mathrm{an}}$. Then h is the *height* of ψ, that is the smallest number such that $\dim\psi_h \leq 1$.

The form ψ is excellent iff all forms ψ_i are defined over F (that is, for each i there exists a form η_i over F such that $\psi_i \simeq (\eta_i)_{K_i}$), cf. [Kn2], 7.14.

Now assume that $\dim\psi$ is even. Then $\psi_{h-1} \simeq a\pi$ for some $a \in K_{h-1}^*$ and some d-fold Pfister form π over K_{h-1} by Theorem 1.9. The form π is called the *leading form* of ψ and the number $d =: \deg\psi$ the *degree* of ψ. We say that ψ is a *good* form if π is defined over F. Then there is, up to isomorphism, a unique d-fold Pfister form τ over F such that $\pi \simeq \tau_{K_{h-1}}$, cf. [Kn2], 9.2, and we will refer to this Pfister form over F as the *leading form* of a good form.

Saying that all odd-dimensional forms have degree 0 and that the zero form has degree ∞ we get a *degree function*, cf. [Kn1], p. 88:

$$\deg : W(F) \to \mathbb{N} \cup \{0\} \cup \{\infty\}\,, \ \ [\psi] \mapsto [\deg \psi]\,.$$

For every $n \geq 0$ let $J_n(F) := \{\, [\psi] \in W(F) \mid \deg \psi \geq n \,\}$. Then $J_n(F)$ is an ideal in the Witt ring $W(F)$ and $J_1(F) = I(F)$ is the fundamental ideal. We are now prepared to formulate the next result we will need later.

Theorem 1.10. (Knebusch [Kn2], 9.6, 7.14, and 10.1; Hoffmann [H2]) *Let ψ be an anisotropic good form of degree $d \geq 1$ with leading form τ. Then*

$$\psi \equiv \tau \bmod J_{d+1}(F)\,.$$

If, in addition, ψ is of height 2 then one of the following conditions holds.

- *The form ψ is excellent. In this case, ψ is a Pfister neighbor whose complementary form is similar to τ. In particular,* $\dim \psi = 2^N - 2^d$ *with $N \geq d+2$, and $\psi_{F(\tau)}$ is hyperbolic.*
- *The form ψ is non-excellent and good. In this case* $\dim \psi = 2^{d+1}$ *and $\psi_{F(\tau)}$ is similar to an anisotropic $(d+1)$-fold Pfister form.*

We denote by $P_d(F)$ (resp. $GP_d(F)$) the set of all quadratic forms over F which are isomorphic (resp. similar) to d-fold Pfister forms.

Finally, we mention the following well-known facts (e.g., [L], IX.1.1, X.1.6).

Remark 1.11. (i) *Anisotropic forms over F remain anisotropic over purely transcendental extensions of F.*

(ii) *Isotropic Pfister forms are hyperbolic.*

2. Elementary properties of quasi-excellent forms and sequences

Lemma 2.1. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence. Then*

- *all forms φ_i are forms of even dimension,*
- *all forms φ_i are anisotropic,*
- $\dim \varphi_0 > \dim \varphi_1 > \cdots > \dim \varphi_h = 0$,
- *for all $s = 1, \dots, h$, we have*

$$((\varphi_0)_{F_s})_{\mathrm{an}} \sim ((\varphi_1)_{F_s})_{\mathrm{an}} \sim \cdots \sim ((\varphi_{s-1})_{F_s})_{\mathrm{an}} \sim (\varphi_s)_{F_s}$$

where $F_s = F(\varphi_0, \dots, \varphi_{s-1})$.

Proof. Obvious from Definition 0.1. □

Lemma 2.2. *Let φ be an anisotropic even-dimensional quasi-excellent form over F. Then there exists a quasi-excellent sequence $(\varphi_0, \varphi_1, \dots, \varphi_h)$ such that $\varphi_0 = \varphi$.*

Proof. Let us define the forms φ_i recursively. We set $\varphi_0 = \varphi$. Now, we suppose that $i > 0$ and that all forms $\varphi_0, \dots, \varphi_{i-1}$ are already defined. Also, we can suppose that these forms are of dimension > 0. Put $F_i = F(\varphi_0, \dots, \varphi_{i-1})$. Since φ is quasi-excellent, there exists a form ψ over F such that $(\varphi_{F_i})_{\mathrm{an}}$ is similar to ψ_{F_i}. We put $\varphi_i = \psi$. If $\varphi_i = 0$ then we are done by setting $h = i$. If $\varphi_i \neq 0$ we repeat the above procedure. $\square$

Lemma 2.3. *Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence. Let E/F be a field extension such that $(\varphi_0)_E$ is isotropic, and let i be the maximal integer such that all forms $(\varphi_0)_E, \dots, (\varphi_{i-1})_E$ are isotropic. Then $((\varphi_0)_E)_{\mathrm{an}} \sim (\varphi_i)_E$.*

Proof. Since the forms $(\varphi_0)_E, \dots, (\varphi_{i-1})_E$ are isotropic, the field extension $E_i := E(\varphi_0, \dots, \varphi_{i-1})$ is purely transcendental over E. Since $F_i \subset E_i$ and $((\varphi_0)_{F_i})_{\mathrm{an}} \sim (\varphi_i)_{F_i}$, it follows that $((\varphi_0)_{E_i})_{\mathrm{an}} \sim ((\varphi_i)_{E_i})_{\mathrm{an}}$. Since E_i/E is purely transcendental we can use Springer's theorem (e.g., [L], 6.1.7) to obtain $((\varphi_0)_E)_{\mathrm{an}} \sim ((\varphi_i)_E)_{\mathrm{an}}$. By definition of i, the form $(\varphi_i)_E$ is anisotropic. $\square$

Corollary 2.4. *Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence. Then the form φ_0 is a quasi-excellent even-dimensional form.*

Proof. Obvious from Lemmas 2.1 and 2.3. $\square$

Lemma 2.5. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence. Then the form φ_{h-1} is similar to a Pfister form.*

Proof. By Definition 0.1, we have $\varphi_h = 0$ and $((\varphi_{h-1})_{F_h})_{\mathrm{an}} \sim (\varphi_h)_{F_h}$, where $F_h = F(\varphi_0, \dots, \varphi_{h-1})$. Therefore, $(\varphi_{h-1})_{F_h}$ is hyperbolic. Note that $F_h \simeq F(\varphi_{h-1})(\varphi_0, \dots, \varphi_{h-2})$. Since the dimensions of the forms $\varphi_0, \dots, \varphi_{h-2}$ are strictly greater than $\dim \varphi_{h-1}$ and $(\varphi_{h-1})_{F(\varphi_{h-1})(\varphi_0, \dots, \varphi_{h-2})}$ is hyperbolic, it follows from Theorem 1.4 that $(\varphi_{h-1})_{F(\varphi_{h-1})}$ is hyperbolic. By Theorem 1.9, φ_{h-1} is similar to a Pfister form. $\square$

Definition 2.6. Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence.

- By Lemma 2.5, the form φ_{h-1} is similar to some Pfister form $\tau \in P_d(F)$. We say that τ is the *leading form* and d is the *degree* of the sequence. Besides, we say that h is the *height* of the sequence.
- The form φ_{h-2} is called the *pre-leading form* of the sequence. Clearly, here we assume that $h \geq 2$.

Remark 2.7. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence. Then its leading form is the leading form of φ_0 as well. In particular, φ_0 is a good form whose height and degree coincide with the height and degree of the sequence.*

We finish this section with a lemma which we will need for the classification of quasi-excellence sequences of the second and third type.

Lemma 2.8. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence with $h \geq 2$. Let E/F be an extension such that $(\varphi_0)_E$ is an anisotropic Pfister neighbor whose complementary form is similar to $(\varphi_2)_E$. Then $\dim \varphi_1$ is a power of 2 and $\dim \varphi_0 = 2 \dim \varphi_1 - \dim \varphi_2$.*

Proof. Let us write $\dim\varphi_0$ in the form $\dim\varphi_0 = 2^n + m$ with $0 < m \leq 2^n$. Since $(\varphi_2)_E$ is similar to the complementary form of $(\varphi_0)_E$, we have $\dim\varphi_2 = 2^{n+1} - \dim\varphi_0 = 2^n - m$. Since $\dim\varphi_1 = \dim((\varphi_0)_{F(\varphi_0)})_{\mathrm{an}}$, Theorem 1.7 shows that either $\dim\varphi_1 \geq 2^n$ or $\dim\varphi_1 = 2^n - m$. The equality $\dim\varphi_1 = 2^n - m$ is obviously false because $\dim\varphi_2 = 2^n - m$. Therefore, $\dim\varphi_1 \geq 2^n$. If $\dim\varphi_1 = 2^n$ then $\dim\varphi_0 = 2^n + m = 2\cdot 2^n - (2^n - m) = 2\dim\varphi_1 - \dim\varphi_2$ and the proof is complete. Hence, we can assume that $\dim\varphi_1 > 2^n$. Then $2^n < \dim\varphi_1 < \dim\varphi_0 = 2^n + m$. Therefore $\dim\varphi_1$ can be written in the form $2^n + m_1$ with $0 < m_1 < m \leq 2^n$. Let $K = F(\varphi_0)$. Then Lemma 2.1 shows that $((\varphi_1)_{K(\varphi_1)})_{\mathrm{an}}$ is similar to $(\varphi_2)_{K(\varphi_1)}$. Hence, $\dim((\varphi_1)_{K(\varphi_1)})_{\mathrm{an}} = \dim\varphi_2$. Since $\dim\varphi_1 = 2^n + m_1$, Theorem 1.6 shows that $\dim\varphi_2 = \dim((\varphi_1)_{K(\varphi_1)})_{\mathrm{an}} \geq 2^n - m_1$. Since $\dim\varphi_2 = 2^n - m$, we get $m_1 \geq m$. This contradicts to the inequality $m_1 < m$ proved earlier. □

3. Inductive properties of quasi-excellent sequences

In this section we study further properties of a quasi-excellent sequence $(\varphi_0, \ldots, \varphi_h)$ of degree d with leading form τ. Then we derive some results on its pre-leading form $\gamma := \varphi_{h-2}$. In particular, we show that $\dim\gamma$ is either 2^{d+1} or $2^N - 2^d$ with $N \geq d+2$ and that $(\varphi_i)_{F(\gamma,\tau)}$ is hyperbolic for all $i = 0, \ldots, h-1$.

Lemma 3.1. *Let $(\varphi_0, \ldots, \varphi_h)$ be a quasi-excellent sequence and let $E = F(\varphi_1)$.*

- *If $(\varphi_0)_E$ is isotropic then $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is a quasi-excellent sequence and $((\varphi_0)_E)_{\mathrm{an}} \sim (\varphi_2)_E$.*
- *If $(\varphi_0)_E$ is anisotropic then $\big((\varphi_0)_E, (\varphi_2)_E, (\varphi_3)_E, \ldots, (\varphi_h)_E\big)$ is a quasi-excellent sequence.*

Proof. Let $F_i = F(\varphi_0, \ldots, \varphi_{i-1})$ and $F_{0,i} = F(\varphi_1, \ldots, \varphi_{i-1})$. Assume that φ_0 is isotropic over $F(\varphi_1)$. Then the extension $F_i/F_{0,i}$ is purely transcendental for all $i \geq 2$. By Lemma 2.1, we have $((\varphi_1)_{F_i})_{\mathrm{an}} \sim (\varphi_i)_{F_i}$ for all $i \geq 1$. Since $F_i/F_{0,i}$ is purely transcendental for $i \geq 2$, we have $((\varphi_1)_{F_{0,i}})_{\mathrm{an}} \sim (\varphi_i)_{F_{0,i}}$ for all $i \geq 2$. This means that the sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is quasi-excellent. Now Lemma 2.3 implies that $((\varphi_0)_E)_{\mathrm{an}} \sim (\varphi_2)_E$. The last statement is obvious from Definition 0.1. □

Lemma 3.2. *Let $(\varphi_0, \ldots, \varphi_h)$ be a quasi-excellent sequence. Suppose that φ_0 is a Pfister neighbor whose complementary form is similar to φ_1. Then the sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is quasi-excellent.*

Proof. By Lemma 3.1, it suffices to show that $(\varphi_0)_{F(\varphi_1)}$ is isotropic. By assumption, there is a form $\eta \sim \varphi_1$ and a Pfister form π such that $\varphi_0 \perp \eta \sim \pi$. Since $\eta_{F(\varphi_1)}$ is isotropic, the form $\pi_{F(\varphi_1)}$ must be hyperbolic. Since $\dim\varphi_0 > \dim\eta$ it follows that $(\varphi_0)_{F(\varphi_1)}$ is isotropic. □

Lemma 3.3. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence of height $h \geq 2$ with leading form $\tau \in P_d(F)$ and let $F_i = F(\varphi_0, \dots, \varphi_{i-1})$. Then the sequence*

$$\big((\varphi_i)_{F_i}, (\varphi_{i+1})_{F_i}, \dots, (\varphi_h)_{F_i}\big)$$

is quasi-excellent of height $h - i$ with leading form $\tau_{F_i} \in P_d(F_i)$ for $1 \leq i < h$.

Proof. By Lemma 2.1, the forms $(\varphi_s)_{F_s}$ are anisotropic for $s = 1, \dots, h$. Thus $(\varphi_s)_{F_i}$ is anisotropic for fixed $i < h$ and $s = i, \dots, h$. In particular, τ_{F_i} is anisotropic since $\tau_{F_i} \sim (\varphi_{h-1})_{F_i}$ by Definition 2.6. Now the result is obvious. □

Lemma 3.4. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence of degree d with leading form τ. Then for all $i = 0, \dots, h-1$, we have $\varphi_i \equiv \tau \bmod J_{d+1}(F)$. In particular, $\deg(\varphi_i) = \deg(\tau) = d$ for all $i = 0, \dots, h-1$.*

Proof. By Remark 2.7 and Theorem 1.10, we have $\varphi_0 \equiv \tau \bmod J_{d+1}(F)$. Using Lemma 3.3 we obtain from Remark 2.7 and Theorem 1.10 that

$$(\varphi_i)_{F_i} \equiv \tau_{F_i} \bmod J_{d+1}(F_i)$$

for all $i = 1, \dots, h-1$. Since $\dim \varphi_0 > \dots > \dim \varphi_{h-2} > 2^d = \dim \varphi_{h-1}$, the canonical map $J_d(F_{i-1})/J_{d+1}(F_{i-1}) \to J_d(F_i)/J_{d+1}(F_i)$ is injective for $i = 1, \dots, h-1$ and $F_0 = F$ as Knebusch [Kn1], 6.11, has shown. Thus the composed map $J_d(F)/J_{d+1}(F) \to J_d(F_i)/J_{d+1}(F_i)$ is also injective. Hence $\varphi_i \equiv \tau \bmod J_{d+1}(F)$ for $i = 1, \dots, h-1$. The second statement follows from the first since $J_d(F)$ is an ideal in the Witt ring $W(F)$. □

Lemma 3.5. *Let γ and τ be anisotropic forms. Suppose that $\dim \gamma = 2^{d+1}$ and $\tau \in P_d(F)$ for suitable d. Suppose also that $\gamma_{F(\gamma)}$ is not hyperbolic and $\gamma_{F(\gamma,\tau)}$ is hyperbolic. Then the form $(\gamma_{F(\gamma)})_{\mathrm{an}}$ is similar to $\tau_{F(\gamma)}$.*

Proof. Let $K = F(\gamma)$. By assumption, the form $\gamma_{K(\tau)}$ is hyperbolic. Thus Theorem 1.3 implies that there exists a K-form μ such that $(\gamma_K)_{\mathrm{an}} \simeq \tau_K \otimes \mu$. Since $\dim \tau = 2^d$ and $\dim(\gamma_K)_{\mathrm{an}} = \dim(\gamma_{F(\gamma)})_{\mathrm{an}} < \dim \gamma = 2^{d+1}$, it follows that $\dim \mu < 2^{d+1}/2^d = 2$. Hence, $\dim \mu = 0$ or 1.

If $\dim \mu = 0$ then $(\gamma_K)_{\mathrm{an}} = 0$. Then $\gamma_{F(\gamma)} = \gamma_K$ is hyperbolic. We get contradiction to the hypothesis of the lemma.

If $\dim \mu = 1$, then the isomorphism $(\gamma_K)_{\mathrm{an}} \simeq \tau_K \otimes \mu$ shows that $(\gamma_K)_{\mathrm{an}}$ is similar to τ_K. The lemma is proved. □

Proposition 3.6. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence with leading form $\tau \in P_d(F)$ and pre-leading form $\gamma = \varphi_{h-2}$. Then*

1. *$\dim \gamma = 2^{d+1}$ or $\dim \gamma = 2^N - 2^d$ with $N \geq d+2$.*
2. *If $\dim \gamma = 2^{d+1}$ then γ is a good non-excellent form of height 2 and degree d with leading form τ.*
3. *If $\dim \gamma \neq 2^{d+1}$ then γ is excellent and $\gamma_{F(\tau)}$ is hyperbolic.*

Proof. (1). Let $E = F_{h-2} = F(\varphi_0, \dots, \varphi_{h-3})$. By Lemma 3.3 with $i = h-2$, the sequence $(\gamma_E, (\varphi_{h-1})_E, (\varphi_h)_E)$ is quasi-excellent of height 2 with leading form $\tau_E \in P_d(E)$. Thus Remark 2.7 implies that γ_E is a good form of height 2, degree d, and leading form τ_E. By Theorem 1.10, there are two types of good forms of height 2, *non-excellent* and *excellent*.

If γ_E is good non-excellent of height 2 and degree d, then $\dim\gamma = 2^{d+1}$.

If γ_E is excellent form of height 2 and degree d, then $\dim\gamma = 2^N - 2^d$ with $N \geq d+2$.

(2). Assume that $\dim\gamma = 2^{d+1}$. We have to prove that $(\gamma_{F(\gamma)})_{\mathrm{an}}$ is similar to $\tau_{F(\gamma)}$. By Lemma 3.5, it suffices to verify that $\gamma_{F(\gamma)}$ is not hyperbolic and $\gamma_{F(\gamma,\tau)}$ is hyperbolic.

Since $E(\gamma) = F_{h-1}$, we have $(\gamma_{E(\gamma)})_{\mathrm{an}} \sim (\varphi_{h-1})_{E(\gamma)} \sim \tau_{E(\gamma)}$ by Lemma 2.1 and Definition 2.6. This shows that $\gamma_{F(\gamma)}$ is not hyperbolic and that $\gamma_{E(\gamma,\tau)}$ is hyperbolic. Since $E = F(\varphi_0, \dots, \varphi_{h-3})$ is the function field of forms of dimension $> \dim\varphi_{h-2} = \dim\gamma$ and $\gamma_{E(\gamma,\tau)}$ is hyperbolic, it follows from Theorem 1.4 that $\gamma_{F(\gamma,\tau)}$ is also hyperbolic. By Lemma 3.5, we are done.

(3). If $\dim\gamma \neq 2^{d+1}$ then γ_E is an excellent form of height 2 and degree d with the leading form τ_E. In this case $\gamma_{E(\tau)}$ is hyperbolic by Theorem 1.10. Hence $\gamma_{F(\tau)}$ is also hyperbolic by Theorem 1.4. □

Proposition 3.7. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence with leading form $\tau \in P_d(F)$ and pre-leading form $\gamma = \varphi_{h-2}$. Then $(\varphi_i)_{F(\gamma,\tau)}$ is hyperbolic for all $i = 0, \dots, h-1$.*

Proof. If $h = 1$ then $\varphi_0 \sim \tau$ by Definition 2.6, hence $(\varphi_0)_{F(\tau)}$ is hyperbolic. If $h = 2$ then the statement is obvious as well. Thus, we can assume that $h \geq 3$. We use induction on h.

Let $E = F(\varphi_0)$. By Lemma 3.3, $((\varphi_1)_E, (\varphi_2)_E, \dots, (\varphi_h)_E)$ is quasi-excellent. By induction assumption, $(\varphi_i)_{E(\gamma,\tau)}$ is hyperbolic for all $i = 1, \dots, h-1$. Since $E(\gamma,\tau) = F(\gamma,\tau,\varphi_0)$ and $\dim\varphi_0$ is strictly greater than the dimensions of all forms $\varphi_1, \dots, \varphi_{h-1}$, Theorem 1.4 shows that the forms $(\varphi_i)_{F(\gamma,\tau)}$ are hyperbolic for all $i = 1, \dots, h-1$.

Now, it suffices to prove that $(\varphi_0)_{F(\gamma,\tau)}$ is hyperbolic. We consider three cases and use the following *observation.* Since $(\varphi_1)_{F(\gamma,\tau)}$ is hyperbolic, hence isotropic, it follows that $F(\gamma,\tau,\varphi_1)$ is purely transcendental over $F(\gamma,\tau)$.

Case 1. The form $(\varphi_0)_{F(\varphi_1)}$ is isotropic.

Then $(\varphi_0)_{F(\gamma,\tau,\varphi_1)}$ is isotropic. Thus $(\varphi_0)_{F(\gamma,\tau)}$ is isotropic too by the above observation. Since the forms $(\varphi_i)_{F(\gamma,\tau)}$ are hyperbolic for all $i = 1, \dots, h-1$, Lemma 2.3 applies with $i = h$ so that $((\varphi_0)_{F(\gamma,\tau)})_{\mathrm{an}} \sim (\varphi_h)_{F(\gamma,\tau)} = 0$.

Case 2. The form $(\varphi_0)_{F(\varphi_1)}$ is anisotropic and $h = 3$.

In this case $\gamma = \varphi_1$. Let $E = F(\varphi_1) = F(\gamma)$. By Lemma 3.1, the sequence $((\varphi_0)_E, (\varphi_2)_E, 0)$ is quasi-excellent of height 2.

Clearly, $\dim\varphi_2 = \dim\tau = 2^d$, and $\dim\varphi_0 > \dim\varphi_1 = \dim\gamma \geq 2^{d+1}$ by Proposition 3.6. Since $\dim(\varphi_0)_E \neq 2^{d+1}$ it follows that $(\varphi_0)_E$ is excellent of height 2 with leading form τ_E. Hence $(\varphi_0)_{E(\tau)}$ is hyperbolic (see Theorem 1.10). Since $E(\tau) = F(\gamma, \tau)$, we are done.

Case 3. The form $(\varphi_0)_{F(\varphi_1)}$ is anisotropic and $h \geq 4$.

Let $E = F(\varphi_1)$. By Lemma 3.1, the sequence

$$((\varphi_0)_E, (\varphi_2)_E, \dots, (\varphi_{h-2})_E, (\varphi_{h-1})_E, 0)$$

is a quasi-excellent of height $h-1$. Clearly, τ_E is the leading form and $\gamma_E = (\varphi_{h-2})_E$ is the pre-leading form of this sequence (we note, that here we use the condition $h \geq 4$). Applying the induction hypothesis, we see that the form $(\varphi_0)_{E(\gamma,\tau)}$ is hyperbolic. Therefore, $(\varphi_0)_{F(\gamma,\tau)}$ is hyperbolic by the above observation. □

4. Classification theorem for sequences of the first type

Recall that a quasi-excellent sequence $(\varphi_0, \dots, \varphi_h)$ of degree d is of the *first type* if $\dim \varphi_{h-2} \neq 2^{d+1}$ or if $h = 1$.

Lemma 4.1. *Let $(\varphi_0, \dots, \varphi_h)$ be a quasi-excellent sequence of the first type with leading form $\tau \in P_d(F)$. Then*

(i) *the form $(\varphi_i)_{F(\tau)}$ is hyperbolic for all $i = 0, \dots, h-1$,*

(ii) *for every $i = 0, \dots, h-1$ there exists an odd-dimensional form μ_i such that $\varphi_i \simeq \mu_i \otimes \tau$,*

(iii) *φ_0 is a Pfister neighbor, whose complementary form is similar to φ_1.*

Proof. (i). For $h = 1$ the statement follows from Remark 1.11. Now assume that $h \geq 2$, and put $\gamma = \varphi_{h-2}$. By Proposition 3.6 (3), the form $\gamma_{F(\tau)}$ is isotropic. Hence, the extension $F(\gamma, \tau)/F(\tau)$ is purely transcendental. This implies, since $(\varphi_i)_{F(\gamma,\tau)}$ is hyperbolic by 3.7, that $(\varphi_i)_{F(\tau)}$ is hyperbolic.

(ii). By Theorem 1.3 and (i), there exists a form μ_i such that $\varphi_i \simeq \mu_i \otimes \tau$. Thus it suffices to prove that $\dim \mu_i$ is odd. If we assume that μ_i is an even-dimensional form, then we get $[\varphi_i] \in I(F) \cdot I^d(F) = I^{d+1}(F)$. This contradicts to Lemma 3.4, where we have proved that $\deg(\varphi_i) = d$ for all $i = 0, \dots, h-1$, since $I^{d+1}(F) \subset J_{d+1}(F)$ by [Kn1], 6.6.

(iii). Let $K = F(\varphi_0)$. By Definition 0.1, there exists $x \in K^*$ such that $((\varphi_0)_K)_{\text{an}} \simeq x\,(\varphi_1)_K$. By (ii), this implies

$$(*) \qquad (\mu_0 \otimes \tau)_K \sim_{\text{w}} x\,(\mu_1 \otimes \tau)_K\,.$$

Let $s_0 = d(\mu_0)$ and $s_1 = d(\mu_1)$. Since μ_0 and μ_1 are both of odd dimension, we have $\mu_0 \equiv \langle s_0 \rangle \mod I^2(F)$ and $\mu_1 \equiv \langle s_1 \rangle \mod I^2(F)$ by Remark 1.2. Thus $s_0\tau_K \equiv (\mu_0 \otimes \tau)_K \mod I^{d+2}(K)$ and $x(\mu_1 \otimes \tau)_K \equiv xs_1\tau_K \mod I^{d+2}(K)$, since $\tau \in I^d(F)$. This yields $s_0\tau_K \equiv xs_1\tau_K \mod I^{d+2}(K)$ by $(*)$. Setting $s = s_0s_1$ we obtain $s\tau_K \equiv x\tau_K \mod I^{d+2}(K)$. Theorem 1.1 now shows that $s\tau_K \simeq x\tau_K$.

Therefore, (ii) and the above yield $((\varphi_0)_K)_{\text{an}} \simeq x(\varphi_1)_K \simeq x(\mu_1 \otimes \tau)_K \simeq (\mu_1)_K \otimes x\tau_K \simeq s(\mu_1 \otimes \tau)_K \simeq (s\varphi_1)_K$. Theorem 1.8 now shows that φ_0 is a Pfister neighbor whose complementary form is isomorphic to $-s\varphi_1$. □

The following theorem proves Theorem 0.6.

Theorem 4.2. *Let $(\varphi_0, \dots, \varphi_h)$ be a sequence of anisotropic forms. Then this sequence is a quasi-excellent sequence of the first type if and only if the following two conditions hold.*

- *For any $i = 0, \dots, h-1$, the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1}.*
- *The form φ_h is zero.*

Proof. Obvious induction by using Lemma 3.2 and Lemma 4.1 (iii). The converse direction follows from Example 0.5 starting with the sequence $\{\varphi_{h-1},\, \varphi_h = 0\}$. ☐

5. Quasi-excellent sequences modulo some ideals

Let $I(F)$ be the ideal of classes of even-dimensional forms in the Witt ring $W(F)$, and let $I^n(F)$ denote the nth power of $I(F)$. We need the following proposition for the classification of sequences of the second and third type.

Proposition 5.1. *Let $(\varphi_0, \varphi_1, \dots, \varphi_h)$ be a quasi-excellent sequence. Suppose that there exists an integer k such that $1 \le k < h$ with the following property: for all $i = 0, \dots, k-1$, there exists $f_i \in F^*$ such that $\varphi_i \equiv f_i\, \varphi_k \mod I^{m+1}(F)$ where m is a minimal integer such that $\dim \varphi_k < 2^m$.*

Then φ_0 is a Pfister neighbor whose complementary form is similar to φ_1.

Proof. For convenience, we will include in our consideration also the case where $k = 0$ and prove, by induction, the following two properties:

(a) If $k \ge 1$ then φ_0 is a Pfister neighbor whose complementary form is similar to φ_1.
(b) If $k \ge 0$ then for any $x \in F^*$ the conditions $\varphi_0 \equiv x\varphi_0 \mod I^{m+1}(F)$ and $\dim \varphi_k < 2^m$ imply that $\varphi_0 \simeq x\varphi_0$.

For a given k we denote properties (a) and (b) by $(a)_k$ and $(b)_k$ correspondingly. The plan of the proof of Proposition 5.1 will be the following.

- We start with the proof of property $(b)_0$.
- For $k \ge 1$, we prove that $(b)_{k-1} \Rightarrow (a)_k$.
- For $k \ge 1$, we prove that $(b)_{k-1} \Rightarrow (b)_k$.

Lemma 5.2. *Condition (b) holds in the case $k = 0$.*

Proof. If $k = 0$ we have in (b) the conditions $[\varphi_0 \perp -x\varphi_0] \in I^{m+1}(F)$ and $\dim \varphi_0 < 2^m$, in particular $\dim(\varphi_0 \perp -x\varphi_0) < 2^{m+1}$. By Theorem 1.1, the form $\varphi_0 \perp -x\varphi_0$ is hyperbolic. Therefore $\varphi_0 \simeq x\varphi_0$. ☐

Lemma 5.3. *Let $k \ge 1$. Then property $(b)_{k-1}$ (stated for all quasi-excellent sequences over all fields of characteristic $\ne 2$) implies property $(a)_k$.*

Proof. Consider a sequence $(\varphi_0, \dots, \varphi_h)$ as in Proposition 5.1. Then, by assumption, there exist $f_0, f_1 \in F^*$ such that $\varphi_0 \equiv f_0\varphi_k \bmod I^{m+1}(F)$ and $\varphi_1 \equiv f_1\varphi_k \bmod I^{m+1}(F)$. Hence $\varphi_0 \equiv f_0f_1\varphi_1 \bmod I^{m+1}(F)$.

Let $E = F(\varphi_0)$. By Definition 0.1, there exists $x \in E^*$ such that $((\varphi_0)_E)_{\mathrm{an}} \simeq x(\varphi_1)_E$. Hence $x(\varphi_1)_E \equiv (\varphi_0)_E \equiv f_0f_1(\varphi_1)_E \bmod I^{m+1}(E)$. Property $(b)_{k-1}$ stated for the quasi-excellent sequence $((\varphi_1)_E, (\varphi_2)_E, \dots, (\varphi_k)_E, \dots, (\varphi_h)_E)$ shows that $x(\varphi_1)_E \simeq f_0f_1(\varphi_1)_E$. Hence $((\varphi_0)_E)_{\mathrm{an}} \simeq x(\varphi_1)_E \simeq (f_0f_1\varphi_1)_E$. By Theorem 1.8, φ_0 is a Pfister neighbor whose complementary form is similar to φ_1. □

Lemma 5.4. *Let $k \geq 1$. Then property $(b)_{k-1}$ (stated for all quasi-excellent sequences over all fields of characteristic $\neq 2$) implies property $(b)_k$.*

Proof. Consider a sequence $(\varphi_0, \dots, \varphi_h)$ as in Proposition 5.1. We assume that property $(b)_{k-1}$ holds. Then property $(a)_k$ also holds (see previous lemma). This means that there exist $a, s \in F^*$, an integer $n > 0$, and an n-fold Pfister form π such that $a\pi \simeq \varphi_0 \perp -s\varphi_1$ and $\dim\varphi_1 < 2^{n-1}$.

Since $2^{n-1} > \dim\varphi_1 \geq \dim\varphi_k$, the definition of m yields $n-1 \geq m$. Therefore $[a\pi] \in I^n(F) \subset I^{m+1}(F)$. Hence, $\varphi_0 \equiv s\varphi_1 \bmod I^{m+1}(F)$. Now, let $x \in F^*$ be as in $(b)_k$. In other words, $\varphi_0 \equiv x\varphi_0 \bmod I^{m+1}(F)$. Then $s\varphi_1 \equiv \varphi_0 \equiv x\varphi_0 \equiv sx\varphi_1 \bmod I^{m+1}(F)$. Hence, $\varphi_1 \equiv x\varphi_1 \bmod I^{m+1}(F)$. Property $(b)_{k-1}$, applied to the quasi-excellent sequence

$$((\varphi_1)_{F(\varphi_0)}, (\varphi_2)_{F(\varphi_0)}, \dots, (\varphi_k)_{F(\varphi_0)}, \dots, (\varphi_h)_{F(\varphi_0)})$$

shows that $(\varphi_1)_{F(\varphi_0)} \simeq x(\varphi_1)_{F(\varphi_0)}$. Hence the form $\varphi_1 \otimes \langle\langle x\rangle\rangle$ is hyperbolic over $F(\varphi_0)$. Since φ_0 is a Pfister neighbor of π, it follows that $F(\varphi_0, \pi)/F(\pi)$ is purely transcendental. Thus $\varphi_1 \otimes \langle\langle x\rangle\rangle$ is also hyperbolic over $F(\pi)$. Since $\dim(\varphi_1 \otimes \langle\langle x\rangle\rangle) < 2^{n-1} \cdot 2 = \dim\pi$, Theorem 1.4 shows that $\varphi_1 \otimes \langle\langle x\rangle\rangle$ is hyperbolic, hence $\varphi_1 \simeq x\varphi_1$. Moreover, $a\pi \otimes \langle\langle x\rangle\rangle \simeq (\varphi_0 \perp -s\varphi_1) \otimes \langle\langle x\rangle\rangle$ is isotropic and therefore hyperbolic (since π is a Pfister form). Hence $a\pi \simeq xa\pi$. Since $a\pi \simeq \varphi_0 \perp -s\varphi_1$ and $\varphi_1 \simeq x\varphi_1$, it follows that $\varphi_0 \simeq x\varphi_0$. □

Clearly, the three lemmas complete the proof of Proposition 5.1. □

6. Five classical conjectures

Let $H^n(F) := H^n(F, \mathbb{Z}/2\mathbb{Z})$ be the nth Galois cohomology group. Let $I^0(F) := W(F)$ be the Witt ring and $I^1(F) := I(F)$ be the fundamental ideal in $W(F)$ of classes of even-dimensional forms. In §1, we have considered the homomorphisms

$$e^0 : I^0(F) \to \mathbb{Z}/2\mathbb{Z} \simeq H^0(F) \quad \text{and} \quad e^1 : I^1(F) \to F^*/F^{*2} \simeq H^1(F)$$

defined by the *dimension* and the *discriminant* respectively. Denoting by ${}_2\mathrm{Br}(F)$ the 2-torsion part of the Brauer group of F we obtain a homomorphism $e^2 : I^2(F) \to {}_2\mathrm{Br}(F) \simeq H^2(F)$ defined by the *Clifford algebra.* We have $\ker(e^n) = I^{n+1}(F)$ for $n = 0, 1, 2$.

For each integer $n > 0$ let $(a_1) \cdot \ldots \cdot (a_n)$ denote the cup-product where (a_i) is the class of $a_i \in F^*$ in $H^1(F)$ for $i = 1, \ldots, n$. The following five conjectures are believed to be true for all fields F of characteristic $\neq 2$.

Conjecture 6.1. (Milnor conjecture). *Let $n \geq 0$ be an integer. Then there exists a homomorphism*

$$e^n : I^n(F) \to H^n(F)$$

such that $\langle\langle a_1, \ldots, a_n \rangle\rangle \mapsto (a_1) \cdot \ldots \cdot (a_n)$. Moreover, the homomorphism e^n induces an isomorphism

$$e^n : I^n(F)/I^{n+1}(F) \simeq H^n(F).$$

Conjecture 6.2. *For any $\pi \in P_m(F)$ and all integers $n \geq m \geq 0$, we have*

$$\ker(H^n(F) \to H^n(F(\pi))) = e^m(\pi)\, H^{n-m}(F).$$

Conjecture 6.3. *We have $J_n(F) = I^n(F)$ for all integers $n \geq 0$.*

Conjecture 6.4. *Let φ be an anisotropic form over F. If $[\varphi] \in I^n(F)$ and $2^n \leq \dim \varphi < 2^n + 2^{n-1}$ then $\dim \varphi = 2^n$.*

Conjecture 6.5. *Let γ be an even-dimensional anisotropic form. Assume that γ is a good non-excellent form of height 2 with leading form $\tau \in P_n(F)$. Then there exists $\tau_0 \in P_{n-1}(F)$, ($\tau_0 = \langle 1 \rangle$ if $n = 1$), and $a, b, c \in F^*$ such that*

- *γ is similar to $\tau_0 \otimes \langle -a, -b, ab, c \rangle$,*
- *$\tau \simeq \tau_0 \otimes \langle\langle c \rangle\rangle$.*

Remark 6.6. *For proving Conjecture 6.5 it suffices to show that there exists $\tau_0 \in P_{n-1}(F)$ such that $\gamma \simeq \tau_0 \otimes \gamma'$ where $\dim \gamma' = 4$.*

Proof. Write $\gamma' = \langle r, s, t, u \rangle$ with $r, s, t, u \in F^*$. Then setting $d = rst$, we obtain $d\gamma' \simeq \langle -a, -b, ab, c \rangle$ with $a = -st$, $b = -rt$, and $c = du$. This shows $\gamma \sim \tau_0 \otimes \psi$ where $\psi = \langle -a, -b, ab, c \rangle$. Since $\langle d(\psi) \rangle \simeq \langle c \rangle$ by definition of $d(\psi) = (-1)^{\binom{4}{2}} \det(\psi)$ and since τ is the leading form of γ, it follows from [Kn1], 6.12, that $\tau \simeq \tau_0 \otimes \langle\langle c \rangle\rangle$. ☐

Remark 6.7. Recent results of Voevodsky [Vo] and Orlov-Vishik-Voevodsky [OVV] show that Conjectures 6.1, 6.2, and 6.3 hold for all fields of characteristic 0. These three conjectures were proved earlier in the cases $n \leq 4$ and characteristic $\neq 2$, (cf. [Pf2], [KRS], and [Kahn]).

Conjecture 6.4 is proved for all fields of characteristic 0 by Vishik [Vi]. In the case $n \leq 4$, it is proved for all fields of characteristic $\neq 2$ (see [H4]).

Conjecture 6.5 is proved in the case $n \leq 3$ for all fields of characteristic $\neq 2$, (see Remark 6.6 and [Kahn]). Moreover, it follows from Proposition 3.6 and from [Kahn], Proposition 4.3 (b), that Conjectures 6.3 and 6.4 for degree $n+1$ imply Conjecture 6.5 for degree n.

Definition 6.8. Let $d \geq 0$ be an integer, and let F be a field. We say that *condition A_d holds for F* if F is of characteristic $\neq 2$ and if for all field extensions F'/F the following conjectures hold:

- Conjecture 6.1 for all $n \leq d+2$,
- Conjecture 6.2 for $n \leq d+2$,
- Conjecture 6.3 for $n = d+1$,
- Conjecture 6.4 for $n = d+2$,
- Conjecture 6.5 for $n = d > 0$.

Theorem 6.9. *Let F be a field of characteristic $\neq 2$. If $d=0$ or $d=1$ then condition A_d holds for F. If $d=2$ then condition A_d holds for F, possibly with the exception of the bijectivity of the homomorphism $e^4 : I^4(F)/I^5(F) \to H^4(F)$.*

Proof. Conjecture 6.1 holds for $n=0$ by definition of the ideal $I(F)$. It has been proved by Pfister for $n=1$, cf. [Pf1], 2.3.6, and by Merkurjev for $n=2$, cf. [M1]. The existence of e^3 has been proved by Arason [Ara], Satz 5.7, and the bijectivity of e^3 by Rost [R] and independently by Merkurjev-Suslin [MS]. The existence of e^4 has been proved by Jacob-Rost [JR] and independently by Szyjewsi [Sz]. The bijectivity of e^4 was claimed by Rost (unpublished).

Conjecture 6.2 holds $n \leq 4$, cf. [KRS].

Conjecture 6.3. For $n = 1, 2$ see [Kn1], 6.2; for $n = 3$ (and $n = 4$), see [Kahn], Théorème 2.8.

Conjecture 6.4 is trivial for $n = 2$. For $n = 3$ it is due to Pfister and for $n = 4$ to Hoffmann, see [H4], Main Theorem for $n = 4$, and 2.9 for $n = 3$.

Conjecture 6.5. For $n = 1$ see [Kn1], 5.10. For for $n = 2$ see Remark 6.6 and [Kahn], Corollaire 2.1. □

Remark 6.7 gives rise to the following theorem.

Theorem 6.10. *Let $d \geq 0$ be an integer. Modulo results proved in* [Vo, OVV] *condition A_d holds for all fields of characteristic* 0.

We are going to prove some consequences of the above conjectures.

Let $GP_n(F)$ denote the set of quadratics forms over F which are similar to n-fold Pfister forms, and let $H^n(K/F) = \ker(H^n(F) \to H^n(K))$.

Lemma 6.11. *Let F be a field of characteristic $\neq 2$ and let $\pi \in GP_r(F)$ for some integer $r \geq 0$. Suppose that Conjecture 6.2 holds for $n = m = r$ and for all field extensions of F. Then*

(1) *for any extension K/F, we have*
$H^r(K(\pi)/F) = H^r(K/F) + H^r(F(\pi)/F)$;
(2) *for any form φ over F, we have*
$H^r(F(\varphi,\pi)/F) = H^r(F(\varphi)/F) + H^r(F(\pi)/F)$.

Proof. (1) Let $u \in H^r(K(\pi)/F)$. Then $u_K \in H^r(K(\pi)/K)$. Conjecture 6.2 applied with $n = m = r$ shows that $u_K = \ell \cdot e^r(\pi_K)$ with $\ell \in H^0(K) \simeq \mathbb{Z}/2\mathbb{Z}$.

Hence $(u - \ell \cdot e^r(\pi)) \in H^r(K/F)$. Therefore, $u \in H^r(K/F) + \ell \cdot e^r(\pi) \subset H^r(K/F) + H^r(F(\pi)/F)$.

(2) It suffices to set $K = F(\varphi)$ in (1). $\square$

Lemma 6.12. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Let γ be an even-dimensional anisotropic form which is good non-excellent of degree d and height 2. Then*

$$H^{d+2}(F(\gamma)/F) = \{e^{d+2}(\gamma \otimes \langle\langle f\rangle\rangle) \mid f \in F^* \textit{ with } \gamma \otimes \langle\langle f\rangle\rangle \in GP_{d+2}(F)\}\,.$$

Proof. Let τ, τ_0 and $a, b, c \in F^*$ be as in Conjecture 6.5. We can assume that $\gamma = \tau_0 \otimes \langle -a, -b, ab, c\rangle$. Let $\pi = \tau_0 \otimes \langle\langle a, b\rangle\rangle$.

Clearly, $\gamma \sim_{\mathrm{w}} \pi \perp -\tau$. Hence $(\gamma_{F(\pi)})_{\mathrm{an}} \simeq (-\tau_{F(\pi)})_{\mathrm{an}}$. Since $\dim\gamma > \dim\tau$, it follows that $\gamma_{F(\pi)}$ is isotropic. Hence $F(\gamma,\pi)/F(\pi)$ is purely transcendental, forcing $H^{d+2}(F(\gamma)/F) \subset H^{d+2}(F(\pi)/F)$. Conjecture 6.2 shows that $H^{d+2}(F(\pi)/F) = e^{d+1}(\pi)H^1(F)$. Hence, an arbitrary element of the group $H^{d+2}(F(\gamma)/F)$ is of the form $e^{d+1}(\pi) \cdot (s) = e^{d+2}(\pi \otimes \langle\langle s\rangle\rangle)$ with $s \in F^*$. Let $\rho = \pi \otimes \langle\langle s\rangle\rangle$. Then $\rho \in P_{d+2}(F)$ and $e^{d+2}(\rho_{F(\gamma)}) = 0$. By Conjecture 6.1, the form $\rho_{F(\gamma)}$ is hyperbolic. Hence γ is similar to a subform of ρ of by Theorem 1.4. Let γ^* and $t \in F^*$ be such that $t\gamma \perp -\gamma^* \simeq \rho$. The forms γ and γ^* are *half-neighbors*. By [H3, Prop. 2.8], there exists $k \in F^*$ such that $\gamma^* \simeq k\gamma$. Then $\gamma \otimes \langle\langle tk\rangle\rangle \simeq t\rho \in GP_{d+2}(F)$. To complete the proof, it suffices to notice that $e^{d+2}(t\rho) = e^{d+2}(\rho)$. $\square$

Lemma 6.13. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Assume that γ is an even-dimensional anisotropic form which is good non-excellent of height 2 with leading form $\tau \in P_d(F)$. Now, let φ be a form such that $\varphi \equiv \tau \bmod I^{d+1}(F)$ and $\varphi_{F(\gamma,\tau)}$ is hyperbolic. Also assume that there exists an extension E/F such that $\dim(\varphi_E)_{\mathrm{an}} = 2^{d+1}$. Then the following is true.*

(1) *There exists $f \in F^*$ such that*

$$\varphi \equiv f\gamma \mod I^{d+2}(F)\,.$$

(2) *If we suppose additionally that $\dim(\varphi_{F(\gamma)})_{\mathrm{an}} < \dim\gamma$, then there exists $f \in F^*$ such that*

$$\varphi \equiv f\gamma \mod I^{d+3}(F)\,.$$

Proof. (1). Let $\psi = \varphi \perp -\tau$. By assumption, we have $[\psi] \in I^{d+1}(F)$. Hence, we can consider the element $e^{d+1}(\psi) \in H^{d+1}(F)$. Since $\varphi_{F(\gamma,\tau)}$ and $\tau_{F(\tau)}$ are hyperbolic, it follows that $\psi_{F(\gamma,\tau)}$ is also hyperbolic. Hence, $e^{d+1}(\psi) \in H^{d+1}(F(\gamma,\tau)/F)$. By Conjecture 6.5, we can assume that $\gamma = \tau_0 \otimes \langle -a, -b, ab, c\rangle$ and $\tau = \tau_0 \otimes \langle\langle c\rangle\rangle$ where $\tau_0 \in P_{d-1}(F)$.

Let $\pi = \tau_0 \otimes \langle\langle a, b\rangle\rangle \in P_{d+1}(F)$. Then $\gamma \sim_{\mathrm{w}} \pi\perp -\tau$. Hence $\gamma_{F(\tau)} \sim_{\mathrm{w}} \pi_{F(\tau)}$. Therefore, by Lemma 6.11(2) and Conjecture 6.2 we have

$$\begin{aligned} H^{d+1}(F(\gamma,\tau)/F) &= H^{d+1}(F(\pi,\tau)/F) \\ &= H^{d+1}(F(\pi)/F) + H^{d+1}(F(\tau)/F) \\ &= e^{d+1}(\pi)H^0(F) + e^d(\tau)H^1(F). \end{aligned}$$

Since $e^d(\tau)H^1(F) = \{e^{d+1}(\tau \otimes \langle\langle s\rangle\rangle) \mid s \in F^*\}$, it follows that any element of the group $H^{d+1}(F(\gamma,\tau)/F)$ has one of the following forms:

- either $e^{d+1}\big[\pi\perp(\tau \otimes \langle\langle s\rangle\rangle)\big]$
- or $e^{d+1}(\tau \otimes \langle\langle s\rangle\rangle)$.

Since $e^{d+1}(\psi) \in H^{d+1}(F(\gamma,\tau)/F)$ and $\psi = \varphi\perp -\tau$, Conjecture 6.1 shows that

- either $\varphi\perp -\tau \equiv \pi\perp(\tau \otimes \langle\langle s\rangle\rangle) \bmod I^{d+2}(F)$
- or $\varphi\perp -\tau \equiv \tau \otimes \langle\langle s\rangle\rangle \bmod I^{d+2}(F)$.

Consider the first case where $\varphi\perp -\tau \equiv \pi\perp(\tau \otimes \langle\langle s\rangle\rangle) \bmod I^{d+2}(F)$. Clearly, we can compute $[\varphi]$ modulo $I^{d+2}(F)$. In our computation, we note that $[\pi]$ and $[\tau \otimes \langle\langle s\rangle\rangle]$ belong to $I^{d+1}(F)$. Hence for any $x \in F^*$, we have $x\pi \equiv \pi$ and $x\tau \otimes \langle\langle s\rangle\rangle \equiv \tau \otimes \langle\langle s\rangle\rangle \bmod I^{d+2}(F)$. Besides, we recall that $\gamma \sim_{\mathrm{w}} \pi\perp -\tau$. Now, we have the following calculation $\varphi \equiv \tau\perp\pi\perp(\tau\otimes\langle\langle s\rangle\rangle) \equiv \tau\perp -\pi\perp -(\tau\otimes\langle\langle s\rangle\rangle) \equiv -\gamma\perp((\gamma\perp -\pi)\otimes\langle\langle s\rangle\rangle) \equiv s\pi\perp -\pi\perp -s\gamma \equiv -s\gamma \bmod I^{d+2}(F)$. Hence $\varphi \equiv f\gamma \bmod I^{d+2}(F)$ with $f = -s$.

Now we consider the second case where $\varphi\perp -\tau \equiv \tau \otimes \langle\langle s\rangle\rangle \bmod I^{d+1}(F)$. Here, we get $\varphi \equiv \tau\perp(\tau \otimes \langle\langle s\rangle\rangle) \equiv \tau\perp -(\tau \otimes \langle\langle s\rangle\rangle) \equiv s\tau \bmod I^{d+2}(F)$. By the assumption of the lemma there exists a field extension E/F such that $\dim(\varphi_E)_{\mathrm{an}} = 2^{d+1}$. Since $\varphi \equiv s\tau \bmod I^{d+2}(F)$, we have $(\varphi_E)_{\mathrm{an}} \equiv s\tau_E \bmod I^{d+2}(E)$. Since $\dim(\varphi_E)_{\mathrm{an}} + \dim\tau_E = 2^{d+1} + 2^d < 2^{d+2}$, Theorem 1.1 shows that $(\varphi_E)_{\mathrm{an}} \simeq s(\tau_E)_{\mathrm{an}}$. This contradicts to the inequality $\dim(\varphi_E)_{\mathrm{an}} = 2^{d+1} > 2^d \geq \dim(\tau_E)_{\mathrm{an}}$.

(2). Let f be as in (1). Set $\psi = \varphi \perp -f\gamma$. We have $\psi \in I^{d+2}(F)$. Since $\dim(\varphi_{F(\gamma)})_{\mathrm{an}} < \dim\gamma$, we have $\dim(\psi_{F(\gamma)})_{\mathrm{an}} < 2\dim\gamma = 2\cdot 2^{d+1} = 2^{d+2}$. By Theorem 1.1, the form $\psi_{F(\gamma)}$ is hyperbolic. Hence $e^{d+2}(\psi) \in H^{d+2}(F(\gamma)/F)$. By Lemma 6.12, there exists $s \in F^*$ such that $\gamma \otimes \langle\langle s\rangle\rangle \in GP_{d+2}(F)$ and $e^{d+2}(\psi) = e^{d+2}(\gamma\otimes\langle\langle s\rangle\rangle)$. Thus $\psi \equiv \gamma\otimes\langle\langle s\rangle\rangle \bmod I^{d+3}(F)$ by Conjecture 6.1. Since $\gamma\otimes\langle\langle s\rangle\rangle \in GP_{d+2}(F)$, we have $\gamma\otimes\langle\langle s\rangle\rangle \equiv -f\gamma\otimes\langle\langle s\rangle\rangle \bmod I^{d+3}(F)$. Therefore, $\varphi \equiv \psi\perp f\gamma \equiv (\gamma\otimes\langle\langle s\rangle\rangle)\perp f\gamma \equiv -f(\gamma\otimes\langle\langle s\rangle\rangle)\perp f\gamma \equiv fs\gamma \bmod I^{d+3}(F)$. □

Proposition 6.14. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Let $(\varphi_0, \varphi_1, \ldots, \varphi_h)$ be a quasi-excellent sequence with leading form $\tau \in P_d(F)$ and pre-leading form $\gamma = \varphi_{h-2}$. Suppose that this sequence is of the second or third type (in particular $h \geq 2$). Let $\varphi = \varphi_i$ with $i \leq h-2$.*

(1) *We have $\varphi \equiv \tau \bmod I^{d+1}(F)$.*
(2) *There exists $f \in F^*$ such that $\varphi \equiv f\gamma \bmod I^{d+2}(F)$.*
(3) *If $\dim(\varphi_{F(\gamma)})_{\mathrm{an}} < \dim\gamma$ then there exists $f \in F^*$ such that*
$\varphi \equiv f\gamma \bmod I^{d+3}(F)$.

Proof. (1). Follows from Lemma 3.4 and Conjecture 6.3.

(2) and (3). Definition 0.3 shows that $\dim\gamma = 2^{d+1}$. By Proposition 3.6(2), γ is a good non-excellent form of height 2 and degree d. By Proposition 3.7, $\varphi_{F(\gamma,\tau)}$ is hyperbolic. Lemma 2.1 yields $\dim(\varphi_E)_{\mathrm{an}} = \dim\varphi_{h-2} = 2^{d+1}$ where $E = F_{h-2} = F(\varphi_0,\dots,\varphi_{h-3})$. Now, Lemma 6.13 completes the proof. □

Corollary 6.15. *Let $(\varphi,\gamma,\tau,0)$ be a quasi-excellent sequence with $\tau\in P_d(F)$ and $\dim\gamma = 2^{d+1}$. Then $\dim\varphi \geq 3\cdot 2^d$. Moreover, if $\dim\varphi = 3\cdot 2^d$ then $\varphi_{F(\gamma)}$ is anisotropic.*

Proof. Let $E = F(\gamma)$. If φ_E is anisotropic, then the sequence $(\varphi_E,\tau_E,0)$ is quasi-excellent by Lemma 3.1. Since $\dim\varphi_E = \dim\varphi > \dim\gamma = 2^{d+1}$, it follows from Proposition 3.6 that $\dim\varphi = 2^N - 2^d$ for some $N\geq d+2$. Therefore, $\dim\varphi \geq 2^{d+2}-2^d = 3\cdot 2^d$.

Now, we assume that φ_E is isotropic. Then $(\varphi_E)_{\mathrm{an}}\sim\tau_E$ by Lemma 3.1 and hence $\dim(\varphi_E)_{\mathrm{an}} = 2^d < \dim\gamma$. By Proposition 6.14, there exists $f\in F^*$ such that $\varphi\equiv f\gamma \mod I^{d+3}(F)$. Suppose that $\dim\varphi\leq 3\cdot 2^d$. Then $\dim\varphi+\dim\gamma\leq 3\cdot 2^d + 2^{d+1} < 2^{d+3}$. By Theorem 1.1, we get $\varphi\simeq f\gamma$. This is a contradiction because $\dim\varphi > \dim\gamma$. □

7. Classification theorem for sequences of the second type

In Definition 6.8 we formulated the condition A_d for a field F. We showed that A_d is true for $d = 0,1$ and $\mathrm{char}(F)\neq 2$ and that (based on results in [Vo, OVV]) A_d is true for all $d\geq 0$ and all fields of characteristic 0, cf. Theorems 6.9 and 6.10.

The main purpose of this section is to prove the following

Theorem 7.1. *Let $d>0$ be an integer and F be a field such that condition A_d holds for F. Let $(\varphi_0,\varphi_1,\dots,\varphi_h)$ be a quasi-excellent sequence of the second type and of degree d. Then*

- *for all $i<h-2$ the form φ_i is a Pfister neighbor whose complementary form is similar to φ_{i+1},*
- *the sequence $(\varphi_{h-2},\varphi_{h-1},\varphi_h)$ is quasi-excellent of the second type.*

First of all, we state the following corollary (which proves Theorems 0.9 and 0.10 for the quasi-excellent sequences of second type).

Corollary 7.2. *Let $d>0$ be an integer and F be a field such that condition A_d holds for F. Then Conjecture 0.7 holds for all quasi-excellent sequences of degree d over F.*

Proof. By Theorem 7.1, it suffices to consider the case $h=2$. In this case, the required result follows immediately from Proposition 3.6(2) and Conjecture 6.5. □

Now, we return to Theorem 7.1. We will prove this theorem by using induction on h. In the case where $h = 2$ the statement is obvious. Thus we can assume that $h \geq 3$. In what follows we will suppose that $h \geq 3$ and Theorem 7.1 holds for all quasi-excellent sequences of height $< h$.

We start with the following lemma.

Lemma 7.3. *If $(\varphi_0)_{F(\varphi_1)}$ is anisotropic then $(\varphi_0)_{F(\varphi_1)}$ is a Pfister neighbor whose complementary form is similar to $(\varphi_2)_{F(\varphi_1)}$.*

Proof. Let $E = F(\varphi_1)$. By Lemma 3.1, the sequence $((\varphi_0)_E, (\varphi_2)_E, \ldots, 0)$ is quasi-excellent of height $h-1$. Let us consider two cases, $h \geq 4$ and $h = 3$.

If $h \geq 4$ then the sequence $((\varphi_0)_E, (\varphi_2)_E, \ldots, (\varphi_h)_E)$ is of the second type. Then Theorem 7.1 (stated for sequences of height $< h$) completes the proof.

If $h = 3$ then the quasi-excellent sequence $((\varphi_0)_E, (\varphi_2)_E, 0)$ is of the first type because $\dim \varphi_0 > \dim \varphi_1 = 2^{d+1}$. In this case, Theorem 4.2 completes the proof. □

The following lemma shows that the situation described in Lemma 7.3 is actually impossible.

Lemma 7.4. *The form $(\varphi_0)_{F(\varphi_1)}$ is isotropic.*

Proof. Assume the contrary, $(\varphi_0)_{F(\varphi_1)}$ is anisotropic. Then Lemmas 7.3 and 2.8 show that $\dim \varphi_1$ is a power of 2 and $\dim \varphi_0 = 2 \dim \varphi_1 - \dim \varphi_2$. Let $K = F(\varphi_0)$. The sequence $((\varphi_1)_K, \ldots, (\varphi_h)_K)$ is quasi-excellent of height $h-1$ by Lemma 3.3. Clearly, this sequence is of the second type. We consider the two cases $h \geq 4$ and $h = 3$. If $h \geq 4$ then Theorem 7.1 (stated for sequences of height $< h$) shows that $(\varphi_1)_K$ is a Pfister neighbor whose complementary form is similar to the non-zero form $(\varphi_2)_K$. This in particular shows that $\dim \varphi_1$ is not a power of 2. We get a contradiction. Now, we assume that $h = 3$. In other words, we have the sequence $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ of the second type. By Definitions 0.3 and 2.6, we have $\dim \varphi_1 = 2^{d+1}$, $\dim \varphi_2 = 2^d$ and $\dim \varphi_0 \neq 3 \cdot 2^d$. On the other hand, $\dim \varphi_0 = 2 \dim \varphi_1 - \dim \varphi_2 = 2 \cdot 2^{d+1} - 2^d = 3 \cdot 2^d$. We get a contradiction. □

Corollary 7.5. *The sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is a quasi-excellent sequence of the second type and of degree d.*

Proof. Follows from Lemmas 7.4, and 3.1. □

Lemma 7.6. *Let $\gamma = \varphi_{h-2}$. Then $\dim((\varphi_i)_{F(\gamma)})_{\mathrm{an}} \leq 2^d$ for all $i = 0, \ldots, h$.*

Proof. Using induction and Corollary 7.5 we see that $\dim((\varphi_i)_{F(\gamma)})_{\mathrm{an}} \leq 2^d$ for all $i \geq 1$. Now, it suffices to prove that $\dim((\varphi_0)_{F(\gamma)})_{\mathrm{an}} \leq 2^d$. Since $h \geq 3$, we have $\dim((\varphi_1)_{F(\gamma)})_{\mathrm{an}} \leq 2^d = \dim \varphi_{h-1} < \dim \varphi_1$. Hence $(\varphi_1)_{F(\gamma)}$ is isotropic forcing that $F(\gamma, \varphi_1)/F(\gamma)$ is purely transcendental. Thus 7.4 yields that $(\varphi_0)_{F(\gamma)}$ is isotropic. By Lemma 2.3, there exists $i > 0$ such that $((\varphi_0)_{F(\gamma)})_{\mathrm{an}} \sim (\varphi_i)_{F(\gamma)} = ((\varphi_i)_{F(\gamma)})_{\mathrm{an}}$. Hence $\dim((\varphi_0)_{F(\gamma)})_{\mathrm{an}} = \dim((\varphi_i)_{F(\gamma)})_{\mathrm{an}} \leq 2^d$. □

Corollary 7.7. *There exists* $f_i \in F^*$ *such that* $\varphi_i \equiv f_i\gamma \mod I^{d+3}(F)$ *for all* $i = 0, \dots, h-2$.

Proof. Obvious consequence of Proposition 6.14 and Lemma 7.6. □

Corollary 7.8. *If* $h \geq 3$ *then* φ_0 *is a Pfister neighbor whose complementary form is similar to* φ_1.

Proof. Corollary 7.7 shows that the condition of Proposition 5.1 holds in the case $k = h-2$ and $m = d+2$. □

Proof of Theorem 7.1. If $h \geq 3$, Corollaries 7.8 and 7.5 show that

- φ_0 is a Pfister neighbor whose complementary form is similar to φ_1,
- the sequence $(\varphi_1, \varphi_2, \dots, \varphi_h)$ is quasi-excellent of the second type.

After that, an evident induction completes the proof. □

8. Classification theorem for sequences of the third type

We proceed similarly as in the previous section. The main purpose is to prove the following theorem.

Theorem 8.1. *Let* $d > 0$ *be an integer and* F *be a field such that condition* A_d *holds for* F. *Let* $(\varphi_0, \varphi_1, \dots, \varphi_h)$ *be a quasi-excellent sequence of the third type and of degree* d. *Then*

- *for all* $i < h-3$ *the form* φ_i *is a Pfister neighbor whose complementary form is similar to* φ_{i+1},
- *the sequence* $(\varphi_{h-3}, \varphi_{h-2}, \varphi_{h-1}, 0)$ *is quasi-excellent of the third type.*

We will prove this theorem by using induction on h. In the case where $h = 3$ the statement is obvious. Thus we can assume that $h \geq 4$. In what follows we will suppose that $h \geq 4$ and Theorem 8.1 holds for all quasi-excellent sequences of height $< h$.

Lemma 8.2. *If* $(\varphi_0)_{F(\varphi_1)}$ *is anisotropic then* $(\varphi_0)_{F(\varphi_1)}$ *is a Pfister neighbor whose complementary form is similar to* $(\varphi_2)_{F(\varphi_1)}$.

Proof. Let $E = F(\varphi_1)$. By Lemma 3.1, the sequence $((\varphi_0)_E, (\varphi_2)_E, \dots, 0)$ is quasi-excellent of height $h-1$. Let us consider two cases, $h \geq 5$ and $h = 4$.

If $h \geq 5$ then the sequence $((\varphi_0)_E, (\varphi_2)_E, \dots, (\varphi_h)_E)$ is of the third type. Then Theorem 8.1 (stated for sequences of height $< h$) completes the proof.

If $h = 4$ then the quasi-excellent sequence $((\varphi_0)_E, (\varphi_2)_E, (\varphi_3)_E, 0)$ has the second type because $\dim\varphi_0 > \dim\varphi_1 = 3 \cdot 2^d$. In this case, Theorem 7.1 completes the proof. □

Lemma 8.3. *The form* $(\varphi_0)_{F(\varphi_1)}$ *is isotropic.*

Proof. Assume the contrary, $(\varphi_0)_{F(\varphi_1)}$ is anisotropic. Then Lemmas 8.2 and 2.8 show that $\dim\varphi_1$ is a power of 2. Let $K = F(\varphi_0)$. The sequence $((\varphi_1)_K, \ldots, (\varphi_h)_K)$ is quasi-excellent of height $h-1$ by Lemma 3.3. Clearly, this sequence is of the third type. We consider two cases, $h \geq 5$ and $h = 4$. If $h \geq 5$ then Theorem 8.1 (stated for sequences of height $< h$) shows that $(\varphi_1)_K$ is a Pfister neighbor whose complementary form is similar to the non-zero form $(\varphi_2)_K$. This in particular shows that $\dim\varphi_1$ is not a power of 2. We get a contradiction. Now, we assume that $h = 4$. Then $\dim\varphi_1 = \dim\varphi_{h-3} = 3\cdot 2^d$ is not a power of 2, a contradiction. □

Corollary 8.4. *The sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is a quasi-excellent sequence of the third type and of degree d.*

Proof. Obvious in view of Lemmas 8.3, and 3.1. □

In what follows we use the following notation:
τ is the leading form. Clearly, we can assume that $\varphi_{h-1} = \tau$;
$\gamma = \varphi_{h-2}$ is the pre-leading form;
$\lambda = \varphi_{h-3}$ is "pre-pre-leading" form.
Thus, our quasi-excellent sequence looks as follows: $(\varphi_0, \ldots, \varphi_{h-4}, \lambda, \gamma, \tau, 0)$

Lemma 8.5. *For all $i = 0, \ldots, h-1$, we have* $\dim((\varphi_i)_{F(\lambda,\gamma)})_{\mathrm{an}} = 2^d$.

Proof. It follows from Lemma 2.1 that $(\tau)_{F(\lambda,\gamma)}$ is anisotropic.

Using induction and Corollary 8.4, we see that $\dim((\varphi_i)_{F(\lambda,\gamma)})_{\mathrm{an}} = 2^d$ for all $i = 1, \ldots, h-1$. In particular, $(\varphi_1)_{F(\lambda,\gamma)}$ is isotropic. Hence $F(\lambda,\gamma,\varphi_1)/F(\lambda,\gamma)$ is purely transcendental. Since $(\varphi_0)_{F(\varphi_1)}$ is isotropic by Lemma 8.3, it follows that $(\varphi_0)_{F(\lambda,\gamma)}$ is also isotropic. By Lemma 2.3, there exists $i > 0$ such that $((\varphi_0)_{F(\lambda,\gamma)})_{\mathrm{an}} \sim (\varphi_i)_{F(\lambda,\gamma)} = ((\varphi_i)_{F(\lambda,\gamma)})_{\mathrm{an}}$.

Hence $\dim((\varphi_0)_{F(\lambda,\gamma)})_{\mathrm{an}} = \dim((\varphi_i)_{F(\lambda,\gamma)})_{\mathrm{an}} = 2^d$. □

Proposition 8.6. *For any $i = 0, \ldots, h-3$ there exists $f_i \in F^*$ such that $\varphi_i \equiv f_i\lambda \bmod I^{d+3}(F)$.*

Proof. There is $s_i \in F^*$ such that $\varphi_i \equiv s_i\gamma \bmod I^{d+2}(F)$ for $i = 0, \ldots, h-3$ by Proposition 6.14(2).

Changing notation $\varphi_i := s_i\varphi_i$, we can assume that $\varphi_i \equiv \gamma \bmod I^{d+2}(F)$ for all $i = 0, \ldots, h-3$. In particular, $\lambda = \varphi_{h-3} \equiv \gamma \bmod I^{d+2}(F)$. Hence, we get the element $e^{d+2}(\lambda\perp -\gamma) \in H^{d+2}(F)$.

Now, we fix an integer $i \leq h-3$ and set $\varphi = \varphi_i$. We have $\varphi \equiv \gamma \equiv \lambda \bmod I^{d+2}(F)$. Hence, we get the elements $e^{d+2}(\varphi\perp -\gamma)$ and $e^{d+2}(\varphi\perp -\lambda)$ in $H^{d+2}(F)$. Recall that $H^n(F'/F) := \ker(H^n(F) \to H^n(F'))$.

Lemma 8.7. (1) $e^{d+2}(\varphi\perp -\gamma) \in H^{d+2}(F(\lambda,\gamma)/F)$,
(2) $e^{d+2}(\varphi\perp -\gamma) \notin H^{d+2}(F(\gamma)/F)$,
(3) $e^{d+2}(\varphi\perp -\lambda) \in H^{d+2}(F(\gamma)/F)$.

Proof. To prove item (1), it suffices to verify that $(\varphi \perp -\gamma)_{F(\lambda,\gamma)}$ is hyperbolic. By Lemma 8.5, we have $\dim(\varphi_{F(\lambda,\gamma)})_{\mathrm{an}} \le 2^d$ and $\dim(\gamma_{F(\lambda,\gamma)})_{\mathrm{an}} \le 2^d$. Hence, $\dim((\varphi \perp -\gamma)_{F(\lambda,\gamma)})_{\mathrm{an}} \le 2^{d+1} < 2^{d+2}$. Since $[\varphi\perp -\gamma] \in I^{d+2}(F)$, Theorem 1.1 shows that $(\varphi \perp -\gamma)_{F(\lambda,\gamma)}$ is hyperbolic.

(2) Assume that $e^{d+2}(\varphi\perp-\gamma) \in H^{d+2}(F(\gamma)/F)$. Let $E = F(\varphi_0, \dots, \varphi_{h-4})$. Then $(\varphi_E)_{\mathrm{an}} \sim (\varphi_{h-3})_E = \lambda_E$ by Lemma 2.1. Hence, there exists $s \in E^*$ such that $(\varphi_E)_{\mathrm{an}} \simeq s\lambda_E$. Therefore, $e^{d+2}(s\lambda_E\perp -\gamma_E) \in H^{d+2}(E(\gamma)/E)$. Hence $e^{d+2}(s\lambda_{E(\gamma)}\perp -\gamma_{E(\gamma)}) = 0$.

Then Conjecture 6.1 implies that $[s\lambda_{E(\gamma)} \perp -\gamma_{E(\gamma)}] \in I^{d+3}(E(\gamma))$. Since $\dim\lambda + \dim\gamma = 3\cdot 2^d + 2^{d+1} < 2^{d+3}$, Theorem 1.1 shows that $s\lambda_{E(\gamma)} \perp -\gamma_{E(\gamma)}$ is hyperbolic. Thus $\dim(\lambda_{E(\gamma)})_{\mathrm{an}} \le \dim\gamma < \dim\lambda$. Hence, $\lambda_{E(\gamma)}$ is isotropic. By Lemma 3.3, the sequence $(\lambda_E, \gamma_E, \tau_E, 0)$ is quasi-excellent. By Corollary 6.15, the form $\lambda_{E(\gamma)}$ is anisotropic. We get a contradiction.

(3). Set $K = F(\gamma)$. Then we have a non-zero element $e^{d+2}(\varphi_K \perp -\gamma_K)$ in the group $H^{d+2}(K(\lambda)/K)$ by (1) and (2).

Since $(\gamma_K)_{\mathrm{an}}$ is similar to τ_K by Lemma 3.5 and Proposition 3.7, there exists $s \in K^*$ such that $(\gamma_K)_{\mathrm{an}} \simeq s\tau_K$. Since $[\lambda \perp -\gamma] \in I^{d+2}(F)$ we obtain that $[\lambda_K \perp -s\tau_K] \in I^{d+2}(K)$. Computing $\dim\lambda + \dim\tau = 3\cdot 2^d + 2^d = 2^{d+2}$, we conclude from the *Arason-Pfister Hauptsatz* that there is a form $\pi \in GP_{d+2}(K)$ such that $\pi \simeq \lambda_K \perp -s\tau_K$, (see Theorem 1.1, and [AP], p. 174, Korollar 3). It follows that $\lambda_{K(\pi)}$ is isotropic, since $\dim\lambda > \dim\tau$, and $\pi_{K(\pi)}$ is hyperbolic. Hence, $K(\pi,\lambda)/K(\pi)$ is purely transcendental. Since by (1) and (2) we have $0 \ne e^{d+2}(\varphi_K \perp -\gamma_K) \in H^{d+2}(K(\pi,\lambda)/K)$ we see that

$$0 \ne e^{d+2}(\varphi_K \perp -\gamma_K) \in H^{d+2}(K(\pi)/K)\,.$$

Thus Conjecture 6.2 shows that $e^{d+2}(\varphi_K \perp -\gamma_K) = e^{d+2}(\pi)$. Clearly, this yields $e^{d+2}(\varphi_K \perp -\gamma_K \perp -\pi) = 0$. Since $\pi \simeq \lambda_K \perp -s\tau_K \sim_{\mathrm{w}} \lambda_K \perp -\gamma_K$ we have $\varphi_K \perp -\gamma_K \perp -\pi \sim_{\mathrm{w}} (\varphi \perp -\lambda)_K$. Hence, $e^{d+2}(\varphi \perp -\lambda)_K = 0$. Therefore, $e^{d+2}(\varphi \perp -\lambda) \in H^{d+2}(K/F)$. □

By Proposition 3.6(2), γ is a good non-excellent form of height 2 and degree d. Now, Lemma 6.12 and item (3) of Lemma 8.7 show that there exists $f \in F^*$ such that $e^{d+2}(\lambda \perp -\varphi) = e^{d+2}(\gamma \otimes \langle\langle f\rangle\rangle)$. By Conjecture 6.1, we have

$$\lambda \perp -\varphi \equiv \gamma \otimes \langle\langle f\rangle\rangle \mod I^{d+3}(F)\,.$$

Since $\gamma \equiv \lambda \bmod I^{d+2}(F)$, it follows that $\lambda \perp -\varphi \equiv \lambda \otimes \langle\langle f\rangle\rangle \simeq \lambda \perp -f\lambda \pmod{I^{d+3}(F)}$. Therefore, $\varphi \equiv f\lambda \bmod I^{d+3}(F)$. This completes the proof of Proposition 8.6. □

Corollary 8.8. *If $h \ge 3$ then φ_0 is a Pfister neighbor whose complementary form is similar to φ_1.*

Proof. Proposition 8.6 shows that the condition of Proposition 5.1 holds in the case $k = h-3$ and $m = d+2$. □

Proof of Theorem 8.1. In case $h \ge 4$, Corollaries 8.8 and 8.4 show that

- φ_0 is a Pfister neighbor whose complementary form is similar to φ_1 ,
- the sequence $(\varphi_1, \varphi_2, \ldots, \varphi_h)$ is quasi-excellent of the third type.

After that, an evident induction completes the proof. □

9. Quasi-excellent sequences of type 3 and height 3

The main purpose of this section is to complete the classification of quasi-excellent sequences of the third type. The results of the previous section show that it suffices to consider only quasi-excellent sequences of height 3.

Lemma 9.1. *The sequence $(\phi_0, \phi_1, \phi_2, \phi_3)$ in Example 0.2 is quasi-excellent.*

Proof. Set $\varrho = \langle\langle a_1, \ldots, a_{d-1}\rangle\rangle$ and $K = F(\phi_0)$. Then $(\phi_1)_K$ is anisotropic by Theorem 1.5. We have $i((\phi_0)_K) \geq \dim\rho = 2^{d-1}$ by [HR], Lemma 2.5 ii, hence $\dim((\phi_0)_K)_{\mathrm{an}} \leq 2^{d+1} = \dim(\phi_1)_K$. Set $\eta = k_0\rho\otimes\langle\langle u, v, c\rangle\rangle$. Then

$$\begin{aligned}\phi_0 \perp -c\, k_0 k_1 \phi_1 &\simeq k_0\rho\otimes(\langle\langle u,v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle \perp -c\,\langle -u,-v,uv,a_d\rangle)\\ &\simeq k_0\rho\otimes(\langle\langle u,v,c\rangle\rangle \perp \langle c\,a_d, -c\,a_d\rangle)\\ &\simeq \eta \perp \rho\otimes\langle 1,-1\rangle\ .\end{aligned}$$

The form $\eta_{F(\eta)}$ is hyperbolic. But $(\phi_1)_{F(\eta)}$ is anisotropic by Theorem 1.5, hence $((\phi_0)_{F(\eta)})_{\mathrm{an}} \sim (\phi_1)_{F(\eta)}$. Since there is an F-place $K \to F(\eta)\cup\{\infty\}$ it follows that $\dim((\phi_0)_K)_{\mathrm{an}} \geq \dim(\phi_1)_{F(\eta)} = 2^{d+1}$, cf [Kn1], Proposition 3.1. Thus $\dim((\phi_0)_K)_{\mathrm{an}} = 2^{d+1} = \dim(\phi_1)_K$.

If η_K is hyperbolic then it follows that $((\phi_0)_K)_{\mathrm{an}} \sim (\phi_1)_K$. Otherwise, η_K is anisotropic, and $((\phi_0)_K)_{\mathrm{an}} \perp -c\,k_0k_1(\phi_1)_K \simeq \eta_K$. This shows that for every $x\in K^*$, the forms $(x(\phi_0)_K)_{\mathrm{an}}$ and $(\phi_1)_K$ are *half-neighbors* in the sense of [H3], p. 258. Since $(\phi_0)_K$ is isotropic there is an $x\in K^*$ such that $x(\phi_0)_K \simeq (\rho\otimes\langle\langle u,v\rangle\rangle)_K \perp (-\rho\otimes\langle\langle a_d\rangle\rangle)_K$, e.g., [HR], Lemma 2.5 i. Thus $(x(\phi_0)_K)_{\mathrm{an}}$ is a $(2^{d+1}, 2^d)$-Pfister form in the sense of [H3], p. 262. Now, [H3], Proposition 2.8, shows that $((\phi_0)_K)_{\mathrm{an}} \sim (\phi_1)_K$.

By Theorem 1.5, the forms ϕ_1, ϕ_2 remain anisotropic over $K = F(\phi_0)$. Thus we consider ϕ_1, ϕ_2 as forms over K and show that $((\phi_1)_{K(\phi_1)})_{\mathrm{an}}) \sim (\phi_2)_{K(\phi_1)}$. We have

$$\begin{aligned}\phi_1 \perp k_1\,k_2\,\phi_2 &\simeq k_1\rho\otimes(\langle -u,-v,uv,a_d\rangle \perp \langle\langle a_d\rangle\rangle)\\ &\sim_{\mathrm{w}} k_1\rho\otimes\langle\langle u,v\rangle\rangle\ .\end{aligned}$$

Set $\psi = k_1\rho\otimes\langle\langle u,v\rangle\rangle$. Then $\dim\psi = 2^{d+1} = \dim\phi_1$. Since $(\phi_2)_{K(\psi)}$ is anisotropic by Theorem 1.5 and since $\psi_{K(\psi)}$ is hyperbolic, the form $(\phi_1)_{K(\psi)}$ is not hyperbolic. There is a K-place $K(\phi_1) \to K(\psi)\cup\{\infty\}$ forcing that $(\phi_1)_{K(\phi_1)}$ is not hyperbolic. But the form $(\phi_1)_{K(\phi_1,\phi_2)}$ is hyperbolic, for otherwise, since $(\phi_2)_{K(\phi_1,\phi_2)}$ is hyperbolic, we would have the contradiction $\dim((\phi_1)_{(K(\phi_1,\phi_2)})_{\mathrm{an}} = \dim\psi = 2^{d+1} = \dim(\phi_1)_{K(\phi_1,\phi_2)}$. Now Lemma 3.5 yields that $((\phi_1)_{K(\phi_1)})_{\mathrm{an}}) \sim (\phi_2)_{K(\phi_1)}$. □

The main result of this section is the following

Proposition 9.2. *Let $d > 0$ be an integer and F be a field such that condition A_d holds for F. Let $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ be a quasi-excellent sequence of degree d and of the third type. Then this sequence looks as in Example 0.2.*

Clearly, this proposition together with the results of the previous sections completes the proof of Theorems 0.9 and 0.10.

We say that a form φ is *divisible* by a form ρ if there is a form χ such that $\varphi \simeq \rho \otimes \chi$.

Lemma 9.3. *Let φ and ψ be anisotropic forms. Suppose that φ and ψ are divisible by a Pfister form ρ (including the case $\rho = \langle 1 \rangle$). Then there exist forms φ_0, ψ_0, μ such that*

- *φ_0, ψ_0 and μ are divisible by ρ,*
- *$\varphi \simeq \varphi_0 \perp \mu$ and $\psi \simeq \psi_0 \perp \mu$,*
- *$(\varphi \perp -\psi)_{\mathrm{an}} \simeq \varphi_0 \perp -\psi_0$.*

Proof. Let μ be a form of maximal dimension satisfying the following conditions:

(a) μ is divisible by ρ,
(b) $\mu \subset \varphi$ and $\mu \subset \psi$.

Then there exist forms φ_0 and ψ_0 such that $\varphi \simeq \varphi_0 \perp \mu$ and $\psi \simeq \psi_0 \perp \mu$. Since φ, ψ and μ are divisible by ρ, it follows from Theorem 1.3 that φ_0 and ψ_0 are also divisible by ρ. Now, it suffices to prove that $(\varphi \perp -\psi)_{\mathrm{an}} \simeq \varphi_0 \perp -\psi_0$.

