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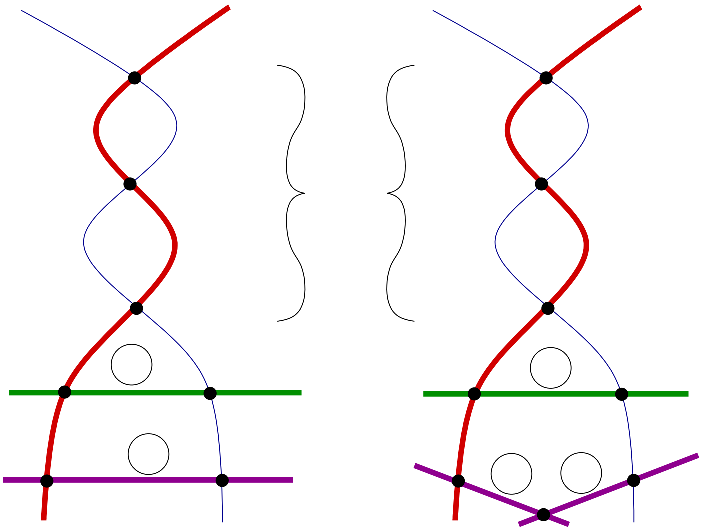

Minimal regular resolution $X_H(p)'$ of $X_H(p)$
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On Families of Pure Slope L-Functions

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Abstract. Let R be the ring of integers in a finite extension K of $\mathbb{Q}_p$, let k be its residue field and let $\chi : \pi_1(X) \to R^\times = GL_1(R)$ be a "geometric" rank one representation of the arithmetic fundamental group of a smooth affine k-scheme X. We show that the locally K-analytic characters $\kappa : R^\times \to \mathbb{C}_p^\times$ are the $\mathbb{C}_p$-valued points of a K-rigid space $\mathcal{W}$ and that

$$L(\kappa\circ\chi, T) = \prod_{\overline{x}\in X} \frac{1}{1-(\kappa\circ\chi)(Frob_{\overline{x}})T^{\deg(\overline{x})}},$$

viewed as a two variable function in T and κ, is meromorphic on $\mathbb{A}^1_{\mathbb{C}_p}\times\mathcal{W}$. On the way we prove, based on a construction of Wan, a slope decomposition for ordinary overconvergent (finite rank) σ-modules, in the Grothendieck group of nuclear σ-modules.

2000 Mathematics Subject Classification: Primary 14F30; Secondary 14G10, 14G13, 14G15, 14G22

Introduction

In a series of remarkable papers [14] [15] [16], Wan recently proved a long outstanding conjecture of Dwork on the p-adic meromorphic continuation of unit root L-functions arising from an ordinary family of algebraic varieties defined over a finite field k. We begin by illustrating his result by a concrete example. Fix $n \geq 0$ and let Y be the affine $n+1$-dimensional $\mathbb{F}_p$-variety in $\mathbb{A}^1\times\mathbb{G}_m^{n+1}$ defined by

$$z^p - z = x_0 + \ldots + \ldots x_n.$$

Define $u : Y \to \mathbb{G}_m$ by sending $(z, x_0, \ldots, x_n)$ to $x_0x_1\cdots x_n$. For $r \geq 1$ and $y \in \mathbb{F}_{p^r}^\times$ let $Y_y/\mathbb{F}_{p^r}$ be the fibre of u above y. For $m \geq 1$ let $Y_y(\mathbb{F}_{p^{rm}})$ be the set of $\mathbb{F}_{p^{rm}}$-rational points and $(Y_y)_0$ the set of closed points of $Y_y/\mathbb{F}_{p^r}$ (a closed point z is an orbit of an $\overline{\mathbb{F}}_{p^r}$-valued point under the p^r-th power Frobenius map σ_{p^r}; its degree $\deg_r(z)$ is the smallest positive integer d such that $\sigma_{p^r}^d$ fixes the orbit pointwise). The zeta function of $Y_y/\mathbb{F}_{p^r}$ is

$$Z(Y_y/\mathbb{F}_{p^r}, T) = \exp(\sum_{m=1}^{\infty} \frac{|Y_y(\mathbb{F}_{p^{rm}})|}{m} T^m) = \prod_{z \in (Y_y)_0} \frac{1}{1 - T^{\deg_r(z)}}.$$

On the other hand for a character $\Psi : \mathbb{F}_p \to \mathbb{C}$ define the Kloosterman sum

$$K_m(y) = \sum_{\substack{x_i \in \mathbb{F}_{p^{rm}}^\times \\ x_0 x_1 \cdots x_n = y}} \Psi(\mathrm{Tr}_{\mathbb{F}_{p^{rm}}/\mathbb{F}_p}(x_0 + x_1 + \ldots + x_n))$$

and let $L_\Psi(Y, T)$ be the series such that

$$T\mathrm{dlog} L_\Psi(y, T) = \sum_{m=1}^{\infty} K_m(y) T^m.$$

Then, as series,

$$\prod_\Psi L_\Psi(Y, T) = Z(Y_y/\mathbb{F}_{p^r}, T),$$

hence to understand $Z(Y_y/\mathbb{F}_{p^r}, T)$ we need to understand all the $L_\Psi(y, T)$. Suppose Ψ is non-trivial. It is known that $L_\Psi(y, T)$ is a polynomial of degree $n+1$: there are algebraic integers $\alpha_0(y), \ldots, \alpha_n(y)$ such that

$$L_\Psi(y, T)^{(-1)^{n-1}} = (1 - \alpha_0(y)T) \cdots (1 - \alpha_n(y)T).$$

These $\alpha_i(y)$ have complex absolute value $p^{rn/2}$ and are ℓ-adic units for any prime $\ell \neq p$. We ask for their p-adic valuation and their variation with y. Embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ we have $\alpha_i(y) \in \mathbb{Q}_p(\pi)$ where $\pi^{p-1} = -p$. Sperber has shown that we may order the $\alpha_i(y)$ such that $\mathrm{ord}_p(\alpha_i(y)) = i$ for any $0 \leq i \leq n$. Fix such an i and for $k \in \mathbb{Z}$ consider the L-function

$$\prod_{y \in (\mathbb{G}_m)_0/\mathbb{F}_p} \frac{1}{1 - \alpha_i^k(y) T^{\deg_1(y)}}$$

(here $\deg_1(y)$ is the minimal r such that $y \in \mathbb{F}_{p^r}^\times$, and $(\mathbb{G}_m)_0/\mathbb{F}_p$ is the set of closed points of $\mathbb{G}_m/\mathbb{F}_p$ defined similarly as before). A priori this series defines a holomorphic function only on the open unit disk. Dwork conjectured and Wan proved that it actually extends to a meromorphic function on $\mathbb{A}^1_{\mathbb{C}_p}$, and varies uniformly with k in some sense. Now let $\mathcal{W}$ be the rigid space of locally $\mathbb{Q}_p(\pi)$-analytic characters of the group of units in the ring of integers of $\mathbb{Q}_p(\pi)$. In this paper we show that

$$L(T, \kappa) = \prod_{y \in (\mathbb{G}_m)_0/\mathbb{F}_p} \frac{1}{1 - \kappa(\alpha_i(y)) T^{\deg_1(y)}}$$

defines a meromorphic function on $\mathbb{A}^1_{\mathbb{C}_p} \times \mathcal{W}$. Specializing $\kappa \in \mathcal{W}$ to the character $r \mapsto r^k$ for $k \in \mathbb{Z}$ we recover Wan's result. The conceptual way to think of this example is in terms of σ-modules: $\mathbb{F}_p$ acts on Y via $z \mapsto z + a$ for $a \in \mathbb{F}_p$. It induces an action of $\mathbb{F}_p$ on the relative n-th rigid cohomology $\mathbb{R}^n u_{rig,*}\mathcal{O}_Y$ of u, and over $\mathbb{Q}_p(\pi)$ the latter splits up into its eigencomponents for the various characters of $\mathbb{F}_p$. The Ψ-eigencomponent $(\mathbb{R}^n u_{rig,*}\mathcal{O}_Y)^\Psi$ is an overconvergent σ-module and $L_\Psi(y,T)^{(-1)^{n-1}}$ is the characteristic polynomial of Frobenius acting on its fibre in y. Crucial is the slope decomposition of $(\mathbb{R}^n u_{rig,*}\mathcal{O}_Y)^\Psi$: it means that for fixed i the $\alpha_i(y)$ vary rigid analytically with y in some sense. We are thus led to consider Dwork's conjecture, i.e. Wan's theorem, in the following general context.

Let R be the ring of integers in a finite extension K of $\mathbb{Q}_p$, let π be a uniformizer and k the residue field. Let X be a smooth affine k-scheme, let A be the coordinate ring of a lifting of X to a smooth affine weak formal R-scheme (so A is a wcfg-algebra) and let $\widehat{A}$ be the p-adic completion of A. Let σ be an R-algebra endomorphism of A lifting the q-th power Frobenius endomorphism of X, where $q = |k|$. A finite rank σ-module over $\widehat{A}$ (resp. over A) is a finite rank free $\widehat{A}$-module (resp. A-module) together with a σ-linear endomorphism ϕ. A finite rank σ-module over $\widehat{A}$ is called overconvergent if it arises by base change $A \to \widehat{A}$ from a finite rank σ-module over A. Let the finite rank overconvergent σ-module Φ over $\widehat{A}$ be ordinary, in the strong sense that it admits a Frobenius stable filtration such that on the j-th graded piece we have: the Frobenius is divisible by π^j and multiplied with π^{-j} it defines a unit root σ-module Φ_j, i.e. a σ-module whose linearization is bijective. (Recall that unit root σ-modules over $\hat{A}$ are the same as continuous representations of $\pi_1(X)$ on finite rank free R-modules.) Although Φ is overconvergent, Φ_j will in general not be overconvergent; and this is what prevented Dwork from proving what is now Wan's theorem: the L-function $L(\Phi_j, T)$ is meromorphic on $\mathbb{A}^1_{\mathbb{C}_p}$. Moreover he proved the same for powers (=iterates of the σ-linear endomorphism) Φ_j^k of Φ_j and showed that in case Φ_j is of rank one the family $\{L(\Phi_j^k, T)\}_{k\in\mathbb{Z}}$ varies uniformly with $k \in \mathbb{Z}$ in a certain sense. At the heart of Wan's striking method lies his "limiting σ-module" construction which allows him to reduce the analysis of the not necessarily overconvergent Φ_j to that of overconvergent σ-modules — at the cost of now working with overconvergent σ-modules of infinite rank, but which are nuclear. To the latter a generalization of the Monsky trace formula can be applied which expresses $L(\Phi_j^k, T)$ as an alternating sum of Fredholm determinants of completely continuous Dwork operators.

The first aim of this paper is to further explore the significance of the limiting σ-module construction which we think to be relevant for the search of good p-adic coefficients on varieties in characteristic p. Following an argument of Coleman [4] we give a functoriality result for this construction. This is then used to prove (Theorem 7.2) a slope decomposition for ordinary overconvergent finite rank σ-modules, in the Grothendieck group $\Delta(\widehat{A})$ of nuclear σ-modules over $\widehat{A}$. More precisely, we show that any Φ_j as above, not necessarily overconvergent, can be written, in $\Delta(\widehat{A})$, as a sum of virtual nuclear *overconvergent* σ-modules. (This is the global version of the decomposition of the corresponding L-function found by Wan.) Our second aim is to strengthen Wan's uniform results on the family $\{L(\Phi_j^k, T)\}_{k\in\mathbb{Z}}$ in case Φ_j is of rank one. More generally we replace Φ_j by the rank one unit root σ-module $\det(\Phi_j)$ if Φ_j has rank >1. Let $\det\Phi_j$ be given by the action of $\alpha\in\widehat{A}^\times$ on a basis element. For $\overline{x}\in X$ a closed point of degree f let $x:\widehat{A}\to R_f$ be its Teichmüller lift, where R_f denotes the unramified extension of R of degree f. Then

$$\alpha_{\overline{x}} = x(\alpha\sigma(\alpha)\dots\sigma^{f-1}(\alpha))$$

lies in $R^\times$. We prove that for any locally K-analytic character $\kappa: R^\times\to\mathbb{C}_p^\times$ the twisted L-function

$$L(\alpha,T,\kappa)=\prod_{\overline{x}\in X}\frac{1}{1-\kappa(\alpha_{\overline{x}})T^{\deg(\overline{x})}}$$

is p-adic meromorphic on $\mathbb{A}^1_{\mathbb{C}_p}$, and varies rigid analytically with κ. More precisely, building on work of Schneider and Teitelbaum [13], we use Lubin-Tate theory to construct a smooth $\mathbb{C}_p$-rigid analytic variety $\mathcal{W}$ whose $\mathbb{C}_p$-valued points are in natural bijection with the set $\mathrm{Hom}_{K\text{-}an}(R^\times,\mathbb{C}_p^\times)$ of locally K-analytic characters of $R^\times$. Then our main theorem is:

Theorem 0.1. *On the $\mathbb{C}_p$-rigid space $\mathbb{A}^1_{\mathbb{C}_p}\times\mathcal{W}$ there exists a meromorphic function L_α whose pullback to $\mathbb{A}^1_{\mathbb{C}_p}$ via $\mathbb{A}^1_{\mathbb{C}_p}\to\mathbb{A}^1_{\mathbb{C}_p}\times\mathcal{W}$, $t\mapsto(t,\kappa)$ for any $\kappa\in\mathrm{Hom}_{K\text{-}an}(R^\times,\mathbb{C}_p^\times)=\mathcal{W}(\mathbb{C}_p)$ is a continuation of $L(\alpha,T,\kappa)$.*

The statement in the abstract above follows by the well known correspondence between representations of the fundamental group and unit-root σ-modules. The analytic variation of the L-series $L(\alpha,T,\kappa)$ with the weight κ makes it meaningful to vastly generalize the eigencurve theme studied by Coleman and Mazur [5] in connection with the Gouvêa-Mazur conjecture. Namely, we can ask for the divisor of the two variable meromorphic function L_α on $\mathbb{A}^1_{\mathbb{C}_p}\times\mathcal{W}$. From a general principle in [3] we already get: for fixed $\lambda\in\mathbb{R}_{>0}$, the difference between the numbers of poles and zeros of L_α on the annulus $|T|=\lambda$ is locally constant on $\mathcal{W}$. We hope for better qualitative results if the σ-module over A giving rise to the σ-module Φ over $\widehat{A}$ carries an overconvergent integrable connection, i.e. is an overconvergent F-isocrystal on X in the sense of Berthelot. The eigencurve from [5] comes about in this context as follows: The Fredholm determinant of the U_p-operator acting on overconvergent p-adic modular forms is a product of certain power rank one unit root L-functions arising from the universal ordinary elliptic curve, see [3]. Also, again in the general case, the p-adic L-function on $\mathcal{W}$ which we get by specializing $T=1$ in L_α should be of particular interest.

The proof of Theorem 0.1 consists of two steps. First we prove (this is essentially Corollary 4.12) the meromorphic continuation to $\mathbb{A}^1_{\mathbb{C}_p}\times\mathcal{W}^0$ for a certain open subspace $\mathcal{W}^0$ of $\mathcal{W}$ which meets every component of $\mathcal{W}$: the subspace of characters of the type $\kappa(r) = r^\ell u(r)^x$ for $\ell \in \mathbb{Z}$ and small $x \in \mathbb{C}_p$, with $u(r)$ denoting the one-unit part of $r \in R^\times$. (In particular, $\mathcal{W}^0$ contains the characters $r \mapsto \kappa_k(r) = r^k$ for $k \in \mathbb{Z}$; for these we have $L(\Phi_j^k, T) = L(\alpha, T, \kappa_k)$.) For this we include $\det(\Phi_j)$ in a *family* of nuclear σ-modules, parametrized by $\mathcal{W}^0$: namely, the factorization into torsion part and one-unit part and then exponentiation with $\ell \in \mathbb{Z}$ resp. with small $x \in \mathbb{C}_p$ makes sense not just for $R^\times$-elements but also for α, hence an analytic family of rank one unit root σ-modules parametrized by $\mathcal{W}^0$. In the Grothendieck group of $\mathcal{W}^0$-parametrized *families* of nuclear σ-modules, we write this deformation family of $\det(\Phi_j)$ as a sum of virtual families of nuclear *overconvergent* σ-modules. In each fibre $\kappa \in \mathcal{W}^0$ we thus obtain, by an infinite rank version of the Monsky trace formula, an expression of the L-function $L(\alpha, T, \kappa)$ as an alternating product of characteristic series of nuclear Dwork operators. While this is essentially an "analytic family version" of Wan's proof (at least if $X = \mathbb{A}^n$), the second step, the extension to the whole space $\mathbb{A}^1_{\mathbb{C}_p} \times \mathcal{W}$, needs a new argument. We use a certain integrality property (w.r.t. $\mathcal{W}$) of the coefficients of (the logarithm of) L_α which we play out against the already known meromorphic continuation on $\mathbb{A}^1_{\mathbb{C}_p} \times \mathcal{W}^0$. However, we are not able to extend the limiting modules from $\mathcal{W}^0$ to all of $\mathcal{W}$; as a consequence, for $\kappa \in \mathcal{W} - \mathcal{W}^0$ we have no interpretation of $L(\alpha, T, \kappa)$ as an alternating product of characteristic series of Dwork operators. Note that for $K = \mathbb{Q}_p$, the locally K-analytic characters of $R^\times = \mathbb{Z}_p^\times$ are precisely the *continuous* ones; the space $\mathcal{W}^0$ in that case is the weight space considered in [3] while $\mathcal{W}$ is that of [5].

Now let us turn to some technical points. Wan develops his limiting σ-module construction and the Monsky trace formula for nuclear overconvergent infinite rank σ-modules only for the base scheme $X = \mathbb{A}^n$. General base schemes X he embeds into $\mathbb{A}^n$ and treats (the pure graded pieces of) finite rank overconvergent σ-modules on X by lifting them with the help of Dwork's F-crystal to σ-modules on $\mathbb{A}^n$ having the same L-functions. We work instead in the infinite rank setting on arbitrary X. Here we need to overcome certain technical difficulties in extending the finite rank Monsky trace formula to its infinite rank version. The characteristic series through which we want to express the L-function are those of certain Dwork operators ψ on spaces of overconvergent functions with *non fixed* radius of overconvergence. To get a hand on these ψ's one needs to write these overconvergent function spaces as direct limits of appropriate affinoid algebras on which the restrictions of the ψ's are completely continuous. Then statements on the ψ's can be made if these affinoid algebras have a common system of orthogonal bases. Only for $X = \mathbb{A}^n$ we find such bases; but we show how one can pass to the limit also for general X. An important justification for proving the trace formula in this form (on general X, with function spaces with *non fixed* radius of overconvergence) is that in the future it will allow us to make full use of the overconvergent connection in case the σ-module over A giving rise to the σ-module Φ over $\widehat{A}$ underlies

an overconvergent F-isocrystal on X (see above) — then the limiting module also carries an overconvergent connection. Deviating from [14] [15], instead of working with formally free nuclear σ-modules with fixed formal bases we work, for concreteness, with the infinite square matrices describing them. This is of course only a matter of language.
A brief overview. In section 1 we show the existence of common orthogonal bases in overconvergent ideals which might be of some independent interest. In section 2 we define the L-functions and prove the trace formula. In section 3 we introduce the Grothendieck group of nuclear σ-modules (and their deformations). In section 4 we concentrate on the case where ϕ_j is the unit root part of ϕ and is of rank one: here we need the limiting module construction. In section 5 we introduce the weight space $\mathcal{W}$, in section 6 we prove (an infinite rank version of) Theorem 0.1, and in section 7 (which logically could follow immediately after section 4) we give the overconvergent representation of Φ_j.

ACKNOWLEDGMENTS: I wish to express my sincere thanks to Robert Coleman and Daqing Wan. Manifestly this work heavily builds on ideas of them, above all on Wan's limiting module construction. Wan invited me to begin further elaborating his methods, and directed my attention to many interesting problems involved. Coleman asked me for the meromorphic continuation to the whole character space and provided me with some helpful notes [4]. In particular the important functoriality result 4.10 for the limiting module and the suggestion of varying it rigid analytically is due to him. Thanks also to Matthias Strauch for discussions on the weight space.