Since $\varphi \perp -\psi \simeq (\varphi_0 \perp \mu) \perp -(\psi_0 \perp \mu) \sim_{\mathrm{w}} \varphi_0 \perp -\psi_0$, it suffices to prove that the form $\varphi_0 \perp -\psi_0$ is anisotropic. Suppose the contrary. Then the forms φ_0 and ψ_0 have a common value, say $\ell \in F^*$. By Theorem 1.3, we have $\ell\rho \subset \varphi_0$ and $\ell\rho \subset \psi_0$. Setting $\tilde{\mu} = \mu \perp \ell\rho$, we see that $\tilde{\mu}$ satisfies conditions (a) and (b). Since $\dim \tilde{\mu} > \dim \mu$, we get a contradiction to the definition of μ. ☐

Lemma 9.4. *Let φ and ψ be anisotropic forms being divisible by a Pfister form $\rho \in P_{d-1}(F)$ (where $\rho = \langle 1 \rangle$ if $d = 1$). Suppose that $\dim \varphi = 3 \cdot 2^d$, $\dim \psi = 2^{d+1}$, and $(\varphi \perp -\psi)_{\mathrm{an}} \in GP_{d+2}(F)$. Then there exist $u, v, a_d, c \in F^*$ such that*

$$\varphi \sim \rho \otimes (\langle\langle u, v \rangle\rangle \perp -c \langle\langle a_d \rangle\rangle) \quad \textit{and} \quad \psi \sim \rho \otimes \langle -u, -v, uv, a_d \rangle .$$

Proof. Let φ_0, ψ_0 and μ be as in Lemma 9.3. We have

$$\begin{aligned} 2 \dim \mu &= (\dim \varphi + \dim \psi) - (\dim \varphi_0 + \dim \psi_0) \\ &= \dim \varphi + \dim \psi - \dim(\varphi \perp -\psi)_{\mathrm{an}} \\ &= 3 \cdot 2^d + 2^{d+1} - 2^{d+2} = 2^d. \end{aligned}$$

Hence, $\dim \mu = 2^{d-1}$. Since μ is divisible by ρ, there exists $s \in F^*$ such that $\mu \simeq s\rho$. Clearly, $\dim \psi_0 = \dim \psi - \dim \mu = 2^{d+1} - 2^{d-1} = 3 \cdot 2^{d-1}$. Since ψ_0 is divisible by $\rho \in P_{d-1}(F)$ there exist $k, u, v \in F^*$ such that

$$\psi_0 \simeq k\rho \otimes \langle 1, -u, -v \rangle .$$

Then $\psi \simeq (k\rho \otimes \langle 1, -u, -v \rangle) \perp s\rho \simeq kuv\rho \otimes \langle uv, -v, -u, a_d \rangle$ with $a_d = skuv$. Thus $\psi \sim \rho \otimes \langle -u, -v, uv, a_d \rangle$.

Put $\pi := \rho \otimes \langle\langle u, v\rangle\rangle \in P_{d+1}(F)$. Since $\psi_0 \simeq k\rho \otimes \langle 1, -u, -v\rangle$, it is easily checked that $k\pi \simeq \psi_0 \perp kuv\rho$. Hence, $(\psi_0)_{F(\pi)}$ is isotropic.

Let $\eta := -k(\varphi \perp -\psi)_{\mathrm{an}} \simeq -k(\varphi_0 \perp -\psi_0)$. By the hypotheses of the lemma, $\eta \in GP_{d+2}(F)$. Since $k\psi_0 \simeq \rho \otimes \langle 1, -u, -v\rangle$ represents 1, it follows that η represents 1. Hence, $\eta \in P_{d+2}(F)$. Since $k\psi_0 \subset \eta$ and $(\psi_0)_{F(\pi)}$ is isotropic, it follows that $\eta_{F(\pi)}$ is isotropic. Thus $\eta \simeq \pi \otimes \eta_0$ for some form η_0 by Theorem 1.3. Since $\eta \in P_{d+2}(F)$ and $\pi \in P_{d+1}(F)$, it follows that $\eta \simeq \pi \otimes \langle\langle c\rangle\rangle$ for suitable $c \in F^*$. Hence, $\eta \simeq \rho \otimes \langle\langle u, v, c\rangle\rangle$. Clearly, $uv \in D(\eta)$. Since η is a Pfister form, we obtain $\eta \simeq uv\eta$, (see [S], 2.10.4). By definition of η we have $k\eta \sim_{\mathrm{w}} \psi \perp -\varphi$. Hence,

$$\begin{aligned}\varphi &\sim_{\mathrm{w}} \psi \perp -k\eta \simeq \psi \perp -kuv\eta \simeq kuv\rho \otimes \langle uv, -v, -u, a_d\rangle \perp -kuv\rho \otimes \langle\langle u, v, c\rangle\rangle \\ &\sim_{\mathrm{w}} kuv\rho \otimes (\langle\langle u, v\rangle\rangle \perp -\langle\langle a_d\rangle\rangle) \perp -kuv\rho \otimes (\langle\langle u, v\rangle\rangle \perp -c\,\langle\langle u, v\rangle\rangle) \\ &\sim_{\mathrm{w}} kuv\rho \otimes (c\,\langle\langle u, v\rangle\rangle \perp -\langle\langle a_d\rangle\rangle) \simeq kuvc\rho \otimes (\langle\langle u, v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle).\end{aligned}$$

Since $\dim \varphi = 3 \cdot 2^d = \dim \rho \otimes (\langle\langle u, v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle)$, we get
$\varphi \simeq kuvc\rho \otimes (\langle\langle u, v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle)$. □

Proof of Proposition 9.2. Let $\tau \in P_d(F)$ be the leading form. Clearly, we can assume that $\varphi_2 = \tau$. In other words, we have a quasi-excellent sequence of the form $(\lambda, \gamma, \tau, 0)$ with $\dim \lambda = 3 \cdot 2^d$, $\dim \gamma = 2^{d+1}$ and $\dim \tau = 2^d$. By Proposition 6.14, there exists $s \in F^*$ such that $\lambda \equiv s\gamma \bmod I^{d+2}(F)$. Replacing λ by $s\lambda$, we can assume that $\lambda \equiv \gamma \bmod I^{d+2}(F)$. Let $\xi := (\lambda \perp -\gamma)_{\mathrm{an}}$. Clearly, $\xi \not\sim_{\mathrm{w}} 0$ and $[\xi] \in I^{d+2}(F)$. By Theorem 1.1, we have $\dim \xi \geq 2^{d+2}$. Since $\dim \xi \leq \dim \lambda + \dim \gamma = 3 \cdot 2^d + 2^{d+1} = 5 \cdot 2^d < 3 \cdot 2^{d+1}$, Conjecture 6.4 yields $\dim \xi = 2^{d+2}$. Hence $(\lambda \perp -\gamma)_{\mathrm{an}} = \xi \in GP_{d+2}(F)$, cf. [AP], Kor. 3.

By Proposition 3.6, γ is a good non-excellent form of height 2 with leading form τ. By Conjecture 6.5, there exists $\rho \in P_{d-1}(F)$ such that γ and τ are divisible by ρ. Then $\gamma_{F(\rho)}$ and $\tau_{F(\rho)}$ are isotropic. Hence $F(\rho, \gamma, \tau)/F(\rho)$ is purely transcendental. Since the form $\lambda_{F(\gamma,\tau)}$ is hyperbolic by Proposition 3.7, it follows that $\lambda_{F(\rho)}$ is hyperbolic. Therefore λ is divisible by ρ (see Theorem 1.3). Applying Lemma 9.4 to the forms ρ, $\varphi = \lambda$, and $\psi = \gamma$, we see that there exists $u, v, a_d \in F^*$ such that

$$\lambda \sim \rho \otimes (\langle\langle u, v\rangle\rangle \perp -c\,\langle\langle a_d\rangle\rangle) \quad \text{and} \quad \gamma \sim \rho \otimes \langle -u, -v, uv, a_d\rangle .$$

It follows that $\tau \sim \rho \otimes \langle\langle a_d\rangle\rangle$ by [Kn1], 6.12. The proof is complete. □

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Realizing Countable Groups As Automorphism Groups of Riemann Surfaces

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Abstract. Every countable group can be realized as the full automorphism group of a Riemann surface as well as the full group of isometries of a Riemannian manifold.

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Saerens and Zame [4], and independently Bedford and Dadok [2] proved that every compact real Lie group K can be realized as the group of holomorphic automorphisms of a complex manifold as well as the group of isometries of a Riemannian manifold.
Here we deduce a similar result for countable discrete groups.
Thus the purpose of this paper is to prove the following theorem:

Theorem. *Let G be a (finite or infinite) countable group.*
Then there exists a (connected) Riemann surface M such that G is isomorphic to the group $\mathrm{Aut}_{\mathcal{O}}(M)$ of all holomorphic automorphisms of M.
Moreover, there exists a Riemannian metric h on M such that $\mathrm{Aut}_{\mathcal{O}}(M)$ equals the group of all isometries of (M, h).

Our strategy is as follows: Using Galois theory of coverings, we first construct a Riemann surface M_1 on which G acts. Then we remove a discrete subset $S \subset M_1$ to kill excess automorphisms. However, we have to show that passing from M_1 to $M_1 \setminus S$ we do not risk enlarging the automorphism group, i.e.,

we will show that every automorphism from $M_1 \setminus S$ extends to M_1. For this purpose we employ the Freudenthal's theory of topological ends.
Finally, hyperbolicity of the Riemann surface is exploited to ensure that there is a hermitian metric of constant negative curvature such that the group of all holomorphic automorphisms coincides with the group of all isometries.
Let us remark that by uniformization theory it is well-known that the following is the list of all Riemann surface with positive-dimensional automorphism group and that their automorphism groups are well-known:

- $\mathbb{P}_1(\mathbb{C})$,
- $\mathbb{C}$,
- $\mathbb{C}^*$,
- $H^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$,
- $E_\tau = \mathbb{C}/\langle 1, \tau\rangle_{\mathbb{Z}}$ with $\tau \in H^+$,
- $A(r, 1) = \{z \in \mathbb{C} : r < |z| < 1\}$ with $0 \le r < +\infty$.

(See [1] for this and other basic facts on Riemann surfaces.)
Therefore our result yields a complete characterization which groups may occur as automorphism group of a Riemann surface.
The above list furthermore has the following consequence which we will use later on:

Fact. *Let M be a Riemann surface with non-commutative fundamental group $\pi_1(M)$. Then $\mathrm{Aut}_{\mathcal{O}}(M)$ is discrete and acting properly discontinuously on M. In particular, every orbit is closed.*

1 Hyperbolic Riemannian Surfaces

A Riemannian surface M is "hyperbolic" (in the sense of Kobayashi) if and only if its universal covering is isomorphic to the unit disk. In this case the Poincaré metric on the unit disk induces a unique hermitian metric of constant Gaussian curvature -1 on M. Every holomorphic automorphism of M is an isometry and conversely every isometry is either holomorphic or antiholomorphic. Thus the group of holomorphic automorphisms of M is a subgroup of index 1 or 2 in the group of isometries and the group of isometries coincides with the group of all holomorphic or antiholomorphic diffeomorphisms of M.

2 Galois theory of coverings

Proposition 1. *Let G be a countable group. Let M_0 be a Riemann surface whose fundamental group is not finitely generated.*
Then there exists an unramified covering $M_1 \to M_0$ such that there is an effective G-action on M_1, $\mathrm{Aut}_{\mathcal{O}}(M_1)$ is discrete and acting properly discontinuously on M_1.

Proof. Let F_∞ be a free group with countably infinitely many generators $\alpha_1, \alpha_2, \ldots$. By standard results on Riemann surfaces (see e.g. [1]) we have $\pi_1(M_0) \simeq F_\infty$. Since G is countable, there is a surjective group homomorphism $\zeta : F_\infty \to G$. Furthermore, we may require that $\alpha_1, \alpha_2 \in \ker \zeta$. Then we obtain a short exact sequence of groups

$$\{e\} \to N \to F_\infty \xrightarrow{\zeta} G \to \{e\}$$

where N is non-commutative, because it contains a free group with two generators (viz. α_1 and α_2).
By Galois theory of coverings, this implies that there exists an unramified covering $M_1 \to M_0$ with $\pi_1(M_1) \simeq N$ and an effective G-action on M_1.
Finally, discreteness of $\mathrm{Aut}_{\mathcal{O}}(M_1)$ as well as the action of $\mathrm{Aut}_{\mathcal{O}}(M_1)$ being properly discontinuous is implied by the "Fact" established above. □

3 Topological ends

Let us recall the basic facts from the theory of ends as developed by Freudenthal [3].
Let X be a locally compact topological space. Then the set of "ends" $e(X)$ is defined by

$$e(X) = \lim_K \pi_0(X \setminus K).$$

Thus, if K_n is an exhaustion of X by an increasing sequence of compact subsets, then every end $\epsilon \in e(X)$ can be represented by a sequence U_n of connected components of $X \setminus K_n$ with $U_n \supset U_{n+1}$. For every connected component $W_{n,i}$ of $X \setminus K_n$ we now define $E_{n,i}$ as the set of ends ϵ which can be represented with $U_n = W_{n,i}$. Now $\bar{X} = X \cup e(X)$ becomes a compact topological space as follows: As a basis of the topology we take the family of all open subsets of X together with $V_{n,i} = W_{n,i} \cup E_{n,i}$ for all n, i.
Then every proper continuous map between locally compact topological spaces X and Y extends to a continuous maps between $\bar{X}$ and $\bar{Y}$. In particular, every homeomorphism of X extends to a homeomorphism of $\bar{X}$.

Definition. *An end ϵ of a Riemann surface X is called a "*puncture*" if there is an open neighbourhood W of ϵ in $\bar{X}$ such that*

- *$W \setminus \{\epsilon\} \subset X$ and*
- *there is a homeomorphism $\xi : W \to D = \{z \in \mathbb{C} : |z| < 1\}$ such that $\xi|_{W \setminus \{\epsilon\}}$ is holomorphic.*

We now prove that the ends of a certain special class of Riemann surfaces cannot be punctures.

Proposition 2. *Let a_n be a diverging sequence in $D = \{z \in \mathbb{C} : |z| < 1\}$ and $r_n \in \mathbb{R}^{>0}$ such that all the closed balls $B_n = \{z \in \mathbb{C} : |z - a_n| \le r_n\}$ are disjoint subsets of D. Then $A = \cup_n B_n$ is a closed subset of D.*

Let $X_0 = D \setminus A$ and let $X_1 \to X_0$ be an unramified covering.
Then no end of X_1 is a puncture.

Proof. First we show that A is indeed closed. Let $vol(B_n) = \pi r_n^2$ denote the euclidean volume. Since the balls B_n are disjoint, we have $\sum vol(B_n) \leq \pi$ and therefore $\lim r_n = 0$. It follows that for every $s < 1$ there is a natural number N such that $|a_n| - r_n > s$ for all $n \geq N$. Therefore $A = \cup_n B_n$ is actually a locally finite union, and closedness of the B_n implies that A is closed.
Now let us assume that there exists an end which is a puncture. The natural embeddings $X_0 \hookrightarrow D \hookrightarrow \mathbb{C}$ composed with the projection $\pi : X_1 \to X_0$ yield a bounded holomorphic function f on X_1. Let $\epsilon \in e(X_1)$ be a puncture with a connected open neighbourhood W as in the above definition of the notion "puncture". Then the Riemann extension theorem implies that f extends through ϵ. In other words, $\lim_{x \to \epsilon} f(x) = a$ exists. Evidently a is contained in the closure of X_0 in $\mathbb{C}$. However, the boundary ∂X_0 is given by $\partial X_0 = \partial D \cup (\cup_n \partial B_n)$, and the openness of holomorphic maps implies that a cannot lie on either ∂D or on one of the sets ∂B_n. Therefore $a \in X_0$. Now choose contractible open neighbourhoods U of a in X_0 and V of ϵ in W such that $V \setminus \{\epsilon\} \subset \pi^{-1}(U)$. Since $\pi : X_1 \to X_0$ is an unramified covering and U is simply-connected, we obtain $\pi^{-1}(U) \simeq G \times U$ where G is equipped with the discrete topology. Being connected, $V^* = V \setminus \{\epsilon\}$ must be contained in one connected component of $\pi^{-1}(U)$. This implies the following: If a_n is a sequence in V^* such that $\lim \pi(a_n) = a$, then there is a point $p \in X_1$ with $\lim a_n = p$. But this contradicts the fact that by construction there is sequence a_n in V^* with $\lim a_n = \epsilon \notin X_1$ and $\lim \pi(a_n) = a$. Thus this case can be ruled out as well, i.e., there cannot exist an end which is a puncture. □

4 Proof of the theorem

Proof. Let a_n be a diverging sequence in $D = \{z \in \mathbb{C} : |z| < 1\}$ and $r_n \in \mathbb{R}^{>0}$ such that all the closed balls $B_n = \{z \in \mathbb{C} : |z - a_n| \leq r_n\}$ are disjoint subsets of D. Let $A = \cup_n B_n$. Then A is a closed subset of D and the fundamental group of $X_0 = D \setminus A$ is not finitely generated. Hence, by prop. 1 there is an unramified covering $\pi : X_1 \to X_0$ with an effective G-action on X_1. Let $\mathrm{Aut}_{\mathcal{O}}(X_1)$ denote the group of all holomorphic automorphisms of X_1 and A the group of all diffeomorphisms of X_1 which are either holomorphic or antiholomorphic. Again by prop. 1 we may assume that $\mathrm{Aut}_{\mathcal{O}}(X_1)$ is discrete and acting properly discontinuously. Since $\mathrm{Aut}_{\mathcal{O}}(X_1)$ is of finite index in A, the group A is likewise discrete and acting properly discontinuously on X_1.
Now, for every $g \in A \setminus \{e\}$ the fixed point set $X_1^g = \{x \in X_1 : g(x) = x\}$ is a nowhere dense real analytic subset of X_1. Hence $\Sigma = \cup_{g \in A \setminus \{e\}} X_1^g$ is a set of measure zero. In particular, $\Sigma \neq X_1$. Let $p \in X_1 \setminus \Sigma$, $S = G(p)$ and $X = X_1 \setminus S$.
Note that the conditions $p \in X_1 \setminus \Sigma$, $S = G(p)$ imply that $g(p) \notin S$ if $g \in A \setminus G$.

Therefore

$$G = \{g \in A : g(S) = S\}.$$

Let h be the unique hermitian metric of constant Gaussian curvature -1 on X and I its isomorphism group. We claim that $I = \mathrm{Aut}_{\mathcal{O}}(X) \simeq G$. To show this, it suffices to show that every holomorphic or antiholomorphic automorphism of X extends to a holomorphic or antiholomorphic automorphism of X_1. If ϕ is a holomorphic or antiholomorphic automorphism of X, it is in particular a self-homeomorphism and therefore extends to a homeomorphism $\bar{\phi}$ of the compact topological space $\bar{X} = X \cup e(X)$ (where $e(X)$ is the set of ends as explained in §3. above). Now $e(X) = e(X_1) \cup S$. Evidently every end of X given by a point of S is a puncture as defined in §3. On the other hand, due to prop. 2 none of the ends of X_1 is a puncture. Now $\bar{\phi}|_{e(X)}$ is a permutation of the elements of $e(X)$ which stabilizes the set of those ends which are punctures. Hence $\bar{\phi}(S) = S$. Thus ϕ extends to a continuous self-map of $X_1 = X \cup S$. However, a continuous map which is holomorphic or antiholomorphic everywhere except for some isolated points, is necessarily holomorphic resp. antiholomorphic everywhere (by Riemann extension theorem). Hence every $\phi \in I$ of X extends to a holomorphic or antiholomorphic automorphism of X_1. Consequently

$$\mathrm{Aut}_{\mathcal{O}}(X) = I = \{g \in A : g(S) = S\} = G.$$

□

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Lifting Galois Representations, and a Conjecture of Fontaine and Mazur

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Abstract. Mumford has constructed 4-dimensional abelian varieties with trivial endomorphism ring, but whose Mumford–Tate group is much smaller than the full symplectic group. We consider such an abelian variety, defined over a number field F, and study the associated p-adic Galois representation. For F sufficiently large, this representation can be lifted to $\mathbf{G}_m(\mathbf{Q}_p) \times \mathrm{SL}_2(\mathbf{Q}_p)^3$.

Such liftings can be used to construct Galois representations which are geometric in the sense of a conjecture of Fontaine and Mazur. The conjecture in question predicts that these representations should come from algebraic geometry. We confirm the conjecture for the representations constructed here.

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Introduction

A construction due to Mumford shows that the Mumford–Tate group of a polarized abelian variety $A/\mathbf{C}$ is not, in general, determined by the polarization and the endomorphism ring of A. This construction, which can be found in [Mum69], proves the existence of families of polarized abelian fourfolds where the general fibre has trivial endomorphism algebra but Mumford–Tate group much smaller than the full symplectic group of $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$.

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Previous papers, [Noo95] and [Noo00], brought to light some properties of an abelian variety A arising from this construction. For example, if A can be defined over a number field F, then it can be shown that, after replacing F by a finite extension, A has good reduction at all non-archimedean places of F and ordinary reduction at "most" of those places. Many of these results are derived from the property that the p-adic Galois representations associated to A factor through maps $\rho_p \colon \mathcal{G}_F \to G(\mathbf{Q}_p)$, where G is the Mumford–Tate group of A. They are therefore to a large extent results on the properties of Galois representations of "Mumford's type".

The present work continues the study of the Galois representations in question, in particular their lifting properties. In this respect, this paper is heavily inspired by Wintenberger's wonderful paper [Win95], even though the results proved therein are never actually used. In fact, the conditions of the main theorem (1.1.3) of that paper are not verified in the case considered here and, indeed, our conclusions are weaker as well.

Let G be the Mumford–Tate group of an abelian variety A over a number field F arising from Mumford's construction. It admits a central isogeny $\widetilde{G} \to G$, where the group $\widetilde{G}$ is (geometrically) a product:

$$\widetilde{G}_{\overline{\mathbf{Q}}} \cong \mathbf{G}_m \times \mathrm{SL}_2^3.$$

We fix a prime number p such that $\widetilde{G}$ decomposes over $\mathbf{Q}_p$ and we attempt to lift the p-adic Galois representation ρ associated to A along the isogeny $\widetilde{G}(\mathbf{Q}_p) \to G(\mathbf{Q}_p)$. This is possible after replacing F by a finite extension. The resulting lift $\tilde{\rho}$ naturally decomposes as a direct sum of a character and three 2-dimensional representations. These representations are studied in section 1. They have rather nice properties but for each one of them there exists a finite place of F where it is not potentially semi-stable. To be precise, the 2-dimensional representations are potentially unramified at all finite places of F of residue characteristic different from p, trivial at certain p-adic places of F but not potentially semi-stable (not even of Hodge–Tate type) at the other p-adic places.

In section 1, the properties of the lifting $\tilde{\rho}$ and its direct factors are studied without using the fact that ρ comes from an abelian variety A with Mumford–Tate group G. If we do use that fact, we can say more about the representations in question. This is the subject of section 2, where it is shown in particular how to distinguish the p-adic places where a given 2-dimensional direct factor of $\tilde{\rho}$ is trivial, from the p-adic places where it is not. The results of section 2 are used in section 3 to show that considering two abelian varieties A and A' of Mumford's kind, and taking a tensor product of two representations of the type considered above, one obtains representations which are potentially crystalline at all p-adic places of F and are potentially unramified at all other finite places of F. These representations are unramified outside a finite set of non-archimedean places of F. For a suitable choice of the varieties A and A', one obtains an irreducible representation.

According to a conjecture of Fontaine and Mazur ([FM95, conjecture 1], see section 0), such a representation should "come from algebraic geometry", i. e. be a subquotient of a Tate twist of an étale cohomology group of an algebraic variety over F. Our construction is geometric in some large sense, but not in the sense of the conjecture. We show in the beginning of section 4 that, in general, our representations do not arise (in the sense of Fontaine and Mazur) from the geometric objects having served in their construction.

It is therefore not clear if the conjecture holds in this case. Nevertheless, it is shown in section 4 that our representations do come from geometry, even from the geometry of abelian varieties, in the strong sense of Fontaine and Mazur. The construction of these abelian varieties is based on a construction going back to Shimura, cf. [Del72, §6].

In the final section, some generalizations of the construction of geometric representations are studied. It is shown in particular that any tensor product of the representations constructed in section 2 which is geometric comes from algebraic geometry.

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0 Notations and terminology

0.1 Notations. For any field F, we denote by $\bar{F}$ an algebraic closure of F and we write $\mathcal{G}_F = \mathrm{Gal}(\bar{F}/F) = \mathrm{Aut}_F(\bar{F})$, the group of F-automorphisms of $\bar{F}$. If $\bar{v}$ is a valuation of $\bar{F}$, then $\mathcal{I}_{F,\bar{v}} \subset \mathcal{D}_{F,\bar{v}} \subset \mathcal{G}_F$ are the inertia group and decomposition group of $\bar{v}$. If F is a local field, we write just $\mathcal{I}_F \subset \mathcal{G}_F$ for the inertia subgroup.

In the case of the field $\mathbf{Q}$ of rational numbers, we let $\overline{\mathbf{Q}}$ be its algebraic closure in the field of complex numbers $\mathbf{C}$. As usual, for any prime number p, $\mathbf{Q}_p$ is the p-adic completion of $\mathbf{Q}$ and $\mathbf{C}_p$ is the completion of $\overline{\mathbf{Q}}_p$.

0.2 Representations of Mumford's type. Let K be a field of characteristic 0, let G be an algebraic group over K and let V be a faithful K-linear representation of G. As in [Noo00, 1.2], we will say that the pair (G, V) is *of Mumford's type* if

- $\mathrm{Lie}(G)$ has one dimensional centre $\mathfrak{c}$,
- $\mathrm{Lie}(G)_{\bar{K}} \cong \mathfrak{c}_{\bar{K}} \oplus \mathfrak{sl}_{2,\bar{K}}^3$ and
- $\mathrm{Lie}(G)_{\bar{K}}$ acts on $V_{\bar{K}}$ by the tensor product of the standard representations.

We do not require G to be connected.

0.3 A conjecture of Fontaine and Mazur. Let F be a number field and let $\mathcal{G}_F = \mathrm{Aut}_F(\bar{F})$. Fix a prime number p and suppose that ρ is a finite dimensional $\mathbf{Q}_p$-linear representation of $\mathcal{G}_F$. Following Fontaine and Mazur in [FM95, §1], we will say that ρ is *geometric* if

- it is unramified outside a finite set of non-archimedean places of F and
- for each non-archimedean valuation $\bar{v}$ of $\bar{F}$, the restriction of ρ to the decomposition group $\mathcal{D}_{F,\bar{v}}$ is potentially semi-stable.

The meaning of the second condition depends on $\bar{v}$. If $\bar{v}(p) > 0$, then the notion of potential semi-stability is the one defined by Fontaine, see for example [Fon94]. If $\bar{v}(p) = 0$ then the fact that ρ is potentially semi-stable means that the restriction of ρ to the inertia group $\mathcal{I}_{F,\bar{v}}$ is quasi-unipotent.
By Grothendieck's theorem on semi-stability of p-adic Galois representations, the condition of potential semi-stability at $\bar{v}$ is automatically verified in the case where $\bar{v}(p) = 0$. We can therefore restrict our attention to valuations $\bar{v}$ with $\bar{v}|p$. The only fact needed in what follows is that a crystalline representation of $\mathcal{D}_{F,\bar{v}}$ is semi-stable and hence that a potentially crystalline representation is potentially semi-stable.
We say that a finite dimensional, irreducible, $\mathbf{Q}_p$-linear representation of $\mathcal{G}_F$ *comes from algebraic geometry* if it is isomorphic to a subquotient of a Tate twist of an étale cohomology group of an algebraic variety over F. We cite conjecture 1 from [FM95].

0.4. Conjecture (Fontaine–Mazur). *Let F be a number field. An irreducible p-adic representation of $\mathcal{G}_F$ is geometric if and only if it comes from algebraic geometry.*

0.5 The "if" part of the conjecture is true. A representation coming from a proper and smooth F-variety having potentially semi-stable reduction at all places of residue characteristic p is semi-stable by the C_{pst}-conjecture, [Fon94], proven by Tsuji in [Tsu99]. The general case of this part of the conjecture (for irreducible representations) follows by [dJ96]. The "only if" part is known for potentially abelian representations by [FM95, §6]. In [Tay00], Taylor proves the conjecture for representations of $\mathcal{G}_{\mathbf{Q}}$ with values in $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ satisfying some supplementary conditions. These results can be generalized to certain two-dimensional $\overline{\mathbf{Q}}_p$-linear representations of $\mathcal{G}_F$, for totally real number fields F.

0.6 Remark. Let F' be a finite extension of F. It follows from [FM95, §4], Remark (b) that the conjecture for representations of $\mathcal{G}_F$ is equivalent to the conjecture for representations of $\mathcal{G}_{F'}$.
For the implication which is still open, this can be seen as follows. Let ρ be a representation such that a Tate twist $\rho_{|\mathcal{G}_{F'}}(n)$ of $\rho_{|\mathcal{G}_{F'}}$ is a subquotient of an

étale cohomology group of an algebraic variety X over F'. Then

$$\mathrm{Ind}_{\mathcal{G}_{F'}}^{\mathcal{G}_F}\left(\rho_{|\mathcal{G}_{F'}}(n)\right)$$

is a subquotient of an étale cohomology group of $\mathrm{Res}_{F'/F}(X)$. The representation $\rho(n)$ is a subquotient of this induced representation.

1 Lifting Galois representations of Mumford's type

1.1 Let p be a prime number, $G_p/\mathbf{Q}_p$ an algebraic group and V_p a faithful representation of G_p such that (G_p, V_p) is of Mumford's type in the sense of 0.2. Throughout this section we will assume that F is a number field or a finite extension of $\mathbf{Q}_\ell$ for some prime number ℓ and that $\rho\colon \mathcal{G}_F \to G_p(\mathbf{Q}_p)$ is a polarizable, continuous representation. The condition that ρ is polarizable means, by definition, that there exists a non-degenerate, alternating, bilinear, $\mathcal{G}_F$-equivariant map $V \times V \to \mathbf{Q}_p(-1)$. Here $\mathbf{Q}_p(-1)$ is the 1-dimensional $\mathbf{Q}_p$-linear representation where $\mathcal{G}_F$ acts through χ^{-1}, the inverse of the cyclotomic character.
We will assume moreover that G_p is connected and that $\mathrm{Lie}(G_p) \cong \mathfrak{c} \oplus \mathfrak{sl}_2^3$. In this case, there exists a central isogeny

$$N\colon \widetilde{G}_p = \mathbf{G}_m \times \mathrm{SL}_2^3 \longrightarrow G_p$$

such that the induced representation of $\widetilde{G}_p$ on V_p is isomorphic to the tensor product $V_0 \otimes V_1 \otimes V_2 \otimes V_3$ of the standard representations of the factors.
We fix a $\mathcal{G}_F$-stable $\mathbf{Z}_p$-lattice $V_{\mathbf{Z}_p} \subset V_p$ such that there exists lattices $V_{i,\mathbf{Z}_p} \subset V_i$ with $V_{\mathbf{Z}_p} = \otimes_{i=1}^4 V_{i,\mathbf{Z}_p}$ and such that, on $V_{\mathbf{Z}_p}$, the polarization form takes values in $\mathbf{Z}_p(-1)$.

1.2 Definition. We say that $\rho(\mathcal{G}_F)$ is *sufficiently small* if it stabilizes $V_{\mathbf{Z}_p}$ and if all elements of $\rho(\mathcal{G}_F)$ are congruent to 1 mod p if $p > 2$ and congruent to 1 mod 4 if $p = 2$.

1.3 Remarks.

1.3.1 For any polarizable, continuous representation $\rho\colon \mathcal{G}_F \to G_p(\mathbf{Q}_p)$ as in the beginning of 1.1, there exist a lattice $V_{\mathbf{Z}_p}$ as in 1.1 and a finite extension $F' \supset F$ such that $\rho(\mathcal{G}_{F'})$ is sufficiently small.

1.3.2 If $\rho(\mathcal{G}_F)$ is sufficiently small, then it does not contain any non-trivial elements of finite order. This implies in particular that ρ is unramified at a finite place v of F of residue characteristic different from p if and only if it is potentially unramified at v.

1.4. Lemma. *If $\rho(\mathcal{G}_F)$ is sufficiently small, then ρ lifts uniquely to a continuous group homomorphism $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}_p(\mathbf{Q}_p)$ with the property that all elements of $\tilde{\rho}(\mathcal{G}_F)$ are congruent to* 1 mod p *if* $p > 2$ *and to* 1 mod 4 *if* $p = 2$.

Proof. The map $\mathrm{GL}_2^3 \to G_p$ obtained by taking the tensor product of the standard representations is surjective on $\mathbf{Q}_p$-valued points and even on $\mathbf{Q}_p$-valued points of GL_2^3 and G_p satisfying the above congruence condition. By Hensel's lemma, the determinant of an element of $\mathrm{GL}_2(\mathbf{Q}_p)$ which satisfies this congruence condition is the square of an element of $\mathbf{G}_m(\mathbf{Q}_p)$ satisfying the same condition, which proves that if $\rho(\mathcal{G}_F)$ is sufficiently small, then any element lifts to an element of $\widetilde{G}_p(\mathbf{Q}_p)$, congruent to 1 mod $2p$.
One has $\ker(N) = \{(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) \mid \epsilon_i = \pm 1, \epsilon_0\epsilon_1\epsilon_2\epsilon_3 = 1\}$. As the congruence condition implies that $-1 \not\equiv 1$, the lifting with this property is unique so we obtain the unique lift $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}_p(\mathbf{Q}_p)$ from the lemma, for the moment just a continuous map. The uniqueness of the lifting from $G_p(\mathbf{Q}_p)$ to $\widetilde{G}_p(\mathbf{Q}_p)$ implies that $\tilde{\rho}$ is a group homomorphism. □

1.5 DEFINITION. If $\rho(\mathcal{G}_F)$ is sufficiently small, then we call the lifting $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}_p(\mathbf{Q}_p)$ of the lemma the *canonical lifting* of ρ.

1.6 THE REPRESENTATIONS ρ_i. For any $\rho\colon \mathcal{G}_F \to G_p(\mathbf{Q}_p)$ and any lifting $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}_p(\mathbf{Q}_p)$, there exists a finite extension F' of F such that $\rho(\mathcal{G}_{F'})$ is sufficiently small and such that the restriction of $\tilde{\rho}$ to $\mathcal{G}_{F'}$ is its canonical lifting. Replacing F by such a finite extension F', we will assume in the sequel that this is already the case over F.
By composing $\tilde{\rho}$ with the projections on the factors of $\widetilde{G}_p(\mathbf{Q}_p)$, we get a character $\rho_0\colon \mathcal{G}_F \to \mathbf{G}_m(\mathbf{Q}_p)$ and representations $\rho_i\colon \mathcal{G}_F \to \mathrm{SL}_2(\mathbf{Q}_p)$ for $i = 1, 2, 3$. The facts that the image of ρ is sufficiently small and that $\tilde{\rho}$ is its canonical lift imply that the $\rho_i(\mathcal{G}_F)$ do not contain any elements of finite order other than the identity.
The following lemma is obvious.

1.7. LEMMA. *Let notations and hypotheses be as in 1.6. In particular, $\rho(\mathcal{G}_F)$ is sufficiently small and $\tilde{\rho}$ is its canonical lifting. If ρ is unramified at some finite place of F of residue characteristic different from p, then so are $\tilde{\rho}$ and the ρ_i.*

1.8. LEMMA. *Let ρ, $\tilde{\rho}$ and the ρ_i (for $i = 0, 1, 2, 3$) be as in 1.6 and lemma 1.7. Then the character ρ_0 satisfies $\rho_0^2 = \chi^{-1}$, where $\chi\colon \mathcal{G}_F \to \mathbf{Z}_p^*$ is the cyclotomic character.*

Proof. The condition that ρ is polarizable means that it respects a symplectic form up to a scalar and that the multiplier is the inverse of the cyclotomic character. This means that the determinant of ρ is χ^{-4}. The representation $\rho_1 \otimes \rho_2 \otimes \rho_3$ has trivial determinant, so the determinant of its product by ρ_0 is ρ_0^8 and hence $\rho_0^8 = \chi^{-4}$. The lemma follows because $\rho_0(\mathcal{G}_F)$ and $\rho(\mathcal{G}_F)$ and hence $\chi(\mathcal{G}_F)$ are congruent to 1 mod $2p$. □

1.9. Proposition. *Under the assumptions of the lemmas, let v be a p-adic valuation of F and let $\mathcal{I} = \mathcal{I}_{F,\bar{v}}$ be the inertia group of a valuation $\bar{v}$ of $\bar{F}$ lying over v. Assume that the restriction of ρ to $\mathcal{I}$ is Hodge–Tate (resp. crystalline). For $i \in \{1, 2, 3\}$, either*

- *the restriction of ρ_i to $\mathcal{I}$ is trivial, or*
- *the restriction of $\rho_0 \cdot \rho_i$ to $\mathcal{I}$ is Hodge–Tate (resp. crystalline).*

Proof. One has an isomorphism $G_p^{\mathrm{ad}}(\mathbf{Q}_p) \cong \mathrm{PSL}_2(\mathbf{Q}_p)^3$ and it can be shown as in [Noo00, 3.5] that the projection of $\rho(\mathcal{I})$ on exactly two of the factors is trivial. We sketch the argument.
Let $\mu_{\mathrm{HT}} \colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{p,\mathbf{C}_p}$ be the cocharacter associated to the Hodge–Tate decomposition of $\rho_{|\mathcal{I}}$. As the kernel of the isogeny $N \colon \widetilde{G}_p \to G_p$ is annihilated by 2, the square

$$\mu_{\mathrm{HT}}^2 \colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{p,\mathbf{C}_p}$$

lifts to a cocharacter $\tilde{\mu} \colon \mathbf{G}_{m,\mathbf{C}_p} \to \widetilde{G}_{p,\mathbf{C}_p}$. One has $\tilde{\mu} = (\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)$, where $\tilde{\mu}_0 \colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathbf{G}_{m,\mathbf{C}_p}$ and $\tilde{\mu}_i \colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathrm{SL}_{2,\mathbf{C}_p}$ for $i = 1, 2, 3$. Since $V_{p,\mathbf{C}_p}$ is the direct sum of two eigenspaces for μ_{HT} and hence for $\tilde{\mu}$, exactly one of the maps $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3$ is non-trivial. The Zariski closure of the image of μ_{HT} therefore projects non-trivially on exactly one of the factors of $G_{p,\mathbf{Q}_p}^{\mathrm{ad}} \cong (\mathrm{PSL}_{2,\mathbf{Q}_p})^3$. The theorem of Sen, [Ser78, Théorème 2], implies the statement concerning the projection of $\rho(\mathcal{I})$.
To end the proof, we can assume that the projection of $\rho(\mathcal{I})$ on the first factor of $G_p^{\mathrm{ad}}(\mathbf{Q}_p)$ is non-trivial. The hypotheses that $\rho(\mathcal{G}_F)$ is sufficiently small and that $\tilde{\rho}$ is the canonical lifting of ρ imply that $(\rho_2)_{|\mathcal{I}}$ and $(\rho_3)_{|\mathcal{I}}$ are trivial. Since the tensor product $\rho_{|\mathcal{I}} = (\rho_0\rho_1)_{|\mathcal{I}} \otimes (\rho_2 \otimes \rho_3)_{|\mathcal{I}}$ is Hodge–Tate (resp. crystalline), this implies that $(\rho_0\rho_1)_{|\mathcal{I}}$ is Hodge–Tate (resp. crystalline) as well. ☐

1.10 Definition. Notations being as above, let $1 \leq i \leq 3$ and let $\bar{v}$ be a p-adic valuation of $\bar{F}$. We say that $(\rho_i)_{|\mathcal{I}_{F,\bar{v}}}$ is *of the first kind* if $(\rho_0\rho_i)_{|\mathcal{I}_{F,\bar{v}}}$ is potentially crystalline. We say that $(\rho_i)_{|\mathcal{I}_{F,\bar{v}}}$ is *of the second kind* if its image is trivial.

1.11 Remark. Let (G_p, V_p) be of Mumford's type as in 1.1, let F be a number field and assume that A/F is a polarizable abelian fourfold such that, for some identification $V_p = \mathrm{H}^1_{\mathrm{\acute{e}t}}(A_{\bar{F}}, \mathbf{Q}_p)$, the Galois representation of $\mathcal{G}_F$ on V_p factors through a map $\rho \colon \mathcal{G}_F \to G_p(\mathbf{Q}_p)$. Then ρ is a representation of the type considered in 1.1. This is the setting in which the results of this section will be applied.
Abelian varieties with this property exist by [Mum69, §4] and [Noo95, 1.7]. By [Noo00, 2.2], such varieties have potentially good reduction at all non-archimedean places of F. This implies that for each p-adic place $\bar{v}$ of $\bar{F}$, the restriction of ρ to the decomposition group $\mathcal{D}_{F,\bar{v}} \subset \mathcal{G}_F$ is potentially crystalline. At all other finite places, ρ is potentially unramified in any case and unramified if $\rho(\mathcal{G}_F)$ is sufficiently small.

2 Galois representations of abelian varieties of Mumford's type

2.1 Fix an abelian fourfold A over a number field F and an embedding $F \subset \mathbf{C}$. Let $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and let $G \subset \mathrm{GL}(V)$ be an algebraic subgroup such that (G, V) is of Mumford's type, with G connected. We assume that the morphism $h\colon S \to \mathrm{GL}(V \otimes \mathbf{R})$ determining the Hodge structure of $A_{\mathbf{C}}$ factors through $G_{\mathbf{R}}$. This condition is equivalent to the condition that the Mumford–Tate group of $A_{\mathbf{C}}$ is contained in G.
Such abelian varieties exist by [Mum69, §4], and [Noo95] implies that there also exist abelian varieties of this type with the additional property that the Mumford–Tate group is equal to G. It is explained in [Noo01, 1.5] that any abelian variety satisfying the above condition whose Mumford–Tate group is equal to G can be obtained as a fibre of one of the families constructed in [Mum69, §4]. As in [Noo01] one can draw the following conclusions.
There is a central isogeny $\widetilde{G} \to G$, where $\widetilde{G} = \mathbf{G}_m \times \widetilde{G}^{\mathrm{der}}$ and $\widetilde{G}^{\mathrm{der}}$ is an algebraic group over $\mathbf{Q}$ such that $\widetilde{G}^{\mathrm{der}}_{\overline{\mathbf{Q}}} \cong \mathrm{SL}^3_{2,\overline{\mathbf{Q}}}$. The group $\widetilde{G}^{\mathrm{der}}$ can be described in the following way. There exist a totally real cubic number field K and a quaternion division algebra D with centre K, with $\mathrm{Cor}_{K/\mathbf{Q}}(D) \cong \mathrm{M}_8(\mathbf{Q})$ and with $D \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathrm{M}_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H}$ such that

$$\widetilde{G}^{\mathrm{der}} = \{x \in D \mid \mathrm{Nrd}(x) = 1\},$$

considered as an algebraic group over $\mathbf{Q}$. Here $\mathbf{H}$ is the algebra of real quaternions and $\mathrm{Nrd}\colon D \to K$ is the reduced norm. The group G can be identified with the image of $D^{\times}$ in $\mathrm{Cor}_{K/\mathbf{Q}}(D) \cong \mathrm{M}_8(\mathbf{Q})$ under the norm map $N\colon D \to \mathrm{Cor}_{K/\mathbf{Q}}(D)$ and therefore has a natural 8-dimensional representation. This representation is isomorphic to V. Note that for any field L containing a normal closure of K, one has

$$\widetilde{G}^{\mathrm{der}}_L \cong \prod_{\varphi\colon K \hookrightarrow L} G_{\varphi},$$

where each G_{φ} is an L-form of SL_2.
In what follows, we will refer to K and D as the number field and the division algebra *associated to G*.

2.2 The Hodge decomposition on $V_{\mathbf{C}} = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{C}) \cong \mathrm{H}^1_{\mathrm{dR}}(A_{\mathbf{C}}/\mathbf{C})$ is determined by a cocharacter

$$\mu_{\mathrm{HdR}}\colon \mathbf{G}_{m,\mathbf{C}} \longrightarrow G_{\mathbf{C}}$$

such that $\mathbf{G}_m$ acts by the character $\cdot^i$ on the subspace $H^{i,1-i} \subset \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{C})$. Let $C_{\mathrm{HdR}} \subset G_{\mathbf{C}}$ be the conjugacy class of μ_{HdR}. It follows from [Noo01, 1.2] (and is easy to check) that the field of definition in $\mathbf{C}$ of C_{HdR} is isomorphic to K. In what follows we will identify K with this subfield of $\mathbf{C}$ and thus assume that $K \subset \mathbf{C}$. One deduces that C_{HdR} can be defined over $F \subset \mathbf{C}$ if and only if $K \subset F$, and hence that, up to replacing F by a finite extension, one can

assume that C_{HdR} can be defined over F. From now on, we will assume that this is the case.
The arguments used in the proof of proposition 1.9 also apply to μ_{HdR}. Thus μ_{HdR}^2 lifts to a map $\tilde{\mu}\colon \mathbf{G}_{m,\mathbf{C}} \to \widetilde{G}_{\mathbf{C}}$. As in *loc. cit.*, the fact that there are two eigenspaces in $V_{\mathbf{C}}$ for the action of $\mathbf{G}_m$ implies that the projection of $\tilde{\mu}$ on one and only one factor $\mathrm{SL}_{2,\mathbf{C}}$ of $\widetilde{G}_{\mathbf{C}}$ is non trivial. This implies that C_{HdR} projects non-trivially on exactly one factor $\mathrm{PSL}_{2,\mathbf{C}}$ of $G^{\mathrm{ad}}_{\mathbf{C}}$, namely the factor corresponding to the embedding $K \subset \mathbf{C}$ fixed above.

2.3 Let p be a prime number such that $\widetilde{G}_{\mathbf{Q}_p} \cong \mathbf{G}_m \times \mathrm{SL}_2^3$. If K and D are the number field and the division algebra associated to G, this condition is equivalent to p being completely split in K and D being split at all places of K above p. The factors SL_2 of the above product correspond to the p-adic valuations of K. As $K \subset F$ by assumption, all Hodge classes on $A_{\mathbf{C}}$ that are invariant for the $G(\mathbf{Q})$-action on the rational cohomology are defined over F, so the Galois representation associated to A factors through a map $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$. This implies that we find ourselves in the situation of 1.1, with F a number field and with $G_{\mathbf{Q}_p}$ playing the role of the group G_p.
After further enlarging F, the image of ρ is sufficiently small. The constructions of 1.6 provide a character $\rho_0\colon \mathcal{G}_F \to \mathbf{G}_m(\mathbf{Q}_p)$ and representations $\rho_1, \rho_2, \rho_3\colon \mathcal{G}_F \to \mathrm{SL}_2(\mathbf{Q}_p)$. The above remarks give rise to a bijective correspondence between the ρ_i $(i = 1, 2, 3)$ and the p-adic valuations v_1, v_2, v_3 of K, with ρ_i corresponding to v_i.

2.4 We summarize the notations and hypotheses in effect at this point.

- (G, V) is a pair of Mumford's type (as defined in 0.2) with G connected.
- K is the number field associated to G.
- p is a prime number which splits completely in K and such that the group G is split at p.
- $F \subset \mathbf{C}$ is a number field A/F an abelian variety.
- $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ is an identification such that the Mumford–Tate group of A is contained in G.
- Inclusions $K \subset F \subset \mathbf{C}$ are fixed such that the field of definition in $\mathbf{C}$ of the conjugacy class $C_{\mathrm{HdR}} \subset G_{\mathbf{C}}$ of μ_{HdR} is equal to K.
- The image of the Galois representation $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ is sufficiently small.

2.5. Proposition. *Under the above hypotheses, let v be a p-adic valuation of F and $\mathcal{I} = \mathcal{I}_{F,\bar{v}}$ be the inertia group of a valuation $\bar{v}$ of $\bar{F}$ lying over v. Suppose that $i \in \{1, 2, 3\}$. Then $(\rho_i)_{|\mathcal{I}}$ is of the first kind if and only if $v_{|K} = v_i$.*

Proof. We already saw (cf. proposition 1.9 and its proof) that the projection of $\rho(\mathcal{I})$ on one and only one factor of $G^{\mathrm{ad}}(\mathbf{Q}_p)$ is non-trivial. This is the factor corresponding to v_i if and only if $(\rho_i)_{|\mathcal{I}}$ is of the first kind. After renumbering the v_i and the ρ_i, we can assume that $v_{|K} = v_1$ and it suffices to show that the projection of $\rho(\mathcal{I})$ on the factor of $G^{\mathrm{ad}}(\mathbf{Q}_p)$ corresponding to v_1 is non-trivial. By 2.2, the conjugacy class $C_{\mathrm{HdR}} \subset G_{\mathbf{C}}$ projects non-trivially to exactly one factor $\mathrm{PSL}_{2,\mathbf{C}}$ of

$$G^{\mathrm{ad}}_{\mathbf{C}} = \prod_{K \hookrightarrow \mathbf{C}} \mathrm{PSL}_{2,\mathbf{C}}.$$

Since the field of definition of C_{HdR} is equal to K, this must be the factor corresponding to the embedding $K \subset F \subset \mathbf{C}$ fixed in 2.2 (cf. 2.4).
Let $\iota\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ be an embedding such that the composite

$$F \subset \overline{\mathbf{Q}} \xrightarrow{\ \iota\ } \mathbf{C}_p$$

induces the valuation v on F. For each K-variety Y, we denote by $Y \otimes_K \mathbf{C}_p$ the base extension of Y to $\mathbf{C}_p$ via the embedding induced by ι and the inclusion $K \subset \overline{\mathbf{Q}}$.
The Hodge–Tate decomposition associated to $\rho_{|\mathcal{I}}$ is determined by a cocharacter $\mu_{\mathrm{HT}}\colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{\mathbf{C}_p}$ and we let C_{HT} be its conjugacy class in $G_{\mathbf{C}_p}$. Since [Bla94, theorem 0.3] implies conjecture 1 of [Win88], it follows from [Win88, proposition 7] that $C_{\mathrm{HdR}} \otimes_K \mathbf{C}_p = C_{\mathrm{HT}}$, as subvarieties of $(G_K) \otimes_K \mathbf{C}_p = G_{\mathbf{C}_p}$. The conjugacy class C_{HT} thus projects non-trivially onto the factor of $G^{\mathrm{ad}}_{\mathbf{C}_p}$ corresponding to the inclusion ι. As $\iota\colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ restricts to an inclusion $K \hookrightarrow \mathbf{Q}_p$ inducing the valuation v_1 on K, it follows from proposition 1.9 and its proof that $\rho(\mathcal{I})$ projects non-trivially to the factor of $G^{\mathrm{ad}}_{\mathbf{Q}_p}$ corresponding to v_1. □

2.6 The special case. We keep the above notations. In this section, it was not assumed that the Mumford–Tate group of A is equal to G. We investigate the case where these two groups are not equal, so let $T \subset G$ the Mumford–Tate group of A and assume that $T \neq G$. As T is a reductive $\mathbf{Q}$-group containing the scalars, it has to be a torus of G. This means that A corresponds to a special point of an appropriate Shimura variety associated to (G, X), where X denotes the $G(\mathbf{R})$-conjugacy class of the morphism $h\colon S \to G_{\mathbf{R}}$ determining the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{R})$.
It follows from [Noo01, §3] that there exists a totally imaginary extension L of K which splits D (with D the division algebra associated to G as in 2.1) and such that T is the image in G of $L^\times \subset D^\times$. If $\widetilde{T} \subset \widetilde{G}$ denotes the connected component of the inverse image of T, one has $\widetilde{T} = \mathbf{G}_m \times (\widetilde{G}^{\mathrm{der}} \cap L^\times)$. There is an identification

$$\widetilde{G}^{\mathrm{der}} \cap L^\times = \{x \in L^\times \mid x\bar{x} = 1\},$$

where the right hand side is considered as an algebraic group over $\mathbf{Q}$. Again by [Noo01, §3], the field L is the reflex field of the CM-type of A.

Assume that p is completely split in L. Then

$$\widetilde{T}_{\mathbf{Q}_p} \cong \mathbf{G}_m \times (\mathbf{G}_m)^3,$$

where the first factor lies in the centre of $\widetilde{G}$ and the other factors identify with maximal tori of the factors $\mathrm{SL}_{2,\mathbf{Q}_p}$ of $\widetilde{G}_{\mathbf{Q}_p}$. In this case, the representations ρ_1, ρ_2 and ρ_3 each decompose as a direct sum of two 1-dimensional representations given by opposite characters, so $\tilde{\rho}$ decomposes as a sum of 1-dimensional representations.

3 Construction of geometric Galois representations

3.1 Fix two representations (G, V) and (G', V') of Mumford's type and assume that the number field associated (as in 2.1) to both groups G and G' is the same totally real cubic field K. We will assume that both G and G' are connected. Also fix a number field $F \subset \mathbf{C}$, two 4-dimensional abelian varieties A/F and A'/F and identifications $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and $V' = \mathrm{H}^1_{\mathrm{B}}(A'(\mathbf{C}), \mathbf{Q})$ such that the Mumford–Tate groups of A and A' are contained in G and G' respectively. In analogy with 2.2, let $\mu_{\mathrm{HdR}}\colon \mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{C}}$ (resp. $\mu'_{\mathrm{HdR}}\colon \mathbf{G}_{m,\mathbf{C}} \to G'_{\mathbf{C}}$) be the morphism determining the Hodge decomposition on $V \otimes \mathbf{C}$ (resp. $V' \otimes \mathbf{C}$) and let C_{HdR} (resp. C'_{HdR}) be the conjugacy class of μ_{HdR} (resp. μ'_{HdR}). We assume that the field of definition in $\mathbf{C}$ of C_{HdR} is equal to that of C'_{HdR} and we identify K with this field, just as in 2.2. Replacing F by a finite extension if necessary, we will assume that $K \subset F$.
Let D and D' be the division algebras associated to G and G' respectively and assume that p is a prime number which splits completely in K and such that

$$D \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong D' \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong \mathrm{M}_2(\mathbf{Q}_p)^3.$$

As in 2.3, the Galois representations associated to A and A' factor through morphisms $\rho\colon \mathcal{G}_F \longrightarrow G(\mathbf{Q}_p)$ and $\rho'\colon \mathcal{G}_F \longrightarrow G'(\mathbf{Q}_p)$. After replacing F by a finite extension, we may assume that ρ and ρ' have sufficiently small image in the sense of 1.2.
Let v_1, v_2 and v_3 be the p-adic valuations of K and order the factors of

$$\widetilde{G}_{\mathbf{Q}_p} \cong \mathbf{G}_{m,\mathbf{Q}_p} \times \prod_{K \hookrightarrow \mathbf{Q}_p} \mathrm{SL}_{2,\mathbf{Q}_p} \cong \widetilde{G}'_{\mathbf{Q}_p}$$

in such a way that the ith factor SL_2 in each product corresponds to the embedding $K \hookrightarrow \mathbf{Q}_p$ inducing v_i. Applying the construction of 1.6 to ρ and ρ', we obtain a character $\rho_0 = \rho'_0$ and Galois representations

$$\rho_i, \rho'_i\colon \mathcal{G}_F \longrightarrow \mathrm{SL}_2(\mathbf{Q}_p)$$

for $i = 1, 2, 3$. We will consider the representation

$$\tau = \rho_1 \otimes \rho'_1\colon \mathcal{G}_F \longrightarrow \mathrm{SL}_2(\mathbf{Q}_p)^2 \longrightarrow \mathrm{GL}_4(\mathbf{Q}_p), \tag{3.1.$*$}$$

where the last arrow is defined by the action of $\mathrm{SL}_2(\mathbf{Q}_p)^2$ on $\mathbf{Q}_p^2 \otimes \mathbf{Q}_p^2 \cong \mathbf{Q}_p^4$.

3.2 Let $\bar{v}$ be a p-adic place of $\bar{F}$ and let $\mathcal{I} = \mathcal{I}_{\bar{F},\bar{v}} \subset \mathcal{G}_F$ be the corresponding inertia group. If the restriction of $\bar{v}$ to K is equal to v_1, then the representations $(\rho_1)_{|\mathcal{I}}$ and $(\rho'_1)_{|\mathcal{I}}$ are both of the first kind by proposition 2.5. It follows that the tensor product $(\rho_1 \otimes \rho'_1)_{|\mathcal{I}}$ is potentially crystalline, as it is the twist by χ of the tensor product of the two potentially crystalline representations $\rho_0\rho_1$ and $\rho_0\rho'_1$.
If the restriction of $\bar{v}$ to K is different from v_1, then the representations $(\rho_1)_{|\mathcal{I}}$ and $(\rho'_1)_{|\mathcal{I}}$ are both of the second kind by proposition 2.5. In this case, the condition that $\rho(\mathcal{G}_F)$ and $\rho'(\mathcal{G}_F)$ are sufficiently small implies that $(\rho_1 \otimes \rho'_1)_{|\mathcal{I}}$ is trivial. It follows that $(\rho_1 \otimes \rho'_1)_{|\mathcal{I}}$ is potentially crystalline in this case as well. In both cases, it follows from [Fon94, 5.1.5] that $(\rho_1 \otimes \rho'_1)_{|\mathcal{D}_{F,\bar{v}}}$ is potentially crystalline. This proves the following theorem.

3.3. Theorem. *Assume that the hypotheses of 2.4 are verified for $K \subset F \subset \mathbf{C}$, A/F, G, p and $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ and for $K \subset F \subset \mathbf{C}$, A'/F, G', p and $\rho'\colon \mathcal{G}_F \to G'(\mathbf{Q}_p)$. Let*

$$\tau = \rho_1 \otimes \rho'_1 \colon \mathcal{G}_F \longrightarrow \mathrm{SL}_2(\mathbf{Q}_p)^2 \longrightarrow \mathrm{GL}_4(\mathbf{Q}_p)$$

be the representation of (3.1.∗). Then τ is a geometric Galois representation in the sense of 0.3. If F is sufficiently large then τ has good reduction everywhere.

3.4. Proposition. *Let p, K, G and G' be as in 3.1, subject to the condition that the associated division algebras D and D' both satisfy*

$$D \otimes \mathbf{R} \cong D' \otimes \mathbf{R} \cong \mathrm{M}_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H}$$

and are both split at the same real place of K. Then one can choose the abelian varieties A and A' used in the construction of τ in 3.1 such that $\mathrm{Lie}(\tau(\mathcal{G}_F)) = \mathfrak{sl}_2^2$. *In that case τ is an irreducible representation of $\mathcal{G}_F$.*

Proof. From the surjections $\mathbf{G}_{m,\mathbf{Q}} \times G^{\mathrm{der}} \to G$ and $\mathbf{G}_{m,\mathbf{Q}} \times (G')^{\mathrm{der}} \to G'$ one deduces a surjection $\mathbf{G}_m^2 \times G^{\mathrm{der}} \times (G')^{\mathrm{der}} \to G \times G'$. Let $\Delta \subset \mathbf{G}_m^2$ be the diagonal and let $G'' \subset G \times G'$ be the image of $\Delta \times G^{\mathrm{der}} \times (G')^{\mathrm{der}}$. The representations V and V' and the projections $G'' \to G$ and $G'' \to G'$ induce a representation of G'' on $V'' = V \oplus V'$. The representations V and V' both carry a bilinear form which is G- resp. G'-invariant up to a scalar. These forms are also G''-invariant up to a scalar and their multipliers are equal. It follows that V'' can be endowed with a bilinear form, G''-invariant up to scalars.
Fix isomorphisms $D \otimes \mathbf{R} \cong \mathrm{M}_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H} \cong D' \otimes \mathbf{R}$. Any pair of morphisms $h_D \colon S \to D_{\mathbf{R}}^{\times}$ and $h_{D'} \colon S \to (D')_{\mathbf{R}}^{\times}$ both conjugate to

$$\begin{aligned} S &\longrightarrow \mathrm{GL}_{2,\mathbf{R}} \times \mathbf{H}^{\times} \times \mathbf{H}^{\times} \\ z = a + bi &\mapsto \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, 1, 1 \right) \end{aligned}$$

for the above identifications defines a morphism $h''\colon S \to G''_{\mathbf{R}}$. As in [Mum69, §4], one shows that these data define the Hodge structure of an abelian variety. One can choose h_D and $h_{D'}$ in such a way that the image of h'' is Zariski dense in G'', i. e. is not contained in a proper subgroup of G'' defined over $\mathbf{Q}$. This implies that there exists an abelian variety $A''_{\mathbf{C}}$ over $\mathbf{C}$ with Mumford–Tate group equal to G'' and such that the representation of its Mumford–Tate group on $\mathrm{H}^1_{\mathrm{B}}(A''_{\mathbf{C}}(\mathbf{C}), \mathbf{Q})$ is isomorphic to the representation of G'' on V''.
It follows from [Noo95, 1.7] and its proof that there exists an abelian variety A'' over a number field $F'' \subset \mathbf{C}$ with Mumford–Tate group G'', such that the representation of its Mumford–Tate group on $\mathrm{H}^1_{\mathrm{B}}(A''(\mathbf{C}), \mathbf{Q})$ is isomorphic to (G'', V'') and with the property that that the image of the Galois representation $\rho''\colon \mathcal{G}_{F''} \to G''(\mathbf{Q}_p)$ is open. By construction of the representation (G'', V'') as a direct sum, A'' is isogenous to a product $A \times A'$, where the Mumford–Tate groups of A and A' are G and G' respectively. The image of

$$(\rho, \rho')\colon \mathcal{G}_F \longrightarrow G(\mathbf{Q}_p) \times G'(\mathbf{Q}_p)$$

is open in $G''(\mathbf{Q}_p)$. This implies that the image of $\tau\colon \mathcal{G}_F \to \mathrm{GL}_4(\mathbf{Q}_p)$ is open in $H(\mathbf{Q}_p)$, where $H \subset \mathrm{GL}_{4,\mathbf{Q}_p}$ is the image of $(\mathrm{SL}_{2,\mathbf{Q}_p})^2$ acting on $\mathbf{Q}_p^2 \otimes \mathbf{Q}_p^2 \cong \mathbf{Q}_p^4$ by the tensor product of the standard representations. Since the representation of H on $\mathbf{Q}_p^4$ is irreducible, the same thing is true for τ. □

3.5 Remark. As noted in 2.1, the condition that

$$D \otimes \mathbf{R} \cong D' \otimes \mathbf{R} \cong \mathrm{M}_2(\mathbf{R}) \times \mathbf{H} \times \mathbf{H}$$

is equivalent to the conditions that there are abelian varieties A and A' such that D and D' are the algebras associated to the Mumford–Tate groups of A and A' respectively.

3.6 A special case. Let us return to the notations of theorem 3.3, i. e. we assume that the hypotheses of 2.4 are verified for $K \subset F \subset \mathbf{C}$, A/F, G, p and $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ and for $K \subset F \subset \mathbf{C}$, A'/F, G', p and $\rho'\colon \mathcal{G}_F \to G'(\mathbf{Q}_p)$.
From now on we moreover assume that the Mumford–Tate group of A is equal to G whereas that of A' is strictly contained in G'. The condition on A implies that $\mathrm{End}(A_{\bar{F}}) = \mathbf{Z}$. Since the representation ρ is semisimple, as $\rho(\mathcal{G}_F) \subset G(\mathbf{Q}_p)$ and because

$$\mathrm{End}_{\mathcal{G}_F}\left(\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)\right) = \mathbf{Q}_p,$$

one easily deduces that $\rho(\mathcal{G}_F)$ is open in $G(\mathbf{Q}_p)$.
As was seen in 2.6, the condition on A' implies that, after replacing F by a finite extension again if necessary, A' is of CM-type and its Mumford–Tate group is a maximal torus $T' \subset G'$. Let L' be the reflex field of the CM-type of A'. The assumption that F is sufficiently large for the Mumford–Tate group of A' to be a torus is equivalent to the condition that $L' \subset F$. By 2.6, the map

$\tilde{\rho}' \colon \mathcal{G}_F \to \widetilde{G}'(\mathbf{Q}_p)$ factors through $\widetilde{T}'(\mathbf{Q}_p)$, where $\widetilde{T}' \subset \widetilde{G}'$ is a maximal torus which identifies with a subgroup of $\mathbf{G}_m \times (L')^\times$.

Suppose that p splits completely in L'. Then the representation ρ_1' decomposes as a direct sum of two opposite characters ψ_1 and ψ_1^{-1} and it follows from 3.3 that the product $\psi_1\rho_1$ is geometric with potentially good reduction everywhere. By construction, the image of $\psi_1\rho_1$ is Zariski dense in GL_2, so $\psi_1\rho_1$ is an irreducible representation. Note that, as F contains the reflex field L' of the CM-type of A' (cf. 2.6), it is a totally imaginary field. We have thus constructed a 2-dimensional geometric representation of the absolute Galois group of a totally imaginary field F.

Of course, the same statement holds for $\psi_1^{-1}\rho_1$ and similar constructions can be carried out using the decompositions of ρ_2' as $\psi_2 \oplus \psi_2^{-1}$ and of ρ_3' as $\psi_3 \oplus \psi_3^{-1}$. All the above statements are true for the 6 representations $\psi_i^{\pm 1}\rho_i$ (for $i = 1, 2, 3$). The representation that will be of interest in section 4 is the direct sum

$$\sigma = \psi_1\rho_1 \oplus \psi_1^{-1}\rho_1 \oplus \psi_2\rho_2 \oplus \psi_2^{-1}\rho_2 \oplus \psi_3\rho_3 \oplus \psi_3^{-1}\rho_3. \tag{3.6.$*$}$$

Let $T_{L'}$ be the kernel of the map $(L')^\times \to K^\times$ induced by the field norm and define $T_K = K^\times \cap T_{L'}$, seen as group schemes over $\mathbf{Q}$. This implies that $T_{L'} = \widetilde{T}' \cap (\widetilde{G}')^{\mathrm{der}}$ and $T_K = \{x \in K^\times \mid x^2 = 1\}$. Let H' be the algebraic $\mathbf{Q}$-group defined by the short exact sequence

$$1 \longrightarrow T_K \xrightarrow{x \mapsto (x, x^{-1})} \widetilde{G}^{\mathrm{der}} \times T_{L'} \longrightarrow H' \longrightarrow 1. \tag{3.6.$\dagger$}$$

The Galois representation on $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$ factors through $(G \times T'/\mathbf{G}_m)(\mathbf{Q}_p)$, where

$$\begin{aligned} \mathbf{G}_{m,\mathbf{Q}} &\longrightarrow G \times T' \\ z &\mapsto (z \cdot \mathrm{id}, z^{-1} \cdot \mathrm{id}). \end{aligned}$$

The maps $\widetilde{G}^{\mathrm{der}} \to G$ and $T_{L'} \to T'$ induce a map $H' \to (G \times T'/\mathbf{G}_m)$ and the representation σ defined in (3.6.$*$) is a lifting to $H'(\mathbf{Q}_p)$ of the representation of $\mathcal{G}_F$ on the Tate twist

$$\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1).$$

The facts that $\rho(\mathcal{G}_F)$ is open in $G(\mathbf{Q}_p)$ and that $\rho'(\mathcal{G}_F)$ is open in $T'(\mathbf{Q}_p)$ imply that $\sigma(\mathcal{G}_F)$ is open in $H'(\mathbf{Q}_p)$.

4 The geometric origin

4.1. Proposition. *Let $\tau \colon \mathcal{G}_F \to \mathrm{GL}_4(\mathbf{Q}_p)$ be a representation as in theorem 3.3 and assume that* $\mathrm{Lie}(\mathrm{im}(\tau)) \cong \mathfrak{sl}_2^2$ *(resp. let $\psi_i^{\pm 1}\rho_i$ be as in 3.6). Let A/F and A'/F be the abelian varieties serving in the construction of τ (resp. $\psi_i^{\pm 1}\rho_i$). Then there do not exist $n, m \in \mathbf{Z}$ such that τ (resp. $\psi_i^{\pm 1}\rho_i$) is isomorphic to a subquotient of a Tate twist of*

$$\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)^{\otimes n} \otimes \mathrm{H}^1_{\text{ét}}(A'_{\bar{F}}, \mathbf{Q}_p)^{\otimes m}.$$

Proof. We first give the proof for $\psi_1\rho_1$, the other $\psi_i^{\pm}\rho_i$ are handled by identical arguments.
Assume that $n, m \in \mathbf{Z}$ such that $\psi_1\rho_1$ is isomorphic to a subquotient of a Tate twist of $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)^{\otimes n} \otimes \mathrm{H}^1_{\text{ét}}(A'_{\bar{F}}, \mathbf{Q}_p)^{\otimes m}$. Then ρ_1 is a twist by a character of a subquotient of $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)^{\otimes n}$ and hence a twist by a character of a subquotient of

$$\rho_1^{\otimes n} \otimes \rho_2^{\otimes n} \otimes \rho_3^{\otimes n}.$$

It follows from 3.6 that $\tilde{\rho}(\mathcal{G}_F)$ is open in $\mathbf{G}_m(\mathbf{Q}_p) \times \mathrm{SL}_2(\mathbf{Q}_p)^3$. The representation theory of SL_2^3 therefore implies that $\rho_1^{\otimes n}$ contains an irreducible factor isomorphic to ρ_1 and that $\rho_2^{\otimes n}$ and $\rho_3^{\otimes n}$ both contain an invariant line. The first condition implies that n is even and the second one that n is odd, a contradiction which proves the proposition.
To prove the result for τ, we use the notations of theorem 3.3. The condition on $\mathrm{im}(\tau)$ implies that $\rho(\mathcal{G}_F)$ and $\rho'(\mathcal{G}_F)$ are not commutative, so G and G' are the Mumford–Tate groups of A and A' respectively. As noted in 3.6, this implies that $\rho(\mathcal{G}_F)$ (resp. $\rho'(\mathcal{G}_F)$) is open in $G(\mathbf{Q}_p)$ (resp. $G'(\mathbf{Q}_p)$).
The Mumford–Tate group of $A \times A'$ is the group $G'' \subset G \times G'$ introduced in the proof of proposition 3.4, so $(\rho, \rho')(\mathcal{G}_F) \subset G''(\mathbf{Q}_p)$. The Zariski closure H_p of the image of (ρ, ρ') is a reductive algebraic subgroup of $G''_{\mathbf{Q}_p}$ containing the centre.
We will show that $H_p = G''$ by proving that its rank is equal to 6. Considering the restrictions to the different inertia subgroups, it is clear that ρ'_1 is isomorphic to neither ρ_2 nor ρ_3. The condition on $\mathrm{im}(\tau)$ implies that ρ'_1 is not isomorphic to ρ_1 either. Together with the fact that $\rho(\mathcal{G}_F)$ is open in $G(\mathbf{Q}_p)$, this implies that the rank of H_p is at least 4.
Let ℓ be a prime number which is inert in the cubic number field K. The Zariski closure H_ℓ of the ℓ-adic Galois representation associated to $A \times A'$ is a reductive algebraic subgroup of $G''_{\mathbf{Q}_\ell}$. It follows from [Ser81, §3] (cf. [LP92, 6.12, 6.13]) that the ranks of H_p and H_ℓ are equal, so H_ℓ is of rank at least 4. As the adjoint group $G''_{\mathbf{Q}_\ell}$ is a product of two $\mathbf{Q}_\ell$-simple groups of rank 3, this gives $H_\ell = G''_{\mathbf{Q}_\ell}$. We conclude that H_ℓ and hence H_p are of rank 6.
The fact that $H_p = G''_{\mathbf{Q}_p}$ implies that $(\rho, \rho')(\mathcal{G}_F)$ is open in $G''(\mathbf{Q}_p)$. The statement about τ can now be proved using the representation theory of SL_2^6 in an argument similar to the one used for $\psi_1\rho_1$. □

4.2 Remark. In the proposition, the condition that $\mathrm{Lie}(\mathrm{im}(\tau)) \cong \mathfrak{sl}_2^2$ implies that τ is irreducible. This is essential for the conclusion of the proposition to hold. To see this, consider the representation $\rho_1 \otimes \rho_1$, which is reducible and decomposes as the sum of the trivial representation $\wedge^2\rho_1$ and $\mathrm{Sym}^2\rho_1$. The only interesting representation of the two is $\mathrm{Sym}^2\rho_1$. It is a quotient of $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)^{\otimes 2}$.

4.3 THE SPECIAL CASE CONSIDERED IN 3.6. From now on and up to and including theorem 4.12, we place ourselves in the situation of 3.6. In particular, the abelian varieties A and A' have Mumford–Tate groups G and $T' \subset G'$ respectively. Moreover, we let D be the division algebra associated to G, as in 2.1, and let L' be the reflex field of the CM-type of A'. We saw in in 2.6 that the Mumford–Tate group of A' is the image of $(L')^\times$ in G'.

4.4 THE CONSTRUCTION OF THE ABELIAN VARIETY B. By [Noo01, proposition 1.5], the morphism $h\colon S \to G_{\mathbf{R}}$ determining the Hodge structure on $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ lifts uniquely to a map

$$h_D\colon S \longrightarrow D^\times_{\mathbf{R}} \cong \mathrm{GL}_{2,\mathbf{R}} \times \mathbf{H}^\times \times \mathbf{H}^\times$$

conjugate to

$$z = a + bi \mapsto \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, 1, 1 \right).$$

As $D \otimes_{\mathbf{Q}} \mathbf{R} = \prod_{K \hookrightarrow \mathbf{R}} D \otimes_K \mathbf{R}$, the factors in the above product correspond to the real embeddings φ_1, φ_2 and φ_3 of K (in that order) and the composite of φ_1 with the inclusion $\mathbf{R} \subset \mathbf{C}$ is the embedding $K \hookrightarrow \mathbf{C}$ fixed in 2.2. On the $'$-side, there is an isomorphism

$$(L')^\times_{\mathbf{R}} \cong \prod_{K \hookrightarrow \mathbf{R}} S, \tag{4.4.$*$}$$

the factors still being indexed by the φ_i. Let

$$h'_{L'} = (1, \mathrm{id}, \mathrm{id})\colon S \longrightarrow \prod_{K \hookrightarrow \mathbf{R}} S \cong (L')^\times_{\mathbf{R}}, \tag{4.4.†}$$

where the trivial component is the one corresponding to φ_1.
Let H denote the algebraic group defined by the short exact sequence

$$1 \longrightarrow K^\times \xrightarrow{x \mapsto (x, x^{-1})} D^\times \times (L')^\times \longrightarrow H \longrightarrow 1. \tag{4.4.‡}$$

The group H' defined by (3.6.†) identifies with a subgroup of this group and H and H' naturally act on $W_B = D \otimes_K L'$. For any pair of maps $h_1\colon S \to D^\times_{\mathbf{R}}$ and $h_2\colon S \to (L')^\times_{\mathbf{R}}$, we will write $h_1 \cdot h_2$ for the composite of the product (h_1, h_2) with the projection onto $H_{\mathbf{R}}$.
The morphisms h_D and $h'_{L'}$ give rise to $h_H = h_D \cdot h'_{L'}\colon S \to H_{\mathbf{R}}$ and via the action of H on W_B, this gives a Hodge structure on W_B, cf. [Del72, §6]. By [Del72, pp. 161–162], this is the Hodge structure on the Betti cohomology of some polarizable complex abelian variety $B_{\mathbf{C}}$. By construction, $W_B = \mathrm{H}^1_{\mathrm{B}}(B_{\mathbf{C}}(\mathbf{C}), \mathbf{Q})$ and $\dim(B_{\mathbf{C}}) = 12$. The totally real field K is contained in the centre of D, so L' is contained in the centre of $D \otimes_K L'$. It follows that L' acts on $B_{\mathbf{C}}$ by isogenies.

4.5. Lemma. *Let $B_{\mathbf{C}}$ be as above. There exist a number field $F' \subset \mathbf{C}$ and an abelian variety B over F' such that $B_{\mathbf{C}} = B \otimes_{F'} \mathbf{C}$ and $L' \subset \mathrm{End}^0(B)$.*

Proof. It suffices to show that $B_{\mathbf{C}}$ admits a model over $\overline{\mathbf{Q}}$. Let X be the conjugacy class of the morphism $h\colon S \to G_{\mathbf{R}}$ defining the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$, let X_D be the conjugacy class of $h_D\colon S \to D^\times$ and Y that of h_H. By [Del72, §6], this gives rise to Shimura data (G, X), $(D^\times, X_D)$, $((L')^\times, \{h'_{L'}\})$ and (H, Y) and morphisms $(D^\times, X_D) \to (G, X)$ and

$$(D^\times \times (L')^\times, X_D \times \{h'_{L'}\}) \to (H, Y).$$

For appropriate compact open subgroups $C \subset G(\mathbf{A}^f)$, $C_D \subset D^\times(\mathbf{A}^f)$, $C_H \subset H(\mathbf{A}^f)$ and $C_{D,L} \subset (D^\times \times (L')^\times)(\mathbf{A}^f)$ we obtain a diagram

$$\begin{array}{ccc} {}_{C_{D,L}}M_{\overline{\mathbf{Q}}}(D^\times \times (L')^\times, X_D \times \{h'_{L'}\}) & \longrightarrow & {}_{C_H}M_{\overline{\mathbf{Q}}}(H, Y) \\ \downarrow & & \\ {}_{C_D}M_{\overline{\mathbf{Q}}}(D^\times, X_D) & & \\ \downarrow & & \\ {}_{C}M_{\overline{\mathbf{Q}}}(G, X) & & \end{array}$$

of morphisms between the weakly canonical models of the associated Shimura varieties.

We dispose of faithful representations of G and of H, and via these representations, the $\mathbf{C}$-valued points of the Shimura varieties ${}_{C}M_{\overline{\mathbf{Q}}}(G, X)$ and ${}_{C_H}M_{\overline{\mathbf{Q}}}(H, Y)$ correspond to isogeny classes of polarized abelian varieties endowed with extra structure. If the above compact subgroups are sufficiently small, all the Shimura varieties in question are smooth and both ${}_{C}M_{\overline{\mathbf{Q}}}(G, X)$ and ${}_{C_H}M_{\overline{\mathbf{Q}}}(H, Y)$ carry families of abelian varieties such that the fibre over a $\mathbf{C}$-valued point of the Shimura variety lies in the isogeny class corresponding to that point. One can moreover choose the compact subgroups above such that the vertical maps are finite.

After fixing a level structure on $A_{\overline{\mathbf{Q}}}$, it corresponds to a point

$$a \in {}_{C}M_{\overline{\mathbf{Q}}}(G, X)(\overline{\mathbf{Q}}).$$

By construction, there is a $\mathbf{C}$-valued point

$$\tilde{a} \in {}_{C_{D,L}}M_{\overline{\mathbf{Q}}}(D^\times \times (L')^\times, X_D \times \{h'_{L'}\})(\mathbf{C})$$

in the fibre over a of the vertical map such that $\tilde{a}$ maps to a point $b \in {}_{C_H}M_{\overline{\mathbf{Q}}}(H, Y)(\mathbf{C})$ corresponding to the isogeny class of $B_{\mathbf{C}}$ endowed with an appropriate level structure. Because the vertical arrows are finite, it follows that $\tilde{a}$ and hence b are defined over $\overline{\mathbf{Q}}$ and this implies that $B_{\mathbf{C}}$ admits a model over $\overline{\mathbf{Q}}$. ∎

4.6 The construction of the abelian variety C. We keep the notations of 4.3 and 4.4. The morphism $h\colon S \to T'_{\mathbf{R}}$ determining the Hodge structure of $A'_{\mathbf{C}}$ on $V' = \mathrm{H}^1_{\mathrm{B}}(A'(\mathbf{C}), \mathbf{Q})$ lifts to a map $h_{L'}\colon S \to (L')^{\times}_{\mathbf{R}}$ inducing an isomorphism of S with the factor of $(L')^{\times}_{\mathbf{R}} \cong \prod_{K \hookrightarrow \mathbf{R}} S$ corresponding to $\varphi_1\colon K \hookrightarrow \mathbf{R}$ and inducing the trivial map on the other factors.
Let $h'_{L'}$ be as in (4.4.†) and denote by $\overline{h'_{L'}}$ its composite with the involution of S induced by complex conjugation. We define yet another map $h''_{L'}\colon S \to (L')^{\times}_{\mathbf{R}}$ as the product $\overline{h'_{L'}} h_{L'}$, where the product is taken for the commutative group structure on $(L')^{\times}_{\mathbf{R}}$. Together with the natural action of L' on $W_C = L'$, this defines a Hodge structure on W_C. This Hodge structure is the Hodge structure of $C_{\mathbf{C}}$ for an abelian variety C of CM-type over a number field $F'' \subset \mathbf{C}$. One has $\dim(C) = 3$ and $L' \subset \mathrm{End}^0(C)$. By construction, $W_C = \mathrm{H}^1_{\mathrm{B}}(C(\mathbf{C}), \mathbf{Q})$ and $\dim_{L'}(W_C) = 1$.

4.7 Remark. Possibly after replacing the identification of equation (4.4.∗) on the first factor by its complex conjugate, one can assume that $h_{L'}$ is given by

$$h_{L'} = (\mathrm{id}, 1, 1)\colon S \longrightarrow \prod_{K \hookrightarrow \mathbf{R}} S \cong (L')^{\times}_{\mathbf{R}}.$$

Doing so, the map $h''_{L'}$ is given by $z \mapsto (z, \bar{z}, \bar{z})$.

4.8 The construction of the motives m' and m. In what follows, we will work in the category $\mathcal{C}$ of motives for absolute Hodge cycles as described in [DM82], especially section 6 of that paper. Recall that this category is constructed as Grothendieck's category of motives except where it concerns the morphisms. These are defined to be given by absolute Hodge classes, not by cycle classes as usual.
We keep the assumptions of 4.3. Replacing F by a finite extension and extending the base field of the varieties in question, we will assume from now on that the abelian varieties A and A' (of 3.6), B (of 4.5) and C (of 4.6) are F-varieties and that L' acts on B and C (over F). The motives $h^1(B)$ and $h^1(C)$ belong to the category $\mathcal{C}_{(L')}$ of objects of $\mathcal{C}$ endowed with L'-action. One can thus form the tensor product

$$m' = h^1(B) \otimes_{L'} h^1(C),$$

still belonging to $\mathcal{C}_{(L')}$, as in [DM82, pp. 155–156].
Let $\widetilde{K}$ be the normal closure of K in $\mathbf{C}$. As K is totally real, one has $\widetilde{K} \subset \mathbf{R}$. The decomposition of the algebra $K \otimes_{\mathbf{Q}} \widetilde{K}$ as a product of fields gives rise to a system (u_1, u_2, u_3) of orthogonal idempotents in $K \otimes_{\mathbf{Q}} \widetilde{K}$, indexed be the real embeddings $\varphi_1, \varphi_2, \varphi_3$ of K. For $1 \le i < j \le 3$, let $u_{i,j} = u_i + u_j$.
Let $m' \otimes_{\mathbf{Q}} \widetilde{K}$ be the external tensor product as in [DM82, pp. 155–156]. As $K \subset L'$, it acts on m' and one deduces a $\widetilde{K}$-linear action of $K \otimes_{\mathbf{Q}} \widetilde{K}$ on $m' \otimes_{\mathbf{Q}} \widetilde{K}$,

given by a $\mathrm{Gal}(\widetilde{K}/K)$-equivariant $\widetilde{K}$-algebra morphism

$$K \otimes_{\mathbf{Q}} \widetilde{K} \longrightarrow \mathrm{End}_{\widetilde{K}}(m' \otimes_{\mathbf{Q}} \widetilde{K}) = \mathrm{End}(m') \otimes_{\mathbf{Q}} \widetilde{K}.$$

This action induces an action of $K \otimes_{\mathbf{Q}} \widetilde{K}$ on $\wedge^3_{\widetilde{K}}(m' \otimes_{\mathbf{Q}} \widetilde{K})$ given by a $\mathrm{Gal}(\widetilde{K}/K)$-equivariant multiplicative map (not a morphism of algebras)

$$K \otimes_{\mathbf{Q}} \widetilde{K} \longrightarrow \mathrm{End}_{\widetilde{K}}(\wedge^3_{\widetilde{K}}(m' \otimes_{\mathbf{Q}} \widetilde{K})) = \mathrm{End}(\wedge^3 m') \otimes_{\mathbf{Q}} \widetilde{K}.$$

For $1 \leq i < j \leq 3$, let $u'_{i,j}$ be the image of $u_{i,j}$ in $\mathrm{End}(\wedge^3 m') \otimes_{\mathbf{Q}} \widetilde{K}$. As

$$\{u_{2,3}, u_{1,3}, u_{1,2}\} \subset K \otimes_{\mathbf{Q}} \widetilde{K}$$

is a $\mathrm{Gal}(\widetilde{K}/K)$-invariant subset, the sum $u'_{2,3} + u'_{1,3} + u'_{1,2} \in \mathrm{End}(\wedge^3 m') \otimes_{\mathbf{Q}} \widetilde{K}$ is also $\mathrm{Gal}(\widetilde{K}/K)$-invariant. This element therefore determines an element $u' \in \mathrm{End}(\wedge^3 m')$. The AH-motive m is defined to be the kernel of u' on $\wedge^3 m'$.

4.9 Remark. Intuitively, the aim of this construction is to pass from an object m' which can be expressed, after tensoring with $\widetilde{K}$, as a direct sum, to the tensor product of the direct factors. This tensor product descends to an object m over $\mathbf{Q}$.

4.10. Proposition. *Let notations and assumptions be as in 4.3–4.8. In particular, the Mumford–Tate group of A is equal to G, that of A' is T', the varieties A, A', B and C are defined over F and L' acts on B and on C. Then there is an isomorphism*

$$m \cong (h^1(A) \otimes h^1(A')(-2))^8$$

of absolute Hodge motives.

Proof. The main theorem (2.11) of Deligne's paper [Del82] states that the spaces of Hodge cycles and of absolute Hodge cycles on an abelian variety coincide. As noted in [DM82, 6.25], this implies that to prove the proposition, it suffices to show that the Hodge structures on the Betti realizations of the motives m and $(h^1(A) \otimes h^1(A')(-2))^8$ are isomorphic.

First consider $h^1(A)$, the Betti realization of which is $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$. The Mumford–Tate group G of A admits a morphism $D^\times \to G$, which makes V into a representation of $D^\times$. The Hodge structure on V is determined by the action of S on $V_{\mathbf{R}}$ given by the morphism $h_D \colon S \to D^\times_{\mathbf{R}}$ of 4.4. Similarly, $V' = \mathrm{H}^1_{\mathrm{B}}(A'(\mathbf{C}), \mathbf{Q})$, the Betti realization of $h^1(A')$, is made into a representation of $(L')^\times$ by the morphism $(L')^\times \to T'$, and the map $h_{L'} \colon S \to (L')^\times_{\mathbf{R}}$ of 4.6 determines the Hodge structure on V'.

In these representations, the subgroup $K^\times \subset D^\times$, resp. $K^\times \subset (L')^\times$ acts through the map $K^\times \to \mathbf{G}_{m,\mathbf{Q}}$ induced by the field norm $\mathrm{N}_{K/\mathbf{Q}}$. It follows that the action of $D^\times \times (L')^\times$ on $V \otimes_{\mathbf{Q}} V'$, the Betti realization of $h^1(A) \otimes h^1(A')$, factors through the group H defined by (4.4.‡). Passing to $\overline{\mathbf{Q}}$, one has

$$H_{\overline{\mathbf{Q}}} \cong \prod_{K \hookrightarrow \overline{\mathbf{Q}}} \left(\mathrm{GL}_2 \times \mathbf{G}_m^2 / \mathbf{G}_m \right),$$

where the map $\mathbf{G}_m \to \mathrm{GL}_2 \times \mathbf{G}_m^2$ on each factor is given by $z \mapsto (z \cdot \mathrm{id}, z^{-1}, z^{-1})$. The representation of $H_{\overline{\mathbf{Q}}}$ on $V \otimes V' \otimes \overline{\mathbf{Q}}$ induces a representation of

$$\left(\mathrm{GL}_{2,\overline{\mathbf{Q}}} \times \mathbf{G}_{m,\overline{\mathbf{Q}}}^2\right)^3$$

on this space which is isomorphic to the tensor product of the standard representations of its factors GL_2 and $\mathbf{G}_m^2$ on $\overline{\mathbf{Q}}^2$.

The Hodge structure on $V \otimes V'$, considered as the Betti realization of $h^1(A) \otimes h^1(A')$, is given by $h_D \cdot h_{L'} \colon S \to H_{\mathbf{R}}$. Multiplying the composite map

$$S \xrightarrow{h_D \cdot h_{L'}} H_{\mathbf{R}} \longrightarrow \mathrm{GL}(V \otimes V' \otimes \mathbf{R})$$

by the square of the norm map $S \to \mathbf{G}_{m,\mathbf{R}}$, one gets the Hodge structure of the Betti realization of the Tate twist $h^1(A) \otimes h^1(A')(-2)$.

The Betti realization of m' is $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}) \otimes_{L'} \mathrm{H}^1_{\mathrm{B}}(C(\mathbf{C}), \mathbf{Q}) = W_B \otimes_{L'} W_C$ and the group H of (4.4.‡) naturally acts on this space. Over $\overline{\mathbf{Q}}$ we have a decomposition

$$(W_B \otimes_{L'} W_C) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} = (W_1 \oplus W_2 \oplus W_3)^2$$

of the induced representation of $H_{\overline{\mathbf{Q}}}$. As before, the factors correspond to the embeddings $K \hookrightarrow \overline{\mathbf{Q}}$. The representation of $H_{\overline{\mathbf{Q}}}$ on W_i is isomorphic to the one induced by the representation of the ith factor $\mathrm{GL}_2 \times \mathbf{G}_m^2$ on $\overline{\mathbf{Q}}^2 \otimes_{\overline{\mathbf{Q}}} \overline{\mathbf{Q}}^2$ by tensor product.

The Hodge structure on the tensor product $W_B \otimes_{L'} W_C$ is obtained by multiplying the map h_H defining the Hodge structure of $B_{\mathbf{C}}$ by $h''_{L'}$, the map determining the Hodge structure of C. Since h_H is defined in 4.4 as the product $h_D \cdot h'_{L'}$ and since $h''_{L'} = h_{L'} \overline{h'_{L'}}$, it follows that $h_H h''_{L'} = h_D \cdot h_{L'} h'_{L'} \overline{h'_{L'}}$. Writing $N' = h'_{L'} \overline{h'_{L'}}$, this implies that the Hodge structure on $W_B \otimes_{L'} W_C$ is determined by

$$h_H h''_{L'} = h_D \cdot h_{L'} N' \colon S \longrightarrow H_{\mathbf{R}}.$$

The image of $N' \colon S \to (L')^\times_{\mathbf{R}}$ lies in $K^\times_{\mathbf{R}}$ and N' is given by

$$\begin{aligned} N' \colon S &\longrightarrow \prod_{K \hookrightarrow \mathbf{R}} \mathbf{G}_{m,\mathbf{R}} \cong K^\times_{\mathbf{R}} \\ z &\mapsto (1, z\bar{z}, z\bar{z}). \end{aligned}$$

With this information, the Betti realization W_m of m can be computed. It is the kernel of the endomorphism of $\wedge^3(W_B \otimes_{L'} W_C)$ induced by the map u' of 4.8. For $1 \le i < j \le 3$, the endomorphism $u_{i,j} \in \mathrm{End}_{\overline{\mathbf{Q}}}(W_B \otimes_{L'} W_C \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})$ induces the identity on W_i and W_j and zero on the remaining factor. Since $W_m \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ is the kernel of $u'_{\overline{\mathbf{Q}}}$ on $\wedge^3(W_B \otimes_{L'} W_C) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$, it follows that there is an isomorphism

$$W_m \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \cong W_1^2 \otimes W_2^2 \otimes W_3^2 = (W_1 \otimes W_2 \otimes W_3)^8$$

of representations of $H_{\overline{\mathbf{Q}}}$. It follows that $V \otimes V' \cong W_m$ as representations of H.
We still have to show that the Hodge structures are the same. In view of what we already know about these Hodge structures, it suffices to show that the action of S on $W_m \otimes \mathbf{R}$ defined by $N' \colon S \to K_{\mathbf{R}}^{\times} \subset H_{\mathbf{R}}$ and the representation of H on W_m is equal to scalar multiplication by the square of the norm $S \to \mathbf{G}_{m,\mathbf{R}}$. This in turn follows immediately from the above description of N' and the fact that $K^{\times} \subset H$ acts on W_m through the map $K^{\times} \to \mathbf{G}_{m,\mathbf{Q}}$ induced by the field norm $\mathrm{N}_{K/\mathbf{Q}}$. □

4.11. Corollary. *Keep the hypotheses of the proposition. For any prime number p, there is an isomorphism of $\mathcal{G}_F$-modules between*

$$\Big(\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(-2)\Big)^8$$

and the étale p-adic realization of m. This representation is a subquotient of $\mathrm{H}^6_{\text{ét}}(B_{\overline{\mathbf{Q}}} \times C_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$.

4.12. Theorem. *Let notations and assumptions be as in 4.3. In particular, the Mumford–Tate group of A is equal to G and that of A' is T'. Assume moreover that A, A', B and C are defined over F and that $\rho(\mathcal{G}_F)$ and $\rho'(\mathcal{G}_F)$ are sufficiently small. Let the representation*

$$\sigma \colon \mathcal{G}_F \to H'(\mathbf{Q}_p) \subset H(\mathbf{Q}_p)$$

be as in (3.6.∗). Then there exists a finite extension F' of F such that the restriction of $\sigma \oplus \sigma$ to $\mathcal{G}_{F'}$ is isomorphic to the representation of $\mathcal{G}_{F'}$ on

$$\mathrm{H}^1_{\text{ét}}(B_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes_{L' \otimes \mathbf{Q}_p} \mathrm{H}^1_{\text{ét}}(C_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1).$$

In particular, the representations $\psi_i^{\pm}\rho_i$ (for $i = 1, 2, 3$) defined in 3.6 come from algebraic geometry in the sense of 0.3.

Proof. The first statement of the theorem implies that $\sigma \colon \mathcal{G}_F \to H'(\mathbf{Q}_p)$ differs from a representation of $\mathcal{G}_F$ on a subquotient of an étale cohomology group of an algebraic variety by a finite character. As σ is the direct sum of the irreducible representations $\psi_i^{\pm}\rho_i$, the second statement of the theorem follows. This leaves the first statement to be proven. From the constructions of B and C and from the proof of proposition 4.10, it is clear that the Mumford–Tate group of the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}) \otimes_{L'} \mathrm{H}^1_{\mathrm{B}}(C(\mathbf{C}), \mathbf{Q})$ is contained in H and that the Mumford–Tate group of $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}) \otimes_{L'} \mathrm{H}^1_{\mathrm{B}}(C(\mathbf{C}), \mathbf{Q})(1)$ is contained in H'. By [Del82, 2.9, 2.11], this implies that there is a finite extension F' of F such that L' acts on $B_{F'}$ and on $C_{F'}$ and such that the representation of $\mathcal{G}_{F'}$ on

$$W^{(p)} = \mathrm{H}^1_{\text{ét}}(B_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes_{L' \otimes \mathbf{Q}_p} \mathrm{H}^1_{\text{ét}}(C_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1)$$

factors through a morphism $\sigma'\colon \mathcal{G}_{F'} \to H'(\mathbf{Q}_p)$. The hypotheses that $D \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong \mathrm{M}_2(\mathbf{Q}_p)^3$ and that p splits completely in L' imply that $W^{(p)}$ decomposes as the direct sum of two isomorphic representations of the group $H'_{\mathbf{Q}_p}$. We contend that, possibly after replacing F' by a finite extension, $W^{(p)}$ is isomorphic to $V^{(p)}$, the $\mathcal{G}_{F'}$ module underlying $\sigma \oplus \sigma$.
The decomposition

$$\sigma = \bigoplus_{i=1,2,3} \left(\psi_i\rho_i \oplus \psi_i^{-1}\rho_i\right)$$

gives a decomposition $V^{(p)} = (V_1 \oplus V_2 \oplus V_3)^2$. As σ lifts the Galois representation on $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1)$ from $(G \times T'/\mathbf{G}_m)(\mathbf{Q}_p)$ to $H'(\mathbf{Q}_p)$, one has

$$\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(A'_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(1) = V_1 \otimes V_2 \otimes V_3.$$

If $u' \in \mathrm{End}(\wedge^3 V^{(p)})$ is constructed as in 4.8, then $\ker(u') = (V_1 \otimes V_2 \otimes V_3)^8$. In analogy with the proof of proposition 4.10, one has $W^{(p)} = (W_1 \oplus W_2 \oplus W_3)^2$ as representations of $H_{\mathbf{Q}_p}$ and $(W_1 \otimes W_2 \otimes W_3)^8$ is the étale p-adic realization of the motive m. It follows from the corollary 4.11 that

$$(V_1 \otimes V_2 \otimes V_3)^8 \cong (W_1 \otimes W_2 \otimes W_3)^8$$

as representations of $\mathcal{G}_{F'}$. Hence σ and σ' have the same projection to $H'(\mathbf{Q}_p)/\ker(N)(\mathbf{Q}_p)$. Here $N\colon T_K \to \{\pm 1\}$ is the map induced by the norm $\mathrm{N}_{K/\mathbf{Q}}$, where T_K is, as in 3.6, the group scheme $\{x \in K^\times \mid x^2 = 1\}$. As $T_K(\mathbf{Q}_p)$ is finite, this implies that $\sigma_{|\mathcal{G}_{F'}}$ and $\sigma'_{|\mathcal{G}_{F'}}$ differ by a finite character and thus proves the theorem. $\square$

4.13. Corollary. *We return to the hypotheses of theorem 3.3, i. e. we only assume that the Mumford–Tate groups of A and A' are contained in G and G' respectively. Then the representation $\tau\colon \mathcal{G}_F \to \mathrm{GL}_4(\mathbf{Q}_p)$ of (3.1.∗), which is geometric by theorem 3.3, comes from algebraic geometry.*

Proof. By remark 0.6, it is sufficient to prove this after replacing F by a finite extension. Apart from the varieties A and A' serving in the construction of τ in 3.1, we choose, after enlarging F if necessary, an auxiliary abelian variety A'' with Mumford–Tate group contained in a group of Mumford's type, which is of CM-type, and such that p splits completely in the reflex field L'' of its CM-type. As in 3.6, let ψ_1 be one of the characters in which the representation ρ''_1 decomposes. For F sufficiently large, the above theorem implies that $\psi_1\rho_1$ and $\psi_1^{-1}\rho'_1$ come from algebraic geometry. It follows that the same is true for $\tau = (\psi_1\rho_1) \otimes (\psi_1^{-1}\rho'_1)$. $\square$

5 Other geometric representations

5.1 Let p be a prime number, F a number field and let $\mathrm{Rep}(\mathcal{G}_F)$ be the tannakian category of finite dimensional, continuous $\mathbf{Q}_p$-linear representations of the absolute Galois group $\mathcal{G}_F$. Consider all abelian fourfolds A/F such that the conditions of 2.4 are verified for some identification $V_A = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and some algebraic group $G_A \subset \mathrm{GL}(V_A)$. We denote the p-adic Galois representation associated to A by $\rho_A \colon \mathcal{G}_F \to G_A(\mathbf{Q}_p)$ and define $\mathcal{M}$ to be the full tannakian subcategory of $\mathrm{Rep}(\mathcal{G}_F)$ generated by the ρ_A for A running through the above set of abelian varieties and the Tate object $\mathbf{Q}_p(1)$.
For any A as above, we denote by $\tilde{\rho}_A$ the canonical lifting of ρ_A constructed in lemma 1.4. Let $\mathcal{L}$ be the full tannakian subcategory of $\mathrm{Rep}(\mathcal{G}_F)$ generated by all liftings $\tilde{\rho}_A$ and $\mathbf{Q}_p(1)$. Obviously, $\mathcal{M}$ is a full subcategory of $\mathcal{L}$.
Let $\tilde{\rho}$ be an object of $\mathcal{L}$, it is a representation of $\mathcal{G}_F$ on a finite dimensional $\mathbf{Q}_p$-vector space V. We let $\widetilde{G}$ be the Zariski closure of the image of $\tilde{\rho}$ in $\mathrm{GL}(V)$. The category generated by $\tilde{\rho}$ contains an object ρ of $\mathcal{M}$ such that there is a central isogeny $\widetilde{G} \to G$, where G denotes the Zariski closure of the image of ρ.

5.2. Theorem. *Let τ be an irreducible representation of $\mathcal{G}_F$ contained in $\mathcal{L}$ which is geometric. Then τ comes from algebraic geometry.*

Proof. Each $\tilde{\rho}_A$ is a direct sum of a (fixed) character ρ_0 and three 2-dimensional representations $\rho_{A,1}$, $\rho_{A,2}$ and $\rho_{A,3}$. It follows that $\mathcal{L}$ is generated by ρ_0 and the $\rho_{A,i}$, for $i = 1, 2, 3$ and A running through the abelian varieties considered in 5.1. This implies that each object of $\mathcal{L}$ is a subquotient of a representation of the form

$$\tau = \rho_0^k \cdot \bigotimes_{j=1}^{r} \rho_j, \tag{5.2.$*$}$$

with $k \in \mathbf{Z}$, and where $\rho_j = \rho_{A_j, i_j}$ for some an abelian variety A_j of the type considered above and some integer $i_j \in \{1, 2, 3\}$. It therefore suffices to consider representations of this form.
Let τ be as in (5.2.$*$). It follows from 3.6 and 4.12 that, for each j, there exist a finite extension F_j of F and a character ψ_j of $\mathcal{G}_{F_j}$ such that $\psi_j \tau_{|\mathcal{G}_{F_j}}$ comes from algebraic geometry. Let F' be a finite extension of F containing all F_j, define the character ψ of $\mathcal{G}_{F'}$ by $\psi = (\rho_0^{-k}) \prod \psi_j$ and let

$$\tau' = \bigotimes_{j=1}^{r} (\psi_i \rho_j)_{|\mathcal{G}_{F'}} = \psi \tau_{|\mathcal{G}_{F'}}.$$

This defines a geometric representation $\mathcal{G}_{F'}$. After replacing F' by a finite extension, τ' is a Tate twist of a subquotient of an étale cohomology group of an algebraic (even abelian) variety over F'.
Now assume that τ and hence $\tau_{|\mathcal{G}_{F'}}$ are geometric. This implies that ψ is a geometric representation of $\mathcal{G}_{F'}$ as well. If follows from [FM95, §6], that ψ

comes from algebraic geometry and this in turn implies that $\tau_{|\mathcal{G}_{F'}} = \psi^{-1}\tau'$ is a Tate twist of a subquotient of an étale cohomology group of an algebraic variety. By remark 0.6, the same thing is true for τ. ☐

5.3 Using the proof of theorem 5.2, one can describe the geometric representations contained in $\mathcal{L}$. We keep the notations used above. Let A/F be an abelian variety as in 5.1, G_A its Mumford–Tate group and v a p-adic valuation of F. Let

$$\mu_{\mathrm{HT},A,v}\colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{A,\mathbf{C}_p}$$

be the cocharacter defined by the Hodge–Tate decomposition of $\rho_{|\mathcal{I}_{F,\bar{v}}}$ for some valuation $\bar{v}$ of $\bar{F}$ with $v = \bar{v}_{|F}$. We saw in the proof of proposition 1.9 that $\mu^2_{\mathrm{HT},A,v}$ lifts to $\tilde{\mu}_{A,v}\colon \mathbf{G}_{m,\mathbf{C}_p} \to \widetilde{G}_{A,\mathbf{C}_p}$. This lifting is conjugate to the map

$$\begin{aligned} \mathbf{G}_{m,\mathbf{C}_p} &\longrightarrow \mathbf{G}_{m,\mathbf{C}_p} \times \mathrm{SL}_{2,\mathbf{C}_p} \times \mathrm{SL}_{2,\mathbf{C}_p} \times \mathrm{SL}_{2,\mathbf{C}_p} \cong \widetilde{G}_{A,\mathbf{C}_p} \\ z &\mapsto \left(z, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \mathrm{id}, \mathrm{id}\right). \end{aligned}$$

The factor SL_2 on which $\tilde{\mu}_{A,v}$ is non trivial is the factor to which the image of $\mathcal{I}_{F,\bar{v}}$ projects non-trivially.
The representations

$$\psi_j\rho_j\colon \mathcal{G}_F \to \mathrm{GL}_2(\mathbf{Q}_p)$$

occurring in the expression for τ' are geometric and the maps $\tilde{\mu}_{A,v}$ define cocharacters $\tilde{\mu}_{j,v}\colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathrm{GL}_{2,\mathbf{C}_p}$. The above description of the $\tilde{\mu}_{A,v}$ and the choice of the ψ_j imply that each $\tilde{\mu}_{j,v}$ is a square of a cocharacter of $\mathrm{GL}_{2,\mathbf{C}_p}$. Similarly, the liftings $\tilde{\mu}_{A,v}$ and the expression of the character

$$\psi\colon \mathcal{G}_F \to \mathbf{G}_m(\mathbf{Q}_p)$$

as a product of ρ_0^{-k} and the ψ_j define a map $\tilde{\mu}_v\colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathbf{G}_{m,\mathbf{C}_p}$. It follows from proposition 1.9, its proof, and the description of the $\tilde{\mu}_{A,v}$, that $\psi_{|\mathcal{I}_{F,\bar{v}}}$ is the nth power of $(\rho_0)_{|\mathcal{I}_{F,\bar{v}}}$ for some integer n and that $\tilde{\mu}_v(z) = z^n$. This means that ψ is potentially semi-stable at v if and only if n is even, which is the case if and only if $\tilde{\mu}_v$ is a square.
The condition that ψ is potentially semi-stable at v is equivalent to τ being potentially semi-stable at v. This proves the following proposition, characterizing the geometric representations in $\mathcal{L}$.

5.4. PROPOSITION. *For each object $\tilde{\rho}\colon \mathcal{G}_F \to \mathrm{GL}(V)$ of $\mathcal{L}$, there exists a central isogeny $N\colon \widetilde{G} \to G$ of algebraic groups over $\mathbf{Q}_p$ such that $\widetilde{G}$ is the Zariski closure of the image of $\tilde{\rho}$ and such that $N \circ \tilde{\rho}$ belongs to $\mathcal{M}$. The representation ρ is semi-stable at v if and only if the cocharacter*

$$\mu_{\mathrm{HT},v}\colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{\mathbf{C}_p},$$

defined by the Hodge–Tate decomposition of $\rho_{|\mathcal{I}_{F,\bar v}}$ *(for a place* $\bar v$ *of* $\bar F$ *lying over* v*), lifts to* $\widetilde{G}_{\mathbf{C}_p}$*. The representation* ρ *is geometric if and only if* $\mu_{\mathrm{HT},v}$ *lifts for each valuation* v *of* F*.*

5.5. Corollary. *Let* F *be a number field,* p *a prime number and* m *be a motive belonging to the category of absolute Hodge motives generated by the motives of the abelian varieties considered in 5.1. Assume that* G *is the Mumford–Tate group of* m *and let* $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ *the associated* p*-adic Galois representation. Suppose that* $\widetilde{G} \to G$ *is a central isogeny such that* ρ *lifts to a representation* $\tilde\rho\colon \mathcal{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ *belonging to* $\mathcal{L}$*. Then the following statements are equivalent.*

1. $\tilde\rho$ *is geometric.*
2. *For some* p*-adic valuation* $\bar v$ *of* $\bar F$*, the restriction* $\tilde\rho_{|\mathcal{I}_{F,\bar v}}$ *is potentially semi-stable.*
3. *The cocharacter* $\mu_{\mathrm{HdR}}\colon \mathbf{G}_{m,\mathbf{C}} \to G_{\mathbf{C}}$ *defining the Hodge filtration on the Betti realization of* m *lifts to* $\widetilde{G}_{\mathbf{C}}$*.*
4. $\tilde\rho$ *comes from algebraic geometry.*

Proof. By hypothesis, the conjugacy class C_{HdR} of μ_{HdR} can be defined over F. It was noted in the proof of proposition 2.5 that, for any embedding $\bar F \hookrightarrow \mathbf{C}_p$ inducing the valuation $\bar v$, the conjugacy class of $\mu_{\mathrm{HT},v}$ is equal to $C_{\mathrm{HdR}} \otimes_F \mathbf{C}_p$. Using proposition 5.4, this proves that 2 implies 3 and that 3 implies 1.
The theorem 5.2 shows that 1 implies 4. The remaining implications are left to the reader. □

5.6 Remarks.

5.6.1 If condition 3 of the corollary is verified, then it follows from [Win95, Théorème 2.1.7] that there exists a finite extension F' of F such that the representations $\mathcal{G}_{F'} \to G(\mathbf{Q}_\ell)$, for ℓ ranging through the prime numbers, simultaneously lift to $\widetilde{G}(\mathbf{Q}_\ell)$.

5.6.2 The geometric representations considered in this paper are obtained by lifting representations coming from algebraic geometry along isogenies of algebraic groups $\widetilde{H} \to H$ over $\mathbf{Q}$. Conjecturally, a representation $\mathcal{G}_F \to H(\mathbf{Q}_p)$ which comes from algebraic geometry corresponds to a morphism $f\colon G^0_M \to H$, where G^0_M is the connected component of the motivic Galois group, see for example [Ser94]. The geometric representations considered in this paper are obtained by lifting along an isogeny $\widetilde{H} \to H$ with the property that the morphism $\mathbf{G}_m \times \mathbf{G}_m \to H_{\mathbf{C}}$ determining the (mixed) Hodge structure on the corresponding Betti cohomology group lifts to $\widetilde{H}$.
In [Ser94, 8.1], Serre asks whether the derived group of G^0_M is simply connected. He notes that an argument due to Deligne shows that if this is the case, then

the fact that the morphism determining the Hodge structure lifts to $\widetilde{H}$ implies that $f\colon G_M^0 \to H$ lifts to a morphism $G_M^0 \to \widetilde{H}$. This in turn should imply that the lifted Galois representation comes from algebraic geometry. An affirmative answer to Serre's question, in conjunction with the standard conjectures involved in the theory of motives, therefore proves the Fontaine–Mazur conjecture for the representations considered in this paper. The above discussion also indicates that, conversely, the conjecture of Fontaine and Mazur is unlikely to be true if the answer to Serre's question is negative.

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p-Adic Fourier Theory

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Abstract. In this paper we generalize work of Amice and Lazard from the early sixties. Amice determined the dual of the space of locally Q_p-analytic functions on Z_p and showed that it is isomorphic to the ring of rigid functions on the open unit disk over C_p. Lazard showed that this ring has a divisor theory and that the classes of closed, finitely generated, and principal ideals in this ring coincide. We study the space of locally L-analytic functions on the ring of integers in L, where L is a finite extension of Q_p. We show that the dual of this space is a ring isomorphic to the ring of rigid functions on a certain rigid variety X. We show that the variety X is isomorphic to the open unit disk over C_p, but not over any discretely valued extension field of L; it is a "twisted form" of the open unit disk. In the ring of functions on X, the classes of closed, finitely generated, and invertible ideals coincide, but unless L=Q_p not all finitely generated ideals are principal.

The paper uses Lubin-Tate theory and results on p-adic Hodge theory. We give several applications, including one to the construction of p-adic L-functions for supersingular elliptic curves.

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In the early sixties, Amice ([Am1], [Am2]) studied the space of K-valued, locally analytic functions on $\mathbb{Z}_p$ and formulated a complete description of its dual, the ring of K-valued, locally $\mathbb{Q}_p$-analytic distributions on $\mathbb{Z}_p$, when K is a complete subfield of $\mathbb{C}_p$. She found an isomorphism between the ring of distributions and the space of global functions on a rigid variety over K parameterizing K-valued, locally analytic characters of $\mathbb{Z}_p$. This rigid variety is in fact the open unit disk, a point z of $\mathbb{C}_p$ with $|z| < 1$ corresponding to the locally $\mathbb{Q}_p$-analytic character $\kappa_z(a) = (1+z)^a$ for $a \in \mathbb{Z}_p$. The rigid function F_λ corresponding to a distribution λ is determined by the formula $F_\lambda(z) = \lambda(\kappa_z)$. Amice's description of the ring of $\mathbb{Q}_p$-analytic distributions was complemented by results of Lazard ([Laz]). He described a divisor theory for the ring of functions on the open disk and proved that, when K is spherically complete, the classes of closed, finitely generated, and principal ideals in this ring coincide.

In this paper we generalize the work of Amice and Lazard by studying the space $C^{an}(o, K)$ of K-valued, locally L-analytic functions on o, and the corresponding ring of distributions $D(o, K)$, when $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ with L finite over $\mathbb{Q}_p$ and K complete and $o = o_L$ the additive group of the ring of integers in L. To clarify this, observe that, as a $\mathbb{Q}_p$-analytic manifold, the ring o is a product of $[L : \mathbb{Q}_p]$ copies of $\mathbb{Z}_p$. The K-valued, $\mathbb{Q}_p$-analytic functions on o are thus given locally by power series in $[L : \mathbb{Q}_p]$ variables, with coefficients in K. The L-analytic functions in $C^{an}(o, K)$ are given locally by power series in *one* variable; they form a subspace of the $\mathbb{Q}_p$-analytic functions cut out by a set of "Cauchy-Riemann" differential equations. These facts are treated in Section 1.

Like Amice, we develop a Fourier theory for the locally L-analytic functions on o. We construct (Section 2) a rigid group variety $\widehat{o}$, defined over L, whose closed points z in a field K parameterize K-valued locally L-analytic characters κ_z of o. We then show that, for K a complete subfield of $\mathbb{C}_p$, the ring of rigid functions on $\widehat{o}/K$ is isomorphic to the ring $D(o, K)$, where the isomorphism $\lambda \mapsto F_\lambda$ is defined by $\lambda(\kappa_z) = F_\lambda(z)$, just as in Amice's situation.

The most novel aspect of this situation is the variety $\widehat{o}$. We prove (Section 3) that $\widehat{o}$ is a rigid variety defined over L that becomes isomorphic over $\mathbb{C}_p$ to the open unit disk, but is not isomorphic to a disk over any discretely valued extension field of L. The ring of rigid functions on $\widehat{o}$ has the property that the classes of closed, finitely generated, and invertible ideals coincide; but we show that unless $L = \mathbb{Q}_p$ (Lazard's situation) there are non-principal, finitely generated ideals, even over spherically complete coefficient fields.

The "uniformization" of $\widehat{o}$ by the open unit disk follows from a result of Tate's in his famous paper on p-divisible groups ([Tat]). We show that over $\mathbb{C}_p$ the group $\widehat{o}$ becomes isomorphic to the group of $\mathbb{C}_p$-valued points of a Lubin-Tate formal group associated to L. The Galois cocycle that gives the descent data on the open unit disk yielding the twisted form $\widehat{o}$ comes directly out of the Lubin-Tate group. The period of the Lubin-Tate group plays a crucial role in the explicit form of our results; in an Appendix we use results of Fontaine [Fon] to obtain information on the valuation of this period, generalizing work of Boxall ([Box]).

We give two applications of our Fourier theory. The first is a generalized Mahler expansion for locally L-analytic functions on o (Section 4). The second is a construction of a p-adic L-function for a CM elliptic curve at a supersingular prime (Section 5). Although the method by which we obtain it is more natural, and we obtain stronger analyticity results, the L-function we construct is essentially that studied by Katz ([Kat]) and by Boxall ([Box]). The paper by Katz in particular was a major source of inspiration in our work.

Our original motivation for studying this problem came from our work on locally analytic representation theory. In the paper [ST] we classified the locally analytic principal series representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. The results of Amice and Lazard played a key role in the proof, and in seeking to generalize those results to the principal series of $\mathrm{GL}_2(L)$ we were led to consider the problems discussed in this paper. The results of this paper are sufficient to extend the methods of [ST] to the groups $\mathrm{GL}_2(L)$, though to keep the paper self-contained we do not give the proof here.

The relationship between formal groups and p-adic integration has been known and exploited in some form by many authors. We have already mentioned the work of Katz [Kat] and Boxall [Box]. Height one formal groups and their connection to p-adic integration is systematically used in [dS] and we have adapted this approach to the height two case in Section 5 of our paper. Some other results of a similar flavor were obtained in [SI]. Finally, we point out that the appearance of p-adic Hodge theory in our work raises the interesting question of relating our results to the work of Colmez ([Col]).

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1. Preliminaries on restriction of scalars

We fix fields $\mathbb{Q}_p \subseteq L \subseteq K$ such that $L/\mathbb{Q}_p$ is finite and K is complete with respect to a nonarchimedean absolute value $|\ |$ extending the one on L. We also fix a commutative d-dimensional locally L-analytic group G. Then the locally convex K-vector space $C^{an}(G,K)$ of all K-valued locally analytic functions on G is defined ([Fe2] 2.1.10).

We consider now an intermediate field $\mathbb{Q}_p \subseteq L_0 \subseteq L$ and let G_0 denote the locally L_0-analytic group obtained from G by restriction of scalars ([B-VAR] §5.14). The dimension of G_0 is $d[L : L_0]$. There is the obvious injective continuous K-linear map

$$(*) \qquad C^{an}(G,K) \longrightarrow C^{an}(G_0,K)\ .$$

We want to describe the image of this map. The Lie algebra $\mathfrak{g}$ of G can naturally be identified with the Lie algebra of G_0 ([B-VAR] 5.14.5). We fix an exponential map $\exp : \mathfrak{g} - - - > G$ for G; it, in particular, is a local isomorphism, and can be viewed as an exponential map for G_0 as well. The Lie algebra $\mathfrak{g}$ acts in a compatible way on both sides of the above map by continuous endomorphisms defined by

$$(\mathfrak{x}f)(g) := \frac{d}{dt} f(\exp(-t\mathfrak{x})g)_{|t=0}$$

([Fe2] 3.1.2 and 3.3.4). By construction the map $\mathfrak{x} \rightarrow \mathfrak{x}f$ on $\mathfrak{g}$, for a fixed $f \in C^{an}(G_0, K)$, resp. $f \in C^{an}(G, K)$, is L_0-linear, resp. L-linear.

Lemma 1.1: *The image of* $(*)$ *is the closed subspace of all* $f \in C^{an}(G_0, K)$ *such that* $(t\mathfrak{x})f = t \cdot (\mathfrak{x}f)$ *for any* $\mathfrak{x} \in \mathfrak{g}$ *and any* $t \in L$.

Proof: We fix an L-basis $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ of $\mathfrak{g}$ as well an orthonormal basis $v_1 = 1, v_2, \ldots, v_e$ of L as a normed L_0-vector space. Then $v_1\mathfrak{x}_1, \ldots, v_e\mathfrak{x}_d$ is an L_0-basis of $\mathfrak{g}$. Using the corresponding canonical coordinates of the second kind ([B-GAL] Chap.III,§4.3) we have, for a given $f \in C^{an}(G_0, K)$ and a given $g \in G$, the convergent expansion

$$f(\exp(t_{11}v_1\mathfrak{x}_1 + \ldots + t_{ed}v_e\mathfrak{x}_d)g) = \sum_{n_{11},\ldots,n_{ed} \geq 0} c_{\underline{n}} t_{11}^{n_{11}} \ldots t_{ed}^{n_{ed}} \ ,$$

with $c_{\underline{n}} \in K$, in a neighborhood of g (i.e., for $t_{ij} \in L_0$ small enough). We now assume that

$$(v_i\mathfrak{x}_j)f = v_i \cdot (\mathfrak{x}_j f) = v_i \cdot ((v_1\mathfrak{x}_j)f)$$

holds true for all i and j. Computing both sides in terms of the above expansion and comparing coefficients results in the equations

$$(n_{ij} + 1)c_{(n_{11},\ldots,n_{ij}+1,\ldots,n_{ed})} = v_i(n_{1j} + 1)c_{(n_{11},\ldots,n_{1j}+1,\ldots,n_{ed})} \ .$$

Introducing the tuple $\underline{m}(\underline{n}) = (m_1, \ldots, m_d)$ defined by $m_j := n_{1j} + \ldots + n_{ej}$ and the new coefficients $b_{\underline{m}(\underline{n})} := c_{(m_1,0,\ldots,m_2,0,\ldots,m_d,0,\ldots)}$ we deduce from this by induction that

$$c_{\underline{n}} = b_{\underline{m}(\underline{n})} \frac{m_1!}{n_{11}! \ldots n_{e1}!} \cdots \frac{m_d!}{n_{1d}! \ldots n_{ed}!} v_1^{n_{11}+\ldots n_{1d}} \ldots v_e^{n_{e1}+\ldots+n_{ed}} \ .$$

Inserting this back into the above expansion and setting $t_j := t_{1j}v_1 + \ldots + t_{ej}v_e$ we obtain the new expansion

$$f(\exp(t_1\mathfrak{x}_1 + \ldots + t_d\mathfrak{x}_d)g) = \sum_{m_1,\ldots,m_d \geq 0} b_{\underline{m}} t_1^{m_1} \ldots t_d^{m_d}$$

which shows that f is locally analytic on G.

Lemma 1.2: *The map* $(*)$ *is a homeomorphism onto its (closed) image.*

Proof: Let $H \subseteq G$ be a compact open subgroup. According to [Fe2] 2.2.4) we then have

$$C^{an}(G,K) = \prod_{g\in G/H} C^{an}(H,K) \ .$$

A corresponding decomposition holds for G_0. This shows that it suffices to consider the case where G is compact. In this case $(*)$ is a compact inductive limit of isometries between Banach spaces ([Fe2] 2.3.2), and the assertion follows from [GKPS] 3.1.16.

The continuous dual $D(G,K) := C^{an}(G,K)'$ is the algebra of K-valued distributions on G. The multiplication is the convolution product $*$ ([Fe1] 4.4.2 and 4.4.4).

We assume from now on that G is compact. To describe the correct topology on $D(G,K)$ we need to briefly recall the construction of $C^{an}(G,K)$. Let $G \supseteq H_0 \supseteq H_1 \supseteq \ldots \supseteq H_n \supseteq \ldots$ be a fundamental system of open subgroups such that each H_n corresponds under the exponential map to an L-affinoid disk. We then have, for each $g \in G$ and $n \in \mathbb{N}$, the K-Banach space $\mathcal{F}_{gH_n}(K)$ of K-valued L-analytic functions on the coset gH_n viewed as an L-affinoid disk. The space $C^{an}(G,K)$ is the locally convex inductive limit

$$C^{an}(G,K) = \varinjlim_n \mathcal{F}_n(G,K)$$

of the Banach spaces

$$\mathcal{F}_n(G,K) := \prod_{g\in G/H_n} \mathcal{F}_{gH_n}(K) \ .$$

The dual $D(G,K)$ therefore coincides as a vector space with the projective limit

$$D(G,K) = \varprojlim_n \mathcal{F}_n(G,K)'$$

of the dual Banach spaces. We always equip $D(G,K)$ with the corresponding projective limit topology. (Using [GKPS] 3.1.7(vii) and the open mapping theorem one can show that this topology in fact coincides with the strong dual topology.) In particular, $D(G,K)$ is a commutative K-Fréchet algebra. The dual of the map $(*)$ is a continuous homomorphism of Fréchet algebras

$$(*)' \qquad D(G_0,K) \longrightarrow\!\!\!\!\!\rightarrow D(G,K) \ .$$

It is surjective since $C^{an}(G_0,K)$ as a compact inductive limit is of countable type ([GKPS] 3.1.7(viii)) and hence satisfies the Hahn-Banach theorem ([Sh] 4.2 and 4.4). By the open mapping theorem $(*)'$ then is a quotient map.

The action of $\mathfrak{g}$ on $C^{an}(G_0, K)$ induces an action of $\mathfrak{g}$ on $D(G_0, K)$ by $(\mathfrak{x}\lambda)(f) := \lambda(-\mathfrak{x}f)$. This action is related to the algebra structure through the L_0-linear inclusion

$$\begin{array}{rccl} \iota: & \mathfrak{g} & \longrightarrow & D(G_0, K) \\ & \mathfrak{x} & \longmapsto & [f \mapsto (-\mathfrak{x}(f))(1)] \end{array}$$

which satisfies

$$\iota(\mathfrak{x}) * \lambda = \mathfrak{x}\lambda \quad \text{for } \mathfrak{x} \in \mathfrak{g} \text{ and } \lambda \in D(G_0, K)$$

(see the end of section 2 in [ST]). Followed by $(*)'$ this inclusion becomes L-linear.

Let $\widehat{G_0}(K) \subseteq C^{an}(G_0, K)$ denote the subset of all K-valued locally analytic characters on G_0. Any $\chi \in \widehat{G_0}(K)$ induces the L_0-linear map

$$\begin{array}{rccl} d\chi: & \mathfrak{g} & \longrightarrow & K \\ & \mathfrak{x} & \longmapsto & \frac{d}{dt}\chi(\exp(t\mathfrak{x}))_{|t=0} \ . \end{array}$$

Lemma 1.3: $\widehat{G}(K) = \{\chi \in \widehat{G_0}(K) : d\chi \text{ is } L\text{-linear}\}$.

Proof: Because of

$$(-\mathfrak{x}\chi)(g) = \chi(g) \cdot d\chi(\mathfrak{x})$$

this is a consequence of Lemma 1.1.

The lemma says that the diagram

$$\begin{tikzcd} \widehat{G}(K) \arrow[r, "\subseteq"] \arrow[d, "d"'] & \widehat{G_0}(K) \arrow[d, "d"] \\ \mathrm{Hom}_L(\mathfrak{g}, K) \arrow[r, "\subseteq"] & \mathrm{Hom}_{L_0}(\mathfrak{g}, K) \end{tikzcd}$$

is cartesian.

We suppose from now on that K is a subfield of $\mathbb{C}_p$ (= the completion of an algebraic closure of $\mathbb{Q}_p$). There is the natural strict inclusion

$$\begin{array}{rcl} D(G_0, K) & = & \varprojlim_n \mathrm{Hom}_K^{cont}(\mathcal{F}_n(G_0, K), K) \\ & & \downarrow \\ D(G_0, \mathbb{C}_p) & = & \varprojlim_n \mathrm{Hom}_{\mathbb{C}_p}^{cont}(\mathcal{F}_n(G_0, \mathbb{C}_p), \mathbb{C}_p) \ . \end{array}$$

The <u>*Fourier transform*</u> F_λ of a $\lambda \in D(G_0, K)$, by definition, is the function

$$\begin{array}{rccl} F_\lambda: & \widehat{G_0}(\mathbb{C}_p) & \longrightarrow & \mathbb{C}_p \\ & \chi & \longmapsto & \lambda(\chi) \ . \end{array}$$

Proposition 1.4: *i. For any $\lambda \in D(G_0, K)$ we have $\lambda = 0$ if and only if $F_\lambda = 0$;*

ii. $F_{\mu * \lambda} = F_\mu F_\lambda$ for any two $\mu, \lambda \in D(G_0, K)$.

Proof: [Fe1] Thm. 5.4.8 (recall that G is assumed to be compact). For the convenience of the reader we sketch the proof of the first assertion in the case of the additive group $G_0 = G = o_L$. (This is the only case in which we actually will use this result in the next section. Moreover, the general proof is just an elaboration of this special case.) Let $\mathfrak{b} \subseteq o_L$ be an arbitrary nonzero ideal viewed as an additive subgroup. We use the convention to denote by $f|a + \mathfrak{b}$, for any function f on o_L and any coset $a + \mathfrak{b} \subseteq o_L$, the function on o_L which is equal to f on the coset $a + \mathfrak{b}$ and which vanishes elsewhere. Suppose now that $F_\lambda = 0$, i.e., that $\lambda(\chi) = 0$ for any $\chi \in \widehat{G}(\mathbb{C}_p)$. Using the character theory of finite abelian groups one easily concludes that

$$\lambda(\chi|a + \mathfrak{b}) = 0 \quad \text{for any } \chi \in \widehat{G}(\mathbb{C}_p) \text{ and any coset } a + \mathfrak{b} \subseteq o_L \ .$$

We apply this to the character $\chi_y(x) := \exp(yx)$ where $y \in o_{\mathbb{C}_p}$ is small enough and obtain by continuity that

$$0 = \lambda(\chi_y|a + \mathfrak{b}) = \sum_{n \geq 0} \frac{y^n}{n!} \lambda(x^n|a + \mathfrak{b}) \ .$$

Viewing the right hand side as a power series in y in a small neighbourhood of zero it follows that

$$\lambda(x^n|a + \mathfrak{b}) = 0 \quad \text{for any } n \geq 0 \text{ and any coset } a + \mathfrak{b} \subseteq o_L \ .$$

Again from continuity we see that $\lambda = 0$.

Corollary 1.5: *i.* $\widehat{G}(\mathbb{C}_p) = \{\chi \in \widehat{G_0}(\mathbb{C}_p) : F_{\iota(t\mathfrak{x}) - t\iota(\mathfrak{x})}(\chi) = 0$ for any $\mathfrak{x} \in \mathfrak{g}$ and $t \in L\}$*;*

ii. the kernel of $(*)'$ is the ideal $I(G) := \{\lambda \in D(G_0, K) : F_\lambda | \widehat{G}(\mathbb{C}_p) = 0\}$.

Proof: The assertion i. is a consequence of Lemma 1.3 and the identity

$$F_{\iota(t\mathfrak{x}) - t\iota(\mathfrak{x})}(\chi) = ((-t\mathfrak{x})\chi)(1) - t(-\mathfrak{x}\chi)(1) = d\chi(t\mathfrak{x}) - t \cdot d\chi(\mathfrak{x}) \ .$$

The assertion ii. follows from Prop. 1.4.i (applied to G).

2. The Fourier transform for $G = o_L$

Let $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ again be a chain of complete fields and let $o := o_L$ denote the ring of integers in L. The aim of this section is to determine the image of the Fourier transform for the compact additive group $G := o$. The restriction of scalars G_0 will always be understood with respect to the extension $L/\mathbb{Q}_p$.

First we have to discuss briefly a certain way to write rigid analytic polydisks in a coordinate free manner. Let $\mathbf{B}_1$ denote the rigid L-analytic open disk of radius one around the point $1 \in L$; its K-points are $\mathbf{B}_1(K) = \{z \in K : |z-1| < 1\}$. We note that the group $\mathbb{Z}_p$ acts on $\mathbf{B}_1$ via the rigid analytic automorphisms

$$\begin{array}{ccc} \mathbb{Z}_p \times \mathbf{B}_1 & \longrightarrow & \mathbf{B}_1 \\ (a, z) & \longmapsto & z^a := \sum\limits_{n \geq 0} \binom{a}{n}(z-1)^n \end{array}$$

(compare [Sch] §§32 and 47). Hence, given any free $\mathbb{Z}_p$-module M of finite rank r, we can in an obvious sense form the rigid L-analytic variety $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M$ whose K-points are $\mathbf{B}_1(K) \otimes_{\mathbb{Z}_p} M$. Any choice of an $\mathbb{Z}_p$-basis of M defines an isomorphism between $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M$ and an r-dimensional open polydisk over L. In particular, the family of all affinoid subdomains in $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M$ has a countable cofinal subfamily. Writing the ring $\mathcal{O}(\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M)$ of global holomorphic functions on $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M$ as the projective limit of the corresponding affinoid algebras we see that $\mathcal{O}(\mathbf{B}_1 \otimes_{\mathbb{Z}_p} M)$ in a natural way is an L-Fréchet algebra.

After this preliminary discussion we recall that the maps

$$\begin{array}{ccc} \widehat{\mathbb{Z}_p}(K) & \longleftrightarrow & \mathbf{B}_1(K) \\ \chi & \longmapsto & \chi(1) \\ \chi_z(a) := z^a & \longleftarrow\!\!\shortmid & z \end{array}$$

are bijections inverse to each other (compare [Am2] 1.1 and [B-GAL] III.8.1). They straightforwardly generalize to the bijection

$$\begin{array}{ccc} \mathbf{B}_1(K) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) & \xrightarrow{\sim} & \widehat{G_0}(K) \\ z \otimes \beta & \longmapsto & \chi_{z\otimes\beta}(g) := z^{\beta(g)} \ . \end{array}$$

By transport of structure the right hand side therefore can and will be considered as the K-points of a rigid analytic group variety $\widehat{G_0}$ over L (which is non-canonically isomorphic to an open polydisk of dimension $[L : \mathbb{Q}_p]$). By construction the Lie algebra of $\widehat{G_0}$ is equal to $\mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{g}, L)$. One easily checks that

$$d\chi_{z\otimes\beta} = \log(z) \cdot \beta \ .$$

If we combine this identity with the commutative diagram after Lemma 1.3 we arrive at the following fact which is recorded here for use in the next section.

Lemma 2.1: *The diagram*

$$\begin{tikzcd} \widehat{G}(K) \arrow[r, "\subseteq"] \arrow[d, "d"] & \mathbf{B}_1(K) \otimes \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \arrow[d, "\log \otimes \mathrm{id}"] \\ \mathrm{Hom}_L(\mathfrak{g}, K) \arrow[r, "\subseteq"] & \mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{g}, K) = K \otimes \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \end{tikzcd}$$

is cartesian.

We denote by $\mathcal{O}(\widehat{G_0}/K)$ the K-Fréchet algebra of global holomorphic functions on the base extension of the variety $\widehat{G_0}$ to K. The main result of Fourier analysis over the field $\mathbb{Q}_p$ is the following.

Theorem 2.2 (Amice): *The Fourier transform is an isomorphism of K-Fréchet algebras*

$$\begin{array}{rcl} \mathcal{F} : D(G_0, K) & \xrightarrow{\cong} & \mathcal{O}(\widehat{G_0}/K) \\ \lambda & \longmapsto & F_\lambda \ . \end{array}$$

Proof: This is a several variable version of [Am2] 1.3 (compare also [Sc]) based on [Am1].

Next we want to compute the ideal $J(o) := \mathcal{F}(I(o))$ in $\mathcal{O}(\widehat{G_0}/K)$. Let $\mathfrak{x}_1 := 1 \in \mathfrak{g} = L$ and $F_t := F_{\iota(t\mathfrak{x}_1) - t\iota(\mathfrak{x}_1)} \in \mathcal{O}(\widehat{G_0}/K)$ for $t \in L$. A straightforward computation shows that

$$F_t(\chi_{z \otimes \beta}) = (\beta(t) - t \cdot \beta(1)) \cdot \log(z) \ .$$

By Cor. 1.5 we have the following facts:

1) $\widehat{G}(\mathbb{C}_p)$ is the analytic subset of the variety $\widehat{G_0}/K$ defined by $F_t = 0$ for $t \in L$.

2) $J(o)$ is the ideal of all global holomorphic functions which vanish on $\widehat{G}(\mathbb{C}_p)$.

In 1) one can replace the family of all F_t by finitely many $F_{t_1}, \ldots, F_{t_e}$ if $t_1, \ldots, t_e$ runs through a $\mathbb{Q}_p$-basis of L.

According to [BGR] 9.5.2 Cor.6 the sheaf of ideals $\mathcal{J}$ in the structure sheaf $\mathcal{O}_{\widehat{G_0}}$ of the variety $\widehat{G_0}$ consisting of all germs of functions vanishing on the analytic subset $\widehat{G}(\mathbb{C}_p)$ is coherent. Moreover, [BGR] 9.5.3 Prop.4 says that the analytic subset $\widehat{G}(\mathbb{C}_p)$ carries the structure of a reduced closed L-analytic subvariety $\widehat{G} \subseteq \widehat{G_0}$ such that for the structure sheaves we have $\mathcal{O}_{\widehat{G}} = \mathcal{O}_{\widehat{G_0}}/\mathcal{J}$. Since $\widehat{G_0}$ is a Stein space the global section functor is exact on coherent sheaves. All this remains true of course after base extension to K. Hence, if $\mathcal{O}(\widehat{G}/K)$ denotes the ring of global holomorphic functions on the base extension of the variety $\widehat{G}$ to K then, by 2), we have

$$(+) \qquad \mathcal{O}(\widehat{G}/K) = \mathcal{O}(\widehat{G_0}/K)/J(o) \ .$$

The ideal $J(o)$ being closed $\mathcal{O}(\widehat{G}/K)$ in particular is in a natural way a K-Fréchet algebra as well. It is clear from the open mapping theorem that this quotient topology on $\mathcal{O}(\widehat{G}/K)$ coincides with the topology as a projective limit of affinoid algebras.

Theorem 2.3: *The Fourier transform is an isomorphism of K-Fréchet algebras*

$$\begin{array}{rcl} \mathcal{F} : D(G,K) & \xrightarrow{\cong} & \mathcal{O}(\widehat{G}/K) \\ \lambda & \longmapsto & F_\lambda \ . \end{array}$$

Proof: This follows from Thm. 2.2, (+), and the surjection $(*)'$ in section 1.

We remark that by construction we (noncanonically) have a cartesian diagram of rigid L-analytic varieties of the form

$$\begin{tikzcd} \widehat{G} \arrow[r] \arrow[d, "d"] & (\mathbf{B}_1)^d \arrow[d, "\log"] \\ \mathbb{A}^1 \arrow[r] & \mathbb{A}^d \end{tikzcd}$$

with $\mathbb{A}^m$ denoting affine m-space where the horizontal arrows are closed immersions and the vertical arrows are etale. Hence the variety $\widehat{G}$ is smooth and quasi-Stein (in the sense of [Kie]).

Lemma 2.4: *$\widehat{G}$ is a smooth rigid analytic group variety over L.*

Proof: We have constructed $\widehat{G}$ as a reduced closed subvariety of the rigid analytic group variety $\widehat{G_0}$ over L. With $\widehat{G}$ also $\widehat{G} \times \widehat{G}$ is smooth. In particular, $\widehat{G} \times \widehat{G}$ is a reduced closed subvariety of $\widehat{G_0} \times \widehat{G_0}$. Since the multiplication and the inverse on $\widehat{G_0}$ preserve $\widehat{G}(\mathbb{C}_p)$ they restrict to morphisms between these reduced subvarieties.

3. Lubin-Tate formal groups and twisted unit disks

Keeping the notations of the previous section we will give in this section a different description of the rigid variety $\widehat{G}$. We will show that the character variety $\widehat{G}$ becomes isomorphic to the open unit disk after base change to $\mathbb{C}_p$. As a corollary, the ring of functions $\mathcal{O}(\widehat{G}/\mathbb{C}_p)$ is the same for any group $G = o$. This result originates in the observation that the character group $\widehat{G}(\mathbb{C}_p)$ can be parametrized with the help of Lubin-Tate theory.

Fix a prime element π of o and let $\mathcal{G} = \mathcal{G}_\pi$ denote the corresponding Lubin-Tate formal group over o. It is commutative and has dimension one and height $[L : \mathbb{Q}_p]$. Most importantly, $\mathcal{G}$ is a formal o-module which means that the ring o acts on $\mathcal{G}$ in such a way that the induced action of o on the tangent space

$t_{\mathcal{G}}$ is the one coming from the natural o-module structure on the latter ([LT]). We always identify $\mathcal{G}$ with the rigid L-analytic open unit disk $\mathbf{B}$ around zero in L. In this way $\mathbf{B}$ becomes an o-module object, and we will denote the action $o \times \mathbf{B} \longrightarrow \mathbf{B}$ by $(g,z) \longmapsto [g](z)$. This identification, of course, also trivializes the tangent space $t_{\mathcal{G}}$.

Let $\mathcal{G}'$ denote the p-divisible group dual to $\mathcal{G}$ and let $T' = T(\mathcal{G}')$ be the Tate module of $\mathcal{G}'$. Lubin-Tate theory tells us that T' is a free o-module of rank one and that the Galois action on T' is given by a continuous character $\tau : \mathrm{Gal}(\mathbb{C}_p/L) \longrightarrow o^\times$. From [Tat] p.177 we know that, by Cartier duality, T' is naturally identified with the group of homomorphisms of formal groups over $o_{\mathbb{C}_p}$ between $\mathcal{G}$ and the formal multiplicative group. This gives rise to natural Galois equivariant and o-invariant pairings

$$< \, , \, > : \; T' \underset{o}{\otimes} \mathbf{B}(\mathbb{C}_p) \longrightarrow \mathbf{B}_1(\mathbb{C}_p)$$

and on tangent spaces

$$(\, , \,) \; : \; T' \underset{o}{\otimes} \mathbb{C}_p \longrightarrow \mathbb{C}_p \; .$$

To describe them explicitly we will denote by $F_{t'}(Z) = \Omega_{t'} Z + \ldots \in Z o_{\mathbb{C}_p}[[Z]]$, for any $t' \in T'$, the power series giving the corresponding homomorphism of formal groups. Then

$$< t', z > = 1 + F_{t'}(z) \quad \text{and} \quad (t', x) = \Omega_{t'} x \; .$$

Proposition 3.1: *The map*

$$(\diamond) \qquad \begin{array}{ccc} \mathbf{B}(\mathbb{C}_p) \otimes_o T' & \longrightarrow & \widehat{G}(\mathbb{C}_p) \\ z \otimes t' & \longmapsto & \kappa_{z \otimes t'}(g) := < t', [g](z) > \end{array}$$

is a well defined isomorphism of groups.

Proof: We will study the following diagram:

$$\begin{tikzcd}
\mathbf{B}(\mathbb{C}_p) \otimes_o T' \arrow[rr, "\log_{\mathcal{G}} \otimes \mathrm{id}"] \arrow[dr, dashed] \arrow[dd, "\alpha"] & & t_{\mathcal{G}}(\mathbb{C}_p) \otimes_o T' \arrow[dr, dashed] \arrow[dd, "d\alpha"] & \\
& \widehat{G}(\mathbb{C}_p) \arrow[rr, "d"] \arrow[dd, "\subseteq"] & & \mathrm{Hom}_L(\mathfrak{g}, \mathbb{C}_p) \arrow[dd, "\subseteq"] \\
\mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbf{B}_1(\mathbb{C}_p)) \arrow[rr, "\mathrm{Hom}(.,\log)"] \arrow[dr, dashed] & & \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{C}_p) \arrow[dr, dashed] & \\
& \widehat{G}_0(\mathbb{C}_p) \arrow[rr, "d"] & & \mathrm{Hom}_{\mathbb{Q}_p}(\mathfrak{g}, \mathbb{C}_p)
\end{tikzcd}$$

Here:

a. The rear face of the cube in this diagram is the tensorization by T' of a portion of the map of exact sequences labelled $(*)$ in [Tat] §4. We use for this the identification

$$\mathrm{Hom}_{\mathbb{Z}_p}(T', .) \otimes_o T' = \mathrm{Hom}_{\mathbb{Z}_p}(o, .) \ .$$

By $\log_{\mathcal{G}}$ we denote the logarithm map of the formal group $\mathcal{G}$. The map α, resp. $d\alpha$, associates to an element $z \otimes t'$, resp. $\mathfrak{x} \otimes t'$, the map $g \mapsto < gt', z >$, resp. $g \mapsto (gt', \mathfrak{x})$, for $g \in o$.

b. The front face of the cube is the diagram after Lemma 1.3.

c. The dashed arrows on the bottom face of the cube come from the discussion before Lemma 2.1.

d. The dashed arrow in the upper left of the diagram is the one we want to establish.

e. The formal o-module property of $\mathcal{G}$ says that the induced o-action on $t_{\mathcal{G}}(\mathbb{C}_p)$ is the same as the action by linearity and the inclusion $o \subseteq \mathbb{C}_p$. It means that we have $(gt', \mathfrak{x}) = (t', g\mathfrak{x}) = g \cdot (t', \mathfrak{x})$ for $g \in o$ and hence that any map in the image of $d\alpha$ is o-linear. This defines the dashed arrow in the upper right of the diagram.

The back face of the cube is commutative by [Tat] §4. The front and bottom faces are commutative by Lemma 1.3 and Lemma 2.1. Furthermore, the dashed arrows on the bottom of the diagram are isomorphisms, the left one by the discussion before Lemma 2.1 and the right one for trivial reasons.

Consider now the right side of the cube. It is commutative by construction. Since, by [Tat] Prop. 11, $d\alpha$ is injective and since the lower dashed arrow is bijective the upper dashed arrow must at least be injective. But by a comparison of dimensions we see that the dashed arrow in the upper right of the cube is an isomorphism as well.

In this situation we now may use the fact that, by Lemma 1.3, the front of the cube is cartesian to obtain that the upper left dashed arrow is well defined (making the whole cube commutative) and is given by $(\diamond)$. But according to [Tat] Prop. 11 the back of the cube also is cartesian. Therefore the map $(\diamond)$ in fact is an isomorphism. $\square$

Fixing a generator t'_{o} of the o-module T' the isomorphism $(\diamond)$ becomes

$$(\diamond\diamond) \qquad \begin{array}{ccc} \mathbf{B}(\mathbb{C}_p) & \xrightarrow{\cong} & \widehat{G}(\mathbb{C}_p) \\ z & \longmapsto & \kappa_z := \kappa_{z \otimes t'_{\mathrm{o}}} \ . \end{array}$$

The main purpose of this section is to see that this latter isomorphism derives from an isomorphism $\mathbf{B}/\mathbb{C}_p \xrightarrow{\cong} \widehat{G}/\mathbb{C}_p$ between rigid $\mathbb{C}_p$-analytic varieties. In fact, we will exhibit compatible admissible coverings by affinoid open subsets on both sides.

Let us begin with the left hand side. For any $r \in p^{\mathbb{Q}}$ we have the affinoid disk

$$\mathbf{B}(r) := \{z : |z| \leq r\}$$

over L. Clearly the disks $\mathbf{B}(r)$ for $r < 1$ form an admissible covering of $\mathbf{B}$. It actually will be convenient to normalize the absolute value $|\ |$ and we do this by the requirement that $|p| = p^{-1}$. The numerical invariants of the finite extension $L/\mathbb{Q}_p$ which will play a role are the ramification index e and the cardinality q of the residue class field of L. Recall that o acts on $\mathbf{B}$ since we identify $\mathbf{B}$ with the formal group $\mathcal{G}_\pi$. We need some information how this covering behaves with respect to the action of π.

LEMMA 3.2: *For any $r \in p^{\mathbb{Q}}$ such that $p^{-q/e(q-1)} \leq r < 1$ we have*

$$[\pi]^{-1}(\mathbf{B}(r)) = \mathbf{B}(r^{1/q}) \quad \textit{and} \quad [p]^{-1}(\mathbf{B}(r)) = \mathbf{B}(r^{1/q^e}) \ ;$$

further, in this situation the map $[\pi^n] : \mathbf{B}(r^{1/q^n}) \to \mathbf{B}(r)$, for any $n \in \mathbb{N}$, is a finite etale affinoid map.

Proof: The second identity is a consequence of the first since π^e and p differ by a unit in o, and for any unit $g \in o^\times$ one has $|[g](z)| = |z|$. Moreover, up to isomorphism, we may assume ([Lan] §8.1) that the action of the prime element π on $\mathbf{B}$ is given by

$$[\pi](z) = \pi z + z^q \ .$$

In this case the first identity follows by a straightforward calculation of absolute values. The finiteness and etaleness of the map $[\pi^n]$ also follows from this explicit formula together with the fact that a composition of finite etale affinoid maps is finite etale.

Now we consider the right side of $(\diamond\diamond)$. The disk $\mathbf{B}_1$ has the admissible covering by the L-affinoid disks $\mathbf{B}_1(r) := \{z : |z-1| \leq r\}$ for $r \in p^{\mathbb{Q}}$ such that $r < 1$. They are $\mathbb{Z}_p$-submodules so that the L-affinoids $\mathbf{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p)$ form an admissible covering of $\mathbf{B}_1 \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p) \cong \widehat{G_0}$. We therefore have the admissible covering of $\widehat{G}$ by the L-affinoids $\widehat{G}(r) := \widehat{G} \cap (\mathbf{B}_1(r) \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(o, \mathbb{Z}_p))$. We emphasize that on both sides the covering is defined over L.

LEMMA 3.3: *For any $r \in p^{\mathbb{Q}}$ such that $p^{-p/(p-1)} \leq r < 1$ we have*

$$\{\chi \in \widehat{G} : \chi^p \in \widehat{G}(r)\} = \widehat{G}(r^{1/p}) \ ;$$

further, in this situation the map $[p^n] : \widehat{G}(r^{1/p^n}) \to \widehat{G}(r)$, for any $n \in \mathbb{N}$, is a finite etale affinoid map.

Proof: This follows from a corresponding identity between the affinoids $\mathbf{B}_1(r)$. It is, in fact, a special case of the previous lemma.

In order to see in which way the isomorphism $(\diamond\diamond)$ respects these coverings, we need more detailed information on the power series $F_{t'}$ representing $t' \in T'$. We summarize the facts that we require in the following lemma.

Lemma 3.4: *Suppose $t' \in T'$ is non-zero; the power series $F_{t'}(Z) = \Omega_{t'}Z + \ldots \in o_{\mathbb{C}_p}[[Z]]$ has the following properties:*

a. $\Omega_{gt'} = \Omega_{t'}g$ for $g \in o$;

b. if t' generates T' as an o-module, then

$$|\Omega_{t'}| = p^{-s} \quad \textit{with} \quad s = \frac{1}{p-1} - \frac{1}{e(q-1)} \ ;$$

c. for any $r < p^{-1/e(q-1)}$, the power series $F_{t'}(Z)$ gives an analytic isomorphism between $\mathbf{B}(r)$ and $\mathbf{B}(r|\Omega_{t'}|)$.

Proof: Part (a) is a restatement of the o-linearity of the pairing (,) introduced at the start of this section. Part (b) follows from work of Fontaine ([Fon]) on p-adic Hodge theory. We give a proof in the appendix. For part (c), recall that $F_{t'}(Z)$ is a formal group homomorphism. Therefore if $F_{t'}(z) = 0$, $F_{t'}$ vanishes on the entire subgroup of $\mathbf{B}(\mathbb{C}_p)$ generated by z. The point z belongs to some $\mathbf{B}(r)(\mathbb{C}_p)$, and therefore so does the entire subgroup generated by z. If this subgroup were infinite, $F_{t'}$ would have infinitely many zeroes in the affinoid $\mathbf{B}(r)(\mathbb{C}_p)$ and would therefore be zero. Consequently z must be a torsion point of the group $\mathcal{G}$. But other than zero, there are no torsion points inside the disk $\mathbf{B}(r)(\mathbb{C}_p)$ if $r < p^{-1/e(q-1)}$ ([Lan] §8.6 Lemma 4 and 5). It follows that the power series $F_{t'}(Z)/\Omega_{t'}Z = 1 + c_1Z + c_2Z^2 + \ldots$ has no zeroes inside $\mathbf{B}(r)(\mathbb{C}_p)$. Suppose that some coefficient c_n in this expansion satisfies $|c_n| > p^{n/e(q-1)}$. Then by considering the Newton polygon of the power series $F_{t'}(Z)/\Omega_{t'}Z$ one sees that the power series in question must have a zero of absolute value less than $p^{-1/e(q-1)}$, which we have seen is impossible. Therefore $|c_n| \leq p^{n/e(q-1)}$, from which part (c) follows immediately.

To simplify the notation, we write $\Omega = \Omega_{t'_o}$ for the "period" of the Lubin-Tate group associated to our fixed generator of T'.

By trivializing the tangent space as well as identifying $\mathrm{Hom}_L(\mathfrak{g}, \mathbb{C}_p)$ with $\mathbb{C}_p$ by evaluation at 1 we may simplify the upper face of the cubical diagram in the proof of Prop. 3.1 to the following:

$$(*) \qquad \begin{tikzcd} \mathbf{B}(\mathbb{C}_p) \arrow[r, "\log_{\mathcal{G}}"] \arrow[d, "z\mapsto\kappa_z"'] & \mathbb{C}_p \arrow[d, "x\mapsto\Omega x"] \\ \widehat{G}(\mathbb{C}_p) \arrow[r, "\kappa_z\mapsto\log\kappa_z(1)"'] & \mathbb{C}_p \end{tikzcd}$$

Let us examine the map $\kappa(z) := \kappa_z$ in coordinates. Choose for the moment a $\mathbb{Z}_p$-basis $e_1, \ldots, e_d$ for o and let $e_1^*, \ldots, e_d^*$ be the dual basis. In coordinates, the map $\kappa : \mathbf{B} \to \widehat{G}_0$ is given by

$$(**) \qquad \kappa_z = \sum_{i=1}^{d} (1 + F_{e_i t'_o}(z)) \otimes e_i^* \ .$$

Note first that this map is explicitly rigid $\mathbb{C}_p$-analytic and we know by Prop. 3.1 that this map factorizes through the subvariety $\widehat{G}/\mathbb{C}_p \subset \widehat{G}_0/\mathbb{C}_p$. We now also see that, if $r = p^{-q/e(q-1)} < p^{-1/e(q-1)}$, the three parts of Lemma 3.4 together imply that this map carries $\mathbf{B}(r)/\mathbb{C}_p$ into $\widehat{G}(r|\Omega|)/\mathbb{C}_p$.

Lemma 3.5: *Let $r = p^{-q/e(q-1)}$; the map*

$$\begin{array}{ccc} \mathbf{B}(r)/\mathbb{C}_p & \xrightarrow{\cong} & \widehat{G}(r|\Omega|)/\mathbb{C}_p \\ z & \longmapsto & \kappa_z \end{array}$$

is a rigid isomorphism.

Proof: In the discussion preceeding the statement of the lemma we saw that this is a well-defined rigid map. Consider now the other maps in the diagram $(*)$.

– For $r = p^{-q/e(q-1)} < p^{-1/e(q-1)}$, the logarithm $\log_{\mathcal{G}}$ of the formal group $\mathcal{G}$ restricts to a rigid isomorphism

$$\log_{\mathcal{G}} : \mathbf{B}(r) \xrightarrow{\cong} \mathbf{B}(r) \ .$$

([Lan] §8.6 Lemma 4).

– Because $|\Omega| r = p^{-1/(p-1)-1/e} < p^{-1/(p-1)}$, the usual logarithm restricts to a rigid isomorphism

$$\log : \mathbf{B}_1(r|\Omega|) \xrightarrow{\cong} \mathbf{B}(r|\Omega|) \ .$$

All of this information, together with the diagram $(*)$, tells us that the following diagram of rigid morphisms commutes:

$$\begin{tikzcd} \mathbf{B}(r)/\mathbb{C}_p \arrow[r, "\cong"] \arrow[d, "z \mapsto \kappa_z"'] & \mathbf{B}(r)/\mathbb{C}_p \arrow[d, "\cong"] \\ \widehat{G}(r|\Omega|)/\mathbb{C}_p \arrow[r, "\kappa_z \mapsto \log \kappa_z(1)"'] & \mathbf{B}(r|\Omega|)/\mathbb{C}_p \end{tikzcd}$$

We claim that the lower arrow in this diagram is injective on $\mathbb{C}_p$-points. Assume that $\log \kappa_z(1) = 0$; we then have $\kappa_z(1) = 1$ which, by the local L-analyticity of κ_z, means that κ_z is locally constant and hence of finite order. But for our value of r we know that $\mathbf{B}_1(r)(\mathbb{C}_p)$ is torsionfree so it follows that κ_z must be the trivial character.

Because the upper horizontal and the right vertical map are rigid isomorphisms and the other two maps at least are injective on $\mathbb{C}_p$-points, all the maps in this diagram must be isomorphisms on $\mathbb{C}_p$-points. Because $\widehat{G}$ is reduced, it follows that the other arrows are rigid isomorphisms as well.

This lemma provides a starting point for the proof of the main theorem of this section.

Theorem 3.6: *The map*

$$\kappa : \mathbf{B}/\mathbb{C}_p \xrightarrow{\cong} \widehat{G}/\mathbb{C}_p$$

is an isomorphism of rigid varieties over $\mathbb{C}_p$; more precisely, if $r = p^{-q/e(q-1)}$ and $n \in \mathbb{N}_0$, then κ is a rigid isomorphism between the affinoids

$$\kappa : \mathbf{B}(r^{1/q^{en}})/\mathbb{C}_p \xrightarrow{\cong} \widehat{G}((r|\Omega|)^{1/p^n})/\mathbb{C}_p \ .$$

Proof: We remark first that the second statement is in fact stronger than the first, because as n runs through $\mathbb{N}_0$ the given affinoids form admissible coverings of $\mathbf{B}/\mathbb{C}_p$ and $\widehat{G}/\mathbb{C}_p$ respectively.

Lemma 3.5 is the case $n = 0$. To obtain the result for all n, fix $n > 0$ and consider the diagram:

$$\begin{tikzcd}
\mathbf{B}(r^{1/q^{en}})/\mathbb{C}_p \arrow[r, "z\mapsto\kappa_z"] \arrow[d, "{[p^n]}"] & \widehat{G}((r|\Omega|)^{1/p^n})/\mathbb{C}_p \arrow[d, "\chi\mapsto\chi^{p^n}"] \\
\mathbf{B}(r)/\mathbb{C}_p \arrow[r, "z\mapsto\kappa_z"] & \widehat{G}(r|\Omega|)/\mathbb{C}_p
\end{tikzcd}$$

By Lemma 3.2, the left-hand vertical arrow is a well-defined finite etale affinoid map of degree $q^{ne} = p^{nd}$. By Lemma 3.4, part (b), $1 > r|\Omega| = p^{-1/(p-1)-1/e} \geq p^{-p/(p-1)}$ so that Lemma 3.3 applies to the right-hand vertical arrow and it enjoys the same properties. Lemma 3.5 shows that the lower arrow is a rigid analytic isomorphism. The upper horizontal arrow then is a well-defined bijective map on points because of Proposition 3.1 and the first assertions in Lemma 3.2 and Lemma 3.3. It is a rigid morphism because it is given in coordinates by the same formula as in the $n = 0$ case (see $(**)$).

To complete the argument, let A and B be the affinoid algebras of $\mathbf{B}(r^{1/q^{en}})/\mathbb{C}_p$ and $\widehat{G}((r|\Omega|)^{1/p^n})/\mathbb{C}_p$ respectively. Let D be the affinoid algebra of $\widehat{G}(r|\Omega|)/\mathbb{C}_p$. The rings A and B are finite etale D-algebras of the same rank. The map $f : B \to A$ induced by the upper arrow in the diagram is a map of D-algebras.

Because f is bijective on maximal ideals and B is reduced (because $\widehat{G}$ is reduced), this map is injective. To see that f also is surjective it suffices, by [B-CA] II §3.3 Prop. 11, to check that the induced map $B/\mathfrak{m}B \longrightarrow A/\mathfrak{m}A$ is surjective for any maximal ideal $\mathfrak{m} \subseteq D$. But the latter is a map of finite etale algebras over $D/\mathfrak{m} = \mathbb{C}_p$ of the same dimension which is bijective on points. Hence it clearly must be bijective.

Corollary 3.7: *The ring of functions $\mathcal{O}(\widehat{G}/\mathbb{C}_p)$ is isomorphic to the ring $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ of $\mathbb{C}_p$-analytic functions on the open unit disk in $\mathbb{C}_p$; in particular, the distribution algebra $D(G,K)$ is an integral domain.*

Let us remark that a careful examination of the proofs shows that these results in fact hold true over any complete intermediate field between L and $\mathbb{C}_p$ which contains the period Ω.

The ring $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ is the ring of power series $F(z) = \sum_{n\geq 0} a_n z^n$ over $\mathbb{C}_p$ which converge on $\{z : |z| < 1\}$. Let $G_L := \mathrm{Gal}(\overline{L}/L)$ be the absolute Galois group of the field L and let G_L act on $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ by

$$F^\sigma(z) := \sum_{n\geq 0} \sigma(a_n) z^n \quad \text{for } \sigma \in G_L \ .$$

By Tate's theorem ([Tat]) we have

$$\mathbb{C}_p^{G_L} = L \ .$$

Hence the ring $\mathcal{O}(\mathbf{B})$ coincides with the ring of Galois fixed elements

$$\mathcal{O}(\mathbf{B}) = \mathcal{O}(\mathbf{B}/\mathbb{C}_p)^{G_L}$$

with respect to this action. This principle which here can be seen directly on power series in fact holds true for any quasi-separated rigid L-analytic variety $\mathcal{X}$; i.e., one has

$$\mathcal{O}(\mathcal{X}) = \mathcal{O}(\mathcal{X}/\mathbb{C}_p)^{G_L} \ .$$

By the way the base extension $\mathcal{X}/\mathbb{C}_p$ is constructed by pasting the base extension of affinoids ([BGR] 9.3.6) this identity immediately is reduced to the case of an affinoid variety. But for any L-affinoid algebra A we may consider an orthonormal base of A and apply Tate's theorem to the coefficients to obtain that

$$A = (A \mathbin{\widehat{\otimes}}_L \mathbb{C}_p)^{G_L} \ .$$

Since, according to our above theorem, $\mathbf{B}/\mathbb{C}_p$ also is the base extension of $\widehat{G}$ the ring $\mathcal{O}(\widehat{G})$ must be isomorphic to the subring of Galois fixed elements in the power series ring $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ with respect to a certain twisted Galois action. To work this out we first note that the natural Galois action on $\widehat{G}(\mathbb{C}_p)$ is given by composition $\kappa \mapsto \sigma \circ \kappa$ for $\sigma \in G_L$ and $\kappa \in \widehat{G}(\mathbb{C}_p)$ viewed as a character $\kappa : G \longrightarrow \mathbb{C}_p^\times$. Suppose that $\kappa = \kappa_z$ is the image of $z \in \mathbf{B}(\mathbb{C}_p)$ under the map ($\diamond\diamond$). The twisted Galois action $z \mapsto \sigma * z$ on $\mathbf{B}(\mathbb{C}_p)$ which we want to consider then is defined by

$$\kappa_{\sigma * z} = \sigma \circ \kappa_z$$

and we have

$$\mathcal{O}(\widehat{G}) \cong \{F \in \mathcal{O}(\mathbf{B}/\mathbb{C}_p) : F = F^\sigma(\sigma(\sigma^{-1} * z)) \text{ for any } \sigma \in G_L\} \ .$$

Recalling that $\tau : G_L \longrightarrow o^\times$ denotes the character which describes the Galois action on T' we compute

$$\begin{aligned} \sigma^{-1} \circ \kappa_z(g) &= \ < \sigma^{-1}(t'_\mathrm{o}), \sigma^{-1}([g](z)) > \ = \ < \tau(\sigma^{-1}) \cdot t'_\mathrm{o}, [g](\sigma^{-1}(z)) > \\ &= \ < t'_\mathrm{o}, [g]([\tau(\sigma^{-1})](\sigma^{-1}(z))) > \ = \ \kappa_{[\tau(\sigma^{-1})](\sigma^{-1}(z))}(g) \end{aligned}$$

for any $g \in G = o$. Hence

$$\sigma^{-1} * z = [\tau(\sigma^{-1})](\sigma^{-1}(z)) \quad \text{and} \quad \sigma(\sigma^{-1} * z) = [\tau(\sigma^{-1})](z) \ .$$

Corollary 3.8: $\mathcal{O}(\widehat{G}) \cong \{F \in \mathcal{O}(\mathbf{B}/\mathbb{C}_p) : F = F^\sigma \circ [\tau(\sigma^{-1})] \text{ for any } \sigma \in G_L\}$.

The following two negative facts show that our above results cannot be improved much.

Lemma 3.9: *Suppose that K is discretely valued; If $L \neq \mathbb{Q}_p$ then $\widehat{G}/K$ and $\mathbf{B}/K$ are not isomorphic as rigid K-analytic varieties.*

Proof: We consider the difference $\delta_1 - \delta_0 \in D(G, K)$ of the Dirac distributions in the elements 1 and 0 in $G = o$, respectively. Since the image of any character in $\widehat{G}(\mathbb{C}_p)$ lies in the 1-units of $o_{\mathbb{C}_p}$ we see that the Fourier transform of $\delta_1 - \delta_0$ as a function on $\widehat{G}(\mathbb{C}_p)$ is bounded. On the other hand every torsion point in $\widehat{G}(\mathbb{C}_p)$ corresponding to a locally constant character on the quotient $o/\mathbb{Z}_p$ is a zero of this function. Since $o \neq \mathbb{Z}_p$ we therefore have a nonzero function in $\mathcal{O}(\widehat{G}/K)$ which is bounded and has infinitely many zeroes. If $\widehat{G}/K$ and $\mathbf{B}/K$ were isomorphic this would imply the existence of a nonzero power series in $\mathcal{O}(\mathbf{B}/K)$ which as a function on the open unit disk in $\mathbb{C}_p$ is bounded and has infinitely many zeroes. By the maximum principle the former means that the coefficients of this power series are bounded in K. But according to the Weierstrass preparation theorem ([B-CA] VII§3.8 Prop. 6) a nonzero bounded power series over a discretely valued field can have at most finitely many zeroes. So we have arrived at a contradiction.