Notations: By $|.|$ we denote an absolute value of K and by $e \in \mathbb{N}$ the absolute ramification index of K. By $\mathbb{C}_p$ we denote the completion of a fixed algebraic closure of K and by ord_π and ord_p the homomorphisms $\mathbb{C}_p^\times \to \mathbb{Q}$ with $\mathrm{ord}_\pi(\pi) = \mathrm{ord}_p(p) = 1$. For R-modules E with $\pi E \neq E$ we set

$$\mathrm{ord}_\pi(x) := \sup\{r \in \mathbb{Q};\, r = \frac{n}{m} \text{ for some } n \in \mathbb{N}_0, m \in \mathbb{N} \text{ such that } x^m \in \pi^n E\}$$

for $x \in E$. Similarly we define ord_p on such E. For $n \in \mathbb{N}$ we write $\mu_n = \{x \in \mathbb{C}_p;\, x^n = 1\}$. We let $\mathbb{N}_0 = \mathbb{Z}_{\geq 0}$. For an element g in a free polynomial ring $A[X_1, \ldots, X_n]$ over a ring A we denote by $\deg(g)$ its (total) degree.

1 ORTHONORMAL BASES OF OVERCONVERGENT IDEALS

In this preparatory section we determine explicit orthonormal K-bases of ideals in overconvergent K-Tate algebras T_n^c (1.5). Furthermore we recall the complete continuity of certain Dwork operators (1.7).

1.1 For $c \in \mathbb{N}$ we let

$$T_n^c := \{ \sum_{\alpha \in \mathbb{N}_0^n} b_\alpha \pi^{[\frac{|\alpha|}{c}]} X^\alpha; \quad b_\alpha \in K, \lim_{|\alpha| \to \infty} |b_\alpha| = 0\}$$

where as usual $|\alpha| = \sum_{i=1}^n \alpha_i$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and where $[r] \in \mathbb{Z}$ for a given $r \in \mathbb{Q}$ denotes the unique integer with $[r] \leq r < [r] + 1$. This is the ring of power series in $X_1, \ldots, X_n$ with coefficients in K, convergent on the polydisk

$$\{x \in \mathbb{C}_p^n; \quad \mathrm{ord}_\pi(x_i) \geq -\frac{1}{c} \text{ for all } 1 \leq i \leq n\}.$$

We view T_n^c as a K-Banach module with the unique norm $|.|_c$ for which $\{\pi^{[\frac{|\alpha|}{c}]} X^\alpha\}_{\alpha \in \mathbb{N}_0^n}$ is an orthonormal basis (this norm is not power multiplicative). Suppose we are given elements $g_1, \ldots, g_r \in R[X_1, \ldots, X_n] - \pi R[X_1, \ldots, X_n]$. Let $\overline{g}_j \in k[X_1, \ldots, X_n]$ be the reduction of g_j, let $d_j = \deg(\overline{g}_j) \leq \deg(g_j)$ be its degree.

Lemma 1.2. *For each $1 \leq j \leq r$ and each $c > \max_j \deg(g_j)$ we have*

$$|\pi^{[\frac{|\alpha|+d_j}{c}]} X^\alpha g_j|_c = 1.$$

Proof: Write $g_j = \sum_{\beta \in \mathbb{N}_0^n} b_\beta X^\beta$ with $b_\beta \in K$. There exists a $\beta_1 \in \mathbb{N}_0^n$ with $|\beta_1| = d_j$ and $|b_{\beta_1}| = 1$. Hence

$$|\pi^{[\frac{|\alpha|+d_j}{c}]} X^\alpha b_{\beta_1} X^{\beta_1}|_c = |\pi^{[\frac{|\alpha|+|\beta_1|}{c}]} X^{\alpha+\beta_1}|_c = 1.$$

Now let $\beta \in \mathbb{N}_0^n$ be arbitrary, with $b_\beta \neq 0$. If $|\beta| > d_j$ then $|b_\beta| \leq |\pi|$. Hence

$$|\pi^{[\frac{|\alpha|+d_j}{c}]} X^\alpha b_\beta X^\beta|_c \leq |\pi^{[\frac{|\alpha|+d_j}{c}] - [\frac{|\alpha|+|\beta|}{c}] + 1} \pi^{[\frac{|\alpha|+|\beta|}{c}]} X^{\alpha+\beta}|_c.$$

But $[\frac{|\alpha|+d_j}{c}] - [\frac{|\alpha|+|\beta|}{c}] + 1 \geq 0$ because $b_\beta \neq 0$, hence $c > |\beta|$. Thus,

$$|\pi^{[\frac{|\alpha|+d_j}{c}]} X^\alpha b_\beta X^\beta|_c \leq 1.$$

On the other hand, if $|\beta| \leq d_j$, then $[\frac{|\alpha|+d_j}{c}] \geq [\frac{|\alpha|+|\beta|}{c}]$ and $|b_\beta| \leq 1$, and again we find

$$|\pi^{[\frac{|\alpha|+d_j}{c}]} X^\alpha b_\beta X^\beta|_c \leq 1.$$

We are done.

1.3 The Tate algebra in n variables over K is the algebra

$$T_n := \{ \sum_{\alpha \in \mathbb{N}_0^n} b_\alpha X^\alpha; \quad b_\alpha \in K, \lim_{|\alpha| \to \infty} |b_\alpha| = 0\}.$$

Let I_∞ (resp. I_c) be the ideal in T_n (resp. in T_n^c) generated by $g_1, \ldots, g_r$. As all ideals in T_n^c, the ideal I_c is closed in T_n^c. We view I_c as a K-Banach module with the norm $|.|_c$ induced from T_n^c.

Lemma 1.4. *If $I_\infty \subset T_n$ is a prime ideal, $I_\infty \neq T_n$, then $I_c = I_\infty \cap T_n^c$ for $c >> 0$.*

Proof: For $c >> 0$ also I_c is a prime ideal in T_n^c. The open immersion of K-rigid spaces $\mathrm{Sp}(T_n) \to \mathrm{Sp}(T_n^c)$ induces an open immersion $V(I_\infty) \to V(I_c)$ of the respective zero sets of $g_1, \ldots, g_r$. That I_c is prime means that $V(I_c)$ is irreducible, and $I_\infty \neq T_n$ means that $V(I_\infty)$ is non empty. Hence an element of $I_\infty \cap T_n^c$, since it vanishes on $V(I_\infty)$, necessarily also vanishes on $V(I_c)$. By Hilbert's Nullstellensatz ([2]) it is then an element of I_c.

Now we fix an integer $c' > \max_j \deg(g_j)$. By 1.2 we find a subset E of $\mathbb{N}_0^n \times \{1, \ldots, r\}$ such that $\{\pi^{[\frac{|\alpha|+d_j}{c'}]} X^\alpha g_j\}_{(\alpha,j)\in E}$ is an orthonormal basis of $I_{c'}$ over K.

Theorem 1.5. *For integers $c \geq c'$, the set $\{\pi^{[\frac{|\alpha|+d_j}{c}]} X^\alpha g_j\}_{(\alpha,j)\in E}$ is an orthonormal basis of I_c over K.*

Proof: Let K^c be a finite extension of K containing a c-th root $\pi^{\frac{1}{c}}$ and a c'-th root $\pi^{\frac{1}{c'}}$ of π. The absolute value $|.|$ extends to K^c. Any norm on a K-Banach module M extends uniquely to a K^c-Banach module norm on $M \otimes_K K^c$, and we keep the same name for it. It is enough to show that $\{\pi^{[\frac{|\alpha|+d_j}{c}]} X^\alpha g_j\}_{(\alpha,j)\in E}$ is an orthonormal basis of $I_c \otimes_K K^c$ over K^c. Let $|.|_c'$ be the supremum norm on $T_n^c \otimes_K K^c$. This is the norm for which $\{X^{\frac{\alpha}{c}}\}_{\alpha\in\mathbb{N}_0^n}$ is an orthonormal basis over K^c. For $j \in \{1, \ldots, r\}$ write $g_j = \sum_{\beta\in\mathbb{N}_0^n} b_\beta X^\beta$ with $b_\beta \in K$. Then, by a computation similar to that in 1.2 we find

$$|\pi^{\frac{|\alpha|+d_j}{c}} X^\alpha b_\beta X^\beta|_c' = 1 \quad \text{if } |\beta| = d_j \text{ and } |b_\beta| = 1,$$

$$|\pi^{\frac{|\alpha|+d_j}{c}} X^\alpha b_\beta X^\beta|_c' < 1 \quad \text{otherwise.}$$

In particular it follows that $|\pi^{\frac{|\alpha|+d_j}{c}} X^\alpha g_j|_c' = 1$. Now a comparison of expansions shows that $\{\pi^{[\frac{|\alpha|+d_j}{c}]} X^\alpha g_j\}_{(\alpha,j)\in E}$ is an orthonormal basis of $I_c \otimes_K K^c$ over K^c with respect to $|.|_c$ if and only if $\{\pi^{\frac{|\alpha|+d_j}{c}} X^\alpha g_j\}_{(\alpha,j)\in E}$ is an orthonormal basis of $I_c \otimes_K K^c$ over K^c with respect to $|.|_c'$. In particular it follows on the one hand that we only need to show that $\{\pi^{\frac{|\alpha|+d_j}{c}} X^\alpha g_j\}_{(\alpha,j)\in E}$ is an orthonormal basis of $I_c \otimes_K K^c$ over K^c with respect to $|.|_c'$, and on the other hand it follows (applying the above with c' instead of c) that $\{\pi^{\frac{|\alpha|+d_j}{c'}} X^\alpha g_j\}_{(\alpha,j)\in E}$ is an orthonormal basis of $I_{c'} \otimes_K K^c$ over K^c with respect to $|.|_{c'}'$. Consider the isomorphism

$$T_n^c \otimes_K K^c \cong T_n^{c'} \otimes_K K^c, \qquad \pi^{\frac{\alpha}{c}} X^\alpha \mapsto \pi^{\frac{\alpha}{c'}} X^\alpha$$

which is isometric with respect to $|.|_c'$ resp. $|.|_{c'}'$. It does not necessarily map $I_c \otimes_K K^c$ to $I_{c'} \otimes_K K^c$. However, from our above computations of the values

$|\pi^{\frac{|\alpha|+d_j}{c}}X^{\alpha}b_{\beta}X^{\beta}|'_c$ it follows that this isomorphism identifies the reductions of the elements of the set $\{\pi^{\frac{|\alpha|+d_j}{c}}X^{\alpha}g_j\}_{(\alpha,j)\in\mathbb{N}_0^n\times\{1,\ldots,r\}}$ with the reductions of the elements of the set $\{\pi^{\frac{|\alpha|+d_j}{c'}}X^{\alpha}g_j\}_{(\alpha,j)\in\mathbb{N}_0^n\times\{1,\ldots,r\}}$ (here by reduction we mean reduction modulo elements of absolute value < 1). The K^c-vector subspaces spanned by these sets are dense in $I_c\otimes_K K^c$ resp. in $I_{c'}\otimes_K K^c$. Since for a subset of $|.|=1$ elements in an orthonormizable K^c-Banach module the property of being an orthonormal basis is equivalent to that of inducing an (algebraic) basis of the reduction, the theorem follows.

1.6 Let B_K be a reduced K-affinoid algebra, i.e. a quotient of a Tate algebra T_m over K (for some m), endowed with its supremum norm $|.|_{\sup}$. Let

$$B=(B_K)^0:=\{b\in B_K;\quad |b|_{\sup}\leq 1\}.$$

For positive integers m and c let

$$[m,c]:=[m,c]_R:=\{z\in R[[X_1,\ldots,X_n]];$$

$$z=\sum_{j=0}^{\infty}\pi^jp_j \text{ with } p_j\in R[X_1,\ldots,X_n] \text{ and } \deg(p_j)\leq m+cj\}$$

and

$$[m,c]_B:=[m,c]\widehat{\otimes}_R B$$

(the π-adically completed tensor product). Note that for $m,c_1,c_2\in\mathbb{N}$ with $c_1<c_2$ we have $[m,c_1]_B\subset T_n^{c_2}\widehat{\otimes}_K B_K$ and also $(\cup_{m,c}[m,c]_B)\otimes_R K=\cup_c(T_n^c\widehat{\otimes}_K B_K)$. Let

$$R[X_1,\ldots,X_n]^{\dagger}:=R[X]^{\dagger}:=\bigcup_{m,c}[m,c].$$

Fix a Frobenius endomorphism σ of $R[X]^{\dagger}$ lifting the q-th power Frobenius endomorphism of $k[X]$. Also fix a Dwork operator θ (with respect to σ) on $R[X]^{\dagger}$, i.e. an R-module endomorphism with $\theta(\sigma(x)y)=x\theta(y)$ for all $x,y\in R[X]^{\dagger}$. By [8] 2.4 we have $\theta(T_n^c)\subset T_n^c$ for all $c>>0$, thus we get a B_K-linear endomorphism $\theta\otimes 1$ on $T_n^c\widehat{\otimes}_K B_K$.

Proposition 1.7. *Let I be a countable set, m',c' positive integers and $\mathcal{M}=(a_{i_1,i_2})_{i_1,i_2\in I}$ an $I\times I$-matrix with entries a_{i_1,i_2} in $[m',c']_B$. Suppose that $\mathcal{M}$ is nuclear, i.e. that for each $M>0$ there are only finitely many $i_2\in I$ such that $\inf_{i_1}\mathrm{ord}_{\pi}a_{i_1,i_2}<M$. For $c>>0$ and $\beta\in\mathbb{N}_0^n$ develop $(\theta\otimes 1)(\pi^{[\frac{|\beta|}{c}]}X^{\beta}a_{i_1,i_2})\in T_n^c\widehat{\otimes}_K B_K$ in the orthonormal basis $\{\pi^{[\frac{|\alpha|}{c}]}X^{\alpha}\}_{\alpha}$ of the B_K-Banach module $T_n^c\widehat{\otimes}_K B_K$ and let $G^c_{\{\alpha,i_1\}\{\beta,i_2\}}\in B_K$ for $\alpha\in\mathbb{N}_0^n$ be its coefficients:*

$$(\theta\otimes 1)(\pi^{[\frac{|\beta|}{c}]}X^{\beta}a_{i_1,i_2})=\sum_{\alpha}G^c_{\{\alpha,i_1\}\{\beta,i_2\}}\pi^{[\frac{|\alpha|}{c}]}X^{\alpha}.$$

Then for all $c >> 0$ and all $M > 0$ there are only finitely many pairs $(\alpha, i_2) \in \mathbb{N}_0^n \times I$ such that

$$\inf_{\beta, i_1} \mathrm{ord}_\pi G^c_{\{\alpha, i_1\}\{\beta, i_2\}} < M.$$

Proof: For simplicity identify I with $\mathbb{N}$. By [8] 2.3 we find integers r and c_0 such that $(\theta \otimes 1)([qm, qc]_B) \subset [m+r, c]_B$ for all $c \geq c_0$, all m. Increasing c_0 and r we may assume that $a_{i_1,i_2} \in [q(r-1), c_0]_B$ for all i_1, i_2. Now let c be so large that for $c' = c - 1$ we have $qc' \geq c_0$. Then one easily checks that $X^\beta a_{i_1 i_2} \in [q(r + [\frac{|\beta|}{q}]), qc']_B$ for all β, i_1, i_2. Hence $(\theta \otimes 1)(X^\beta a_{i_1,i_2}) \in [r + [\frac{|\beta|}{q}], c']$. This means

$$|\alpha| \leq r + [\frac{|\beta|}{q}] + c'(\mathrm{ord}_\pi(G^c_{\{\alpha,i_1\}\{\beta,i_2\}}) + [\frac{|\alpha|}{c}] - [\frac{|\beta|}{c}])$$

for all α, and thus

$$\mathrm{ord}_\pi(G^c_{\{\alpha,i_1\}\{\beta,i_2\}}) \geq [\frac{|\beta|}{c}] - [\frac{|\alpha|}{c}] + \frac{|\alpha| - r - [\frac{|\beta|}{q}]}{c'}.$$

Here the right hand side tends to infinity as $|\alpha|$ tends to infinity, uniformly for all β — independently of i_1 and i_2 — because $c/q \leq c' \leq c$. Now let $M \in \mathbb{N}$ be given. By the above we find $N'(M) \in \mathbb{N}$ such that for all α with $|\alpha| \geq N'(M)$ we have $\mathrm{ord}_\pi(G^c_{\{\alpha,i_1\}\{\beta,i_2\}}) \geq M$. Now fix α. We have

$$\mathrm{ord}_\pi(G^c_{\{\alpha,i_1\}\{\beta,i_2\}}) \geq [\frac{|\beta|}{c}] - [\frac{|\alpha|}{c}] + \mathrm{ord}_\pi(\theta \otimes 1)(X^\beta a_{i_1,i_2}).$$

By nuclearity of $\mathcal{M}$ the right hand side tends to zero as i_2 tends to infinity, uniformly for all i_1, all β. In other words, there exists $N(\alpha, M)$ such that $\mathrm{ord}_\pi(G^c_{\{\alpha,i_1\}\{\beta,i_2\}}) \geq M$ for all $i_2 \geq N(\alpha, M)$, for all i_1, all β. Now set

$$N(M) = N'(M) + \max\{N(\alpha, M); |\alpha| < N'(M)\}.$$

Then we find $\inf_{\beta,i_1} \mathrm{ord}_\pi G^c_{\{\alpha,i_1\}\{\beta,i_2\}} \geq M$ whenever $|\alpha| + i_2 \geq N(M)$. We are done.

2 L-functions

This section introduces our basic setting. We define nuclear (overconvergent) matrices (which give rise to nuclear (overconvergent) σ-modules), their associated L-functions and Dwork operators and give the Monsky trace formula (2.13).

2.1 Let $q \in \mathbb{N}$ be the number of elements of k, i.e. $k = \mathbb{F}_q$. Let $X = \mathrm{Spec}(\overline{A})$ be a smooth affine connected k-scheme of dimension d. So $\overline{A}$ is a smooth k-algebra. By [6] it can be represented as $\overline{A} = A/\pi A$ where

$$A = \frac{R[X_1, \ldots, X_n]^\dagger}{(g_1, \ldots, g_r)}$$

with polynomials $g_j \in R[X_1,\dots,X_n] - \pi R[X_1,\dots,X_n]$ such that A is R-flat. By [10] we can lift the q-th power Frobenius endomorphism of $\overline{A}$ to an R-algebra endomorphism σ of A. Then A, viewed as a $\sigma(A)$-module, is locally free of rank q^d. Shrinking X if necessary we may assume that A is a finite free $\sigma(A)$-module of rank q^d. As before, B_K denotes a reduced K-affinoid algebra, and $B = (B_K)^0$.

2.2 Let I be a countable set. An $I\times I$-matrix $\mathcal{M} = (a_{i_1,i_2})_{i_1,i_2\in I}$ with entries in an R-module E with $E \neq \pi E$ is called *nuclear* if for each $M > 0$ there are only finitely many i_2 such that $\inf_{i_1} \mathrm{ord}_\pi(a_{i_1,i_2}) < M$ (thus $\mathcal{M}$ is nuclear precisely if its transpose is the matrix of a completely continuous operator, or in the terminology of other authors (e.g. [8]): a compact operator). An $I \times I$-matrix $\mathcal{M} = (a_{i_1,i_2})_{i_1,i_2\in I}$ with entries in $A\widehat{\otimes}_R B$ is called *nuclear overconvergent* if there exist positive integers m, c and a nuclear matrix $I \times I$-matrix $\widetilde{\mathcal{M}}$ with entries in $[m, c]_B$ which maps (coefficient-wise) to $\mathcal{M}$ under the canonical map

$$[m, c]_B \hookrightarrow R[X]^\dagger\widehat{\otimes}_R B \to A\widehat{\otimes}_R B.$$

Clearly, if $\mathcal{M}$ is nuclear overconvergent then it is nuclear.
Example: Let $B_K = K$. Nuclear overconvergence implies that the matrix entries are in the subring A of its completion $A\widehat{\otimes}_R R = \widehat{A}$. Conversely, *if I is finite*, an $I\times I$-matrix with entries in A is automatically nuclear overconvergent. Similarly, *if I is finite*, any $I \times I$-matrix with entries in $\widehat{A}$ is automatically nuclear.

2.3 For nuclear matrices $\mathcal{N} = (c_{h_1,h_2})_{h_1,h_2\in H}$ and $\mathcal{N}' = (d_{g_1,g_2})_{g_1,g_2\in G}$ with entries in $A\widehat{\otimes}_R B$ define the $(G \times H) \times (G \times H)$-matrix

$$\mathcal{N} \otimes \mathcal{N}' := (e_{(h_1,g_1),(h_2,g_2)})_{(h_1,g_1),(h_2,g_2)\in(G\times H)},$$

$$e_{(h_1,g_1),(h_2,g_2)} := c_{h_1,h_2} d_{g_1,g_2}.$$

Now choose an ordering of the index set H. For $k \in \mathbb{N}_0$ let $\bigwedge^k(H)$ be the set of k-tuples $(h_1,\dots,h_k) \in H^k$ with $h_1 < \dots < h_k$. Define the $\bigwedge^k(H) \times \bigwedge^k(H)$-matrix

$$\bigwedge^k(\mathcal{N}) := \mathcal{N}^{\wedge k} := (f_{\vec{h}_1,\vec{h}_2})_{\vec{h}_1,\vec{h}_2\in\bigwedge^k(H)},$$

$$f_{\vec{h}_1,\vec{h}_2} = f_{(h_{11},\dots,h_{1k}),(h_{21},\dots,h_{2k})} := \prod_{i=1}^k c_{h_{1,i}h_{2,i}}.$$

It is straightforward to check that $\mathcal{N} \otimes \mathcal{N}'$ and $\bigwedge^k(\mathcal{N})$ are again nuclear, and even nuclear overconvergent if $\mathcal{N}$ and $\mathcal{N}'$ are nuclear overconvergent.