According to Lazard ([Laz]) the ring $\mathcal{O}(\mathbf{B}/K)$, for K spherically complete, is a so called Bezout domain which is the non-noetherian version of a principal ideal domain and which by definition means that any finitely generated ideal is principal. We show that this also fails in our setting as soon as $L \neq \mathbb{Q}_p$.

Lemma 3.10: *Suppose that K is discretely valued; if $L \neq \mathbb{Q}_p$ then the ideal of functions in $\mathcal{O}(\widehat{G}/K)$ vanishing in the trivial character $\kappa_0 \in \widehat{G}(L)$ is finitely generated but not principal.*

Proof: The ideal in question is a quotient of the corresponding ideal for the polydisk $\widehat{G_0}/K$ which visibly is finitely generated. Reasoning by contradiction let f be a generator of the ideal in the assertion. As a consequence of Theorems A and B ([Kie] Satz 2.4) for the quasi-Stein variety $\widehat{G}/K$ we then have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\widehat{G}/K} \xrightarrow{f\cdot} \mathcal{O}_{\widehat{G}/K} \longrightarrow K \longrightarrow 0$$

on $\widehat{G}/K$ where the third term is a skyscraper sheaf in the point κ_0. The corresponding sequence of sections in any affinoid subdomain is split-exact and hence remains exact after base extension to $\mathbb{C}_p$. It follows that we have a corresponding exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\widehat{G}/\mathbb{C}_p} \xrightarrow{f\cdot} \mathcal{O}_{\widehat{G}/\mathbb{C}_p} \longrightarrow \mathbb{C}_p \longrightarrow 0$$

on $\widehat{G}/\mathbb{C}_p$. Using Theorem B we deduce from it that f also generates the ideal of functions vanishing in κ_0 in $\mathcal{O}(\widehat{G}/\mathbb{C}_p)$. Consider f now as a rigid map $f : \widehat{G}/K \longrightarrow \mathbb{A}^1/K$ into the affine line. The composite $f \circ \kappa : \mathbf{B}/\mathbb{C}_p \longrightarrow \widehat{G}/\mathbb{C}_p \longrightarrow \mathbb{A}^1/\mathbb{C}_p$ then is given by a power series $F \in \mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ which generates the maximal ideal of functions vanishing in the point 0. Hence F is of the form

$$F(z) = az(1 + b_1 z + b_2 z^2 + \ldots) \quad \text{with } a \in \mathbb{C}_p \text{ and } b_i \in o_{\mathbb{C}_p}$$

and gives an isomorphism

$$B/\mathbb{C}_p \xrightarrow{\simeq} \mathbf{B}^-(|a|)/\mathbb{C}_p$$

between the open unit disk and the open disk $\mathbf{B}^-(|a|)$ of radius $|a|$ over $\mathbb{C}_p$. We see that f in fact is a rigid map

$$\widehat{G}/K \longrightarrow \mathbf{B}^-(|a|)/K$$

which becomes an isomorphism after base extension to $\mathbb{C}_p$. It follows from the general descent principle we have noted earlier that f induces an isomorphism of rings

$$\mathcal{O}(\mathbf{B}^-(|a|)/K) \xrightarrow{\cong} \mathcal{O}(\widehat{G}/K)$$

which obviously respects bounded functions. This leads to a contradiction by repeating the argument in the proof of the previous lemma. By a more refined descent argument one can in fact show that f already is an isomorphism of rigid K-analytic varieties which is in direct contradiction to Lemma 3.9.

We close this section by remarking that, because $\widehat{G}/K$ is a smooth 1-dimensional quasi-Stein rigid variety, one has, for any K, the following positive results about the integral domain $\mathcal{O}(\widehat{G}/K)$:

1. For ideals I of $\mathcal{O}(\widehat{G}/K)$, the three properties "$I$ is closed", "I is finitely generated", and "I is invertible" are equivalent.

2. The closed ideals in $\mathcal{O}(\widehat{G}/K)$ are in bijection with the divisors of $\widehat{G}/K$. (A divisor is an infinite sum of closed points, having only finite support in any affinoid subdomain). In addition a Baire category theory argument shows that given a divisor D there is a function $F \in \mathcal{O}(\widehat{G}/K)$ whose divisor is of the form $D + D'$ where D and D' have disjoint support.

3. Any finitely generated submodule in a finitely generated free $\mathcal{O}(\widehat{G}/K)$-module is closed.

In particular, $\mathcal{O}(\widehat{G}/K)$ is a Prüfer domain and consequently a coherent ring. We omit the proofs which consist of rather standard applications of Theorems A and B ([Kie]).

4. Generalized Mahler expansions

In this section we apply the Fourier theory to obtain a generalization of the Mahler expansion ([Am1] Cor. 10.2) for locally L-analytic functions. Crucial to our computations is the observation that the power series $F_{t'_o}(Z)$ introduced before Prop. 3.1, which gives the formal group homomorphism $\mathcal{G} \to \mathbb{G}_m$ associated to t'_o, is given as a formal power series by the formula

$$F_{t'_o}(Z) = \exp(\Omega \log_{\mathcal{G}}(Z)) - 1.$$

Throughout the following, we let ∂ denote the invariant differential on the formal group $\mathcal{G}$.

Definition 4.1: *For $m \in \mathbb{N}_0$, let $P_m(Y) \in L[Y]$ be the polynomial defined by the formal power series expansion*

$$\sum_{m=0}^{\infty} P_m(Y) Z^m = \exp(Y \log_{\mathcal{G}}(Z)) \ .$$

Observe that in the case $\mathcal{G} = \mathbb{G}_m$, we have

$$\exp(Y \log(1+Z)) = \sum_{m=0}^{\infty} \binom{Y}{m} Z^m$$

so in that case $P_m(Y) = \binom{Y}{m}$.

Lemma 4.2: *The polynomials $P_m(Y)$ satisfy the following properties:*

1. $P_0(Y) = 1$ *and* $P_1(Y) = Y$*;*
2. $P_m(0) = 0$ *for all* $m \geq 1$*;*
3. *the degree of* P_m *is exactly* m*, and the leading coefficient of* P_m *is* $1/m!$*;*
4. $P_m(Y + Y') = \sum_{i+j=m} P_i(Y)P_j(Y')$*;*
5. $P_m(a\Omega) \in o_{\mathbb{C}_p}$ *for all* $a \in o_L$*;*
6. *for* $f(x) \in \mathbb{C}_p[[x]]$*, we have the identity*

$$(P_m(\partial)f(x))|_{x=0} = \frac{1}{m!}\frac{d^m f}{dx^m}|_{x=0} \ .$$

Proof: The first four properties are clear from the definition. The fifth property follows from the fact that, for $a \in o_L$, the power series

$$F_{at'_o}(Z) = F_{t'_o}([a](Z)) = \exp(a\Omega \log_{\mathcal{G}}(Z)) = \sum_{m=0}^{\infty} P_m(a\Omega)Z^m$$

has coefficients in $o_{\mathbb{C}_p}$. For the last property, let $\delta = \frac{d}{dx}$ be the invariant differential on the additive formal group. Then Taylor's formula says that

$$\exp(\delta b)h(a) = \sum_m (\frac{\delta^m}{m!}h(a))b^m = h(a+b) \ .$$

Using the fact that $\log_{\mathcal{G}}$ and $\exp_{\mathcal{G}}$ are inverse isomorphisms between $\mathcal{G}$ and the additive group over L, Taylor's formula for $\mathcal{G}$ can be obtained by making the substitutions

$$\begin{aligned} a &= \log_{\mathcal{G}}(x) \\ b &= \log_{\mathcal{G}}(y) \\ h &= f \circ \exp_{\mathcal{G}} \ . \end{aligned}$$

It follows easily that $\delta h(a) = \partial f(x)$, and Taylor's formula becomes

$$\exp(\partial \log_{\mathcal{G}}(y))f(x) = f(x +_{\mathcal{G}} y) \ .$$

Comparing coefficients after expanding both sides in y and setting $x = 0$ gives the result.

Remark: The identity (6) is part of the theory of Cartier duality, as sketched for example in Section 1 of [Kat]. Corollary 1.8 of [Kat] shows that $P_m(\partial)$ is the invariant differential operator called there $D(m)$. Comparing this fact with

Formula 1.1 of [Kat] yields the claim in part (6) for the functions $f(x) = x^n$, and the general fact then follows by linearity.

Our goal now is to study the functions $P_m(Y\Omega)$ as elements of the locally convex vector space $C^{an}(G, \mathbb{C}_p)$, where as always $G = o_L$ as a locally L-analytic group. The Banach space $\mathcal{F}_{a+\pi^n o_L}(K)$, for any complete intermediate field $L \subseteq K \subseteq \mathbb{C}_p$ and any coset $a + \pi^n o_L$ in o_L, is equipped with the norm

$$\| \sum_{i=0}^{\infty} c_i (x-a)^i \|_{a,n} := \max_i \{ |c_i \pi^{ni}| \} \ .$$

This Banach space is the same as the Tate algebra of K-valued rigid analytic functions on the disk $a + \pi^n o_{\mathbb{C}_p}$, and the norm, by the maximum principle, has the alternative definition

$$\|f\|_{a,n} = \max_{x \in a + \pi^n o_{\mathbb{C}_p}} |f(x)|$$

which is sometimes more convenient for computation. We recall that

$$\mathcal{F}_n(o_L, K) = \prod_{a \bmod \pi^n o_L} \mathcal{F}_{a+\pi^n o_L}(K) \quad \text{and} \quad C^{an}(G, K) = \varinjlim_n \mathcal{F}_n(o_L, K) \ .$$

Lemma 4.3: *For all $a \in o_L$ and all $m \geq 0$, we have*

$$\|P_m(Y\Omega)\|_{a,n} \leq \max_{0 \leq i \leq m} \|P_i(Y\Omega)\|_{0,n} \ .$$

Proof: By Property (4) of Lemma 4.2, we have

$$P_m((a + \pi^n x)\Omega) = \sum_{i+j=m} P_i(a\Omega) P_j(\pi^n \Omega x) \ .$$

Therefore, using Property (5) of Lemma 4.2, we have

$$\begin{aligned} \|P_m(Y\Omega)\|_{a,n} &= \max\nolimits_{z \in a+\pi^n o_{\mathbb{C}_p}} |P_m(z\Omega)| \\ &\leq \max\nolimits_{0 \leq i \leq m} \max\nolimits_{x \in o_{\mathbb{C}_p}} |P_i(\pi^n \Omega x)| \\ &= \max\nolimits_{0 \leq i \leq m} \|P_i(Y\Omega)\|_{0,n} \ . \end{aligned}$$

Lemma 4.4: *The following estimate holds for $P_m(Y\Omega)$ and $m \geq 1$:*

$$||P_m(Y\Omega)||_{0,n} < p^{-1/(p-1)} p^{\frac{m}{eq^{n-1}(q-1)}} .$$

Proof: Let $r = p^{-q/e(q-1)}$. As shown in [Lan] §8.6 Lemma 4, and as we have used earlier, the functions $\log_{\mathcal{G}}$ and $\exp_{\mathcal{G}}$ are inverse isomorphisms from $\mathbf{B}(r)$ to itself, and $\log_{\mathcal{G}}$ is rigid analytic on all of $\mathbf{B}$. Furthermore by Lemma 3.2, $[\pi^n]^{-1}\mathbf{B}(r) = \mathbf{B}(r^{1/q^n})$ for any $n \in \mathbb{N}$. It follows that

$$\| \log_{\mathcal{G}}(x) \|_{\mathbf{B}(r^{1/q^n})} = rp^{n/e}$$

where $\|f\|_{\mathcal{X}}$ denotes the spectral semi-norm on an affinoid $\mathcal{X}$. The function $H(x,y) = y\Omega \log_{\mathcal{G}}(x)$ is a rigid function of two variables on the affinoid domain $\mathbf{B}(r^{1/q^n}) \times \mathbf{B}(p^{-n/e})$ satisfying

$$\begin{aligned} \|H(x,y)\|_{\mathbf{B}(r^{1/q^n})\times\mathbf{B}(p^{-n/e})} &\leq p^{-n/e}p^{-1/(p-1)+1/e(q-1)}rp^{n/e} \\ &= p^{-1/(p-1)-1/e} < p^{-1/(p-1)} . \end{aligned}$$

We conclude from this that $\exp(H(x,y))$ is rigid analytic on $\mathbf{B}(r^{1/q^n}) \times \mathbf{B}(p^{-n/e})$ and that

$$\| \exp(H(x,y)) - 1 \|_{\mathbf{B}(r^{1/q^n})\times\mathbf{B}(p^{-n/e})} < p^{-1/(p-1)}.$$

The power series expansion of $\exp(H(x,y))$ is

$$\exp(H(x,y)) = \sum_{m=0}^{\infty} P_m(y\Omega)x^m,$$

and so we conclude that, for all $m \geq 1$ and for all $y \in \mathbf{B}(p^{-n/e})$, we have

$$|P_m(y\Omega)|r^{m/q^n} < p^{-1/(p-1)}.$$

Therefore we obtain

$$\|P_m(Y\Omega)\|_{0,n} < p^{-1/(p-1)}p^{\frac{m}{eq^{n-1}(q-1)}}$$

as desired.

From these lemmas we may deduce the following proposition.

Proposition 4.5: *Given a sequence $\{c_m\}_{m\geq 0}$ of elements of $\mathbb{C}_p$, the series*

$$\sum_{m=0}^{\infty} c_m P_m(y\Omega)$$

converges to an element of $\mathcal{F}_n(o_L, \mathbb{C}_p)$ provided that $|c_m|p^{m/eq^{n-1}(q-1)} \to 0$ as $m \to \infty$. More generally, this series converges to an element of $C^{an}(G, \mathbb{C}_p)$ provided that there exists a real number r, with $r > 1$, such that $|c_m|r^m \to 0$ as $m \to \infty$.

Theorems 2.3 and 3.6 together imply the existence of a pairing

$$\{\ ,\ \} : \mathcal{O}(\mathbf{B}/\mathbb{C}_p) \times C^{an}(G, \mathbb{C}_p) \to \mathbb{C}_p$$

that identifies $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ and the continuous dual of $C^{an}(G, \mathbb{C}_p)$, both equipped with their projective limit topologies. The following lemma gives some basic computational formulae for this pairing; we will use some of these in the proof of our main theorem in this section.

Lemma 4.6: *The following formulae hold for the pairing* $\{\ ,\ \}$*, given* $F \in \mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ *and* $f \in C^{an}(G, \mathbb{C}_p)$*:*

1. $\{1, f\} = f(0)$*;*
2. $\{F_{at'_o}, f\} = f(a) - f(0)$ *for* $a \in o_L$*;*
3. $\{F, \kappa_z\} = F(z)$ *for* $z \in \mathbf{B}$*;*
4. $\{F_{at'_o} F, f\} = \{F, f(a + .) - f\}$ *for* $a \in o_L$*;*
5. $\{F, \kappa_z f\} = \{F(z +_{\mathcal{G}} .), f\}$ *for* $z \in \mathbf{B}$*;*
6. $\{F, f(a.)\} = \{F \circ [a], f\}$ *for* $a \in o_L$*;*
7. $\{F, f'\} = \{\Omega \log_{\mathcal{G}} \cdot F, f\}$*;*
8. $\{F, xf(x)\} = \{\Omega^{-1} \partial F, f\}$*;*
9. $\{F, P_m(.\Omega)\} = (1/m!) \frac{d^m F}{dZ^m}(0)$.

Proof: These properties follow from the definition of the Fourier transform and from the density of the subspace generated by the characters in $C^{an}(G, \mathbb{C}_p)$. For example to see property (5): if $\lambda'(f) := \lambda(\kappa_z f)$, then

$$F_{\lambda'}(z') = \lambda(\kappa_z \kappa_{z'}) = \lambda(\kappa_{z +_{\mathcal{G}} z'}) = F_\lambda(z +_{\mathcal{G}} z')\ .$$

For property (7), using (4) we have

$$\begin{aligned} \{F, f'\} &= \lim_{\epsilon \to 0} \epsilon^{-1} \{F, f(. + \epsilon) - f\} \\ &= \lim_{\epsilon \to 0} \epsilon^{-1} \{F_{\epsilon t'_o} F, f\} \\ &= \{\Omega \log_{\mathcal{G}} \cdot F, f\} \end{aligned}$$

using continuity and the fact that $F_{\epsilon t'_o} = \exp(\epsilon \Omega \log_{\mathcal{G}}) - 1$ for small ϵ. An analogous computation based on (5) gives (8). The last property (9) follows from Lemma 4.2.6 and (8).

We may now prove the main result of this section.

Theorem 4.7: *Any function $f \in C^{an}(G, \mathbb{C}_p)$ has a unique representation in the form*

$$f = \sum_{m=0}^{\infty} c_m P_m(.\Omega)$$

as in Prop. 4.5; in this representation, $c_m = \{Z^m, f\}$.

Proof: Part (9) of Lemma 4.6, along with continuity, shows that if f has a representation in the given form then $c_m = \{Z^m, f\}$. The functions Z^m generate a dense subspace in $\mathcal{O}(\widehat{G}/\mathbb{C}_p)$, and so a function f with all $c_m = 0$ must be zero (for example all Dirac distributions pair to zero against f). This implies that this type of representation, if it exists, is unique. Suppose we show that, for any $f \in C^{an}(G, \mathbb{C}_p)$, there exists an $r > 1$ such that $|\{Z^m, f\}|\, r^m \to 0$ as $m \to \infty$. Then by Prop. 4.5, the series

$$\overline{f}(x) := \sum_{m=0}^{\infty} \{Z^m, f\} P_m(x\Omega)$$

converges to a locally analytic function and by Lemma 4.6, Part 9, we have $\{Z^m, f\} = \{Z^m, \overline{f}\}$ for all m. Therefore $\overline{f} = f$.

Thus we have reduced our main theorem to the claim that $|\{Z^m, f\}|\, r^m \to 0$ as $m \to \infty$ for some $r > 1$. The function f being locally analytic it belongs to one of the Banach spaces $\mathcal{F}_n(o_L, \mathbb{C}_p)$. Using the topological isomorphism between the Fréchet spaces $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$ and $D(G, \mathbb{C}_p) = C^{an}(G, \mathbb{C}_p)'$, there is a rational number $s > 0$ such that the map $\mathcal{O}(\mathbf{B}/\mathbb{C}_p) \to \mathcal{F}_n(o_L, \mathbb{C}_p)'$ factors through the Tate algebra $\mathcal{O}(\mathbf{B}(p^{-s})/\mathbb{C}_p)$. If we choose another rational number s' so that $0 < s' < s$, then in the Tate algebra $\mathcal{O}(\mathbf{B}(p^{-s})/\mathbb{C}_p)$, the set of rigid functions $\{(Z/p^{-s'})^m\}_{m \geq 0}$ goes to zero and therefore so does $|\{Z^m, f\}|\, p^{s'm}$. This proves the existence of the desired expansion.

Remark: In [Kat], Katz discusses what he calls "Gal-continuous" functions. These are continuous functions on G that satisfy (in our notation) $\sigma(f(x)) = f(\tau(\sigma)x)$ for all $\sigma \in G_L$. If $\{c_m\}$ is a sequence of elements of L such that $|c_m| \to 0$, then $f(x) := \sum_{m=0}^{\infty} c_m P_m(x\Omega)$ is continuous by Part (5) of Lemma 4.2, and by the Galois properties of Ω it is even Gal-continuous.

5. p-adic L-functions

In this section we will illustrate how the integration theory developed in this paper applies to yield p-adic L-functions for CM elliptic curves E at supersingular primes. In fact, our method allows us to apply the Coleman power series approach described in [dS] directly in the supersingular case. We will content ourselves with proving a supersingular analogue of a weak version of Theorem II.4.11 of [dS]; this should demonstrate sufficiently the nature of our construction, without requiring too much of a diversion into global arithmetic.

We should emphasize that the L-functions we will construct in the supersingular case come from locally analytic distributions on Galois groups, not measures. A character on the Galois group is integrable provided that its restriction to a small open subgroup is a power of $\overline{\varphi}$, where $\overline{\varphi}$ gives the representation on the dual Tate module of E. We therefore cannot make any immediate connection to Iwasawa module structure of, for example, elliptic units. This is an interesting problem for the future.

Our results are closely related to those of Boxall ([Box]). See the Remark after Prop. 5.2 for more discussion of the relationship.

Before discussing p-adic L-functions we will develop Fourier theory for the multiplicative group; this will be useful because the p-adic L-functions we construct arise as locally analytic distributions on Galois groups that are naturally isomorphic to multiplicative, rather than additive groups. Let H be $o_L^\times$ as L-analytic group and let H_1 be the subgroup $1 + \pi o_L$. Using the Teichmüller character ω, we have

$$H = H_1 \times k^\times$$

where k is the residue field of o_L. For $x \in H$, let $< x >$ be the projection of x to H_1.

As always, G is the additive group o_L. Let us assume that the absolute ramification index e of the field L satisfies $e < p - 1$. We define $\ell := \pi^{-1} \cdot \log$ so that $\ell : H_1 \xrightarrow{\cong} G$ is an L-analytic isomorphism.

This map induces an isomorphism between the distribution algebras $D(H_1, K)$ and $D(G, K)$. The group $\widehat{H}(\mathbb{C}_p)$ of locally L-analytic, $\mathbb{C}_p$-valued characters of H is isomorphic to a product of $q - 1$ copies of the open unit disk $\mathbf{B}$ using the results of section 3, indexed by the (finite) character group of $k^\times$. For $z \in \mathbf{B}(\mathbb{C}_p)$, let ψ_z be the corresponding character of H_1. Then for any distribution $\lambda \in D(H, K)$, and any character $\omega^i \psi_z$ with $z \in \mathbf{B}(\mathbb{C}_p)$, and $0 \le i \le q - 1$, we have the "Mellin transform"

$$M_\lambda(z, \omega^i) = \lambda(\omega^i \psi_z) \ .$$

For each fixed value of the second variable, M_λ is a rigid function in $\mathcal{O}(\mathbf{B}/\mathbb{C}_p)$.

Now let us compare the Fourier transforms for G and H in a different way. The group $o_L^\times$, as an L-analytic manifold, is an open submanifold of o_L. If we have a distribution λ in $D(G, K)$ that vanishes on functions with support in πo_L, then λ gives a distribution on $H = o_L^\times \subset o_L$. It follows easily from Lemma 4.6.5 that λ is supported on H precisely when its Fourier transform F_λ satisfies

$$\sum_{[\pi](z)=0} F_\lambda(. +_{\mathcal{G}} z) = 0 \ .$$

We have the following result comparing the Fourier and Mellin transforms.

Proposition 5.1: *Let λ be a distribution in $D(G, \mathbb{C}_p)$ supported on H, let F_λ be its Fourier transform, and let M_λ be its Mellin transform; suppose that $n \in \mathbb{N}$ satisfies $n \equiv i \pmod{q-1}$; then*

$$M_\lambda(\exp_{\mathcal{G}}(n\pi/\Omega), \omega^i) = \int_{o_L^\times} x^n d\lambda(x) = \Omega^{-n}(\partial^n F_\lambda(z)|_{z=0}) .$$

Note that the hypothesis $e < p-1$ guarantees that $M_\lambda(\exp_{\mathcal{G}}(x\pi/\Omega), \omega^i)$ is a (globally) analytic function of $x \in o_L$. Thus the left hand side of these equations gives a (globally) L-analytic interpolation of the values on the right side.

Proof: Let $z(n) = \exp_{\mathcal{G}}(n\pi/\Omega)$. By definition,

$$M_\lambda(z(n), \omega^i) = \lambda(\omega^i \psi_{z(n)}) .$$

Now

$$\begin{aligned} \psi_{z(n)}(< x >) &= \kappa_{z(n)}(\ell(< x >)) \\ &= t'_{\mathrm{o}}([\ell(< x >)](z(n))) \\ &= \exp(\Omega\ell(< x >) \log_{\mathcal{G}}(z(n))) \end{aligned}$$

because $|[\ell(< x >)](z(n))| < p^{-1/e(q-1)}$ (by Lemma 3.4.b and the hypothesis $e < p-1$). But

$$\exp(\Omega\ell(< x >) \log_{\mathcal{G}}(z(n))) = \exp(n \log(< x >)) = < x >^n$$

so $(\omega^i \psi_{z(n)})(x) = x^n$. But

$$\lambda(x \mapsto x^n) = \Omega^{-n}(\partial^n F_\lambda(z)|_{z=0})$$

by Lemma 4.6.8/9.

Now we will embark on a digression into the theory of CM elliptic curves, following the notation and the logic of Chap. II in [dS]. Let $\mathbf{K}$ be an imaginary quadratic field, and let $\mathfrak{f}$ be an integral ideal of $\mathbf{K}$ such that the roots of unity in $\mathbf{K}$ are distinct mod $\mathfrak{f}$. Let p be a rational prime that is relatively prime to $6\mathfrak{f}$ and inert in $\mathbf{K}$. Let $\mathbf{F}$ be the ray class field $\mathbf{K}(\mathfrak{f})$ and let $\mathbf{F}_n := \mathbf{K}(p^n\mathfrak{f})$ and $\mathbf{F}_\infty := \bigcup_{n\in\mathbb{N}} \mathbf{F}_n$. Assume for technical reasons that will become clear in a moment that p as a prime of $\mathbf{K}$ *splits completely in* $\mathbf{F}$. Let $\wp$ be a prime above p in $\mathbf{F}$. The prime $\wp$ ramifies totally in $\mathbf{F}_\infty$; let F_∞ be the completion of $\mathbf{F}_\infty$ at the unique prime above $\wp$. Let o be the ring of integers in the local field $\mathbf{K}_p$.

Fix an elliptic curve E over $\mathbf{F}$ with CM by the ring of integers in $\mathbf{K}$ and with associated Hecke character of the form $\psi_{E/\mathbf{F}} = \varphi \circ N_{\mathbf{F}/\mathbf{K}}$, where φ is a Hecke character of $\mathbf{K}$ of type $(1,0)$ and conductor dividing $\mathfrak{f}$; we moreover assume that there is a complex period Ω_∞ so that the period lattice $\mathcal{L}$ of E is $\Omega_\infty\mathfrak{f}$.

We view φ also as a character of $\Gamma_{\mathbf{K}} := Gal(\mathbf{F}_\infty/\mathbf{K})$; it is $\mathbf{K}_p^\times$-valued on the subgroup $\Gamma_{\mathbf{F}} = Gal(\mathbf{F}_\infty/\mathbf{F})$. If $\mathfrak{a}$ is an integral ideal of $\mathbf{K}$ such that the Artin symbol $\sigma_{\mathfrak{a}}$ belongs to $\Gamma_{\mathbf{F}}$, then $\sigma_{\mathfrak{a}}$ acts on the p-adic Tate module of E through multiplication by $\varphi(\mathfrak{a})$. We let $\overline{\varphi}$ be the Hecke character giving the action of $\Gamma_{\mathbf{F}}$ on the dual Tate module of E. The character φ gives us an isomorphism

$$\varphi : \Gamma_{\mathbf{F}} \to o^\times .$$

We use this isomorphism to equip $\Gamma_{\mathbf{F}}$ with an o-analytic structure. We let $\mathbf{N}$ denote the absolute norm.

Our assumption that p splits completely in $\mathbf{F}$ means that that the formal group $\widehat{E}_\wp$ of E at $\wp$ is a Lubin-Tate group over o of height two. (To handle general p, deShalit works with what he calls "relative" Lubin-Tate groups. Presumably one can generalize our results to this situation as well.) Furthermore, the field $\mathbf{F}_\infty$ contains all of the p-power torsion points of this formal group, as well as (by the Weil pairing) all of the p-power roots of unity. Thus our uniformization result holds over this field. Choose an o-generator t'_o of the (global) dual Tate module $\mathrm{Hom}(T_p(E), T_p(\mathbb{G}_m))$ defined over $\mathbf{F}_\infty$. Then the pairing $\{\ ,\ \}$ from section 4 looks like:

$$\mathcal{O}(\widehat{E}_\wp/F_\infty) \times C^{an}(o, F_\infty) \to F_\infty\ .$$

Now we introduce the machinery of Coleman power series and elliptic units. Let $\mathfrak{a}$ be an integral ideal relatively prime to $p\mathfrak{f}$ and let $\Theta(y; \mathcal{L}, \mathfrak{a})$ be the elliptic function from [dS] II.2.3 (10). Let $P(z)$ be the Taylor expansion of $\Theta(\Omega_\infty - z; \mathcal{L}, \mathfrak{a})$ and let $Q_{\mathfrak{a}}(Z) := P(\log_{\widehat{E}_\wp}(Z))$. This power series belongs to $o[[Z]]$, as shown in [dS] Prop. II.4.9; note that this proposition is true for inert primes as well as split ones, as is clear from its proof. The power series $Q_{\mathfrak{a}}(Z)$ is the Coleman power series associated to a norm-compatible sequence of elliptic units, as deShalit explains.

Define

$$g_{\mathfrak{a}}(Z) = \log Q_{\mathfrak{a}}(Z) - \frac{1}{p^2} \sum_{\substack{z \in \widehat{E}_\wp \\ [p](z)=0}} \log Q_{\mathfrak{a}}(Z +_{\widehat{E}_\wp} z)\ .$$

Proposition 5.2: *The power series $g_{\mathfrak{a}}(Z) \in \mathcal{O}(\widehat{E}_\wp/F_\infty)$ is the Fourier transform of an F_∞-valued, locally analytic distribution on o supported on $o^\times$. By means of the isomorphism φ from $\Gamma_{\mathbf{F}}$ to $o^\times$, it defines a locally analytic distribution on $\Gamma_{\mathbf{F}}$ with the interpolation property*

$$\Omega^m\{g_{\mathfrak{a}}(Z), \varphi^m\}$$

$$= -12(1 - \varphi(p)^m p^{-2})\Omega_\infty^{-m}(m-1)!(\mathbf{N}(\mathfrak{a})L_{\mathfrak{f}}(\overline{\varphi}^m, m, 1) - \varphi(\mathfrak{a})^m L_{\mathfrak{f}}(\overline{\varphi}^m, m, \mathfrak{a}))$$

for any $m \in \mathbb{N}$. Here $L_{\mathfrak{f}}(\overline{\varphi}, s, \mathfrak{c})$ denotes the "partial" Hecke L-function of conductor $\mathfrak{f}$, equal to $\sum_{\mathfrak{b}} \overline{\varphi}(\mathfrak{b})\mathbf{N}(\mathfrak{b})^{-s}$ over ideals $\mathfrak{b}$ prime to $\mathfrak{f}$ and such that $(\mathfrak{b}, \mathbf{F}/\mathbf{K}) = (\mathfrak{c}, \mathbf{F}/\mathbf{K})$.

Proof: The first assertion follows easily from the formulae in Lemma 4.6. The interpolation property comes from the formula (Lemma 4.6 again):

$$\{g_{\mathfrak{a}}(Z), \varphi^m\} = \Omega^{-m}\partial^m g_{\mathfrak{a}}(Z)|_{Z=0} \ .$$

The rest of the computation is just a version of [dS] II.4.10. The invariant differential ∂ pulls back to d/dy on the complex uniformization of E, so

$$\Omega^m\{g_{\mathfrak{a}}(Z), \varphi^m\} = \left(\frac{d}{dy}\right)^m (\log\Theta(\Omega_\infty - y; \mathcal{L}, \mathfrak{a}) - p^{-2}\log\Theta(\Omega_\infty - y; p^{-1}\mathcal{L}, \mathfrak{a}))|_{y=0}$$

and the claimed formula then follows from the equivalent in our situation of [dS] II.4.7 (17), along with II.3.1 (7) and Prop. II.3.5.

Remark: From Prop. 5.1, we see that the Mellin transform $M_{\mathfrak{a}}$ of the distribution $\{g_{\mathfrak{a}}(Z), \cdot\}$ is an o-analytic function on o interpolating the special values $\Omega^{-m}\partial^m g_{\mathfrak{a}}(Z)|_{Z=0}$. This function (on $\mathbb{Z}_p$) was constructed by Boxall ([Box]). Other than the fact that our construction is arguably more natural, the principal new results here are that our function is $\mathbf{K}_p$-analytic on o rather than $\mathbb{Q}_p$-analytic on $\mathbb{Z}_p$. The existence of such an analytic interpolating function implies congruences among the special values.

We may give the slightly larger Galois group $\Gamma_{\mathbf{K}}$ a locally analytic structure by transporting that of $\Gamma_{\mathbf{F}}$ to its finitely many cosets in $\Gamma_{\mathbf{K}}$. To extend the integration pairing to $\Gamma_{\mathbf{K}}$, recall that, along with E we have finitely many other elliptic curves E^σ as σ runs through $Gal(\mathbf{F}/\mathbf{K})$. We also have, for each $\sigma \in \Gamma_{\mathbf{K}}$, an isogeny $\iota(\sigma) : E \to E^\sigma$ as in [dS] Prop. II.1.5. If $\mathfrak{a}$ is an ideal prime to p, then the associated isogeny $\iota(\sigma_{\mathfrak{a}})$ has degree $\mathbf{N}(\mathfrak{a})$, which is prime to p and therefore induces an isomorphism between the formal groups $\widehat{E}_\wp$ and $\widehat{E^{\sigma_{\mathfrak{a}}}}_\wp$.

A typical locally analytic function $\overline{f}$ on $\Gamma_{\mathbf{K}}$ may be written

$$\overline{f}(\sigma) = f_i(\varphi(\sigma_i^{-1}\sigma)) \quad \text{when } \sigma \in \sigma_i\Gamma_{\mathbf{F}}$$

where $\mathfrak{c}_i$ is a collection of integral ideals of $\mathbf{K}$ so that the Artin symbols $\sigma_i = \sigma_{\mathfrak{c}_i}$ form the set $Gal(\mathbf{F}/\mathbf{K})$, and $f_i \in C^{an}(o, F_\infty)$, supported on $o^\times$.

Let $\mathcal{O}(\widehat{E}_\wp/F_\infty)^0$ denote the subspace of functions $F \in \mathcal{O}(\widehat{E}_\wp/F_\infty)$ satisfying $\sum_{[p](z)=0} F(. +_{\widehat{E}_\wp} z) = 0$. These are the distributions supported on $o^\times$. We define an extended integration pairing

$$\{\ ,\ \} : \bigoplus_{\sigma_{\mathfrak{a}} \in Gal(\mathbf{F}/\mathbf{K})} \mathcal{O}(\widehat{E^{\sigma_{\mathfrak{a}}}}_\wp/F_\infty)^0 \times C^{an}(\Gamma_{\mathbf{K}}, F_\infty) \longrightarrow F_\infty$$

by setting

$$\{\overline{h}, \overline{f}\} := \sum_i \{h_i \circ \iota(\sigma_i), f_i\} \ .$$

Lemma 5.3: *This pairing is well-defined (i.e., it is independent of the choice of coset representatives), and identifies the left hand space with the continuous dual of the right hand space.*

Proof: The duality is clear; the key point is that the pairing is well defined. Suppose we replace σ_i with $\sigma_i\tau_{\mathfrak{b}}$, where $\tau_{\mathfrak{b}} \in \Gamma_{\mathbf{F}}$. Then $h_i \circ \iota(\sigma_i\tau_{\mathfrak{b}}) = h_i \circ \iota(\sigma_i) \circ [\varphi(\tau_{\mathfrak{b}})]$ (see [dS] II.4.5). The decomposition of $\overline{f}$ also changes, with f_i replaced by $f_i' = f_i(\varphi(\tau_{\mathfrak{b}})^{-1}.)$. Then, using Lemma 4.6 as usual, the pairing satisfies

$$\{h_i \circ \iota(\sigma_i) \circ [\varphi(\tau_{\mathfrak{b}})], f_i(\varphi(\tau_{\mathfrak{b}})^{-1}.)\} = \{h_i \circ \iota(\sigma_i), f_i\} \ .$$

Theorem 5.4: *(Compare [dS] Thm. II.4.11) Let $\overline{h} = \{h_i\}$ where $h_i := \sigma_i(g_{\mathfrak{a}})$ with $g_{\mathfrak{a}}$ the (formal) elliptic function over $\mathbf{F}$ used for the construction of the partial L-function in Prop. 5.2. Let ϵ be any locally analytic character on $\Gamma_{\mathbf{K}}$, whose restriction to $\Gamma_{\mathbf{F}}$ is φ^m for some $m \in \mathbb{N}$. Then the locally analytic distribution $\overline{h}$ on $\Gamma_{\mathbf{K}}$ has the interpolation property*

$$\Omega^m\{\overline{h}, \epsilon\} = 12(m-1)!\,\Omega_\infty^{-m}(1-\epsilon(p)p^{-2})(\epsilon(\mathfrak{a}) - \mathbf{N}(\mathfrak{a}))L_{\mathfrak{f}}(\epsilon^{-1}, 0) \ .$$

Proof: The proof is a long computation very much in the spirit of [dS] Thm. II.4.11 (though we have cheated in the statement of the Theorem and avoided the case where p divides the conductor). Choose coset representatives $\sigma_i = \sigma_{\mathfrak{c}_i}$. The point of the computation is that ([dS] II.2.4 (ii) and II.4.5 (iv))

$$\sigma_i(g_{\mathfrak{a}}) \circ \iota(\sigma_i) = g_{\mathfrak{a}\mathfrak{c}_i} - \mathbf{N}(\mathfrak{a})g_{\mathfrak{c}_i}$$

and

$$f_i(a) = \epsilon(\sigma_i)a^m \ .$$

Then the partial terms in the pairing are

$$\epsilon(\sigma_i)\Omega^{-m}(\partial^m g_{\mathfrak{a}\mathfrak{c}_i} - \mathbf{N}(\mathfrak{a})\partial^m g_{\mathfrak{c}_i}) \ .$$

These terms may then be evaluated using Prop. 5.2, and when the results are combined one obtains the statement of the theorem.

Remark: One can compute an interpolation result for more general locally analytic characters ϵ – explicitly, characters which restrict to φ^m on an open subgroup of $\Gamma_{\mathbf{F}}$ – by following the same line of argument as in [dS] II.4.11.

Appendix. p-adic periods of Lubin-Tate groups

In the analysis in section 3 of the behavior of the isomorphism ($\diamond\diamond$) relative to the affinoid coverings on $\widehat{G}$ and on $\mathbf{B}$ we needed rather exact information about the "period" Ω of the Lubin-Tate group $\mathcal{G}$. In this appendix, we apply results of Fontaine [Fon] to obtain this information.

All of the significant ideas in this section come from the article [Fon], and we follow the notation of that article with the following exceptions. We will use the letter X for the module of differentials $\Omega_{o_L}(o_{\overline{L}})$ (called Ω by Fontaine). We also do not distinguish between $\mathcal{G}$ as a formal group or p-divisible group thanks to [Tat] Prop. 1.

As before, we let $\mathcal{G}$ be the Lubin-Tate group over o associated to the uniformizer π and let $\mathcal{G}'$ be the dual p-divisible group. We denote by q and e respectively the number of elements in $o/\pi o$ and the ramification index of $L/\mathbb{Q}_p$. Furthermore, T and T' are the Tate modules of $\mathcal{G}$ and $\mathcal{G}'$ respectively, and $F_{t'}(Z) \in Zo_{\mathbb{C}_p}[[Z]]$ is the power series corresponding to $t' \in T'$ as in section 3. We let ω be the invariant differential on $\mathcal{G}$ such that $\omega = (1 + \ldots)dZ$. We define $\Omega_{t'}$ so that $F_{t'}(Z) = \Omega_{t'}Z + \cdots$.

We write $\mathcal{G}_n$ and $\mathcal{G}'_n$ for the group schemes of p^n torsion points on $\mathcal{G}$ and $\mathcal{G}'$ respectively. Let L_n be the finite extension field of L generated by the $\overline{L}$-points of $\mathcal{G}'_n$.

The various maps that are denoted by decorated forms of the letter ϕ are those defined in [Fon].

We remark that, in the case that $e \leq p-1$, the result of part (c) of the following Theorem was obtained by Boxall ([Box]) by power series computations.

Theorem: *a. For $t' \in T'$, we have*

$$\phi^0_{\mathcal{G}'}(t') = \frac{dF_{t'}}{1+F_{t'}} = \Omega_{t'}\omega \ ;$$

b. there is a sequence of elements $\Omega_{t'}(n) \in o_{L_n}$ for integers $n \geq 1$ such that $\Omega_{t'}(n+1) \equiv \Omega_{t'}(n) \pmod{p^n o_{L_{n+1}}}$ and such that the sequence of $\Omega_{t'}(n)$ converges in $\mathbb{C}_p$ to $\Omega_{t'}$;

c. let t'_o be any generator of the o module T' ; then the fundamental period $\Omega = \Omega_{t'_o}$ satisfies

$$|\Omega| = p^{-s}$$

where

$$s = \frac{1}{p-1} - \frac{1}{e(q-1)} \ .$$

Proof: We begin with parts (a) and (b). First of all, $F_{t'}(Z)$ being a formal group homomorphism from $\mathcal{G}$ to $\mathbb{G}_m$, the pullback of the invariant differential $dZ/(1+Z)$ on $\mathbb{G}_m$ must be a multiple of ω. Comparing leading coefficients shows that $dF_{t'}(Z)/(1+F_{t'}(Z)) = \Omega_{t'}\omega$. Fontaine's map

$$\phi^0_{\mathcal{G}'} : o_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} T' \to t^*_{\mathcal{G}}(o_{\mathbb{C}_p})$$

as defined on p. 406 of [Fon], is the limit of maps

$$\phi^0_{\mathcal{G}'_n} : o_{\overline{L}} \otimes \mathcal{G}'_n(o_{\overline{L}}) \to t^*_{\mathcal{G}_n}(o_{\overline{L}})$$

defined on p. 396. To compute these maps, recall that the affine algebra of $\mathcal{G}_n$ is $R_n = o_L[[Z]]/J_n$ where J_n is the ideal generated by $[p^n](Z)$. The element t' is represented explicitly as $(t'(n))_n$ where the $t'(n)$ are a compatible sequence of homomorphisms $\mathcal{G}_n \to \mu_{p^n}$ over $o_{\mathbb{C}_p}$. Further, the element $t'(n)$ is given explicitly by the class $1 + F_{t'}(Z) + J_n$ in $o_{\mathbb{C}_p} \otimes R_n$. But each homomorphism $\mathcal{G}_n \to \mu_{p^n}$ is defined over o_{L_n}. Therefore $1 + F_{t'}(Z) \equiv g_n(Z) \pmod{J_n}$ for some $g_n \in o_{L_n} \otimes R_n$. Now on the one hand Fontaine's map is given by the formula

$$\phi^0_{\mathcal{G}'_n}(t'(n)) = \frac{dg_n}{g_n} \in t^*_{\mathcal{G}_n}(o_{L_n}) \ .$$

By Prop. 10 of [Fon] we know that

$$t^*_{\mathcal{G}_n}(o_{L_n}) = t^*_{\mathcal{G}}(o_{L_n})/p^n t^*_{\mathcal{G}}(o_{L_n}) \ .$$

Therefore we may choose $\Omega_{t'}(n) \in o_{L_n}$ so that

$$dg_n/g_n \equiv \Omega_{t'}(n)\omega \pmod{p^n t^*_{\mathcal{G}}(o_{L_n})} \ .$$

But both g_n and $1 + F_{t'}$ represent the same map over $o_{\mathbb{C}_p}$ from $\mathcal{G}_n$ to μ_{p^n}, and therefore we must have

$$\Omega_{t'}(n) \equiv \Omega_{t'} \pmod{p^n t^*_{\mathcal{G}}(o_{\mathbb{C}_p})} \ .$$

By definition $\phi^0_{\mathcal{G}'}(t')$ is the limit of the $\Omega_{t'}(n)\omega$, which we have just shown is $\Omega_{t'}\omega$.

Now consider the following commutative diagram, obtained from Prop. 8 of [Fon] by applying section 5.9 to pass to the inverse limit over multiplication by p:

$$\begin{tikzcd}
o_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} T \times o_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} T' \arrow[r, "\phi"] \arrow[d, "\theta"] & t^*_{\mathcal{G}'}(o_{\mathbb{C}_p}) \oplus t_{\mathcal{G}}(T_p(X)) \ \times \ t^*_{\mathcal{G}}(o_{\mathbb{C}_p}) \oplus t_{\mathcal{G}'}(T_p(X)) \arrow[d, "\nu"] \\
o_{\mathbb{C}_p} \otimes T_p(\mathbb{G}_m) \arrow[r, "\xi"] & T_p(X)
\end{tikzcd}$$

Here the map ϕ is

$$\phi = \phi_{\mathcal{G}, o_{\mathbb{C}_p}} \times \phi_{\mathcal{G}', o_{\mathbb{C}_p}}$$

as defined in Prop. 11 of [Fon], where it is shown to be injective, and to induce an isomorphism upon tensoring with $\mathbb{C}_p$. The vertical arrows are the natural pairings, and the lower horizontal arrow is induced by the map ξ of Thm. 1' of [Fon].

Each of the spaces $\mathbb{C}_p \otimes_{\mathbb{Z}_p} T$ and $\mathbb{C}_p \otimes_{\mathbb{Z}_p} T'$ decompose into a direct sum of one-dimensional eigenspaces corresponding to distinct embeddings of $L \hookrightarrow \mathbb{C}_p$. The map ϕ is o-linear and therefore must respect this decomposition. On the upper right, the o-actions on the spaces $t^*_{\mathcal{G}}(o_{\mathbb{C}_p})$ and $t_{\mathcal{G}}(T_p(X))$ are given by the given embedding $o \subseteq o_{\mathbb{C}_p}$ (while the o-actions on the corresponding spaces for $\mathcal{G}'$ are given by the other $[L:\mathbb{Q}_p]-1$ embeddings of o in $o_{\mathbb{C}_p}$). Therefore the above diagram can be "reduced" to the following:

$$\begin{tikzcd}
(o_{\mathbb{C}_p} \otimes_o T) \times \mathrm{Hom}_o(T, o_{\mathbb{C}_p}(1)) \arrow[r, "\phi"] \arrow[d, "\theta"] & t_{\mathcal{G}}(T_p(X)) \times t^*_{\mathcal{G}}(o_{\mathbb{C}_p}) \arrow[d, "\nu"] \\
o_{\mathbb{C}_p} \otimes T_p(\mathbb{G}_m) \arrow[r, "\xi"] & T_p(X)
\end{tikzcd}$$

We choose a generator u of T, a generator ϵ of $T_p(\mathbb{G}_m)$, and we let $f \in \mathrm{Hom}_o(T, o_{\mathbb{C}_p}(1))$ be the unique o-linear map such that $f(u) = \epsilon$. Trace the pairing (u, f) both ways through the square, using the explicit formulae for the maps involved from [Fon], and accounting for the fact that Fontaine writes $\mathbb{G}_m$ multiplicatively. If we write $\phi^0_{\mathcal{G}'}(f) = \Omega_f \omega$, then

$$\nu\phi(u, f) = \Omega_f u^* \omega$$

while

$$\xi\theta(u, f) = f(u)^* dZ/(1+Z) \ .$$

Comparing these formulae with the explicit isomorphisms $\xi_{L,\mathcal{G}}$ and ξ_L defined in section 1 of [Fon], we see that the commutativity of the square means that

$$\Omega_f \xi_{L,\mathcal{G}}(u \otimes \omega) = \xi_L(f(u) \otimes dZ/(1+Z)) \ .$$

This fact, when combined with Thm. 1 and Cor. 1 of [Fon], tells us that

$$\Omega_f \mathfrak{a}_L = \mathfrak{a}_{L,\mathcal{G}} \ .$$

We conclude that

$$|\Omega_f| = p^{-r} \quad \text{with} \quad r = \frac{1}{p-1} - \frac{1}{e(q-1)} + \mathrm{ord}_p(\mathcal{D}_{L/\mathbb{Q}_p})$$

where $\mathcal{D}_{L/\mathbb{Q}_p}$ is the different of the extension $L/\mathbb{Q}_p$.

To complete the calculation let t'_0 be our chosen generator for the o-module T'. Some elementary linear algebra using properties of the different shows that there is a generator x of $\mathcal{D}_{L/\mathbb{Q}_p}$ such that we have

$$x t'_0 = f + f' \quad \text{in } o_{\mathbb{C}_p} \otimes_{\mathbb{Z}_p} T'$$

with f' vanishing on the eigenspace in $\mathbb{C}_p \otimes_o T$ corresponding to the given embedding $L \subseteq \mathbb{C}_p$. This means that $\phi^0_{\mathcal{G}'}(f') = 0$ so that

$$x \Omega_{t'_0} = \Omega_f \ .$$

In other words, the valuation of $\Omega = \Omega_{t'_0}$ is $\mathrm{ord}_p(\Omega_f) - \mathrm{ord}_p(x)$ as claimed.

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On the Derived Category of Sheaves on a Manifold

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Abstract. Let M be a non–compact, connected manifold of dimension ≥ 1. Let $D(\text{sheaves}/M)$ be the unbounded derived category of chain complexes of sheaves of abelian groups on M. We prove that $D(\text{sheaves}/M)$ is not a compactly generated triangulated category, but is well generated.

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0. Introduction

We remind the reader. Let $\mathcal{T}$ be a triangulated category in which arbitrary coproducts exist. For example, we may take $\mathcal{T} = D(\text{sheaves}/M)$, the unbounded derived category of chain complexes of sheaves of abelian groups on a topological space M. An object $c \in \mathcal{T}$ is called *compact* if, for every collection $\{t_\lambda \mid \lambda \in \Lambda\}$ of objects in $\mathcal{T}$,

$$\mathcal{T}\left(c, \coprod_{\lambda\in\Lambda} t_\lambda\right) \;=\; \bigoplus_{\lambda\in\Lambda} \mathcal{T}(c, t_\lambda).$$

A triangulated category $\mathcal{T}$ is called *compactly generated* if arbitrary coproducts exist in $\mathcal{T}$, and there are plenty of compact objects. [The precise definition is that every non–zero object in $\mathcal{T}$ admits a non–zero map from a compact object, and there is a set of isomorphism classes of compact objects.] In this article, we will see the following

Theorem 0.1. *Let M be a non–compact, connected manifold of dimension $\geq$ 1. Let $D(\text{sheaves}/M)$ be the unbounded derived category of chain complexes of sheaves of abelian groups on M. Then the only compact object in $D(\text{sheaves}/M)$ is the zero object.*

In a recent book [6], the author defined a generalisation of compactly generated categories, the well generated triangulated categories. We will not repeat the definition here. Assuming the reader is familiar with the definitions, we state our next result.

Theorem 0.2. *Let M and $D(\text{sheaves}/M)$ be as in Thoerem 0.1. Then the category $D(\text{sheaves}/M)$ is well generated. More generally, for any Grothendieck abelian category $\mathcal{A}$, the derived category $D(\mathcal{A})$ is well generated.*

The two theorems above should perhaps explain the point of the book [6]. Surely the category $D(\text{sheaves}/M)$ is natural enough, we want to prove something about it. The fact that it is not compactly generated says that the old theorems of the subject cannot be applied. Well generated triangulated categories is an attempt to pick out a class of triangulated categories which includes $D(\text{sheaves}/M)$, but is restrictive enough so that we can prove good theorems, for example Brown representability. For much more information about well generated triangulated categories the reader is referred both to the book [6], as well as several beautiful insights in Krause's paper [4].

In Section 1 we give a fairly detailed and self contained account of the proof of Theorem 0.1. Compactly generated triangulated categories have been around for many years. The people who have worked with them might understandably want a good account of why $D(\text{sheaves}/M)$ is not compactly generated. It seems only right to make the presentation easily accessible. In Section 2 we give a very terse proof of Theorem 0.2. The argument makes no attempt to be self-contained. The proof relies heavily on results from Alonso, Jeremías and Souto's [1], and from [6].

I would like to thank Marco Schlichting, who asked me to provide proofs for the two theorems above. In the case of Theorem 0.2 I know several proofs, and the one presented here was chosen mostly because it is only a paragraph long. In the case of Theorem 0.1, until Schlichting prompted me I had carelessly assumed it was a true fact, without ever checking the details.

1. The proof of Theorem 0.1

In this section we give a fairly detailed and complete proof of Theorem 0.1. The proof attempts to be reasonably self-contained. We will, however, assume that the reader is familiar with the six gluing functors. Let M be a topological space. Suppose $M = U \cup Z$ is the disjoint union of an open set U and its complement Z. Let $i : Z \longrightarrow M$, $j : U \longrightarrow M$ be the inclusions. Then there are six functors on chain complexes of sheaves of abelian groups, denoted $j_!$, j^*, j_*, i^*, i_* and $i^!$, which allow us to glue complexes of sheaves on U and Z to form complexes of sheaves on M. There are many excellent accounts of this in the literature, for example in Beilinson, Bernstein and Deligne's [2].

Before plunging into the proof, we remind the reader of a key property of compact objects.

Reminder 1.1. Let $\mathcal{A}$ be a Grothendieck abelian category, $\mathcal{T} = D(\mathcal{A})$ its derived category. Suppose c is compact in $D(\mathcal{A})$, and suppose we are given a sequence of chain complexes

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

with $\underrightarrow{\operatorname{colim}} X_i = X$. Then $\mathcal{T}(c, X) = \underrightarrow{\operatorname{colim}} \mathcal{T}(c, X_i)$.

Proof. The proof may be found in Lemma 2.8 of [5] combined with Remark 2.2 of [3]. □

Lemma 1.2. *Suppose M is a manifold, and that $c \in D(\text{sheaves}/M)$ is a compact object. Then there is a compact set $K \subset M$, so that c is acyclic outside K.*

Proof. Choose an increasing sequence of open subsets $U_\ell \subset M$, $\ell \in \mathbb{N}$ so that

(i) $M = \cup U_\ell$ is the union of the U_ℓ.

(ii) The closure $\overline{U}_\ell$ of U_ℓ is compact, and $\overline{U}_\ell \subset U_{\ell+1}$.

Let $j_\ell : U_\ell \longrightarrow M$ be the inclusion. Because the U_ℓ are increasing, we have a sequence of chain complexes of sheaves

$$\{j_1\}_! j_1^* c \longrightarrow \{j_2\}_! j_2^* c \longrightarrow \{j_3\}_! j_3^* c \longrightarrow \cdots$$

with direct limit c. The identity map $c \longrightarrow c$ is a map from a compact object c to a direct limit. By Remark 1.1, it must factor through $\{j_\ell\}_! j_\ell^* c$ for some ℓ. And the complex $\{j_\ell\}_! j_\ell^* c$ is acyclic outside the compact set $\overline{U}_\ell$. □

Definition 1.3. *The* support *of a chain complex c of sheaves on M is the set of all points $p \in M$ so that the stalk at p of c is not acyclic.*

Lemma 1.4. *Suppose M is a manifold, and that $c \in D(\text{sheaves}/M)$ is a compact object. Then the support of c is compact.*

Proof. Let p be point outside the support of c; that is, the stalk of c at p is acyclic. Let $U = M - \{p\}$ be the complement of p. Let $j : U \longrightarrow M$ be the inclusion of the open set, $i : \{p\} \longrightarrow M$ the inclusion of its complement. Consider now the triangle

$$j_! j^* c \longrightarrow c \longrightarrow i_* i^* c \longrightarrow \Sigma j_! j^* c.$$

Since the stalk of c at p vanishes, we know that $i^* c = 0$. Thus c is quasi-isomorphic to $j_! j^* c$. In particular $j_! j^* c$ is compact. We assert that $j^* c$ must also be compact. This is because

$$\begin{aligned}
&\mathrm{Hom}\left(j^*c\,,\ \coprod_{\lambda\in\Lambda} t_\lambda\right) && \\
&= \mathrm{Hom}\left(j^*c\,,\ j^* j_! \coprod_{\lambda\in\Lambda} t_\lambda\right) && \text{since } j^* j_! = 1 \\
&= \mathrm{Hom}\left(j_! j^*c\,,\ j_! \coprod_{\lambda\in\Lambda} t_\lambda\right) && \text{by adjunction} \\
&= \mathrm{Hom}\left(j_! j^*c\,,\ \coprod_{\lambda\in\Lambda} j_! t_\lambda\right) && \text{as } j_! \text{ respects coproducts} \\
&= \bigoplus_{\lambda\in\Lambda} \mathcal{T}(j_! j^*c\,,\ j_! t_\lambda) && \text{since } c = j_! j^* c \text{ is compact} \\
&= \bigoplus_{\lambda\in\Lambda} \mathcal{T}(j^*c\,,\ j^* j_! t_\lambda) && \text{by adjunction} \\
&= \bigoplus_{\lambda\in\Lambda} \mathcal{T}(j^*c\,,\ t_\lambda) && \text{since } j^* j_! = 1.
\end{aligned}$$

Lemma 1.2, applied to the complex j^*c on $M - \{p\}$, now tells us that the support of j^*c is contained in a compact subset of $M - \{p\}$. Hence so is the support of $c \cong j_!j^*c$. What this proves is that any point p not in the support of c is in the interior of the complement of the support. The support is closed. By Lemma 1.2, the support is contained in a compact subset of M. Being a closed subset of a compact subset, the support of c must be compact. $\square$

Notation 1.5. From now on, let c be a compact object in $D(\text{sheaves}/M)$, with M a non-compact, connected manifold. Let K be the support of c. By Lemma 1.4 we know that K is compact. To prove Theorem 0.1 we must show that K is empty. In the rest of this section we assume $K \neq \emptyset$, and deduce a contradiction.

Lemma 1.6. *Let the notation be as in Notation 1.5. There is a compact set $L \subset M$ so that*

(i) *The support of c is contained in L; that is, $K \subset L$.*
(ii) *L is a deformation retract of a neighbourhood.*
(iii) *The boundary of L contains a point in K.*

Proof. Choose a Morse function on the manifold M; that is, a proper function $\varphi : M \longrightarrow [0, \infty)$ with only non-degenerate critical points. Now $\varphi(K)$ is a compact subset of $\mathbb{R}$. Let $k \in \mathbb{R}$ be the maximum of $\varphi(K)$. By jiggling φ a little, we may assume that k is a regular value of φ. Now let $L = \varphi^{-1}[0, k]$. Because φ is proper, L must be compact. Because k is the maximum of φ on $K \subset M$, there must be a point $x \in K$ with $\varphi(x) = k$. Since φ is regular at x, x must be a boundary point of L. Now, for any Riemannian metric on M, the gradient flow along φ deformation retracts $\varphi^{-1}[0, k+\varepsilon]$ to $L = \varphi^{-1}[0, k]$. $\square$

Theorem 1.7. *With the notation as above, the object c vanishes.*

Proof. With the notation as in the proof of Lemma 1.6, let $L = \varphi^{-1}[0, k]$, and let $\widehat{L} = \varphi^{-1}[0, k+\varepsilon]$. Denote by $i : L \longrightarrow M$ and $\widehat{\imath} : \widehat{L} \longrightarrow M$ the inclusions. Let $p : \widehat{L} \longrightarrow L$ be the retraction, and $j : U = \{M - L\} \longrightarrow M$ the open inclusion.

We wish to consider the complex $b = \widehat{\imath}_* p^* c$ on M. The fact that $M = L \cup U$ is a disjoint union of an open set and its complement gives a triangle

$$j_!j^*b \longrightarrow b \longrightarrow i_*i^*b \longrightarrow \Sigma j_!j^*b.$$

Now

$$\begin{aligned} i_*i^*b &= i_*i^*\widehat{\imath}_*p^*c && \text{by definition of } b \\ &= i_*i^*c && \text{since } p \text{ and } \widehat{\imath} \text{ are the identity on } L \\ &= c && \text{since } c \text{ is supported on } K \subset L. \end{aligned}$$

It follows that the map

$$i_*i^*b \longrightarrow \Sigma j_!j^*b$$

is a map from a compact object $c = i_* i^* b$. Now write $U = M - L$ as an increasing union of $U_\ell \subset U$, with $\overline{U}_\ell$ compact and $\overline{U}_\ell \subset U_{\ell+1}$. As in the proof of Lemma 1.2, we write

$$j_! j^* b = \underrightarrow{\mathrm{colim}} \{j_\ell\}_! j_\ell^* b$$

with $j_\ell : U_\ell \longrightarrow U$ the inclusion. The compactness of $i_* i^* b$ guarantees that the map

$$i_* i^* b \longrightarrow \Sigma j_! j^* b$$

must factor as

$$i_* i^* b \longrightarrow \Sigma \{j_\ell\}_! j_\ell^* b \longrightarrow \Sigma j_! j^* b.$$

But now $i_* i^* b$ is supported on L, while $\{j_\ell\}_! j_\ell^* b$ is supported on the compact set $\overline{U}_\ell \subset M - L$. The map between them must vanish. In the triangle

$$j_! j^* b \longrightarrow b \longrightarrow i_* i^* b \xrightarrow{\beta} \Sigma j_! j^* b.$$

we have shown that the map β vanishes. We conclude that there is an isomorphism in the derived category

$$b \cong i_* i^* b \oplus j_! j^* b.$$

Now let x be a point in K on the boundary of L; in the notation of the proof of Lemma 1.6 this means that $\varphi(x) = k$. Then $p^{-1}(x)$ is the interval $[k, k+\varepsilon]$. If we pull back the isomorphism $b \cong i_* i^* b \oplus j_! j^* b$ to $p^{-1}(x) = [k, k+\varepsilon]$, we have that b is the complex of constant sheaves on the interval $[k, k+\varepsilon]$, whose value is the stalk of c at x, which by hypothesis is not acyclic. This complex is quasi-isomorphic to a direct sum $b \cong i_* i^* b \oplus j_! j^* b$, where i is the inclusion of the endpoint k and j the inclusion of the complement. But this is absurd; it is easy to see that any map $b \longrightarrow j_! j^* b$ must vanish. □

2. The proof of Theorem 0.2

We need to prove that, for any Grothendieck abelian category $\mathcal{A}$, the derived category is well generated. By Proposition 5.1 in Alonso, Jeremías and Souto's paper [1], we know that there exists a ring R and a set of objects $L \subset D(R)$ for which the following is true.

Proposition 2.1. (=Proposition 5.1 in [1]) *Let $\mathcal{L}_\mathcal{A}$ be the smallest localising subcategory of $D(R)$ containing L. Then the derived category $D(\mathcal{A})$ of $\mathcal{A}$ is equivalent to the quotient $D(\mathcal{A}) \cong D(R)/\mathcal{L}_\mathcal{A}$.*

The category $D(R)$ is compactly generated, hence well generated. By Proposition 8.4.2 of [6] (more precisely by part (8.4.2.3) of the Proposition),

$$D(R) = \bigcup_\alpha \{D(R)\}^\alpha.$$

Since L is a set of objects, the coproduct of all the objects in L is an object of $D(R)$, and therefore must lie in $\{D(R)\}^\alpha$ for some regular cardinal α. Now apply Theorem 4.4.9 of [6]. We have that, for any regular cardinal $\beta \geq \alpha$, $\mathcal{L}_\mathcal{A}^\beta$ is just $\langle L\rangle^\beta$, the smallest β-localising subcategory containing L, while $\{D(R)\}^\beta = \langle R\rangle^\beta$ is the smallest β-localising subcategory containing R. And if $\beta > \aleph_0$, then $\{D(\mathcal{A})\}^\beta = \{D(R)\}^\beta/\mathcal{L}_{\mathcal{A}}^\beta$. The categories $\mathcal{L}_{\mathcal{A}}^\beta$, $\{D(R)\}^\beta$ and $\{D(\mathcal{A})\}^\beta$ are all essentially small, and generate $\mathcal{L}_{\mathcal{A}}$, $D(R)$ and $D(\mathcal{A})$ respectively. It follows that $D(\mathcal{A})$ is well generated.

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Divisible Subgroups of Brauer Groups and Trace Forms of Central Simple Algebras

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Abstract.

Let F be a field of characteristic different from 2 and assume that F satisfies the strong approximation theorem on orderings (F is a SAP field) and that $I^3(F)$ is torsion-free. We prove that the 2-primary component of the torsion subgroup of the Brauer group of F is a divisible group and we prove a structure theorem on the 2-primary component of the Brauer group of F. This result generalizes well-known results for algebraic number fields. We apply these results to characterize the trace form of a central simple algebra over such a field in terms of its determinant and signatures.

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1 Introduction and Preliminaries

Let A be a central simple algebra over a field F of characteristic different from 2. The quadratic form $q : A \to F$ given by $x \mapsto \mathrm{Trd}_A(x^2) \in F$ is called *the trace form of* A, and is denoted by $\mathcal{T}_A$. This trace form has been studied by many authors (cf.[Le], [LM], [Ti] and [Se], Annexe §5 for example). In particular, its classical invariants are well-known (*loc.cit.*).

In this article, we prove some divisibility results for the Brauer group of fields F under the assumption that F satisfies the strong approximation theorem on orderings (F is a SAP field) and $I^3(F)$ is torsion-free. Then we apply these results to characterize the trace form of a central simple algebra over such a field in terms of its determinant and signatures.

First we review the necessary background for this article. For a field F, $\mathrm{Br}(F)$ denotes the Brauer group of F. If p is a prime number, ${}_p\mathrm{Br}(F)$ denotes the p-primary component of $\mathrm{Br}(F)$. If $n \geq 1$, $\mathrm{Br}_n(F)$ denotes the kernel of multiplication by n in the Brauer group. If A is a central simple algebra over F, *the exponent of* A, denoted by $\exp A$, is the order of $[A]$ in $\mathrm{Br}(F)$ and *the index of* A, denoted by $\operatorname{ind} A$, is the degree of the division algebra which corresponds to A. We know that $\exp A$ divides $\operatorname{ind} A$. If $a, b \in F^\times$, we denote by $(a,b)_F$ the corresponding quaternion algebra, or simply (a,b) if no confusion is possible. We also use the same symbol to denote its class in the Brauer group.
We refer to [D], [J], or [Sc] for more information on central simple algebras over general fields.
In the following, all quadratic forms are nonsingular. If q is a quadratic form over F, $\dim q$ is the dimension of q and $\det q \in F^\times/F^{\times 2}$ is the determinant of q. We denote by $\mathbb{H}$ the hyperbolic plane.
If $q \simeq \langle a_1, \cdots, a_n\rangle$, the *Hasse-Witt invariant of* q is given by $w_2(q) = \sum\limits_{i<j}(a_i, a_j) \in \mathrm{Br}_2(F)$.
If $a_1, \cdots, a_n \in F^\times$, the quadratic form $\langle\langle a_1, \cdots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ is called an *n-fold Pfister form.*
If F is a formally real field, the space of orderings of F is denoted by Ω_F. We let $\mathrm{sign}_v(q) \in \mathbb{Z}$ denote the signature of q relative to an ordering $v \in \Omega_F$. Thus $\mathrm{sign}_v(q)$ is the difference between the number of positive elements and the number of negative elements in any diagonalization of q.
If $n \geq 1$, $I^n(F)$ is the n^{th} power of the fundamental ideal of the Witt ring $W(F)$ of F. We denote by $I^n(F)_t$ the kernel of the map $I^n(F) \to \prod\limits_{v \in \Omega_F} I^n(F_v)$. We will say that $I^n(F)$ is torsion-free if $I^n(F)_t = 0$. A field F satisfies property A_n if every torsion n-fold Pfister form defined over F is hyperbolic over F. See [EL2], section 4, for more details on property A_n. The absolute stability index of F, denoted $st_a(F)$ is the smallest nonnegative integer n such that $I^{n+1}(F) = 2I^n(F)$ (or ∞, if no such integer exists). See [EP], p. 1248 for more details. The reduced stability index of F, denoted $st_r(F)$ is the smallest nonnegative integer n such that $I^{n+1}(F) \equiv 2I^n(F) \mod W(F)_t$. See [La2], Chapter 13, for more details.
A field F satisfies the strong approximation property (SAP) if for every clopen set X of Ω_F there exists $a \in F^\times$ such that $a >_v 0$ if $v \in X$ and $a <_v 0$ otherwise. See [La2] for various equivalent definitions and basic properties of SAP fields. If q is a quadratic form defined over F, then $\hat{q} \in C(\Omega_F, \mathbb{Z})$ is the continuous function $\hat{q} : \Omega_F \longrightarrow \mathbb{Z}$ defined by $\hat{q}(v) = \mathrm{sign}_v(q)$ for every $v \in \Omega_F$.
If M is a discrete torsion Galois-module of exponent m, prime to the characteristic of F , $H^n(F, M)$ denotes the n-th cohomology group
$H^n(Gal(F^{sep}/F), M)$. The group $H^n(F,M)_t$ denotes the kernel of the map $H^n(F,M) \to \prod\limits_{v \in \Omega_F} H^n(F_v, M)$. If L/F is any field extension, $\mathrm{Res}_{L/F}$ denotes the restriction map. We then have $\mathrm{Res}_{L/F}(w_2(q)) = w_2(q_L)$ for any quadratic form q over F. If L/F is a finite Galois extension, $\mathrm{Cor}_{L/F}$ denotes the core-

striction map.
In this paper, we deal only with the case when n is even, because we know that $\mathcal{T}_A \simeq n < 1 > \perp \frac{n(n-1)}{2}\mathbb{H}$ when n is odd (cf.[Se], Annexe §5 for example).
An abelian group G is *divisible* if for all $n \geq 1$, we have $G = nG$. If J is any set, $G^{(J)}$ is the group of families of elements of G indexed by J, with finite supports.
In the following, F always denotes a field of characteristic different from 2, and $K = F(\sqrt{-1})$.
We now recall some results about the classical invariants of trace forms of central simple algebras:

Theorem 1.1. *Let A be a central simple algebra over F of degree n. Then we have:*

1. $\dim \mathcal{T}_A = n^2$
2. $\det \mathcal{T}_A = (-1)^{\frac{n(n-1)}{2}}$
3. *We have* $\operatorname{sign}_v \mathcal{T}_A = \pm n$ *for each* $v \in \Omega_F$*, and* $\operatorname{sign}_v \mathcal{T}_A = n$ *if and only if* $\operatorname{Res}_{F_v/F}([A]) = 0$*, where F_v is the real closure of (F, v).*
4. *If $n = 2m \geq 2$, then* $w_2(\mathcal{T}_A) = \frac{m(m-1)}{2}(-1,-1) + m[A]$

The three first statements can be found in [Le], and the last one is proved in [LM] or [Ti] for example.

2 Divisibility results in the Brauer group

Proposition 2.1. *Let* $\theta : I^3(F) \longrightarrow \prod_{v \in \Omega_F} I^3(F_v)/I^4(F_v)$*. If $st_r(F) \leq 4$, then* $\ker(\theta) = I^3(F)_t + I^4(F)$.

Proof. It is clear that $\ker(\theta) \supseteq I^3(F)_t + I^4(F)$. Now let $q \in I^3(F)$ and assume $q \in \ker(\theta)$. Then $q_v \in I^4(F_v)$ and this implies $16|\operatorname{sign}_v(q)$ for each $v \in \Omega_F$. Thus $\hat{q} \in C(\Omega_F, 16\mathbb{Z})$. Since $st_r(F) \leq 4$, Theorem 13.1 of [La2] applied to the preorder $T = \sum F^2$ implies there exists $q_0 \in I^4(F)$ such that $\hat{q} = \hat{q_0}$. Then $q - q_0 \in I^3(F) \cap W(F)_t = I^3(F)_t$ and hence $q \in I^3(F)_t + I^4(F)$.
□

Corollary 2.2. *Let* $\bar{\theta} : I^3(F)/I^4(F) \longrightarrow \prod_{v \in \Omega_F} I^3(F_v)/I^4(F_v)$*. If $I^3(F)_t = 0$ and $st_r(F) \leq 4$, then $\bar{\theta}$ is injective and $H^3(F, \mu_2)_t = 0$.*

Proof. The hypothesis and Proposition 2.1 imply $\ker(\theta) = I^4(F)$. Therefore $\ker(\bar{\theta}) = (0)$ and $\bar{\theta}$ is injective. Since $I^3(F)/I^4(F) \simeq H^3(F, \mu_2)$, and $I^3(F_v)/I^4(F_v) \simeq H^3(F_v, \mu_2)$ by [MS1] and [MS2], it follows $H^3(F, \mu_2)_t = 0$. □

Proposition 2.3. *Assume that* $I^3(F)_t = 0$ *and* $st_r(F) \leq 4$. *Let* $\alpha \in H^2(F, \mu_{2^r})_t$ $(r \geq 1)$. *Then there exists* $\beta \in H^2(F, \mu_{2^{r+1}})$ *such that* $\alpha = 2\beta$.

Proof. The exact sequence

$$1 \to \mu_2 \to \mu_{2^{r+1}} \to \mu_{2^r} \to 1$$

(where the last map is squaring) induces the following commutative diagram with exact rows

$$\begin{tikzcd} H^2(F, \mu_{2^{r+1}}) \arrow[r] \arrow[d] & H^2(F, \mu_{2^r}) \arrow[r] \arrow[d] & H^3(F, \mu_2) \arrow[d] \\ \prod\limits_{v \in \Omega_F} H^2(F_v, \mu_{2^{r+1}}) \arrow[r] & \prod\limits_{v \in \Omega_F} H^2(F_v, \mu_{2^r}) \arrow[r] & \prod\limits_{v \in \Omega_F} H^3(F_v, \mu_2). \end{tikzcd}$$

Since the third vertical map is injective by Corollary 2.2, a diagram chase gives the conclusion. □

In Theorem 2.7 below, we need a hypothesis that is slightly stronger than the one occurring in Proposition 2.3. The following result gives a characterization of this hypothesis.

Proposition 2.4. *Let* $K = F(\sqrt{-1})$. *The following statements are equivalent.*

1. *F satisfies property* A_3 *and* $st_a(F) \leq 2$.
2. $I^3(F)_t = 0$ *and* $st_r(F) \leq 2$.
3. $st_a(K) \leq 2$.
4. $I^3(K) = 0$.
5. $H^3(K, \mu_2) = 0$.

Proof. (4) $\iff$ (5): $I^3(K) = 0$ if and only if $I^3(K)/I^4(K) = 0$ by the Arason-Pfister Hauptsatz, and $I^3(K)/I^4(K) \simeq H^3(K, \mu_2)$ by [MS1] and [MS2].
(3) $\iff$ (4): $st_a(K) \leq 2$ means $I^3(K) = 2I^2(K)$ and this holds if and only if $I^3(K) = 0$, since $\langle 1, 1\rangle = 0$ implies $2I^2(K) = 0$.
(1) $\iff$ (3): This is [EP], Theorem 3.3.
(1) $\implies$ (2): Property A_3 implies $I^3(F)_t = 0$, by [EL1], Theorem 3 and Corollary 3. It is clear that $st_a(F) \leq 2$ implies $st_r(F) \leq 2$, by [La2], Theorem 13.1(3).
(2) $\implies$ (1): Clearly $I^3(F)_t = 0$ implies F satisfies property A_3. Let q be a 3-fold Pfister form defined over F. Then there exists $q' \in I^2(F)$ such that $q - 2q' \in I^3(F)_t = 0$. Thus $q = 2q'$ with $q' \in I^2(F)$ and it follows $I^3(F) = 2I^2(F)$. □

Proposition 2.5. *If* $st_r(F) \leq 2$, *then for every* $\beta \in H^2(F, \mu_{2^{r+1}})$, *there exists* $\beta' \in H^2(F, \mu_{2^{r+1}})_t$ *such that* $2\beta' = 2\beta$.

Proof. Since the characteristic of F is not 2, we have $H^2(F, \mu_{2^{r+1}}) \simeq \mathrm{Br}_{2^{r+1}}(F)$. Let A be a central simple algebra over F such that $\beta = [A]$, and set $X = \{v \in \Omega_F, \mathrm{sign}_v \mathcal{T}_A = n\}$, where $n = \deg A$. Then $X^c = \{v \in \Omega_F, \mathrm{sign}_v \mathcal{T}_A = -n\}$ by Theorem 1.1. Since the total signature map is continuous with respect to the topology on Ω_F, the set X is clopen. Since $st_r(F) \leq 2$ and X is clopen, there exists $q \in I^2(F)$ such that $\mathrm{sign}_v(q) = \begin{cases} 4, & \text{if } v \notin X \\ 0, & \text{if } v \in X. \end{cases}$ In the Witt ring WF we have $q = \sum_{i=1}^n \langle\langle a_i, b_i \rangle\rangle$, with $a_i, b_i \in F^\times$. Let B be a central simple algebra over F such that $[B] = \sum_{i=1}^n (a_i, b_i)_F$. Let $\gamma \in H^2(F, \mu_{2^{r+1}})$ be such that $\gamma = [B]$ under the isomorphism $H^2(F, \mu_{2^{r+1}}) \simeq \mathrm{Br}_{2^{r+1}}(F)$.
Now set $\beta' = \beta + \gamma$. We clearly have $2\beta' = 2\beta$. Moreover, if $v \in X$, then $\mathrm{Res}_{F_v/F}(\beta) = 0$ by Theorem 1.1 and $\mathrm{Res}_{F_v/F}(\gamma) = 0$ by the choice of B. Similar arguments show that $\mathrm{Res}_{F_v/F}(\beta') = 0$ for all $v \notin X$. It follows that $\beta' \in H^2(F, \mu_{2^{r+1}})_t$. $\square$

Remark 2.6. In Proposition 2.5, a stronger conclusion is possible if we also assume that F is a SAP field. This is equivalent to assuming $st_r(F) \leq 1$. (See [La2].) In this case there exists an element $a \in F^\times$ such that $a >_v 0$ if $v \in X$ and $a <_v 0$ if $v \notin X$. Let $\gamma \in H^2(F, \mu_{2^{r+1}})$ be such that $\gamma = (-1, a)_F$ under the isomorphism $H^2(F, \mu_{2^{r+1}}) \simeq \mathrm{Br}_{2^{r+1}}(F)$. Now set $\beta' = \beta + \gamma$. We clearly have $2\beta' = 2\beta$. We finish as before. This observation will be used in the proof of Theorem 2.8.

Theorem 2.7. *Assume $I^3(F)_t = 0$ and $st_r(F) \leq 2$. Then ${}_2\mathrm{Br}(F)_t$ is a divisible group.*

Proof. It suffices to check that for all $[B] \in {}_2\mathrm{Br}(F)_t$ and all primes p, there exists $[A] \in {}_2\mathrm{Br}(F)_t$ such that $p[A] = [B]$. Let $[B] \in {}_2\mathrm{Br}(F)_t$. Then, there exists $r \geq 1$ such that $2^r[B] = 0$. Assume first that p is odd. Then $\gcd(p, 2^r) = 1$, so there exist $n, m \in \mathbb{Z}$ such that $np + m2^r = 1$. Then $[B] = (np + m2^r)[B] = p(n[B])$. If $p = 2$, apply Proposition 2.3 and Proposition 2.5. $\square$

We now give a structure theorem on the 2-primary component of the Brauer group. We denote by $\sum F^2$ the multiplicative subgroup of $F^\times$ of nonzero sums of squares. We use the notation of [K].

Theorem 2.8. *Assume that $I^3(F)_t = 0$ and F is SAP. Let T (resp. Λ) be an index set of a $\mathbb{Z}/2\mathbb{Z}$-basis of $\mathrm{Br}_2(F)_t$ (resp. of $F^\times / \sum F^{\times 2}$). Then we have the following group isomorphism*

$$ {}_2\mathrm{Br}(F) \simeq \mathbb{Z}(2^\infty)^{(T)} \times (\mathbb{Z}/2\mathbb{Z})^{(\Lambda)}. $$

Proof. Theorem 2.7 implies that ${}_2\mathrm{Br}(F)_t$ is a divisible group. Since every element of ${}_2\mathrm{Br}(F)_t$ has 2-power order, the structure theorems on divisible groups (see [K] for example) imply that this group is isomorphic to $\mathbb{Z}(2^\infty)^{(T)}$, where T is an index set of a basis of the 2-torsion part of ${}_2\mathrm{Br}(F)_t$, namely $\mathrm{Br}_2(F)_t$.

Let $[A] \in {}_2\operatorname{Br}(F)$. Remark 2.6 shows that there exists $a \in F^\times$ such that $[A'] := [A] + (-1, a)$ is a torsion element. Choose elements $a_\lambda \in F^\times$ such that $(a_\lambda \sum F^{\times 2})_{\lambda \in \Lambda}$ is a $\mathbb{Z}/2\mathbb{Z}$-basis of $F^\times / \sum F^{\times 2}$. Then $a = b \prod\limits_{\lambda \in \Lambda} a_\lambda^{r_\lambda}$, where $b \in \sum F^2$ and $r_\lambda = 0$ or 1. Since b is a sum of squares, $(-1, b)$ is a torsion element, so we have a decomposition $[A] = [B] + \sum r_\lambda(-1, a_\lambda)$, where $[B] = [A'] + (-1, b)$ is a torsion element. Now we show that $[B]$ and the r_λ's are uniquely determined. Assume that $[B] + \sum r_\lambda(-1, a_\lambda) = 0$. Then $(-1, \prod a_\lambda^{r_\lambda}) = -[B]$ is a torsion element. This implies that $\prod a_\lambda^{r_\lambda}$ is positive at all orderings of F, so $\prod a_\lambda^{r_\lambda}$ is a sum of squares. By choice of the a_λ's, this implies that $r_\lambda = 0$ for all $\lambda \in \Lambda$ and hence that $[B] = 0$. $\square$

3 Trace forms of central simple algebras

In this section, we give realization theorems for trace forms of central simple algebras.

Theorem 3.1. *Let $n = 2m \geq 2$ be an even integer. Assume that F is SAP and $I^2(F)$ is torsion-free. Then a quadratic form q is isomorphic to the trace form of a central simple algebra of degree n if and only if the following conditions are satisfied :*

1. $\dim q = n^2$
2. $\det q = (-1)^{\frac{n(n-1)}{2}}$
3. $\operatorname{sign}_v q = \pm n$, *for all* $v \in \Omega_F$.