2.4 We will use the term "nuclear" also for another concept. Namely, suppose ψ is an operator on a vector space V over K. For $g = g(X) \in K[X]$ let

$$F(g) := \cap_n g(\psi)^n V \quad \text{and} \quad N(g) := \cup_n \ker g(\psi)^n.$$

Let us call a subset S of $K[X]$ bounded away from 0 if there is an $r \in \mathbb{Q}$ such that $g(a) \neq 0$ for all $\{a \in \mathbb{C}_p; \mathrm{ord}_p(a) \geq r\}$. We say ψ is nuclear if for any subset S of $K[X]$ bounded away from 0 the following two conditions hold:
(i) $F(g) \oplus N(g) = V$ for all $g \in S$
(ii) $N(S) := \sum_{g \in S} N(g)$ is finite dimensional.
(In particular, if $g \notin (X)$, we can take $S = \{g\}$ and as a consequence of (ii) get $N(g) = \ker g(\psi)^n$ for some n.) Suppose ψ is nuclear. Then we can define $P_S(X) = \det(1 - X\psi|_{N(S)})$ for subsets S of $K[X]$ bounded away from 0. These S from a directed set under inclusion, and in [8] it is shown that

$$P(X) := \lim_S P_S(X)$$

(coefficient-wise convergence) exists in $K[[X]]$: the characteristic series of ψ.

2.5 Let $(N_c)_{c\in\mathbb{N}}$ be an inductive system of B_K-Banach modules with injective (but not necessarily isometric) transition maps $\rho_{c,c'} : N_c \to N_{c'}$ for $c' \geq c$. Suppose this system has a countable common orthogonal B_K-basis, i.e. there is a subset $\{q_m; m \in \mathbb{N}\}$ of N_1 such that for all c and $m \in \mathbb{N}$ there are $\lambda_{m,c} \in K^\times$ such that $\{\lambda_{m,c}\rho_{1,c}(q_m); m \in \mathbb{N}\}$ is an orthonormal B_K-basis of N_c. Let

$$N := \lim_{\overrightarrow{c}} N_c\text{"} = \text{"}\bigcup_c N_c$$

and let $N' \subset N$ be a B_K-submodule such that $N'_c = N' \cap N_c$ is closed in N_c for all c. Endow N'_c with the norm induced from N_c and suppose that also the inductive system $(N'_c)_{c\in\mathbb{N}}$ has a countable common orthogonal B_K-basis. Let u be a B_K-linear endomorphism of N with $u(N') \subset N'$ and restricting to a completely continuous endomorphism $u : N_c \to N_c$ for each c. In that situation we have:

Proposition 2.6. *u induces a completely continuous B_K-endomorphism u of $N''_c = N_c/N'_c$ for each c, and* $\det(1 - uT; N''_c)$ *is independent of c. If $B_K = K$, the induced endomorphism u of $N'' = N/N'$ is nuclear in the sense of 2.4, and its characteristic series coincides with* $\det(1 - uT; N''_c)$ *for each c.*

Proof: From [3] A2.6.2 we get that u on N'_c and u on N''_c are completely continuous (note that N''_c is orthonormizable, as follows from [3] A1.2), and that

$$\det(1 - uT; N_c) = \det(1 - uT; N'_c)\det(1 - uT; N''_c)$$

for each c. The assumption on the existence of common orthogonal bases implies (use [5] 4.3.2)

$$\det(1 - uT; N_c) = \det(1 - uT; N_{c'}), \qquad \det(1 - uT; N'_c) = \det(1 - uT; N'_{c'})$$

for all c, c'. Hence

$$\det(1 - uT; N''_c) = \det(1 - uT; N''_{c'})$$

for all c, c'. Also note that for $c' \geq c$ the maps $N''_c \to N''_{c'}$ are injective. The additional assumptions in case $B_K = K$ now follow from [8] Theorem 1.3 and Lemma 1.6.

2.7 Shrinking X if necessary we may assume that the module of (p-adically separated) differentials $\Omega^1_{A/R}$ is free over A. Fix a basis $\omega_1, \ldots, \omega_d$. With respect to this basis, let $\mathcal{D}$ be the $d \times d$-matrix of the σ-linear endomorphism of $\Omega^1_{A/R}$ which the R-algebra endomorphism σ of A induces. Then $\mathcal{D}^{\wedge k} = \bigwedge^k(\mathcal{D})$ is the matrix of the σ-linear endomorphism of $\Omega^k_{A/R} = \bigwedge^k(\Omega^1_{A/R})$ which σ induces.
Let $\theta = \sigma^{-1} \circ \mathrm{Tr}$ be the endomorphism of $\Omega^d_{A/R}$ constructed in [7] Theorem 8.5. It is a Dwork operator: we have $\theta(\sigma(a)y) = a\theta(y)$ for all $a \in A$, $y \in \Omega^d_{A/R}$. Denote also by θ the Dwork operator on A which we get by transport of structure from θ on $\Omega^d_{A/R}$ via the isomorphism $A \cong \Omega^d_{A/R}$ which sends $1 \in A$ to our distinguished basis element $\omega_1 \wedge \ldots \wedge \omega_d$ of $\Omega^d_{A/R}$.
For $c \in \mathbb{N}$ define the subring A^c of $A_K = A \otimes_R K$ as the image of

$$T^c_n \hookrightarrow R[X]^\dagger \otimes_R K \to A_K.$$

This is again a K-affinoid algebra, and we have

$$\theta(A^c) \subset A^c$$

for $c >> 0$. To see this, choose an R-algebra endomorphism $\widetilde{\sigma}$ of $R[X]^\dagger$ which lifts both σ on A and the q-th power Frobenius endomorphism on $k[X]$. With respect to this $\widetilde{\sigma}$ choose a Dwork operator $\widetilde{\theta}$ on $R[X]^\dagger$ lifting θ on A (as in the beginning of the proof of [8] Theorem 2.3). Then apply [8] Lemma 2.4 which says $\widetilde{\theta}(T^c_n) \subset T^c_n$.

2.8 Let $\mathcal{M} = (a_{i_1,i_2})_{i_1,i_2 \in I}$ be a nuclear overconvergent $I \times I$-matrix with entries in $A \widehat{\otimes}_R B$. For $c \in \mathbb{N}$ let $\check{M}^c_I$ be the $A^c \widehat{\otimes}_K B_K$-Banach module for which the set of symbols $\{\check{e}_i\}_{i \in I}$ is an orthonormal basis. For $c \geq c'$ we have the continuous inclusion of B_K-algebras $A^{c'} \widehat{\otimes}_K B_K \subset A^c \widehat{\otimes}_K B_K$, hence a continuous inclusion of B_K-modules $\check{M}^{c'}_I \subset \check{M}^c_I$. Since $\mathcal{M}$ is nuclear overconvergent we have $a_{i_1,i_2} \in A^c \widehat{\otimes}_K B_K$ for all $c >> 0$, all i_1, i_2. We may thus define for all $c >> 0$ the B_K-linear endomorphism $\psi = \psi[\mathcal{M}]$ of $\check{M}^c_I$ by

$$\psi(\sum_{i_1 \in I} b_{i_1} \check{e}_{i_1}) = \sum_{i_1 \in I} \sum_{i_2 \in I} (\theta \otimes 1)(b_{i_1} a_{i_1,i_2}) \check{e}_{i_2}$$

($b_{i_1} \in A^c \widehat{\otimes}_K B_K$). Clearly these endomorphisms extend each other for increasing c, hence we get an endomorphism $\psi = \psi[\mathcal{M}]$ on

$$\check{M}_I := \bigcup_{c >> 0} \check{M}^c_I.$$

2.9 Suppose $B_K = K$ and I is finite, and $\mathcal{M}$ is the matrix of the σ-linear endomorphism ϕ acting on the basis $\{e_i\}_{i\in I}$ of the free A-module M. Then we define $\psi[\mathcal{M}]$ as the Dwork operator

$$\psi[\mathcal{M}] : \mathrm{Hom}_A(M, \Omega^d_{A/R}) \to \mathrm{Hom}_A(M, \Omega^d_{A/R}), \quad f \mapsto \theta \circ f \circ \phi.$$

This definition is compatible with that in 2.8: Consider the canonical embedding

$$\mathrm{Hom}_A(M, \Omega^d_{A/R}) \to \mathrm{Hom}_A(M, \Omega^d_{A/R}) \otimes_R K \overset{w}{\cong} \check{M}_I$$

where the inverse of the A_K-linear isomorphism w sends $\check{e}_i \in \check{M}_I$ to the homomorphism which maps $e_i \in M$ to $\omega_1 \wedge \ldots \wedge \omega_d$ and which maps $e_{i'}$ for $i' \neq i$ to 0. This embedding commutes with the operators $\psi[\mathcal{M}]$.

THEOREM 2.10. *For each $c >> 0$, the endomorphism $\psi = \psi[\mathcal{M}]$ on $\check{M}_I^c$ is a completely continuous B_K-Banach module endomorphism. Its Fredholm determinant* $\det(1 - \psi T; \check{M}_I^c)$ *is independent of c. Denote it by* $\det(1 - \psi T; \check{M}_I)$. *If $B_K = K$, the endomorphism $\psi = \psi[\mathcal{M}]$ on $\check{M}_I$ is nuclear in the sense of [8], and its characteristic series as defined in [8] coincides with* $\det(1 - \psi T; \check{M}_I)$.

PROOF: Choose a lifting of $\mathcal{M} = (a_{i_1,i_2})_{i_1,i_2\in I}$ to a nuclear matrix $(\widetilde{a}_{i_1,i_2})_{i_1,i_2\in I}$ with entries in $[m,c]_B$. Also choose a lifting of θ on A to a Dwork operator $\widetilde{\theta}$ on $R[X]^\dagger$ (with respect to a lifting of σ, as in 2.7). Let N_I^c be the $T_n^c\widehat{\otimes}_K B_K$-Banach module for which the set of symbols $\{(\check{e}_i)\widetilde{}\}_{i\in I}$ is an orthonormal basis, and define the B_K-linear endomorphism $\widetilde{\psi}$ of N_I^c by

$$\widetilde{\psi}(\sum_{i_1\in I} \widetilde{b}_{i_1}(\check{e}_{i_1})\widetilde{}) = \sum_{i_1\in I}\sum_{i_2\in I} (\widetilde{\theta}\otimes 1)(\widetilde{b}_{i_1}\widetilde{a}_{i_1,i_2})(\check{e}_{i_2})\widetilde{}$$

($\widetilde{b}_{i_1} \in T_n^c\widehat{\otimes}_K B$). An orthonormal basis of N_I^c as a B_K-Banach module is given by

$$\{\pi^{[\frac{|\alpha|}{c}]}X^\alpha(\check{e}_i)\widetilde{}\}_{\alpha\in\mathbb{N}_0^n, i\in I}. \tag{1}$$

By 1.7 the matrix for $\widetilde{\psi}$ in this basis is completely continuous; that is, $\widetilde{\psi}$ is completely continuous. If $I_c \subset T_n^c$ and $I_\infty \subset T_n$ denote the respective ideals generated by the elements $g_1, \ldots, g_r$ from 2.1, then $I_\infty \cap T_n^c$ is the kernel of $T_n^c \to A^c$, so by 1.4 the sequences

$$0 \to I_c \to T_n^c \to A^c \to 0 \tag{2}$$

are exact for $c >> 0$. Let H be the B_K-Banach module with orthonormal basis the set of symbols $\{h_i\}_{i\in I}$. From (2) we derive an exact sequence

$$0 \to I_c\widehat{\otimes}_K H \to T_n^c\widehat{\otimes}_K H \to A^c\widehat{\otimes}_K H \to 0 \tag{3}$$

(To see exactness of (3) on the right note that one of the equivalent norms on A^c is the residue norm for the surjective map of K-affinoid algebras $T_n^c \to A^c$ (this surjection even has a continuous K-linear section as the proof of [3] A2.6.2 shows)). We use the following isomorphisms of $T_n^c\widehat{\otimes}_K B_K$-Banach modules (in (i)) resp. of $A^c\widehat{\otimes}_K B_K$-Banach modules (in (ii)):

$$T_n^c\widehat{\otimes}_K H = (T_n^c\widehat{\otimes}_K B_K)\widehat{\otimes}_{B_K} H \cong N_I^c, \quad 1\otimes h_i \mapsto (\check{e}_i)\breve{} \qquad (i)$$

$$A^c\widehat{\otimes}_K H = (A^c\widehat{\otimes}_K B_K)\widehat{\otimes}_{B_K} H \cong \check{M}_I^c, \quad 1\otimes h_i \mapsto \check{e}_i \qquad (ii)$$

By 1.5 we find a subset E of $\mathbb{N}_0^n\times\{1,\ldots,r\}$ such that $\{\pi^{[\frac{|\alpha|+d_j}{c}]}X^\alpha g_j\}_{(\alpha,j)\in E}$ is an orthonormal basis of I_c over K for all $c>>0$. For the B_K-Banach modules $I_c\widehat{\otimes}_K H = (I_c\widehat{\otimes}_K B_K)\widehat{\otimes}_{B_K} H$ we therefore have the orthonormal basis

$$\{\pi^{[\frac{|\alpha|+d_j}{c}]}X^\alpha g_j\otimes h_i\}_{(\alpha,j)\in E, i\in I}. \qquad (4)$$

It is clear that the systems of orthonormal bases (1) resp. (4) make up systems of common orthonormal bases when c increases. (This is why we took pains to prove 1.5; the present argument could be simplified if we could prove the existence of a common orthogonal basis for the system $(\check{M}_I^c)_{c>>0}$.) Now let $N_I = \cup_c N_I^c$. From the exactness of the sequences (3) and from the injectivity of the maps $\check{M}_I^c \to \check{M}_I^{c'}$ for $c\le c'$ we get $I_c\widehat{\otimes}_K H = T_n^c\widehat{\otimes}_K H \cap \mathrm{Ker}(N_I \to \check{M}_I)$. Thus the theorem follows from 2.6.

Corollary 2.11.

$$\prod_{r=0}^{d} \det(1-\psi[\mathcal{M}\otimes\mathcal{D}^{\wedge r}]T;\check{M}_I)^{(-1)^{r-1}}$$

is the quotient of entire power series in the variable T with coefficients in B_K; in other words, it is a meromorphic function on $\mathbb{A}_K^1\times_{\mathrm{Sp}(K)}\mathrm{Sp}(B_K)$.

2.12 Let $B_K = K$. We want to define the L-function of a nuclear matrix $\mathcal{M} = (a_{i_1,i_2})_{i_1,i_2\in I}$ (with entries in $\widehat{A}$). For $f\in\mathbb{N}$ define the f-fold σ-power $\mathcal{M}^{(\sigma)^f}$ of $\mathcal{M}$ to be the matrix product

$$\mathcal{M}^{(\sigma)^f} := ((a_{i_1,i_2})_{i_1,i_2\in I})(\sigma(a_{i_1,i_2})_{i_1,i_2\in I})\ldots((\sigma^{f-1}(a_{i_1,i_2}))_{i_1,i_2\in I}).$$

Let $\overline{x}\in X$ be a geometric point of degree f over k, that is, a surjective k-algebra homomorphism $\overline{A}\to\mathbb{F}_{q^f}$. Let R_f be the unramified extension of R with residue field $\mathbb{F}_{q^f}$, and let $x:\widehat{A}\to R_f$ be the Teichmüller lifting of $\overline{x}$ with respect to σ (the unique σ^f-invariant surjective R-algebra homomorphism lifting $\overline{x}$). By (quite severe) abuse of notation we write

$$\mathcal{M}_{\overline{x}} := x(\mathcal{M}^{(\sigma)^f}),$$

the $I\times I$-matrix with R_f-entries obtained by applying x to the entries of $\mathcal{M}^{(\sigma)^f}$ — the "fibre of $\mathcal{M}$ in $\overline{x}$". The nuclearity condition implies that $\mathcal{M}_{\overline{x}}$ is nuclear; equivalently, its transpose is a completely continuous matrix over R_f in the sense of [12]. It turns out that the Fredholm determinant $\det(1-\mathcal{M}_{\overline{x}}T^{\deg(\overline{x})})$ has coefficients in R, not just in R_f. We define the $R[[T]]$-element

$$L(\mathcal{M},T):=\prod_{\overline{x}\in X}\frac{1}{\det(1-\mathcal{M}_{\overline{x}}T^{\deg(\overline{x})})}.$$

It is trivially holomorphic on the open unit disk. Let T be the set of k-valued points $\overline{x}:\overline{A}\to k$ of X. For a completely continuous endomorphism ψ of an orthonormizable K-Banach module we denote by $\mathrm{Tr}_K(\psi)\in K$ its trace.

Theorem 2.13. *Let $\mathcal{M}$ be a nuclear overconvergent matrix over $\hat{A}$.*
(1) For each $\overline{x}\in T$ the element

$$S_{\overline{x}}:=\sum_{0\le j\le d}(-1)^j\mathrm{Tr}((\mathcal{D}^{\wedge d-j})_{\overline{x}})$$

is invertible in R. For $0\le i\le d$, we have

$$\mathrm{Tr}_K(\psi[\mathcal{M}\otimes\mathcal{D}^{\wedge i}])=\sum_{\overline{x}\in T}\frac{\mathrm{Tr}((\mathcal{D}^{\wedge d-i})_{\overline{x}})\mathrm{Tr}(\mathcal{M}_{\overline{x}})}{S_{\overline{x}}}.$$

(2)

$$L(\mathcal{M},T)=\prod_{r=0}^{d}\det(1-\psi[\mathcal{M}\otimes\mathcal{D}^{\wedge r}]T;\check{M}_I)^{(-1)^{r-1}}.$$

In particular, by 2.11, $L(\mathcal{M},T)$ is meromorphic on $\mathbb{A}^1_K$.

Proof: Let $J\subset A$ be the ideal generated by all elements of the form $a-\sigma(a)$ with $a\in A$. Then $\mathrm{Spec}(A/J)$ is a direct sum of copies of $\mathrm{Spec}(R)$, indexed by T: It is the direct sum of all Teichmüller lifts of elements in T (or rather, their restrictions from $\widehat{A}$ to A; cf. [8] Lemma 3.3). Let $C(A,\sigma)$ be the category of finite (not necessarily projective) A-modules (M,ϕ) with a σ-linear endomorphism ϕ, let $m(A,\sigma)$ be the free abelian group generated by the isomorphism classes of objects of $C(A,\sigma)$, and let $n(A,\sigma)$ be the subgroup of $m(A,\sigma)$ generated by the following two types of elements. The first type is of the form $(M,\phi)-(M_1,\phi_1)-(M_2,\phi_2)$ where

$$0\to(M_1,\phi_1)\to(M,\phi)\to(M_2,\phi_2)\to 0$$

is an exact sequence in $C(A,\sigma)$. The second type is of the form $(M,\phi_1+\phi_2)-(M,\phi_1)-(M,\phi_2)$ for σ-linear operators ϕ_1, ϕ_2 on the same M. Set $K(A,\sigma)=m(A,\sigma)/n(A,\sigma)$. By the analogous procedure define the group

$K^*(A,\sigma)$ associated with the category of finite A-modules with a Dwork operator relative to σ. (Here we follow the notation in [16]. The notation in [8] is the opposite one !). By [8], both $K(A,\sigma)$ and $K^*(A,\sigma)$ are free A/J-modules of rank one. For a finite square matrix $\mathcal{N}$ over A we denote by $\mathrm{Tr}_{A/J}(\mathcal{N}) \in A/J$ the trace of the matrix obtained by reducing modulo J the entries of $\mathcal{N}$. Moreover, for such $\mathcal{N}$ we view $\psi[\mathcal{N}]$ always as a Dwork operator on a (finite) A-module as in 2.9, i.e. we do not invert π. From [16] sect.3 it follows that $\psi[\mathcal{D}^{\wedge i}]$ can be identified with the standard Dwork operator ψ_i on $\Omega^i_{A/R}$ from [8]. By [8] sect.5 Cor.1 we have

$$[\psi[\mathcal{D}^{\wedge d}]]\sum_{0\leq j\leq d}(-1)^j\mathrm{Tr}_{A/J}(\mathcal{D}^{\wedge d-j}) = [(A,\mathrm{id})] \tag{1}$$

in $K^*(A,\sigma)$, and $\sum_{0\leq j\leq d}(-1)^j\mathrm{Tr}_{A/J}(\mathcal{D}^{\wedge d-j})$ is invertible in A/J. By [8] Theorem 5.2 we also have

$$[\psi[\mathcal{D}^{\wedge i}]] = \mathrm{Tr}_{A/J}(\mathcal{D}^{\wedge d-i})[\psi[\mathcal{D}^{\wedge d}]] \tag{2}$$

in $K^*(A,\sigma)$. To prove the theorem suppose first that $\mathcal{M}$ is a finite square matrix. It then gives rise to an element $[\mathcal{M}]$ of $K(A,\sigma)$. By [16] 10.8 we have

$$[\mathcal{M}] = \mathrm{Tr}_{A/J}(\mathcal{M})[(A,\mathrm{id})]$$

in $K(A,\sigma)$. Application of the homomorphism of A/J-modules

$$\lambda_{d-i} : K(A,\sigma) \to K^*(A,\sigma)$$

of [16] p.42 gives

$$[\psi[\mathcal{M}\otimes\mathcal{D}^{\wedge i}]] = \mathrm{Tr}_{A/J}(\mathcal{M})[\psi[\mathcal{D}^{\wedge i}]] \tag{3}$$

in $K^*(A,\sigma)$. From $(1),(2),(3)$ we get

$$[\psi[\mathcal{M}\otimes\mathcal{D}^{\wedge i}]] = \frac{\mathrm{Tr}_{A/J}(\mathcal{M})\mathrm{Tr}_{A/J}(\mathcal{D}^{\wedge d-i})}{\sum_{0\leq j\leq d}(-1)^j\mathrm{Tr}_{A/J}(\mathcal{D}^{\wedge d-j})}[(A,\mathrm{id})]$$

in $K^*(A,\sigma)$. Taking the R-trace proves (1) in case $\mathcal{M}$ is a finite square matrix. Then taking the alternating sum over $0\leq i\leq d$ gives the additive formulation of (2) in case $\mathcal{M}$ is a finite square matrix (see also [16] Theorem 3.1).
The case where the index set I for $\mathcal{M}$ is infinite follows by a limiting argument from the case where I is finite. We explain this for (2), leaving the easier (1) to the reader. Let $\mathcal{P}(I)$ be the set of *finite* subsets of I. For $I'\in\mathcal{P}(I)$, the $I'\times I'$-sub-matrix $\mathcal{M}^{I'} = (a_{i_1,i_2})_{i_1,i_2\in I'}$ of $\mathcal{M}$ is again nuclear overconvergent. Hence, in view of the finite square matrix case it is enough to show

$$L(\mathcal{M},T) = \lim_{I'\in\mathcal{P}(I)} L(\mathcal{M}^{I'},T) \tag{1}$$

and for any $0 \leq r \leq d$ also

$$\det(1 - \psi[\mathcal{M} \otimes \mathcal{D}^{\wedge r}]T; \check{M}_I) = \lim_{I' \in \mathcal{P}(I)} \det(1 - \psi[\mathcal{M}^{I'} \otimes \mathcal{D}^{\wedge r}]T; \check{M}_I) \quad (2)$$

(coefficient-wise convergence). For $I' \in \mathcal{P}(I)$ define the $I \times I$-matrix $\mathcal{M}[I'] = (a^{I'}_{i_1,i_2})_{i_1,i_2 \in I}$ by $a^{I'}_{i_1,i_2} = a_{i_1,i_2}$ if $i_2 \in I'$ and $a^{I'}_{i_1,i_2} = 0$ otherwise. For a geometric point $\overline{x} \in X$ we may view the fibre matrices $\mathcal{M}_{\overline{x}}$ resp. $\mathcal{M}[I']_{\overline{x}}$ for $I' \in \mathcal{P}(I)$ as the transposed matrices of completely continuous operators $\lambda_{\overline{x}}$ resp. $\lambda[I']_{\overline{x}}$ acting all on one single K-Banach space $E_{\overline{x}}$ with orthonormal basis indexed by I. And we may view the fibre matrix $\mathcal{M}^{I'}_{\overline{x}}$ as the transposed matrix of the restriction of $\lambda[I']_{\overline{x}}$ to a $\lambda[I']_{\overline{x}}$-stable subspace of $E_{\overline{x}}$, spanned by a finite subset of our given orthonormal basis and containing $\lambda[I']_{\overline{x}}(E_{\overline{x}})$. For the norm topology on the space $L(E_{\overline{x}}, E_{\overline{x}})$ of continuous K-linear endomorphisms of $E_{\overline{x}}$ we find, using the nuclearity of $\mathcal{M}$, that $\lim_{I'} \lambda[I']_{\overline{x}} = \lambda_{\overline{x}}$. Hence it follows from [12] prop.7,c) that

$$\det(1 - \mathcal{M}_{\overline{x}} T^{\deg(\overline{x})}) = \lim_{I' \in \mathcal{P}(I)} \det(1 - \mathcal{M}[I']_{\overline{x}} T^{\deg(\overline{x})}).$$

But by [12] prop.7,d) we have

$$\det(1 - \mathcal{M}[I']_{\overline{x}} T^{\deg(\overline{x})}) = \det(1 - \mathcal{M}^{I'}_{\overline{x}} T^{\deg(\overline{x})}).$$

Together we get (1). The proof of (2) is similar: By the proof of 1.7 we have indeed

$$\lim_{I' \in \mathcal{P}(I)} \psi[\mathcal{M}[I'] \otimes \mathcal{D}^{\wedge r}] = \psi[\mathcal{M} \otimes \mathcal{D}^{\wedge r}]$$

in the space of continuous K-linear endomorphisms of $\check{M}^c_I$, so [12] prop.7,c) gives

$$\det(1 - \psi[\mathcal{M} \otimes \mathcal{D}^{\wedge r}]T; \check{M}^c_I) = \lim_{I' \in \mathcal{P}(I)} \det(1 - \psi[\mathcal{M}[I'] \otimes \mathcal{D}^{\wedge r}]T; \check{M}^c_I).$$

Now the $\psi[\mathcal{M}[I'] \otimes \mathcal{D}^{\wedge r}]$ do not have finite dimensional image in general, but clearly an obvious generalization of [12] prop.7,d) shows

$$\det(1 - \psi[\mathcal{M}[I'] \otimes \mathcal{D}^{\wedge r}]T; \check{M}^c_I) = \det(1 - \psi[\mathcal{M}^{I'} \otimes \mathcal{D}^{\wedge r}]T; \check{M}^c_I)$$

for $I' \in \mathcal{P}(I)$. We are done.

3 The Grothendieck group

In this section we introduce the Grothendieck group $\Delta(A\widehat{\otimes}_R B)$ of nuclear σ-modules. It is useful since on the one hand, formation of the L-function of a given nuclear σ-module factors over this group, and on the other hand, many natural nuclear σ-modules which are not nuclear overconvergent can be represented in this group through nuclear overconvergent ones.

3.1 We will write σ also for the endomorphism $\sigma\otimes 1$ of $A\widehat{\otimes}_R B=\widehat{A}\widehat{\otimes}_R B$. For $\ell=1,2$ let $\mathcal{M}_\ell$ be $I_\ell\times I_\ell$-matrices with entries in $A\widehat{\otimes}_R B$, for countable index sets I_ℓ. We say $\mathcal{M}_1$ is *σ-similar* to $\mathcal{M}_2$ over $A\widehat{\otimes}_R B$ if there exist a $I_1\times I_2$-matrix $\mathcal{S}$ and a $I_2\times I_1$-matrix $\mathcal{S}'$, both with entries in $A\widehat{\otimes}_R B$, such that $\mathcal{S}\mathcal{S}'$ (resp. $\mathcal{S}'\mathcal{S}$) is the identity $I_1\times I_1$ (resp. $I_2\times I_2$) -matrix, and such that $\mathcal{S}'\mathcal{M}_1\mathcal{S}^\sigma=\mathcal{M}_2$ (in particular it is required that all these matrix products converge coefficient-wise in $A\widehat{\otimes}_R B$). Clearly, σ-similarity is an equivalence relation.

3.2 Let $m(A\widehat{\otimes}_R B)$ be the free abelian group generated by the σ-similarity classes of nuclear matrices (over arbitrary countable index sets) with entries in $A\widehat{\otimes}_R B$. Let $\Delta(A\widehat{\otimes}_R B)$ be the quotient of $m(A\widehat{\otimes}_R B)$ by the subgroup generated by all the elements $[\mathcal{M}]-[\mathcal{M}']-[\mathcal{M}'']$ for matrices $\mathcal{M}=(a_{i_1,i_2})_{i_1,i_2\in I}$, $\mathcal{M}'=(a_{i_1,i_2})_{i_1,i_2\in I'}$ and $\mathcal{M}''=(a_{i_1,i_2})_{i_1,i_2\in I''}$ where $I=I'\sqcup I''$ is a partition of I such that $a_{i_1,i_2}=0$ for all pairs $(i_1,i_2)\in I'\times I''$ (in other words, $\mathcal{M}$ is in block triangular form and $\mathcal{M}'$, $\mathcal{M}''$ are the matrices on the block diagonal).
Elements $z\in\Delta(A\widehat{\otimes}_R B)$ can be written as $z=[\mathcal{M}_+]-[\mathcal{M}_-]$ with nuclear matrices $\mathcal{M}_+,\mathcal{M}_-$. If $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ is a collection of nuclear matrices such that $\mathrm{ord}_\pi(\mathcal{M}_n)\to\infty$ (where $\mathrm{ord}_\pi(\mathcal{M})=\min_{i_1,i_2}\{\mathrm{ord}_\pi a_{i_1,i_2}\}$ for a matrix $\mathcal{M}=(a_{i_1,i_2})_{i_1,i_2\in I}$) and if $\{\nu_n\}_{n\in\mathbb{N}}$ are integers, then the infinite sum $\sum_{n\in\mathbb{N}}\nu_n[\mathcal{M}_n]$ can be viewed as an element of $\Delta(A\widehat{\otimes}_R B)$ as follows: Sorting the ν_n according to their signs means breaking up this sum into a positive and a negative summand, so we may assume $\nu_n\geq 1$ for all n. Replacing $\mathcal{M}_n$ by the block diagonal matrix $diag(\mathcal{M}_n,\mathcal{M}_n,\ldots,\mathcal{M}_n)$ with ν_n copies of $\mathcal{M}_n$ we may assume $\nu_n=1$ for all n. Since all $\mathcal{M}_n$ are nuclear and $\mathrm{ord}_\pi(\mathcal{M}_n)\to\infty$ the block diagonal matrix $\mathcal{M}=diag(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\ldots)$ is nuclear. It represents the desired element of $\Delta(A\widehat{\otimes}_R B)$. Matrix tensor product (see 2.2) defines a multiplication in $\Delta(A\widehat{\otimes}_R B)$: One checks that

$$([\mathcal{M}_{1,+}]-[\mathcal{M}_{1,-}])\otimes([\mathcal{M}_{2,+}]-[\mathcal{M}_{2,-}])$$

$$=[\mathcal{M}_{1,+}\otimes\mathcal{M}_{2,+}]-[\mathcal{M}_{1,+}\otimes\mathcal{M}_{2,-}]-[\mathcal{M}_{1,-}\otimes\mathcal{M}_{2,+}]+[\mathcal{M}_{1,-}\otimes\mathcal{M}_{2,-}]$$

is independent of the chosen representations.

3.3 A more suggestive way to think of $\Delta(A\widehat{\otimes}_R B)$ is the following. We say that a subset $\{e_i\}_{i\in I}$ of an $A\widehat{\otimes}_R B$-module M is a *formal basis* if there is an isomorphism of $A\widehat{\otimes}_R B$-modules

$$\{(d_i)_{i\in I};\, d_i\in A\widehat{\otimes}_R B\}\cong M$$

mapping for any $j\in I$ the sequence $(d_i)_i$ with $d_j=1$ and $d_i=0$ for $i\neq j$ to e_j. A *nuclear σ-module over $A\widehat{\otimes}_R B$* is an $A\widehat{\otimes}_R B$-module M together with a σ-linear endomorphism ϕ such that there exists a formal basis $\{e_i\}_{i\in I}$ of M

such that the action of ϕ on $\{e_i\}_{i\in I}$ is described by a nuclear matrix $\mathcal{M}$ with entries in $A\widehat{\otimes}_R B$, i.e. $\phi e_i = \mathcal{M}e_i$ if we think of e_i as the i-th column of the identity $I \times I$ matrix. We usually think of a nuclear σ-module over $A\widehat{\otimes}_R B$ as a *family of nuclear σ-modules over A, parametrized by the rigid space* $\mathrm{Sp}(B_K)$. In the above situation, if $\mathcal{S}$ is a (topologically) invertible $I \times I$-matrix with entries in $A\widehat{\otimes}_R B$, then $\mathcal{S}^{-1}\mathcal{M}\mathcal{S}^{\sigma}$ is the matrix of ϕ in the new formal basis consisting of the elements $\mathcal{S}e_i = e_i'$ of M (if now we think of e_i' as the i-th column of the identity $I\times I$-matrix). Hence we can view $\Delta(A\widehat{\otimes}_R B)$ as the Grothendieck group of nuclear σ-modules over $A\widehat{\otimes}_R B$, i.e. as the quotient of the free abelian group generated by (isomorphism classes of) nuclear σ-modules over $A\widehat{\otimes}_R B$, divided out by the relations $[(M,\phi)]-[(M',\phi')]-[(M'',\phi'')]$ coming from short exact sequences

$$0 \to (M',\phi') \to (M,\phi) \to (M'',\phi'') \to 0$$

which are $A\widehat{\otimes}_R B$-linearly (but not necessarily ϕ-equivariantly) split.

Proposition 3.4. *Let $B_K = K$. Let $x \in \Delta(\widehat{A})$ be represented by a convergent series $x = \sum_{\ell\in\mathbb{N}} \nu_\ell[\mathcal{M}_\ell]$ with nuclear matrices $\mathcal{M}_\ell$ over $\widehat{A}$. Then the L-series*

$$L(x,T) := \prod_{\ell\in\mathbb{N}} L(\mathcal{M}_\ell,T)^{\nu_\ell}$$

is independent of the chosen representation of x. If all $\mathcal{M}_\ell$ are nuclear overconvergent, then $L(x,T)$ represents a meromorphic function on $\mathbb{A}^1_K$.

Proof: One checks that σ-similar nuclear matrices over $\widehat{A}$ have the same L-function. Indeed, even the Euler factors at closed points of X are the same: they are given by Fredholm determinants of similar (in the ordinary sense) completely continuous matrices. Now let $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$ give rise to a typical relation $[\mathcal{M}] = [\mathcal{M}']+[\mathcal{M}'']$ as in our definition of $\Delta(A\widehat{\otimes}_R B)$. Then one checks that

$$L(\mathcal{M},T) = L(\mathcal{M}',T)L(\mathcal{M}'',T),$$

again by comparing Euler factors. And finally it also follows from the Euler product definition that $\mathrm{ord}_\pi(1 - L(\mathcal{M}_\ell,T)) \to \infty$ if $\mathrm{ord}_\pi(\mathcal{M}_\ell) \to \infty$. Altogether we get the well definedness of $L(x,T)$. If the $\mathcal{M}_l$ are nuclear overconvergent, then the $L(\mathcal{M}_\ell,T)$ are meromorphic by 2.13 and we get the second assertion.

4 Resolution of unit root parts of rank one

In this section we describe a family version of the limiting module construction. Given a rank one unit root σ-module (M_{unit},ϕ_{unit}) which is the unit root part of a (unit root ordinary) nuclear σ-module (M,ϕ) and such that

ϕ_{unit} acts by a 1-unit $a_{i_0,i_0} \in \widehat{A}$ on a basis element of M_{unit}, we choose an affinoid rigid subspace $\mathrm{Sp}(B_K)$ of $\mathbb{A}^1_K$ such that for each $\mathbb{C}_p$-valued point $x \in \mathrm{Sp}(B_K) \subset \mathbb{C}_p$ the exponentiation $a^x_{i_0,i_0}$ is well defined. Hence we get a rank one σ-module over $A\widehat{\otimes}_R B$. We express its class in $\Delta(\widehat{A}\widehat{\otimes}_R B)$ through a set (indexed by $r \in \mathbb{Z}$) of nuclear σ-modules $(B^r(M), B^r(\phi))$ over $A\widehat{\otimes}_R B$ which are overconvergent if (M,ϕ) is overconvergent, even if (M_{unit},ϕ_{unit}) is not overconvergent. Later $\mathrm{Sp}(B_K)$ will be identified with the set of characters $\kappa : U_R^{(1)} \to \mathbb{C}_p^\times$ of the type $\kappa(u) = \kappa_x(u) = u^x$ for small $x \in \mathbb{C}_p$, where $U_R^{(1)}$ denotes the group of 1-units in R. To obtain the optimal parameter space for the $B^r(M)$ (i.e. the maximal region in $\mathbb{C}_p$ of elements x for which κ_x occurs in the parameter space) one needs to go to the union of all these $\mathrm{Sp}(B_K)$. This K-rigid space is not affinoid any more; in the case $K = \mathbb{Q}_p$ it is the parameter space $\mathcal{B}^*$ from [3]. We will however not pass to this limit here, since for an extension of the associated unit root L-function even to the *whole* character space we will have another method available in section 6.

Lemma 4.1. *Let E be a p-adically separated and complete ring such that $E \to E \otimes \mathbb{Q}$ is injective and denote again by* ord_p *the natural extension of* ord_p *from E to $E \otimes \mathbb{Q}$.*
(i) Let $x \in E$. If $\mathrm{ord}_p(x) > \frac{1}{p-1}$*, then* $\mathrm{ord}_p(\frac{x^n}{n!}) \geq 0$ *for all $n \geq 0$, and*

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$$

converges.
(ii) Let $x \in E$. If $\mathrm{ord}_p(x) > 0$*, then* $\mathrm{ord}_p(\frac{x^n}{n}) \geq 0$ *for all $n \geq 1$, and*

$$\log(1+x) = (-1)^{n-1} \sum_{n\geq 1} \frac{x^n}{n}$$

converges. Moreover, if $\mathrm{ord}_p(x) > \beta \geq \frac{1}{p-1}$*, then* $\mathrm{ord}_p(\log(1+x)) > \beta$*; if* $\mathrm{ord}_p(x) \geq \frac{1}{p-1}\frac{1}{p^b}$ *for some $b \in \mathbb{N}_0$, then* $\mathrm{ord}_p(\log(1+x)) \geq \frac{1}{p-1} - b$.

Proof: Proceed as in [11], p.252, p.356.

4.2 Fix a countable non empty set I and an element $i_0 \in I$, let $I_1 = I - \{i_0\}$. Let $\mathcal{M} = (a_{i_1,i_2})_{i_1,i_2\in I}$ be a nuclear $I \times I$-matrix over $\widehat{A}$. It is called 1-*normal* if $1 - a_{i_0,i_0} \in \pi\widehat{A}$ and if $a_{i_1,i_2} \in \pi\widehat{A}$ for all $(i_1,i_2) \neq (i_0,i_0)$. It is called *standard normal* if $a_{i_1,i_0} = 0$ for all $i_1 \in I_1$, if a_{i_0,i_0} is invertible in $\widehat{A}$ and if $a_{i_1,i_2} \in \pi\widehat{A}$ for all $(i_1,i_2) \neq (i_0,i_0)$. It is called *standard* 1-*normal* if it is both standard normal and 1-normal.
That $\mathcal{M}$ is standard normal means that the associated σ-module (M,ϕ) has a unique ϕ-stable submodule of rank one on which ϕ acts on a basis element by multiplication with a unit in $\widehat{A}$: the *unit root part* (M_{unit},ϕ_{unit})

of (M,ϕ). In general, (M_{unit},ϕ_{unit}) will not be overconvergent even if (M,ϕ) is overconvergent. The purpose of this section is to present another construction of σ-modules departing from (M,ϕ) which does preserve overconvergence and allows us to recapture (M_{unit},ϕ_{unit}) in $\Delta(\widehat{A})$, and even certain of its twists.