Proof. The necessity follows from Theorem 1.1. Conversely, let q be a quadratic form satisfying the conditions above. Since $I^2(F)$ is torsion-free, it is well-known that quadratic forms are classified by dimension, determinant and signatures (see [EL1]). Let $X = \{v \in \Omega_F, \operatorname{sign}_v q = n\}$. This is a clopen set, so the *SAP* property of F implies there exists $a \in F^\times$ such that $a >_v 0$ if $v \in X$ and $a <_v 0$ otherwise. Set $A = M_m((-1, a))$. Then $\operatorname{Res}_{F_v/F}([A]) = 0$ if and only if $\operatorname{sign}_v q = n$, so $\mathcal{T}_A$ and q have the same signatures. Since they also have equal dimension and determinant, they are isomorphic. $\square$

The following proposition gives a characterization of fields that satisfy the hypotheses of Theorem 3.1. Note the similarity to Proposition 2.4.

Proposition 3.2. *Let $K = F(\sqrt{-1})$. The following statements are equivalent.*

1. *F satisfies property A_2 and F is a SAP field ($st_a(F) \leq 1$).*
2. *$I^2(F)_t = 0$ and F is a SAP field ($st_r(F) \leq 1$).*
3. *$st_a(K) \leq 1$.*
4. *$I^2(K) = 0$.*

5. $u(K) \leq 2$.

6. $\tilde{u}(F) \leq 2$.

7. $I^2(F)_t = 0$ *and* F *is linked.*

Proof. The proof of the equivalence of (1)-(4) is very similar to the proof of the equivalence of the corresponding statements in Proposition 2.4. The equivalence of (4) and (5) is well-known. The equivalence of (6) and (7) appears in [E]. The equivalence of (2) and (6) appears in [ELP]. □

We now give a characterization of fields F such that $I^2(F)$ is torsion-free in terms of Brauer groups.

Proposition 3.3. *$I^2(F)$ is torsion-free if and only if $\mathrm{Br}(F)$ has no element of order 4.*

Proof. Assume that $[A] \in \mathrm{Br}(F)$ has order 4, so $[A] \in H^2(F, \mu_4)$. Then $2[A] \in H^2(F, \mu_2)$ has order 2. Moreover, it is well-known that the image of $[A] \in H^2(F, \mu_4) \mapsto 2[A] \in H^2(F, \mu_2)$ is the kernel of
$[B] \in H^2(F, \mu_2) \mapsto (-1) \cup [B] \in H^3(F, \mu_2)$ (see for example [LLT], Proposition A2 and Remark A3). So $(-1) \cup 2[A] = 0$, that is $2[A] = \mathrm{Cor}_{K/F}([B])$ for some $[B] \in H^2(K, \mu_2)$. Since $H^2(K, \mu_2)$ is generated by elements of the form $(a, b), a \in F^\times, b \in K^\times$, the transfer formula shows that $2[A] = \sum (a_i, \mathrm{N}_{K/F}(b_i))$ for some $a_i \in F^\times$ and $b_i \in K^\times$. Since $2[A]$ has order 2, it is not split, so there exists i such that $(a_i, \mathrm{N}_{K/F}(b_i))$ is not split. Then the norm form of this quaternion algebra is not hyperbolic, and it is a torsion 2-fold Pfister form, since $\mathrm{N}_{K/F}(b_i)$ is the sum of 2 squares.

Conversely, assume that $I^2(F)$ is not torsion-free. Then property A_2 fails (see [EL2], section 4). Theorem 4.3(3) in [EL2] (with $x = 1$) implies that there exists a binary form $\langle 1, -a \rangle$ and an element $b = u^2 + v^2$, with $u, v \in F$, such that $\langle 1, -a \rangle$ does not represent b. This means $\langle\langle a, b \rangle\rangle$ is an anisotropic 2-fold Pfister form and b is not a square. Let $L := F(\sqrt{b + v\sqrt{b}})$. Then L/F is a cyclic quartic extension which contains $F(\sqrt{b})$. Denote by σ a generator of $Gal(L/F)$ and let A be the cyclic algebra $(a, L/F, \sigma)$ (see [Sc] for the definition and basic properties of cyclic algebras). It is not difficult to show that $2[A] = (a, b)$ (for example use [J], Corollary 2.13.20). By construction, the norm form of this quaternion algebra is not hyperbolic, so $2[A]$ is not split, and $[A]$ has order 4. □

We now apply the results of section 2 to prove the following theorem:

Theorem 3.4. *Let $n = 2m \geq 2$ be an even integer.*
Write $n = 2^{r+1}s, r \geq 0, s \geq 1$ odd. Assume that F satisfies the following conditions:

(a) *$I^3(F)$ is torsion-free*

(b) *For every $[A] \in \mathrm{Br}(F)$ such that $2^{r+1}[A] = 0$, there exists A', $\deg A' = 2^{r+1}$ such that $[A'] = [A]$*

(c) If $r \geq 1$, assume $st_r(F) \leq 2$.

Then a quadratic form q is isomorphic to the trace form of a central simple algebra of degree n if and only if the following conditions are satisfied :

1. $\dim q = n^2$
2. $\det q = (-1)^{\frac{n(n-1)}{2}}$
3. $\operatorname{sign}_v q = \pm n$, *for all* $v \in \Omega_F$.

Before we begin the proof of this theorem, we need the following calculation.

Lemma 3.5. *Let $n = 2m$, $m \geq 1$, and assume F is a real closed field. Let $q_+ = n\langle 1\rangle \perp \dfrac{n(n-1)}{2}\mathbb{H}$ and let $q_- = n\langle -1\rangle \perp \dfrac{n(n-1)}{2}\mathbb{H}$. Then $w_2(q_+) = \dfrac{m(m-1)}{2}(-1,-1)_F$ and $w_2(q_-) = \left(\dfrac{m(m-1)}{2} + m\right)(-1,-1)_F$. In particular, if m is odd, then $w_2(q_+) \neq w_2(q_-)$.*

Proof. Let $A = M_n(F)$ and let $B = M_m((-1,-1))$. Then $\deg A = \deg B = n$ and hence Theorem 1.1 implies $\operatorname{sign}(\mathcal{T}_A) = n$ and $\operatorname{sign}(\mathcal{T}_B) = -n$. This implies $\mathcal{T}_A \simeq q_+$ and $\mathcal{T}_B \simeq q_-$. In addition, Theorem 1.1 implies

$$w_2(q_+) = w_2(\mathcal{T}_A) = \frac{m(m-1)}{2}(-1,-1) + m[A] = \frac{m(m-1)}{2}(-1,-1)$$

and

$$\begin{aligned} w_2(q_-) = w_2(\mathcal{T}_B) &= \frac{m(m-1)}{2}(-1,-1) + m(-1,-1) \\ &= \left(\frac{m(m-1)}{2} + m\right)(-1,-1). \end{aligned}$$

The last statement of this Lemma is clear since $(-1,-1)_F \neq 0$ if F is real closed. □

Proof of Theorem 3.4 Notice that property (a) implies that quadratic forms are classified by dimension, determinant, Hasse-Witt invariant and signatures (see [EL1]).

The necessity follows from Theorem 1.1. Now suppose q satisfies (1)-(3). Assume first that $r = 0$, so m is odd. By hypothesis, there exists a quaternion algebra Q such that $[Q] = w_2(q) + \dfrac{m(m-1)}{2}(-1,-1)_F$. Let $A = M_m(Q)$. Then

$$w_2(\mathcal{T}_A) = \frac{m(m-1)}{2}(-1,-1)_F + m[Q] = \frac{m(m-1)}{2}(-1,-1)_F + [Q] = w_2(q).$$

We have $\operatorname{sign}_v(\mathcal{T}_A) = n$ if and only if $\operatorname{Res}_{F_v/F}([Q]) = 0$, by Theorem 1.1, which is equivalent to $w_2(q_{F_v}) = \dfrac{m(m-1)}{2}(-1,-1)_{F_v}$. This occurs if and only if

$q_{F_v} \simeq q_+$, by Lemma 3.5, since m is odd and $\mathrm{sign}_v(q) = \pm n$. Thus q and $\mathcal{T}_A$ have the same signatures. Since q and $\mathcal{T}_A$ also have the same dimension, determinant and Hasse-Witt invariant, it follows that they are isomorphic.
Assume now that $r \geq 1$. Let B be a central simple algebra over F such that $[B] = w_2(q) + \dfrac{m(m-1)}{2}(-1,-1)_F$. Since m is even and $\mathrm{sign}_v(q) = \pm n$, it follows from Lemma 3.5 that

$$\mathrm{Res}_{F_v/F}([B]) = \mathrm{Res}_{F_v/F}(w_2(q) + \frac{m(m-1)}{2}(-1,-1)_{F_v}) = 0$$

for all $v \in \Omega_F$. By Theorem 2.7, there exists $[A_1] \in {}_2\,\mathrm{Br}(F)_t$ such that $2^r[A_1] = [B]$. Let $X = \{v \in \Omega_F, \mathrm{sign}_v\, q = n\}$. Since X is clopen and $st_r(F) \leq 2$, we can use the ideas in the proof of Proposition 2.5 to find a central simple algebra D over F such that $2[D] = 0$ and such that $[A_2] = [A_1] + [D]$ satisfies $\mathrm{Res}_{F_v/F}[A_2] = 0$ if and only if $\mathrm{sign}_v(q) = n$. Then $2^r[A_2] = [B]$ since $r \geq 1$. Since $2[B] = 0$, we have $2^{r+1}[A_2] = 0$, and so by assumption there exists a central simple algebra A_3, $\deg A_3 = 2^{r+1}$, such that $[A_3] = [A_2]$. Now set $A = M_s(A_3)$, and note that A has degree n. Since A and A_2 are Brauer equivalent, q and $\mathcal{T}_A$ have equal signatures by construction of A_2. Since

$$m[A] = 2^r s[A_2] = s[B] = [B] = \frac{m(m-1)}{2}(-1,-1)_F + w_2(q),$$

it follows that $w_2(\mathcal{T}_A) = \dfrac{m(m-1)}{2}(-1,-1)_F + m[A] = w_2(q)$. Thus q and $\mathcal{T}_A$ are isomorphic, since they have the same dimension, determinant, Hasse-Witt invariant, and signature. □

Corollary 3.6. *Assume F satisfies the following conditions.*

(a) $I^3(F)$ is torsion-free

(b′) For every $r \geq 0$ and for every $[A] \in \mathrm{Br}(F)$ such that $2^{r+1}[A] = 0$, there exists A', $\deg A' = 2^{r+1}$ such that $[A'] = [A]$.

Then a quadratic form q is isomorphic to the trace form of a central simple algebra of degree n if and only if the following conditions are satisfied :

1. $\dim q = n^2$
2. $\det q = (-1)^{\frac{n(n-1)}{2}}$
3. $\mathrm{sign}_v\, q = \pm n$, *for all* $v \in \Omega_F$.

Proof. This follows immediately from the Theorem 3.4 and the following observation. Condition (b′) with $r = 0$ implies that F is a linked field. That is, a sum of quaternion algebras defined over F is similar to another quaternion algebra defined over F. A theorem of Elman ([E]) states that a field F is linked and has $I^3(F)_t = 0$ if and only if $\tilde{u}(F) \leq 4$. It is known that if $\tilde{u}(F) < \infty$, then F is a SAP field (see [ELP]). Thus condition (c) in Theorem 3.4 holds automatically in the situation of Corollary 3.6. □

Remark 3.7. Condition (b) is realized for example when $\exp A = \operatorname{ind} A$ for every central simple algebra. In particular, it is the case when every central simple algebra is cyclic. For example, condition (b) holds for local fields, global fields or quotient fields of excellent two-dimensional local domains with algebraically closed residue fields of characteristic zero, e.g. finite extensions of $\mathbb{C}((X,Y))$ (see [CTOP], Theorem 2.1 for the last example and [CF] for the others). Such fields also satisfy condition (a). This is well-known for local fields and global fields (see [CF]). If F is a field of the last type, then $I^3(F) = 0$ (see [CTOP], Corollary 3.3).

We finish this paper giving a local-global principle for trace forms over global fields.

Corollary 3.8. *Let F be a global field of characteristic different from 2, and let $n = 2m \geq 2$ be an even integer. Then a quadratic form q over F is isomorphic to the trace form of a central simple algebra of degree n defined over F if and only if q is isomorphic to the trace form of a central simple algebra of degree n defined over all completions of F.*

Proof. Assume that q is a trace form over all completions of F. Then $\dim q = n^2$. By assumption, $(-1)^{\frac{n(n-1)}{2}} \det q$ is a nonzero square over all completions of F, so it is a nonzero square in F, and hence $\det q = (-1)^{\frac{n(n-1)}{2}} \in F^\times/F^{\times 2}$. Since q is a trace form over all real completions of F, we have $\operatorname{sign}_v q = \pm n$ for all real places v of F, according to whether q_{F_v} is isomorphic to the trace form of the split algebra or that of $M_m((-1,-1)_{F_v})$. Now apply Theorem 3.4. The other implication is clear, since $(\mathcal{T}_A)_L \simeq \mathcal{T}_{A\otimes L}$ for every central simple algebra over F, and every field extension L/F. □

The fact that $q_{F_\mathfrak{p}} \simeq \mathcal{T}_{A_\mathfrak{p}}$ for all places $\mathfrak{p}$ implies that $q \simeq \mathcal{T}_A$ does not mean that $A \otimes F_\mathfrak{p} \simeq A_\mathfrak{p}$ for all places. We sketch below the construction of a counterexample.

Example 3.9. We refer to [CF] for the definition of $\operatorname{inv}_\mathfrak{p}$ and the theorems concerning central simple algebras over global fields.

Assume $n \equiv 0$ [8]. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be two places of F. For $i = 1, 2$, let A_i be a central simple of degree n over $F_{\mathfrak{p}_i}$ such that $\operatorname{inv}_{\mathfrak{p}_i}[A_i] = \dfrac{1}{n}$, and let $A_\mathfrak{p}$ be $M_n(F_\mathfrak{p})$ for the other places over F. Now let $q_\mathfrak{p}$ be the trace form of $A_\mathfrak{p}$. We have $w_2(q_\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2$. Moreover $\det q_\mathfrak{p} = (-1)^{\frac{n(n-1)}{2}}$ for all $\mathfrak{p}$, so by [Sc], 6.6.10, there exists a quadratic form q over F such that $q_{F_\mathfrak{p}} \simeq q_\mathfrak{p}$. So q is locally a trace form, then q is the trace form of some central simple algebra A over F, but we can never have $A \otimes F_\mathfrak{p} \simeq A_\mathfrak{p}$ for all $\mathfrak{p}$. Otherwise, we will have $\sum \operatorname{inv}_\mathfrak{p}([A]) = 0 \in \mathbb{Q}/\mathbb{Z}$, which is not the case by choice of the $A_\mathfrak{p}$'s.

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Tamagawa Numbers for Motives with (Non-Commutative) Coefficients

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Abstract. Let M be a motive which is defined over a number field and admits an action of a finite dimensional semisimple $\mathbb{Q}$-algebra A. We formulate and study a conjecture for the leading coefficient of the Taylor expansion at 0 of the A-equivariant L-function of M. This conjecture simultaneously generalizes and refines the Tamagawa number conjecture of Bloch, Kato, Fontaine, Perrin-Riou et al. and also the central conjectures of classical Galois module theory as developed by Fröhlich, Chinburg, M. Taylor et al. The precise formulation of our conjecture depends upon the choice of an order $\mathfrak{A}$ in A for which there exists a 'projective $\mathfrak{A}$-structure' on M. The existence of such a structure is guaranteed if $\mathfrak{A}$ is a maximal order, and also occurs in many natural examples where $\mathfrak{A}$ is non-maximal. In each such case the conjecture with respect to a non-maximal order refines the conjecture with respect to a maximal order. We develop a theory of determinant functors for all orders in A by making use of the category of virtual objects introduced by Deligne.

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1. Introduction

The study of values of L-functions attached to varieties over number fields occupies a prominent place in number theory and has led to some remarkably general conjectures. A seminal step was made by Bloch and Kato who conjecturally described up to sign the leading coefficient at zero of L-functions attached to motives of negative weight [4]. A little later, Fontaine and Perrin-Riou and (independently) Kato used the determinant functor to extend this conjecture to motives of any weight and with commutative coefficients, thereby taking into account the action of endomorphisms of the variety under consideration (cf. [19, 20, 27, 28]). In this article we shall formulate and study a yet more general conjecture which deals with motives with coefficients which need not be commutative and which in the commutative case recovers all of the above conjectures. We remark that the motivation for such a general conjecture is that in many natural cases, ranging from the central conjectures of classical Galois module theory to the recent attempts to develop an Iwasawa theory for elliptic curves which do not possess complex multiplication, it is necessary to consider motives with respect to coefficients which are not commutative.

We now fix a motive M which is defined over a number field K and carries an action of a finite dimensional semisimple $\mathbb{Q}$-algebra A. The precise formulation of our conjecture depends upon the choice of an order $\mathfrak{A}$ in A for which there exists a 'projective $\mathfrak{A}$-structure' on M (as defined in §3.3). We observe that if $\mathfrak{A}$ is any maximal order in A (as in the case considered by Fontaine and Perrin-Riou in [19]), then there always exists a projective $\mathfrak{A}$-structure on M, and in addition that if M arises by base change of a motive through a finite Galois extension L/K and $A := \mathbb{Q}[G]$ with $G := \mathrm{Gal}(L/K)$ (as in the case considered by Kato in [27]), then there exists a projective $\mathbb{Z}[G]$-structure on M. In general, we find that if there exists a projective $\mathfrak{A}$-structure on M, then there also exists a projective $\mathfrak{A}'$-structure on M for any order $\mathfrak{A} \subset \mathfrak{A}' \subset A$ but that the conjecture which we formulate for the pair $(M, \mathfrak{A}')$ is (in general strictly) weaker than that for the pair $(M, \mathfrak{A})$. This observation is important since we shall show that there are several natural examples (such as the case $\mathfrak{A} = \mathbb{Z}[G]$ described above) in which projective structures exist with respect to orders which are not maximal.

The key difficulty encountered when attempting to formulate Tamagawa number conjectures with respect to non-commutative coefficients is the fact that there is no determinant functor over non-commutative rings. In this article we circumvent this difficulty by making systematic use of the notion of 'categories of virtual objects' as described by Deligne in [17]. In our approach Tamagawa numbers are then elements of a relative algebraic K-group $K_0(\mathfrak{A}, \mathbb{R})$ and the Tamagawa number conjecture is an identity in this group. The group $K_0(\mathfrak{A}, \mathbb{R})$ is the relative K_0 which arises from the inclusion of rings $\mathfrak{A} \to A_{\mathbb{R}} := A \otimes_{\mathbb{Q}} \mathbb{R}$ and hence lies in a natural long exact sequence

$$K_1(\mathfrak{A}) \to K_1(A_{\mathbb{R}}) \to K_0(\mathfrak{A}, \mathbb{R}) \to K_0(\mathfrak{A}) \to K_0(A_{\mathbb{R}}).$$

We remark that in the non-equivariant setting originally considered by Bloch and Kato [4] one has $\mathfrak{A} = \mathbb{Z}$, $A = \mathbb{Q}$ and $K_0(\mathfrak{A}, \mathbb{R}) \cong \mathbb{R}^\times/\mathbb{Z}^\times$. This latter quotient identifies with the group of positive real numbers, and hence in this case Tamagawa numbers can be interpreted as volumes. For motives with non-commutative coefficients however, the only way we have at present been able to formulate a conjecture is by use of the group $K_0(\mathfrak{A}, \mathbb{R})$.

The basic content of this article is as follows. Various algebraic preliminaries relating to determinant functors, categories of virtual objects and relative algebraic K-theory, which may themselves be of some independent interest, are given in §2. In §3 we recall preliminaries on motives, define the notion of a 'projective $\mathfrak{A}$-structure' on M and give several natural examples of this notion. We henceforth assume that $\mathfrak{A}$ is an order in A for which there exists a projective $\mathfrak{A}$-structure on M. In the remainder of §3 we combine the results of §2 with certain standard assumptions on motives to define a canonical element $R\Omega(M,\mathfrak{A})$ of $K_0(\mathfrak{A},\mathbb{R})$. In §4 we review the A-equivariant L-function $L({}_AM,s)$ of M. This is a meromorphic function of the complex variable s which takes values in the center $\zeta(A_{\mathbb{C}})$ of $A_{\mathbb{C}} := A\otimes_{\mathbb{Q}}\mathbb{C}$, and the leading coefficient $L^*({}_AM,0)$ in its Taylor expansion at $s=0$ belongs to the group of units $\zeta(A_{\mathbb{R}})^\times$ of $\zeta(A_{\mathbb{R}})$. We also define a canonical 'extended boundary homomorphism' $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}} : \zeta(A_{\mathbb{R}})^\times \to K_0(\mathfrak{A},\mathbb{R})$ which has the property that the composite of $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ with the reduced norm map $K_1(A_{\mathbb{R}}) \to \zeta(A_{\mathbb{R}})^\times$ is equal to the boundary homomorphism $K_1(A_{\mathbb{R}}) \to K_0(\mathfrak{A},\mathbb{R})$ which occurs in the above long exact sequence. We then formulate the central conjecture of this article (Conjecture 4) which states that

$$\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(L^*({}_AM,0)) = -R\Omega(M,\mathfrak{A}) \text{ in } K_0(\mathfrak{A},\mathbb{R}).$$

We remark that our use of the map $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ in this context is motivated by the central conjectures of classical Galois module theory. In the remainder of §4 we review some of the current evidence for our conjecture, establish its standard functorial properties and also derive several interesting consequences of these functorial properties. Finally, in §5 we use the Artin-Verdier Duality Theorem to investigate the compatibility of our conjecture with the functional equation of $L({}_AM,s)$.

In a sequel to this article [11] we shall give further evidence for our general conjectures by relating them to classical Galois module theory (in particular, to certain much studied conjectures of Chinburg [13, 14]) and by proving them in several nontrivial cases. In particular, we prove the validity of our central conjecture for pairs $(M,\mathfrak{A}) = (h^0(\mathrm{Spec}(L)), \mathbb{Z}[\mathrm{Gal}(L/K)])$ where $K=\mathbb{Q}$ and $L/\mathbb{Q}$ belongs to an infinite family of Galois extensions for which $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the Quaternion group of order 8.

This article together with its sequel [11] subsumes the contents of an earlier preprint of the same title, and also of the preprint [10]. In the preprint [5] the first named author described an earlier approach to formulating Tamagawa number conjectures with respect to non-commutative coefficients, by using the notions of 'trivialized perfect complex' and 'refined Euler characteristic' (cf. Remark 4 in §2.8 in this regard). However, by making systematic use of virtual objects the approach adopted here seems to be both more flexible and transparent, and in particular allows us to prove the basic properties of our construction in a very natural manner.

We would like to thank P. Deligne for a number of very valuable comments.

2. Determinant functors for orders in semisimple algebras

2.1. Picard categories. This section introduces the natural target categories for our generalized determinant functors. Recall that a groupoid is a nonempty category in which all morphisms are isomorphisms. A *Picard category* $\mathcal{P}$ is a groupoid equipped with a bifunctor $(L, M) \to L \boxtimes M$ with an associativity constraint [32] and so that all the functors $-\boxtimes M$, $M \boxtimes -$ for a fixed object M are autoequivalences of $\mathcal{P}$. In a Picard category there exists a unit object $\mathbf{1}_{\mathcal{P}}$, unique up to unique isomorphism, and for each object M an inverse M^{-1}, unique up to unique isomorphism, with an isomorphism $M \boxtimes M^{-1} \xrightarrow{\sim} \mathbf{1}_{\mathcal{P}}$. For a Picard category $\mathcal{P}$ define $\pi_0(\mathcal{P})$ to be the group of isomorphism classes of objects of $\mathcal{P}$ (with product induced by $\boxtimes$), and set $\pi_1(\mathcal{P}) := \mathrm{Aut}_{\mathcal{P}}(\mathbf{1}_{\mathcal{P}})$. We shall only have occasion to consider *commutative* Picard categories in which $\boxtimes$ also satisfies a commutativity constraint [32] and for which $\pi_0(\mathcal{P})$ is therefore abelian. The group $\pi_1(\mathcal{P})$ is always abelian. A monoidal functor $F : \mathcal{P}_1 \to \mathcal{P}_2$ between Picard categories induces homomorphisms $\pi_i(F) : \pi_i(\mathcal{P}_1) \to \pi_i(\mathcal{P}_2)$ for $i \in \{0, 1\}$, and F is an equivalence of categories if and only if $\pi_i(F)$ is an isomorphism for both $i \in \{0, 1\}$ (by a monoidal functor we mean a strong monoidal functor as defined in [31][Ch. XI.2]).

2.2. The fibre product of categories. Let $F_i : \mathcal{P}_i \to \mathcal{P}_3$, $i \in \{1, 2\}$ be functors between categories and consider the *fibre product category* $\mathcal{P}_4 := \mathcal{P}_1 \times_{\mathcal{P}_3} \mathcal{P}_2$ [2, Ch. VII, §3]

$$
\begin{array}{ccc}
\mathcal{P}_4 & \xrightarrow{G_2} & \mathcal{P}_2 \\
\Big\downarrow{\scriptstyle G_1} & & \Big\downarrow{\scriptstyle F_2} \\
\mathcal{P}_1 & \xrightarrow{F_1} & \mathcal{P}_3.
\end{array}
\tag{1}
$$

Explicitly, $\mathcal{P}_4$ is the category with objects (L_1, L_2, λ) with $L_i \in \mathrm{Ob}(\mathcal{P}_i)$ for $i \in \{1, 2\}$ and $\lambda : F_1(L_1) \xrightarrow{\sim} F_2(L_2)$ an isomorphism in $\mathcal{P}_3$, and where morphisms $\alpha : (L_1, L_2, \lambda) \to (L'_1, L'_2, \lambda')$ are pairs $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in \mathrm{Hom}_{\mathcal{P}_i}(L_i, L'_i)$ so that the diagram

$$
\begin{array}{ccc}
F_1(L_1) & \xrightarrow{F_1(\alpha_1)} & F_1(L'_1) \\
\Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle \lambda'} \\
F_2(L_2) & \xrightarrow{F_2(\alpha_2)} & F_2(L'_2)
\end{array}
$$

in $\mathcal{P}_3$ commutes. If $\mathcal{P}_5$ is a category, $H_i : \mathcal{P}_5 \to \mathcal{P}_i$ for $i \in \{1, 2\}$ functors and $\beta : F_1 \circ H_1 \cong F_2 \circ H_2$ a natural isomorphism, then there exists a unique functor $H : \mathcal{P}_5 \to \mathcal{P}_4$ with $H_i = G_i \circ H$ for $i \in \{1, 2\}$ and such that β is induced by the natural isomorphism $F_1 \circ G_1 \cong F_2 \circ G_2$. If $\mathcal{P}_i$ for $i \in \{1, 2, 3\}$ are Picard categories and F_1 and F_2 are monoidal functors, then the fibre product category $\mathcal{P}_4$ is a Picard category with product $(L_1, L_2, \lambda) \boxtimes (L'_1, L'_2, \lambda') = (L_1 \boxtimes L'_1, L_2 \boxtimes L'_2, \lambda \boxtimes \lambda')$ and the functors $G_i : \mathcal{P}_4 \to \mathcal{P}_i$ for $i \in \{1, 2\}$ are both monoidal.

Lemma 1. *(Mayer-Vietoris sequence) For a fibre product diagram (1) of Picard categories one has an exact sequence*

$$0 \to \pi_1(\mathcal{P}_4) \xrightarrow{(\pi_1(G_1),\pi_1(G_2))} \pi_1(\mathcal{P}_1) \oplus \pi_1(\mathcal{P}_2) \xrightarrow{\pi_1(F_1)-\pi_1(F_2)} \pi_1(\mathcal{P}_3) \xrightarrow{\delta}$$
$$\to \pi_0(\mathcal{P}_4) \xrightarrow{(\pi_0(G_1),\pi_0(G_2))} \pi_0(\mathcal{P}_1) \oplus \pi_0(\mathcal{P}_2) \xrightarrow{\pi_0(F_1)-\pi_0(F_2)} \pi_0(\mathcal{P}_3).$$

Proof. The map δ is defined by $\delta(\beta) = (\mathbf{1}_{\mathcal{P}_1}, \mathbf{1}_{\mathcal{P}_2}, \beta)$. Given the explicit description of $\pi_0(-)$ and $\pi_1(-)$ it is an elementary computation to establish the exactness of this sequence. A general Mayer-Vietoris sequence for categories with product can be found in [2, Ch. VII, Th. (4.3)]. For Picard categories this sequence specialises to our Lemma (except for the injectivity of the first map). □

2.3. Determinant functors and virtual objects. Let $\mathcal{E}$ be an exact category [38, p. 91] and $(\mathcal{E}, \mathrm{is})$ the subcategory of all isomorphisms in $\mathcal{E}$. The main example we have in mind is the category $\mathrm{PMod}(R)$ of finitely generated projective modules over a (not necessarily commutative) ring R. By a *determinant functor* we mean a Picard category $\mathcal{P}$ together with the following data.

a) A functor $[\ ] : (\mathcal{E}, \mathrm{is}) \to \mathcal{P}$.
b) For each short exact sequence
$$\Sigma : 0 \to E' \to E \to E'' \to 0$$
a morphism $[\Sigma] : [E] \xrightarrow{\sim} [E'] \boxtimes [E'']$ in $\mathcal{P}$, functorial for isomorphisms of short exact sequences.
c) For each zero object 0 in $\mathcal{E}$ an isomorphism
$$\zeta(0) : [0] \xrightarrow{\sim} \mathbf{1}_{\mathcal{P}}.$$

This data is subject to the following axioms.

d) For an isomorphism $\phi : E \to E'$ and Σ the exact sequence $0 \to E \to E'$ (resp. $E \to E' \to 0$) $[\phi]$ (resp. $[\phi^{-1}]$) is the composite map
$$[E] \xrightarrow{[\Sigma]} [0] \boxtimes [E'] \xrightarrow{\zeta(0)\boxtimes\mathrm{id}} [E']$$
(resp.
$$[E'] \xrightarrow{[\Sigma]} [E] \boxtimes [0] \xrightarrow{\zeta(0)\boxtimes\mathrm{id}} [E]).$$
e) For admissible subobjects $0 \subseteq E'' \subseteq E' \subseteq E$ of an object E of $\mathcal{E}$ the diagram
$$\begin{array}{ccc} [E] & \longrightarrow & [E''] \boxtimes [E/E''] \\ \downarrow & & \downarrow \\ [E'] \boxtimes [E/E'] & \longrightarrow & [E''] \boxtimes [E'/E''] \boxtimes [E/E'] \end{array}$$
in $\mathcal{P}$ commutes.

The terminology here is borrowed from the key example in which $\mathcal{E}$ is the category of vector bundles on a scheme, $\mathcal{P}$ is the category of line bundles and the functor is taking the highest exterior power (see §2.5 below). However, as was shown by Deligne in [17, §4], there exists a universal determinant functor for any given exact category $\mathcal{E}$. More precisely, there exists a Picard category $V(\mathcal{E})$, called the 'category of virtual objects' of $\mathcal{E}$, together with data a)-c) which in addition to d) and e) also satisfies the following universal property.

f) For any Picard category $\mathcal{P}$ the category of monoidal functors $\mathrm{Hom}^{\boxtimes}(V(\mathcal{E}),\mathcal{P})$ is naturally equivalent to the category of determinant functors $(\mathcal{E},\mathrm{is}) \to \mathcal{P}$.

Although comparatively inexplicit it is this construction which works best for the purposes of this paper.
We recall that the category $V(\mathcal{E})$ has a commutativity constraint defined as follows. Let

$$\tau_{E',E''} : [E'] \boxtimes [E''] \xleftarrow{[\Sigma_1]} [E' \oplus E''] \xrightarrow{[\Sigma_2]} [E''] \boxtimes [E']$$

be the isomorphism induced by the short exact sequences

$$\Sigma_1 : 0 \to E' \to E' \oplus E'' \to E'' \to 0$$

$$\Sigma_2 : 0 \to E'' \to E' \oplus E'' \to E' \to 0.$$

Replacing $[\Sigma]$ by $\tau_{E',E''} \circ [\Sigma]$ yields a datum a), b), c) with values in $V(\mathcal{E})^{\boxtimes - op}$, the Picard category with product $(L,M) \mapsto M \boxtimes L$, and satisfying d),e). By the universal property f) of $V(\mathcal{E})$, this corresponds to a monoidal functor $F : V(\mathcal{E}) \to V(\mathcal{E})^{\boxtimes - op}$. Since we have only changed the value of [ ] on short exact sequences, F is the identity on objects and morphisms and so the monoidality of F gives a commutativity constraint on $V(\mathcal{E})$.
The proof of the existence of $V(\mathcal{E})$ in [17, §4.2-5] also gives a topological model of $V(\mathcal{E})$ which in turn implies that there are isomorphisms

$$K_i(\mathcal{E}) \xrightarrow{\sim} \pi_i(V(\mathcal{E})) \tag{2}$$

with the algebraic K-groups of the exact category $\mathcal{E}$ (see [38]) for $i \in \{0,1\}$. An exact functor $F : \mathcal{E} \to \mathcal{E}'$ induces a datum a),b),c) on $\mathcal{E}$ with values in $V(\mathcal{E}')$ and hence by f) a monoidal functor $V(F) : V(\mathcal{E}) \to V(\mathcal{E}')$. The isomorphism (2) then commutes with the maps induced by F on $K_i(\mathcal{E})$ and by $V(F)$ on $\pi_i(V(\mathcal{E}))$ for $i \in \{0,1\}$. Moreover, for $i = 0$ the isomorphism (2) is the map induced by the functor [ ], and for $i = 1$ the element in $K_1(\mathcal{E})$ represented by $\phi \in \mathrm{Aut}_{\mathcal{E}}(P)$ is sent to $[\phi] \boxtimes \mathrm{id}([P]^{-1})$ under (2).

2.4. Projective modules and extension to the derived category. For a ring R denote by $\mathrm{PMod}(R)$ the exact category of finitely generated projective left-R-modules and put $V(R) := V(\mathrm{PMod}(R))$. For a ring homomorphism $R \to R'$ we denote by $R' \otimes_R -$ both the scalar extension functor $\mathrm{PMod}(R) \to \mathrm{PMod}(R')$ and also the induced functor $V(R) \to V(R')$. It is known that the Whitehead group $K_1(R) := K_1(\mathrm{PMod}(R))$ of R is generated by automorphisms of objects of $\mathrm{PMod}(R)$.

We write $D(R)$ for the derived category of the homotopy category of complexes of R-modules, and $D^p(R)$ for the full triangulated subcategory of $D(R)$ which consists of perfect complexes. We say that an R-module X is *perfect* if the associated complex $X[0]$ belongs to $D^p(R)$, and we write $D^{p,p}(R)$ for the full subcategory of $D^p(R)$ consisting of those objects for which the cohomology modules are perfect in all degrees. The association $X \mapsto X[0]$ gives a full embedding of $\mathrm{PMod}(R)$ into $D^p(R)$.
In what follows we use the term 'true triangle' as synonymous for 'short exact sequence of complexes'. By a 'true nine term diagram' we shall mean a commutative diagram of complexes of the form

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' \\ {\scriptstyle f'}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle h'}\downarrow \\ X'' & \xrightarrow{u''} & Y'' & \xrightarrow{v''} & Z'' \end{array} \tag{3}$$

in which all of the rows and columns are true triangles.

Proposition 2.1. *The functor* $[\,] : (\mathrm{PMod}(R), \mathrm{is}) \to V(R)$ *extends to a functor* $[\,] : (D^p(R), \mathrm{is}) \to V(R)$. *Moreover, for each true triangle*

$$E = E(u,v) : X \xrightarrow{u} Y \xrightarrow{v} Z$$

in which X, Y, Z *are objects of* $D^p(R)$ *there exists an isomorphism* $[E] : [Y] \xrightarrow{\sim} [X] \boxtimes [Z]$ *in* $V(R)$ *which satisfies all of the following conditions:*

a) *If*

$$\begin{array}{ccccc} X & \xrightarrow{\quad u \quad} & Y & \xrightarrow{\quad v \quad} & Z \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ X' & \xrightarrow{\quad u' \quad} & Y' & \xrightarrow{\quad v' \quad} & Z' \end{array}$$

is a commutative diagram of true triangles and f, g, h *are all quasi-isomorphisms, then* $[f] \boxtimes [h] \circ [E(u,v)] \circ [g]^{-1} = [E(u',v')]$.

b) *If* u *(resp.* v*) is a quasi-isomorphism, then* $[E] = [u]^{-1}$ *(resp.* $[E] = [v]$*).*

c) $[\,]$ *commutes with the functors induced by any ring extension* $R \to R'$ *and for any true triangle* E *we have* $R' \otimes_R [E] = [R' \otimes_R E]$.

d) *For any true nine term diagram (3) in which all terms are objects of $D^p(R)$, the diagram*

$$\begin{array}{ccc} [Y'] & \xrightarrow{[E(u',v')]} & [X']\boxtimes[Z'] \\ \Big\downarrow{\scriptstyle [E(g,g')]} & & \Big\downarrow{\scriptstyle [E(f,f')]\boxtimes[E(h,h')]} \\ [Y]\boxtimes[Y''] & \xrightarrow{[E(u,v)]\boxtimes[E(u'',v'')]} & [X]\boxtimes[Z]\boxtimes[X'']\boxtimes[Z''] \end{array}$$

in $V(R)$ commutes. (Note that we have suppressed any explicit reference to commutativity constraints in the above diagram).

e) *For any object X of $D^{p,p}(R)$ there exists a canonical isomorphism*

$$[X] \xrightarrow{\sim} \boxtimes_{i\in\mathbb{Z}} [H^i(X)]^{(-1)^i} \qquad (4)$$

which is functorial with respect to quasi-isomorphisms.

Proof. This follows directly from [30, Prop. 4, Th. 2] where the same statement is proved for the determinant functor over a commutative ring R (see §2.5 below). Indeed, since the only properties of the determinant functor used in that proof are those listed in [loc. cit., Prop. 1] and all of these properties are satisfied by the functor [ ] (see [17, Lem. 4.8] for nine term diagrams) these arguments apply to give the desired extension of [ ] to $D^p(R)$ with properties a)-d). For e) see [30, Rem. b) after Th. 2]. □

Remark 1. As pointed out in [30, Rem. before Prop. 6], it is not possible to construct isomorphisms $[E]$ for all exact triangles E in $D^p(R)$ in such a way that the obvious generalisations of properties a)-d) hold. On the subcategory $D^{p,p}(R)$, however, one can at least construct isomorphisms so that a)-c) hold and so that d) holds under further assumptions (for example, that one of Y, X', Z' or Y'' is acyclic, or that X'' acyclic and $\mathrm{Hom}_{D^p(R)}(X,w)=0$ where w is such that $Z[-1]\xrightarrow{w} X \xrightarrow{u} Y \xrightarrow{v} Z$ is an exact triangle).

2.5. Commutative rings. If R is a commutative ring, then one can consider the Picard category $\mathcal{P}(R)$ of *graded line bundles* on $\mathrm{Spec}(R)$ [30]. Recall that a graded line bundle is a pair (L,α) consisting of an invertible (that is, projective rank one) R-module L and a locally constant function $\alpha : \mathrm{Spec}(R)\to\mathbb{Z}$. A homomorphism $h:(L,\alpha)\to(M,\beta)$ is a module homomorphism $h: L\to M$ such that $\alpha(\mathfrak{p})\neq\beta(\mathfrak{p})$ implies $h_{\mathfrak{p}}=0$ for all $\mathfrak{p}\in\mathrm{Spec}(R)$, and $\mathcal{P}(R)$ is the category of graded line bundles and isomorphisms of such. The category $\mathcal{P}(R)$ is a symmetric monoidal category with tensor product $(L,\alpha)\otimes(M,\beta):=(L\otimes_R M,\alpha+\beta)$, the usual associativity constraint, unit object $(R,0)$, and commutativity constraint

$$\psi(l\otimes m) := \psi_{(L,\alpha),(M,\beta)}(l\otimes m) = (-1)^{\alpha(\mathfrak{p})\beta(\mathfrak{p})} m\otimes l \qquad (5)$$

for local sections $l \in L_{\mathfrak{p}}$ and $m \in M_{\mathfrak{p}}$. For a finitely generated projective R-module P one defines

$$\mathrm{Det}_R(P) := (\bigwedge_R^{\mathrm{rank}_R(P)} P, \mathrm{rank}_R(P)) \in \mathrm{Ob}(\mathcal{P}(R)).$$

The functor $\mathrm{Det}_R : (\mathrm{PMod}(R), \mathrm{is}) \to \mathcal{P}(R)$ is equipped with the data b) and c) of §2.3 and satisfies d) and e). Hence by f) there exists a unique monoidal functor

$$\mathrm{VDet}_R : V(R) \to \mathcal{P}(R),$$

which is also compatible with the commutativity constraints (this is the reason for the choice of signs in (5)). The functor VDet_R is an equivalence of categories if and only if the natural maps

$$K_0(R) \to \mathrm{Pic}(R) \times H^0(\mathrm{Spec}(R), \mathbb{Z}) \tag{6}$$

$$K_1(R) \to R^\times$$

are both bijective. For any commutative ring R, these maps are split surjections by [24, Exp. I, 6.11-6.14; Exp. X, Th. 5.3.2], [43, (1.8)]. They are known to be bijective if, for example, R is either a local ring, a semisimple ring or the ring of integers in a number field.

2.6. Semisimple rings. We recall here some facts about $K_0(R)$ and $K_1(R)$ for semisimple rings R. For the moment we let F be any field and assume that R is a central simple algebra over F. We fix a finite extension F'/F so that $R' := R \otimes_F F' \cong M_n(F')$ and an indecomposable idempotent e of R'.
The map $V \mapsto \dim_{F'} e(V \otimes_F F')$ is additive in $V \in \mathrm{Ob}(\mathrm{PMod}(R))$ and therefore induces a homomorphism

$$\mathrm{rr}_R : K_0(R) \to \mathbb{Z}.$$

This 'reduced rank' homomorphism is injective and has image $[\mathrm{End}_R(S) : F]^{\frac{1}{2}}\mathbb{Z}$ where here S is the unique simple R-module. Similarly, if $\phi \in \mathrm{End}_R(V)$, then we set $\mathrm{detred}(\phi) := \det_{F'}(\phi \otimes 1 | e(V \otimes_F F'))$. This is an element of F which is independent of the choices of both F' and e. Recalling that $K_1(R)$ is generated by pairs (V, ϕ) with $\phi \in \mathrm{Aut}_R(V)$ it is not hard to show that detred induces a homomorphism

$$\mathrm{nr}_R : K_1(R) \to F^\times$$

(cf. [15, §45A]). This 'reduced norm' homomorphism is in general neither injective nor surjective.

Proposition 2.2. *If F is either a local or a global field, then nr_R is injective. If F is a local field different from $\mathbb{R}$, then nr_R is bijective. If $F = \mathbb{R}$, then $\mathrm{im}(\mathrm{nr}_R) = (\mathbb{R}^\times)^2$ if R is a matrix algebra over the division ring of real quaternions, and $\mathrm{im}(\mathrm{nr}_R) = \mathbb{R}^\times$ otherwise. Finally, if F is a number field, then*

$$\mathrm{im}(\mathrm{nr}_R) = \{f \in F^\times : f_v > 0 \text{ for all } v \in S_A(F)\} \tag{7}$$

where $S_A(F)$ denotes the set of places v of F such that $F_v = \mathbb{R}$ and $A \otimes_{F,v} \mathbb{R}$ is a matrix algebra over the division ring of real quaternions.

Proof. See [15, (45.3)] □

If now R is a general semisimple ring, then the above considerations apply to each of the Wedderburn factors of R. The center $\zeta(R)$ of R is a product of fields and we obtain maps

$$\mathrm{rr}_R : K_0(R) \to H^0(\mathrm{Spec}\,(\zeta(R)), \mathbb{Z}), \qquad \mathrm{nr}_R : K_1(R) \to \zeta(R)^\times.$$

If R is finite dimensional over either a local or a global field, then both of these maps are injective.

Lemma 2. *If R is any semisimple ring, then the maps* rr_R *and* nr_R *are both induced by a determinant functor* $(\mathrm{PMod}(R), \mathrm{is}) \to \mathcal{P}(\zeta(R))$.

Proof. We first observe that the target group of rr_R (resp. nr_R) does indeed coincide with $\pi_0(\mathcal{P}(\zeta(R)))$ (resp. $\pi_1(\mathcal{P}(\zeta(R)))$).
To construct a determinant functor it is clearly sufficient to restrict attention to each Wedderburn factor of R. Such a factor is isomorphic to $M_n(D)$, say, where D is a division ring with center F. By fixing an exact (Morita) equivalence $\mathrm{PMod}(M_n(D)) \to \mathrm{PMod}(D)$, it therefore suffices to construct a determinant functor for D. To this end we suppose that F'/F is a field extension such that $D \otimes_F F' \cong M_d(F')$, that e is an indecomposable idempotent of $M_d(F')$ and that $e_1, .., e_d$ is an ordered F'-basis of $eM_d(F')$. Any finitely generated projective D-module V is free, and for any D-basis $v_1, .., v_r$ of V the wedge product $b := \bigwedge e_i v_j$ (with the e_i in the fixed ordering) is an F'-basis of $\mathrm{Det}_{F'}(e(V \otimes_F F'))$. Since any change of basis v_i multiplies b by an element of $\mathrm{im}(\mathrm{nr}_D) \subseteq F^\times$, the F-space spanned by b yields a well defined graded F-line bundle. □

This result shows that if the maps rr_R and nr_R are both injective, then one can dispense with virtual objects and instead use an explicit functor to graded line bundles over $\zeta(R)$. However, this approach no longer seems to be possible when one considers orders in non-commutative semisimple algebras, and it is in this setting that the existence of virtual objects will be most useful for us.

2.7. Orders in finite-dimensional $\mathbb{Q}$-algebras. Let A be a finite-dimensional $\mathbb{Q}$-algebra (associative and unital but not necessarily commutative) and put $A_F := A \otimes_{\mathbb{Q}} F$ for any field F of characteristic zero. For brevity we write A_p for $A_{\mathbb{Q}_p}$. Let R be a finitely generated subring of $\mathbb{Q}$. We call an R-subalgebra $\mathfrak{A}$ of A an *R-order* if $\mathfrak{A}$ is a finitely generated R-module and $\mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Q} = A$. We shall refer to a $\mathbb{Z}$-order more simply as an *order*. For any order $\mathfrak{A}$ we set $\mathfrak{A}_p := \mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $\hat{\mathfrak{A}} := \mathfrak{A} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \cong \prod_p \mathfrak{A}_p$ and $\hat{A} := A \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. The diagram of exact categories and exact (scalar extension) functors

$$\begin{array}{ccc} \mathrm{PMod}(\mathfrak{A}) & \longrightarrow & \mathrm{PMod}(A) \\ \downarrow & & \downarrow \\ \mathrm{PMod}(\hat{\mathfrak{A}}) & \longrightarrow & \mathrm{PMod}(\hat{A}) \end{array}$$

induces a corresponding diagram of Picard categories and monoidal functors. These diagrams commute up to a natural equivalence of functors. By the universal property of the fibre product category we therefore obtain a monoidal functor

$$V(\mathfrak{A}) \to V(\hat{\mathfrak{A}}) \times_{V(\hat{A})} V(A) =: \mathbb{V}(\mathfrak{A}). \tag{8}$$

We use the notation $\mathbb{V}(\mathfrak{A})$ in an attempt to stress the adelic nature of $\mathbb{V}(-)$.

Proposition 2.3. *The functor (8) induces an isomorphism on π_0 and a surjection on π_1.*

Proof. There is a map of long exact Mayer-Vietoris sequences

$$\begin{array}{ccccccccc} & \longrightarrow & K_1(\mathfrak{A}) & \longrightarrow & K_1(\hat{\mathfrak{A}}) \oplus K_1(A) & \longrightarrow & K_1(\hat{A}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(\mathbb{V}(\mathfrak{A})) & \longrightarrow & \pi_1(V(\hat{\mathfrak{A}})) \oplus \pi_1(V(A)) & \longrightarrow & \pi_1(V(\hat{A})) & \longrightarrow & \dots \end{array}$$

$$\begin{array}{ccccccc} \dots & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(\hat{\mathfrak{A}}) \oplus K_0(A) & \longrightarrow & K_0(\hat{A}) \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \pi_0(\mathbb{V}(\mathfrak{A})) & \longrightarrow & \pi_0(V(\hat{\mathfrak{A}})) \oplus \pi_0(V(A)) & \longrightarrow & \pi_0(V(\hat{A})) \end{array}$$

where the top sequence can be found in [15, (42.19)], the bottom sequence arises from Lemma 1 and the vertical maps are the isomorphisms (2) or, in the case of $K_i(\mathfrak{A})$, the isomorphisms (2) composed with the map induced by (8). The commutativity of the diagram follows from the naturality of (2) and, in the case of the boundary map, from an elementary computation using the explicit description of $K_1(-)$ (cf. [15, (38.28), (40.6)]). The statement of the proposition is then an easy consequence of the Five Lemma. □

Remark 2. The functor (8) may fail to be an equivalence of categories even if $\mathfrak{A}$ is commutative. Indeed, the map $K_1(\mathfrak{A}) \to \pi_1(\mathbb{V}(\mathfrak{A}))$ is an isomorphism if and only if the map $K_1(\mathfrak{A}) \to K_1(\hat{\mathfrak{A}}) \oplus K_1(A)$ is injective. This injectivity condition fails for $\mathfrak{A} = \mathbb{Z}[G]$ where G is any finite abelian group for which $SK_1(\mathbb{Z}[G])$ is nontrivial (see [15, Rem. after (48.8)] for examples of such groups G) because $\mathbb{Z}_p[G]$ is then a product of local rings and therefore $SK_1(\mathbb{Z}_p[G]) = 0$. However, if A is semisimple and $\mathfrak{A}$ is a maximal order, then the functor (8) is an equivalence of Picard categories. Indeed, under these hypotheses one can use the Wedderburn decompositions of A and $\mathfrak{A}$ and the Morita invariance of each functor $K_i(-)$ in order to reduce to the case in which A is a division ring. In this case the injectivity of $K_1(\mathfrak{A}) \to K_1(\hat{\mathfrak{A}}) \oplus K_1(A)$ is a consequence of [15, (45.15)].

Proposition 2.3 is crucial in what follows because it allows us to work in a Picard category $\mathbb{V}(\mathfrak{A})$ which has the 'correct' π_0 and in which objects localize in a similar manner to graded line bundles. Indeed, as the following result shows, it is quite reasonable to regard $\mathbb{V}(\mathfrak{A})$ as a generalisation of the category $\mathcal{P}(\mathfrak{A})$ to orders which need not be commutative.

Proposition 2.4. *If $\mathfrak{A}$ is a finite flat commutative $\mathbb{Z}$-algebra, then there is a natural equivalence of Picard categories $\mathcal{P}(\mathfrak{A}) \xrightarrow{\sim} \mathbb{V}(\mathfrak{A})$.*

Proof. We shall show that the natural monoidal functors in the diagram

$$
\begin{array}{ccc}
 & & \mathbb{V}(\mathfrak{A}) = V(\hat{\mathfrak{A}}) \times_{V(\hat{A})} V(A) \\
 & & \Big\downarrow {\scriptstyle \hat{D}:=\mathrm{VDet}_{\hat{\mathfrak{A}}} \times \mathrm{VDet}_A} \\
\mathcal{P}(\mathfrak{A}) & \longrightarrow & \mathcal{P}(\hat{\mathfrak{A}}) \times_{\mathcal{P}(\hat{A})} \mathcal{P}(A)
\end{array} \tag{9}
$$

are equivalences of Picard categories.

We recall that a ring R is said to be local (resp. semilocal) if $R/J(R)$ is a division ring (resp. is semisimple) where $J(R)$ denotes the Jacobson radical of R.

Lemma 3. *a) Suppose $R = \prod R_\iota$ is a (possibly infinite) product of semilocal rings R_ι. Then the natural map $K_i(R) \to \prod K_i(R_\iota)$ is injective for $i = 0$ and bijective for $i = 1$.*
b) If $R = \prod R_\iota$ with each R_ι local and commutative, then the functor VDet_R *is an equivalence.*

Proof. For a semilocal ring R_ι finitely generated projective modules P_ι and P'_ι are isomorphic if and only if their classes in $K_0(R_\iota)$ agree (see the discussion after [15, (40.35)]). In addition, for any finitely generated projective R-module P the natural map $P \to \prod_\iota P \otimes_R R_\iota$ is an isomorphism since both sides are additive and the map is an isomorphism for $P = R$. Hence the isomorphism class of P can be recovered from its image in $\prod K_0(R_\iota)$, and this implies a) for $i = 0$.

For any ring R we have $K_1(R) = \varinjlim_n GL_n(R)/E_n(R)$ where $E_n(R)$ is the subgroup generated by elementary matrices [15, (40.26)]. If R is semilocal, then the map $GL_n(R)/E_n(R) \to K_1(R)$ is an isomorphism for $n \geq 2$ [15, (40.31), (40.44)]. In addition, the minimal number of generators needed to express a matrix in $E_n(R_\iota)$ as a product of elementary matrices is bounded, depending on n but not on R_ι [15, (40.31)]. Hence $E_n(\prod R_\iota) = \prod E_n(R_\iota)$ and this implies that

$$K_1(R) = K_1(\prod R_\iota) \cong \varinjlim_n \prod GL_n(R_\iota)/E_n(R_\iota)$$
$$\cong \varinjlim_n \prod K_1(R_\iota) \cong \prod K_1(R_\iota),$$

i.e. statement a) for $i = 1$.
We remark now that b) follows from a) by using the decomposition $R^\times = \prod R_\iota^\times$, the fact that the functors VDet_{R_ι} are equivalences, and the fact that the image of the map

$$K_0(R) \to \prod K_0(R_\iota) = \prod H^0(\mathrm{Spec}(R_\iota), \mathbb{Z})$$

lies in (and is therefore isomorphic to) the subgroup $H^0(\mathrm{Spec}(R), \mathbb{Z})$. $\square$

The functor $\mathrm{VDet}_{\hat{\mathfrak{A}}}$ in (9) is an equivalence by Lemma 3b) since $\hat{\mathfrak{A}}$ is a finite continuous commutative $\hat{\mathbb{Z}}$-algebra and hence a product of local rings. Similarly, VDet_A is an equivalence since A is Artinian and commutative and hence a product of local rings. The ring $\hat{A}$ is a filtered direct limit of rings $\mathfrak{A}_S := \prod_{p\in S} A_p \times \prod_{p\notin S} \mathfrak{A}_p$ for finite sets of primes S. As the ring $\mathfrak{A}_S$ is likewise a product of local rings the functor $\mathrm{VDet}_{\mathfrak{A}_S}$ is also an equivalence. It follows that in the commutative diagram

$$\begin{array}{ccc}
\varinjlim_S \pi_i(V(\mathfrak{A}_S)) & \longrightarrow & \pi_i(V(\hat{A})) \\
\downarrow & & \downarrow{\scriptstyle \pi_i(\mathrm{VDet}_{\hat{A}})} \\
\varinjlim_S \pi_i(\mathcal{P}(\mathfrak{A}_S)) & \longrightarrow & \pi_i(\mathcal{P}(\hat{A}))
\end{array}$$

the left hand vertical map is an isomorphism. Furthermore, the upper horizontal map is an isomorphism by (2) and [43, Lem. 5.9], and the lower horizontal map is an isomorphism for $i = 0$ (resp. $i = 1$) since $H^0(\mathrm{Spec}(\hat{A}), \mathbb{Z}) = \varinjlim_S H^0(\mathrm{Spec}(\mathfrak{A}_S), \mathbb{Z})$ by [23, 8.2.11] and the fact that an affine scheme is quasi-compact (resp. since $\hat{A}^\times = \varinjlim_S \mathfrak{A}_S^\times$).
We deduce that $\mathrm{VDet}_{\hat{A}}$, and as a consequence also $\hat{D}$, is an equivalence of Picard categories. It is known that the map (6) is an isomorphism for $R = \mathfrak{A}$ because $\mathfrak{A}$ is Noetherian of dimension 1 [24, Exp. VI 6.9]. Using Proposition 2.3 we find an isomorphism

$$\pi_0(\mathcal{P}(\mathfrak{A})) = \mathrm{Pic}(\mathfrak{A}) \times H^0(\mathrm{Spec}(\mathfrak{A}), \mathbb{Z}) \xleftarrow{\sim} K_0(\mathfrak{A})$$
$$\xrightarrow{\sim} \pi_0(\mathbb{V}(\mathfrak{A})) \xrightarrow{\pi_0(\hat{D})} \pi_0(\mathcal{P}(\hat{\mathfrak{A}}) \times_{\mathcal{P}(\hat{A})} \mathcal{P}(A)).$$

This isomorphism coincides with π_0 of the lower horizontal functor in (9). Moreover, from the Mayer-Vietoris sequence for the fibre product $\mathcal{P}(\hat{\mathfrak{A}}) \times_{\mathcal{P}(\hat{A})} \mathcal{P}(A)$ one easily deduces that

$$\pi_1(\mathcal{P}(\mathfrak{A})) = \mathfrak{A}^\times \cong \pi_1(\mathcal{P}(\hat{\mathfrak{A}}) \times_{\mathcal{P}(\hat{A})} \mathcal{P}(A)).$$

Hence the lower horizontal functor in (9) is an equivalence, and this finishes the proof of Proposition 2.4. □

This proposition makes it reasonable to think of objects of $\mathbb{V}(\mathfrak{A})$ as generalized graded line bundles. We invite the reader to do so when following the rest of this paper.

Lemma 4. *Assume that A is semisimple. Then the natural functor*

$$\mathbb{V}(\mathfrak{A}) = V(\hat{\mathfrak{A}}) \times_{V(\hat{A})} V(A) \to \prod_p V(\mathfrak{A}_p) \times_{\prod_p V(A_p)} V(A)$$

induces an injection on π_0 and an isomorphism on π_1.

Proof. By using Mayer-Vietoris sequences and the Five Lemma it suffices to show that the maps $K_1(\hat{A}) \to \prod_p K_1(A_p)$ and $K_0(\hat{\mathfrak{A}}) \to \prod_p K_0(\mathfrak{A}_p)$ are injective, and that the map $K_1(\hat{\mathfrak{A}}) \to \prod_p K_1(\mathfrak{A}_p)$ is bijective. For the latter two maps this is immediate from Lemma 3a). Since

$$K_1(\hat{A}) = \varinjlim_S K_1(\prod_{p\in S} A_p \times \prod_{p\notin S} \mathfrak{A}_p) \cong \prod_{p\in S} K_1(A_p) \times \prod_{p\notin S} K_1(\mathfrak{A}_p)$$

is a limit over finite sets S it therefore suffices to show that the map $K_1(\mathfrak{A}_p) \to K_1(A_p)$ is injective for any sufficiently large p. To prove this we may assume that A_p is a product of matrix algebras over finite field extensions F of $\mathbb{Q}_p$ and that $\mathfrak{A}_p$ is the corresponding product of matrix algebras over integer rings $\mathcal{O}_F$. By Morita equivalence the required result thus follows from the injectivity of the natural map $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times \to K_1(F^\times) = F^\times$. □

Remark 3. We believe that the assertion of Lemma 4 may well continue to hold without the assumption that A is semisimple, but we have no need for such additional generality in what follows.

2.8. The relative K_0. Let $\mathcal{P}_0$ be the Picard category with unique object $\mathbf{1}_{\mathcal{P}_0}$ and $\mathrm{Aut}_{\mathcal{P}_0}(\mathbf{1}_{\mathcal{P}_0}) = 0$. For A and $\mathfrak{A}$ as in §2.7 and an extension field F of $\mathbb{Q}$ we define $\mathbb{V}(\mathfrak{A}, F)$ to be the fibre product category in the diagram

$$\begin{array}{ccc}
\mathbb{V}(\mathfrak{A}, F) := \mathbb{V}(\mathfrak{A}) \times_{V(A_F)} \mathcal{P}_0 & \longrightarrow & \mathcal{P}_0 \\
\big\downarrow & & \big\downarrow{\scriptstyle F_2} \\
\mathbb{V}(\mathfrak{A}) & \xrightarrow{F_1} & V(A_F)
\end{array}$$

where here F_2 is the unique monoidal functor and $F_1((L, M, \lambda)) = M \otimes_A A_F$ for each object (L, M, λ) of $\mathbb{V}(\mathfrak{A})$. We define the category $V(\mathfrak{A}_p, \mathbb{Q}_p) := V(\mathfrak{A}_p) \times_{V(A_p)} \mathcal{P}_0$ in a similar manner.

Proposition 2.5. *For any field extension F of $\mathbb{Q}$ one has an isomorphism*

$$\pi_0\mathbb{V}(\mathfrak{A},F)\xrightarrow{\sim}K_0(\mathfrak{A},F),$$

and for any prime p an isomorphism

$$\pi_0 V(\mathfrak{A}_p,\mathbb{Q}_p)\xrightarrow{\sim}K_0(\mathfrak{A}_p,\mathbb{Q}_p),$$

where the respective right hand sides are the relative algebraic K-groups as defined in [44, p. 215].

Proof. We recall that $K_0(\mathfrak{A},F)$ is an abelian group with generators (X,g,Y), where X and Y are finitely generated projective $\mathfrak{A}$-modules and $g:X\otimes_{\mathbb{Z}}F\to Y\otimes_{\mathbb{Z}}F$ is an isomorphism of A_F-modules. For the defining relations we refer to [44, p.215]. By using these relations one checks that the map

$$(X,g,Y)\mapsto([X]\boxtimes[Y]^{-1},[g]\boxtimes\mathrm{id}([Y\otimes_{\mathbb{Z}}F]^{-1})\in\pi_0\mathbb{V}(\mathfrak{A},F)$$

induces a homomorphism $c:K_0(\mathfrak{A},F)\to\pi_0\mathbb{V}(\mathfrak{A},F)$. This homomorphism fits into a natural map of the relative K-theory exact sequence [44, Th. 15.5] to the Mayer-Vietoris sequence of the fibre product defining $\mathbb{V}(\mathfrak{A},F)$

$$\begin{array}{ccccccccc}
K_1(\mathfrak{A}) & \longrightarrow & K_1(A_F) & \xrightarrow{\delta^1_{\mathfrak{A},F}} & K_0(\mathfrak{A},F) & \xrightarrow{\delta^0_{\mathfrak{A},F}} & K_0(\mathfrak{A}) & \longrightarrow & K_0(A_F)\\
\downarrow & & \downarrow & & c\downarrow & & \downarrow & & \downarrow\\
\pi_1\mathbb{V}(\mathfrak{A}) & \longrightarrow & \pi_1 V(A_F) & \longrightarrow & \pi_0\mathbb{V}(\mathfrak{A},F) & \longrightarrow & \pi_0\mathbb{V}(\mathfrak{A}) & \longrightarrow & \pi_0 V(A_F).
\end{array}$$

The commutativity of this diagram is easy to check, given the explicit nature of all of the maps involved. For example, $\delta^0_{\mathfrak{A},F}((X,g,Y))=[X]-[Y]$, $\delta^1_{\mathfrak{A},F}$ sends the element in $K_1(A_F)$ represented by an $n\times n$-matrix g to $(\mathfrak{A}^n,g,\mathfrak{A}^n)$ and the vertical maps are as described above. Given the commutativity of this diagram, the isomorphisms (2) combine with Proposition 2.3 and the Five Lemma to imply that c is bijective. The proof for $V(\mathfrak{A}_p,\mathbb{Q}_p)$ is entirely similar using the long exact relative K-theory sequence

$$K_1(\mathfrak{A}_p)\to K_1(A_p)\xrightarrow{\delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}}K_0(\mathfrak{A}_p,\mathbb{Q}_p)\xrightarrow{\delta^0_{\mathfrak{A}_p,\mathbb{Q}_p}}K_0(\mathfrak{A}_p)\to K_0(A_p). \tag{10}$$

□

Remark 4. For any isomorphism $\lambda:X\to Y$ of A_F-modules we write λ_{Triv} for the isomorphism $[\lambda]\boxtimes\mathrm{id}([Y]^{-1}):[X]\boxtimes[Y]^{-1}\xrightarrow{\sim}\mathbf{1}_{V(A_F)}$. For any $\mathbb{Z}$-graded module or morphism $X^\bullet$ we write X^+, resp. X^-, for the direct sum of X^i over all even, resp. odd, indices i.

If $P^\bullet$ is an object of the category $\mathrm{PMod}(\mathfrak{A})^\bullet$ of bounded complexes of objects of $\mathrm{PMod}(R)$ and ψ an A_F-equivariant isomorphism from $H^+(P^\bullet)\otimes F$ to $H^-(P^\bullet)\otimes F$, then we set

$$\begin{aligned}\langle P^\bullet,\psi\rangle:=&([P^+]\boxtimes[P^-]^{-1},[H^+(P^\bullet)\otimes F]\boxtimes[H^-(P^\bullet)\otimes F]^{-1},h;\psi_{\mathrm{Triv}})\\ \in&\mathbb{V}(\mathfrak{A},F),\end{aligned}$$

where here h denotes the composite of the canonical isomorphism

$$[P^+ \otimes F] \boxtimes [P^- \otimes F]^{-1} \xrightarrow{\sim} [P^\bullet \otimes F],$$

the isomorphism (4) for $X = P^\bullet \otimes F$ and the canonical isomorphism

$$\boxtimes_{i\in\mathbb{Z}}[H^i(P^\bullet) \otimes F]^{(-1)^i} \xrightarrow{\sim} [H^+(P^\bullet) \otimes F] \boxtimes [H^-(P^\bullet) \otimes F]^{-1}.$$

If now Y is any object of $D^p(\mathfrak{A})$ and ψ is an A_F-equivariant isomorphism from $H^+(Y) \otimes F$ to $H^-(Y) \otimes F$, then the pair (Y, ψ^{-1}) constitutes, in the terminology of [5, §1.2], a 'trivialized perfect complex (of $\mathfrak{A}$-modules)'. Choose an $\mathfrak{A}$-equivariant quasi-isomorphism $\xi : P^\bullet \to Y$ with $P^\bullet$ an object of $\mathrm{PMod}(\mathfrak{A})^\bullet$, and write ψ_ξ for the composite isomorphism $H^-(\xi \otimes F)^{-1} \circ \psi \circ H^+(\xi \otimes F)$: $H^+(P^\bullet) \otimes F \xrightarrow{\sim} H^-(P^\bullet) \otimes F$. Then under the isomorphism $\pi_0\mathbb{V}(\mathfrak{A}, F) \xrightarrow{\sim} K_0(\mathfrak{A}, F)$ of Proposition 2.5, the image of the class of $\langle P^\bullet, \psi_\xi\rangle$ in $\pi_0(\mathbb{V}(\mathfrak{A}, F))$ is equal to the inverse of the 'refined Euler characteristic' class $\chi_{\mathfrak{A}}(Y, \psi^{-1})$ which is defined in [loc. cit., Th. 1.2.1].

In the remainder of this section we recall some useful facts concerning the groups $K_0(\mathfrak{A}, F)$.

If F is a field of characteristic 0, then one has a commutative diagram of long exact relative K-theory sequences (cf. [44, Th. 15.5])

$$\begin{array}{ccccccccc} K_1(\mathfrak{A}) & \longrightarrow & K_1(A_F) & \longrightarrow & K_0(\mathfrak{A}, F) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(A_F) \\ \| & & \uparrow{\scriptstyle\beta} & & \uparrow & & \| & & \uparrow \\ K_1(\mathfrak{A}) & \longrightarrow & K_1(A) & \longrightarrow & K_0(\mathfrak{A}, \mathbb{Q}) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(A). \end{array} \tag{11}$$

The scalar extension morphism β is injective and so, as a consequence of the Five Lemma, this diagram induces an inclusion

$$K_0(\mathfrak{A}, \mathbb{Q}) \subseteq K_0(\mathfrak{A}, F). \tag{12}$$

Furthermore, the map

$$(X, g, Y) \mapsto \prod_p (X_p, g_p, Y_p),$$

where $X_p := X \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $Y_p := Y \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $g_p := g \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for each prime p, induces an isomorphism

$$K_0(\mathfrak{A}, \mathbb{Q}) \xrightarrow{\sim} \bigoplus_p K_0(\mathfrak{A}_p, \mathbb{Q}_p) \tag{13}$$

where the sum is taken over all primes p (see the discussion following [15, (49.12)]).

2.9. The locally free class group. For any field F of characteristic 0 we define

$$\mathrm{Cl}(\mathfrak{A}, F) := \ker\big(K_0(\mathfrak{A}, F) \to K_0(\mathfrak{A}) \to \prod_p K_0(\mathfrak{A}_p)\big)$$

and

$$\mathrm{Cl}(\mathfrak{A}) := \ker\big(K_0(\mathfrak{A}) \to \prod_p K_0(\mathfrak{A}_p)\big).$$

The motivic invariants that we construct will belong to groups of the form $\mathrm{Cl}(\mathfrak{A}, \mathbb{R})$. The group $\mathrm{Cl}(\mathfrak{A})$ is the 'locally free class group' of $\mathfrak{A}$, as discussed in [15, §49].

We observe that the diagram (11) restricts to give a commutative diagram with exact rows

$$\begin{array}{ccccccc} K_1(A_F) & \xrightarrow{\delta^1_{\mathfrak{A},F}} & \mathrm{Cl}(\mathfrak{A}, F) & \xrightarrow{\delta^0_{\mathfrak{A},F}} & \mathrm{Cl}(\mathfrak{A}) & \longrightarrow & 0 \\ \uparrow{\scriptstyle \beta} & & \uparrow & & \| & & \\ K_1(A) & \xrightarrow{\delta^1_{\mathfrak{A},\mathbb{Q}}} & \mathrm{Cl}(\mathfrak{A}, \mathbb{Q}) & \xrightarrow{\delta^0_{\mathfrak{A},\mathbb{Q}}} & \mathrm{Cl}(\mathfrak{A}) & \longrightarrow & 0, \end{array} \tag{14}$$

and hence that (12) restricts to give an inclusion $\mathrm{Cl}(\mathfrak{A}, \mathbb{Q}) \subseteq \mathrm{Cl}(\mathfrak{A}, F)$. In addition, the restriction of the isomorphism (13) to $\mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$ combines with the exact sequence (10) to induce an isomorphism

$$\mathrm{Cl}(\mathfrak{A}, \mathbb{Q}) \xrightarrow{\sim} \bigoplus_p K_1(A_p)/\mathrm{im}(K_1(\mathfrak{A}_p)). \tag{15}$$

In many cases of interest the maps

$$K_1(A_p)/\mathrm{im}(K_1(\mathfrak{A}_p)) \to K_0(\mathfrak{A}_p, \mathbb{Q}_p)$$

are bijective for all p, and hence one has

$$\mathrm{Cl}(\mathfrak{A}, F) = K_0(\mathfrak{A}, F).$$

For example, this is the case if $\mathfrak{A}$ is commutative, if $\mathfrak{A} = \mathbb{Z}[G]$ where G is any finite group [15, Rem. (49.11)(iv)] or if $\mathfrak{A}$ is a maximal order in A [loc. cit., Th. 49.32].

3. Motives

3.1. Motivic structures. We fix a number field K and denote by S_∞ the set of archimedean places of K. For each $\sigma \in \mathrm{Hom}(K, \mathbb{C})$ we write $v(\sigma)$ for the corresponding element of S_∞. We also fix an algebraic closure $\bar{K}$ of K and let G_K denote the Galois group $\mathrm{Gal}(\bar{K}/K)$.

The category of (pure Chow) motives over K is a $\mathbb{Q}$-linear category with a functor to the category of realisations [26] and on which motivic cohomology functors are well defined. As is common in the literature on L-functions we shall treat motives in a formal sense: they are to be regarded as given by their realisations, motivic cohomology and the usual maps between these groups (that is, by a motivic structure in the sense of [20]). For example, if X is a smooth, projective variety over K, n a non-negative integer and r any integer, then $M := h^n(X)(r)$ is not in general known to exist as a Chow motive. However, the realisations of M are

$H_{dR}(M) := H^n_{dR}(X/K)$,

a filtered K-space, with its natural decreasing filtration $\left\{F^i H^n_{dR}(X/K)\right\}_{i\in\mathbb{Z}}$ shifted by r;

$H_l(M) := H^n_{et}(X \times_K \overline{K}, \mathbb{Q}_l(r))$,

a compatible system of l-adic representations of G_K;

$H_\sigma(M) := H^n(\sigma X(\mathbb{C}), (2\pi i)^r\mathbb{Q})$,

for each $\sigma \in \mathrm{Hom}(K, \mathbb{C})$ a $\mathbb{Q}$-Hodge structure over $\mathbb{R}$ or $\mathbb{C}$ according to whether $v(\sigma)$ is real or complex. If c denotes complex conjugation, then there is an obvious isomorphism of manifolds $\sigma X(\mathbb{C}) \xrightarrow{\sim} (c \circ \sigma)X(\mathbb{C})$ which we use to identify $H_\sigma(M)$ with $H_{c\circ\sigma}(M)$ if $v = v(\sigma)$ is complex. We then denote either of the two Hodge structures by $H_v(M)$, and we shall subsequently only make constructions which are independent of this choice.

One possible definition of the motivic cohomology of $M = h^n(X)(r)$ is

$$H^0(K, M) := \begin{cases} (CH^r(X)/CH^r(X)_{\mathrm{hom}\sim 0}) \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } n = 2r \\ 0, & \text{if } n \neq 2r \end{cases}$$

and

$$H^1(K, M) := \begin{cases} (K_{2r-n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(r)}, & \text{if } 2r - n - 1 \neq 0 \\ CH^r(X)_{\mathrm{hom}\sim 0} \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{if } 2r - n - 1 = 0. \end{cases}$$

Here $(K_{2r-n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(r)}$ is the eigenspace for the k-th Adams operator with eigenvalue k^r. One also defines a subspace

$$H^1_f(K, M) \subseteq H^1(K, M)$$

consisting of classes which are called 'finite' (or 'integral') at all non-archimedean places of K and puts $H^0_f(K, M) := H^0(K, M)$. In the K-theoretical version $H^1_f(K, M)$ is defined just as $H^1(K, M)$ but with $(K_{2r-n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(r)}$ replaced by

$$\mathrm{im}((K_{2r-n-1}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q})^{(r)} \to (K_{2r-n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(r)})$$

where $\mathcal{X}$ is a regular proper model of X over $\mathrm{Spec}\,(\mathcal{O}_K)$ (see [41] for the definition if such a model does not exist). The spaces $H^i_f(K, M)$ are expected to be finite dimensional, but this is not yet known to be true in general.

Let A be a finite dimensional semisimple $\mathbb{Q}$-algebra. From now on we shall be interested in motives with coefficients in A, i.e. in pairs (M, ϕ) where $\phi : A \to \mathrm{End}(M)$ is a ring homomorphism. It suffices here to understand $\mathrm{End}(M)$ as endomorphisms of motivic structures. However, in all of the explicit examples considered in [11] A is in fact an algebra of correspondences, i.e. consists of endomorphisms in the category of Chow motives. If M has coefficients in A, then the dual motive M^* has coefficients in A^{op}.

3.2. Basic exact sequences. In this section we recall relevant material from [4] and [20].
We write K_v for the completion of K at a place v, and we fix an algebraic closure $\bar{K}_v$ of K_v and an embedding of $\bar{K}$ into $\bar{K}_v$. We denote by $G_v \subseteq G_K$ the corresponding decomposition group and, if v is non-archimedean, by $I_v \subset G_v$ and $f_v \in G_v/I_v$ the inertia subgroup and Frobenius automorphism respectively. For $v \in S_\infty$ and an $\mathbb{R}$-Hodge structure H over K_v (what we call) the Deligne cohomology of H is by definition the cohomology of the complex

$$R\Gamma_{\mathcal{D}}(K_v, H) := \left(H^{G_v} \xrightarrow{\alpha_v} (H \otimes_{\mathbb{R}} \bar{K}_v)^{G_v}/F^0\right),$$

where here G_v acts diagonally on $H \otimes_{\mathbb{R}} \bar{K}_v$, and α_v is induced from the obvious inclusion $H \hookrightarrow H \otimes_{\mathbb{R}} \bar{K}_v$. Now if $H = H_v(M) \otimes_{\mathbb{Q}} \mathbb{R}$, then there is a canonical comparison isomorphism

$$H \otimes_{\mathbb{R}} \bar{K}_v \cong H_{dR}(M) \otimes_{K,v} \bar{K}_v$$

which is G_v-equivariant (the right hand side having the obvious G_v action). It follows that the complex

$$R\Gamma_{\mathcal{D}}(K, M) := \bigoplus_{v \in S_\infty} R\Gamma_{\mathcal{D}}(K_v, H_v(M) \otimes_{\mathbb{Q}} \mathbb{R})$$

can also be written as

$$\bigoplus_{v \in S_\infty} (H_v(M) \otimes_{\mathbb{Q}} \mathbb{R})^{G_v} \xrightarrow{\alpha_M} \left(\bigoplus_{v \in S_\infty} H_{dR}(M) \otimes_K K_v/F^0\right) = \left(H_{dR}(M)/F^0\right) \otimes_{\mathbb{Q}} \mathbb{R}. \tag{16}$$

For an $\mathbb{R}$-vector space W we write W^* for the linear dual $\mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$. If W is an A-module, then we always regard W^* as an A^{op}-module in the natural way.

Conjecture 1. *(cf.* [20]*[Prop. III.3.2.5]): There exists a long exact sequence of finite-dimensional $A_{\mathbb{R}}$-spaces*

$$\begin{aligned} 0 \longrightarrow H^0(K, M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\epsilon} \ker(\alpha_M) &\xrightarrow{r_B^*} \left(H^1_f(K, M^*(1)) \otimes_{\mathbb{Q}} \mathbb{R}\right)^* \\ \xrightarrow{\delta} H^1_f(K, M) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{r_B} \mathrm{coker}(\alpha_M) &\xrightarrow{\epsilon^*} \left(H^0(K, M^*(1)) \otimes_{\mathbb{Q}} \mathbb{R}\right)^* \longrightarrow 0 \end{aligned} \tag{17}$$

where here ϵ is the cycle class map into singular cohomology; r_B is the Beilinson regulator map; and (if both $H^1_f(K, M)$ and $H^1_f(K, M^*(1))$ are non-zero so that M has weight -1, then) δ is a height pairing. Moreover, the $\mathbb{R}$-dual of (17) identifies with the corresponding sequence for $M^*(1)$ where the isomorphisms

$$\ker(\alpha_M)^* \cong \mathrm{coker}(\alpha_{M^*(1)}), \quad \mathrm{coker}(\alpha_M)^* \cong \ker(\alpha_{M^*(1)})$$

are constructed in Lemma 18 below.

For each prime number p we set $V_p := H_p(M)$. Following [8, (1.8)] we shall now construct for each place v a true triangle in $D^p(A_p)$

$$0 \to R\Gamma_f(K_v, V_p) \to R\Gamma(K_v, V_p) \to R\Gamma_{/f}(K_v, V_p) \to 0 \tag{18}$$

in which all of the terms will be defined as specific complexes, rather than only to within unique isomorphism in $D^p(A_p)$ as the notation would perhaps suggest (we have however chosen to keep the traditional notation for mnemonic purposes.)
For a profinite group Π and a continuous Π-module N we denote by $C^\bullet(\Pi, N)$ the standard complex of continuous cochains.
If $v \in S_\infty$, then we set

$$R\Gamma_f(K_v, V_p) := R\Gamma(K_v, V_p) := C^\bullet(G_v, V_p).$$

We also define $R\Gamma_{/f}(K_v, V_p) := 0$ and we take (18) to be the obvious true triangle (with second arrow equal to the identity map).
If $v \notin S_\infty$ and $v \nmid p$, then we set

$$\begin{aligned} R\Gamma(K_v, V_p) &:= C^\bullet(G_v, V_p), \\ R\Gamma_f(K_v, V_p) &:= C^\bullet(G_v/I_v, V_p^{I_v}) \subseteq C^\bullet(G_v, V_p). \end{aligned}$$

We define $R\Gamma_{/f}(K_v, V_p)$ to be the complex which in each degree i is equal to the quotient of $C^i(G_v, V_p)$ by $C^i(G_v/I_v, V_p^{I_v})$ (with the induced differential), and we take (18) to be the tautological true triangle. We observe that there is a canonical quasi-isomorphism

$$\begin{aligned} R\Gamma_f(K_v, V_p) &\xrightarrow{\pi} \big(V_p^{I_v} \xrightarrow{1-f_v^{-1}} V_p^{I_v}\big) \\ &=: \big(V_{p,v} \xrightarrow{\phi_v} V_{p,v}\big) \end{aligned} \tag{19}$$

where in the latter two complexes the spaces are placed in degrees 0 and 1, and π is equal to the identity map in degree 0 and is induced by evaluating a 1-cocycle at f_v^{-1} in degree 1.
If now $v \mid p$, then by [4, Prop. 1.17] one has an exact sequence of continuous G_v-modules

$$0 \to \mathbb{Q}_p \to B^0 \xrightarrow{\beta_1 - \beta_2} B^1 \to 0 \tag{20}$$

where here $B^0 := B_{cris} \times B_{dR}^+$ and $B^1 := B_{cris} \times B_{dR}$ are certain canonical algebras and $\beta_1(x, y) = (x, x)$ and $\beta_2(x, y) = (\phi(x), y)$ are algebra homomorphisms. We write $B^\bullet$ for the complex $B^0 \xrightarrow{\beta_1 - \beta_2} B^1$ where the modules are placed in degrees 0 and 1, and we set

$$\begin{aligned} R\Gamma(K_v, V_p) &:= \mathrm{Tot}\, C^\bullet(G_v, B^\bullet \otimes_{\mathbb{Q}_p} V_p), \\ R\Gamma_f(K_v, V_p) &:= H^0(K_v, B^\bullet \otimes_{\mathbb{Q}_p} V_p). \end{aligned}$$

We observe that, since $V_p \to B^\bullet \otimes_{\mathbb{Q}_p} V_p$ is a resolution of V_p, the natural map

$$C^\bullet(G_v, V_p) \to R\Gamma(K_v, V_p) \tag{21}$$

is a quasi-isomorphism. Also, since $R\Gamma_f(K_v, V_p)$ is a subcomplex of $R\Gamma(K_v, V_p)$ we can define $R\Gamma_{/f}(K_v, V_p)$ to be the complex which in each degree i is equal to the quotient of the i-th term of $R\Gamma(K_v, V_p)$ by the i-th term of $R\Gamma_f(K_v, V_p)$ (with the induced differential). With these definitions we take (18) to be the tautological true triangle. Further, using the notation

$$\begin{aligned} D_{cris}(V_p) &:= H^0(K_v, B_{cris} \otimes_{\mathbb{Q}_p} V_p), \\ D_{dR}(V_p) &:= H^0(K_v, B_{dR} \otimes_{\mathbb{Q}_p} V_p), \\ F^0 D_{dR}(V_p) &:= H^0(K_v, B_{dR}^+ \otimes_{\mathbb{Q}_p} V_p), \\ t_v(V_p) &:= D_{dR}(V_p)/F^0 D_{dR}(V_p) \end{aligned}$$

there is a commutative diagram of complexes

$$\begin{array}{ccc} 0 & \longrightarrow & t_v(V_p) \\ \uparrow & & \uparrow \\ F^0 D_{dR}(V_p) & \xrightarrow{\subseteq} & D_{dR}(V_p) \\ {\scriptstyle (0,-\mathrm{id})}\downarrow & & \downarrow{\scriptstyle (0,\mathrm{id})} \\ D_{cris}(V_p) \oplus F^0 D_{dR}(V_p) & \xrightarrow{d} & D_{cris}(V_p) \oplus D_{dR}(V_p) \end{array}$$

where d is induced by $(\beta_1 - \beta_2) \otimes \mathrm{id}_{V_p}$ so that the lower row is $R\Gamma_f(K_v, V_p)$, and the vertical maps are both quasi-isomorphisms. With $t_v^\bullet(V_p)$ denoting the central complex in the above diagram we obtain a canonical quasi-isomorphism

$$t_v^\bullet(V_p) \xrightarrow{\sim} t_v(V_p)[-1]$$

and a canonical true triangle

$$t_v^\bullet(V_p) \to R\Gamma_f(K_v, V_p) \to \left(V_{p,v} \xrightarrow{\phi_v} V_{p,v}\right) \tag{22}$$

where here $\left(V_{p,v} \xrightarrow{\phi_v} V_{p,v}\right) := \left(D_{cris}(V_p) \xrightarrow{1-\varphi_v} D_{cris}(V_p)\right)$ and the spaces are placed in degrees 0 and 1. In addition, from Faltings' fundamental comparison theorem between V_p and $H_{dR}(M)$ over K_v there exists a canonical A_p-equivariant isomorphism

$$(H_{dR}(M)/F^0) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \bigoplus_{v|p} t_v(V_p). \tag{23}$$

Let now V be any finitely generated (projective) A_p-module. If $\phi \in \mathrm{End}_{A_p}(V)$ and C denotes the (perfect) complex $V \xrightarrow{\phi} V$ (with the modules placed in degrees 0 and 1), then there is an isomorphism in $V(A_p)$

$$[C] = [V] \boxtimes [V]^{-1} \cong \mathbf{1}_{V(A_p)} \tag{24}$$

which corresponds to the canonical isomorphism $X \boxtimes X^{-1} \cong \mathbf{1}$ for any object X in a Picard category. Note however that if ϕ is an automorphism and C is therefore acyclic, then the isomorphism (24) differs from the isomorphism $[C] \cong \mathbf{1}_{V(A_p)}$ induced by the quasi-isomorphism $C \to 0$ (cf. also [8, Rem.

after (1.16)] in this regard). In the notation which was introduced in Remark 4 after Proposition 2.5 this latter isomorphism is denoted by ϕ_{Triv} whereas the isomorphism (24) is denoted by $\mathrm{id}_{V,\mathrm{Triv}}$.

We now fix a finite set S of places of K containing S_∞ and the places where M has bad reduction. We denote by S_p the union of S and the set of places of K above p, and we set $S_{p,f} := S_p \setminus S_\infty$. We denote by $\mathcal{O}_{K,S_p}$ the ring of S_p-integers in K and by G_{S_p} its étale fundamental group with respect to the previously chosen base point $\bar{K}$. For any continuous G_{S_p}-module N we set

$$R\Gamma(\mathcal{O}_{K,S_p},N) := C^\bullet(G_{S_p},N),$$

$$R\Gamma_c(\mathcal{O}_{K,S_p},N) := \mathrm{Cone}\left(R\Gamma(\mathcal{O}_{K,S_p},N) \to \bigoplus_{v\in S_p} C^\bullet(G_v,N)\right)[-1]$$

where the morphism here is induced by the natural maps $G_v \subseteq G_K \to G_{S_p}$. For $N = V_p$ we set

$${}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) := \mathrm{Cone}\left(R\Gamma(\mathcal{O}_{K,S_p},V_p) \to \bigoplus_{v\in S_p} R\Gamma(K_v,V_p)\right)[-1],$$

$$R\Gamma_f(K,V_p) := \mathrm{Cone}\left(R\Gamma(\mathcal{O}_{K,S_p},V_p) \to \bigoplus_{v\in S_p} R\Gamma_{/f}(K_v,V_p)\right)[-1]$$

where in both cases we have used the morphism (21) for each place $v \mid p$. Then there is a natural quasi-isomorphism

$$R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \xrightarrow{\sim} {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p)$$

and the maps

$$R\Gamma(\mathcal{O}_{K,S_p},V_p) \to \bigoplus_{v\in S_p} R\Gamma(K_v,V_p) \hookleftarrow \bigoplus_{v\in S_p} R\Gamma_f(K_v,V_p) \tag{25}$$

induce a true nine term diagram

(26)

$$\begin{tikzcd}
\bigoplus_{v\in S_p} R\Gamma_f(K_v,V_p)[-1] \arrow[r, equal] \arrow[d] & \bigoplus_{v\in S_p} R\Gamma_f(K_v,V_p)[-1] \arrow[d] & \\
\bigoplus_{v\in S_p} R\Gamma(K_v,V_p)[-1] \arrow[r] \arrow[d] & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \arrow[r] \arrow[d] & R\Gamma(\mathcal{O}_{K,S_p},V_p) \arrow[d, equal] \\
\bigoplus_{v\in S_p} R\Gamma_{/f}(K_v,V_p)[-1] \arrow[r] & R\Gamma_f(K,V_p) \arrow[r] & R\Gamma(\mathcal{O}_{K,S_p},V_p).
\end{tikzcd}$$

Note that in what follows we will systematically use the lower left numbering to distinguish between different but naturally quasi-isomorphic versions of a complex. When there is no danger of confusion we shall simply drop this numbering and leave implicit the resulting identifications.
The complex $R\Gamma_f(K, V_p)$ is acyclic outside degrees $0, 1, 2$ and 3 and in Lemma 19 below we will define a natural isomorphism in $D^p(A_p)$

$$\mathrm{AV}_f : R\Gamma_f(K, V_p) \cong R\Gamma_f(K, V_p^*(1))^*[-3].$$

Conjecturally therefore, the cohomology of $R\Gamma_f(K, V_p)$ is completely described by applying the following to both M and $M^*(1)$.

Conjecture 2. *For both $i \in \{0, 1\}$ there exists a canonical A_p-equivariant isomorphism*

$$c_p^i(M) : H_f^i(K, M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} H^i R\Gamma_f(K, V_p). \tag{27}$$

In addition, we recall that for each $v \in S_\infty$ the comparison isomorphism between $H_v(M)$ and V_p induces an isomorphism in $D^p(A_p)$

$$R\Gamma_f(K_v, V_p) = R\Gamma(K_v, V_p) \cong V_p^{G_v}[0] \cong (H_v(M)^{G_v} \otimes_{\mathbb{Q}} \mathbb{Q}_p)[0]. \tag{28}$$

3.3. Projective $\mathfrak{A}$-structures. Let M be a motive which is defined over K and admits an action of the finite dimensional semisimple $\mathbb{Q}$-algebra A. If $\mathfrak{A}$ is an R-order in A (cf. §2.7) and V is an A-module, then an $\mathfrak{A}$-submodule T of V will be said to be an '$\mathfrak{A}$-lattice (in V)' if it is both finitely generated and full (i.e., satisfies $V = A \otimes_{\mathfrak{A}} T$).

Definition 1. *Let $\mathfrak{A}$ be an R-order in A. An* $\mathfrak{A}$-structure *T on M is a set $\{T_v : v \in S_\infty\}$ where, for each $v \in S_\infty$, T_v is an $\mathfrak{A}$-lattice in $H_v(M)$ and for each prime $l \in \mathrm{Spec}(R)$ the image T_l of $T_v \otimes_{\mathbb{Z}} \mathbb{Z}_l$ under the comparison isomorphism $H_v(M) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong H_l(M)$ is both independent of v and G_K-stable. An $\mathfrak{A}$-structure T on M is* projective*, resp.* free*, if each T_v is a projective, resp. free, $\mathfrak{A}$-module.*

If M is a motive with A-action, then there always exist $\mathfrak{A}$-structures on M. For example, if $M = h^n(X)$ for a smooth projective variety X defined over K, then there is an $\mathfrak{A}$-structure $h^n(X, \mathfrak{A})$ on M such that, for each $v \in S_\infty$, $h^n(X, \mathfrak{A})_v$ is the $\mathfrak{A}$-lattice in $H_v(M)$ which is generated by the image of $H^n(\sigma X(\mathbb{C}), \mathbb{Z})$ for an embedding $\sigma : K \to \mathbb{C}$ which corresponds to v. However, there need not exist projective $\mathfrak{A}$-structures on M. Indeed, even if there are full projective $\mathfrak{A}$-modules T_v in each space $H_v(M)$, it can occur that none of the corresponding modules T_l is G_K-stable (this is the case if, for example, $M = h^1(E)$ for an elliptic curve E defined over an imaginary quadratic field A for which $\mathrm{End}_A(E)$ is the maximal order in A and $\mathfrak{A}$ is any proper suborder of $\mathrm{End}_A(E)$). Nevertheless, the following examples show that a projective $\mathfrak{A}$-structure on M naturally exists in a variety of interesting cases.

Examples.

a) If $\mathfrak{A}$ is an hereditary R-order, and hence *a fortiori* if it is a maximal R-order, then there always exists a projective $\mathfrak{A}$-structure on M (cf. [15, Th. (26.12)]).

b) ('The Galois case') Let L/K be a finite Galois extension, and set $G := \mathrm{Gal}(L/K)$. If M_K is any motive which is defined over K, then the motive $M := h^0(\mathrm{Spec}(L)) \otimes M_K$ has a natural action of the semisimple algebra $\mathbb{Q}[G]$ via the first factor. Furthermore, if T_K is any $\mathbb{Z}$-structure on M_K, then $H^0(\mathrm{Spec}(L \otimes_{K,\sigma} \mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} T_K$ is a free $\mathbb{Z}[G]$-structure on M. Recall here that for any embedding $\sigma : K \to \mathbb{C}$ the scheme $\mathrm{Spec}(L \otimes_{K,\sigma} \mathbb{C})$ naturally identifies with the G-set $\Sigma := \{\tau \in \mathrm{Hom}(L, \mathbb{C}) : \tau|_K = \sigma\}$ and hence that $H^0(\mathrm{Spec}(L \otimes_{K,\sigma} \mathbb{C}), \mathbb{Z}) = \mathrm{Maps}(\Sigma, \mathbb{Z})$ is a free $\mathbb{Z}[G]$-module of rank one.

c) (Cf. [42, §4, Rem. following Cor. 2]). If X is a simple abelian variety defined over K which admits complex multiplication over K by a CM-field A, then the motive $h^1(X)$ has a natural A-action. In addition, if the order $\mathfrak{A} = \mathrm{End}_K(X) \subseteq A$, consisting of those elements which preserve each lattice $H^1(\sigma X(\mathbb{C}), \mathbb{Z})$, is Gorenstein then each $h^1(X, \mathfrak{A})_v$ is a projective $\mathfrak{A}$-lattice by [15, (37.13)] and hence there exists a projective $\mathfrak{A}$-structure on M. We note in particular that if X is an elliptic curve, then $\mathfrak{A}$ is automatically Gorenstein as a consequence of [1, 6.3].

d) Continuing the previous example, we assume now that X is an elliptic curve defined over K and so that $\mathfrak{A} := \mathrm{End}_K(X)$ is an order in an imaginary quadratic field A. Then A is contained in K and one can consider the $[K : A]$-dimensional abelian variety Y over A which is defined as the Weil restriction of X from K to A. If moreover K/A is an abelian Galois extension and X is isogenous to all of its Galois conjugates over A then $T := \mathrm{End}_A(Y) \otimes \mathbb{Q}$ is an algebra of dimension $[K : A]$ over A and $\mathfrak{T} := \mathrm{End}_A(Y)$ is an order in T, nonmaximal at primes dividing $[K : A][\mathcal{O}_A : \mathfrak{A}]$. This is shown in [21][15.1.6] for maximal $\mathfrak{A}$ but the arguments there extend to general $\mathfrak{A}$. The description of $\mathrm{End}_A(Y)$ in [21][15.1.5] also shows that $H^1(Y(\mathbb{C}), \mathbb{Z})$ is a projective $\mathfrak{T}$-module, and hence that the motive $M = h^1(Y)$ over the base field A admits a projective $\mathfrak{T}$-structure.

e) Let N be a prime number and consider the modular curve $X = X_0(N)$ defined over $K = \mathbb{Q}$. The *Hecke algebra* A is a finite dimensional commutative semisimple $\mathbb{Q}$-algebra consisting of correspondences which act on X, and $H^1(X(\mathbb{C}), \mathbb{Q})$ is known to be a free rank two A-space. If $\mathfrak{A}$ denotes the integral Hecke algebra (i.e., the subring of A which is generated by the Hecke correspondences over $\mathbb{Z}$), then $\mathfrak{A}$ is an order in A and Mazur shows in [34, II, (14.2), (16.3), (15.1)] that $H^1(X(\mathbb{C}), \mathbb{Z})_{\mathfrak{m}}$ is a free module over $\mathfrak{A}_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of $\mathfrak{A}$ which do not contain 2. This implies that $H^1(X(\mathbb{C}),\mathbb{Z}[\frac{1}{2}])$ is a projective module over $\mathfrak{A}[\frac{1}{2}]$, and hence that there exists a projective $\mathfrak{A}[\frac{1}{2}]$-structure on M.

We shall henceforth assume that $\mathfrak{A}$ is an order (leaving to the reader the obvious modifications which are necessary for R-orders as discussed above) and that we are given a projective $\mathfrak{A}$-structure T on M. We assume that p is a prime number which satisfies neither of the following conditions:

(P1) The motive $\mathrm{Res}^K_{\mathbb{Q}} M$ has bad reduction at p.
(P2) $p-2 < \min\{i \geq 0 | F^{i+1}H_{dR}(M) = 0\} - \max\{i \leq 0 | F^i H_{dR}(M) = H_{dR}(M)\}$.

Then, as explained in [8, 1.5.1], one can use the theory of Fontaine-Laffaille to define complexes $R\Gamma_f(K_v, T_p)$ for each place v in exactly the same way as for V_p. The same arguments which lead to diagram (26) can then be used to derive an analogous diagram in $D(\mathfrak{A}_p)$ in which V_p is replaced by T_p, and which naturally identifies with (26) after tensoring with $\mathbb{Q}_p$.
For any finite set of primes $\mathcal{S}$ we write $\mathfrak{A}_{\mathcal{S}}$ for the localisation of $\mathfrak{A}$ at the multiplicative set generated by the primes in $\mathcal{S}$. For $i \in \{2,3\}$ we set $H^i_f(K,M) := H^{3-i}_f(K, M^*(1))^*$ and $c^i_p(M) := H^i(\mathrm{AV}_f)^{-1} \circ (c^{3-i}_p(M^*(1))^*)^{-1}$. We now introduce an additional hypothesis on the pair (M,A).

Coherence hypothesis: For each $i \in \{0,1,2,3\}$ there exists a finitely generated $\mathfrak{A}$-module $H^i_f(K,M;T)$ and an $\mathfrak{A}$-equivariant map $\tau^i : H^i_f(K,M;T) \to H^i_f(K,M)$ such that $\tau^i \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism. In addition, there exists a finite set $\mathcal{S}$ of primes containing all primes satisfying either (P1) or (P2) and such that for each $p \notin \mathcal{S}$ and $i \in \{0,1,2,3\}$ there is a commutative diagram of $\mathfrak{A}_p$-modules

$$\begin{array}{ccc} H^i_f(K,M;T) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \xrightarrow{\tau^i \otimes_{\mathbb{Z}} \mathbb{Q}_p} & H^i_f(K,M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \\ {\scriptstyle c^i_p(T)}\downarrow & & \downarrow{\scriptstyle c^i_p(M)} \\ H^i R\Gamma_f(K,T_p) & \xrightarrow{\otimes_{\mathbb{Z}_p} \mathbb{Q}_p} & H^i R\Gamma_f(K,V_p) \end{array}$$

in which $c^i_p(T)$ is an isomorphism.

Remark 5. This hypothesis is identical to an assumption on integral structures in motivic cohomology which is made in [8, §1.5], and is independent of the choices of both $\mathfrak{A}$ and T. As we shall see below, the hypothesis is not actually required in order to formulate conjectures on special values of motivic L-functions and is correspondingly not made in either of [4] or [20]. However, under the Coherence hypothesis one can define an invariant in $\mathrm{Cl}(\mathfrak{A})$ without reference to the L-function of M, and this ties in well with the approach of classical Galois module theory. Indeed, in concrete cases, the $\mathfrak{A}$-module structure of the groups $H^i_f(K,M;T)$ can be of considerable interest (see for example

[8, §1.6]), and this structure can be studied via the conjectures formulated in §4 below.

3.4. Virtual objects attached to motives. Let M be a motive which is defined over K and admits an action of a finite dimensional semisimple $\mathbb{Q}$-algebra A. In this section we fix an order $\mathfrak{A}$ in A and a projective $\mathfrak{A}$-structure T on M (always assuming that such a structure exists).
We shall henceforth use the following notational convention. When referring to the individual triangles in a true nine term diagram with equation number (n) we denote by $(n)_?$ with ? equal to 'top', 'bot', 'left', 'rght', 'hor' or 'vert' the top, bottom, left, right, central horizontal and central vertical triangle respectively.
We define an object $\Xi(M)$ of $V(A)$ by setting

$$\begin{aligned}\Xi(M) :=&[H^0_f(K,M)]\boxtimes[H^1_f(K,M)]^{-1}\boxtimes[H^1_f(K,M^*(1))^*]\\ &\boxtimes[H^0_f(K,M^*(1))^*]^{-1}\boxtimes\boxtimes_{v\in S_\infty}[H_v(M)^{G_v}]^{-1}\boxtimes[H_{dR}(M)/F^0].\end{aligned}\tag{29}$$

Note that this is the inverse of the space Ξ used in both [8] and [27] (because our choice of normalisation for the virtual object associated to a perfect complex is the inverse of that of [8, (0.2)]).
Applying the functor [ ] to the isomorphisms (27), (28), (23), the isomorphisms (24), (19) or the triangle (22) for all $v\in S_{p,f}$ and finally to the triangle $(26)_{\mathrm{vert}}$, we obtain for each prime p an isomorphism in $V(A_p)$

$$\vartheta_p(M,S): A_p\otimes_A\Xi(M)\xrightarrow{\sim}[R\Gamma_c(\mathcal{O}_{K,S_p},V_p)]\cong A_p\otimes_{\mathfrak{A}_p}[R\Gamma_c(\mathcal{O}_{K,S_p},T_p)]$$

which we shall also abbreviate as $\vartheta_p(M)$ or even ϑ_p if there is no danger of confusion. We note here that $R\Gamma_c(\mathcal{O}_{K,S_p},T_p)$ is a perfect complex of $\mathfrak{A}_p$-modules by [18, Th. 5.1], and hence we obtain an object

$$\Xi(M,T_p,S):=([R\Gamma_c(\mathcal{O}_{K,S_p},T_p)],\Xi(M),\vartheta_p)$$

of $V(\mathfrak{A}_p)\times_{V(A_p)}V(A)$.

Lemma 5. *For another choice T' of projective $\mathfrak{A}$-structure on M and another choice of the finite set of places S' the objects $\Xi(M,T_p,S)$ and $\Xi(M,T'_p,S')$ are isomorphic in $V(\mathfrak{A}_p)\times_{V(A_p)}V(A)$.*

Proof. By embedding S and S' into the union $S\cup S'$ we can assume that $S\subseteq S'$, and by induction we can then reduce to the case that $S'=S\cup\{w\}$ and $w\nmid p$. For any continuous G_{S_p}-module N one has a commutative diagram of complexes

$$\begin{array}{ccc}
C^\bullet(G_{S_p},N) & \longrightarrow & \bigoplus_{v\in S_p} C^\bullet(G_v,N) \\
\downarrow & & \downarrow \\
C^\bullet(G_{S'_p},N) & \xrightarrow{r} & C^\bullet(G_w,N)/C^\bullet(G_w/I_w,N)\oplus \bigoplus_{v\in S_p} C^\bullet(G_v,N) \\
\| & & \uparrow \\
C^\bullet(G_{S'_p},N) & \longrightarrow & \bigoplus_{v\in S'_p} C^\bullet(G_v,N)
\end{array}$$

which induces a quasi-isomorphism $R\Gamma_c(\mathcal{O}_{K,S_p},N)\xrightarrow{\sim}\mathrm{Cone}(r)[-1]$ and a true triangle

$$R\Gamma_f(K_w,N)[-1]\to R\Gamma_c(\mathcal{O}_{K,S'_p},N)\to \mathrm{Cone}(r)[-1] \tag{30}$$

where $R\Gamma_f(K_w,N)=C^\bullet(G_w/I_w,N)$ is naturally quasi-isomorphic to

$$N\xrightarrow{1-f_v^{-1}}N \tag{31}$$

(compare [36, Chap. II, Prop. 2.3d]). For $N=T_p$ the true triangle (30) lies in $D(\mathfrak{A}_p)$. In conjunction with isomorphisms of the form (24), it therefore induces an isomorphism $\iota:[R\Gamma_c(\mathcal{O}_{K,S_p},T_p)]\xrightarrow{\sim}[R\Gamma_c(\mathcal{O}_{K,S'_p},T_p)]$ in $V(\mathfrak{A}_p)$.
We have a natural map from diagram (25) to the diagram

(32)
$$R\Gamma(\mathcal{O}_{K,S'_p},V_p)\to\bigoplus_{v\in S_p}R\Gamma(K_v,V_p)\oplus R\Gamma_{/f}(K_w,V_p)\leftarrow\bigoplus_{v\in S_p}R\Gamma_f(K_v,V_p).$$

We now denote by T9(S), resp. T9, the diagram (26), resp. the true nine term diagram which is induced by (32). Then we obtain a map $\phi:\mathrm{T9}(S)\to\mathrm{T9}$ which restricts to give quasi-isomorphisms on all terms in the central column. In a similar way, there is a map $\psi:\mathrm{T9}(S')\to\mathrm{T9}$ which is moreover a termwise surjection. The kernel of ψ is naturally quasi-isomorphic to a sum of complexes (31) and hence naturally trivialized by isomorphisms of the form (24). Since the same trivializations are used in the construction of $\vartheta_p(M,S')$, we have $(A_p\otimes_{\mathfrak{A}_p}\iota)\circ\vartheta_p(M,S)=\vartheta_p(M,S')$. Hence the pair (ι,id) defines an isomorphism

$$(\iota,\mathrm{id}):([R\Gamma_c(\mathcal{O}_{K,S_p},T_p)],\Xi(M),\vartheta_p)\xrightarrow{\sim}([R\Gamma_c(\mathcal{O}_{K,S'_p},T_p)],\Xi(M),\vartheta'_p)$$

in the category $V(\mathfrak{A}_p)\times_{V(A_p)}V(A)$.
Replacing T_p by $p^nT_p\subseteq T_p\cap T'_p$ we can assume that $T_p\subseteq T'_p$. Then there is a true triangle of perfect complexes of $\mathfrak{A}_p$-modules

$$R\Gamma_c(\mathcal{O}_{K,S_p},T_p)\to R\Gamma_c(\mathcal{O}_{K,S_p},T'_p)\to R\Gamma_c(\mathcal{O}_{K,S_p},T'_p/T_p). \tag{33}$$

Since T'_p/T_p is finite $R\Gamma_c(\mathcal{O}_{K,S_p},T'_p/T_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is acyclic and hence there is a canonical isomorphism $\tau_{\mathbb{Q}}:[R\Gamma_c(\mathcal{O}_{K,S_p},T'_p/T_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p]\cong\mathbf{1}_{V(A_p)}$. By [18, Th. 5.1] the class of $(R\Gamma_c(\mathcal{O}_{K,S_p},T'_p/T_p),\mathbf{1}_{V(\mathfrak{A}_p)},\tau_{\mathbb{Q}})$ in $\pi_0(V(\mathfrak{A}_p,\mathbb{Q}_p))\cong$

$K_0(\mathfrak{A}_p, \mathbb{Q}_p)$ is 0. Upon unraveling the definition of $V(\mathfrak{A}_p, \mathbb{Q}_p)$ this means that $\tau_{\mathbb{Q}}$ is induced by an isomorphism $\tau : [R\Gamma_c(\mathcal{O}_{K,S_p}, T'_p/T_p)] \cong \mathbf{1}_{V(\mathfrak{A}_p)}$. Hence the isomorphism induced by the triangle (33)

$$\begin{aligned}\iota : [R\Gamma_c(\mathcal{O}_{K,S_p}, T'_p)] \cong & [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)] \boxtimes [R\Gamma_c(\mathcal{O}_{K,S_p}, T'_p/T_p)] \\ \xrightarrow{\mathrm{id}\boxtimes\tau} & [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)]\end{aligned}$$

is part of an isomorphism

$$(\iota, \mathrm{id}) : ([R\Gamma_c(\mathcal{O}_{K,S_p}, T'_p)], \Xi(M), \vartheta_p) \xrightarrow{\sim} ([R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], \Xi(M), \vartheta_p)$$

in the category $V(\mathfrak{A}_p) \times_{V(A_p)} V(A)$. □

By taking the product over all primes p we now obtain an object

$$\Xi(M, T, S)_{\mathbb{Z}} := (\prod_p [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], \Xi(M), \prod_p \vartheta_p)$$

of the fibre product category $\prod_p V(\mathfrak{A}_p) \times_{\prod_p V(A_p)} V(A)$.

Lemma 6. *If the Coherence hypothesis is satisfied, then* $\Xi(M, T, S)_{\mathbb{Z}}$ *is isomorphic to the image of an object of* $\mathbb{V}(\mathfrak{A})$ *under the functor of Lemma 4.*

Proof. Assume that $\mathcal{S}$ is a finite set of primes as in the Coherence hypothesis and also containing all primes p at which $\mathfrak{A}_p$ is not a maximal $\mathbb{Z}_p$-order in A_p. Then $\mathfrak{A}_{\mathcal{S}}$ is a (left) regular ring [15, Th. (26.12)], and so any finitely generated $\mathfrak{A}_{\mathcal{S}}$-module is of finite projective dimension. As in [8, (1.24)] there exists a full $\mathfrak{A}_{\mathcal{S}}$-sublattice D_{dR} of $H_{dR}(M)$ so that for $p \notin \mathcal{S}$ the isomorphism (23) is induced by an isomorphism

$$(D_{dR}/F^0) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \bigoplus_{v|p} D_{cr,v}(T_p)/F^0 \tag{34}$$

where here $D_{cr,v}(-)$ is an integral version of the functor $D_{cris}(-)$ for K_v [8, p. 82]. We define an object $\Xi_{\mathcal{S}}$ of $V(\mathfrak{A}_{\mathcal{S}})$ by setting

$$\begin{aligned}\Xi_{\mathcal{S}} := \Xi_{\mathcal{S}}(M, T, S) :=& [H^0_f(K, M; T)_{\mathcal{S}}] \boxtimes [H^1_f(K, M; T)_{\mathcal{S}}]^{-1} \boxtimes [H^2_f(K, M; T)_{\mathcal{S}}] \\ &\boxtimes [H^3_f(K, M; T)_{\mathcal{S}}]^{-1} \boxtimes \boxtimes_{v\in S_\infty} [(T_v)^{G_v}_{\mathcal{S}}]^{-1} \boxtimes [D_{dR}/F^0].\end{aligned}$$

We set $\hat{\mathfrak{A}}' := \prod_{p\notin\mathcal{S}} \mathfrak{A}_p$. Then the finite product decomposition $\hat{\mathfrak{A}} \cong \prod_{p\in\mathcal{S}} \mathfrak{A}_p \times \hat{\mathfrak{A}}'$ induces a decomposition $V(\hat{\mathfrak{A}}) \cong \prod_{p\in\mathcal{S}} V(\mathfrak{A}_p) \times V(\hat{\mathfrak{A}}')$, and via this we define an object

$$\Xi' := ((\prod_{p\in\mathcal{S}} [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], \hat{\mathfrak{A}}' \otimes_{\mathfrak{A}_{\mathcal{S}}} \Xi_{\mathcal{S}}), A \otimes_{\mathfrak{A}_{\mathcal{S}}} \Xi_{\mathcal{S}}, \prod_{p\in\mathcal{S}} \vartheta_p \times \mathrm{id}_{\Xi_{\mathcal{S}}})$$

of $\mathbb{V}(\mathfrak{A})$. Under the Coherence hypothesis, there exists a natural isomorphism $A \otimes_{\mathfrak{A}_{\mathcal{S}}} \Xi_{\mathcal{S}}(M, T, S) \xrightarrow{\sim} \Xi(M)$. The image of Ξ' under the functor of Lemma 4 is isomorphic to $\Xi(M, T, S)_{\mathbb{Z}}$ because for each $p \notin \mathcal{S}$ the isomorphism ϑ_p is induced by an isomorphism

$$\vartheta^T_p : \mathfrak{A}_p \otimes_{\mathfrak{A}_{\mathcal{S}}} \Xi_{\mathcal{S}}(M, T, S) \xrightarrow{\sim} [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)]$$

in $V(\mathfrak{A}_p)$ (see [8] for more details). This finishes the proof of Lemma 6. □

Lemma 6, Lemma 5 and Lemma 4 now combine to imply that

$$\Xi(M)_{\mathbb{Z}} := \Xi(M, T, S)_{\mathbb{Z}}$$

is an object of $\mathbb{V}(\mathfrak{A})$ which is independent to within isomorphism in $\mathbb{V}(\mathfrak{A})$ of the choices of both S and T. The conjectural exact sequence (17) combines with (16) to induce an isomorphism in $V(A_{\mathbb{R}})$

$$\vartheta_\infty : A_{\mathbb{R}} \otimes_A \Xi(M) \cong \mathbf{1}_{V(A_{\mathbb{R}})}.$$

Under the Coherence hypothesis, we therefore obtain an object

$$(\Xi(M)_{\mathbb{Z}}, \vartheta_\infty) := (\prod_p [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)], \Xi(M), \prod_p \vartheta_p; \vartheta_\infty)$$

of $\mathbb{V}(\mathfrak{A}, \mathbb{R})$. We let $R\Omega(M, \mathfrak{A})$ denote the class of this element in $\pi_0(\mathbb{V}(\mathfrak{A}, \mathbb{R})) \cong K_0(\mathfrak{A}, \mathbb{R})$.

Lemma 7. $R\Omega(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$.

Proof. We need to show that the class of $R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)$ in $K_0(\mathfrak{A}_p)$ vanishes, and this follows as an easy consequence of results in [18]. More precisely, if Γ denotes the image of G_S in $\mathrm{Aut}(T_p)$ and $\mathbb{Z}_p[[\Gamma]]$ the profinite group algebra of Γ, then [18, Prop. 5.1] shows that there exists a bounded complex $P_\bullet$ of finitely generated projective $\mathbb{Z}_p[[\Gamma]]$-modules and an isomorphism $R\Gamma_c(\mathcal{O}_{K,S_p}, N) \cong \mathrm{Hom}_{\mathbb{Z}_p[[\Gamma]]}(P_\bullet, N)$ in $D(\mathfrak{A}_p)$ for any continuous, profinite or discrete, $\mathfrak{A}_p[\Gamma]$-module N. If Σ is a set of representatives for the isomorphism classes of simple $\mathbb{Z}_p[[\Gamma]]$-modules and $P_I \to I$ a projective hull for each $I \in \Sigma$, then we have isomorphisms of $\mathbb{Z}_p[[\Gamma]]$-modules $P_i \cong \prod_{I\in\Sigma} P_I^{n_{I,i}}$ for some integers $n_{I,i}$ (note that Σ is finite since Γ contains a pro-p group of finite index). The $\mathfrak{A}_p$-module $N_I := \mathrm{Hom}_{\mathbb{Z}_p[[\Gamma]]}(P_I, N)$ is a direct summand of N, and hence is projective if N is projective.
We write $\mathrm{cl}_{\mathfrak{A}_p}(X)$, resp. $\mathrm{cl}_{\mathbb{Z}_p}(Y)$, for the class in $K_0(\mathfrak{A}_p)$ of any perfect complex of $\mathfrak{A}_p$-modules, resp. for the class in the Grothendieck group $K_0(\mathbb{Z}_p, \mathbb{Q}_p) \cong \mathbb{Z}$ of the category of finite $\mathbb{Z}_p$-modules of any bounded complex of finite $\mathbb{Z}_p$-modules Y. Then if either $\Lambda = \mathfrak{A}_p$, or if $\Lambda = \mathbb{Z}_p$ and N is finite, there is an identity

$$\mathrm{cl}_\Lambda(R\Gamma_c(\mathcal{O}_{K,S_p}, N)) = \sum_{I\in\Sigma}\left(\sum_{i\in\mathbb{Z}}(-1)^i n_{I,i}\right)\mathrm{cl}_\Lambda(N_I). \tag{35}$$

Assume now that $N \in \Sigma$. Then $N_I = 0$ for each $I \in \Sigma$ with $I \neq N$, and hence (35) with $\Lambda = \mathbb{Z}_p$ implies that

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(-1)^i n_{N,i} \cdot \mathrm{cl}_{\mathbb{Z}_p}(N) &= \mathrm{cl}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{K,S_p}, N)) \\ &= \sum_{i\in\mathbb{Z}}(-1)^i \, \mathrm{cl}_{\mathbb{Z}_p}(H^i_c(\mathcal{O}_{K,S_p}, N)) = 0,\end{aligned}$$

where the last equality follows from Tate's formula for the Euler characteristic of a finite G_S-module. Since $\mathrm{cl}_{\mathbb{Z}_p}(N) \neq 0$, it follows that $\sum_{i\in\mathbb{Z}}(-1)^i n_{I,i} = 0$ for all $I \in \Sigma$.
From (35) with $\Lambda = \mathfrak{A}_p$ and $N = T_p$ we now deduce that $\mathrm{cl}_{\mathfrak{A}_p}(R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)) = 0$, as required. □

3.5. Functorialities. Let $\rho : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism between orders $\mathfrak{A}$ and $\mathfrak{B}$ in finite-dimensional, semisimple $\mathbb{Q}$-algebras A and B respectively. We denote by $\rho_{\mathbb{Q}} : A \to B$ the induced homomorphism of algebras. For any field F of characteristic 0, the scalar extension functor $\mathfrak{B} \otimes_{\mathfrak{A}} -$ induces a natural homomorphism

$$\rho_* : K_0(\mathfrak{A}, F) \to K_0(\mathfrak{B}, F)$$

which sends the class of (X, g, Y) to that of $(\mathfrak{B} \otimes_{\mathfrak{A}} X, 1 \otimes g, \mathfrak{B} \otimes_{\mathfrak{A}} Y)$. If $\mathfrak{B}$ is a projective $\mathfrak{A}$-module via ρ, then there also exists a homomorphism in the reverse direction

$$\rho^* : K_0(\mathfrak{B}, F) \to K_0(\mathfrak{A}, F)$$

which is simply induced by restriction of scalars. If $\mathfrak{A}$ is commutative and $\mathfrak{B} = M_n(\mathfrak{A})$ is a matrix algebra over $\mathfrak{A}$, then we set

$$e := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{pmatrix} \in \mathfrak{B}. \tag{36}$$

In this case the exact functor $V \mapsto \mathrm{Im}(e) \subset V$ induces an equivalence of exact categories $\mu : \mathrm{PMod}(\mathfrak{B}) \to \mathrm{PMod}(\mathfrak{A})$ and hence also an isomorphism

$$\mu_* : K_0(\mathfrak{B}, F) \xrightarrow{\sim} K_0(\mathfrak{A}, F). \tag{37}$$

If M is a motive over K with A-action, then we define $B \otimes_A M$ to be the motive over K with B-action which occurs as the largest direct factor of $B \otimes_{\mathbb{Q}} M$ upon which the left action of A on M and the right action of A on B coincide. Here $B \otimes_{\mathbb{Q}} M$ is the direct sum of $[B : \mathbb{Q}]$ copies of M (see [16, 2.1]). With this definition one has

$$H(B \otimes_A M) \cong B \otimes_A H(M) \tag{38}$$

for $H(-)$ equal to any of the functors $H_v(-)$,$H_v(-)^{G_v}$, $H_{dR}(-)$, $F^n H_{dR}(-)$, $H_l(-)$, $H_l(-)^{I_v}$, $H^0_f(K, -)$ or $H^1_f(K, -)$. If now T is a projective $\mathfrak{A}$-structure in M (as defined in §3.3), then each $\mathfrak{B} \otimes_{\mathfrak{A}} T_v$ is a projective $\mathfrak{B}$-module and hence a lattice in $B \otimes_A H_v(M) \cong H_v(B \otimes_A M)$. It follows that if M admits a projective $\mathfrak{A}$-structure, then $B \otimes_A M$ admits a projective $\mathfrak{B}$-structure.
If M is a motive over K with B-action, then it can be regarded as a motive with A-action via $\rho_{\mathbb{Q}}$. Assuming that $\mathfrak{B}$ is a projective $\mathfrak{A}$-module via ρ, any projective $\mathfrak{B}$-lattice T_v in $H_v(M)$ is also a projective $\mathfrak{A}$-lattice (via ρ). Hence, if in this case M admits a projective $\mathfrak{B}$-structure, then it also admits a projective $\mathfrak{A}$-structure.

Suppose now that A is commutative, that $B = M_n(A)$ for a natural number n and that M is a motive over K with B-action. Since the category of motives is pseudo-abelian (i.e., contains images of idempotents) eM is a motive with A-action. Also, if T_v is a projective $M_n(\mathfrak{A})$-lattice in $H_v(M)$, then eT_v is a projective $\mathfrak{A}$-lattice in $eH_v(M) = H_v(eM)$. Hence, if M admits a projective $M_n(\mathfrak{A})$-structure, then eM admits a projective $\mathfrak{A}$-structure.

Theorem 3.1. *a) If M admits a projective $\mathfrak{A}$-structure, then $B \otimes_A M$ admits a projective $\mathfrak{B}$-structure and*

$$\rho_*(R\Omega(M, \mathfrak{A})) = R\Omega(B \otimes_A M, \mathfrak{B}).$$

b) If M admits a projective $\mathfrak{B}$-structure and $\mathfrak{B}$ is a projective $\mathfrak{A}$-module via ρ, then M admits a projective $\mathfrak{A}$-structure (via $\rho_{\mathbb{Q}}$) and

$$\rho^*(R\Omega(M, \mathfrak{B})) = R\Omega(M, \mathfrak{A}).$$

c) If $\mathfrak{A}$ is commutative and M admits a projective $M_n(\mathfrak{A})$-structure, then eM admits a projective $\mathfrak{A}$-structure and

$$\mu_*(R\Omega(M, M_n(\mathfrak{A}))) = R\Omega(eM, \mathfrak{A}).$$

Proof. In case a) the exact functor $B \otimes_A - : \mathrm{PMod}(A) \to \mathrm{PMod}(B)$ induces a monoidal functor $B \otimes_A - : V(A) \to V(B)$ and hence a natural isomorphism $[B \otimes_A -] \cong B \otimes_A [-]$. Together with (38) this yields an isomorphism of virtual B-modules

$$B \otimes_A \Xi(M) \cong \Xi(B \otimes_A M). \tag{39}$$

The map ϑ_p is induced by the A-equivariant isomorphisms and exact sequences (28), (27), (19), (22), (23), (24) for all $v \in S_{p,f}$ and $(26)_{\mathrm{vert}}$, all of which transform into the corresponding isomorphisms and exact sequences for $B \otimes_A M$ when tensored over A with B: this follows from the canonical isomorphisms (38) together with 'projection formula' isomorphisms of the type

$$B_p \otimes_{A_p} R\Gamma_?(X, H_p(M)) \cong R\Gamma_?(X, B_p \otimes_{A_p} H_p(M)) \tag{40}$$

for $(?, X)$ equal to any of the pairs $(c, \mathcal{O}_{K,S_p}), (f, K)$ or $(\acute{e}t, K_v)$. Hence, if $\vartheta_p^{B \otimes_A M}$ denotes the isomorphism ϑ_p for the B-equivariant motive $B \otimes_A M$, then one has a commutative diagram

$$\begin{array}{ccc} B \otimes_A \Xi(M) & \xrightarrow{1 \otimes \vartheta_p} & B_p \otimes_{A_p} R\Gamma_c(\mathcal{O}_{K,S_p}, H_p(M)) \\ {\scriptstyle (39)}\downarrow & & {\scriptstyle (40)}\downarrow \\ \Xi(B \otimes_A M) & \xrightarrow{\vartheta_p^{B \otimes_A M}} & R\Gamma_c(\mathcal{O}_{K,S_p}, B_p \otimes_{A_p} H_p(M)). \end{array}$$

Moreover, the isomorphism (40) for the pair $(?, X) = (c, \mathcal{O}_{K,S_p})$ is induced by an isomorphism

$$\mathfrak{B}_p \otimes_{\mathfrak{A}_p} R\Gamma_c(\mathcal{O}_{K,S_p}, T_p) \cong R\Gamma_c(\mathcal{O}_{K,S_p}, \mathfrak{B}_p \otimes_{\mathfrak{A}_p} T_p)$$

where T_p is a projective $\mathfrak{A}_p$-lattice in $H_p(M)$ [18, Prop. 4.2]. We deduce that there exists an isomorphism in $\mathbb{V}(\mathfrak{B})$

$$\Xi(B \otimes_A M)_{\mathbb{Z}} \cong \mathfrak{B} \otimes_{\mathfrak{A}} \Xi(M)_{\mathbb{Z}}.$$

The map ϑ_∞ is induced by the A-equivariant exact sequence (17) which, as a consequence of (38), is transformed into the corresponding B-equivariant exact sequence for $B \otimes_A M$ when one applies $B \otimes_A -$. So the map ϑ_∞ for $B \otimes_A M$, which we denote by $\vartheta_\infty^{B\otimes_A M}$, is equal to the composite

$$\Xi(B \otimes_A M) \otimes_{\mathbb{Q}} \mathbb{R} \cong B_{\mathbb{R}} \otimes_{A_{\mathbb{R}}} (\Xi(M) \otimes_{\mathbb{Q}} \mathbb{R}) \xrightarrow{1\otimes\vartheta_\infty} B_{\mathbb{R}} \otimes_{A_{\mathbb{R}}} \mathbf{1}_{V(A_{\mathbb{R}})} \cong \mathbf{1}_{V(B_{\mathbb{R}})}.$$

Hence one has

$$\begin{aligned} \rho_*(R\Omega(M, \mathfrak{A})) = \rho_*((\Xi(M)_{\mathbb{Z}}, \vartheta_\infty)) &= (\mathfrak{B} \otimes_{\mathfrak{A}} \Xi(M)_{\mathbb{Z}}, 1 \otimes \vartheta_\infty) \\ &= (\Xi(B \otimes_A M)_{\mathbb{Z}}, \vartheta_\infty^{B\otimes_A M}) = R\Omega(B \otimes_A M, \mathfrak{B}). \end{aligned}$$

This proves a).

We now simply observe that the proofs of b) and c) follow along exactly the same lines with the role of the functor $B \otimes_A -$ being played by the exact functor $\mathrm{Res}_A^B : \mathrm{PMod}(B) \to \mathrm{PMod}(A)$ which is restriction of scalars in b) and restriction of scalars and passage to the direct summand cut out by the idempotent e in c). The analogues of the isomorphisms (38) and (40) for the functor Res_A^B are in both of these cases obvious. $\square$

4. L-functions

4.1. Equivariant L-factors and ϵ-factors. Let A be a finite-dimensional semisimple $\mathbb{Q}$-algebra and W a pseudo-abelian, $\mathbb{C}$-linear category. We define W_A to be the category of A-modules in W. Thus the objects of W_A are pairs $(V, A \to \mathrm{End}_W(V))$, and morphisms in W_A are morphisms in W which commute with the A-actions.

We fix a maximal set $i(A)$ of non-conjugate indecomposable idempotents in $A_{\mathbb{C}}$. More concretely, if

$$A_{\mathbb{C}} \cong \prod_{i=1}^{r} M_{n_i}(\mathbb{C}),$$

then we can take $i(A) = \{e_1, \cdots, e_r\}$ where e_i is the matrix (36) of size n_i in the i-th factor, and 0 in all others factors. The functors $V \mapsto (\mathrm{im}(e_i))_{1\le i\le r}$ and $(V_i)_{1\le i\le r} \mapsto \prod_{i=1}^r V_i \otimes_{\mathbb{C}} \mathbb{C}^{n_i}$ set up an equivalence of pseudo-abelian categories

(41) $$W_A \cong \prod_{i=1}^{r} W.$$

If C is a set and $\epsilon : \mathrm{Ob}(W) \to C$ is any map which is constant on isomorphism classes, then we get a well defined induced map

(42) $$\epsilon : \mathrm{Ob}(W_A) \to C^{i(A)} \qquad V \mapsto \epsilon(\mathrm{im}(e_i))$$

which does not depend on the choice of $i(A)$ and is constant on isomorphism classes.
We now suppose given a motive M over K with an action of A. For any non-archimedean, resp. archimedean, place v of K we let W_v be the category of complex representations of the Weil-Deligne group, resp. of the Weil group, of K_v [45]. In order to apply the preceding considerations to W_v we need the following

Conjecture 3. *(Compatibility) For any finite place v of K, any rational prime $l \nmid v$ and any embedding $\tau : \mathbb{Q}_l \to \mathbb{C}$ consider the object $H_l(M) \otimes_{\mathbb{Q}_l,\tau} \mathbb{C}$ of $W_{v,A}$. Then the isomorphism class of the Frobenius semisimplification* [45, (4.1.3)] *of $H_l(M) \otimes_{\mathbb{Q}_l,\tau} \mathbb{C}$ in $W_{v,A}$ is independent of the choices of l and τ.*

Remark 6. If A is a number field, then this reduces to the compatibility conjecture formulated in [45, (4.2.4)].

Let $\mathcal{K}(\mathbb{C})$ be the multiplicative group of meromorphic functions on $\mathbb{C}$. As in [45] one attaches to any $V \in \mathrm{Ob}(W_v)$ an L-factor $L_v(V,s) \in \mathcal{K}(\mathbb{C})$ and an ϵ-factor $\epsilon_v(V,s,\psi_v,dx_v) \in \mathcal{K}(\mathbb{C})$ (also depending upon a choice of Haar measure dx_v on K_v and of an additive character $\psi_v : K_v \to \mathbb{C}^\times$). Assuming Conjecture 3 we use (42) to associate to the pair (M,A) equivariant L-factors $L_v({}_AM,s)$ and equivariant ϵ-factors $\epsilon_v({}_AM,s,\psi_v,dx_v)$ in $\mathcal{K}(\mathbb{C})^{i(A)}$, and we view these as meromorphic functions with values in

$$\mathbb{C}^{i(A)} \cong \zeta(A_\mathbb{C}) \cong \zeta(A) \otimes_\mathbb{Q} \mathbb{C} \cong \prod_{\sigma \in \mathrm{Hom}(\zeta(A),\mathbb{C})} \mathbb{C}. \tag{43}$$

We then define

$$\epsilon({}_AM,s) := \prod_v \epsilon_v({}_AM,s,\psi_v,dx_v)$$
$$\Lambda({}_AM,s) := \prod_v L_v({}_AM,s)$$

where the products are taken over all places v of K and ψ_v, dx_v are chosen as in [45, (3.5)]. We also set

$$L_\infty({}_AM,s) := \prod_{v \in S_\infty} L_v({}_AM,s)$$

and for any finite set S of places of K

$$L_S({}_AM,s) := \prod_{v \notin S} L_v({}_AM,s).$$

We usually abbreviate $L_{S_\infty}({}_AM,s)$ to $L({}_AM,s)$. We observe that the product for $L({}_AM,s)$ converges in a half plane $\mathrm{Re}(s) >> 0$ and that in the product for $\epsilon({}_AM,s)$ almost all of the terms are equal to 1. If there is no danger of confusion we shall often suppress the dependence on A and so write $L(M,s)$ etc.

Remark 7. Following [16, Rem. 2.12] one can define the L-factors $L_v({}_AM, s)$ in a more direct way than the above, and this allows one to assume a slightly weaker compatibility than that of Conjecture 3. To be more precise, for each finite place v of K of residue characteristic p and any prime $l \neq p$, one considers the A_l-module $V_l := H_l(M)^{I_v}$ together with its action of the Frobenius automorphism $f_v \in \mathrm{End}_{A_l}(V_l)$. Under the assumption that

$$P_v(H_l({}_AM), X) := \mathrm{detred}_{A_l}\big(1 - f_v^{-1} \cdot X | V_l\big) \in \zeta(A_l)[X].$$

belongs to $\zeta(A)[X]$ and is independent of the choice of l, one can define $L_v({}_AM, s)$ to be equal to $P_v(H_l({}_AM), \mathrm{N}\, v^{-s}) \in \zeta(A_{\mathbb{C}})$ for each $s \in \mathbb{C}$. We observe that the above assumption on $P_v(H_l({}_AM), X)$ is a consequence of Conjecture 3, and conversely that it implies Conjecture 3 if $H_l(M)$ is unramified at v.

Lemma 8. *If s is real, then $L_v({}_AM, s)$, $\epsilon({}_AM, s)$ and $L_S({}_AM, s)$ all belong to $\zeta(A) \otimes_{\mathbb{Q}} \mathbb{R} \cong \zeta(A_{\mathbb{R}})$.*

Proof. For any $\alpha \in \zeta(A_{\mathbb{C}})$ we denote by α_σ its σ-component under the isomorphism (43). If c denotes complex conjugation the isomorphism (43) identifies $\zeta(A) \otimes_{\mathbb{Q}} \mathbb{R} = \zeta(A_{\mathbb{R}}) \subset \zeta(A_{\mathbb{C}})$ with the set $\{(\alpha_\sigma) | \alpha_{c\circ\sigma} = c(\alpha_\sigma)\}$. If v is non-archimedean and s is real, then $L_v({}_AM, s)$ belongs to this set because $P_v(H_l({}_AM), X)$ has coefficients in $\zeta(A)$. Since the action of c is continuous, the same is therefore true for $L_S({}_AM, s)$. If v is archimedean and s is real, then $L_v({}_AM, s)_\sigma \in \mathbb{R}$ since the Γ-function is real valued for real arguments. On the other hand, if $V \in \mathrm{Ob}(W_v)$ arises from a $\mathbb{R}$-Hodge structure then there is an isomorphism $V^c \cong V$ in W_v given by complex conjugation of the coefficients. Finally for any v, $V \in \mathrm{Ob}(W_v)$ and $s \in \mathbb{R}$ one has $\epsilon(V^c, s, \psi_v^c, dx_v) = \epsilon(V, s, \psi_v, dx_v)^c$ by [45, 3.6]. If $V = V_\sigma$ arises from M and $\sigma \in \mathrm{Hom}(\zeta(A), \mathbb{C})$ we have $V_\sigma^c = V_{c\circ\sigma}$ and

$$\epsilon({}_AM, s)_\sigma^c := \prod_v \epsilon(V_\sigma, s, \psi_v, dx_v)^c = \prod_v \epsilon(V_\sigma^c, s, \psi_v^c, dx_v) = \epsilon({}_AM, s)_{c\circ\sigma}$$

where this last equality follows because ψ_v^c and dx_v also satisfy the conditions of [45, (3.5)]. $\square$

4.2. The extended boundary homomorphism. Recall that the reduced norm homomorphism $\mathrm{nr}_{A_{\mathbb{R}}} : K_1(A_{\mathbb{R}}) \to \zeta(A_{\mathbb{R}})^\times$ is injective but not in general surjective (cf. Proposition 2.2). In this section we define a canonical homomorphism $\zeta(A_{\mathbb{R}})^\times \to \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$ which upon restriction to $\mathrm{im}(\mathrm{nr}_{A_{\mathbb{R}}})$ is equal to the composite $\delta^1_{\mathfrak{A},\mathbb{R}} \circ \mathrm{nr}_{A_{\mathbb{R}}}^{-1}$, where here $\delta^1_{\mathfrak{A},\mathbb{R}}$ is the homomorphism $K_1(A_{\mathbb{R}}) \to \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$ which occurs in diagram (14). This construction plays a key role in the formulation of conjectures in the next section.

Lemma 9. *There exists a canonical homomorphism*

$$\hat{\delta}^1_{\mathfrak{A},\mathbb{R}} : \zeta(A_{\mathbb{R}})^\times \to \mathrm{Cl}(\mathfrak{A}, \mathbb{R})$$

which satisfies $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(\mathrm{nr}_{A_\mathbb{R}}(x)) = \delta^1_{\mathfrak{A},\mathbb{R}}(x)$ *for each* $x \in K_1(A_\mathbb{R})$.

Proof. In conjunction with the equality (7), the Weak Approximation Theorem guarantees that for each $y \in \zeta(A_\mathbb{R})^\times$ there exists an element λ of $\zeta(A)^\times$ such that $\lambda y \in \mathrm{im}(\mathrm{nr}_{A_\mathbb{R}})$. For each prime p we also view λ as an element of $\zeta(A_p)^\times = \mathrm{im}(\mathrm{nr}_{A_p})$, and we then set

$$\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y) := \delta^1_{\mathfrak{A},\mathbb{R}}(\mathrm{nr}^{-1}_{A_\mathbb{R}}(\lambda y)) - \sum_p \delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}(\mathrm{nr}^{-1}_{A_p}(\lambda)) \in \mathrm{Cl}(\mathfrak{A},\mathbb{R}). \tag{44}$$

Here we view $K_1(A_p)/\mathrm{im}(K_1(\mathfrak{A}_p))$ as a subgroup of $\mathrm{Cl}(\mathfrak{A},\mathbb{R})$ via the isomorphism (15) and the inclusion (12). The sum is taken over all primes p but is finite since for almost all p both $\lambda \in \zeta(\mathfrak{A}_p)^\times$ and $\mathrm{nr}^{-1}_{A_p}(\zeta(\mathfrak{A}_p)^\times)$ is contained in the image of $K_1(\mathfrak{A}_p)$. If λ' is any other element of $\zeta(A)^\times$ such that $\lambda' y \in \mathrm{im}(\mathrm{nr}_{A_\mathbb{R}})$, then (7) implies that $\lambda/\lambda' \in \mathrm{im}(\mathrm{nr}_A)$. Hence, if $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y)'$ is the element (44) formed with respect to λ' rather than λ, then

$$\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y) - \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y)' = \delta^1_{\mathfrak{A},\mathbb{R}}(\mathrm{nr}^{-1}_{A_\mathbb{R}}(\lambda/\lambda')) - \sum_p \delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}(\mathrm{nr}^{-1}_{A_p}(\lambda/\lambda'))$$

and this difference is zero since both terms on the right hand side are equal to $\delta^1_{\mathfrak{A},\mathbb{Q}}(\mathrm{nr}^{-1}_A(\lambda/\lambda'))$. It now only remains to check that the assignment $y \mapsto \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y)$ is a homomorphism, and this is easy to verify directly. □

4.3. The main conjectures. We can now formulate the central conjecture of this paper. This conjecture is a generalisation to non-commutative coefficients of [8, Conj. 4] (which in turn generalized the central conjectures of [4, 20, 27]).

Conjecture 4. *Let* M *be a motive which carries an action of the finite-dimensional semisimple* $\mathbb{Q}$*-algebra* A*, and let* $\mathfrak{A}$ *be any order in* A *for which* M *admits a projective* $\mathfrak{A}$*-structure. Assume that* (M,A) *satisfies the Coherence hypothesis.*

(i) $L({}_AM,s)$ *can be analytically continued to* $s=0$.

(ii) *Regarding* $\mathrm{ord}_{s=0}L({}_AM,s)$ *as a locally constant function on* $\mathrm{Spec}(\zeta(A_\mathbb{C}))$ *one has*

$$\mathrm{ord}_{s=0}L({}_AM,s) \;=\; \mathrm{rr}_A(H^1_f(K,M^*(1))^*) - \mathrm{rr}_A(H^0_f(K,M^*(1))^*)$$

where the map rr_A *is as defined in §2.6.*

(iii) *(Rationality) Set*

$$L^*({}_AM,0) := \lim_{s\to 0} s^{-\mathrm{ord}_{s=0}L({}_AM,s)}L({}_AM,s) \in \zeta(A_\mathbb{R})^\times,$$
$$L(M,\mathfrak{A}) := \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(L^*({}_AM,0)) \in \mathrm{Cl}(\mathfrak{A},\mathbb{R})$$

and

$$T\Omega(M,\mathfrak{A}) := L(M,\mathfrak{A}) + R\Omega(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{R}).$$

Then $T\Omega(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$.

(iv) *(Integrality)* $T\Omega(M,\mathfrak{A})=0$.

Remark 8. It is possible to formulate an equivalent conjecture without assuming the Coherence hypothesis (which was required to define $R\Omega(M,\mathfrak{A})$). To do this, we note that the pair $(\Xi(M),\vartheta_\infty)$ represents an object of $V(A,\mathbb{R}) := V(A)\times_{V(A_\mathbb{R})}\mathcal{P}_0$, and we consider the Mayer-Vietoris sequence for this fibre product

$$\cdots\to K_1(A_\mathbb{R})\xrightarrow{\delta}\pi_0(V(A,\mathbb{R}))\to K_0(A)\to K_0(A_\mathbb{R})\to\cdots$$

Just as in the proof of Lemma 9, we let $\lambda\in\zeta(A)^\times$ be any element such that $\lambda L^*({}_AM,0)$ belongs to $\mathrm{im}(\mathrm{nr}_{A_\mathbb{R}})$. Then Conjecture 4(iii) is equivalent to

Conjecture 5. *In* $\pi_0(V(A,\mathbb{R}))$ *one has* $[\Xi(M),\vartheta_\infty]+\delta(\mathrm{nr}_{A_\mathbb{R}}^{-1}(\lambda L^*({}_AM,0)))=0$.

It is clear that this conjecture does not involve $\Xi(M)_\mathbb{Z}$. Further, as a consequence of the definition of δ and the definition of isomorphism in the category $V(A,\mathbb{R})$, Conjecture 5 implies the existence of an isomorphism in $V(A)$

$$\vartheta^{(\lambda)}:\Xi(M)\cong\mathbf{1}_{V(A)}$$

which maps to $-\,\mathrm{nr}_{A_\mathbb{R}}^{-1}(\lambda L^*({}_AM,0))\circ\vartheta_\infty$ in $V(A_\mathbb{R})$. Since the map $K_1(A)\cong\pi_1(V(A))\to\pi_1(V(A_\mathbb{R}))\cong K_1(A_\mathbb{R})$ is injective, the isomorphism $\vartheta^{(\lambda)}$ is unique. One can therefore define an object

$$\xi(M,\mathfrak{A}_p,\lambda):=([R\Gamma_c(\mathcal{O}_{K,S_p},T_p)],(\vartheta^{(\lambda)}\otimes\mathbb{Q}_p)\circ\vartheta_p^{-1})$$

of $V(\mathfrak{A}_p,\mathbb{Q}_p)$ and formulate the following

Conjecture 6. *Assuming Conjecture 5, the class*

$$T\Omega(M,\mathfrak{A}_p):=[\xi(M,\mathfrak{A}_p,\lambda)]-\delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}(\mathrm{nr}_{A_p}^{-1}(\lambda))$$

vanishes in $\pi_0(V(\mathfrak{A}_p,\mathbb{Q}_p))\cong K_0(\mathfrak{A}_p,\mathbb{Q}_p)$.

Under the Coherence hypothesis (and Conjecture 5) one can show that $T\Omega(M,\mathfrak{A}_p)$ is equal to the p-component of the element $T\Omega(M,\mathfrak{A})$ of $K_0(\mathfrak{A},\mathbb{Q})$ under the decomposition (13). This implies that, under the Coherence hypothesis, Conjecture 6 is valid for all but finitely many primes p, and that its validity for all p is equivalent to the validity of Conjecture 4(iv).

Remark 9. In this remark we assume that $\mathfrak{A}$ is commutative. Then Proposition 2.4 implies that the Picard category $\mathbb{V}(\mathfrak{A})$ is equivalent to the category $\mathcal{P}(\mathfrak{A})$ of graded invertible $\mathfrak{A}$-modules. Hence one can work with the graded determinant functor and the category $\mathcal{P}(\mathfrak{A})$ to formulate conjectures which are equivalent to those of Conjecture 4. This is the approach taken in [8], and also in [20] and [28], except that in each of these references ordinary rather than graded determinants are used.

We recall that, for any commutative ring R, an isomorphism in $\mathcal{P}(R)$ of the form

$$\mathrm{Det}_R(\bigoplus_{i\in I} P_i) \cong \bigotimes_{i\in I} \mathrm{Det}_R(P_i) \tag{45}$$

is well defined because one can define an isomorphism for any given ordering of I and the isomorphisms so obtained are compatible with reordering I in the same way on both sides. (This is a consequence of standard coherence theorems for symmetric monoidal categories [32]). However, if one ignores rank data, then (45) depends upon an ordering of I. As a consequence, for example, unless an ordering of the set S_p is specified the definition of the isomorphism ϑ_p in [8, §1.4] is ambiguous to within multiplication by an element η of $A_p^\times$ which corresponds to a locally constant map $\mathrm{Spec}(A_p) \to \{\pm 1\}$ (see also the remarks in [20, 0.4] or [28, Rem. 3.2.3(3) and 3.2.6] to this effect). It is clear that such ambiguity cannot be permitted in the formulation of Conjecture 4(iv) because in general $\eta \notin \mathfrak{A}_p^\times$. By working in $\mathcal{P}(A)$ the definition of ϑ_p in [8] becomes unambiguous and the same is true for all of the other determinant computations in loc. cit. All computations involving the determinant functor in both loc. cit. and [9], and also in the work [20, 28] of other authors, should therefore be understood to take place in categories of the form $\mathcal{P}(R)$.

Remark 10. We quickly review some of the current evidence for Conjecture 4. At the outset, we remark that any proven case of the central conjectures of [4, 20] provides evidence for Conjecture 4 for pairs of the form $(M, \mathbb{Z})$ (in this regard see also Remark 11 in §4.5). Moreover, in [11] it is shown that Conjecture 4 implies the central conjecture of Kato's paper [27] (in all cases to which the latter applies), and that in the context of Tate motives Conjecture 4(iv) refines a number of previously formulated (and much studied) conjectures. For example, if L/K is a finite Galois extension of number fields, then it is shown in [11] that Conjecture 4(iv) for $M = h^0(\mathrm{Spec}(L))$ and with $\mathfrak{A}$ equal to $\mathbb{Z}[\mathrm{Gal}(L/K)]$, resp. equal to any maximal order in $\mathbb{Q}[\mathrm{Gal}(L/K)]$ which contains $\mathbb{Z}[\mathrm{Gal}(L/K)]$, is a refinement of the main conjecture formulated by Chinburg in [13], resp. is equivalent to the so called 'Strong Stark Conjecture' (that is, [loc. cit., Conj. 2.2]). In this direction, the reader can also consult [6].
We now fix a Galois extension L of $\mathbb{Q}$ and set $G := \mathrm{Gal}(L/\mathbb{Q})$. The main result of [25] is equivalent to the validity of Conjecture 4(iv) for pairs $(h^0(\mathrm{Spec}\,(L))(r), \mathfrak{M}_{(2)})$ where here G is abelian, $\mathfrak{M}_{(2)}$ denotes the maximal $\mathbb{Z}[\frac{1}{2}]$-order in $\mathbb{Q}[G]$ and r is any integer. In addition, the main result of [12] implies that Conjecture 4(iv) is valid for all pairs $(h^0(\mathrm{Spec}\,(L))(r), \mathbb{Z}[\frac{1}{2}][G])$ with G abelian and r any integer less than 1 (in this regard see also Remark 19 in §5.3). Relaxing the condition that G is abelian, it is also known that Conjecture 4(iv) is valid for the pairs $(h^0(\mathrm{Spec}\,(L)), \mathbb{Z}[G])$ where L ranges over a natural (infinite) family of fields for which G is isomorphic to the quaternion group of order 8 [11].

The above examples can all be regarded as providing evidence for Conjecture 4 in the setting of Example b) in §3.3 ('The Galois case'). The equivariant Birch-Swinnerton Dyer conjecture for elliptic curves with CM by the maximal order $\mathcal{O}$ of the CM-field, as formulated by Gross in [22], is perhaps the earliest integral equivariant special value conjecture in a setting other than the Galois case. Moreover, the relative algebraic K-group $K_0(\mathcal{O},\mathbb{R})$ is introduced in [22] in an ad-hoc manner in order to formulate the conjecture (which can indeed be shown to be equivalent to Conjecture 4(iv) in all relevant cases). Some instances of Gross' conjecture have been proved by Rubin [40]. However, at present we are unaware of any examples in which Conjecture 4(iv) has been verified in a non-Galois case and with $\mathfrak{A}$ non-maximal.

4.4. Functorialities. In this section we shall discuss the behaviour of the element $L(M,\mathfrak{A})$, and hence (given Theorem 3.1) also of $T\Omega(M,\mathfrak{A})$, under the functorialities discussed in §3.5.

We let $\rho : \mathfrak{A} \to \mathfrak{B}$ be as in §3.5, and we use the notation ρ_* for any of the maps induced by the exact functor $\mathfrak{B} \otimes_{\mathfrak{A}} - : \mathrm{PMod}(\mathfrak{A}) \to \mathrm{PMod}(\mathfrak{B})$ or its scalar extensions on algebraic K-groups. These maps ρ_* combine to give a map of the localization sequence (11) into the corresponding sequence with $\mathfrak{A}$ replaced by $\mathfrak{B}$. The same holds for the maps ρ^* (resp. μ_*) induced by the functor $\mathrm{res}^{\mathfrak{B}}_{\mathfrak{A}} : \mathrm{PMod}(\mathfrak{B}) \to \mathrm{PMod}(\mathfrak{A})$ if $\mathfrak{B}$ is a projective $\mathfrak{A}$-module (resp. by the functor $\mu : \mathrm{PMod}(M_n(\mathfrak{A})) \to \mathrm{PMod}(\mathfrak{A})$ if $\mathfrak{A}$ is commutative).

Our first result describes the functorial properties of the extended boundary homomorphism.

Lemma 10. *There exists a homomorphism $\rho_* : \zeta(A_{\mathbb{R}})^\times \to \zeta(B_{\mathbb{R}})^\times$ which fits into a commutative diagram*

$$\begin{array}{ccccc} K_1(A_{\mathbb{R}}) & \xrightarrow{\mathrm{nr}_{A_{\mathbb{R}}}} & \zeta(A_{\mathbb{R}})^\times & \xrightarrow{\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}} & \mathrm{Cl}(\mathfrak{A},\mathbb{R}) \\ \downarrow{\scriptstyle\rho_*} & & \downarrow{\scriptstyle\rho_*} & & \downarrow{\scriptstyle\rho_*} \\ K_1(B_{\mathbb{R}}) & \xrightarrow{\mathrm{nr}_{B_{\mathbb{R}}}} & \zeta(B_{\mathbb{R}})^\times & \xrightarrow{\hat{\delta}^1_{\mathfrak{B},\mathbb{R}}} & \mathrm{Cl}(\mathfrak{B},\mathbb{R}). \end{array} \tag{46}$$

The analogous statements also hold for both ρ^* and μ_*.

Proof. For any field F of characteristic 0 the ring homomorphism $\rho_F : A_F \to B_F$ induces an exact functor $B_F \otimes_{A_F} - : \mathrm{PMod}(A_F) \to \mathrm{PMod}(B_F)$ and hence also a group homomorphism $\rho_{F,*} : K_1(A_F) \to K_1(B_F)$. Although the reduced norm map nr_{A_F} is not in general bijective it identifies $\zeta(A_F)^\times$ with the sheafification of the presheaf $F \mapsto K_1(A_F)$ for the étale topology on $\mathrm{Spec}(F)$. If $\bar{F}$ is an algebraic closure of F and $\Gamma = \mathrm{Gal}(\bar{F}/F)$, then we have

$$H^0(\Gamma, K_1(A_{\bar{F}})) \cong H^0(\Gamma, \zeta(A_{\bar{F}})^\times) \cong H^0(\Gamma, (\zeta(A_F) \otimes_F \bar{F})^\times) \cong \zeta(A_F)^\times,$$

and the map $\rho_{F,*}$ can be defined on $\zeta(A_F)^\times$ via this formula. By construction then, the left hand square in (46) commutes (even with $\mathbb{R}$ replaced by any field of characteristic 0) and we also have a commutative diagram

$$\begin{array}{ccc} \zeta(A_F)^\times & \xrightarrow{\rho_{F,*}} & \zeta(B_F)^\times \\ {\scriptstyle\subseteq}\big\downarrow & & \big\downarrow{\scriptstyle\subseteq} \\ \zeta(A_E)^\times & \xrightarrow{\rho_{E,*}} & \zeta(B_E)^\times \end{array} \tag{47}$$

for any fields $E \supseteq F \supseteq \mathbb{Q}$. The localisation sequences (11) for A and B form a commutative diagram with the maps induced by ρ. With notation as in the proof of Lemma 9, the commutativity of the left hand square in (46) therefore implies that

$$\begin{aligned} \rho_*(\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(y)) &= \rho_*(\delta^1_{\mathfrak{A},\mathbb{R}}(\mathrm{nr}^{-1}_{A_\mathbb{R}}(\lambda y))) - \sum_p \rho_*(\delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}(\mathrm{nr}^{-1}_{A_p}(\lambda))) \\ &= \delta^1_{\mathfrak{B},\mathbb{R}}(\mathrm{nr}^{-1}_{B_\mathbb{R}}(\rho_{\mathbb{R},*}(\lambda y))) - \sum_p \delta^1_{\mathfrak{B}_p,\mathbb{Q}_p}(\mathrm{nr}^{-1}_{B_p}(\rho_{\mathbb{Q}_p,*}(\lambda))). \end{aligned}$$

From the commutativity of (47) with $E/F = \mathbb{Q}_p/\mathbb{Q}$ and $E/F = \mathbb{R}/\mathbb{Q}$ it is clear that one can use the element $\rho_{\mathbb{Q},*}(\lambda)$ of $\zeta(B)^\times$ to compute $\hat{\delta}^1_{\mathfrak{B},\mathbb{R}}(\rho_{\mathbb{R},*}(y))$. It follows that the above displayed formula is equal to $\hat{\delta}^1_{\mathfrak{B},\mathbb{R}}(\rho_{\mathbb{R},*}(y))$, and hence that the right hand square in (46) commutes.
The arguments for ρ^* and μ_* are entirely similar. $\square$

Theorem 4.1. *All assertions of Theorem 3.1 remain valid with $R\Omega(-,-)$ replaced by either $L(-,-)$ or $T\Omega(-,-)$.*

Proof. We have a commutative diagram of exact functors

$$\begin{array}{ccc} \mathrm{PMod}(A_\mathbb{C}) & \xrightarrow{\kappa_A} & \prod_{i(A)} \mathrm{PMod}(\mathbb{C}) \\ {\scriptstyle B\otimes_A -}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\ \mathrm{PMod}(B_\mathbb{C}) & \xrightarrow{\kappa_B} & \prod_{i(B)} \mathrm{PMod}(\mathbb{C}) \end{array} \tag{48}$$

where κ_A and κ_B are given by (41), and $\eta := \kappa_B \circ (B \otimes_A -) \circ \kappa_A^{-1}$ is essentially given by an $|i(A)| \times |i(B)|-$matrix $N = (n_{i,j})$ with non-negative integer entries. Indeed, one checks easily that η sends $(V_i)_{i\in i(A)} \in \mathrm{Ob}(\prod_{i(A)} \mathrm{PMod}(\mathbb{C}))$ to $(\oplus_i V_i^{n_{i,j}})_{j\in i(B)}$. There exists a similar diagram involving the same matrix N for any pseudo-abelian $\mathbb{C}$-linear category W in place of $\mathrm{PMod}(\mathbb{C})$. For an abelian group C and any map ϵ defined as in (42) which is *additive* for direct sums in W, we therefore obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Ob}(W_A) & \xrightarrow{\epsilon} & C^{i(A)} \\ {\scriptstyle B\otimes_A -}\big\downarrow & & \big\downarrow{\scriptstyle N} \\ \mathrm{Ob}(W_B) & \xrightarrow{\epsilon} & C^{i(B)}. \end{array}$$

This observation applies to $L_v(M,s)$ and so by taking into account (38) it follows that $N(L_v({}_AM,s)) = L_v({}_B(B \otimes_A M), s) \in \mathbb{C}^{i(B)}$. Now since

$\mathbb{C}^{i(A)} \xrightarrow{N} \mathbb{C}^{i(B)}$ is an analytic map it commutes with the Euler product and hence $N(L({}_AM,s)) = L({}_B(B\otimes_A M),s) \in \mathbb{C}^{i(B)}$. By analytic continuation it follows that $N(L^*({}_AM,0)) = L^*({}_B(B\otimes_A M),0) \in \zeta(B_\mathbb{R})^\times$. We now recall that the map ρ_* defined on $\zeta(A_\mathbb{C})^\times \cong (\mathbb{C}^\times)^{i(A)}$ in Lemma 10 is compatible with the induced map on $K_1(A_\mathbb{C})$. The diagram of exact categories (48) then shows that ρ_* is given by the matrix N after making the canonical identification $K_1(\mathbb{C}) \cong \mathbb{C}^\times$. Lemma 10 then implies that $\rho_*(L(M,\mathfrak{A})) = L(B\otimes_A M,\mathfrak{B})$, i.e. the precise analogue of Theorem 3.1a).
The analogues of b) and c) in Theorem 3.1 follow by exactly the same argument using the maps ρ^* and μ_*. □

4.5. Consequences of functoriality. In terms of the notation of Theorem 3.1, Theorem 4.1 implies that if Conjecture 4(iv) is valid for the pair $(M,\mathfrak{A})$, then it is also valid for the pair $(B\otimes_A M,\mathfrak{B})$. In addition, if ρ_* is injective, then the converse is also true. Analogous statements also hold for ρ^*. It is therefore of some interest to know when the maps ρ_* and ρ^* are injective. The next result investigates $\ker(\rho_*)$ in the case that ρ is injective.

Lemma 11. *Let $\iota : \mathfrak{A} \to \mathfrak{B}$ denote the inclusion map between orders in finite dimensional semisimple $\mathbb{Q}$-algebras $A \subseteq B$. Assume that $\zeta(B)\cap A = \zeta(A)$.*

a) *The natural map $\iota_* : \mathrm{Cl}(\mathfrak{A},\mathbb{R}) \to \mathrm{Cl}(\mathfrak{B},\mathbb{R})$ has finite kernel contained in $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$. Moreover, $\iota_*^{-1}(\mathrm{Cl}(\mathfrak{B},\mathbb{Q})) = \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$.*
b) *If either $\mathfrak{A}$ is a maximal order, or B is commutative and $\mathfrak{B}\cap A = \mathfrak{A}$, then ι_* is injective.*
c) *The group $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$ is torsion-free if and only if for each prime p the image of the natural map $K_1(\mathfrak{A}_p) \to K_1(A_p) \cong \zeta(A_p)^\times$ is equal to the group of units of the maximal $\mathbb{Z}_p$-order in $\zeta(A_p)$. This condition holds if $\mathfrak{A}$ is a maximal order in A.*
d) *If $\mathfrak{B}$ is a maximal order, then $\ker(\iota_*)$ is the torsion subgroup of $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$.*

Proof. For any finite dimensional semisimple $\mathbb{Q}$-algebra C and field F of characteristic 0 we set $\zeta(C_F)^{\times+} := \mathrm{im}(\mathrm{nr}_{C_F}) \subset \zeta(C_F)^\times$. The map nr_{C_F} induces an isomorphism $K_1(C_F) \cong \zeta(C_F)^{\times+}$, and in what follows we regard this as an identification. We will often use the fact that since $\zeta(B)\cap A = \zeta(A)$ the natural map $K_1(A_F) \to K_1(B_F)$ corresponds under the above identifications to the inclusion $\zeta(A_F)^{\times+} \subseteq \zeta(B_F)^{\times+}$ [15, (45.3)]. We also use the fact that Proposition 2.2 implies an explicit description of $\zeta(C_F)^{\times+}$ in terms of positivity conditions at each quaternion component of C.
We first prove that $\iota_*^{-1}(\mathrm{Cl}(\mathfrak{B},\mathbb{Q})) = \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$. We thus suppose that x is any element of $\mathrm{Cl}(\mathfrak{A},\mathbb{R})$ for which $\iota_*(x) \in \mathrm{Cl}(\mathfrak{B},\mathbb{Q})$. The fact that diagram (14) is exact implies that, after possibly adding to x an element of $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$, we can assume that there exists an element $\tilde{x}$ of $K_1(A_\mathbb{R}) = \zeta(A_\mathbb{R})^{\times+}$ such that $x = \delta^1_{\mathfrak{A},\mathbb{R}}(\tilde{x})$. Since the image of $\tilde{x}$ in $K_0(\mathfrak{B},\mathbb{R})$ lies in $K_0(\mathfrak{B},\mathbb{Q})$ diagram (11) (with A replaced by B) implies that $\tilde{x} \in \zeta(B)^{\times+}$. Now $B\cap A_\mathbb{R} = A$ and so $\zeta(B)^{\times+}\cap\zeta(A_\mathbb{R})^{\times+} = \zeta(A)^{\times+}$ (as a consequence of Proposition 2.2). Hence

$x \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q}) \subset \mathrm{Cl}(\mathfrak{A},\mathbb{R})$, as required. We note in particular that this implies that $\ker(\iota_*) \subseteq \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$.

We write

$$\iota_{*,p} : \zeta(A_p)^\times / \operatorname{im}(K_1(\mathfrak{A}_p)) \to \zeta(B_p)^\times / \operatorname{im}(K_1(\mathfrak{B}_p)) \tag{49}$$

for the natural map which is induced by the inclusion $\zeta(A_p) \subseteq \zeta(B_p)$. We observe that the decomposition (15) induces a corresponding decomposition $\iota_* = \bigoplus_p \iota_{*,p}$, and hence that ι_* is injective if and only if each map $\iota_{*,p}$ is injective.

We now consider b). If firstly B and hence A are commutative, then the image of $K_1(\mathfrak{A}_p)$ in $K_1(A_p) = A_p^\times$ is isomorphic to $\mathfrak{A}_p^\times$ [15, (45.12)] and similarly for B_p. Hence if $\mathfrak{B}_p \cap A_p = \mathfrak{A}_p$, then the map $\iota_{*,p}$ is injective, as required. We assume now that $\mathfrak{A}_p$ is a maximal $\mathbb{Z}_p$-order in A_p. In this case $\zeta(\mathfrak{A}_p)$ is the (unique) maximal $\mathbb{Z}_p$-order in $\zeta(A_p)$ and the map nr_{A_p} induces an identification $\operatorname{im}(K_1(\mathfrak{A}_p)) = \zeta(\mathfrak{A}_p)^\times \subset \zeta(A_p)^\times$ by [15, (45.8)]. To prove injectivity of $\iota_{*,p}$ we embed $\mathfrak{B}_p$ in a maximal $\mathbb{Z}_p$-order $\mathfrak{M}_p$ of B_p. Then $\operatorname{im}(K_1(\mathfrak{B}_p)) \subseteq \operatorname{im}(K_1(\mathfrak{M}_p)) = \zeta(\mathfrak{M}_p)^\times$. In addition, the intersection $C_p := \zeta(\mathfrak{M}_p) \cap \zeta(A_p)$ is a $\mathbb{Z}_p$-order in $\zeta(A_p)$ and is therefore contained in $\zeta(\mathfrak{A}_p)$. Hence one has

$$\operatorname{im}(K_1(\mathfrak{B}_p)) \cap \zeta(A_p)^\times \subseteq C_p^\times \subseteq \zeta(\mathfrak{A}_p)^\times = \operatorname{im}(K_1(\mathfrak{A}_p))$$

and so $\iota_{*,p}$ is indeed injective. This finishes the proof of b).