4.3 For $\nu\in\mathbb{Q}$ we define the $\mathbb{C}_p$-subsets

$$D^{\geq\nu}:=\{x\in\mathbb{C}_p;\quad \mathrm{ord}_p(x)\geq\nu\}$$

$$D^{>\nu}:=\{x\in\mathbb{C}_p;\quad \mathrm{ord}_p(x)>\nu\}.$$

We use these notations also for the natural underlying rigid spaces. Let $B(\nu)_K$ be the reduced K-affinoid algebra consisting of power series in the free variable V, with coefficients in K, convergent on $D^{\geq\nu}$ (viewing V as the standard coordinate). Thus

$$B(\nu)_K=\{\sum_{\alpha\in\mathbb{N}_0}c_\alpha V^\alpha;\quad c_\alpha\in K,\lim_{\alpha\to\infty}(\mathrm{ord}_p(c_\alpha)+\nu\alpha)=\infty\}.$$

4.4 Fix $\nu\in\mathbb{Q}$ and let

$$B:=(B(\nu)_K)^0:=\{\sum_{\alpha\in\mathbb{N}_0}c_\alpha V^\alpha\in B(\nu)_K;\quad \mathrm{ord}_p(c_\alpha)+\nu\alpha\geq 0\text{ for all }\alpha\},$$

$$J:=\{q:I_1\to\mathbb{N}_0;\quad q(i)=0\text{ for almost all }i\in I_1\},$$
$$C:=(A\widehat{\otimes}_R B)^J=\prod_J A\widehat{\otimes}_R B.$$

Define a multiplication in C as follows. Given $\beta=(\beta_q)_{q\in J}$ and $\beta'=(\beta'_q)_{q\in J}$ in C, the component at $q\in J$ of the product $\beta\beta'$ is defined as

$$(\beta\beta')_q=\sum_{\substack{(q_1,q_2)\in J^2\\ q_1+q_2=q}}\beta_{q_1}\beta'_{q_2}.$$

C is p-adically complete. For $c\in\mathbb{N}_0$ we defined $[0,c]_B$ in 1.6, and now we let

$$C_c:=([0,c]_B)^J=\prod_J[0,c]_B,$$

a complete subring of C. We view C as a $A\widehat{\otimes}_R B$-algebra by means of the ring morphism $h:A\widehat{\otimes}_R B\to C$ defined for $y\in A\widehat{\otimes}_R B$ by $h(y)_q=y\in A\widehat{\otimes}_R B$ if $q\in J$ is the zero map $I_1\to\mathbb{N}_0$, and by $h(y)_q=0\in A\widehat{\otimes}_R B$ for all other $q\in J$. In turn,

$$C\cong A\widehat{\otimes}_R B[[I_1]],$$

the free power series ring on the set I_1 (viewed as a set of free variables).

4.5 Let $\mu : S_1 \to S_2$ be a homomorphism of arbitrary R-modules. With I, i_0, I_1 and J from above we now define a homomorphism

$$\lambda(\mu) : (S_1)^I = \prod_I S_1 \to (S_2)^J = \prod_J S_2.$$

Given $a = (a_i)_{i\in I} \in \prod_I S_1$, the q-component $\lambda(\mu)(a)_q$ of $\lambda(\mu)(a)$, for $q \in J$, is defined as follows. If $q \in J$ is the zero map $I_1 \to \mathbb{N}_0$, then $\lambda(\mu)(a)_q = \mu(a_{i_0}) \in S_2$. If there is a $i \in I_1$ such that $q(i) = 1$ and $q(i') = 0$ for all $i' \in I_1 - \{i\}$, then $\lambda(\mu)(a)_q = \mu(a_i) \in S_2$ (for this i). For all other $q \in J$ we let $\lambda(\mu)(a)_q = 0 \in S_2$.
Returning to the situation in 4.4, the natural inclusion $\tau : \widehat{A} \to A\widehat{\otimes}_R B = \widehat{A}\widehat{\otimes}_R B$ gives us an embedding of $\widehat{A}$-modules

$$\lambda = \lambda(\tau) : \widehat{A}^I = \prod_I \widehat{A} \to C = \prod_J A\widehat{\otimes}_R B.$$

It is clear that $\lambda(([0,c]_R)^I) \subset C_c$.

4.6 Now let $\mathcal{M} = (a_{i_1,i_2})_{i_1,i_2\in I}$ be a nuclear and 1-normal $I \times I$-matrix over $\widehat{A}$. Then

$$\mu := \inf(\{\mathrm{ord}_p(a_{i_0,i_0} - 1)\} \cup \{\mathrm{ord}_p(a_{i_1,i_2});\ (i_1,i_2) \neq (i_0,i_0)\}) \geq \frac{1}{e} > 0.$$

If $\mu > \frac{1}{p-1}$ choose $\nu \in \mathbb{Q}$ such that $\nu > \frac{1}{p-1} - \mu$. If only $\mu \geq \frac{1}{p^b}\frac{1}{p-1}$ for some $b \in \mathbb{N}_0$ choose $\nu \in \mathbb{Q}$ such that $\nu > b$. With this ν define B and C as above.
We view $\mathcal{M}$ as the set, indexed by $i_2 \in I$, of its columns

$$a_{(i_2)} := (a_{i_1,i_2})_{i_1\in I} \in \widehat{A}^I.$$

For each $r \in \mathbb{Z}$ we now define a $J \times J$-matrix $\mathcal{B}^r(\mathcal{M}) = (b^{(r)}_{q_1,q_2})_{q_1,q_2\in J}$ over $A\widehat{\otimes}_R B$ associated with $\mathcal{M}$. To define $\mathcal{B}^r(\mathcal{M})$ it is enough to define the set, indexed by $q_2 \in J$, of the columns

$$b^{(r)}_{(q_2)} = (b^{(r)}_{q_1,q_2})_{q_1\in J} \in \prod_J A\widehat{\otimes}_R B = C$$

of $\mathcal{B}^r(\mathcal{M})$. Using the ring structure of C we define

$$b^{(r)}_{(q_2)} := \lambda(a_{(i_0)})^V \lambda(a_{(i_0)})^r \frac{\prod_{i\in I_1} \lambda(a_{(i)})^{q_2(i)}}{\lambda(a_{(i_0)})^{|q_2|}}.$$

Here $|q| = \sum_{i\in I_1} q(i)$ for $q \in J$, and $\lambda(a_{(i_0)})^V \in C$ is defined as

$$\lambda(a_{(i_0)})^V := \exp(V \log(\lambda(a_{(i_0)}))).$$

For this to make sense note that $\mathrm{ord}_p(\lambda(a_{(i_0)}) - 1_C) \geq \mu > \frac{1}{p-1}$ (resp. $\mathrm{ord}_p(\lambda(a_{(i_0)}) - 1_C) \geq \mu \geq \frac{1}{p^b}\frac{1}{p-1}$), hence $\mathrm{ord}_p(\log(\lambda(a_{(i_0)}))) \geq \mu$ (resp.

$\mathrm{ord}_p(\log(\lambda(a_{(i_0)}))) \geq \frac{1}{p-1} - b)$ by 4.1(ii). Thus $V\log(\lambda(a_{(i_0)}))$ is, in view of our choice of ν, indeed an element of $A\widehat{\otimes}_R B$, with $\mathrm{ord}_p(V\log(\lambda(a_{(i_0)}))) \geq \mu+\nu > \frac{1}{p-1}$ (resp. $\mathrm{ord}_p(V\log(\lambda(a_{(i_0)}))) \geq \frac{1}{p-1} - b + \nu > \frac{1}{p-1}$), so we can apply 4.1(i) to it.
If the free variable V specializes to integer values, $\lambda(a_{(i_0)})^V$ specializes to the usual exponentiation by integers of the unit $\lambda(a_{(i_0)})$ in C (just as we use usual exponentiation for the other factors in the above definition of $b^{(r)}_{(q_2)}$). Let $\mathcal{B}^r_-(\mathcal{M})$ be the matrix obtained from $\mathcal{B}^r(\mathcal{M})$ by replacing V with $-V$ (i.e. the matrix defined by the same recipe, but now using $\lambda(a_{(i_0)})^{-V}$ in place of $\lambda(a_{(i_0)})^V$ as the first factor of $b^{(r)}_{(q_2)}$).

4.7 The particular choice of ν made in 4.6 will play no role in the sequel. However, there is some theoretical interest in taking ν as small as possible: the smaller ν, the larger $D^{\geq\nu}$ which is the parameter space for our families of σ-modules defined by the matrices $\mathcal{B}^r(\mathcal{M})$ and $\mathcal{B}^r_-(\mathcal{M})$. The ultimate result 6.11 on the family of twisted unit root L-functions does not depend on the choice (in the prescribed range) of ν here: for 6.11 it is not important how far the family extends, we only need to extend it to $D^{\geq\nu}$ for *some* $\nu<\infty$. But we get trace formulas, which are important for a further qualitative study, only for those members of this family of twisted unit root L-functions whose parameters (=locally K-analytic characters) are in $D^{\geq\nu}$.

Proposition 4.8. *The matrices $\mathcal{B}^r(\mathcal{M})$ and $\mathcal{B}^r_-(\mathcal{M})$ are nuclear. If $\mathcal{M}$ is nuclear overconvergent, then $\mathcal{B}^r(\mathcal{M})$ and $\mathcal{B}^r_-(\mathcal{M})$ are nuclear overconvergent.*

Proof: Nuclearity: Given $M>0$, we need to show $\mathrm{ord}_\pi(b^{(r)}_{(q_2)}) > M$ for all but finitely many $q_2 \in J$. It is clear that $\mathrm{ord}_\pi(\lambda(a_{(i_0)})^V\lambda(a_{(i_0)})^m)=0$ for all $m\in\mathbb{Z}$, therefore we need to concentrate only on the factors $\prod_{i\in I_1}\lambda(a_{(i)})^{q_2(i)}$. By nuclearity of $\mathcal{M}$ we know that $\mathrm{ord}_\pi(\lambda(a_{(i)})) = \mathrm{ord}_\pi(a_{(i)}) > M$ for all but finitely many $i\in I_1$. Therefore we need to concentrate only on those q_2 with support inside this finite exceptional subset of I_1. Among these q_2 we have $|q_2|>M$ for all but finitely many q_2. But $|q_2|>M$ (and $\mathrm{ord}_\pi(a_{(i)})\geq 1$ for all $i\in I_1$) implies

$$\mathrm{ord}_\pi(\prod_{i\in I_1}\lambda(a_{(i)})^{q_2(i)}) = \sum_{i\in I_1} q_2(i)\mathrm{ord}_\pi(a_{(i)}) \geq \sum_{i\in I_1} q_2(i) = |q_2| > M.$$

Nuclearity is established. Now assume $\mathcal{M}$ is nuclear overconvergent. Then it can be lifted to a nuclear, overconvergent and 1-normal matrix $\widetilde{\mathcal{M}} = (\widetilde{a}_{i_1,i_2})_{i_1,i_2\in I}$ with entries in $R[X]^\dagger$. Then, perhaps increasing the c from our nuclearity condition, there is a $c\in\mathbb{N}$ such that $\widetilde{a}_{i_1,i_2}\in[0,c]_R$ for all $(i_1,i_2)\neq(i_0,i_0)$, and also $\widetilde{a}_{i_0,i_0}-1\in[0,c]_R$. Then all entries of $\mathcal{B}^r(\widetilde{\mathcal{M}})$ are in $[0,c]_B$. Hence $\mathcal{B}^r(\widetilde{\mathcal{M}})$ is nuclear and overconvergent. Clearly it is a lifting of

$\mathcal{B}^r(\mathcal{M})$, so we are done.

4.9 Now let us look at the σ-module over $A\widehat{\otimes}_R B$ defined by the matrix $\mathcal{B}^r(\mathcal{M})$. By construction, this is the $A\widehat{\otimes}_R B$-module C (which in fact even is a $A\widehat{\otimes}_R B$-algebra), with the σ-linear endomorphism defined by $\mathcal{B}^r(\mathcal{M})$. We view it as an analytic family, parametrized by the rigid space $\mathrm{Sp}(B(\nu)_K) = D^{\geq\nu}$, of nuclear σ-modules over $\widehat{A}$; its fibres at points $\mathbb{Z} \cap D^{\geq\nu}$ are Wan's "limiting modules" [15]. Yet another description is due to Coleman [4], which we now present (in a slightly generalized form). It will be used in the proof of 4.10. The nuclear matrix $\mathcal{M}$ over $\widehat{A}$ is the matrix in a formal basis $\{e_i\}_{i\in I}$ of a σ-linear endomorphism ϕ on a $\widehat{A}$-module M. The element $e = e_{i_0} \in M$ can also be viewed as an element of the symmetric $\widehat{A}$-algebra $\mathrm{Sym}_{\widehat{A}}(M)$ defined by M, so it makes sense to adjoin its inverse to $\mathrm{Sym}_{\widehat{A}}(M)$. Let D be the subring of degree zero elements in $\mathrm{Sym}_{\widehat{A}}(M)[\frac{1}{e}]\widehat{\otimes}_R B$: the $A\widehat{\otimes}_R B$-sub-algebra of $\mathrm{Sym}_{\widehat{A}}(M)[\frac{1}{e}]\widehat{\otimes}_R B$ generated by all $\frac{m}{e}$ for $m \in M$. Let $\mathcal{I} \subset D$ be the ideal generated by all elements $\frac{m}{e}$ for $m \in M$ with $\phi(m) \in \pi M$, and let $B^r(M)$ be the $(\pi, \mathcal{I})$-adic completion of D. For all $\alpha \in (A\widehat{\otimes}_R B)^\times$, all $m_1, m_2 \in M$, if we set $e' = \alpha e + \pi m_1$, we have

$$\frac{m_2}{e'} = \frac{m_2}{\alpha e}\sum_{i=0}^{\infty}(\frac{\pi}{\alpha}\frac{m_1}{e})^i \qquad (*)$$

in $B^r(M)$. By our assumptions on $\mathcal{M}$ we know $\phi(e) - e \in \pi M$. Therefore there exists a unique σ-linear ring endomorphism ψ of $B^r(M)$ with $\psi(\frac{m}{e}) = \frac{\phi(m)}{\phi(e)}$ for all $m \in M$: Take $(*)$ as a definition, with $e' = \phi(e)$, $m_2 = \phi(m)$ and $\alpha = 1$. Similarly as in 4.6 we can define, for integers $r \in \mathbb{Z}$, the element

$$(\frac{\phi(e)}{e})^{V+r} = \exp(V\log(\frac{\phi(e)}{e}))(\frac{\phi(e)}{e})^r$$

of $B^r(M)$. We define the σ-linear endomorphism $B^r(\phi)$ of $B^r(M)$ by

$$B^r(\phi)(y) = (\frac{\phi(e)}{e})^{V+r}\psi(y)$$

for $y \in B^r(M)$. Clearly $\mathcal{B}^r(\mathcal{M})$ is the matrix of $B^r(\phi)$ acting on the formal basis

$$\{\prod_{i\in I}(\frac{e_i}{e})^{q(i)}\}_{q\in J}$$

of $B^r(M)$ over $A\widehat{\otimes}_R B$. The σ-module defined by $\mathcal{B}^r_-(\mathcal{M})$ is described similarly.

Proposition 4.10. *The σ-similarity classes (over $A\widehat{\otimes}_R B$) of $\mathcal{B}^r(\mathcal{M})$ and $\mathcal{B}^r_-(\mathcal{M})$ depend only on the σ-similarity class (over $\widehat{A}$) of $\mathcal{M}$.*

Proof: We prove this for $\mathcal{B}^r(\mathcal{M})$, the argument for $\mathcal{B}^r_-(\mathcal{M})$ is the same. It is enough to prove that $B^r(M)$, as a $A\widehat{\otimes}_R B$-module together with its σ-linear endomorphism $B^r(\phi)$, depends only on the σ-module M. Let $\mathcal{M}' = (a'_{i_1,i_2})_{i_1,i_2\in I}$ be another 1-normal nuclear matrix over $\widehat{A}$ which is σ-similar to $\mathcal{M}$. We can view $\mathcal{M}'$ as the matrix of the *same* σ-linear endomorphism ϕ on the *same* $\widehat{A}$-module M, but in another formal basis $\{e'_i\}_{i\in I}$. For the element $e' = e'_{i_0}$ our assumptions imply $\phi(e') - e' \in \pi M$. Therefore e' and e both generate the unit root part modulo π of M, hence there is a $\alpha \in \widehat{A}^\times$ with $e' - \alpha e \in \pi M$. Observe that

$$\alpha e \equiv e' \equiv \phi(e') \equiv \phi(\alpha e) \equiv \sigma(a)\phi(e) \equiv \sigma(\alpha)e$$

modulo πM. Since $R^\times$ is the subgroup of $\widehat{A}^\times$ fixed by σ we may and will assume $\alpha \in R^\times$. From $(*)$ in 4.9 it follows that for $m \in M$ the element $\frac{m}{e'}$ of the π-adic completion of $\mathrm{Sym}_{\widehat{A}}(M)[M^{-1}]\widehat{\otimes}_R B$ actually lies in its subring $B^r(M)$. By a symmetry argument we deduce that $B^r(M)$ is the same when constructed with respect to e or with respect to e'. Moreover the endomorphism ψ on $B^r(M)$ is the same when constructed with respect to e or with respect to e': it is uniquely determined by its action on $B^r(M) \cap \mathrm{Sym}_{\widehat{A}}(M)[M^{-1}]\widehat{\otimes}_R B$, where it is characterized by $\psi(\frac{m_1}{m_2}) = \frac{\phi(m_1)}{\phi(m_2)}$ for $m_1, m_2 \in M$. Now let

$$B^r(\phi)'(y) = (\frac{\phi(e')}{e'})^{V+r}\psi(y)$$

for $y \in B^r(M)$. The needed $A\widehat{\otimes}_R B$-linear endomorphism λ_r of $B^r(M)$ satisfying $\lambda_r \circ B^r(\phi)' = B^r(\phi) \circ \lambda_r$ we now define to be the multiplication with $(\frac{e'}{\alpha e})^{V+r} \in B^r(M)$ (by now obviously defined). Here we use that $\alpha \in R^\times$.

Now suppose $\mathcal{M} = (a_{i_1,i_2})_{i_1,i_2\in I}$ is even a standard 1-normal nuclear $\widehat{A}$-matrix. Define $\mathcal{M}_{unit} := a_{i_0,i_0} \in \widehat{A}$ and $(\mathcal{M}_{unit})^V = \exp(V\log(\mathcal{M}_{unit})) \in A\widehat{\otimes}_R B$ as in 4.6.

Theorem 4.11. *For $s \in \mathbb{Z}$ we have the following equalities in $\Delta(A\widehat{\otimes}_R B)$:*

$$[(\mathcal{M}_{unit})^s(\mathcal{M}_{unit})^V] = \bigoplus_{r\geq 1}(-1)^{r-1}r[\mathcal{B}^{s-r}(\mathcal{M}) \otimes \bigwedge^r(\mathcal{M})]$$

$$[(\mathcal{M}_{unit})^s(\mathcal{M}_{unit})^{-V}] = \bigoplus_{r\geq 1}(-1)^{r-1}r[\mathcal{B}^{s-r}_-(\mathcal{M}) \otimes \bigwedge^r(\mathcal{M})].$$

Proof: We prove the first equality, the second is proved similarly. First note that our assumptions imply that π^{r-1} divides $\bigwedge^r(\mathcal{M})$, so the right hand side converges. Since $\mathcal{M}$ is standard 1-normal we have $\mathcal{B}^{s-r}(\mathcal{M}) = (\mathcal{M}_{unit})^s\mathcal{B}^{-r}(\mathcal{M})$ so we may assume $s = 0$. Let $\mathcal{M}'' = (a''_{i_1,i_2})_{i_1,i_2\in I}$ be the $\widehat{A}$-matrix with $a''_{i_1,i_2} = a_{i_1,i_2}$ for all $(i_1,i_2) \in (I_1 \times I_1) \cup \{(i_0,i_0)\}$, and $a''_{i_1,i_2} = 0$ for the other (i_1,i_2). Since $\mathcal{M}$ is standard 1-normal we see that $[\mathcal{B}^{-r}(\mathcal{M})\otimes\bigwedge^r(\mathcal{M})] = [\mathcal{B}^{-r}(\mathcal{M}'')\otimes\bigwedge^r(\mathcal{M}'')]$ in view of the relations divided out in the definition of $\Delta(A\widehat{\otimes}_R B)$. Hence we may assume $\mathcal{M} = \mathcal{M}''$. Suppose that i_0 is minimal in the ordering of I (which we tacitly chose to define $\bigwedge^r(I)$ and $\bigwedge^r(\mathcal{M})$, see 2.2). For $r \geq 1$ let M_r be the $A\widehat{\otimes}_R B$-module $(A\widehat{\otimes}_R B)^{(J\times\bigwedge^r(I))}$. It has the formal basis $(e_{(q,\vec{\imath})})_{q\in J,\vec{\imath}\in\bigwedge^r(I)}$, where $e_{(q,\vec{\imath})}$ is the $(q,\vec{\imath})$-th column of the identity $(J\times\bigwedge^r(I))\times(J\times\bigwedge^r(I))$-matrix. The matrix $\mathcal{B}^{-r}(\mathcal{M})\otimes\bigwedge^r(\mathcal{M})$ describes the action of a σ-linear endomorphism ϕ_r of M_r on this basis. Actually we will need r copies of (M_r,ϕ_r) and its formal basis $(e_{(q,\vec{\imath})})_{q\in J,\vec{\imath}\in\bigwedge^r(I)}$: We denote them by $(M_r^{(\ell)},\phi_r^{(\ell)})$ and $(e^{(\ell)}_{(q,\vec{\imath})})_{(q,\vec{\imath})}$ for $1 \leq \ell \leq r$. We get the σ-module

$$(M_r^{\bullet},\phi_r^{\bullet}) = \oplus_{1\leq\ell\leq r}(M_r^{(\ell)},\phi_r^{(\ell)})$$

with formal basis

$$H_r = (e^{(\ell)}_{(q,\vec{\imath})})_{q\in J,\vec{\imath}\in\bigwedge^r(I),\ell\in\{1,\ldots,r\}}.$$

Define $A\widehat{\otimes}_R B$-linear maps

$$\alpha_r^{(\ell)}: M_r^{(\ell)} \to M_{r+1}^{(\ell)}$$

$$\beta_r^{(\ell)}: M_r^{(\ell)} \to M_{r+1}^{(\ell+1)}$$

as follows. For $\vec{\imath} = (i_1,\ldots,i_r) \in \bigwedge^r(I)$ with $i_1 < \ldots < i_r$, and another $i \in I$, let $\tau(\vec{\imath},i) = \max(\{t \leq r; i_t < i\} \cup \{0\})$, and if in addition $i \neq i_{\tau(\vec{\imath},i)+1}$ let

$$[\vec{\imath},i] = (i_1,\ldots,i_{\tau(\vec{\imath},i)},i,i_{\tau(\vec{\imath},i)+1},\ldots,i_r) \in \bigwedge^{r+1}(I).$$

For $q \in J$ and $i \in I_1$ with $q(i) \neq 0$ define $q^{i-} \in J$ by $q^{i-}(i') = q(i')$ for $i' \in I_1 - \{i\}$, and $q^{i-}(i) = q(i) - 1$. Now set

$$\alpha_r^{(\ell)}(e^{(\ell)}_{(q,\vec{\imath})}) = e^{(\ell)}_{(q,[\vec{\imath},i_0])}$$

if $i_1 \neq i_0$, and set $\alpha_r^{(\ell)}(e^{(\ell)}_{(q,\vec{\imath})}) = 0$ if $i_1 = i_0$. Set

$$\beta_r^{(\ell)}(e^{(\ell)}_{(q,\vec{\imath})}) = \sum_t (-1)^{\tau(\vec{\imath},i_t)} e^{(\ell+1)}_{(q^{i_t-},[\vec{\imath},i_t])}$$

where the sum runs through all $1 \leq t \leq r$ with $i_t \neq i_0$, with $i_t \neq i_{\tau(\vec{\imath},i_t)+1}$ and with $q(i_t) \neq 0$. One checks that $\phi_{r+1}^{(\ell)} \circ \alpha_r^{(\ell)} = \alpha_r^{(\ell)} \circ \phi_r^{(\ell)}$ (use the standard 1-normality of $\mathcal{M}$), and that $\phi_{r+1}^{(\ell+1)} \circ \beta_r^{(\ell)} = \beta_r^{(\ell)} \circ \phi_r^{(\ell)}$ (use $\mathcal{M} = \mathcal{M}''$). Hence for

$$\psi_r^{\bullet} = \oplus_{1\leq\ell\leq r}(\alpha_r^{(\ell)} \oplus \beta_r^{(\ell)}) : M_r^{\bullet} \to M_{r+1}^{\bullet}$$

we have $\phi_{r+1}^{\bullet} \circ \psi_r^{\bullet} = \psi_r^{\bullet} \circ \phi_r^{\bullet}$. Also note that $\phi_1^{\bullet} = \phi_1^{(1)}$ on $M_1^{\bullet} = M_1^{(1)}$ restricts on the rank one $A\widehat{\otimes}_R B$-submodule $M_0^{\bullet}$ spanned by the basis element $e^{(1)}_{(0,i_0)} \in$

$J \times I = J \times \bigwedge^1(I)$ to a σ-linear endomorphism ϕ_0 with matrix $(\mathcal{M}_{unit})^V$. Let $\psi_0^\bullet : M_0^\bullet \to M_1^\bullet$ be the inclusion and consider

$$0 \to M_0^\bullet \stackrel{\psi_0^\bullet}{\to} M_1^\bullet \stackrel{\psi_1^\bullet}{\to} M_2^\bullet \stackrel{\psi_2^\bullet}{\to} \dots \qquad (*)$$

We saw that this sequence is equivariant for the σ-linear endomorphisms $\phi_r^\bullet$ which are described by matrices as occur in the statement of the theorem, so it remains to show that $(*)$ is split exact; more precisely, that for each r there are disjoint subsets G_r^1 and G_r^2 of $M_r^\bullet$ with the following properties: $\psi_r^\bullet$ induces a bijection of sets $G_r^2 \cong G_{r+1}^1$, and the union $G_r^1 \cup G_r^2$ is a formal basis for $M_r^\bullet$ (transforming under an invertible matrix to the formal basis H_r). We let $G_0^1 = \emptyset$, $G_0^2 = H_0 = \{e_{(0,i_0)}^{(1)}\}$. For $r \geq 1$ we let

$$G_r^1 = \{\psi_{r-1}^\bullet(h); h \in H_{r-1}\}.$$

We let G_r^2 be the subset of H_r consisting of those $e_{(q,\vec{\imath})}^{(\ell)}$ with ℓ, q and $\vec{\imath} = (i_1, \dots, i_r) \in \bigwedge^r(I)$ satisfying one of the following conditions: either

$$[\ell = 1 \text{ and } ((i_1 \neq i_0) \text{ or } (i_1 = i_0 \text{ and } \exists(i \in I_1 - \{i_2, \dots, i_r\}) : \quad q(i) \neq 0))]$$

or

$$[\ell \neq 1 \text{ and } ((i_1 \neq i_0 \text{ and } \forall(1 \leq k \leq r)\exists(i \in I_1 - \{i_k\}) : \quad q(i) \neq 0)$$
$$\text{or } (i_1 = i_0 \text{ and } \exists(i \in I_1 - \{i_2, \dots, i_r\}) : \quad q(i) \neq 0))].$$

The desired properties are formally verified, the proof is complete.

Corollary 4.12. *Suppose our $\mathcal{M}$ is also overconvergent nuclear. Then for each $s \in \mathbb{Z}$ the series*

$$\prod_{\overline{x} \in X} \frac{1}{\det(1 - (\mathcal{M}_{unit})^{\frac{s}{\overline{x}}}(\mathcal{M}_{unit})^{\frac{y}{\overline{x}}} T^{\deg(\overline{x})})}$$

defines a meromorphic function in the variables T and y on $\mathbb{A}^1_{\mathbb{C}_p} \times D^{\geq \nu}$, specializing for $y \in D^{\geq \nu}(K)$ to $L(\mathcal{M}_{unit}^{s+y}, T)$.

Proof: The series is trivially holomorphic on $D^{>0} \times D^{\geq \nu}$. We claim that it is equal to

$$\prod_{r \geq 1} \Big(\prod_{i=0}^{d} \det(1 - \psi[\mathcal{B}^{s-r}(\mathcal{M}) \otimes \bigwedge^r(\mathcal{M}) \otimes \mathcal{D}^{\wedge i}]T)^{(-1)^{i-1}}\Big)^{(-1)^{r-1}r}$$

which clearly extends as desired. It suffices to prove equality at all specializations $V = y$ at K-rational points $y \in D^{\geq \nu}(K)$ (since these y are Zariski dense in $D^{\geq \nu}$). But for such y both series coincide with

$$\prod_{r \geq 1} L(\mathcal{B}^{s-r}(\mathcal{M})|_{V=y} \otimes \bigwedge^r(\mathcal{M}), T)^{(-1)^{r-1}r} :$$

For the series in the statement of 4.12 this follows from 3.4 and 4.11, for the first series written in this proof this follows from 2.13.

5 Weight space $\mathcal{W}$

In this section we describe a K-rigid analytic space $\mathcal{W}$ whose set of $\mathbb{C}_p$-valued points can be identified with the set of locally K-analytic characters $\kappa : R^\times \to \mathbb{C}_p$ occuring in Theorem 0.1.

5.1 For a K-analytic group manifold G (see [1]) we denote by $\mathrm{Hom}_{K\text{-}an}(G, \mathbb{C}_p^\times)$ the group of locally K-analytic characters $G \to \mathbb{C}_p^\times$: characters which locally on G can be expanded into power series in $\dim_K(G)$-many variables. If $K = \mathbb{Q}_p$ these are precisely the *continuous* characters $G \to \mathbb{C}_p^\times$. The examples relevant for us are $G = R$, $G = R^\times$, $G = U_R^{(1)}$ and $G = \overline{U}_R^{(1)}$, where we write

$$U_R^{(1)} := 1 + \pi R \quad \text{and} \quad \overline{U}_R^{(1)} := \frac{U_R^{(1)}}{(U_R^{(1)})_{\text{tors}}}.$$

To extinguish any confusion, although in these examples G even carries a natural structure of K-*rigid* group variety, the definition of $\mathrm{Hom}_{K\text{-}an}(G, \mathbb{C}_p^\times)$ does not refer to this (indeed more "rigid") structure: local K-analycity of a character κ requires only that κ, as a $\mathbb{C}_p^\times \subset \mathbb{C}_p$ valued function on G, can be expanded into convergent power series on each member of some open covering of G — an open covering in the naive sense, not necessarily admissible in the sense of rigid geometry.

5.2 Let $\mathcal{G} = \mathcal{G}_\pi$ be the Lubin-Tate formal group over R corresponding to our chosen uniformizer $\pi \in R$ (see [9]). For $x \in R$ denote by $[x] \in U.R[[U]]$ the formal power series which defines the multiplication with x in the formal R-module $\mathcal{G}$. The R-module $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{G}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}, \mathbb{G}_{m,\mathcal{O}_{\mathbb{C}_p}})$ is free of rank one. Fix a generator with corresponding power series $F(Z) \in Z.\mathcal{O}_{\mathbb{C}_p}[[Z]]$. Substitution yields power series $F([x]) \in U.\mathcal{O}_{\mathbb{C}_p}[[U]]$ for $x \in R$. By [13] we have a group isomorphism

$$D^{>0} \xrightarrow{\cong} \mathrm{Hom}_{K\text{-}an}(R, \mathbb{C}_p^\times)$$
$$z \mapsto [x \mapsto 1 + F([x])(z)].$$

Here $D^{>0}$ carries the group structure defined by $\mathcal{G}$. Let $m \in \mathbb{Z}_{\geq -1}$ be minimal such that $\pi^m \log(U_R^{(1)}) \subset R$. Since $(U_R^{(1)})_{\text{tors}} = \mathrm{Ker}(\log)$ we have a well defined injective homomorphism of K-analytic group varieties

$$\overline{U}_R^{(1)} \xrightarrow{\theta} R, \quad u \mapsto \pi^m \log(u) = \theta(u)$$

inducing a homomorphism

$$\mathrm{Hom}_{K\text{-}an}(R, \mathbb{C}_p^\times) \xrightarrow{\delta} \mathrm{Hom}_{K\text{-}an}(\overline{U}_R^{(1)}, \mathbb{C}_p^\times).$$

Note that $\mathrm{Coker}(\theta)$ is finite. Thus $\mathrm{Ker}(\delta)$ is finite, and on the other hand δ is surjective (since $\mathbb{C}_p^\times$ is divisible). In other words, $\mathrm{Hom}_{K\text{-}an}(\overline{U}_R^{(1)}, \mathbb{C}_p^\times)$ is the quotient of $\mathrm{Hom}_{K\text{-}an}(R, \mathbb{C}_p^\times)$ by a finite subgroup $\Delta \subset \mathrm{Hom}_{K\text{-}an}(R, \mathbb{C}_p^\times)$. The formal group law $\mathcal{G}$ defines a structure of $\mathbb{C}_p$-rigid analytic group variety on $D^{>0}$ (with its standard coordinate U). By means of the above isomorphism we view $\mathrm{Hom}_{K\text{-}an}(R, \mathbb{C}_p^\times)$ as its group of $\mathbb{C}_p$-valued points. Accordingly, we view $\mathrm{Hom}_{K\text{-}an}(\overline{U}_R^{(1)}, \mathbb{C}_p^\times)$ as the group $(D^{>0}/\Delta)(\mathbb{C}_p)$ of $\mathbb{C}_p$-valued points of the $\mathbb{C}_p$-rigid group variety $D^{>0}/\Delta$. Let

$$U_R^{(1)} \xleftarrow{u} R^\times \xrightarrow{v} \mu_{q-1}$$

be the natural projections. We have $(U_R^{(1)})_{\mathrm{tors}} = \mu_{p^a}$ for some $a \geq 0$. Let

$$\mathcal{F} = \{0, \ldots, p^a - 1\} \times \{0, \ldots, q-2\}.$$

For each $(s,t) \in \mathcal{F}$ let $\mathcal{W}_{(s,t)}$ be a copy of the $\mathbb{C}_p$-rigid space $D^{>0}/\Delta$, and let

$$\mathcal{W} := \coprod_{(s,t)\in\mathcal{F}} \mathcal{W}_{(s,t)}.$$

For an element $\omega \in \mathcal{W}_{(s,t)}(\mathbb{C}_p) \subset \mathcal{W}(\mathbb{C}_p)$ we define the character

$$\kappa_\omega : R^\times \to \mathbb{C}_p, \quad r \mapsto u(r)^s \widetilde{\omega}(u(r)) v(r)^t =: \kappa_\omega(r)$$

where $\widetilde{\omega} \in \mathrm{Hom}_{K\text{-}an}(U_R^{(1)}, \mathbb{C}_p^\times)$ is the image of ω under the natural map

$$\mathcal{W}_{(s,t)}(\mathbb{C}_p) \cong (D^{>0}/\Delta)(\mathbb{C}_p) \cong \mathrm{Hom}_{K\text{-}an}(\overline{U}_R^{(1)}, \mathbb{C}_p^\times) \to \mathrm{Hom}_{K\text{-}an}(U_R^{(1)}, \mathbb{C}_p^\times).$$

Since $R^\times = \mu_{q-1} \times U_R^{(1)}$ we get:

PROPOSITION 5.3. *The assignment $\omega \mapsto \kappa_\omega$ defines a bijection*

$$\mathcal{W}(\mathbb{C}_p) \cong \mathrm{Hom}_{K\text{-}an}(R^\times, \mathbb{C}_p^\times).$$

Thus $\mathrm{Hom}_{K\text{-}an}(R^\times, \mathbb{C}_p^\times)$ can be viewed as the set of $\mathbb{C}_p$-valued points of the $\mathbb{C}_p$-rigid variety $\mathcal{W}$.

LEMMA 5.4. *For $\nu \in \mathbb{Q}$, $\nu > \frac{m}{e} + \frac{1}{p-1}$, there exists an open embedding of $\mathbb{C}_p$-rigid varieties $\iota : D^{\geq\nu} \to D^{>0}$ such that for all $x \in R$ and all $y \in D^{\geq\nu}$ we have*

$$1 + F([x])(\iota(y)) = \exp(\pi^{-m}xy).$$

PROOF: Let $\log_{\mathcal{G}}$ be the logarithm of $\mathcal{G}$. Write $F(Z) = \Omega.Z + \ldots \in Z.\mathcal{O}_{\mathbb{C}_p}[[Z]]$. Then we have the identity of formal power series (cf. [13] sect.4)

$$1 + F([x]) = \exp(\Omega \log_{\mathcal{G}}([x]))$$

in $\mathcal{O}_{\mathbb{C}_p}[[U]]$. But $\log_{\mathcal{G}}([x]) = x.\log_{\mathcal{G}}(U)$ by [9] 8.6 Lemma 2, therefore it is enough to find ι with

$$\log_{\mathcal{G}}(\iota(y)) = \pi^{-m}\Omega^{-1}y.$$

By [9] 8.6 Lemma 4 the power series inverse to $\log_{\mathcal{G}}$ defines an open embedding $\exp_{\mathcal{G}} : D^{\geq\beta} \to D^{>0}$ for $\beta > \frac{1}{e(q-1)}$. Thus $\iota(y) = \exp_{\mathcal{G}}(\pi^{-m}\Omega^{-1}y)$ is appropriate; it is well defined on $D^{\geq\nu}$ because we have $\mathrm{ord}_p(\Omega) = \frac{1}{p-1} - \frac{1}{e(q-1)}$ by [13], hence $\nu - \frac{m}{e} - \mathrm{ord}_p(\Omega) > \frac{1}{e(q-1)}$.

5.5 Example: Consider the case $K = \mathbb{Q}_p$, $\pi = p$. Then

$$\mathcal{G} = \mathbb{G}_m, \qquad \log_{\mathcal{G}}(Z) = \log(1+Z), \qquad m = -1$$

$$[x] = (1+U)^x - 1 = \sum_{n\geq 1}\binom{x}{n}U^n \qquad (x \in \mathbb{Z}_p).$$

We may choose $F(Z) = Z$, and for $\nu > \frac{2-p}{p-1}$ the associated embedding

$$\iota : D^{\geq\nu} \to D^{>0}; \quad y \mapsto \iota(y) = \exp(py) - 1$$

is an isomorphism $\iota : D^{\geq\nu} \cong D^{\geq\nu+1} \subset D^{>0}$.

6 Meromorphic continuation of unit root L-functions

In this section we prove (the infinite rank version of) Theorem 0.1. Let us give a sketch. For simplicity suppose that $\alpha \in \widehat{A}$ is a matrix of the ordinary unit root part of some nuclear overconvergent σ-module M over A (in the general case, α splits into two factors each of which is of this more special type and can "essentially" be treated separately). An appropriate multiplicative decomposition of α (see 6.3) allows us to assume that α is a 1-unit. Then the results of section 4, together with the trace formula 2.13 already show meromorphy of L_α on $\mathbb{A} \times \mathcal{W}^0$ for some open subspace $\mathcal{W}^0 \subset \mathcal{W}$ meeting each component of $\mathcal{W}$: this is essentially what we proved in 4.12. More precisely we get a decomposition of L_α into holomorphic functions on $\mathbb{A} \times \mathcal{W}^0$ which are Fredholm determinants $\det(\psi)$ of certain completely continuous operators ψ arising from limiting modules. We express the coefficients of the logarithms of these $\det(\psi)$ through the traces $\mathrm{Tr}(\psi^f)$ of iterates ψ^f of these ψ. Then we repeat the limiting module construction in each fibre $\overline{x} \in X$ and prove its commutation with its global counterpart. Together with the trace formula 2.13 and the description of the embedding $\mathcal{W}^0 \to \mathcal{W}$ given in 5.4 this can be used to show that all the functions $\mathrm{Tr}(\psi^f)$, a priori living on $\mathcal{W}^0$, extend to functions on $\mathcal{W}$, bounded by 1. By the general principle 6.1 below this implies the theorem.

Lemma 6.1. *For $m \in \mathbb{N}$ let $g_m(U) \in \mathcal{O}_{\mathbb{C}_p}[[U_1,\ldots,U_g]]$. Suppose there exists a $\tau > 0$ such that*

$$f(T,U) = \exp(-\sum_{m=1}^{\infty} \frac{g_m(U)}{m} T^m) \in \mathbb{C}_p[[T,U_1,\ldots,U_g]]$$

converges on $\mathbb{A}^1_{\mathbb{C}_p} \times (D^{\geq\tau})^g$, where T resp. $U_1,\ldots,U_g$ are the standard coordinates on $\mathbb{A}^1_{\mathbb{C}_p}$, resp. on $(D^{\geq\tau})^g$. Then $f(T,U)$ converges on all of $\mathbb{A}^1_{\mathbb{C}_p} \times (D^{>0})^g$.