We next prove c) and also the first (and only remaining) assertion of a). We observe that

$$\ker(\iota_{*,p}) = (\operatorname{im}(K_1(\mathfrak{B}_p)) \cap \zeta(A_p)^\times) / \operatorname{im}(K_1(\mathfrak{A}_p)),$$

and that this quotient is finite since its numerator and denominator are both of finite index in the unit group of the maximal $\mathbb{Z}_p$-order in $\zeta(A_p)$ (cf. [15, Exer. (45.4)]). In addition if $\mathfrak{A}_p$ is maximal, then $\zeta(A_p)$ is a product of local fields and $\zeta(\mathfrak{A}_p)$ is the corresponding product of valuation rings, and hence $K_0(\mathfrak{A}_p,\mathbb{Q}_p) \cong \zeta(A_p)^\times/\zeta(\mathfrak{A}_p)^\times$ is torsion free (in fact free abelian of finite rank). Hence in this case $\ker(\iota_{*,p})$ is trivial. This implies that $\ker(\iota_*)$ is finite (as claimed in a)) since $\mathfrak{A}_p$ is a maximal $\mathbb{Z}_p$-order for almost all p. This argument also implies that the torsion subgroup of $\zeta(A_p)^\times / \operatorname{im}(K_1(\mathfrak{A}_p))$ is equal to $\mathfrak{M}_p^\times / \operatorname{im}(K_1(\mathfrak{A}_p))$ where $\mathfrak{M}_p$ is the maximal $\mathbb{Z}_p$-order in $\zeta(A_p)$. It follows that $\mathrm{Cl}(\mathfrak{A},\mathbb{Q})$ is indeed torsion free if and only if the condition in c) is satisfied. We remark that if $\mathfrak{A}$ is a maximal order, then this condition is satisfied as a consequence of [15, (45.8)].

We observe finally that d) follows immediately upon combining a) and c). □

Remark 11. The original conjecture of Bloch and Kato (as formulated in [4] and reworked in [20]) is equivalent to Conjecture 4(iv) for the pair $(M,\mathbb{Z})$. Now for any order $\mathfrak{A}$ the unique homomorphism $\mathbb{Z} \to \mathfrak{A}$ is flat. Hence, if $\mathfrak{A}$ is any order in A for which M admits a projective $\mathfrak{A}$-structure, then Conjecture 4(iv) for the pair $(M,\mathfrak{A})$ implies the conjecture of Bloch and Kato.

Remark 12. From Lemma 11a), it follows that Conjecture 4(iii) for any given pair $(M, \mathfrak{A})$ is equivalent to Conjecture 4(iii) for any pair $(B \otimes_A M, \mathfrak{B})$ where $\mathfrak{A} \subseteq \mathfrak{B}$. As a consequence, it suffices to verify Conjecture 4(iii) after an arbitrary extension $A \subseteq B$ of the operating algebra and for any choice of order in B.

Remark 13. In the Galois case (cf. Example b) in §3.3) there is a natural interplay between a change of coefficients and a change of field extension. This situation is described precisely by the following result.

Proposition 4.1. *Let M_K be a motive over K and L/K a Galois extension with group G so that $\mathbb{Q}[G]$ acts on $M := M_L = h^0(\mathrm{Spec}(L)) \otimes M_K$. Let H be a subgroup of G.*

a) Let $K' = L^H$ denote the fixed field of H and $T\Omega(M', \mathbb{Z}[H])$ the element constructed from the base change $M'_{K'}$ of M_K to K' and the extension L/K' with group H. Then

$$\rho_H^{G,*}(T\Omega(M, \mathbb{Z}[G])) = T\Omega(M, \mathbb{Z}[H]) = T\Omega(M', \mathbb{Z}[H]) \tag{50}$$

where $\rho_H^G : \mathbb{Z}[H] \to \mathbb{Z}[G]$ is the natural inclusion morphism (which is flat).

b) Set $Q := G/H$, $L' := L^H$ and $M_{L'} := h^0(\mathrm{Spec}(L')) \otimes M_K$. Then

$$q_{Q,*}^G(T\Omega(M, \mathbb{Z}[G])) = T\Omega(\mathbb{Q}[Q] \otimes_{\mathbb{Q}[G]} M, \mathbb{Z}[Q]) = T\Omega(M_{L'}, \mathbb{Z}[Q]) \tag{51}$$

where $q_Q^G : \mathbb{Z}[G] \to \mathbb{Z}[Q]$ is the natural projection.

Proof. After taking into account Theorem 4.1, we need only prove the second equalities of (50) and (51).

We observe first that the second equality of (51) is an immediate consequence of the isomorphism $\mathbb{Q}[Q] \otimes_{\mathbb{Q}[G]} h^0(\mathrm{Spec}(L)) \cong h^0(\mathrm{Spec}(L'))$ of motives over K with $\mathbb{Q}[Q]$-action.

On the other hand, the second equality of (50) is best understood by thinking of M_K as arising from a variety $X \to \mathrm{Spec}(K)$. Then both M and M' will arise from the same variety $X' = \mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} X$, respectively viewed over K and K' (and with H-action in both cases). It is well known that the L-functions taken over either K or K' are the same [16, Rem 2.9]. In addition, the groups $H^i_f(-,-)$ and $H_{dR}(-,-)$ are the same from both points of view since they only depend on the underlying scheme X'. Since also $H_v(M) = \bigoplus_{v'|v} H_{v'}(M')$ for each $v \in S_\infty$ it follows that $\Xi(M) = \Xi(M')$. The exact sequence (17) is the same for M and M'. Further, if $\pi : \mathrm{Spec}(\mathcal{O}_{K',S_p}) \to \mathrm{Spec}(\mathcal{O}_{K,S_p})$ denotes the natural finite morphism, then $\pi_*(H_p(M')) = H_p(M)$ and so $R\Gamma_c(\mathcal{O}_{K',S_p}, H_p(M')) \cong R\Gamma_c(\mathcal{O}_{K,S_p}, H_p(M))$. The map ϑ_p is therefore the same for both M and M' and hence $\Xi(M)_\mathbb{Z} = \Xi(M')_\mathbb{Z}$. This in turn implies that $T\Omega(M, \mathbb{Z}[H]) = T\Omega(M', \mathbb{Z}[H])$, as required. □

4.6. Reduction to the commutative case. In this section we use Theorem 4.1 to prove that Conjecture 4(iii), and also Conjecture 4(iv) for all pairs $(M,\mathfrak{A})$ with $\mathfrak{A}$ a maximal order, can be verified by restricting to motives with commutative coefficients.

Proposition 4.2. *a) Conjecture 4(iii) holds for all pairs $(M,\mathfrak{A})$ if it holds for all such pairs with $\mathfrak{A}$ commutative and maximal.*
b) Conjecture 4(iv) holds for all pairs $(M,\mathfrak{A})$ where $\mathfrak{A}$ is a maximal order, if it holds for all such pairs with $\mathfrak{A}$ commutative and maximal.

Proof. By remark 12 after Lemma 11 we may assume throughout that $\mathfrak{A}$ is maximal. Consider the Wedderburn decomposition $A \cong \prod_{i=1}^{r} M_{m_i}(D_i)$ of A and put $F_i := \zeta(D_i)$. Pick a splitting field E_i for each i so that $M_{m_i}(D_i)\otimes_{F_i} E_i \cong M_{n_i}(E_i)$ with $n_i = m_i\sqrt{[D_i:F_i]}$. Then $B=\prod_{i=1}^{r} M_{n_i}(E_i)$ contains A and we have $\zeta(B)\cap A=\zeta(A)$. The image of $\mathfrak{A}$ can be embedded into a maximal order $\mathfrak{B}$ in B, and we write $\iota:\mathfrak{A}\to\mathfrak{B}$ for the corresponding morphism. One has $\iota_*(T\Omega(M,\mathfrak{A}))=T\Omega(B\otimes_A M,\mathfrak{B})$ by Theorem 4.1, and so Lemma 11 implies that Conjecture 4(iii), resp. (iv), is valid for $(M,\mathfrak{A})$ if and only if it is valid for $(B\otimes_A M,\mathfrak{B})$.
Now [15, Th. (26.25)] implies that, perhaps after enlarging each field E_i, we can assume that $\mathfrak{B}=b\cdot\mathfrak{B}'\cdot b^{-1}$ with $\mathfrak{B}'=\prod_{i=1}^{r} M_{n_i}(\mathcal{O}_{E_i})$ and $b\in B^\times$. In this case multiplication by b gives an isomorphism of pairs $(B\otimes_A M, B)\cong(B\otimes_A M, b\cdot B\cdot b^{-1})$ which in turn induces an equality

$$\begin{aligned} T\Omega(B\otimes_A M,\mathfrak{B}) &= T\Omega(B\otimes_A M,\mathfrak{B}') \\ &= \prod_{i=1}^{r} T\Omega(\epsilon_i(B\otimes_A M), M_{n_i}(\mathcal{O}_{E_i})) \\ &\in \bigoplus_{i=1}^{r} K_0(M_{n_i}(\mathcal{O}_{E_i}),\mathbb{R}) \cong K_0(\mathfrak{B}',\mathbb{R}) \end{aligned}$$

where ϵ_i are the central idempotents of B. From Theorem 4.1 one has

$$\mu_{i,*}(T\Omega(\epsilon_i(B\otimes_A M), M_{n_i}(\mathcal{O}_{E_i}))) = T\Omega(e_i\epsilon_i(B\otimes_A M),\mathcal{O}_{E_i})$$

where here e_i is the matrix (36) of size n_i, and $\mu_{i,*}: K_0(M_{n_i}(\mathcal{O}_{E_i}),\mathbb{R}) \xrightarrow{\sim} K_0(\mathcal{O}_{E_i},\mathbb{R})$ is the associated isomorphism (37). Since also $K_0(M_{n_i}(\mathcal{O}_{E_i}),\mathbb{Q}) = \mu_{i,*}^{-1}(K_0(\mathcal{O}_{E_i},\mathbb{Q}))$, it is clear that Conjecture 4(iii), resp. (iv), is true for $(B\otimes_A M,\mathfrak{B})$ if and only if it is true for each pair $(e_i\epsilon_i(B\otimes_A M),\mathcal{O}_{E_i})$. This finishes the proof of the proposition. □

Remark 14. Let A be a central simple algebra over a number field F with ring of integers $\mathcal{O}$. If $\mathfrak{A}$ is any maximal order in A, then the reduction to commutative coefficients effected by Proposition 4.2b) implies that Conjecture 4(iv) for the pair $(M,\mathfrak{A})$ can only determine $L^*({}_AM,0)$ to within multiplication by *all* elements of $\mathcal{O}^\times$ (inside $(F\otimes_{\mathbb{Q}}\mathbb{R})^\times=\zeta(A_{\mathbb{R}})^\times$). This reflects the general fact that if $\mathfrak{A}$ is any maximal order in a finite dimensional semisimple $\mathbb{Q}$-algebra

A, then $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ vanishes on all of $\zeta(\mathfrak{A})^\times$ rather than only on $\zeta(\mathfrak{A})^\times \cap \mathrm{im}(\mathrm{nr}_{A_\mathbb{R}})$. (The latter fact follows as an easy consequence of [15, (45.7), (45.8)]).

Remark 15. Non-maximal non-commutative orders $\mathfrak{A}$ arise as natural operating rings in many interesting examples. In general, when attempting to verify Conjecture 4(iv) for any such pair $(M, \mathfrak{A})$ no reduction to commutative coefficients is possible. In [11] we give a detailed discussion of Conjecture 4(iv) for a number of such examples.

5. Kummer duality

We recall that if M is any motive with an action of a semisimple $\mathbb{Q}$-algebra A, then the dual motive M^* is naturally endowed with an action of the opposite algebra A^{op}. After fixing an isomorphism $A^* \cong A^{op}$ of A^{op}-modules [15, (9.8)], we then have a functorial isomorphism of A^{op}-modules

$$\begin{aligned} W^* &= \mathrm{Hom}_\mathbb{Q}(W, \mathbb{Q}) \cong \mathrm{Hom}_\mathbb{Q}(A \otimes_A W, \mathbb{Q}) \\ &\cong \mathrm{Hom}_A(W, \mathrm{Hom}_\mathbb{Q}(A^{op}, \mathbb{Q})) \cong \mathrm{Hom}_A(W, A) \end{aligned} \tag{52}$$

for any A-module W. It follows that if M has a projective $\mathfrak{A}$-structure $\{T_v : v \in S_\infty\}$, then $M^*(1)$ has a projective $\mathfrak{A}^{op}$-structure $\{\mathrm{Hom}_\mathfrak{A}(T_v, \mathfrak{A})(1) : v \in S_\infty\}$. In this section we shall compare the elements $T\Omega(M, \mathfrak{A})$ and $T\Omega(M^*(1), \mathfrak{A}^{op})$. This comparison is naturally motivated by the problem of deciding whether Conjecture 4(iv) is compatible with the functional equation of $L({}_AM, s)$. The comparison result we prove in this section is most conveniently formulated in terms of an element $T\Omega^{loc}(M, \mathfrak{A})$ of $\mathrm{Cl}(\mathfrak{A}, \mathbb{R})$ the theory of which is strikingly parallel to that of $T\Omega(M, \mathfrak{A})$ but involves no assumptions on the motivic cohomology of M. Indeed, $T\Omega^{loc}(M, \mathfrak{A})$ takes the form $L^{loc}(M, \mathfrak{A}) + R\Omega^{loc}(M, \mathfrak{A})$ where the first term is defined in terms of the equivariant archimedean Euler factors and epsilon factors which are attached to M and $M^*(1)$ and the second term is of an algebraic nature, involving the realisations of M.

5.1. Definition of $R\Omega^{loc}(M, \mathfrak{A})$. We first define a virtual A-module

$$\Xi^{loc}(M) := [H_{dR}(M)] \boxtimes [H_B(M)]^{-1}$$

where

$$H_B(M) := \bigoplus_{\sigma \in \mathrm{Hom}(K, \mathbb{C})} H_\sigma(M).$$

Recall that for each $\sigma \in \mathrm{Hom}(K, \mathbb{C})$ we write $v(\sigma)$ for the corresponding element of S_∞. The action of $\mathrm{Gal}(\mathbb{C}/K_{v(\sigma)})$ on each space $H_\sigma(M)$ induces upon $H_B(M)$ an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In addition, by taking the direct sum over the $A \times \mathrm{Gal}(\mathbb{C}/K_{v(\sigma)})$-equivariant period isomorphisms

$$H_\sigma(M) \otimes_\mathbb{Q} \mathbb{C} \cong H_{dR}(M) \otimes_{K,\sigma} \mathbb{C}$$

one obtains an $A \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant isomorphism

$$H_B(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} H_{dR}(M) \otimes_{K,\sigma} \mathbb{C} = H_{dR}(M) \otimes_{\mathbb{Q}} \mathbb{C} \tag{53}$$

and after taking $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants this in turn induces an $A_{\mathbb{R}}$-equivariant isomorphism

$$(H_B(M) \otimes_{\mathbb{Q}} \mathbb{C})^+ \cong H_{dR}(M) \otimes_{\mathbb{Q}} \mathbb{R}. \tag{54}$$

Here and in what follows, for any commutative ring R and $R[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-module X we write X^+ and X^- for the R-submodules of X upon which complex conjugation acts as multiplication by 1 and -1 respectively. There is also an A-equivariant direct sum decomposition

$$(H_B(M) \otimes_{\mathbb{Q}} \mathbb{C})^+ = (H_B(M)^+ \otimes_{\mathbb{Q}} \mathbb{R}) \oplus (H_B(M)^- \otimes_{\mathbb{Q}} \mathbb{R}(2\pi i)^{-1}) \tag{55}$$

and an isomorphism

$$H_B(M)^- \otimes_{\mathbb{Q}} \mathbb{R}(2\pi i)^{-1} \cong H_B(M)^- \otimes_{\mathbb{Q}} \mathbb{R} \tag{56}$$

which is induced by identifying $\mathbb{R}(2\pi i)^{-1}$ with $\mathbb{R}$ by sending $(2\pi i)^{-1}$ to 1. Let ϵ_B (resp. ϵ_{dR}) be the automorphism $[-1]$ in $\pi_1 V(A_{\mathbb{R}}) \cong K_1(A_{\mathbb{R}})$ which is induced by multiplication by -1 on $H_B(M)^+ \otimes_{\mathbb{Q}} \mathbb{R}$ (resp. $F^0 H_{dR}(M) \otimes_{\mathbb{Q}} \mathbb{R}$). We write

$$\vartheta_\infty^{loc} : \Xi^{loc}(M) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbf{1}_{V(A_{\mathbb{R}})} \tag{57}$$

for the isomorphism of virtual $A_{\mathbb{R}}$-modules which is obtained by applying the functor $[\quad]$ to (54), (55) and (56) and then multiplying by $\epsilon_B \epsilon_{dR}$. The reason for the introduction of $\epsilon_B \epsilon_{dR}$ will become clear in the proof of Theorem 5.3 below.

As in previous sections, we now fix a finite set S of places of K which contains S_∞ and all places at which M has bad reduction, and for each rational prime p we set $V := V_p := H_p(M)$. For a finite group Π and a Π-module N we denote by $C^\bullet_{\mathrm{Tate}}(\Pi, N)$ the standard complex computing Tate cohomology. By a slight abuse of notation we also set $C^\bullet_{\mathrm{Tate}}(\Pi, N) := C^\bullet(\Pi, N)$ for any infinite profinite group Π.

For any continuous G_{S_p}-module N we set

$$\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, N) := \mathrm{Cone}\left(C^\bullet(G_{S_p}, N) \to \bigoplus_{v \in S_p} C^\bullet_{\mathrm{Tate}}(G_v, N) \right)[-1]$$

and if $N = V_p$ is a $\mathbb{Q}_p$-vector space we define

$${}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V_p) := \mathrm{Cone}\left(C^\bullet(G_{S_p}, N) \to \bigoplus_{v \in S_{p,f}} R\Gamma(K_v, V_p) \right)[-1]$$

so that there is a natural quasi-isomorphism

$$\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V_p) \to {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V_p). \tag{58}$$

We fix once and for all an injective resolution $\mathfrak{A}_p \to I^\bullet$ of $\mathfrak{A}_p$-$\mathfrak{A}_p$-bimodules, and for any complex N of $\mathfrak{A}_p$-modules (which is cohomologically bounded above) we define a complex of $\mathfrak{A}_p^{op}$-modules by $N^* := \mathrm{Hom}_{\mathfrak{A}_p}(N, I^\bullet)$. Note here that, since the natural map $\mathfrak{A}_p \to \mathfrak{A}_p \otimes_{\mathbb{Z}_p} \mathfrak{A}_p^{op}$ is flat, each I^n is an injective $\mathfrak{A}_p$-module and hence that $N^* = \mathrm{RHom}_{\mathfrak{A}_p}(N, \mathfrak{A}_p)$ in $D(\mathfrak{A}_p^{op})$. We shall moreover assume that $I^0 = A_p$ and that each I^n is torsion if $n \geq 1$. There is then an isomorphism of complexes of A_p-A_p-bimodules

$$I^\bullet_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I^\bullet \cong A_p[0]. \tag{59}$$

If $N = T = T_p \subset V_p$ is a projective $\mathfrak{A}_p$-lattice, and in that case only, we put $T^* = \mathrm{Hom}_{\mathfrak{A}_p}(T, \mathfrak{A}_p)$. The isomorphism (52) then induces an identification

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T^*(1) \cong \mathrm{Hom}_{A_p}(V, A_p(1)) \cong \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1)) = V^*(1) \tag{60}$$

of $T^*(1)$ with an $\mathfrak{A}_p^{op}$-lattice in $V^*(1)$.

Lemma 12. *a) There is a commutative diagram of maps of complexes*

$$\begin{array}{ccccc}
R\Gamma(\mathcal{O}_{K,S_p}, V) & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma(K_v, V) & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma_{/f}(K_v, V) \\
{}_1\mathrm{AV}\downarrow & & \oplus\,\mathrm{AV}_v\downarrow & & \oplus\,\mathrm{AV}_{f,v}\downarrow \\
{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1))^*[-3] & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma(K_v, V^*(1))^*[-2] & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v, V^*(1))^*[-2]
\end{array}$$

in which all of the vertical maps are quasi-isomorphisms. Moreover $[{}_1\mathrm{AV}]$, $[\mathrm{AV}_v]$ *and* $[\mathrm{AV}_{f,v}]$ *are independent of any choices made in the construction of this diagram.*
b) There is a natural quasi-isomorphism

$$R\Gamma(\mathcal{O}_{K,S_p}, T) \xrightarrow{\mathrm{AV}} \tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, T^*(1))^*[-3] \tag{61}$$

so that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{AV} = \nu[-3] \circ {}_1\mathrm{AV}$ *where* ν *is the composite isomorphism*

$$\begin{aligned}
{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1))^* &= \mathrm{Hom}_{\mathbb{Q}_p}({}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1)), \mathbb{Q}_p) \\
&\xrightarrow{(52)} \mathrm{Hom}_{A_p}({}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1)), A_p) \\
&\xrightarrow{(59)} \mathrm{Hom}_{A_p}({}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1)), I^\bullet_{\mathbb{Q}_p}) \\
&\xrightarrow{(58)} \mathrm{Hom}_{A_p}(\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V^*(1)), I^\bullet_{\mathbb{Q}_p}) \\
&\xrightarrow{(60)} \mathrm{Hom}_{\mathfrak{A}_p}(\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, T^*(1)), I^\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\end{aligned}$$

Proof. We first define local pairings for places $v \mid p$. To do this we continue to use the notation introduced in §3.2.
Recall that B^i is an algebra for $i = 0, 1$ and that the differential of $B^\bullet$ is a difference of two algebra homomorphisms β_1 and β_2 (cf. (20)). There therefore exists a natural morphism of complexes $\mu : B^\bullet \otimes_{\mathbb{Q}_p} B^\bullet \to B^\bullet$ for which $\mu^0 : B^0 \otimes_{\mathbb{Q}_p} B^0 \to B^0$ is given by multiplication, $\mu^1 : (B^0 \otimes_{\mathbb{Q}_p} B^1) \oplus (B^1 \otimes_{\mathbb{Q}_p} B^0) \to$

B^1 is defined by $\mu_1(x\otimes y, y'\otimes x') = \beta_2(x)y + \beta_1(x')y'$, and $\mu^2 : B^1 \otimes_{\mathbb{Q}_p} B^1 \to 0$ is the zero map. This morphism induces a commutative diagram of pairings

$$\begin{array}{ccc} V\otimes_{\mathbb{Q}_p} V^*(1) & \longrightarrow & \mathbb{Q}_p(1) \\ \big\downarrow & & \big\downarrow \\ (B^\bullet\otimes_{\mathbb{Q}_p} V)\otimes_{\mathbb{Q}_p}(B^\bullet\otimes_{\mathbb{Q}_p} V^*(1)) & \longrightarrow & B^\bullet\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(1) \end{array}$$

and also a commutative diagram of cup product pairings

$$\begin{array}{ccc} C^\bullet(G_v,V)\times C^\bullet(G_v,V^*(1)) & \xrightarrow{\cup} & C^\bullet(G_v,\mathbb{Q}_p(1)) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Tot} C^\bullet(G_v, B^\bullet\otimes_{\mathbb{Q}_p} V)\times \operatorname{Tot} C^\bullet(G_v,(B^\bullet\otimes_{\mathbb{Q}_p} V^*(1))) & \longrightarrow & \operatorname{Tot} C^\bullet(G_v, B^\bullet\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(1)). \end{array}$$

We thereby obtain a commutative diagram of local and global cup product pairings

$$\begin{array}{ccc} C^\bullet(G_{S_p},V)\times C^\bullet(G_{S_p},V^*(1)) & \xrightarrow{\cup} & C^\bullet(G_{S_p},\mathbb{Q}_p(1)) \\ {\scriptstyle \mathrm{res}_V}\big\downarrow{\scriptstyle \times\,\mathrm{res}_{V^*(1)}} & & \big\downarrow{\scriptstyle \mathrm{res}_{\mathbb{Q}_p(1)}} \\ \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,V)\times \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,V^*(1)) & \xrightarrow{\cup} & \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,\mathbb{Q}_p(1)) \end{array} \tag{62}$$

and hence an induced pairing on the mapping cone

$$C^\bullet(G_{S_p},V)\times {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V^*(1)) \xrightarrow{\cup} {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1)) \xrightarrow{\tilde{\mathrm{Tr}}} \mathbb{Q}_p[-3] \tag{63}$$

so that

$$\mathrm{res}^{ad}_{\mathbb{Q}_p(1)}\big(\mathrm{res}_V(x)\cup y\big) = x\cup \mathrm{res}^{ad}_{V^*(1)}(y) \tag{64}$$

where here

$$\mathrm{res}^{ad}_V : \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,V)\to {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V)[1]$$

is the natural map. The morphism $\tilde{\mathrm{Tr}}$ in (63) is chosen to be a lift of the map

$$\begin{aligned} {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1)) &\supseteq \tau^{\le 3}{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1)) \\ &\to H^3_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1))[-3] \xrightarrow{\mathrm{Tr}} \mathbb{Q}_p[-3] \end{aligned}$$

(such a lift exists because $\mathbb{Q}_p$ is an injective $\mathbb{Q}_p$-module).
In the diagram of claim a) the map ${}_1\,\mathrm{AV}$ is induced by (63) and the maps AV_v by the local cup product pairing composed with $\tilde{\mathrm{Tr}}\circ\mathrm{res}^{ad}_{\mathbb{Q}_p(1)}$. These maps are quasi-isomorphisms by local and global duality and the compatibility of local and global trace maps [36, Chap. II, §3]. In addition, the commutativity of the left hand square of the diagram in a) is a consequence of (64).
The right hand square of the diagram in a) arises as a direct sum of commutative squares over the places in $S_{p,f}$, and for each such place v the existence of the appropriate square will follow directly if we can show that the complexes $R\Gamma_f(K_v,V)$ and $R\Gamma_f(K_v,V^*(1))$ (and not only their cohomology) annihilate

each other under the pairing $\tilde{\mathrm{Tr}}\circ\mathrm{res}^{ad}_{\mathbb{Q}_p(1)}\circ\cup$ constructed above. To prove the required annihilation property, we consider separately the cases $v\nmid p$ and $v\mid p$. If firstly $v\nmid p$, then $R\Gamma_f(K_v,V)$ coincides with the subcomplex $C^\bullet(G_v/I_v,V^{I_v})$ of $C^\bullet(G_v,V)$. In addition, since $H^2(G_v/I_v,\mathbb{Q}_p(1))=0$, we can certainly choose the lifting $\tilde{\mathrm{Tr}}$ in such a way that it vanishes on the subcomplex

$$\mathrm{res}^{ad}_{\mathbb{Q}_p(1)}\left(\bigoplus_{v\in S_{p,f}}C^\bullet(G_v/I_v,\mathbb{Q}_p(1))\right)\subseteq {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1))[1].$$

If now $v\mid p$, then the subcomplex $R\Gamma_f(K_v,V)=H^0(G_v,B^\bullet\otimes_{\mathbb{Q}_p}V)$ is concentrated in degrees 0 and 1 and the cup product of $x\in H^0(G_v,B^1\otimes_{\mathbb{Q}_p}V)$ and $x'\in H^0(G_v,B^1\otimes_{\mathbb{Q}_p}V^*(1))$ is given by $\mu_2(x\otimes x')=0$. Hence $R\Gamma_f(K_v,V)$ and $R\Gamma_f(K_v,V^*(1))$ do indeed annihilate each other.

We observe that, for each $v\in S_{p,f}$, the resulting morphism $\mathrm{AV}_{f,v}$: $R\Gamma_f(K_v,V)\to R\Gamma_f(K_v,V^*(1))^*[-2]$ is a quasi-isomorphism as a consequence of [4, Prop. 3.8].

Also, since all of the maps which are induced on cohomology by ${}_1\,\mathrm{AV}$, AV_v and $\mathrm{AV}_{f,v}$ are independent of the choice of the lift $\tilde{\mathrm{Tr}}$, and their sources and targets all belong to $D^{p,p}(A_p)=D^p(A_p)$, Proposition 2.1e) implies the final assertion of claim a).

To prove claim b), we argue in a similar way starting with the diagram

(65)
$$\begin{tikzcd}
C^\bullet(G_{S_p},T)\times C^\bullet(G_{S_p},T^*(1)) \arrow[r, "\cup"] \arrow[d, "\mathrm{res}_T\times\mathrm{res}_{T^*(1)}"] & C^\bullet(G_{S_p},\mathfrak{A}_p(1)) \arrow[d, "\mathrm{res}_{\mathfrak{A}_p(1)}"] \\
\bigoplus_{v\in S_p}C^\bullet_{\mathrm{Tate}}(G_v,T)\times\bigoplus_{v\in S_p}C^\bullet_{\mathrm{Tate}}(G_v,T^*(1)) \arrow[r, "\cup"] & \bigoplus_{v\in S_p}C^\bullet_{\mathrm{Tate}}(G_v,\mathfrak{A}_p(1))
\end{tikzcd}$$

and using a lift $\tilde{\mathrm{Tr}}_{\mathfrak{A}}$ of the map

$$\begin{aligned}\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathfrak{A}_p(1))&\supseteq\tau^{\leq 3}\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathfrak{A}_p(1))\\&\to H^3_c(\mathcal{O}_{K,S_p},\mathfrak{A}_p(1))[-3]\xrightarrow{\mathrm{Tr}_{\mathfrak{A}}}\mathfrak{A}_p[-3]\to I^\bullet[-3].\end{aligned}$$

The resulting map AV (as in (61)) is a quasi-isomorphism by [9, Lem. 16]. In addition, there is a natural map from diagram (65) to diagram (62), inducing a commutative diagram

$$\begin{tikzcd}
C^\bullet(G_{S_p},T)\times\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},T^*(1)) \arrow[r] \arrow[d] & \tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathfrak{A}_p(1))) \arrow[r, "\tilde{\mathrm{Tr}}_{\mathfrak{A}}"] \arrow[d] & I^\bullet[-3] \arrow[d] \\
C^\bullet(G_{S_p},V)\times{}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V^*(1)) \arrow[r] & {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},\mathbb{Q}_p(1))) \arrow[r, "\tilde{\mathrm{Tr}}"] & \mathbb{Q}_p[-3]
\end{tikzcd}$$

where the left vertical arrow involves (60) and the middle and right vertical arrows involve the map $A \to \mathbb{Q}$ which is the image of $1 \in A^{op}$ under the isomorphism $A^{op} \cong A^* = \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q})$ chosen before (52). The second statement in b) then follows easily from this last commutative diagram. $\square$

We now define a virtual A_p-module $\Lambda_p(S, V_p)$ by setting

$$\Lambda_p(S, V_p) := \boxtimes_{v \in S_{p,f}} [R\Gamma(K_v, V_p)]^{-1} \boxtimes [\mathrm{Ind}_K^{\mathbb{Q}} V_p]^{-1}. \tag{66}$$

We also define

$$\theta_p : A_p \otimes_A \Xi^{loc}(M) \cong \Lambda_p(S, V_p) \tag{67}$$

to be the isomorphism in $V(A_p)$ which results from composing the isomorphisms obtained by applying $[\quad]$ to the canonical $A \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant comparison isomorphism

$$H_B(M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{Ind}_K^{\mathbb{Q}} V_p, \tag{68}$$

to the A-equivariant (Poincaré duality) exact sequence

$$0 \to (H_{dR}(M^*(1))/F^0)^* \to H_{dR}(M) \to H_{dR}(M)/F^0 \to 0, \tag{69}$$

to (23) for both M and $M^*(1)$, to (18) and the maps $\mathrm{AV}_{f,v}$ for each $v \in S_{p,f}$, to (19) for each $v \in S$ with $v \nmid p$ and (22) for each $v \mid p$, and by then using the isomorphisms (24) for $V = V_{p,v}$ and $V = V_p^*(1)_v$ and each $v \in S_{p,f}$.

Given a projective $\mathfrak{A}$-structure T on M we define $C(K, T_p)$ to be the mapping cone of the composite map

(70)
$$\begin{array}{ccc} R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)[-1] \longrightarrow & R\Gamma(\mathcal{O}_{K,S_p}, T_p)[-1] & \\ & \Big\downarrow{\scriptstyle \mathrm{AV}} & \\ & \tilde{R\Gamma}_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*[-4] & \longrightarrow R\Gamma_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*[-4] \end{array}$$

and we set

$$\Lambda_p(S, T_p) := [C(K, T_p)]. \tag{71}$$

We next define a canonical isomorphism in $V(A_p)$

$$\theta_p' : \Lambda_p(S, V_p) \xrightarrow{\sim} A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S, T_p).$$

To do this we first define ${}_1C(K, V_p)$ just as $C(K, T_p)$ but using diagram $A_p \otimes_{\mathfrak{A}_p}$ (70) with $R\Gamma_c(\mathcal{O}_{K,S_p}, V_p)$ replaced by ${}_1R\Gamma_c(\mathcal{O}_{K,S_p}, V_p)$, and we define ${}_2C(K, V_p)$ by also replacing $\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p}, V_p^*(1))$, $R\Gamma_c(\mathcal{O}_{K,S_p}, V_p^*(1))$ and AV by their respective versions indexed by 1. Then there are natural quasi-isomorphisms

$$A_p \otimes_{\mathfrak{A}_p} C(K, T_p) \xrightarrow{\sim} {}_1C(K, V_p) \xleftarrow{\sim} {}_2C(K, V_p) \tag{72}$$

where we have used the last assertion in Lemma 12b) for the second quasi-isomorphism. Setting

$$ {}_2L(S_p,V_p) := $$
$$ \mathrm{Cone}\left({}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \to R\Gamma(\mathcal{O}_{K,S_p},V_p) \xrightarrow{{}_1\,\mathrm{AV}} {}_1\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-3]\right) $$

we obtain a true nine term diagram

$$ (73) \qquad \begin{array}{ccccc} \bigoplus_{v\in S_\infty} R\Gamma(K_v,V_p^*(1))^*[-4] & = & \bigoplus_{v\in S_\infty} R\Gamma(K_v,V_p^*(1))^*[-4] & & \\ \downarrow & & \downarrow & & \\ {}_1\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-4] & \longrightarrow & {}_2L(S_p,V_p)[-1] & \longrightarrow & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \\ \downarrow & & \downarrow & & \| \\ {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-4] & \longrightarrow & {}_2C(K,V_p) & \longrightarrow & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p). \end{array} $$

There is also a commutative diagram of true triangles

$$ (74) \qquad \begin{array}{ccccc} R\Gamma(\mathcal{O}_{K,S_p},V_p)[-1] & \longrightarrow & {}_1L(S_p,V_p)[-1] & \longrightarrow & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \\ {\scriptstyle {}_1\mathrm{AV}}\downarrow & & \lambda\downarrow & & \| \\ {}_1\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-4] & \longrightarrow & {}_2L(S_p,V_p)[-1] & \longrightarrow & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \end{array} $$

where the bottom row coincides with the central row in (73) and

$$ {}_1L(S_p,V_p) := \mathrm{Cone}\left({}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) \to R\Gamma(\mathcal{O}_{K,S_p},V_p)\right). $$

Lemma 13. *Let*

$$ E : 0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0 $$

be any true triangle, and let E' denote the associated canonical true triangle

$$ 0 \longrightarrow C \longrightarrow \mathrm{Cone}(\pi) \longrightarrow B[1] \longrightarrow 0. $$

Then there is a natural quasi-isomorphism $A[1] \xrightarrow{q} \mathrm{Cone}(\pi)$ for which the following diagram commutes

$$ \begin{array}{ccc} [A[1]] & \longrightarrow & [A[1]]\boxtimes[C]\boxtimes[C]^{-1} \\ {\scriptstyle [q]}\downarrow & & {\scriptstyle [E]}\downarrow \\ [\,\mathrm{Cone}(\pi)] & \xrightarrow{[E']} & [C]\boxtimes[B[1]]. \end{array} $$

Proof. This is an immediate consequence of the true nine term diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A[1] & = & A[1] \\ \downarrow & & \downarrow{\scriptstyle q} & & \downarrow \\ C & \longrightarrow & \mathrm{Cone}(\pi) & \longrightarrow & B[1] \\ \| & & \downarrow & & \downarrow \\ C & \longrightarrow & Z & \longrightarrow & C[1], \end{array}$$

where here $q(x) = (0, \iota(x)) \in C \oplus B[1] = \mathrm{Cone}(\pi)$ and Z is acyclic. □

By applying Lemma 13 to the short exact sequence given by the central row in (26) we obtain a canonical quasi-isomorphism

$$L(S_p, V_p) := \bigoplus_{v \in S_p} R\Gamma(K_v, V_p) \xrightarrow{q} {}_1L(S_p, V_p). \tag{75}$$

Upon composing the isomorphisms in $V(A_p)$ which are induced by (72), the central column in (73) and the isomorphisms λ^{-1} from (74), q^{-1} from (75),

$$\bigoplus_{v \in S_\infty} R\Gamma(K_v, V_p^*(1))^* \cong \mathrm{Ind}_K^{\mathbb{Q}} V_p^*(1)^{*+}[0],$$

$$(\mathrm{Ind}_K^{\mathbb{Q}} V_p^*(1))^{*+} \cong (\mathrm{Ind}_K^{\mathbb{Q}} V_p(-1))^+ \cong (\mathrm{Ind}_K^{\mathbb{Q}} V_p)^- \tag{76}$$

and

$$\mathrm{Ind}_K^{\mathbb{Q}} V_p \cong (\mathrm{Ind}_K^{\mathbb{Q}} V_p)^+ \oplus (\mathrm{Ind}_K^{\mathbb{Q}} V_p)^-, \tag{77}$$

we obtain the desired isomorphism

$$\theta_p' : \Lambda_p(S, V_p) = [L(S_{p,f}, V_p)]^{-1} \boxtimes [\mathrm{Ind}_K^{\mathbb{Q}} V_p]^{-1} \cong A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S, T_p). \tag{78}$$

Lemma 14. *If p is odd, then the isomorphism θ_p' is induced by an isomorphism*

$$\boxtimes_{v \in S_{p,f}} [R\Gamma(K_v, T_p)]^{-1} \boxtimes [\mathrm{Ind}_K^{\mathbb{Q}} T_p]^{-1} \cong \Lambda_p(S, T_p). \tag{79}$$

Proof. For each $v \in S_\infty$ and continuous G_{S_p}-module N we define $R\Gamma_\Delta(K_v, N)$ by the short exact sequence

$$0 \to C^\bullet(G_v, N) \to C^\bullet_{\mathrm{Tate}}(G_v, N) \to R\Gamma_\Delta(K_v, N)[1] \to 0$$

where the second map is the natural inclusion. We then have a true nine term diagram

$$\begin{array}{ccccc}
\bigoplus_{v\in S_p} C^\bullet(G_v,N)[-1] & \longrightarrow & R\Gamma_c(\mathcal{O}_{K,S_p},N) & \longrightarrow & R\Gamma(\mathcal{O}_{K,S_p},N) \\
\big\downarrow & & \big\downarrow & & \Vert \\
\bigoplus_{v\in S_p} C^\bullet_{\mathrm{Tate}}(G_v,N)[-1] & \longrightarrow & \tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},N) & \longrightarrow & R\Gamma(\mathcal{O}_{K,S_p},N) \\
\big\downarrow & & \big\downarrow & & \\
\bigoplus_{v\in S_\infty} R\Gamma_\Delta(K_v,N) & = & \bigoplus_{v\in S_\infty} R\Gamma_\Delta(K_v,N). & &
\end{array} \tag{80}$$

Upon defining

$$\begin{aligned}&{}_3L(S_p,T_p) := \\ &\quad \mathrm{Cone}\left(R\Gamma_c(\mathcal{O}_{K,S_p},T_p)\to R\Gamma(\mathcal{O}_{K,S_p},T_p)\xrightarrow{\mathrm{AV}}\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},T_p^*(1))^*[-3]\right)\end{aligned}$$

there is in addition a true nine term diagram

$$\begin{array}{ccccc}
\bigoplus_{v\in S_\infty} R\Gamma_\Delta(K_v,T_p^*(1))^*[-4] & = & \bigoplus_{v\in S_\infty} R\Gamma_\Delta(K_v,T_p^*(1))^*[-4] & & \\
\big\downarrow & & \big\downarrow & & \\
\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},T_p^*(1))^*[-4] & \longrightarrow & {}_3L(S_p,T_p)[-1] & \longrightarrow & R\Gamma_c(\mathcal{O}_{K,S_p},T_p) \\
\big\downarrow & & \big\downarrow & & \Vert \\
R\Gamma_c(\mathcal{O}_{K,S_p},T_p^*(1))^*[-4] & \longrightarrow & C(K,T_p) & \longrightarrow & R\Gamma_c(\mathcal{O}_{K,S_p},T_p)
\end{array} \tag{81}$$

in which the left hand column is the dual of the central column in (80) with $N=T_p^*(1)$. If p is odd, then all terms in the central column of (81) belong to $D^p(\mathfrak{A}_p)$, and $R\Gamma_\Delta(K_v,T_p^*(1))$ is naturally quasi-isomorphic to $R\Gamma(K_v,T_p^*(1))$. In addition, setting

$${}_4L(S_p,T_p) := \mathrm{Cone}\left(R\Gamma_c(\mathcal{O}_{K,S_p},T_p)\to R\Gamma(\mathcal{O}_{K,S_p},T_p)\right)$$

there exists a commutative diagram of true triangles similar to (74) and quasi-isomorphisms

$$\bigoplus_{v\in S_p} R\Gamma(K_v,T_p)\xrightarrow{q}{}_4L(S_p,T_p)\leftarrow{}_3L(S_p,T_p)$$

which together give (79). □

Remark 16. If p is odd, then the isomorphism (79) allows a more direct definition of $\Lambda_p(S,T_p)$ than that given by (71). However, we do not expect the statement of Lemma 14 to hold for $p=2$. More concretely, if for example $\mathfrak{A}=\mathbb{Z}$, $p=2$ and we interpret virtual objects as graded determinants, then the $\mathbb{Z}_2$-lattices in the $\mathbb{Q}_2$-line $\Lambda_p(S,V_p)$ given respectively by $\Lambda_p(S,T_p)$ and $\boxtimes_{v\in S_{p,f}}[R\Gamma(K_v,T_p)]^{-1}\boxtimes[\mathrm{Ind}_K^{\mathbb{Q}}T_p]^{-1}$ may well differ.

We now define

$$\vartheta_p^{loc} := \epsilon(S,p) \circ \theta_p' \circ \theta_p : A_p \otimes_A \Xi^{loc}(M) \cong A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S, T_p) \tag{82}$$

where θ_p was defined in (67), θ_p' in (78) and where $\epsilon(S,p) \in \pi_1(V(A_p))$ is the automorphism $[-1]$ which is induced by multiplication by -1 on $\bigoplus_{v \in S_{p,f}} R\Gamma_{/f}(K_v, V_p)$ (again, the reason for the introduction of $\epsilon(S,p)$ will become clear in the proof of Theorem 5.3 below). We then define an object of the category $V(\mathfrak{A}_p) \times_{V(A_p)} V(A)$ by setting

$$\Xi^{loc}(M, T_p, S) := (\Lambda_p(S, T_p), \Xi^{loc}(M), \vartheta_p^{loc}).$$

The following result is a natural analogue of Lemmas 5 and 6 for $\Xi^{loc}(M, T_p, S)$.

Lemma 15. *a) For a different choice of projective $\mathfrak{A}$-structure T' on M and a different set of places S' the objects $\Xi^{loc}(M, T_p, S)$ and $\Xi^{loc}(M, T_p', S')$ are isomorphic in $V(\mathfrak{A}_p) \times_{V(A_p)} V(A)$.*
b) Let M be a direct factor of $h^n(X)(r)$ for a smooth projective variety X over K. If (M, A) satisfies Conjecture 3, then the object

$$\Xi^{loc}(M)_{\mathbb{Z}} := (\prod_p \Lambda_p(S, T_p), \Xi^{loc}(M), \prod_p \vartheta_p^{loc})$$

of the category $\prod_p V(\mathfrak{A}_p) \times_{\prod_p V(A_p)} V(A)$ is isomorphic to the image of an object of $\mathbb{V}(\mathfrak{A})$ under the functor of Lemma 4.

Proof. Under further assumptions on M both of these claims follow from the proof of Theorem 5.3 given below. For brevity, we shall therefore just sketch a proof here.
For a second lattice $T' \subseteq T$ one has a commutative diagram

$$\begin{array}{ccc} R\Gamma_c(\mathcal{O}_{K,S_p}, T_p')[-1] & \longrightarrow & R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)[-1] \\ {\scriptstyle \mathrm{AV}}\downarrow & & {\scriptstyle \mathrm{AV}}\downarrow \\ R\Gamma_c(\mathcal{O}_{K,S_p}, (T_p')^*(1))^*[-4] & \longrightarrow & R\Gamma_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*[-4]. \end{array}$$

One can then argue just as in the proof of Lemma 5 using [18, Th. 5.1] for both of the modules $\mathcal{F} := T_p/T_p'$ and $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{F}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \mathrm{Ext}^1_{\mathfrak{A}_p}(\mathcal{F}, \mathfrak{A}_p(1))$.
For the independence of S it is enough for us to consider the case $S' = S \cup \{w\}$ with $w \notin S_p$. In this case, the true triangle (18) is induced from a triangle in $D^p(\mathfrak{A}_p)$

$$\left(T_p \xrightarrow{1-f_w^{-1}} T_p\right) \to R\Gamma(K_w, T_p) \to \left(T_p^*(1) \xrightarrow{1-f_w^{-1}} T_p^*(1)\right)^* [-2] \tag{83}$$

and the isomorphism (24) is induced by an isomorphism $[T_p \xrightarrow{1-f_w^{-1}} T_p] \cong \mathbf{1}_{V(\mathfrak{A}_p)}$, and similarly for $T_p^*(1)$. Moreover, $\epsilon(S',p)\epsilon(S,p)^{-1}$ coincides with the automorphism which is induced by multiplication by -1 on $[T_p^*(1) \xrightarrow{1-f_w^{-1}}$

$T_p^*(1)]^*[-2]$ and hence lies in the image of $\pi_1(V(\mathfrak{A}_p))$. This suffices to construct an isomorphism between $\Xi^{loc}(M, T_p, S)$ and $\Xi^{loc}(M, T_p, S')$.
Claim b) is proved by choosing a smooth proper model of X over $\mathrm{Spec}(\mathcal{O}_{K,S})$ and then arguing just as in [8, pp. 81-83]. □

Following the last result, we define $R\Omega^{loc}(M, \mathfrak{A})$ to be the class of $(\Xi^{loc}(M)_{\mathbb{Z}}, \vartheta_\infty^{loc})$ in $\pi_0(\mathbb{V}(\mathfrak{A}, \mathbb{R})) \cong K_0(\mathfrak{A}, \mathbb{R})$.
We observe that, as a consequence of (71), one has

$$R\Omega^{loc}(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R}).$$

5.2. Definition of $L^{loc}(M, \mathfrak{A})$. Recall that $L_\infty({}_AM, s) = \prod_{v \in S_\infty} L_v({}_AM, s)$ and $\Lambda({}_AM, s) = L_\infty({}_AM, s)L({}_AM, s)$. The following conjecture is standard.

Conjecture 7. *There is an identity of meromorphic $\zeta(A_{\mathbb{C}})$-valued functions of the complex variable s*

$$\Lambda({}_AM, s) = \epsilon({}_AM, s)\Lambda({}_{A^{op}}M^*(1), -s).$$

Letting $\rho \in \mathbb{Z}^{\pi_0(\mathrm{Spec}(\zeta(A_{\mathbb{R}})))}$ denote the algebraic order at $s = 0$ of the meromorphic function $\Lambda({}_{A^{op}}M^*(1), s)$, we set

$$\mathcal{E}({}_AM) := (-1)^\rho \epsilon({}_AM, 0) \frac{L_\infty^*({}_{A^{op}}M^*(1), 0)}{L_\infty^*({}_AM, 0)} \in \zeta(A_{\mathbb{R}})^\times$$

and

$$L^{loc}(M, \mathfrak{A}) := \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(\mathcal{E}({}_AM)) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R}).$$

We then set

$$T\Omega^{loc}(M, \mathfrak{A}) := L^{loc}(M, \mathfrak{A}) + R\Omega^{loc}(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{R}).$$

The following result can be proved by mimicking the proofs of Theorems 3.1 and 4.1.

Theorem 5.1. *All assertions of Theorem 3.1 remain valid with $R\Omega(-,-)$ replaced by either $R\Omega^{loc}(-,-)$, $L^{loc}(-,-)$ or $T\Omega^{loc}(-,-)$.* □

We now describe conditions under which $T\Omega^{loc}(M, \mathfrak{A})$ can be shown to belong to $\mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$.

Theorem 5.2. *If Deligne's conjecture* [16, Conj. 6.6] *on the nature of rank one motives over* $\mathbb{Q}$ *is valid, then* $T\Omega^{loc}(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$. *More precisely, if (in the notation of the proof of Proposition 4.2b) in* §4.6*) for each index* $i \in \{1,\dots,r\}$ *there exists an integer* p_i*, an* E_i*-valued Dirichlet character* χ_i *and an isomorphism of motives over* $\mathbb{Q}$ *with coefficients in* E_i

$$\bigwedge\nolimits_{E_i}^{max} e_i\epsilon_i(\mathrm{Res}^K_{\mathbb{Q}}(B\otimes_A M)) \cong E_i(p_i)(\chi_i), \tag{84}$$

then $T\Omega^{loc}(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$.

Proof. Upon combining the functorial behaviour of $T\Omega^{loc}(-,-)$ which is described in Theorem 5.1 together with the arguments used to prove Proposition 4.2 one finds that the containment $T\Omega^{loc}(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$ can be decided by considering the motives which occur on the left hand side of (84). Indeed, it follows that $T\Omega^{loc}(M,\mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A},\mathbb{Q})$ if and only if $T\Omega^{loc}(e_i\epsilon_i(\mathrm{Res}^K_{\mathbb{Q}}(B\otimes_A M)),\mathcal{O}_{E_i}) \in K_0(\mathcal{O}_{E_i},\mathbb{Q})$ for each index $i\in\{1,\dots,r\}$.
We now fix such an index i and set $N := e_i\epsilon_i(\mathrm{Res}^K_{\mathbb{Q}}(B\otimes_A M))$ and $E := E_i$. Following [11, Lemma 1a)], one has $T\Omega^{loc}(N,\mathcal{O}_E) \in K_0(\mathcal{O}_E,\mathbb{Q})$ if and only if

$$\mathcal{E}({}_EN)^{-1}\vartheta^{loc}_\infty(\Xi^{loc}(N)) \subseteq E,$$

where here ϑ^{loc}_∞ is the isomorphism (57) for the pair (N,E). It therefore suffices to prove the displayed inclusion and to do this we adapt the proof of [16, Th. 5.6].
After fixing E-bases of $H_B(N)$ and $H_{dR}(N)$ we let $\delta(N) \in E_{\mathbb{C}}$ denote the corresponding determinant of the isomorphism (53) (with $M = N$ and $K = \mathbb{Q}$). We set $d^{\pm} := \mathrm{rank}_E(H_B(N)^{\pm})$. After adjoining E-bases of $H_B(N)^+$ and $H_B(N)^-$ the isomorphism (56) (with $M=N$) implies that $\vartheta^{loc}_\infty(\Xi^{loc}(N))$ is the E-subspace of $E_{\mathbb{R}}$ which is generated by the element $(2\pi i)^{-d^-}\delta(N)$, and so we need to prove that

$$\mathcal{E}({}_EN)^{-1}(2\pi i)^{-d^-}\delta(N) \in E. \tag{85}$$

Now from [37, Lem. C.3.7] one has

$$\frac{L^*_\infty({}_EN^*(1),0)}{L^*_\infty({}_EN,0)}.(2\pi)^{d^-+t} \in E$$

where here $t := \frac{1}{2}w(d^+ + d^-) \in \mathbb{Z}$ with w equal to the weight of N. On the other hand, by assuming that there is an isomorphism of the form (84) Deligne has proved that

$$\epsilon({}_EN,0)i^{d^-}(2\pi)^{-t}\delta(N)^{-1} \in E$$

([16, second formula on p. 331]). Upon combining the last two displayed containments we obtain (85). □

5.3. The comparison of $T\Omega(M,\mathfrak{A})$ and $T\Omega(M^*(1),\mathfrak{A}^{op})$. The exact functor $P \mapsto P^* := \mathrm{Hom}_{\mathfrak{A}^{op}}(P,\mathfrak{A}^{op})$ induces an equivalence of exact categories $\mathrm{PMod}(\mathfrak{A}^{op}) \to \mathrm{PMod}(\mathfrak{A})^{op}$. We obtain induced equivalences $\mathrm{PMod}(A^{op}) \to \mathrm{PMod}(A)^{op}$ under scalar extension and also induced equivalences of Picard categories

$$\mathbb{V}(\mathfrak{A}^{op}) \xrightarrow{*} \mathbb{V}(\mathfrak{A})^{op} \xrightarrow{\iota} \mathbb{V}(\mathfrak{A}), \quad \mathbb{V}(\mathfrak{A}^{op},\mathbb{R}) \xrightarrow{*} \mathbb{V}(\mathfrak{A},\mathbb{R})^{op} \xrightarrow{\iota} \mathbb{V}(\mathfrak{A},\mathbb{R})$$

where the functor ι sends each morphism to its inverse. We denote each of these composite functors by $X \mapsto X^*$ and we use ψ^* to denote the induced isomorphisms on algebraic K-groups.
If F is any field of characteristic 0, then the maps ψ^* combine to give an isomorphism of localisation sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_1(A_F^{op}) & \xrightarrow{\delta^1_{\mathfrak{A}^{op},F}} & \mathrm{Cl}(\mathfrak{A}^{op},F) & \xrightarrow{\delta^0_{\mathfrak{A}^{op},F}} & \mathrm{Cl}(\mathfrak{A}^{op}) & \longrightarrow & 0 \\ & & {\scriptstyle\psi^*}\downarrow & & {\scriptstyle\psi^*}\downarrow & & {\scriptstyle\psi^*}\downarrow & & \\ \cdots & \longrightarrow & K_1(A_F) & \xrightarrow{\delta^1_{\mathfrak{A},F}} & \mathrm{Cl}(\mathfrak{A},F) & \xrightarrow{\delta^0_{\mathfrak{A},F}} & \mathrm{Cl}(\mathfrak{A}) & \longrightarrow & 0. \end{array} \tag{86}$$

Lemma 16. *One has*

$$\psi^* \circ \hat{\delta}^1_{\mathfrak{A}^{op},\mathbb{R}} = -\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$$

on $\zeta(A^{op}_{\mathbb{R}})^\times = \zeta(A_{\mathbb{R}})^\times$.

Proof. For any field F of characteristic 0 there is a commutative diagram

$$\begin{array}{ccc} K_1(A_F^{op}) & \xrightarrow{\psi^*} & K_1(A_F) \\ {\scriptstyle \mathrm{nr}_{A_F^{op}}}\downarrow & & {\scriptstyle \mathrm{nr}_{A_F}}\downarrow \\ \zeta(A_F)^\times & \xrightarrow{-1} & \zeta(A_F)^\times. \end{array} \tag{87}$$

This is a consequence of the fact that if $V \in \mathrm{Ob}(\mathrm{PMod}(A_F^{op}))$ and $\phi \in \mathrm{Aut}_{A_F^{op}}(V)$, then $\psi^*(\phi) = \mathrm{Hom}_F(\phi,F)^{-1}$.
The claimed equality thus follows from the definition of $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ in terms of $\delta^1_{\mathfrak{A},\mathbb{R}}$ and $\delta^1_{\mathfrak{A}_p,\mathbb{Q}_p}$ by using the commutativity of (86) and (87) (cf. the proof of Lemma 10). $\square$

Theorem 5.3. *Assume that Conjectures 1 and 2 and the Coherence hypothesis are valid for both* (M,A) *and* $(M^*(1),A^{op})$*, and also that Conjecture 7 is valid for* (M,A)*. Let* $\mathfrak{A}$ *be an order in* A *for which* M *has a projective* $\mathfrak{A}$*-structure. Then* $M^*(1)$ *has a projective* $\mathfrak{A}^{op}$*-structure and there is an equality*

$$T\Omega(M,\mathfrak{A}) + \psi^*(T\Omega(M^*(1),\mathfrak{A}^{op})) = T\Omega^{loc}(M,\mathfrak{A}) \tag{88}$$

in $K_0(\mathfrak{A},\mathbb{R})$.

Corollary 1. *Assume that Conjecture 4 is valid for the pair* $(M, \mathfrak{A})$. *Then Conjecture 4(iii), resp. 4(iv), is valid for the pair* $(M^*(1), \mathfrak{A}^{op})$ *if and only if* $T\Omega^{loc}(M, \mathfrak{A}) \in \mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$, *resp.* $T\Omega^{loc}(M, \mathfrak{A}) = 0$.

Proof. This follows as an immediate consequence of Theorem 5.3 and the fact that ψ^* restricts to give an isomorphism $\mathrm{Cl}(\mathfrak{A}^{op}, \mathbb{Q}) \cong \mathrm{Cl}(\mathfrak{A}, \mathbb{Q})$. □

Remark 17. If $\mathfrak{A}$ is commutative, then $\mathfrak{A} = \mathfrak{A}^{op}$ and ψ^* coincides with multiplication by -1 on $K_0(\mathfrak{A}, \mathbb{R})$ and so (88) simplifies to give an equality

$$T\Omega(M, \mathfrak{A}) - T\Omega(M^*(1), \mathfrak{A}) = T\Omega^{loc}(M, \mathfrak{A}).$$

To justify the claim that $\psi^* = -1$ whenever $\mathfrak{A}$ is commutative we first recall that, as a consequence of Propositions 2.5 and 2.4, all elements in $K_0(\mathfrak{A}, \mathbb{R})$ can be represented by pairs $((L, \alpha), g)$ where (L, α) is a graded invertible $\mathfrak{A}$-module and $g : A_{\mathbb{R}} \otimes_{\mathfrak{A}} (L, \alpha) \cong \mathbf{1}_{A_{\mathbb{R}}} = (A_{\mathbb{R}}, 0)$ is an isomorphism in $\mathcal{P}(A_{\mathbb{R}})$. Since the image of $\mathrm{Spec}(A_{\mathbb{R}})$ is dense in $\mathrm{Spec}(\mathfrak{A})$, this implies that $\alpha = 0$ and also that $g : L \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} A_{\mathbb{R}}$ is an isomorphism of ordinary line bundles. Then $\psi^*(L, 0) = (L^*, 0)$ and

$$\psi^*((L, 0)_{\mathbb{R}} \xrightarrow{g} (A_{\mathbb{R}}, 0)) = ((L^*, 0)_{\mathbb{R}} \xrightarrow{(g^*)^{-1}} (\mathfrak{A}_{\mathbb{R}}^*, 0) \cong (\mathfrak{A}_{\mathbb{R}}, 0))$$

where this last isomorphism sends the identity map in $\mathfrak{A}^* = \mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A})$ to the identity element of $\mathfrak{A}$. Now since $(L \otimes_{\mathfrak{A}} L^*)_{\mathbb{R}} \xrightarrow{g \otimes (g^*)^{-1}} \mathfrak{A}_{\mathbb{R}} \otimes \mathfrak{A}_{\mathbb{R}} = \mathfrak{A}_{\mathbb{R}}$ is isomorphic to $\mathfrak{A}_{\mathbb{R}} \xrightarrow{\mathrm{id}} \mathfrak{A}_{\mathbb{R}}$ via the evaluation map $L \otimes_{\mathfrak{A}} L^* \to \mathfrak{A}$, it follows that $\psi^*((L, 0), g)$ does indeed represent the inverse of $((L, 0), g)$ in $K_0(\mathfrak{A}, \mathbb{R})$.

Proof of Theorem 5.3. By applying the (monoidal) functor $(-)^*$ to the object $\Xi(M^*(1))$ of $V(A^{op})$ (as defined in (29)) one finds that there is an isomorphism in $V(A)$

$$\begin{aligned}\Xi(M^*(1))^* \cong\ & [H_f^0(K, M^*(1))^*] \boxtimes [H_f^1(K, M^*(1))^*]^{-1} \\ & \boxtimes [H_f^1(K, M)] \boxtimes [H_f^0(K, M)]^{-1} \\ & \boxtimes \left(\boxtimes_{v \in S_\infty} [H_v(M^*(1))^{G_v,*}]^{-1} \right) \boxtimes [(H_{dR}(M^*(1))/F^0)^*]\end{aligned}$$

and hence also an isomorphism

$$\begin{aligned}\Xi(M) \boxtimes \Xi(M^*(1))^* \cong\ & [\bigoplus_{v \in S_\infty} H_v(M)^{G_v}]^{-1} \boxtimes [(H_{dR}(M)/F^0)] \\ & \boxtimes [\bigoplus_{v \in S_\infty} H_v(M^*(1))^{G_v,*}]^{-1} \boxtimes [(H_{dR}(M^*(1))/F^0)^*].\end{aligned} \tag{89}$$

We now observe that

$$\bigoplus_{v \in S_\infty} H_v(M)^{G_v} = H_B(M)^+ \tag{90}$$

and that there are natural A-equivariant isomorphisms

$$H_B(M^*(1))^{+*} \cong H_B(M^*(1))^{*+} \cong H_B(M)^-(-1) \cong H_B(M)^- \tag{91}$$

where the last map is induced by sending each element $y \otimes (2\pi i)^{-1}$ to y. After applying $[\quad]$ to (90), to the linear dual of (90) for $M^*(1)$, to (91), to the natural isomorphism

$$H_B(M) \cong H_B(M)^+ \oplus H_B(M)^- \tag{92}$$

and to the Poincaré duality sequence (69), the right hand side of (89) identifies with $\Xi^{loc}(M)$, and hence one obtains an isomorphism of virtual A-modules

$$\vartheta^{PD} : \Xi(M) \boxtimes \Xi(M^*(1))^* \cong \Xi^{loc}(M).$$

Lemma 17. *a)* $\vartheta^{loc}_\infty \circ (A_\mathbb{R} \otimes_A \vartheta^{PD}) = \vartheta_\infty(M) \boxtimes \vartheta_\infty(M^*(1))^*$.
b) For each projective $\mathfrak{A}$-structure T on M and each prime p there is a commutative diagram in $V(A_p)$

$$\begin{array}{ccc}
A_p \underset{A}{\otimes} (\Xi(M) \boxtimes \Xi(M^*(1))^*) & \xrightarrow{A_p \underset{A}{\otimes} \vartheta^{PD}} & A_p \underset{A}{\otimes} \Xi^{loc}(M) \\
{\scriptstyle \vartheta_p(M)\boxtimes\vartheta_p(M^*(1))^*}\big\downarrow & & \big\downarrow{\scriptstyle \vartheta_p^{loc}} \\
A_p \underset{\mathfrak{A}_p}{\otimes} [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)] \boxtimes [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*] & \xrightarrow{A_p \underset{\mathfrak{A}_p}{\otimes} \vartheta_p^{AV}} & A_p \underset{\mathfrak{A}_p}{\otimes} \Lambda_p(S, T_p)
\end{array}$$

where

$$\vartheta_p^{AV} : [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p)] \boxtimes [R\Gamma_c(\mathcal{O}_{K,S_p}, T_p^*(1))^*] \xrightarrow{\sim} \Lambda_p(S, T_p)$$

is the isomorphism in $V(\mathfrak{A}_p)$ which is induced by the definition (71) of $\Lambda_p(S, T_p)$.

We assume for the moment that this lemma is true. Then from claim b) we deduce that there is an isomorphism in $\mathbb{V}(\mathfrak{A})$

$$\vartheta^{PD}_\mathbb{Z} : \Xi(M)_\mathbb{Z} \boxtimes \Xi(M^*(1))^*_\mathbb{Z} \cong \Xi^{loc}(M)_\mathbb{Z}.$$

Taken in conjunction with the equality of claim a), this isomorphism in turn implies that there is an equality

$$R\Omega(M, \mathfrak{A}) + \psi^*(R\Omega(M^*(1), \mathfrak{A}^{op})) = R\Omega^{loc}(M, \mathfrak{A}) \tag{93}$$

in $K_0(\mathfrak{A}, \mathbb{R})$.
On the other hand, by taking leading coefficients at $s = 0$ in Conjecture 7 we find that

$$L^*({}_AM, 0) = \mathcal{E}({}_AM) L^*({}_{A^{op}}M^*(1), 0)$$

in $\zeta(A_{\mathbb{R}})^{\times}$. By applying $\hat{\delta}^1_{\mathfrak{A},\mathbb{R}}$ to this equality and then using Lemma 16 we obtain an equality

$$\begin{aligned} L(M,\mathfrak{A}) =& L^{loc}(M,\mathfrak{A}) + \hat{\delta}^1_{\mathfrak{A},\mathbb{R}}(L^*(M^*(1),0)) \\ =& L^{loc}(M,\mathfrak{A}) - \psi^*(L(M^*(1),\mathfrak{A}^{op})) \end{aligned}$$

in $K_0(\mathfrak{A},\mathbb{R})$. Upon comparing this equality to (93) we finally obtain the formula of Theorem 5.3.

It therefore only remains to prove Lemma 17, and our proof of this result will occupy the rest of this section. Before starting the proof however we introduce another useful convention. For any integer n the symbol $(n)^*$ refers to the equality, isomorphism or exact triangle which is obtained by applying the functor * to the displayed formula (n) with $M^*(1)$ in place of M and $V_p^*(1)$ in place of V_p; $(n)^+$ refers to the equality, isomorphism or exact triangle obtained by taking $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants of (n); $(n)_v$ indicates that the formula (n) is to be used for all places v in $S_{p,f}$ to which it applies.

Proof of Lemma 17. We begin with the proof of part a).

Lemma 18. *(cf.* [20]*[Prop. III.1.1.6 iii)]) With notation as in section 3.2 there are natural isomorphisms of $A_{\mathbb{R}}$-modules*

$$\ker(\alpha_M) \cong \mathrm{coker}(\alpha_{M^*(1)})^*, \quad \mathrm{coker}(\alpha_M) \cong \ker(\alpha_{M^*(1)})^*.$$

Proof. For any $\mathbb{Q}$-space W and field of characteristic zero F we set $W_F := W \otimes_F \mathbb{Q}$. There is a commutative diagram of $A_{\mathbb{R}} \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-modules

$$\begin{tikzcd}
H_B(M)_{\mathbb{R}} \arrow[r, "\alpha_{M,\mathbb{C}}"] & H_{dR}(M)_{\mathbb{C}}/F^0 \\
F^0H_{dR}(M)_{\mathbb{C}} \oplus H_B(M)_{\mathbb{R}} \arrow[u] \arrow[r] \arrow[d, equal] & H_{dR}(M)_{\mathbb{C}} \arrow[u] \arrow[d, "(53)"] \\
F^0H_{dR}(M)_{\mathbb{C}} \oplus H_B(M)_{\mathbb{R}} \arrow[r] \arrow[d] & H_B(M)_{\mathbb{C}} \arrow[d] \\
F^0H_{dR}(M)_{\mathbb{C}} \arrow[r] \arrow[d, "\beta_1"'] & H_B(M)_{\mathbb{C}}/H_B(M)_{\mathbb{R}} \arrow[d, "\beta_2"'] \\
(H_{dR}(M^*(1))_{\mathbb{C}}/F^0)^* \arrow[r, "\alpha^*_{M^*(1),\mathbb{C}}"] & H_B(M^*(1))^*_{\mathbb{R}}
\end{tikzcd} \tag{94}$$

where all arrows other than β_1, β_2 are natural projections, inclusions or sum maps, possibly combined with the comparison isomorphism (53). The maps β_1

and β_2 arise as follows. There is a perfect duality of $\mathbb{R}$-vector spaces

$$\begin{array}{ccccc}
H_B(M)_{\mathbb{C}} \times H_B(M^*(1))_{\mathbb{C}} & \longrightarrow & H_B(\mathbb{Q}(1))_{\mathbb{C}} & & \\
\Big\downarrow \cong & & \Big\downarrow \cong & & \\
H_{dR}(M)_{\mathbb{C}} \times H_{dR}(M^*(1))_{\mathbb{C}} & \longrightarrow & H_{dR}(\mathbb{Q}(1))_{\mathbb{C}} & \xrightarrow{\tau} & \mathbb{R}
\end{array}$$

where the vertical isomorphisms are given by (53) for M,$M^*(1)$ and $\mathbb{Q}(1)$, and τ is the $\mathbb{R}$-linear splitting of the inclusion

$$\mathbb{R} = H_{dR}(\mathbb{Q}(1))_{\mathbb{R}} \subset H_{dR}(\mathbb{Q}(1))_{\mathbb{C}} = \mathbb{C}$$

with kernel $H_B(\mathbb{Q}(1))_{\mathbb{R}} = 2\pi i \cdot \mathbb{R}$. One verifies that $H_B(M)_{\mathbb{R}}$ is the orthogonal complement of $H_B(M^*(1))_{\mathbb{R}}$ under this pairing, and it is also well known that $F^0 H_{dR}(M)$ is the orthogonal complement of $F^0 H_{dR}(M^*(1))$. Hence we obtain the isomorphisms β_i. Viewing the rows of (94) as complexes concentrated in degrees zero and one, an easy inspection shows that all rows are quasi-isomorphic. The same is then true for the diagram $(94)^+$ whose top (resp. bottom) row coincides with $R\Gamma_{\mathcal{D}}(K, M)$ (resp. $R\Gamma_{\mathcal{D}}(K, M^*(1))^*[-1]$). This proves the Lemma. □

We shall next establish existence of the following commutative diagram in $V(A_{\mathbb{R}})$

(95)

$$\begin{array}{ccc}
A_{\mathbb{R}} \otimes_A \left(\Xi(M) \boxtimes \Xi(M^*(1))^*\right) & \xrightarrow{A_{\mathbb{R}} \otimes_A \vartheta^{PD}} & A_{\mathbb{R}} \otimes_A \Xi^{loc}(M) \\
{\scriptstyle A_{\mathbb{R}} \otimes_A (89)} \Big\downarrow {\scriptstyle A_{\mathbb{R}} \otimes_A (90)} & & \| \\
[H_B(M)_{\mathbb{R}}^+]^{-1} \boxtimes [(H_{dR}(M)/F^0)_{\mathbb{R}}] \boxtimes & & \\
 & \xrightarrow{\gamma_1} & [H_B(M)_{\mathbb{R}}]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}] \\
[H_B(M^*(1))_{\mathbb{R}}^{*,+}]^{-1} \boxtimes [(H_{dR}(M^*(1))/F^0)_{\mathbb{R}}^*] & & \\
\Big\downarrow {\scriptstyle [\beta_1],[\beta_2]} & & {\scriptstyle (55)} \Big\downarrow {\scriptstyle (56)} \\
[H_B(M)_{\mathbb{R}}^+]^{-1} \boxtimes [(H_{dR}(M)/F^0)_{\mathbb{R}}] \boxtimes & & \\
 & \longrightarrow & [(H_B(M)_{\mathbb{C}})^+]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}] \\
[(H_B(M)_{\mathbb{C}})^+ / H_B(M)_{\mathbb{R}}^+]^{-1} \boxtimes [F^0 H_{dR}(M)_{\mathbb{R}}] & & \\
{\scriptstyle \beta_3} \Big\downarrow & & {\scriptstyle \epsilon_B} \Big\downarrow {\scriptstyle \epsilon_{dR}} \\
[F^0 H_{dR}(M)_{\mathbb{R}}]^{-1} \boxtimes [H_B(M)_{\mathbb{R}}^+]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}] & & \\
 & \xrightarrow{\gamma_2} & [(H_B(M)_{\mathbb{C}})^+]^{-1} \boxtimes [H_{dR}(M)_{\mathbb{R}}] \\
\boxtimes [F^0 H_{dR}(M)_{\mathbb{R}}] \boxtimes [H_B(M)_{\mathbb{R}}^+] \boxtimes [(H_B(M)_{\mathbb{C}})^+]^{-1} & & \\
{\scriptstyle \beta_4} \Big\downarrow & & {\scriptstyle (54)} \Big\downarrow \\
\mathbf{1}_{V(A_{\mathbb{R}})} & = & \mathbf{1}_{V(A_{\mathbb{R}})}.
\end{array}$$

The first square in (95) is commutative by the definition of ϑ^{PD} if we define γ_1 to be induced by equations $A_{\mathbb{R}} \otimes_A (91)$, $A_{\mathbb{R}} \otimes_A (92)$ and $A_{\mathbb{R}} \otimes_A (69)$. The second square in (95) is the $\boxtimes$-product of the two commutative squares

$$
\begin{array}{ccc}
\begin{array}{c}[H_B(M)^+_{\mathbb{R}}]^{-1}\boxtimes\\ [H_B(M^*(1))^{*,+}_{\mathbb{R}}]^{-1}\end{array} & \xrightarrow{A_{\mathbb{R}}\otimes\{(91),(92)\}} & [H_B(M)_{\mathbb{R}}]^{-1} \\
\big\downarrow{\scriptstyle [\beta_2]} & & {\scriptstyle (55)}\big\downarrow{\scriptstyle (56)} \\
\begin{array}{c}[H_B(M)^+_{\mathbb{R}}]^{-1}\boxtimes\\ [(H_B(M)_{\mathbb{C}})^+/H_B(M)^+_{\mathbb{R}}]^{-1}\end{array} & \longrightarrow & [(H_B(M)_{\mathbb{C}})^+]^{-1}
\end{array}
\tag{96}
$$

and

$$
\begin{array}{ccc}
\begin{array}{c}[(H_{dR}(M)/F^0)_{\mathbb{R}}]\boxtimes\\ [(H_{dR}(M^*(1))/F^0)^*_{\mathbb{R}}]\end{array} & \xrightarrow{A_{\mathbb{R}}\otimes_A(69)} & [H_{dR}(M)_{\mathbb{R}}] \\
\big\downarrow{\scriptstyle [\beta_1]} & & \| \\
\begin{array}{c}[(H_{dR}(M)/F^0)_{\mathbb{R}}]\boxtimes\\ [F^0H_{dR}(M)_{\mathbb{R}}]\end{array} & \longrightarrow & [H_{dR}(M)_{\mathbb{R}}].
\end{array}
$$

In both of those squares the bottom horizontal maps are induced by the obvious short exact sequences. The square (96) commutes since the identification $H_B(M)^-_{\mathbb{R}}\cong H_B(M)^-\otimes(2\pi i)^{-1}\mathbb{R}$ used in $A_{\mathbb{R}}\otimes_A(91)$ is inverse to that used in (56).

Concerning the third square in (95), the map β_3 is the $\boxtimes$-product of the isomorphism

$$
\begin{aligned}
[R\Gamma_{\mathcal{D}}(K,M)]^{-1} = [H_B(M)^+_{\mathbb{R}}]^{-1}\boxtimes[(H_{dR}(M)/F^0)_{\mathbb{R}}] &\xrightarrow{\sim} \\
[F^0H_{dR}(M)_{\mathbb{R}}]^{-1}\boxtimes[H_B(M)^+_{\mathbb{R}}]^{-1}&\boxtimes[H_{dR}(M)_{\mathbb{R}}]
\end{aligned}
$$

induced by the quasi-isomorphism between the first and second row in $(94)^+$ and a similar isomorphism induced by the quasi-isomorphism between the fourth and third row in $(94)^+$. The map γ_2 is induced by canceling mutually inverse terms in the first and second row of its source term. Effectively then, the third square in (95) is the $\boxtimes$-product of the squares

$$
\begin{array}{ccc}
\begin{array}{c}\mathbf{1}_{V(A_{\mathbb{R}})}\boxtimes\\ [F^0H_{dR}(M)_{\mathbb{R}}]\end{array} & \longrightarrow & [F^0H_{dR}(M)_{\mathbb{R}}] \\
\big\downarrow & & {\scriptstyle \epsilon_{dR}}\big\downarrow \\
\begin{array}{c}[F^0H_{dR}(M)_{\mathbb{R}}]^{-1}\boxtimes[F^0H_{dR}(M)_{\mathbb{R}}]\\ [F^0H_{dR}(M)_{\mathbb{R}}]\end{array} & \longrightarrow & \mathbf{1}_{V(A_{\mathbb{R}})}\boxtimes[F^0H_{dR}(M)_{\mathbb{R}}]
\end{array}
\tag{97}
$$

and

$$
\begin{tikzcd}
{[H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes \mathbf{1}_{V(A_{\mathbb{R}})}} \arrow[r] \arrow[d] & {[(H_B(M)^+_{\mathbb{R}}]^{-1}} \arrow[d, "\epsilon_B"'] \\
{[H_B(M)^+_{\mathbb{R}}]^{-1} \boxtimes [H_B(M)^+_{\mathbb{R}}] \boxtimes [H_B(M)^+_{\mathbb{R}}]^{-1}} \arrow[r] & {\mathbf{1}_{V(A_{\mathbb{R}})} \boxtimes [(H_B(M)^+_{\mathbb{R}}]^{-1}}
\end{tikzcd}
\tag{98}
$$

with (the identity maps on) $[(H_{dR}(M)/F^0)_{\mathbb{R}}]$ and $[(H_B(M)_{\mathbb{C}})^+/H_B(M)^+_{\mathbb{R}}]^{-1}$. In both diagrams (97) and (98) the left and bottom arrows are respectively induced by the two different ways to parenthesize the lower left term, and we have written this term so that the positions of its factors roughly match with their position in (95). We refer to [17][(4.1.1)] for the commutativity of (97) and (98), with the particular correcting factors ϵ_{dR} and ϵ_B given in [17][(4.9)]. Finally, the map β_4 in (95) is induced by the quasi-isomorphism between the second and third row in $(94)^+$, and the commutativity of the bottom square in (95) simply follows from the identity $(53)^+ = (54)$.
We now observe that the right hand vertical map in (95) coincides with ϑ_∞^{loc} by definition, and that the left hand vertical map

$$
\begin{aligned}
&A_{\mathbb{R}} \otimes_A \left(\Xi(M) \boxtimes \Xi(M^*(1))^*\right) \\
&\qquad \xrightarrow{\sim} [R\Gamma_{\mathcal{D}}(K,M)]^{-1} \boxtimes [R\Gamma_{\mathcal{D}}(K,M^*(1))^*[-1]] \xrightarrow{(94)^+} \mathbf{1}_{V(A_{\mathbb{R}})}
\end{aligned}
$$

does indeed coincide with $\vartheta_\infty(M) \boxtimes \vartheta_\infty(M^*(1))^*$ (using the second condition in Conjecture 1). Lemma 17a) then follows from the commutativity of diagram (95).

We now consider claim b) of Lemma 17. To further shorten notation we henceforth write $R\Gamma_?$ for $R\Gamma_?(\mathcal{O}_{K,S_p}, V_p)$, $R\Gamma_f$ for $R\Gamma_f(K,V_p)$, $\tilde{R}\Gamma_c$ for $\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p}, V_p)$ and $L_?(S)$ for $\bigoplus_{v\in S} R\Gamma_?(K_v,V_p)$. In addition we use $R\Gamma_?^*$ as an abbreviation for $R\Gamma_?(\mathcal{O}_{K,S_p}, V_p^*(1))^*$, and also introduce similar abbreviations $R\Gamma_f^*, \tilde{R}\Gamma_c^*$ and $L_?(S)^*$.

We shall first establish the existence of the following commutative diagram in $V(A_p)$

(99)

$$
\begin{array}{ccc}
A_p \otimes_A (\Xi(M) \boxtimes \Xi(M^*(1))^*) & \xrightarrow{A_p \otimes_A \vartheta^{PD}} & A_p \otimes_A \Xi^{loc}(M) \\
{\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \theta_p}\downarrow \\
\begin{array}{c} [R\Gamma_f] \boxtimes [L_f(S_{p,f})]^{-1} \boxtimes [L(S_\infty)]^{-1} \\ \boxtimes [R\Gamma_f^*] \boxtimes [L_f(S_{p,f})^*]^{-1} \boxtimes [L(S_\infty)^*]^{-1} \end{array} & \xrightarrow{\beta_1} & \begin{array}{c} [L(S_{p,f})]^{-1} \boxtimes [\,\mathrm{Ind}\, V_p]^{-1} \\ =: \Lambda(S, V_p) \end{array} \\
{\scriptstyle (26)_{\mathrm{bot}} \boxtimes}\downarrow{\scriptstyle (100)_{\mathrm{bot}} \circ [\alpha_2]} & & \| \\
\begin{array}{c} [L_{/f}(S_{p,f})]^{-1} \boxtimes [R\Gamma] \boxtimes [L_f(S_{p,f})]^{-1} \boxtimes [L(S_\infty)]^{-1} \\ \boxtimes [L_f(S_{p,f})^*] \boxtimes [{}_1\tilde{R}\Gamma_c^*] \boxtimes [L_f(S_{p,f})^*]^{-1} \boxtimes [L(S_\infty)^*]^{-1} \end{array} & \xrightarrow{\beta_2} & [L(S_{p,f})]^{-1} \boxtimes [\,\mathrm{Ind}\, V_p]^{-1} \\
{\scriptstyle (18)_{v \in S_p}}\downarrow & & {\scriptstyle (76)\,(77)}\downarrow{\scriptstyle \epsilon(S,p)} \\
[L(S_p)]^{-1} \boxtimes [R\Gamma] \boxtimes [{}_1\tilde{R}\Gamma_c^*[-4]] \boxtimes [L(S_\infty)^*]^{-1} & \xrightarrow{1\ \mathrm{AV}_{\mathrm{Triv}}} & [L(S_p)]^{-1} \boxtimes [L(S_\infty)^*]^{-1} \\
{\scriptstyle (26)_{\mathrm{hor}} \boxtimes}\downarrow{\scriptstyle (73)_{\mathrm{left}}} & & {\scriptstyle (73)_{\mathrm{vert}}}\downarrow{\scriptstyle \circ[\lambda]\circ[q]} \\
[{}_1R\Gamma_c] \boxtimes [{}_1R\Gamma_c^*[-4]] & \xrightarrow{(73)_{\mathrm{bot}}} & [{}_2C(K, V_p)] \\
{\scriptstyle \alpha_3}\downarrow & & {\scriptstyle (72)}\downarrow \\
[R\Gamma_c] \boxtimes [R\Gamma_c^*[-4]] & \xrightarrow{\vartheta_p^{\mathrm{AV}}} & A_p \otimes_{\mathfrak{A}_p} \Lambda_p(S, T_p).
\end{array}
$$

The maps $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ which occur in this diagram will all be defined in the course of our proof that the diagram is commutative.