Proof: We reduce the convergence of f at a given point $x \in \mathbb{A}^1_{\mathbb{C}_p} \times (D^{>0})^g$ to the convergence of f at regions – chosen in dependence on x – of $\mathbb{A}^1_{\mathbb{C}_p} \times (D^{\geq\tau})^g$ with possibly much larger T-coordinates than the T-coordinate of x. For $m \geq 1$ let

$$I_m = \{i = (i_1,\ldots,i_m) \in (\mathbb{N}_0)^m; \quad i_1 + 2i_2 + \ldots + mi_m = m\}.$$

We may write

$$f(T,U) = 1 + \sum_{m=1}^{\infty} \alpha_m(U) T^m$$

$$\alpha_m(U) = \sum_{i \in I_m} (-1)^{i_1+\ldots+i_m} \frac{g_1(U)^{i_1}\ldots g_m(U)^{i_m}}{i_1!\ldots i_m! 1^{i_1} 2^{i_2}\ldots m^{i_m}} = \sum_{\ell \in (\mathbb{N}_0)^g} \beta_{m,\ell} U^\ell$$

$$\beta_{m,\ell} = \sum_{i \in I_m} (-1)^{i_1+\ldots+i_m} \frac{\gamma_{m,\ell,(i_1,\ldots,i_m)}}{i_1!\ldots i_m! 1^{i_1} 2^{i_2}\ldots m^{i_m}}$$

for certain $\gamma_{m,\ell,(i_1,\ldots,i_m)} \in \mathcal{O}_{\mathbb{C}_p}$. We have the estimate

$$\begin{aligned}
\mathrm{ord}_p(\beta_{m,l}) &\geq \min_{i \in I_m} -\mathrm{ord}_p(i_1!\ldots i_m! 1^{i_1} 2^{i_2}\ldots m^{i_m}) \\
&\geq \min_{i \in I_m} -\sum_{j=1}^{m} (\mathrm{ord}_p(i_j!) + i_j(\frac{j}{p})) \\
&\geq \min_{i \in I_m} -\sum_{j=1}^{m} (i_j + i_j(\frac{j}{p})) \\
&\geq \min_{i \in I_m} -2\sum_{j=1}^{m} j i_j = -2m.
\end{aligned}$$

Now let $(t,u_1,\ldots,u_g) \in \mathbb{A}^1_{\mathbb{C}_p} \times (D^{>0})^g$ be given. Set

$$0 < \rho = \min\{1, \frac{\mathrm{ord}_p(u_1)}{\tau}, \ldots, \frac{\mathrm{ord}_p(u_g)}{\tau}\} \leq 1$$

$$\lambda = \frac{\mathrm{ord}_p(t) - 2(1-\rho)}{\rho}.$$

Then we find

$$\begin{aligned}\mathrm{ord}_p(\beta_{m,\ell}u^\ell t^m) &\geq -2m(1-\rho)+\rho\mathrm{ord}_p(\beta_{m,\ell})+\rho|\ell|\tau+\rho m\lambda+2m(1-\rho)\\ &= \rho(\mathrm{ord}_p(\beta_{m,l})+|\ell|\tau+m\lambda)\end{aligned}$$

and this term tends to infinity as $|\ell|+m$ tends to infinity since by hypothesis f converges at the points $(\widetilde{t},\widetilde{u}_1,\ldots,\widetilde{u}_g)$ with $\mathrm{ord}_p(\widetilde{t})\geq\lambda$ and $\mathrm{ord}_p(\widetilde{u}_i)\geq\tau$. The lemma follows.

Now let I and $i_0\in I$ be as in 4.2. In particular we can talk about 1-normality and standard normality of $I\times I$-matrices.

Lemma 6.2. *Suppose the nuclear $I\times I$-matrix $\mathcal{M}$ with entries in $\widehat{A}$ is* 1*-normal. Then $\mathcal{M}$ is σ-similar to a standard* 1*-normal nuclear $I\times I$-matrix.*

Proof: In case $A=R[X]^\dagger$, this is the translation of [15] Lemma 6.5 into matrix terminology. But the proof works for general A.

Lemma 6.3. *Let $\mathcal{N}$ be a nuclear overconvergent $I\times I$-matrix over A which is σ-similar to a standard normal nuclear $I\times I$-matrix over $\widehat{A}$. Then there exist a $\xi\in A$ and a nuclear overconvergent* 1*-normal $I\times I$ matrix $\mathcal{M}$ over A, both unique up to σ-similarity, such that*
(i) the 1×1-matrix ξ^{q-1} is σ-similar to $1\in A$, and
(ii) $\xi\mathcal{M}$ is σ-similar to $\mathcal{N}$.

Proof: For the existence see Wan [16] (there I is finite, but at this point this is not important). For the uniqueness (which by the way we do not need in the sequel) we follow Coleman [4]. Let ξ' and $\mathcal{M}'$ be another such pair. Then $\xi'=a\xi$ for some $a\in A^\times$, hence $a^{q-1}=\frac{\sigma(b)}{b}$ for some $b\in A^\times$ by hypothesis (i) for ξ and ξ'. On the other hand, from hypothesis (ii) for $\mathcal{M}$ and $\mathcal{M}'$ it follows that $\mathcal{M}'$ and $\frac{1}{a}\mathcal{M}$ are σ-similar, and by 1-normality of $\mathcal{M}$ and $\mathcal{M}'$ this implies $a=\frac{d\sigma(c)}{c}$ for some $c,d\in A^\times$ with $d-1\in\pi A$. Thus for $e=\frac{b}{c^{q-1}}$ we have $d^{q-1}=\frac{\sigma(e)}{e}$. In particular $(\sigma(e)-e)\in\pi A$, hence $e\in R+\pi A$, so we may assume in addition $e-1\in\pi A$. For (the unique) $f\in A$ with $f^{q-1}=e$ and $f-1\in\pi A$ we then see $d=\frac{\sigma(f)}{f}$. Thus $a=\frac{\sigma(ef)}{ef}$ and it follows that ξ is σ-similar to σ', and $\mathcal{M}$ to $\mathcal{M}'$. We are done.

6.4 Let $\mathcal{M}$ be a standard 1-normal nuclear $I\times I$-matrix over $\widehat{A}$. Define $I_1=I-\{i_0\}$ and J as in 4.4. Let $\overline{x}$ be a closed point of X of degree f and write $\mathcal{M}_{\overline{x}}=(a^{\overline{x}}_{i_1,i_2})_{i_1,i_2\in I}$ for the fibre matrix $\mathcal{M}_{\overline{x}}$ with entries $a^{\overline{x}}_{i_1,i_2}$ in R_f as defined in 2.12. We denote its i_2-column for $i_2\in I$ by

$$a^{\overline{x}}_{(i_2)}:=(a^{\overline{x}}_{i_1,i_2})_{i_1\in I}\in\prod_I R_f.$$

Let

$$\eta = 1 + F([\pi^m \log(a^{\overline{x}}_{i_0,i_0})]) \in \mathcal{O}_{\mathbb{C}_p}[[U]]$$

with F and m as in 5.2. For $r \in \mathbb{Z}$ we now define a nuclear $J \times J$-matrix $\widetilde{\mathcal{B}}^r(\mathcal{M}_{\overline{x}}) = (b^{(r),\overline{x}}_{q_1,q_2})_{q_1,q_2 \in J}$ with entries in $\mathcal{O}_{\mathbb{C}_p}[[U]]$. It is enough to give the columns

$$b^{(r),\overline{x}}_{(q_2)} := (b^{(r),\overline{x}}_{q_1,q_2})_{q_1 \in J} \in \prod_{j \in J} \mathcal{O}_{\mathbb{C}_p}[[U]],$$

indexed by $q_2 \in J$, of $\widetilde{\mathcal{B}}^r(\mathcal{M}_{\overline{x}})$. The natural embedding $\rho : R_f \to \mathcal{O}_{\mathbb{C}_p}[[U]]$ defines a map

$$\lambda = \lambda(\rho) : \prod_I R_f \to \prod_J \mathcal{O}_{\mathbb{C}_p}[[U]]$$

as explained in 4.5. We will also need the $\mathcal{O}_{\mathbb{C}_p}[[U]]$-algebra structure on $\prod_J \mathcal{O}_{\mathbb{C}_p}[[U]]$ analogous to that on C in 4.4. Namely, the one we get from the natural identification

$$\prod_J \mathcal{O}_{\mathbb{C}_p}[[U]] \cong \mathcal{O}_{\mathbb{C}_p}[[U]][[I_1]],$$

the formal power series ring over $\mathcal{O}_{\mathbb{C}_p}[[U]]$ on the set I_1 (viewed as a set of free variables). Using this $\mathcal{O}_{\mathbb{C}_p}[[U]]$-algebra structure we set

$$b^{(r),\overline{x}}_{(q_2)} := \eta \lambda(a^{\overline{x}}_{(i_0)})^r \frac{\prod_{i \in I_1} \lambda(a^{\overline{x}}_{(i)})^{q_2(i)}}{\lambda(a^{\overline{x}}_{(i_0)})^{|q_2|}}.$$

Note that $\lambda(a^{\overline{x}}_{(i_0)}) = a^{\overline{x}}_{i_0,i_0}$ in the $\mathcal{O}_{\mathbb{C}_p}[[U]]$-algebra $\prod_J \mathcal{O}_{\mathbb{C}_p}[[U]]$ since $\mathcal{M}$ is standard normal. Let $\widetilde{\mathcal{B}}^r_-(\mathcal{M}_{\overline{x}})$ be the matrix defined by the same recipe, but now using η^{-1} in place of η.

6.5 Let $\xi \in A$ be a unit, let $(s,t) \in \mathcal{F}$, let $r_1, r_2 \in \mathbb{N}$, for $\ell = 1$ and $\ell = 2$ let $I^{(\ell)}$ be a countable index set, $i_0^{(\ell)} \in I^{(\ell)}$ an element and $\mathcal{M}_\ell$ a standard 1-normal (with respect to $i_0^{(\ell)}$) nuclear $I^{(\ell)} \times I^{(\ell)}$-matrix over $\widehat{A}$. Arguing as in 4.8, where we proved that the matrices $\mathcal{B}^r(\mathcal{M})$ are nuclear, we see that the trace

$$g^{r_1,r_2,s,t}_{\overline{x},\xi,\mathcal{M}_1,\mathcal{M}_2}(U) := \mathrm{Tr}(\xi^t_{\overline{x}} \widetilde{\mathcal{B}}^{s-r_1}(\mathcal{M}_{1,\overline{x}}) \otimes \bigwedge^{r_1}(\mathcal{M}_1)_{\overline{x}} \otimes \widetilde{\mathcal{B}}^{-s-r_2}_-(\mathcal{M}_{2,\overline{x}}) \otimes \bigwedge^{r_2}(\mathcal{M}_2)_{\overline{x}})$$

(the fibre $\xi_{\overline{x}} \in R$ is defined as in 2.12 by viewing ξ as a 1×1-matrix) is a well defined element in $\mathcal{O}_{\mathbb{C}_p}[[U]]$, i.e. the infinite sum of diagonal elements of this tensor product matrix converges in $\mathcal{O}_{\mathbb{C}_p}[[U]]$. We may view it as a function on $D^{>0}$. Let $\nu \in \mathbb{Q}$ satisfy both 5.4 and the condition from 4.6 for both $\mathcal{M}_1$ and $\mathcal{M}_2$ so that we may form the matrices $\mathcal{B}^{s-r_1}(\mathcal{M}_1)$ and $\mathcal{B}^{-s-r_2}_-(\mathcal{M}_2)$ over $A \widehat{\otimes}_R B$ with $B = (B(\nu)_K)^0$. Recall the embedding $\iota : D^{\geq \nu} \to D^{>0}$ from 5.4 and that we view the free variable V as standard coordinate on the source $D^{\geq \nu}$, and the free variable U as standard coordinate on the target $D^{>0}$ of ι.

For a matrix $\mathcal{N}$ with coefficients in $A\widehat{\otimes}_R B$ and for $y \in D^{\geq \nu}$ we denote by $\mathcal{N}|_{V=y}$ the matrix with entries in $\widehat{A}$ obtained from $\mathcal{N}$ by specializing elements $a \otimes V^n \in A\widehat{\otimes}_R B$ (for $n \in \mathbb{N}_0$) to $a \otimes y^n \in \widehat{A}$.

Lemma 6.6. *For K-rational points $y \in D^{\geq \nu}$ we have*

$$g^{r_1,r_2,s,t}_{\overline{x},\xi,\mathcal{M}_1,\mathcal{M}_2}(\iota(y)) =$$

$$\mathrm{Tr}(((\xi^t \mathcal{B}^{s-r_1}(\mathcal{M}_1) \otimes \bigwedge^{r_1}(\mathcal{M}_1) \otimes \mathcal{B}_-^{-s-r_2}(\mathcal{M}_2) \otimes \bigwedge^{r_2}(\mathcal{M}_2))|_{V=y})_{\overline{x}})$$

Proof: Taking $\overline{x}$-fibres commutes with $\otimes$, thus

$$\xi^t_{\overline{x}}(\mathcal{B}^{s-r_1}(\mathcal{M}_1)|_{V=y})_{\overline{x}} \otimes \bigwedge^{r_1}(\mathcal{M}_1)_{\overline{x}} \otimes (\mathcal{B}_-^{-s-r_2}(\mathcal{M}_2)|_{V=y})_{\overline{x}} \otimes \bigwedge^{r_2}(\mathcal{M}_2)_{\overline{x}}$$

$$= ((\xi^t \mathcal{B}^{s-r_1}(\mathcal{M}_1) \otimes \bigwedge^{r_1}(\mathcal{M}_1) \otimes \mathcal{B}_-^{-s-r_2}(\mathcal{M}_2) \otimes \bigwedge^{r_2}(\mathcal{M}_2))|_{V=y})_{\overline{x}}.$$

Therefore it suffices to show

$$(\mathcal{B}^r(\mathcal{M})|_{V=y})_{\overline{x}} = \widetilde{\mathcal{B}}^r(\mathcal{M}_{\overline{x}})|_{U=\iota(y)} \quad \text{and} \quad (\mathcal{B}^r_-(\mathcal{M})|_{V=y})_{\overline{x}} = \widetilde{\mathcal{B}}^r_-(\mathcal{M}_{\overline{x}})|_{U=\iota(y)},$$

for standard 1-normal nuclear matrices $\mathcal{M}$ over $\widehat{A}$ and $r \in \mathbb{Z}$. This is essentially a statement on commutation of the two operations "$\mathcal{M} \mapsto \mathcal{B}^r(\mathcal{M})$" and "taking the f-fold σ-power of a square matrix". In our situation this holds since $\mathcal{M}$ is standard normal, as we will now explain. For such $\mathcal{M}$ we keep the notation from 6.4. From 5.4 it follows that $\widetilde{\mathcal{B}}^r(\mathcal{M}_{\overline{x}})|_{U=\iota(y)}$ is the matrix constructed by the same recipe as $\widetilde{\mathcal{B}}^r(\mathcal{M}_{\overline{x}})$, but using

$$(a^{\overline{x}}_{i_0,i_0})^y = \exp(y \log(a^{\overline{x}}_{i_0,i_0})) \in R_f$$

in place of η. Observe that $a^{\overline{x}}_{i_0,i_0} = (a_{i_0,i_0})_{\overline{x}}$ where $(a_{i_0,i_0})_{\overline{x}}$ is defined as in 2.12 by viewing the (i_0,i_0)-entry a_{i_0,i_0} of $\mathcal{M}$ as a 1×1-matrix — this is because $\mathcal{M}$ is standard. In particular we see $(a^{\overline{x}}_{i_0,i_0})^y = ((a_{i_0,i_0})_{\overline{x}})^y \in R$. Let (M,ϕ) be the σ-module over $\widehat{A}$ such that the action of ϕ on a formal basis $\{e_i\}_{i\in I}$ of M is given by $\mathcal{M}$. As in 4.9 consider the $\widehat{A}$-algebra $D = \mathrm{Sym}_{\widehat{A}}(M)[\frac{1}{e_{i_0}}]^0$ of degree zero elements in $\mathrm{Sym}_{\widehat{A}}(M)[\frac{1}{e_{i_0}}]$. Let $B^r(M)$ be its completion as in 4.9. Denote by ψ the natural σ-linear ring endomorphism of $B^r(M)$ defined by ϕ, as in 4.9. Then $\mathcal{B}^r(\mathcal{M})|_{V=y}$ is the matrix of the σ-linear endomorphism $\psi_{y+r} = B^r(\phi)|_{V=y} = ((a_{i_0,i_0})_{\overline{x}})^{y+r}\psi$ (use that $\mathcal{M}$ is standard). Hence $(\mathcal{B}^r(\mathcal{M})|_{V=y})_{\overline{x}}$ is the matrix of the R_f-linear endomorphism $(\psi^f_{y+r})_{\overline{x}}$ which the f-fold iterate ψ^f_{y+r} of ψ_{y+r} induces on the fibre $B^r(M)_{\overline{x}} = B^r(M) \otimes_{\widehat{A}} R_f$ (formed with respect to the Teichmüller lift $x : \widehat{A} \to R_f$ of $\overline{x}$). On the other hand we can view $B^r(M)_{\overline{x}}$ as the completion (analogously to 4.9) of $\mathrm{Sym}_{R_f}(M_{\overline{x}})[\frac{1}{e_{i_0}}]^0$ (with $M_{\overline{x}} = M \otimes_{\widehat{A}} R_f$). Then

$\widetilde{\mathcal{B}}^r(\mathcal{M}_{\overline{x}})|_{U=\iota(y)}$ is the matrix of the R_f-linear endomorphism $((a_{i_0,i_0})_{\overline{x}})^{y+r}\psi_{f,\overline{x}}$ of $B^r(M)_{\overline{x}}$ where $\psi_{f,\overline{x}}$ is the R_f-linear ring endomorphism of $B^r(M)_{\overline{x}}$ induced by the endomorphism which the f-fold iterate ϕ^f of ϕ induces on $M_{\overline{x}}$. Thus it remains to show $(\psi^f_{y+r})_{\overline{x}} = ((a_{i_0,i_0})_{\overline{x}})^{y+r}\psi_{f,\overline{x}}$. Now we clearly have $(\psi^f_{y+r})_{\overline{x}} = ((a_{i_0,i_0})_{\overline{x}})^{y+r}(\psi^f)_{\overline{x}}$ with $(\psi^f)_{\overline{x}}$ the fibre of ψ^f in $B^r(M)_{\overline{x}}$. Therefore we conclude using the functoriality $(\psi^f)_{\overline{x}} = \psi_{f,\overline{x}}$ of the (σ-linear) functor $\mathrm{Sym}_{\widehat{A}}(?)$.