Concerning the first square, we recall that the construction of ϑ^{PD} involves the set of equations

$$\vartheta^{PD}\colon \quad (29) \quad (29)^* \quad \mathrm{AV}^{\mathrm{mot}} \quad (69) \quad (91) \quad (92)$$

where here we denote by $\mathrm{AV}^{\mathrm{mot}}$ the collection of four isomorphisms

$$[H_f^i(K,M)]^{-1} \boxtimes [H_f^i(K,M)] \cong \mathbf{1},\ [H_f^i(K,M^*(1))^*]^{-1} \boxtimes [H_f^i(K,M^*(1))^*] \cong \mathbf{1}$$

with $i \in \{0,1\}$. The construction of θ_p involves the following equations (with v running through the set $S_{p,f}$):

$$
\begin{array}{lllllllll}
\theta_p: & (68) & (69) & (18)_v & \mathrm{AV}_{f,v} & (19)_{v\nmid p} & (22)_{v|p} & (23) & (24)_v \\
 & & & & & (19)^*_{v\nmid p} & (22)^*_{v|p} & (23)^* & (24)^*_v
\end{array}
$$

Note now that when listing all of the equations that are involved in the composition $\kappa := \theta_p \circ (A_p \otimes_A \vartheta^{PD})$, the equations (91) and (92) are transformed into (76) and (77) respectively as they are 'conjugated' by (68), and that (68) is in turn equivalent to a combination of $(28)^+$ and $(28)^{*,+}$ for $v \in S_\infty$. The isomorphism $\mathrm{AV}^{\mathrm{mot}}$ is a combination of AV_f, (27) and $(27)^*$. Finally note that the isomorphism induced by (69) is used in ϑ^{PD} whilst its inverse is used in

θ_p, so that (69) does not in fact occur in the composition κ. In summary, we find that the isomorphism κ involves the sets of equations

$$\begin{array}{llllllll}\alpha_1: & (27) & (28)_{v|\infty} & (29) & (19)_{v\nmid p} & (22)_{v|p} & (23) & (24)_v \\ & (27)^* & (28)^*_{v|\infty} & (29)^* & (19)^*_{v\nmid p} & (22)^*_{v|p} & (23)^* & (24)^*_v\end{array}$$

and

$$\beta_1: \quad (18)_v \quad \mathrm{AV}_f \quad \mathrm{AV}_{f,v} \quad (76) \quad (77).$$

If we then define α_1 (resp. β_1) as the isomorphism induced by the set of equations carrying the label α_1 (resp. β_1) we have $\kappa = \beta_1 \circ \alpha_1$, i.e. commutativity of the first square in (99).

In order to consider the second square in (99) we first define the map AV_f.

Lemma 19. *There exists a commutative diagram of true triangles*

(100)
$$\begin{array}{ccccc}\bigoplus_{v\in S_{p,f}} R\Gamma_{/f}(K_v,V_p)[-1] & \longrightarrow & R\Gamma_f(K,V_p) & \longrightarrow & R\Gamma(\mathcal{O}_{K,S_p},V_p) \\ {\scriptstyle \oplus \mathrm{AV}_{f,v}[-1]}\downarrow & & {\scriptstyle \mathrm{AV}_f}\downarrow & & {\scriptstyle {}_1\mathrm{AV}}\downarrow \\ \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v,V_p^*(1))^*[-3] & \longrightarrow & {}_1R\Gamma_f(K,V_p^*(1))^*[-3] & \longrightarrow & {}_1\tilde{R}\Gamma_c(\mathcal{O}_{K,S_p},V_p^*(1))^*[-3],\end{array}$$

in which the upper row coincides with $(26)_{\mathrm{bot}}$*, all of the vertical maps are quasi-isomorphisms, and there exists a natural quasi-isomorphism*

$$R\Gamma_f(K,V_p^*(1))^*[-3] \xrightarrow{\alpha_2} {}_1R\Gamma_f(K,V_p^*(1))^*[-3].$$

Proof. In view of Lemma 12a) it suffices to show that the mapping cone of the lower composite map in Lemma 12a) is naturally quasi-isomorphic to $R\Gamma_f(K,V_p^*(1))^*[-2]$. Indeed, if this is true, then (100) is simply induced by taking the mapping cones of the composite horizontal maps in the diagram of Lemma 12a).

We observe that there is a map from diagram (25) into the diagram

$$R\Gamma(\mathcal{O}_{K,S_p},V_p) \to \bigoplus_{v\in S_{p,f}} R\Gamma(K_v,V_p) \hookleftarrow \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v,V_p) \tag{101}$$

and hence an induced map of the true nine term diagram (26) into the true nine term diagram that is induced by (101). In particular on the central columns we obtain a map which coincides with the map between the second and third column in the following true nine term diagram

$$
\begin{array}{ccccc}
\bigoplus_{v\in S_\infty} R\Gamma(K_v,V_p)[-1] & \longrightarrow & \bigoplus_{v\in S_p} R\Gamma_f(K_v,V_p)[-1] & \longrightarrow & \bigoplus_{v\in S_{p,f}} R\Gamma_f(K_v,V_p)[-1] \\
\| & & \big\downarrow & & \big\downarrow \\
\bigoplus_{v\in S_\infty} R\Gamma(K_v,V_p)[-1] & \longrightarrow & {}_1R\Gamma_c(\mathcal{O}_{K,S_p},V_p) & \longrightarrow & {}_1\tilde{R\Gamma}_c(\mathcal{O}_{K,S_p},V_p) \\
 & & \big\downarrow & & \big\downarrow \\
 & & R\Gamma_f(K,V_p) & = & R\Gamma_f(K,V_p).
\end{array} \tag{102}
$$

Upon replacing V_p by $V_p^*(1)$, dualizing the third column of this diagram and shifting by [-4] we obtain a true triangle

$$
R\Gamma_f^*[-4] \to {}_1\tilde{R\Gamma}_c^*[-4] \to L_f(S_{p,f})^*[-3]. \tag{103}
$$

The map α_2 then arises as the map q of Lemma 13 when the latter is applied to the triangle (103). $\square$

The diagram (100) induces a commutative diagram

$$
\begin{array}{ccc}
[R\Gamma_f]\boxtimes[{}_1R\Gamma_f^*] & \xrightarrow{[\,\mathrm{AV}_f\,]_{\mathrm{Triv}}} & \mathbf{1}_{V(A_p)} \\
{\scriptstyle (26)_{\mathrm{bot}}\boxtimes}\big\downarrow{\scriptstyle (100)_{\mathrm{bot}}} & & \| \\
\begin{array}{c}[L_{/f}(S_{p,f})]^{-1}\boxtimes[R\Gamma] \\ \boxtimes[L_f(S_{p,f})^*]\boxtimes[{}_1\tilde{R\Gamma}_c^*]\end{array} & \xrightarrow{[\oplus\mathrm{AV}_{f,v}\,]_{\mathrm{Triv}}\boxtimes[{}_1\,\mathrm{AV}\,]_{\mathrm{Triv}}} & \mathbf{1}_{V(A_p)}.
\end{array} \tag{104}
$$

The second square in (99) is obtained by taking the $\boxtimes$-product of the rows in (104) with the isomorphism

$$
\begin{array}{c}[L_f(S_{p,f})]^{-1}\boxtimes[L(S_\infty)]^{-1} \\ \boxtimes[L_f(S_{p,f})^*]^{-1}\boxtimes[L(S_\infty)^*]^{-1}\end{array} \xrightarrow{\beta_2'} [L(S_{p,f})]^{-1}\boxtimes[\,\mathrm{Ind}\,V_p]^{-1},
$$

where here β_2' involves the equations

$$
\beta_2' : \quad (18)_v \quad \mathrm{AV}_{f,v} \quad (76) \quad (77).
$$

In order to establish the third square in (99) we begin with the commutative diagram

$$
\begin{array}{ccc}
[L(S_\infty)]^{-1}\boxtimes[L(S_\infty)^*]^{-1} & \xrightarrow{(76)\,(77)} & [\,\mathrm{Ind}\,V_p] \\
\big\downarrow & & {\scriptstyle (76)\,(77)}\big\downarrow \\
[L(S_\infty)]^{-1}\boxtimes[L(S_\infty)^*]^{-1} & = & [L(S_\infty)]^{-1}\boxtimes[L(S_\infty)^*]^{-1}.
\end{array} \tag{105}
$$

Next we consider the diagram

$$\begin{array}{ccc} \begin{array}{c}[L_{/f}(S_{p,f})]^{-1} \boxtimes [L_f(S_{p,f})]^{-1} \\ \boxtimes [L_f(S_{p,f})^*] \boxtimes [L_f(S_{p,f})^*]^{-1}\end{array} & \xrightarrow{\beta_2''} & [L(S_{p,f})]^{-1} \\ {\scriptstyle [\alpha_2]}\downarrow & & \downarrow{\scriptstyle \epsilon(S,p)} \\ [L(S_{p,f})]^{-1} & = & [L(S_{p,f})]^{-1}. \end{array} \tag{106}$$

Here β_2'' is induced by applying $\mathrm{AV}_{f,v}$, $(18)_v$ to the two right hand terms in the top left item and by trivializing the two left hand terms via another application of $\mathrm{AV}_{f,v}$. For the map α_2 on the other hand one applies $(18)_v$ to the upper two terms and trivializes the lower two terms. The commutativity of (106) then follows from [17, (4.1.1)] where the particular correcting factor $\epsilon(S,p)$ is as computed in [loc. cit., 4.9]. Indeed, upon writing diagram [loc. cit., (4.1.1)] with $X = [L_{/f}(S_{p,f})]$, taking $\boxtimes$-product with $[L_f(S_{p,f})]^{-1}$ and using $\mathrm{AV}_{f,v}$ twice we obtain (106). The third square in (99) is obtained by taking the $\boxtimes$-product of the diagrams (105) and (106) and then taking the $\boxtimes$-product of both rows of the resulting diagram with the isomorphism

$$[R\Gamma] \boxtimes [{}_1\tilde{R}\Gamma_c^*] \xrightarrow{{}_1\mathrm{AV}_{\mathrm{Triv}}} \mathbf{1}_{V(A_p)}.$$

We now consider the fourth square in (99). We observe first that there is a commutative diagram in $V(A_p)$

$$\begin{array}{ccc} [L(S_p)]^{-1} \boxtimes [R\Gamma] \boxtimes [R\Gamma]^{-1} \boxtimes [L(S_\infty)^*]^{-1} & \xrightarrow{\tau} & [L(S_p)]^{-1} \boxtimes [L(S_\infty)^*]^{-1} \\ {\scriptstyle (26)_{\mathrm{hor}}}\downarrow & & \downarrow{\scriptstyle [q]} \\ [{}_1R\Gamma_c] \boxtimes [R\Gamma]^{-1} \boxtimes [L(S_\infty)^*]^{-1} & \xrightarrow{(74)_{\mathrm{top}}} & [{}_1L(S_p)]^{-1} \boxtimes [L(S_\infty)^*]^{-1} \\ {\scriptstyle [{}_1\mathrm{AV}]}\downarrow & & \downarrow{\scriptstyle [\lambda]} \\ [{}_1R\Gamma_c] \boxtimes [{}_1\tilde{R}\Gamma_c^*[-4]] \boxtimes [L(S_\infty)^*]^{-1} & \xrightarrow{(73)_{\mathrm{hor}}} & [{}_2L(S_p)]^{-1} \boxtimes [L(S_\infty)^*]^{-1} \\ \downarrow{\scriptstyle (73)_{\mathrm{left}}} & & {\scriptstyle (73)_{\mathrm{vert}}}\downarrow \\ [{}_1R\Gamma_c] \boxtimes [{}_1R\Gamma_c^*[-4]] & \xrightarrow{(73)_{\mathrm{bot}}} & [{}_2C(K,V_p)] \end{array}$$

in which the lower square is induced by the true nine term diagram (73), the central square by (74) and the upper square by Lemma 13. The map τ is the canonical isomorphism $[R\Gamma] \boxtimes [R\Gamma]^{-1} \cong \mathbf{1}_{V(A_p)}$. Upon replacing both occurrences of $[R\Gamma]^{-1}$ in this diagram by $[{}_1\tilde{R}\Gamma_c^*[-4]]$ and τ by ${}_1\mathrm{AV}_{\mathrm{Triv}}$, the upper square is still commutative and the resulting total square gives the fourth square in (99).

Finally, in the bottom square in (99) the map α_3 is defined by replacing complexes by their versions indexed by 1 in exactly the same order as in (72). The commutativity of this square is then clear.

It is clear that the right vertical map of (99) is equal to ϑ_p^{loc}. Hence, having now established the commutativity of (99), we shall prove Lemma 17b) if we can show that the left vertical map in (99) is equal to $\vartheta_p(M) \boxtimes \vartheta_p(M^*(1))^*$. In view of the definition of α_1 it therefore suffices for us to show that

$$(26)_{\mathrm{vert}} = (26)_{\mathrm{bot}} \circ (18)_{v \in S_p} \circ (26)_{\mathrm{hor}} \tag{107}$$

and also

$$(26)^*_{\mathrm{vert}} = [\alpha_2] \circ (100)_{\mathrm{bot}} \circ (73)_{\mathrm{left}}. \tag{108}$$

The identity (107) coincides with the identity in $V(A_p)$ induced by the true nine term diagram (26) since $(26)_{\mathrm{left}}$ coincides with the sum of (18) over $v \in S_p$. On the other hand, the identity (108) is a consequence of the following commutative diagram

$$\begin{tikzcd}
{[{}_1R\Gamma_c^*]} \arrow[r, "(26)^*_{\mathrm{vert}}"] \arrow[d, "(73)_{\mathrm{left}}"'] & {[R\Gamma_f^*] \boxtimes [L_f(S_p)^*]^{-1}} \arrow[d, equal] \\
{[{}_1\tilde{R\Gamma}_c^*] \boxtimes [L(S_\infty)^*]^{-1}} \arrow[r, "(103)"] \arrow[d, "(100)_{\mathrm{bot}}"'] & {[R\Gamma_f^*] \boxtimes [L_f(S_{p,f})^*]^{-1} \boxtimes [L(S_\infty)^*]^{-1}} \arrow[d, equal] \\
{[{}_1R\Gamma_f^*] \boxtimes [L_f(S_{p,f})^*]^{-1} \boxtimes [L(S_\infty)^*]^{-1}} \arrow[r, "{[\alpha_2]}"] & {[R\Gamma_f^*] \boxtimes [L_f(S_{p,f})^*]^{-1} \boxtimes [L(S_\infty)^*]^{-1}}
\end{tikzcd}$$

where here the first square is induced by the dual of diagram (102) with V_p replaced by $V_p^*(1)$ and the second square results from applying Lemma 13 to (103) and then taking the $\boxtimes$-product of each vertex of the resulting square with $[L_f(S_{p,f})^*]^{-1}$. $\square$

In view of Corollary 1 we are naturally led to make the following

Conjecture 8. $T\Omega^{loc}(M, \mathfrak{A}) = 0$.

This conjecture is itself of some independent interest, and will be considered in greater detail elsewhere. We therefore restrict ourselves here to a few brief remarks concerning the Galois case. We fix a finite Galois extension of number fields L/K and set $G := \mathrm{Gal}(L/K)$.

Remark 18. In this remark we assume that G is abelian. If M is any motive which is defined over K, then Conjecture 8 for the pair $(h^0(\mathrm{Spec}\,(L)) \otimes M, \mathbb{Z}[G])$ can be interpreted in terms of the 'local epsilon conjecture' formulated by Kato in [29]. In particular, [loc. cit., Th. 4.1] can be used to verify (at least modulo the 'sign ambiguities' discussed in Remark 9 of §4.3) that if $K = \mathbb{Q}$, then for all integers r Conjecture 8 is valid for the pair $(h^0(\mathrm{Spec}\,(L))(r), \mathbb{Z}[\frac{1}{2}][G])$. (Details of this deduction will be given elsewhere.) When combined with Corollary 1 and the main result of [12] (cf. Remark 10 in §4.3) this implies (again modulo the same sign ambiguities as above) that $T\Omega(h^0(\mathrm{Spec}\,(L))(r), \mathbb{Z}[\frac{1}{2}][G]) = 0$ for all integers r.

Remark 19. In [7] it is shown that (for any G)

$$\delta^0_{\mathbb{Z}[G],\mathbb{R}}(T\Omega^{loc}(h^0(\mathrm{Spec}(L))(1), \mathbb{Z}[G])) = \Omega(L/K, 2) - w(L/K)$$

where here $\Omega(L/K, 2)$ and $w(L/K)$ are respectively equal to the 'second Chinburg invariant' and the 'Cassou-Noguès-Fröhlich root number class' as defined in [14]. This implies that Conjecture 8 is compatible with the conjectures formulated by Chinburg in loc. cit. For a further discussion of connections between Theorem 5.3 and the extensive existing theory concerning the conjectures of Chinburg, the reader can consult [11].

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A Generalization of Mumford's Geometric Invariant Theory

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Abstract. We generalize Mumford's construction of good quotients for reductive group actions. Replacing a single linearized invertible sheaf with a certain group of sheaves, we obtain a Geometric Invariant Theory producing not only the quasiprojective quotient spaces, but more generally all divisorial ones. As an application, we characterize in terms of the Weyl group of a maximal torus, when a proper reductive group action on a smooth complex variety admits an algebraic variety as orbit space.

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Introduction

Let the reductive group G act regularly on a variety X. In [19], Mumford associates to every G-linearized invertible sheaf $\mathcal{L}$ on X a set $X^{ss}(\mathcal{L})$ of semistable points. He proves that there is a *good quotient* $p\colon X^{ss}(\mathcal{L}) \to X^{ss}(\mathcal{L})/\!\!/G$, that means p is a G-invariant affine regular map and the structure sheaf of the quotient space is the sheaf of invariants.

Mumford's theory is designed for the quasiprojective category: His quotient spaces are always quasiprojective. Conversely, for connected G and smooth X, if a G-invariant open set $U \subset X$ has a good quotient $U \to U/\!\!/G$ with $U/\!\!/G$ quasiprojective, then U is a saturated subset of a set $X^{ss}(\mathcal{L})$ for some G-linearized invertible sheaf $\mathcal{L}$ on X.

However, there frequently occur good quotients with a non quasiprojective quotient space; even if X is quasiaffine and G is a one dimensional torus, see e.g. [2]. For $X = \mathbb{P}_n$ or X a vector space with linear G-action, the situation is reasonably well understood, see [8] and [9]. But for general X, the picture is still far from being complete.

The purpose of this article is to present a general theory for good quotients with so called *divisorial* quotient spaces. Recall from [12] that an irreducible variety Y is divisorial if every $y \in Y$ admits an affine neighbourhood of the form $Y \setminus \mathrm{Supp}(D)$ with an effective Cartier divisor D on Y. This is a considerable generalization of quasiprojectivity. For example, all smooth varieties are divisorial.

Our approach to divisorial good quotient spaces is to replace Mumford's single invertible sheaf $\mathcal{L}$ with a free finitely generated group Λ of Cartier divisors on X. Then a G-linearization of such a group Λ is a certain G-sheaf structure on the graded $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to Λ; for the precise definitions see Section 1.

In Section 2, we associate to every G-linearized group $\Lambda \subset \mathrm{CDiv}(X)$ a set $X^{ss}(\Lambda) \subset X$ of semistable points and a set $X^{s}(\Lambda) \subset X^{ss}(\Lambda)$ of stable points. Theorem 3.1 generalizes Mumford's result on existence of good quotients:

Theorem 1. *For any G-linearized group Λ of Cartier divisors, there is a good quotient $X^{ss}(\Lambda) \to X^{ss}(\Lambda)/\!\!/G$ with a divisorial quotient space $X^{ss}(\Lambda)/\!\!/G$.*

We note here that our quotient spaces are allowed to be non separated; see also the brief discussion at the end of Section 3. As in the classical situation, the restriction of the above quotient map to the set of stable points separates the orbits. In Theorem 4.1, we give a converse of the above result:

Theorem 2. *For $\mathbb{Q}$-factorial, e.g. smooth, X every G-invariant open subset $U \subset X$ with a good quotient such that $U/\!\!/G$ is divisorial occurs as a saturated subset of a set of semistable points $X^{ss}(\Lambda)$.*

As an application, we discuss actions of connected reductive groups G on normal complex varieties X. The starting point is the reduction theorem of A. Białynicki-Birula and J. Święcicka [6, Theorem 5.1]: If some maximal torus $T \subset G$ admits a good quotient $X \to X/\!\!/T$, then there is a "good quotient" for the action of G on X in the category of algebraic spaces.

Examples show that in general, the quotient space really drops out of the category of algebraic varieties, see [7, page 15]. So, there arises a natural question: When there is a good quotient $X \to X/\!\!/G$ in the category of algebraic varieties?

Our answers to this question are formulated in terms of the normalizer $N(T)$ of a maximal torus $T \subset G$. Recall that the connected component of the unit element of $N(T)$ is just T; in other words $N(T)/T$ is finite. The first result is the following, see Theorem 5.1:

Theorem 3. *Let G be a connected reductive group, and let X be a normal complex G-variety. Then the following statements are equivalent:*

i) *There is a good quotient $X \to X/\!\!/G$ with a divisorial prevariety $X/\!\!/G$.*
ii) *There is a good quotient $X \to X/\!\!/N(T)$ with a divisorial prevariety $X/\!\!/N(T)$.*

Moreover, if one of these statements holds with a separated quotient space then so does the other.

We specialize to proper G-actions. It is an easy consequence of the reduction theorem [6, Theorem 5.1] that such an action always admits a "geometric quotient" in the category of algebraic spaces. Fundamental results of Kollár [18], Keel and Mori [15] extend this fact to a more general framework.

In our second result, Theorem 5.2, the words geometric quotient refer to a good quotient (in the category of algebraic varieties) that separates orbits:

Theorem 4. *Suppose that a connected reductive group G acts properly on a $\mathbb{Q}$-factorial complex variety X. Then the following statements are equivalent:*

i) *There exists a geometric quotient $X \to X/G$.*
ii) *There exists a geometric quotient $X \to X/N(T)$.*

Moreover, if one of these statements holds, then the quotient spaces X/G and $X/N(T)$ are separated and $\mathbb{Q}$-factorial.

So, for proper G-actions on $\mathbb{Q}$-factorial varieties, the answer to the above question is encoded in an action of the Weyl group $W := N(T)/T$: A geometric quotient $X \to X/G$ exists in the category of algebraic varieties if and only if the induced action of W on X/T admits an algebraic variety as orbit space.

1. G-linearization and ample groups

Throughout the whole article, we work in the category of algebraic prevarieties over an algebraically closed field $\mathbb{K}$. In particular, the word point refers to a closed point. First we fix the notions concerning group actions and quotients.

In this section, G denotes a linear algebraic group, and X is an irreducible G-prevariety, that means X is an irreducible (possibly non separated) prevariety (over $\mathbb{K}$) together with a regular group action $\sigma\colon G \times X \to X$.

For reductive G, a *good quotient* of the G-prevariety X is a G-invariant affine regular map $p\colon X \to X/\!\!/G$ of prevarieties such that $p^*\colon \mathcal{O}_{X/\!\!/G} \to p_*(\mathcal{O}_X)^G$ is an isomorphism. By a *geometric quotient* we mean a good quotient that separates orbits. Geometric quotient spaces are denoted by X/G.

Remark 1.1. [22, Theorem 1.1]. Let $p\colon X \to X/\!\!/G$ be a good quotient for an action of a reductive group G. Then we have:

i) For every G-invariant closed set $A \subset X$ the image $p(A) \subset X/\!\!/G$ is closed.
ii) If $A, B \subset X$ are closed G-invariant subsets, then $p(A \cap B)$ equals $p(A) \cap p(B)$.
iii) Each fibre $p^{-1}(y)$ contains exactly one closed G-orbit.
iv) Every G-invariant regular map $X \to X'$ factors uniquely through p.

Now we introduce the basic concepts used in this article, compare also [13] and [14]. When we speak of a subgroup of the group $\mathrm{CDiv}(X)$ of Cartier divisors of X, we always mean a finitely generated free subgroup.

Let $\Lambda \subset \mathrm{CDiv}(X)$ be such a subgroup. Denoting by $\mathcal{A}_D := \mathcal{O}_X(D)$ the sheaf of sections of $D \in \Lambda$, we obtain a Λ-graded $\mathcal{O}_X$-algebra:

$$\mathcal{A} := \bigoplus_{D \in \Lambda} \mathcal{A}_D.$$

The following notion extends Mumford's concept of a G-linearized invertible sheaf to groups of divisors:

Definition 1.2. Fix the canonical G-sheaf structure $(g \cdot f)(x) := f(g^{-1} \cdot x)$ on the structure sheaf $\mathcal{O}_X$.

i) A *G-linearization* of the group Λ is a graded G-sheaf structure on the Λ-graded $\mathcal{O}_X$-algebra $\mathcal{A}$ such that the representation of G on $\mathcal{A}(U)$ is rational for every G-invariant open subset $U \subset X$.
ii) A *strong G-linearization* of the group Λ is a G-linearization of Λ such that on each homogeneous component $\mathcal{A}_D$, $D \in \Lambda$, the G-sheaf structure arises from a G-linearization $\sigma^*(\mathcal{A}_D) \cong \mathrm{pr}_X^*(\mathcal{A}_D)$ in the sense of [19, Definition 1.6].

The reason to introduce besides the straightforward generalization 1.2 ii) also the weaker notion 1.2 i), is that in practice the latter is often much easier to handle. However, in many important cases both notions coincide, for example if the component G^0 of the unit element is a torus:

Proposition 1.3. *If X is covered by G^0-invariant affine open subsets, then every G-linearization of Λ is in fact a strong G-linearization of Λ.*

Proof. Assume that $\Lambda \subset \mathrm{CDiv}(X)$ is G-linearized, and let $\mathcal{A}_D$ be a homogeneous component of the associated graded $\mathcal{O}_X$-algebra. Consider a geometric line bundle $p \colon L \to X$ having $\mathcal{A}_D$ as its sheaf of sections. Then the G-sheaf structure of $\mathcal{A}_D$ gives rise to a set theoretical action, namely

$$G \times L \to L, \qquad (g, z) \mapsto g \cdot z := (g \cdot f)(g \cdot p(z)),$$

where for given $z \in L$ we choose any local section f of $\mathcal{A}_D$ satisfying $f(p(z)) = z$. Note that this well defined. In view of [16, Lemma 2.3], we only have to show that this action is regular. Since for fixed $g \in G$ the map $z \mapsto g \cdot z$ is obviously regular, it suffices to show that $G^0 \times L \to L$ is regular.

According to our assumption on X, it suffices to treat the case that X is affine. But then the rational representation of G^0 on the $\mathcal{O}(X)$-algebra

$$A := \bigoplus_{n \in \mathbb{N}} \mathcal{A}_{nD}(X)$$

defines a regular G^0-action on the dual bundle $L' := \mathrm{Spec}(A)$ such that $L' \to X$ becomes equivariant and G^0 acts linearly on the fibres. It is straightforward to check that this G^0-action on L' is dual to the G^0-action on L. Hence also the latter action is regular. $\square$

Concerning existence of linearizations, we have the following generalization of [19, Corollary 1.6], compare [14, Proposition 3.6]:

Proposition 1.4. *Suppose that G is connected and that X is a normal separated variety. Then every group $\Lambda \subset \mathrm{CDiv}(X)$ admits a strongly G-linearized subgroup $\Lambda' \subset \Lambda$ of finite index.*

Proof. Choose a basis $D_1, \ldots, D_r$ of Λ. According to [16, Proposition 2.4], there is a positive integer n such that each sheaf $\mathcal{A}_{nD_i}$ admits a G-linearization in Mumford's sense. Tensoring these linearizations gives the desired strong linearization of the subgroup $\Lambda' \subset \Lambda$ generated by $nD_1, \ldots, nD_r$. □

A more special existence statement for non connected G will be given in 4.2. There is also a uniqueness statement like [19, Proposition 1.4]. Note that in our version, we do not assume G to be connected:

Proposition 1.5. *Suppose that $\Lambda \subset \mathrm{CDiv}(X)$ admits two strong G-linearizations. If $\mathcal{O}^*(X) = \mathbb{K}^*$ holds and G has only finitely many characters, then the two G-linearizations coincide on a subgroup $\Lambda' \subset \Lambda$ of finite index.*

Proof. To distinguish the two G-sheaf structures on the graded $\mathcal{O}_X$-algebra associated to Λ, we denote them by $(g, f) \mapsto g \cdot f$ and $(g, f) \mapsto g * f$. Consider a homogeneous component $\mathcal{A}_D$, and the tensor product

$$\mathcal{A}_D \otimes_{\mathcal{O}_X} \mathcal{A}_{-D}, \qquad g \bullet (f \otimes h) := g \cdot f \otimes g * h.$$

Since as an $\mathcal{O}_X$-module, $\mathcal{A}_D \otimes_{\mathcal{O}_X} \mathcal{A}_{-D}$ is isomorphic to the structure sheaf itself, we obtain a G-sheaf structure on $\mathcal{O}_X$, also denoted by $(g, f) \mapsto g \bullet f$. As it arises from a G-linearization in the sense of [19, Definition 1.6], this G-sheaf structure is of the form

$$(g \bullet f)(x) = \chi(g, x) f(g^{-1} \cdot x)$$

with a function $\chi \in \mathcal{O}^*(G \times X)$. Since we assumed $\mathcal{O}^*(X) \cong \mathbb{K}^*$, the function χ does not depend on the second variable. In fact, χ even turns out to be a character on G.

Now, replacing in this setting D with a multiple nD amounts to replacing χ with χ^n. Thus, taking n to be the order of the character group of G, we see that for any $D \in \Lambda$, the two G-sheaf structures on $\mathcal{A}_{nD}$ coincide. The assertion follows. □

We look a bit closer to the $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to a group $\Lambda \subset \mathrm{CDiv}(X)$. This algebra gives rise to a prevariety $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ and a canonical map $q \colon \widehat{X} \to X$. We list some basic features of this construction:

Remark 1.6. Let $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ and $q \colon \widehat{X} \to X$ be as above. For an open subset $U \subset X$, set $\widehat{U} := q^{-1}(U)$.

i) For a section $f \in \mathcal{A}_D(U)$, let $Z(f) := \mathrm{Supp}(D|_U + \mathrm{div}(f))$ denote the set of its zeroes. Then we have

$$\widehat{U}_f := \{x \in \widehat{U};\ f(x) \neq 0\} = q^{-1}(U \setminus Z(f)).$$

ii) The algebraic torus $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$ acts regularly on $\widehat{X}$ such that every $f \in \mathcal{A}_D(U)$ is homogeneous with respect to the character χ^D, i.e., we have

$$f(t \cdot x) = \chi^D(t) \cdot f(x).$$

iii) The action of H on $\widehat{X}$ is free and the map $q \colon \widehat{X} \to X$ is a geometric quotient for this action.

For the subsequent constructions, it is important to figure out those groups $\Lambda \subset \mathrm{CDiv}(X)$ for which the associated prevariety $\widehat{X}$ over X is in fact a quasiaffine variety. This leads to the following notion:

Definition 1.7. We call the group $\Lambda \subset \mathrm{CDiv}(X)$ *ample on an open subset* $U \subset X$, if there are homogeneous sections $f_1, \ldots, f_r \in \mathcal{A}(U)$ such that the sets $U \setminus Z(f_i)$ are affine and cover U.

If $\Lambda \subset \mathrm{CDiv}(X)$ is ample on X, then we say for short that Λ is ample. So, the prevariety X admits an ample group $\Lambda \subset \mathrm{CDiv}(X)$ if and only if it is *divisorial* in the sense of Borelli [12], i.e., every $x \in X$ has an affine neighbourhood $X \setminus \mathrm{Supp}(D)$ with some effective $D \in \mathrm{CDiv}(X)$.

Remark 1.8. If X is a divisorial prevariety, then the intersection $U \cap U'$ of any two affine open subsets $U, U' \subset X$ is again affine.

In the following statement, we subsume the consequences of the existence of a G-linearized ample group, compare [13, Section 2]. By an affine closure of a quasiaffine variety Y we mean an affine variety $\overline{Y}$ containing Y as an open dense subvariety.

Proposition 1.9. *Let G be a linear algebraic group and let X be a G-prevariety. Suppose that $\Lambda \subset \mathrm{CDiv}(X)$ is G-linearized and ample on some G-invariant open $U \subset X$. Let $\widehat{U} := q^{-1}(U) \subset \widehat{X}$, where $q \colon \widehat{X} \to X$ is as above.*

i) *$\widehat{U}$ is quasiaffine and the representation of G on $\mathcal{O}(\widehat{U})$ induces a regular G-action on $\widehat{U}$ such that the actions of G and $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$ commute and the canonical map $q \colon \widehat{U} \to X$ becomes G-equivariant.*
ii) *For any collection $f_1, \ldots, f_r \in \mathcal{A}(U)$ satisfying the ampleness condition, there exists a $(G \times H)$-equivariant affine closure $\overline{U}$ of $\widehat{U}$ such that the f_i extend to regular functions on $\overline{U}$ and $q^{-1}(U_{f_i}) = \overline{U}_{f_i}$ holds.*

Proof. Use [13, Lemmas 2.4 and 2.5]. □

2. Stability notions

Generalizing [19, Definitions 1.7 and 1.8] we shall associate to a linearized group of divisors sets of semistable, stable and properly stable points. Moreover, for ample linearized groups, we give a geometric interpretation of semistability in terms of a generalized nullcone.

Let G be a reductive algebraic group, and let X be an irreducible G-prevariety. Suppose that $\Lambda \subset \mathrm{CDiv}(X)$ is a G-linearized (finitely generated free) subgroup. Denote the associated Λ-graded $\mathcal{O}_X$-algebra by

$$\mathcal{A} = \bigoplus_{D \in \Lambda} \mathcal{A}_D.$$

Definition 2.1. Let G, X, Λ and $\mathcal{A}$ be as above. We say that a point $x \in X$ is

i) *semistable*, if x has an affine neighbourhood $U = X \setminus Z(f)$ with some G-invariant $f \in \mathcal{A}_D(X)$ such that the $D' \in \Lambda$ admitting a G-invariant $f_{D'} \in \mathcal{A}_{D'}(U)$ which is invertible in $\mathcal{A}(U)$ form a subgroup of finite index in Λ,
ii) *stable*, if x is semistable, its orbit $G \cdot x$ is of maximal dimension and $G \cdot x$ is closed in the set of semistable points of X,
iii) *properly stable*, if x is semistable, its isotropy group G_x is finite and $G \cdot x$ is closed in the set of semistable points of X.

Following Mumford's notation, we denote the G-invariant open sets corresponding to the semistable, stable and properly stable points by $X^{ss}(\Lambda)$, $X^{s}(\Lambda)$ and $X^{s}_{0}(\Lambda)$ respectively. If we want to specify the acting group G, we also write $X^{ss}(\Lambda, G)$ etc..

Remark 2.2. Let X be complete, let $D \in \mathrm{CDiv}(X)$ be an effective Cartier divisor and suppose that the invertible sheaf $\mathcal{L} := \mathcal{A}_D$ on X is G-linearized in the sense of [19, Definition 1.6]. Then the induced G-sheaf structure of $\mathcal{A}_D$ extends to a G-linearization of $\Lambda := \mathbb{Z}D$. Moreover,

i) $X^{ss}(\Lambda)$ contains precisely the points of X which are semistable in the sense of [19, Definition 1.7 i)],
ii) $X^{s}(\Lambda)$ contains precisely the points of X which are stable in the sense of [19, Definition 1.7 ii)],
iii) $X^{s}_{0}(\Lambda)$ contains precisely the points of X which are properly stable in the sense of [19, Definition 1.8].

The remainder of this section is devoted to giving a geometric interpretation of semistability. For this, let $U \subset X$ denote any G-invariant open subset such that Λ is ample on U and $X^{ss}(\Lambda)$ is contained in U, for example $U = X^{ss}(\Lambda)$. As usual, let

$$\widehat{X} := \mathrm{Spec}(\mathcal{A}), \qquad q \colon \widehat{X} \to X, \qquad \widehat{U} := q^{-1}(U).$$

Recall from Section 1 that the map $q\colon \widehat{X} \to X$ is a geometric quotient for the action of $H := \operatorname{Spec}(\mathbb{K}[\Lambda])$ on $\widehat{X}$ induced by the Λ-grading of $\mathcal{A}$. Moreover, $\widehat{U}$ is a quasiaffine variety and carries a regular G-action making $q\colon \widehat{U} \to X$ equivariant.

Our description involves two choices. First let $f_1, \dots, f_r \in \mathcal{A}(X)$ be homogeneous G-invariant sections such that the sets $X \setminus Z(f_i)$ are as in Definition 2.1 i) and cover $X^{ss}(\Lambda)$.

Next choose a $(G \times H)$-equivariant affine closure $\overline{U}$ of $\widehat{U}$ such that the functions $f_i \in \mathcal{O}(\widehat{U})$ extend regularly to $\overline{U}$ and $\overline{U}_{f_i} = \widehat{U}_{f_i}$ holds for each $i = 1, \dots, r$. Consider the good quotient

$$\overline{p}\colon \overline{U} \to \overline{U}/\!\!/G := \operatorname{Spec}(\mathcal{O}(\overline{U}))^G.$$

Then the quotient variety $\overline{U}/\!\!/G$ inherits a regular action of H such that the map $\overline{p}\colon \overline{U} \to \overline{U}/\!\!/G$ becomes H-equivariant. In this setting, the set $\overline{U} \setminus q^{-1}(X^{ss}(\Lambda))$ takes over the role of the classical nullcone:

Proposition 2.3. *Let $V_0 := \overline{U}/\!\!/G \setminus \overline{p}(\overline{U} \setminus \widehat{U})$, and let $V_1 \subset \overline{U}/\!\!/G$ be the union of all H-orbits with finite isotropy.*

i) *One always has $q^{-1}(X^{ss}(\Lambda)) \subset \overline{p}^{-1}(V_0 \cap V_1)$.*
ii) *If $U = X$, then $q^{-1}(X^{ss}(\Lambda)) = \overline{p}^{-1}(V_0 \cap V_1)$.*

The main point in the proof is to express Condition 2.1 i) in terms of the action of the torus H on the affine variety $\overline{U}/\!\!/G$. Consider more generally an arbitrary algebraic torus T and a quasiaffine T-variety Y.

Lemma 2.4. *The isotropy group T_y of a point $y \in Y$ is finite if and only if there is a homogeneous function $h \in \mathcal{O}(Y)$ such that Y_h is an affine neighbourhood of y and the characters $\chi' \in \operatorname{Char}(T)$ admitting an invertible χ'-homogeneous $h' \in \mathcal{O}(Y_h)$ form a sublattice of finite index in $\operatorname{Char}(T)$.*

Proof. First suppose that T_y is finite. Consider the orbit $B := T \cdot y$. This is a locally closed affine subvariety of Y. The set M consisting of all characters $\chi' \in \operatorname{Char}(T)$ admitting a χ'-homogeneous $h' \in \mathcal{O}(B)$ with $h'(y) = 1$ is a sublattice of $\operatorname{Char}(T)$. We show that M is of full rank:

Otherwise there is a non trivial one parameter subgroup $\lambda\colon \mathbb{K}^* \to T$ such that $\chi \circ \lambda = 1$ holds for every $\chi \in M$. Thus, by the definition of M, all homogeneous functions of $\mathcal{O}(B)$ are constant along $\lambda(\mathbb{K}^*) \cdot y$. As these functions separate the points of B, we conclude $\lambda(\mathbb{K}^*) \subset T_y$. A contradiction.

Now, choose any T-homogeneous function $h \in \mathcal{O}(Y)$ such that Y_h is affine, contains B as a closed subset, and for some base $\chi'_1, \dots, \chi'_d$ of M the associated functions $h'_i \in \mathcal{O}(B)$ extend to invertible regular homogeneous functions on Y_h. Then this $h \in \mathcal{O}(Y)$ is as desired. The "if" part of the assertion is settled by similar arguments. $\square$

Proof of Proposition 2.3. Let $W := X^{ss}(\Lambda)$ and $\widehat{W} := q^{-1}(W)$. We begin with the inclusion "$\subset$" of assertions i) and ii). First note that $\widehat{W}$ is $\overline{p}$-saturated, because this holds for each $\overline{U}_{f_i}$ and, according to Remark 1.6 i), $\widehat{W}$ is covered by these subsets. In particular, it follows $\overline{p}(\widehat{W}) \subset V_0$.

To verify $\overline{p}(\widehat{W}) \subset V_1$, let $z \in \widehat{W}$. Take one of the f_i with $z \in \overline{U}_{f_i}$. As it is G-invariant, f_i descends to an H-homogeneous function $h \in \mathcal{O}(\overline{U}/\!\!/G)$. By the properties of f_i, the function h satisfies the condition of Lemma 2.4 for the point $\overline{p}(z)$. Hence $H_{\overline{p}(z)}$ is finite, which means $\overline{p}(z) \in V_1$.

We come to the inclusion "$\supset$" of assertion ii). Let $y \in V_0 \cap V_1$. Lemma 2.4 provides an $h \in \mathcal{O}(\overline{X}/\!\!/G)$, homogeneous with respect to some $\chi^D \in \operatorname{Char}(H)$, such that $y \in V := (\overline{X}/\!\!/G)_h$ holds and the $D' \in \Lambda$ admitting an invertible $\chi^{D'}$-homogeneous function on V form a subgroup of finite index in Λ. Suitably modifying h, we achieve additionally $V \subset V_0 \cap V_1$.

Now, consider a point $z \in \overline{p}^{-1}(y)$. Since $y \in V_0$, we have $z \in \widehat{X}$. We have to show that $q(z)$ is semistable. For this, consider the G-invariant homogeneous section $f := \overline{p}^*(h)|_{\widehat{X}}$ of $\mathcal{A}_D(X)$. By the choice of h, this f fulfills the conditions of Definition 2.1 i) and thus the point $q(z)$ is in fact semistable. □

Corollary 2.5. *Let $\Lambda \subset \operatorname{CDiv}(X)$ be an ample G-linearized group.*

i) *A point $x \in X^{ss}(\Lambda)$ with an orbit $G\cdot x$ of maximal dimension is stable if and only if for any $z \in q^{-1}(x)$ the orbit $G\cdot z$ is closed in $\widehat{X}$.*
ii) *A point $x \in X^{ss}(\Lambda)$ with finite isotropy group G_x is properly stable if and only if for any $z \in q^{-1}(x)$ the orbit $G\cdot z$ is closed in $\widehat{X}$.* □

3. The quotient of the set of semistable points

Let G be a reductive algebraic group, and let X be a G-prevariety. In this section we show that any set of semistable points admits a good quotient. The result generalizes [19, Theorem 1.10].

Theorem 3.1. *Let $\Lambda \subset \operatorname{CDiv}(X)$ be a G-linearized subgroup. Then there exists a good quotient $p\colon X^{ss}(\Lambda) \to X^{ss}(\Lambda)/\!\!/G$ and the quotient space $X^{ss}(\Lambda)/\!\!/G$ is a divisorial prevariety.*

An immediate consequence of this result is that the set of stable points admits a geometric quotient. More precisely, by the properties of good quotients we have:

Remark 3.2. In the notation of 3.1, the set $X^s(\Lambda)$ is p-saturated and the restriction $p\colon X^s(\Lambda) \to p(X^s(\Lambda))$ is a geometric quotient.

In the proof of Theorem 3.1, we make use of the following observation on geometric quotients for torus actions, compare [1, Proposition 1.5]:

Lemma 3.3. *Let T be an algebraic torus and suppose that Y is an irreducible quasiaffine T-variety with geometric quotient $p\colon Y \to Y/T$. Then Y/T is a divisorial prevariety.*

Proof. We may assume that T acts effectively. Set for short $Z := Y/T$. Given a point $z \in Z$, choose a T-homogeneous $f \in \mathcal{O}(Y)$ such that $U := Y_f$ is an affine neighbourhood of $p^{-1}(z)$. Consider the affine neighbourhood $V := p(U)$ of z. We show that $B := Z \setminus V$ is the support of an effective Cartier divisor on Y.

Let $\chi \in \mathrm{Char}(T)$ be the weight of the above $f \in \mathcal{O}(Y)$. Since T acts effectively with geometric quotient, all isotropy groups T_y are finite. So we can use Lemma 2.4 to cover Y by T-invariant affine open sets U_i admitting invertible functions $g_i \in \mathcal{O}(U_i)$ that are homogeneous with respect to some common multiple $m\chi$.

Each $h_i := f^m/g_i \in \mathcal{O}(U_i)$ is T-invariant and hence we have $h_i = p^*(h_i')$ with a regular function h_i' defined on $V_i := p(U_i)$. By construction, the zero set of h_i' is just $B \cap V_i$. Since every h_i'/h_j' is regular and invertible on $V_i \cap V_j$, the functions h_i' yield local equations for an effective Cartier divisor E on Z having support B. □

Proof of Theorem 3.1. As usual, let $\mathcal{A}$ be the graded $\mathcal{O}_X$-algebra associated to Λ. We consider the corresponding prevariety $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ and the map $q\colon \widehat{X} \to X$. Recall that the latter is a geometric quotient for the action of $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$ on $\widehat{X}$. Set for short $W := X^{ss}(\Lambda)$. Surely, Λ is ample on W. Proposition 1.9 yields that $\widehat{W} := q^{-1}(W)$ is a quasiaffine variety. Moreover, $\widehat{W}$ carries a G-action that commutes with the action of H and makes $q\colon \widehat{W} \to W$ equivariant. Choose $f_1, \ldots, f_r \in \mathcal{A}(X)$ satisfying the conditions of Definition 2.1 such that W is covered by the affine sets $X \setminus Z(f_i)$, and set $h_i := f_i|_W$.

Choose a $(G \times H)$-equivariant affine closure $\overline{W}$ of $\widehat{W}$ such that the above $h_i \in \mathcal{O}(\widehat{W})$ extend regularly to $\overline{W}$ and satisfy $\overline{W}_{h_i} = \widehat{W}_{h_i}$. The set $\widehat{W}$ is saturated with respect to the good quotient $\overline{p}\colon \overline{W} \to \overline{W}/\!\!/G$ because this holds for the sets $\overline{W}_{h_i}$. Consequently, restricting $\overline{p}$ to $\widehat{W}$ yields a good quotient $\widehat{p}\colon \widehat{W} \to \widehat{W}/\!\!/G$.

Moreover, Proposition 2.3 i) tells us that H acts with at most finite isotropy groups on $\widehat{W}/\!\!/G$. Thus, there is a geometric quotient $\widehat{W}/\!\!/G \to (\widehat{W}/\!\!/G)/H$. By Lemma 3.3, the quotient space is a divisorial prevariety. Since good quotients are categorical, we obtain a commutative diagram

$$\begin{tikzcd} \widehat{W} \arrow[r, "\widehat{p}"] \arrow[d, "/H"'] & \widehat{W}/\!\!/G \arrow[d, "/H"] \\ W \arrow[r] & (\widehat{W}/\!\!/G)/H \end{tikzcd}$$

Now it is straightforward to check that the induced map $W \to (\widehat{W}/\!\!/G)/H$ is the desired good quotient for the action of G on W. $\square$

We conclude this section with a short discussion of the question, when the quotient space $X^{ss}(\Lambda)/\!\!/G$ is separated. Translating the usual criterion for separateness in terms on functions on the quotient space to the setting of invariant sections of the $\mathcal{O}_X$-algebra $\mathcal{A}$ of a G-linearized group Λ, we obtain:

Remark 3.4. Let $\Lambda \subset \mathrm{CDiv}(X)$ be a G-linearized group on a G-variety X, and let $X^{ss}(\Lambda)$ be covered by $X \setminus Z(f_i)$ with G-invariant sections $f_1, \dots, f_r \in \mathcal{A}(X)$ as in 2.1 i). The quotient space $X^{ss}/\!\!/G$ is separated if and only if for any two indices i, j the multiplication map defines a surjection in degree zero:

$$\mathcal{A}(X)^G_{(f_i)} \otimes \mathcal{A}(X)^G_{(f_j)} \to \mathcal{A}(X)^G_{(f_i f_j)}.$$

In the classical setting [19, Definition 1.7], the group Λ is of rank one, and the above sections f_i are of positive degree. In particular, for suitable positive powers n_i, all sections $f_i^{n_i}$ are of the same degree, and Remark 3.4 implies that the resulting quotient space is always separated.

As soon as we leave the classical setting, the above reasoning may fail, and we can obtain nonseparated quotient spaces, as the following two simple examples show. Both examples arise from the hyperbolic $\mathbb{K}^*$-action on the affine plane. In the first one we present a group Λ of rank one defining a nonseparated quotient space:

Example 3.5. Let the onedimensional torus $T := \mathbb{K}^*$ act diagonally on the punctured affine plane $X := \mathbb{K}^2 \setminus \{(0,0)\}$ via

$$t \cdot (z_1, z_2) := (t z_1, t^{-1} z_2).$$

Consider the group $\Lambda \subset \mathrm{CDiv}(X)$ generated by the principal divisor $D := \mathrm{div}(z_1)$. Since D is T-invariant, the group Λ is canonically T-linearized. We claim that the corresponding set of semistable points is

$$X^{ss}(\Lambda) = X.$$

To verify this claim, let $\mathcal{A}$ denote the graded $\mathcal{O}_X$-algebra associated to Λ, and consider the T-invariant sections

$$f_1 := 1 \in \mathcal{A}_D(X), \qquad f_2 := z_1 z_2 \in \mathcal{A}_{-D}(X).$$

Then the sets $X \setminus Z(f_1)$ and $X \setminus Z(f_2)$ form an affine cover of X. Moreover, we have T-invariant invertible sections:

$$1 \in \mathcal{A}_D(X \setminus Z(f_1)), \qquad \frac{1}{z_1 z_2} \in A_D(X \setminus Z(f_2)).$$

So, $f_1, f_2 \in \mathcal{A}(X)$ satisfy the conditions of Definition 2.1 i), and the claim is verified. The quotient space $Y := X^{ss}(\Lambda)/\!\!/T$ is the affine line with doubled zero. In particular, Y is a nonseparated prevariety.

In view of Remark 3.4, we obtain always separated quotient spaces when starting with a group $\Lambda = \mathbb{Z}D$, where D is a divisor on a complete G-variety X. In this setting, the lack of enough invariant sections of degree zero on the sets $X \setminus Z(f_i)$ occurs for groups Λ of higher rank:

Example 3.6. Let the onedimensional torus $T := \mathbb{K}^*$ act diagonally on the projective plane $X := \mathbb{P}_2$ via

$$t \cdot [z_0, z_1, z_2] := [z_0, tz_1, t^{-1}z_2].$$

Consider the group $\Lambda \subset \mathrm{CDiv}(X)$ generated by the divisors $D_1 := E_0 + E_1$ and $D_2 := E_0 + E_2$, where E_i denotes the prime divisor $V(X; z_i)$. Since the divisors D_i are T-invariant, the group Λ is canonically T-linearized. We claim that the corresponding set of semistable points is

$$X^{ss}(\Lambda) = X \setminus \{[1,0,0], [0,1,0], [0,0,1]\}.$$

To check this claim, denote the right hand side by U. Let $\mathcal{A}$ again denote the graded $\mathcal{O}_X$-algebra associated to Λ, and consider the T-invariant sections

$$f_1 := 1 \in \mathcal{A}_{D_1}(X), \qquad f_2 := 1 \in \mathcal{A}_{D_2}(X), \qquad f_3 := \frac{z_1 z_2}{z_0^2} \in \mathcal{A}_{D_1+D_2}(X).$$

For the respective zero sets of these sections we have

$$Z(f_1) = V(X; z_0 z_1), \qquad Z(f_2) = V(X; z_0 z_2), \qquad Z(f_3) = V(X; z_1 z_2).$$

So, the set U is indeed the union of the affine sets $X \setminus Z(f_i)$. Moreover, we have invertible sections

$$\begin{aligned} 1 &\in \mathcal{A}_{D_1}(X \setminus Z(f_1)), & \frac{z_0^2}{z_1 z_2} &\in \mathcal{A}_{D_2}(X \setminus Z(f_1)), \\ 1 &\in \mathcal{A}_{D_2}(X \setminus Z(f_2)), & \frac{z_0^2}{z_1 z_2} &\in \mathcal{A}_{D_1}(X \setminus Z(f_2)), \\ \frac{z_1 z_2}{z_0^2} &\in \mathcal{A}_{2D_1}(X \setminus Z(f_3)), & \frac{z_1 z_2}{z_0^2} &\in \mathcal{A}_{2D_2}(X \setminus Z(f_3)). \end{aligned}$$

Thus $f_1, f_2, f_3 \in \mathcal{A}(X)$ satisfy the conditions of Definition 2.1 i). Since the fixed points $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$ occur as limit points of suitable T-orbits through U, they cannot be semistable. The claim is verified.

Note that $X^{ss}(\Lambda)$ equals in fact the set of (properly) stable points. The quotient space $Y := X^{ss}(\Lambda) /\!\!/ T$ is a projective line with doubled zero. In particular, Y is a nonseparated prevariety.

4. Good quotients for $\mathbb{Q}$-factorial G-varieties

Let G be a not necessarily connected reductive group, and let X be an irreducible G-prevariety. In [19, Converse 1.13], Mumford shows that, provided X is a smooth variety and G is connected, every open subset U with a geometric quotient $U \to U/G$ such that U/G is quasiprojective arises in fact from a set of stable points.

Here we generalize this statement to non connected G and open subsets with a divisorial good quotient space. Assume that X is $\mathbb{Q}$-factorial, i.e., X is normal and for each Weil divisor D on X, some multiple of D is Cartier. Moreover, suppose that X is of affine intersection, i.e., for any two open affine subsets of X their intersection is again affine.

To formulate our result, let $U \subset X$ be an open G-invariant set of the G-prevariety X such that there exists a good quotient $U \to U/\!\!/G$. Then we have:

Theorem 4.1. *If $U/\!\!/G$ is divisorial, then there exists a G-linearized group $\Lambda \subset \mathrm{CDiv}(X)$ such that U is contained in $X^{ss}(\Lambda)$ and is saturated with respect to the quotient map $X^{ss}(\Lambda) \to X^{ss}(\Lambda)/\!\!/G$.*

For the proof of this statement, we need two lemmas. The first one is an existence statement on *canonical linearizations*:

Let H be any linear algebraic group. We say that a Weil divisor E on a normal H-prevariety Y is *H-tame*, if $\mathrm{Supp}(E)$ is H-invariant and for any two prime cycles E_1, E_2 of E with $E_2 = h \cdot E_1$ for some $h \in H$ their multiplicities in E coincide.

Lemma 4.2. *Let $\Lambda \subset \mathrm{CDiv}(Y)$ be a group consisting of H-tame divisors. Then Λ admits a canonical H-linearization, namely*

$$\mathcal{A}_E(U) \to \mathcal{A}_E(h \cdot U), \qquad (h \cdot f)(x) := f(h^{-1} \cdot x).$$

Proof. First we note that the canonical action of H on $\mathbb{K}(Y)$ induces indeed a H-sheaf structure on the sheaf $\mathcal{A}_E$ of an H-tame Cartier divisor E on Y. This follows from the fact that for $f \in \mathbb{K}(Y)$, the order of a translate $h \cdot f$ along a prime divisor E_0 of E is given by

$$\mathrm{ord}_{E_0}(h \cdot f) = \mathrm{ord}_{h^{-1} \cdot E_0}(f).$$

We still have to show that for every H-invariant open set $V \subset Y$, the representation of H on $\mathcal{A}_E(V)$ is regular. Consider the maximal separated subsets $V_1, \ldots, V_r$ of $V \setminus \mathrm{Supp}(E)$, see [3, Theorem I]. Their intersection V' is H-invariant, and $\mathcal{A}_E(V)$ injects H-equivariantly into $\mathcal{O}(V')$. Hence [16, Section 2.5] gives the claim. □

Now, consider a normal prevariety Y with effective $E_1, \ldots, E_r \in \mathrm{CDiv}(Y)$ such that the sets $V_i := Y \setminus \mathrm{Supp}(E_i)$ are affine and cover Y. Let $\Gamma \subset \mathrm{CDiv}(Y)$ be the subgroup generated by $E_1, \ldots, E_r$. Denote the associated Γ-graded $\mathcal{O}_Y$-algebra by

$$\mathcal{B} := \bigoplus_{E \in \Gamma} \mathcal{B}_E := \bigoplus_{E \in \Gamma} \mathcal{O}_Y(E).$$

Lemma 4.3. *In the above setting, every open set $V_i = Y \setminus \mathrm{Supp}(E_i)$ is covered by finitely many open affine subsets $V_{ij} \subset V_i$ with the following properties:*

i) *$V_{ij} = Y \setminus Z(h_{ij})$ with some $h_{ij} \in \mathcal{B}_{n_i E_i}(Y)$, where $n_i \in \mathbb{N}$,*

ii) *for each* $k = 1, \dots, r$ *there exists an* $h_{ijk} \in \mathcal{B}_{E_k}(V_{ij})$ *without zeroes in* V_{ij}.

Proof. Let $y \in V_i$ and consider an affine open neighbourhood $V \subset V_i$ of y such that on V we have $E_k = \mathrm{div}(h'_k)$ with some $h'_k \in \mathcal{O}(V)$ for all k. Then each $h_k := 1/h'_k$ is a section of $\mathcal{B}_{E_k}(V)$ without zeroes in V. By suitably shrinking V, we achieve $V = X \setminus Z(h)$ with some $h \in \mathcal{B}_{n_i E_i}(X)$ and some $n_i \in \mathbb{N}$. Since finitely many of such V cover V_i, the assertion follows. □

Proof of Theorem 4.1. Since the quotient space $Y := U /\!\!/ G$ is divisorial, we find effective $E_1, \dots, E_r \in \mathrm{CDiv}(Y)$ such that the sets $V_i := Y \setminus \mathrm{Supp}(E_i)$ are affine and cover Y. Let V_{ij}, h_{ij} and h_{ijk} as in Lemma 4.3. Consider the quotient map $p\colon U \to Y$ and the pullback divisors

$$D'_i := p^*(E_i) \in \mathrm{CDiv}(U).$$

Then every $U_i := p^{-1}(V_i)$ is affine and equals $U \setminus \mathrm{Supp}(D'_i)$. Moreover, since they are locally defined by invariant functions, we see that the divisors D'_i are G-tame. Since X is $\mathbb{Q}$-factorial and of affine intersection, we can construct G-tame effective divisors $D_i \in \mathrm{CDiv}(X)$ with the following properties:

i) $D_i|_U = m_i D'_i$ holds with some $m_i \in \mathbb{N}$ and we have $X \setminus \mathrm{Supp}(D_i) = U_i$,
ii) for some $l_i \in \mathbb{N}$, every $f_{ij} := p^*(h_{ij}^{l_i})$ extends to a global section of $\mathcal{O}_X(D_i)$ and satisfies $X \setminus Z(f_{ij}) = p^{-1}(V_{ij})$.

Let $\Lambda \subset \mathrm{CDiv}(X)$ denote the group generated by the divisors $D_1, \dots, D_r$, and let $\mathcal{A}$ be the associated graded $\mathcal{O}_X$-algebra. Lemma 4.2 tells us that the group Λ is canonically G-linearized by setting $g{\cdot}f(x) := f(g^{-1}{\cdot}x)$ on the homogeneous components of $\mathcal{A}$.

Note that the set $U \subset X$ is covered by the affine open subsets $U_{ij} := p^{-1}(V_{ij})$. Thus, using the pullback data f_{ij} and

$$f_{ijk} := p^*(h_{ijk}^{m_i}) \in \mathcal{A}_{D_i}(U_{ij}),$$

it is straightforward to check $U \subset X^{ss}(\Lambda)$. Moreover, since the U_{ij} are defined by the G-invariant sections f_{ij}, we see that they are saturated with respect to the quotient map $p'\colon X^{ss}(\Lambda) \to X^{ss}(\Lambda) /\!\!/ G$. Hence U is p'-saturated in $X^{ss}(\Lambda)$. □

Corollary 4.4. *Let the algebraic torus T act effectively and regularly on a $\mathbb{Q}$-factorial variety X, and let $U \subset X$ be the union of all T-orbits with finite isotropy group. If* $\dim(X \setminus U) < \dim(T)$, *then U is the set of semistable points of a T-linearized group* $\Lambda \subset \mathrm{CDiv}(X)$.

Proof. By [23, Corollary 3], there is a geometric quotient $U \to U/T$. Using Proposition 1.9 and Lemma 3.3, we see that U/T is a divisorial prevariety. Theorem 4.1 provides a T-linearized group $\Lambda \subset \mathrm{CDiv}(X)$ such that $X^{ss}(\Lambda)$ contains U as a saturated subset with respect to $p\colon X^{ss}(\Lambda) \to X^{ss}(\Lambda)/T$.

Semicontinuity of the fibre dimension of p and $\dim(X \setminus U) < \dim(T)$ imply $U = X^{ss}(\Lambda)$. □

The classical example of a generic $\mathbb{C}^*$-action on the Grassmannian of two dimensional planes in $\mathbb{C}^4$, compare also [5] and [25], fits into the setting of the above observation:

Example 4.5. Realize the complex Grassmannian $X := G(2;4)$ via Plücker relations as a quadric hypersurface in the complex projective space $\mathbb{P}_5$:

$$X = V(\mathbb{P}_5; z_0z_5 - z_1z_4 + z_2z_3).$$

This allows us to define a regular action of the one dimensional torus $T = \mathbb{C}^*$ on X in terms of coordinates:

$$t\cdot[z_0, z_1, z_2, z_3, z_4, z_5] := [tz_0, t^2z_1, t^3z_2, t^3z_3, t^4z_4, t^5z_5].$$

This T-action has six fixed points. Let $U \subset X$ be the complement of the fixed point set. It is well known that the quotient space $Y := U/T$ is a nonseparated prevariety which is covered by four projective open subsets. Moreover, Y contains two nonprojective complete open subsets, see [5, Remark 1.6] and [25, Example 6.4].

According to Corollary 4.4, the set U can be realized as the set of semistable points of a T-linearized group of divisors. Let us do this explicitly. Consider for example the prime divisors $D_1 := V(X; z_1)$ and $D_2 := V(X; z_4)$ and the group

$$\Lambda := \mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \subset \mathrm{CDiv}(X).$$

Then the group Λ is canonically T-linearized. We show that $X^{ss}(\Lambda) = U$ holds. Let $\mathcal{A}$ denote the graded $\mathcal{O}_X$-algebra associated to Λ, and consider the following T-invariant sections $f_{ij} \in \mathcal{A}(X)$:

$$f_{01} := \frac{z_0^2z_4}{z_1^3} \in \mathcal{A}_{4D_1-D_2}(X), \qquad h_{01} := 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{01})),$$

$$f_{02} := \frac{z_0z_2}{z_1^2} \in \mathcal{A}_{2D_1}(X), \qquad h_{02} := \frac{z_2^3}{z_0z_4^2} \in \mathcal{A}_{2D_2}(X \setminus Z(f_{02})),$$

$$f_{03} := \frac{z_0z_3}{z_1^2} \in \mathcal{A}_{2D_1}(X), \qquad h_{03} := \frac{z_3^3}{z_0z_4^2} \in \mathcal{A}_{2D_2}(X \setminus Z(f_{03})),$$

$$f_{04} := \frac{z_0^2z_4}{z_1^3} \in \mathcal{A}_{3D_1}(X), \qquad h_{04} := 1 \in \mathcal{A}_{D_2}(X \setminus Z(f_{04})),$$

$$f_{05} := \frac{z_0^3z_5}{z_1^4} \in \mathcal{A}_{4D_1}(X), \qquad h_{05} := \frac{z_0z_5^3}{z_4^4} \in \mathcal{A}_{4D_2}(X \setminus Z(f_{05})),$$

$$f_{12} := \frac{z_2^2}{z_1z_4} \in \mathcal{A}_{2D_1+D_2}(X), \qquad h_{12} := 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{12})),$$

$$f_{13} := \frac{z_3^2}{z_1z_4} \in \mathcal{A}_{2D_1+D_2}(X), \qquad h_{13} := 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{13})),$$

$$f_{14} := 1 \in \mathcal{A}_{D_1+D_2}(X), \qquad h_{14} := 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{14})),$$

$$
\begin{aligned}
f_{15} &:= \frac{z_1 z_5^2}{z_4^3} \in \mathcal{A}_{3D_2}(X), & h_{15} &:= 1 \in \mathcal{A}_{D_1}(X \setminus Z(f_{15})),\\
f_{24} &:= \frac{z_2^2}{z_1 z_4} \in \mathcal{A}_{D_1+2D_2}(X), & h_{24} &:= 1 \in \mathcal{A}_{D_2}(X \setminus Z(f_{24})),\\
f_{25} &:= \frac{z_2 z_5}{z_4^2} \in \mathcal{A}_{2D_2}(X), & h_{25} &:= \frac{z_2^3}{z_1^2 z_5} \in \mathcal{A}_{2D_1}(X \setminus Z(f_{25})),\\
f_{34} &:= \frac{z_3^2}{z_1 z_4} \in \mathcal{A}_{D_1+2D_2}(X), & h_{34} &:= 1 \in \mathcal{A}_{D_2}(X \setminus Z(f_{34})),\\
f_{35} &:= \frac{z_3 z_5}{z_4^2} \in \mathcal{A}_{2D_2}(X), & h_{35} &:= \frac{z_3^3}{z_1^2 z_5} \in \mathcal{A}_{2D_1}(X \setminus Z(f_{35})),\\
f_{45} &:= \frac{z_1 z_5^2}{z_4^3} \in \mathcal{A}_{4D_2-D_1}(X), & h_{45} &:= 1 \in \mathcal{A}_{D_2}(X \setminus Z(f_{45})).
\end{aligned}
$$

By definition, we have $Z(f_{ij}) = V(X; z_i z_j)$ for the set of zeroes of f_{ij}. Consequently, U is the union of the affine open subsets $X_{ij} := X \setminus Z(f_{ij})$. Moreover, every h_{ij} is invertible over X_{ij}, and the claim follows.

In fact, using $\mathrm{Pic}_T(X) \cong \mathbb{Z}^2$, it is not hard to show that besides the T-invariant open subsets $W \subset X$ admitting a projective quotient variety $W /\!\!/ T$, the subset U is the only open subset of the form $X^{ss}(\Lambda)$ with a T-linearized group $\Lambda \subset \mathrm{CDiv}(X)$.

5. Reduction theorems for good quotients

In this section, G is a connected reductive group and the field $\mathbb{K}$ is of characteristic zero. Fix a maximal torus $T \subset G$ and denote by $N(T)$ its normalizer in G. The first result of this section relates existence of a good quotient by G to existence of a good quotient by $N(T)$:

Theorem 5.1. *For a normal G-prevariety X, the following statements are equivalent:*

i) *There is a good quotient $X \to X /\!\!/ G$ with a divisorial prevariety $X /\!\!/ G$.*
ii) *There is a good quotient $X \to X /\!\!/ N(T)$ with a divisorial prevariety $X /\!\!/ N(T)$.*

Moreover, if one of these statements holds with a separated quotient space, then so does the other.

Note that if X admits a divisorial good quotient space, then X itself is divisorial. In the second result, we specialize to geometric quotients. Recall that an action of G on X is said to be proper, if the map $G \times X \to X \times X$ sending (g, x) to $(g \cdot x, x)$ is proper.

Theorem 5.2. *Suppose that G acts properly on a $\mathbb{Q}$-factorial variety X. Then the following statements are equivalent:*

i) *There exists a geometric quotient $X \to X/G$.*

ii) *There exists a geometric quotient $X \to X/N(T)$.*

Moreover, if one of these statements holds, then the quotient spaces X/G and $X/N(T)$ are separated $\mathbb{Q}$-factorial varieties.

As an immediate consequence, we obtain the following statement on orbit spaces by the special linear group $\mathrm{SL}_2(\mathbb{K})$, which applies for example to the problem of moduli for n ordered points on the projective line, compare [20] and [4, Section 5]:

Corollary 5.3. *Let $\mathrm{SL}_2(\mathbb{K})$ act properly on an open subset $U \subset X$ of a $\mathbb{Q}$-factorial toric variety X such that some maximal torus $T \subset \mathrm{SL}_2(\mathbb{K})$ acts by means of a homomorphism $T \to T_X$ to the big torus $T_X \subset X$. Then there is a geometric quotient $U \to U/\mathrm{SL}_2(\mathbb{K})$.*

Proof. Since $\mathrm{SL}_2(\mathbb{K})$ acts properly, there is a geometric quotient $U \to U/T$. Let $U' \subset X$ be a maximal open subset such that $U \subset U'$ and there is a geometric quotient $U' \to U'/T$. Then the set U' is invariant under the big torus T_X, see e.g. [24, Corollary 2.4]. Thus the geometric quotient space $Y' := U'/T$ is again a toric variety.

In particular, any two points $y, y' \in Y'$ admit a common affine neighbourhood in Y'. But this property is inherited by $Y := U/T$. Thus, since $W := N(T)/T$ is of order two, we obtain a geometric quotient $Y \to Y/W$. The composition of $U \to Y$ and $Y \to Y/W$ is a geometric quotient for the action of $N(T)$ on U. So Theorem 5.2 gives the claim. $\square$

We come to the proof of Theorems 5.1 and 5.2. We make use of the following well known fact on semisimple groups:

Lemma 5.4. *If G is semisimple then the character group of $N(T)$ is finite.*

Proof. It suffices to show that for each $\widetilde{\chi} \in \mathrm{Char}(N(T))$, the restriction $\chi := \widetilde{\chi}|_T$ is trivial. Clearly χ is fixed under the action of the Weyl group $W = N(T)/T$ on $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Char}(T)$ induced by the $N(T)$-action

$$(n \cdot \alpha)(t) := \alpha(n^{-1} t n)$$

on $\mathrm{Char}(T)$. On the other hand, W acts transitively on the set of Weyl chambers associated to the root system determined by $T \subset G$. Consequently, χ lies in the closure of every Weyl chamber and hence is trivial. $\square$

Proof of Theorem 5.1. The implication "i)$\Rightarrow$ii)" is easy, use [21, Lemma 4.1]. To prove the converse, we first reduce to the case that G is semisimple: Let $R \subset G$ be the radical of G. Then R is a torus, and we have $R \subset T$. In particular, there is a good quotient $X \to X'$ for the action of R on X. Thus we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "/\!\!/N(T)"] \arrow[dr, "/\!\!/R"'] & & X/\!\!/N(T) \\ & X' \arrow[ur] & \end{tikzcd}$$

Consider the induced action of the connected semisimple group $G' := G/R$ on X'. The image T' of T under the projection $G \to G'$ is a maximal torus of G'. Moreover, $N(T')$ is the image of $N(T)$ under $G \to G'$. Thus, the upwards arrow of the above diagram is a good quotient for the action of $N(T')$ on X'.

To proceed, we only have to derive from the existence of a good quotient $X' \to X'/\!\!/N(T')$ that there is a good quotient $X' \to X'/\!\!/G'$ with a divisorial prevariety $X'/\!\!/G'$. In other words, we may assume from the beginning that the group G is semisimple.

Let $p\colon X \to X/\!\!/N(T)$ denote the good quotient. Using Lemmas 4.2 and 4.3 we can construct a canonically $N(T)$-linearized ample group $\Lambda \subset X$ consisting of $N(T)$-tame divisors such that we have

$$X^{ss}(\Lambda, N(T)) = X.$$

Note that this equality also holds for any subgroup $\Lambda' \subset \Lambda$ of finite index in Λ. We construct now such a subgroup $\Lambda' \subset \Lambda$ for which the canonical $N(T)$-linearization of Λ' extends to a strong G-linearization. The first step is to realize X as an open G-invariant subset of a certain G-prevariety Y with $\mathcal{O}(Y) = \mathbb{K}$.

Consider the maximal separated open subsets $X_1, \ldots, X_m \subset X$, see [3, Theorem I]. Since G is connected, it leaves these sets invariant. By Sumihiro's Equivariant Completion Theorem [23, Theorem 3], we find G-equivariant open embeddings $X_i \to Z_i$ into complete G-varieties Z_i. Applying equivariant normalization, we achieve that each Z_i is normal.

Let Y_i denote the union of X_i with the set of regular points of Z_i. Note that $\mathcal{O}(Y_i) = \mathbb{K}$. Define Y to be the G-equivariant gluing of the varieties Y_i along the invariant open subsets $X_i \subset Y_i$. Then we have $\mathcal{O}(Y) = \mathbb{K}$. Moreover, all points of $Y \setminus X$ are regular points of Y.

By closing components, every Cartier divisor $D \in \Lambda$ extends to a Cartier divisor on Y. Let $\Gamma \subset \mathrm{CDiv}(Y)$ denote the (free) group of Cartier divisors generated by these extensions. Lemma 4.2 ensures that the canonical $N(T)$-linearization of Λ extends to a canonical $N(T)$-linearization of the group Γ. By [23, Corollary 2] and Proposition 1.3, this linearization is even a strong one.

We claim that some subgroup $\Gamma' \subset \Gamma$ of finite index admits a strong G-linearization. Let $\mathcal{B}$ be the graded $\mathcal{O}_Y$-algebra associated to Γ. For each homogeneous component $\mathcal{B}_{E,i} := \mathcal{B}_E|_{Y_i}$, some power $\mathcal{B}_{nE,i}$ admits a G-linearization as in [16, Proposition 2.4]. Since G is semisimple, these linearizations are unique, see [19, Proposition 1.4]. Thus they define G-sheaf structures on the $\mathcal{O}_{Y_i}$-algebras

$$\mathcal{B}_i := \bigoplus_{E \in \Gamma_i} \mathcal{B}_{E,i},$$

for suitable subgroups $\Gamma_i \subset \Gamma$ of finite index. Again by uniqueness of strong G-linearizations, we can patch the above G-sheaf structures together to the desired strong G-linearization on the intersection $\Gamma' \subset \Gamma$ of the subgroups $\Gamma_i \subset \Gamma$, and our claim is proved.

Now, since the character group of $N(T)$ is finite, Proposition 1.5 tells us that on some subgroup $\Gamma'' \subset \Gamma'$ of finite index, the canonical $N(T)$-linearization and the one induced by the G-linearization coincide. Thus restricting Γ'' to X provides the desired subgroup $\Lambda' \subset \Lambda$ of finite index. We replace Λ with Λ'.

In order to obtain a quotient of X by G, we want to apply Theorem 3.1. So we have to show that $X^{ss}(\Lambda, G)$ equals X. For this, let $\mathcal{A}$ be the graded $\mathcal{O}_X$-algebra associated to Λ, and set $\widehat{X} := \operatorname{Spec}(\mathcal{A})$. Moreover, let $q\colon \widehat{X} \to X$ be the canonical map and $H := \operatorname{Spec}(\mathbb{K}[\Lambda])$ the torus acting on $\widehat{X}$. Note that

$$X^{ss}(\Lambda, T) = X^{ss}(\Lambda, N(T)) = X.$$

Choose G-invariant homogeneous $f_1, \ldots, f_r \in \mathcal{A}(X)$ and T-invariant homogeneous $h_1, \ldots, h_s \in \mathcal{A}(X)$ such that the complements $X \setminus Z(f_i)$ and $X \setminus Z(h_i)$ satisfy the condition of Definition 2.1 i) and

$$\begin{aligned} X^{ss}(\Lambda, G) &= (X \setminus Z(f_1)) \cup \ldots \cup (X \setminus Z(f_r)), \\ X^{ss}(\Lambda, T) &= (X \setminus Z(h_1)) \cup \ldots \cup (X \setminus Z(h_r)). \end{aligned}$$

Since Λ is ample, Proposition 1.9 yields a $(G \times H)$-equivariant affine closure $\overline{X}$ of $\widehat{X}$ such that the f_i and the h_j extend to regular functions on $\overline{X}$ satisfying $\overline{X}_{f_i} = \widehat{X}_{f_i}$ and $\overline{X}_{h_j} = \widehat{X}_{h_j}$. Moreover, we obtain a commutative diagram of H-equivariant maps:

$$\begin{tikzcd}
\overline{X} \arrow[rr, "\overline{p}_G", "/\!\!/ G"'] \arrow[dr, "/\!\!/ T", "\overline{p}_T"'] & & \overline{X} /\!\!/ G \\
& \overline{X} /\!\!/ T \arrow[ur] &
\end{tikzcd}$$

Now, let $x \in X$, and assume that x is not semistable with respect to G. Choose $z \in q^{-1}(x)$, and let $y := \overline{p}_G(z)$. By Proposition 2.3 ii), the assumption $x \notin X^{ss}(\Lambda, G)$ amounts to $y \in \overline{p}_G(\overline{X} \setminus \widehat{X})$ or to an isotropy group H_y of positive dimension.

First suppose that we have $y \in \overline{p}_G(\overline{X} \setminus \widehat{X})$. Let $G \cdot z'$ be the closed orbit in $\overline{p}_G^{-1}(y)$. Then $G \cdot z'$ is contained in $\overline{X} \setminus \widehat{X}$. Moreover, the Hilbert-Mumford-Birkes Lemma [10], provides a maximal torus $T' \subset G$ such that the closure of $T' \cdot z$ intersects $G \cdot z'$.

Let $g \in G$ with $gT'g^{-1} = T$. Then the closure of $T{\cdot}g{\cdot}z$ contains a point $z'' \in G{\cdot}z'$. Surely, $\overline{p}_T(g{\cdot}z)$ equals $\overline{p}_T(z'')$. Thus, since $z'' \in \overline{X} \setminus \widehat{X}$, Proposition 2.3 ii) tells us that $g{\cdot}x = q(g{\cdot}z)$ is not semistable with respect to T. A contradiction.

As the situation $y \in \overline{p}_G(\overline{X} \setminus \widehat{X})$ is excluded, the isotropy group H_y is of positive dimension, and the whole fibre $\overline{p}_G^{-1}(y)$ is contained in $\widehat{X}$. Let $H_0 \subset H_y$ be the connected component of the neutral element. Then H_0 acts freely on the fibre $\overline{p}_G^{-1}(y)$, and the closed orbit $G{\cdot}z' \subset \overline{p}_G^{-1}(y)$ is invariant by H_0.

Let $\mu\colon g \mapsto g{\cdot}z'$ denote the orbit map. Since the actions of G and H_0 commute, $G' := \mu^{-1}(H_0{\cdot}z')$ is a subgroup of G. Since $H_0{\cdot}z' \cong H_0$, there is a torus $S' \subset G'$ with $\mu(S') = H_0{\cdot}z'$, use for example [11, Proposition IV.11.20].

Let $T' \subset G$ be a maximal torus with $S' \subset T'$ and choose $g \in G$ with $T = gT'g^{-1}$. Then $H_0{\cdot}g{\cdot}z'$ equals $(gS'g^{-1}){\cdot}g{\cdot}z'$. According to Proposition 2.3 ii), the point $q(g{\cdot}z')$ is not semistable with respect to T. A contradiction. So, every $x \in X$ is semistable with respect to G, and the implication "ii)⇒i)" is proved.

We come to the supplement concerning separateness. Clearly, existence of a good quotient $X \to X/\!\!/G$ with $X/\!\!/G$ separated implies that also the quotient space $X/\!\!/N(T)$ is separated.

For the converse, suppose that $X \to X/\!\!/N(T)$ exists with a separated divisorial $X/\!\!/N(T)$. Then there is a good quotient $X \to X/\!\!/T$ with a separated quotient space $X/\!\!/T$, and [6, Theorem 5.4] implies that also the quotient space $X/\!\!/G$ is separated. □

In the proof of Theorem 5.2, we shall use that geometric quotient spaces of proper actions inherit $\mathbb{Q}$-factoriality. By the lack of a reference for this presumably well-known fact, we give here a proof:

Lemma 5.5. *Suppose that a reductive group H acts regularly with finite isotropy groups on a variety Y and that there is a geometric quotient $p\colon Y \to Y/H$. If Y is $\mathbb{Q}$-factorial, then so is Y/H.*

Proof. Assume that Y is $\mathbb{Q}$-factorial, and let $E \subset Y/H$ be a prime divisor. Then $p^{-1}(E)$ is a union of prime divisors $D_1, \ldots, D_r$. Some multiple mD of the divisor $D := D_1 + \ldots + D_r$ is Cartier. Using Lemma 4.2 and Proposition 1.3, we see that the group of Cartier divisors generated by mD is canonically strongly H-linearized.

Enlarging m, we achieve that the sheaf $\mathcal{A}_{mD}$ is equivariantly isomorphic to the pullback $p^*(\mathcal{L})$ of some invertible sheaf $\mathcal{L}$ on Y/H, use e.g. [17, Proposition 4.2]. The canonical section $1 \in \mathcal{A}_{mD}(Y)$ is H-invariant and hence induces a section $f \in \mathcal{L}(Y/H)$ having precisely E as its set of zeroes. □

Proof of Theorem 5.2. If one of the quotients exists, then by [19, Section 0.4] and Lemma 5.5, the quotient space is separated and $\mathbb{Q}$-factorial. Now, existence of a geometric quotient $X \to X/G$ surely implies existence of a geometric quotient $X/N(T)$. Conversely, if $X/N(T)$ exists, then it is $\mathbb{Q}$-factorial. Hence Theorem 5.1 yields a geometric quotient $X \to X/G$. □

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