6.7 For $f \in \mathbb{N}$ let T_f be the set of all closed points of X of degree f. Let $A_f = A \otimes_R R_f$. Note that the f-fold σ-power $(\mathcal{D}^{\wedge i})^{(\sigma)^f}$ (as defined in 2.12) is the matrix describing the endomorphism which the R_f-algebra endomorphism $\sigma^f \otimes 1$ of A_f induces on $\Omega^i_{A_f/R_f} = \Omega^i_{A/R} \otimes_R R_f$. Therefore we may apply 2.13 to the situation obtained by base change $\otimes_R R_f$, with $\sigma^f \otimes 1 \in \mathrm{End}(A_f)$ replacing $\sigma \in \mathrm{End}(A)$. We get that

$$S_{\overline{x}} := \sum_{0 \leq i \leq d} (-1)^i \mathrm{Tr}((\mathcal{D}^{\wedge i})_{\overline{x}})$$

for $\overline{x} \in T_f$ is invertible in R_f. For $0 \leq j \leq d$ we may define

$$h^{r_1,r_2,j,s,t}_{f,\xi,\mathcal{M}_1,\mathcal{M}_2}(U) := \sum_{\overline{x} \in T_f} \frac{\mathrm{Tr}((\mathcal{D}^{\wedge d-j})_{\overline{x}})}{S_{\overline{x}}} g^{r_1,r_2,s,t}_{\overline{x},\xi,\mathcal{M}_1,\mathcal{M}_2} \in \mathcal{O}_{\mathbb{C}_p}[[U]],$$

$$D^{r_1,r_2,j,s,t}_{\xi,\mathcal{M}_1,\mathcal{M}_2}(T,U) := \exp(-\sum_{f=1}^{\infty} \frac{h^{r_1,r_2,j,s,t}_{f,\xi,\mathcal{M}_1,\mathcal{M}_2}(U)}{f} T^f) \in \mathbb{C}_p[[T,U]].$$

Theorem 6.8. *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are σ-similar to 1-normal nuclear overconvergent matrices over A, then $D^{r_1,r_2,j,s,t}_{\xi,\mathcal{M}_1,\mathcal{M}_2}(T,U)$ defines a holomorphic function on $\mathbb{A}^1_{\mathbb{C}_p} \times D^{>0}$. There exists a nuclear overconvergent matrix $\mathcal{N}$ over $A\widehat{\otimes}_R B$ which is σ-similar to*

$$\xi^t \mathcal{B}^{s-r_1}(\mathcal{M}_1) \otimes \bigwedge^{r_1}(\mathcal{M}_1) \otimes \mathcal{B}_-^{-s-r_2}(\mathcal{M}_2) \otimes \bigwedge^{r_2}(\mathcal{M}_2),$$

and for K-rational points $y \in D^{\geq \nu}$ we have

$$D^{r_1,r_2,j,s,t}_{\xi,\mathcal{M}_1,\mathcal{M}_2}(T,\iota(y)) = \det(1 - \psi[\mathcal{N}|_{V=y} \otimes \mathcal{D}^{\wedge j}]T).$$

Proof: The existence of $\mathcal{N}$ follows from 4.8 and 4.10. Next let us make a general remark. For a nuclear overconvergent matrix $\mathcal{M}$ over A we defined the completely continuous operator $\psi[\mathcal{M}] = \psi_A[\mathcal{M}]$ in 2.8 relative to the Frobenius endomorphism σ on A. Now consider the f-fold σ-power $\mathcal{M}^{(\sigma)^f}$ of $\mathcal{M}$ from 2.12 and *view it as a matrix over $A_f = A \otimes_R R_f$*. As such we define the $K_f = R_f \otimes \mathbb{Q}$-linear completely continuous operator $\psi_{A_f}[\mathcal{M}^{(\sigma)^f}]$ relative to the Frobenius endomorphism σ^f on A_f. One finds

$$\psi_{A_f}[\mathcal{M}^{(\sigma)^f}] = \psi_A[\mathcal{M}]^f \otimes_K K_f.$$

We apply this to $\mathcal{M} = \mathcal{N}|_{V=y} \otimes \mathcal{D}^{\wedge j}$ and obtain

$$\begin{aligned}\mathrm{Tr}_K(\psi_A[\mathcal{N}|_{V=y} \otimes \mathcal{D}^{\wedge j}]^f) &= \mathrm{Tr}_{K_f}(\psi_A[\mathcal{N}|_{V=y} \otimes \mathcal{D}^{\wedge j}]^f \otimes_K K_f) \\ &= \mathrm{Tr}_{K_f}(\psi_{A^f}[(\mathcal{N}|_{V=y} \otimes \mathcal{D}^{\wedge j})^{(\sigma)^f}]) \\ &= \mathrm{Tr}_{K_f}(\psi_{A^f}[(\mathcal{N}|_{V=y})^{(\sigma)^f} \otimes (\mathcal{D}^{\wedge j})^{(\sigma)^f}]) \\ &= \sum_{\overline{x} \in T_f} \frac{\mathrm{Tr}((\mathcal{D}^{\wedge d-j})_{\overline{x}})\mathrm{Tr}((\mathcal{N}|_{V=y})_{\overline{x}})}{S_{\overline{x}}}.\end{aligned}$$

where for the last equality we applied 2.13. But

$$\mathrm{Tr}((\mathcal{N}|_{V=y})_{\overline{x}}) = \mathrm{Tr}((((\xi^t \mathcal{B}^{s-r_1}(\mathcal{M}_1) \otimes \bigwedge^{r_1}(\mathcal{M}_1) \otimes \mathcal{B}_-^{-s-r_2}(\mathcal{M}_2) \otimes \bigwedge^{r_2}(\mathcal{M}_2))|_{V=y})_{\overline{x}})$$

which by 6.6 is equal to $g^{r_1,r_2,s,t}_{\overline{x},\xi,\mathcal{M}_1,\mathcal{M}_2}(\iota(y))$. Thus the stated formula is proven since its right hand side may be written as

$$\exp(-\sum_{f=1}^{\infty} \frac{\mathrm{Tr}_K(\psi[\mathcal{N}|_{V=y} \otimes \mathcal{D}^{\wedge j}]^f)}{f} T^f).$$

Furthermore the points $\iota(y)$ for K-rational points $y \in D^{\geq \nu}$ are Zariski dense in $\iota(D^{\geq \nu})$, therefore we get the equality of holomorphic functions

$$D^{r_1,r_2,j,s,t}_{\xi,\mathcal{M}_1,\mathcal{M}_2}(T,U) = \det(1 - \psi[\mathcal{N} \otimes \mathcal{D}^{\wedge j}]T)$$

on $D^{>0} \times \iota(D^{\geq \nu})$, where in the function on the right hand side we substitute V by $\iota^{-1}(U)$. But the right hand side extends to a holomorphic function on $\mathbb{A}^1_{\mathbb{C}_p} \times \iota(D^{\geq \nu})$, since $\psi[\mathcal{N} \otimes \mathcal{D}^{\wedge j}]$ is completely continuous by 2.10. The definition of $D^{r_1,r_2,j,s,t}_{\xi,\mathcal{M}_1,\mathcal{M}_2}(T,U)$ and 6.1 now show the holomorphy on all of $\mathbb{A}^1_{\mathbb{C}_p} \times D^{>0}$, completing the proof.

6.9 Let $\alpha \in \widehat{A}$ be a unit. For closed points $\overline{x} \in X$ define $\alpha_{\overline{x}} \in R$ as in 2.12 by viewing α as a 1×1-matrix. For $\kappa \in \mathrm{Hom}_{K\text{-}an}(R^\times, \mathbb{C}_p^\times)$ we ask for the twisted L-function

$$L(\alpha, T, \kappa) := \prod_{\overline{x} \in X} \frac{1}{1 - \kappa(\alpha_{\overline{x}})T^{\deg(\overline{x})}}.$$

It can be written as a power series with coefficients in $\mathcal{O}_{\mathbb{C}_p}$, hence is trivially holomorphic on $D^{>0}$ (in the variable T).

6.10 We say that $\alpha \in \widehat{A}$ is *ordinary geometric* if there exists a nuclear $G \times G$-matrix $\mathcal{H} = (h_{g_1,g_2})_{g_1,g_2 \in G}$ over $\widehat{A}$, a non negative integer $j \in \mathbb{N}_0$ and a nested sequence of $(j+1)$ finite subsets $G_0 \subset G_1 \subset \ldots \subset G_j$ of the (countable) index set G such that:
(i) $\mathcal{H}$ is σ-similar to a nuclear overconvergent matrix over A.

(ii) $h_{g_1,g_2}=0$ whenever there is a $0\le\ell\le j$ with $g_2\in G_\ell$ and $g_1\notin G_\ell$. Thus, $\mathcal{H}$ is in block triangular form.
(iii) $\pi^{\ell+1}$ divides h_{g_1,g_2} whenever $g_2\notin G_\ell$, for all $0\le\ell\le j$.
(iv) For all $0\le\ell\le j$ the element

$$H_\ell:=\pi^{-\sum_{i=1}^{\ell}i(c_i-c_{i-1})}\det((h_{g_1,g_2})_{g_1,g_2\in G_\ell})$$

of $\widehat{A}$ is a unit, where we set $c_\ell=|G_\ell|$. Set $H_{-1}=1$.
(v) We have $\alpha=H_j/H_{j-1}=\pi^{-j(c_j-c_{j-1})}\det((h_{g_1,g_2})_{g_1,g_2\in(G_j-G_{j-1})})$.
The meaning of this definition is that α is the determinant of the pure slope j part (as a unit root σ-module) of a nuclear σ-module over A which is ordinary up to slope j and overconvergent (but neither the pure slope j part itself nor its determinant need to be overconvergent). See [16] for details on the Hodge-Newton decomposition by slopes.

Theorem 6.11. *Suppose α is ordinary geometric. Then there exists a meromorphic function L_α on the $\mathbb{C}_p$-rigid space $\mathbb{A}^1_{\mathbb{C}_p}\times\mathcal{W}$ whose pullback to $\mathbb{A}^1_{\mathbb{C}_p}$ via*

$$\mathbb{A}^1_{\mathbb{C}_p}\to\mathbb{A}^1_{\mathbb{C}_p}\times\mathcal{W},\quad t\mapsto(t,\kappa)$$

for any $\kappa\in\operatorname{Hom}_{K\text{-}an}(R^\times,\mathbb{C}_p^\times)=\mathcal{W}(\mathbb{C}_p)$ is a continuation of $L(\alpha,T,\kappa)$.

Proof: We treat every component $\mathcal{W}_{(s,t)}$ of $\mathcal{W}$ separately, so let us fix $(s,t)\in\mathcal{F}$. Keeping the notation from 6.10 we begin with some definitions. For $0\le\ell\le j$ let $I^{(\ell)}$ be the index set of the nuclear matrix $\bigwedge^{c_\ell}(\mathcal{H})$. Our assumptions on $\mathcal{H}$ imply that $\bigwedge^{c_\ell}(\mathcal{H})$ is standard normal with respect to some $i_0^{(\ell)}\in I_\ell$. Moreover it is σ-similar to a nuclear overconvergent $I^{(\ell)}\times I^{(\ell)}$ matrix. Thus we may apply 6.3 to get a $\xi_\ell\in A$ and a 1-normal (with respect to $i_0^{(\ell)}$) nuclear overconvergent $I^{(\ell)}\times I^{(\ell)}$-matrix $\mathcal{M}_\ell$ over A such that ξ_ℓ^{q-1} is σ-similar to $1\in A$, and $\xi_\ell\mathcal{M}_\ell$ is σ-similar to $\bigwedge^{c_\ell}(\mathcal{H})$. By 6.2 there is a standard 1-normal (with respect to $i_0^{(\ell)}$) nuclear $I^{(\ell)}\times I^{(\ell)}$-matrix $\mathcal{M}'_\ell$ over $\widehat{A}$ which is σ-similar to $\mathcal{M}_\ell$. Let $\mathcal{M}'_{\ell,unit}\in\widehat{A}$ be the $(i_0^{(\ell)},i_0^{(\ell)})$-entry of $\mathcal{M}'_\ell$. This is a 1-unit. Then $\xi_\ell\mathcal{M}'_{\ell,unit}\in\widehat{A}$ is σ-similar to the $(i_0^{(\ell)},i_0^{(\ell)})$-entry of $\bigwedge^{c_\ell}(\mathcal{H})$ which we denote by a_ℓ. We will need these definitions for $\ell=j-1$ and $\ell=j$ if $j>0$. If $j=0$ we set $\xi_{-1}=\mathcal{M}_{-1}=\mathcal{M}'_{-1}=\mathcal{M}'_{-1,unit}=a_{-1}=1\in R$. Our definitions imply $\alpha=a_j/a_{j-1}$, thus if we set $\xi=\xi_j/\xi_{j-1}$ and $\mu=\mathcal{M}'_{j,unit}/\mathcal{M}'_{j-1,unit}$ we find that α is σ-similar to $\xi\mu$. Let

$$H(T,U)=\prod_{\overline{x}\in X}\frac{1}{1-\xi_{\overline{x}}^{t}\mu_{\overline{x}}^{s}(1+F([\pi^m\log(\mu_{\overline{x}})]))T^{\deg\overline{x}}}.$$

This is a holomorphic function on $D^{>0}\times D^{>0}$ where we view T (resp. U) as coordinate for the first (resp. second) factor $D^{>0}$. Recall from 5.2 the finite étale covering of rigid spaces $D^{>0}\to\mathcal{W}_{(s,t)}$ which on $\mathbb{C}_p$-valued points is given by

$$D^{>0}\to\mathcal{W}_{(s,t)}(\mathbb{C}_p)\cong\operatorname{Hom}_{K\text{-}an}(\overline{U}_R^{(1)},\mathbb{C}_p^\times)$$

$$z \longmapsto [\overline{w} \mapsto 1 + F([\pi^m \log(\overline{w})])(z)].$$

We see that for any $w \in U_R^{(1)}$ the holomorphic function $1 + F([\pi^m \log(w)])(U)$ in the variable U on $D^{>0}$ descends to $\mathcal{W}_{(s,t)}$. Thus our $H(T,U)$ descends to a holomorphic function $L_{\alpha,(s,t)}$ on $D^{>0} \times \mathcal{W}_{(s,t)}$. Moreover, for $\kappa \in \mathcal{W}_{(s,t)}(\mathbb{C}_p) \subset \mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{K\text{-}an}(R^\times, \mathbb{C}_p^\times)$ the pullback of $L_{\alpha,(s,t)}$ via

$$D^{>0} \to D^{>0} \times \mathcal{W}_{(s,t)}, \quad t \mapsto (t, \kappa)$$

is $L(\alpha, T, \kappa)$: this is immediate since $\xi_{\overline{x}}\mu_{\overline{x}} = \alpha_{\overline{x}}$ is the decomposition of $\alpha_{\overline{x}} \in R^\times$ according to $R^\times = \mu_{q-1} \times U_R^{(1)}$, for any $\overline{x} \in X$. These considerations also show that for K-rational points $y \in D^{\geq\nu}$ we have

$$H(T, \iota(y)) = L(\xi^t \mu^s \mu^y, T) \tag{1}$$

with $\iota : D^{\geq\nu} \to D^{>0}$ from 5.4. To show that $L_{\alpha,(s,t)}$ is meromorphic on $\mathbb{A}^1_{\mathbb{C}_p} \times \mathcal{W}_{(s,t)}$ it is enough to show that $H(T,U)$ is meromorphic on $\mathbb{A}^1_{\mathbb{C}_p} \times D^{>0}$. Consider the $\mathbb{C}_p[[T,U]]$-element

$$\underline{H}(T,U) = \prod_{r_1,r_2 \geq 1} \Big(\prod_{i=0}^{d} D^{r_1,r_2,i,s,t}_{\xi,\mathcal{M}'_j,\mathcal{M}'_{j-1}}(T,U)^{(-1)^{i-1}}\Big)^{(-1)^{r_1+r_2} r_1 r_2}.$$

By 6.8 each factor $D^{r_1,r_2,i,s,t}_{\xi,\mathcal{M}'_j,\mathcal{M}'_{j-1}}(T,U)$ is holomorphic on $\mathbb{A}^1_{\mathbb{C}_p} \times D^{>0}$. Moreover, since $\bigwedge^{r_\ell}(\mathcal{M}'_\ell)$ is divisible by $\pi^{r_\ell - 1}$ it also follows from 6.8 that $\mathrm{ord}_\pi(1 - D^{r_1,r_2,i,s,t}_{\xi,\mathcal{M}'_j,\mathcal{M}'_{j-1}}(T,U))$ tends to infinity as $r_1 + r_2$ tends to infinity (if the index set G is finite then the above product is even finite). Therefore $\underline{H}(T,U)$ is meromorphic on $\mathbb{A}^1_{\mathbb{C}_p} \times D^{>0}$. We claim

$$\underline{H}(T,U) = H(T,U)$$

as meromorphic functions on $D^{>0} \times D^{>0}$. As in 6.8 it is enough to check this on all subsets $D^{>0} \times \iota(y) \subset D^{>0} \times D^{>0}$ for K-rational points $y \in D^{\geq\nu}$. From 4.11 we get the following equalities in the Grothendieck group $\Delta(\widehat{A})$:

$$[\xi^t (\mathcal{M}'_{j,unit})^s (\mathcal{M}'_{j,unit})^y] = \bigoplus_{r \geq 1} (-1)^{r-1} r [\xi^t \mathcal{B}^{s-r}(\mathcal{M}'_j)|_{V=y} \otimes \bigwedge^r (\mathcal{M}'_j)]$$

$$[(\mathcal{M}'_{j-1,unit})^{-s} (\mathcal{M}'_{j-1,unit})^{-y}] = \bigoplus_{r \geq 1} (-1)^{r-1} r [\mathcal{B}_-^{-s-r}(\mathcal{M}'_{j-1})|_{V=y} \otimes \bigwedge^r (\mathcal{M}'_{j-1})]$$

(with the notation $|_{V=y}$ explained in 6.5 still in force: V is the standard coordinate on $D^{\geq\nu}$). Together

$$[\xi^t \mu^s \mu^y] = [\frac{\xi^t (\mathcal{M}'_{j,unit})^s (\mathcal{M}'_{j,unit})^y}{(\mathcal{M}'_{j-1,unit})^s (\mathcal{M}'_{j-1,unit})^y}] =$$

$$\bigoplus_{r_1,r_2\geq 1}(-1)^{r_1+r_2}r_1r_2[\xi^t(\mathcal{B}^{s-r_1}(\mathcal{M}'_j)\otimes\bigwedge^{r_1}\mathcal{M}'_j\otimes\mathcal{B}_-^{-s-r_2}(\mathcal{M}'_{j-1})\otimes\bigwedge^{r_2}\mathcal{M}'_{j-1})|_{V=y}].$$

Combining with 6.8 and the trace formula 2.13 we get

$$\underline{H}(T,\iota(y))=L(\xi^t\mu^s\mu^y,T). \tag{2}$$

Comparing (1) and (2) completes the proof.

7 Higher rank

7.1 A finite rank σ-module (M,ϕ) over $\widehat{A}$ is called *ordinary* if it admits a separated and exhausting ϕ-stable filtration by free sub-$\widehat{A}$-modules

$$0=M_0\subset M_1\subset M_2\subset\dots$$

of M with free quotients, such that each quotient $(M_i/M_{i+1},\phi)$ is of the form $(U_i,\pi^i.\phi_i)$ where (U_i,ϕ_i) is a unit root σ-module; that is, $\widehat{A}_\sigma\otimes U_i\to U_i, a\otimes u\mapsto a.\phi_i(u)$ is bijective. The (U_i,ϕ_i) are called the *graded pieces of* (M,ϕ), and (U_0,ϕ_0) is called the *unit root part of* (M,ϕ), also denoted by ϕ_{unit}.

Theorem 7.2. *(Hodge-Newton slope decomposition for overconvergent σ-modules) Let $\mathcal{M}$ be the matrix, in some basis, of a graded piece of an ordinary and overconvergent finite rank σ-module (M,ϕ) over $\widehat{A}$. Then there exists a convergent series representation*

$$[\mathcal{M}]=\sum_{r\geq 1}\pm[\mathcal{C}_r]$$

in $\Delta(\widehat{A})$ with nuclear overconvergent matrices $\mathcal{C}_r$ over A.

Proof: (1) By induction on m we prove that for each $m\in\mathbb{N}_0$ there exist finite index sets J^1_m,J^2_m, ordinary overconvergent σ-modules α_t and β_t of finite rank for each $t\in J^1_m\cup J^2_m$, with $(\beta_t)_{unit}$ of rank one, and integers $m_t\geq m$ for each $t\in J^2_m$, such that

$$[\mathcal{M}]=(\sum_{t\in J^2_m}\pm[\pi^{m_t}(\alpha_t)_{unit}\otimes(\beta_t)^{-1}_{unit}])+(\sum_{t\in J^1_m}\pm[\alpha_t\otimes(\beta_t)^{-1}_{unit}]) \tag{$*$}$$

in $\Delta(\widehat{A})$. Here, by abuse of notation, we identify a finite rank σ-module with the σ-similarity class of matrices it corresponds to. For $m=0$ one has $[\mathcal{M}]=[\alpha_{unit}\otimes\beta^{-1}_{unit}]$ for some α,β, by [16] 6.2. Now let us pass from m to $m+1$. Fix $t\in J^2_m$. Let $(\alpha_{t,t'})_{t'\in T_t}$ be the set of higher graded pieces of α_t (i.e. $(\alpha_t)_{unit}$ omitted). By [16] 6.2 again, there exist for each $t'\in T_t$ ordinary overconvergent finite rank σ-modules $\widetilde{\alpha}_{t,t'}$ and $\widetilde{\beta}_{t,t'}$, with $(\widetilde{\beta}_{t,t'})_{unit}$ of rank one, such that $[\alpha_{t,t'}] = [(\widetilde{\alpha}_{t,t'})_{unit} \otimes (\widetilde{\beta}_{t,t'})^{-1}_{unit}]$. Thus

$$[(\alpha_t)_{unit} \otimes (\beta_t)^{-1}_{unit}] = [\alpha_t \otimes (\beta_t)^{-1}_{unit}] - \sum_{t \in T_t} [\pi^{m_{t'}}(\alpha_{t,t'}) \otimes (\beta_t)^{-1}_{unit}]$$

$$= [\alpha_t \otimes (\beta_t)^{-1}_{unit}] - \sum_{t \in T_t} [\pi^{m_{t'}}(\widetilde{\alpha}_{t,t'})_{unit} \otimes (\beta_t \otimes \widetilde{\beta}_{t,t'})^{-1}_{unit}]$$

with integers $m_{t'} \geq 1$ (the higher slopes of α_t). Inserting this into the formula given by induction hypothesis for m gives the formula for $m+1$.
(2) To get the desired convergent series representation for $\mathcal{M}$ it is now enough to express, for $t \in J^1_m$, the terms $(\beta_t)^{-1}_{unit}$ in $(*)$ through overconvergent matrices. This is achieved by factoring β_t according to 6.3 and applying 4.11 (with $V = 0$ and $s = -1$ there) to the 1-normal overconvergent factor of β_t.

7.3 As a corollary of Theorem 7.2 (and 3.4) we recover Wan's result: that $L(\mathcal{M}, T)$ is a meromorphic function on $\mathbb{A}^1$.

References

[1] *N. Bourbaki*, Variétés différentielles et analytiques. Fascicule de résultats. Paris: Hermann 1967

[2] *S.Bosch, U.Günzer, R.Remmert*, Non-Archimedean Analysis, Grundlehren der mathematischen Wissenschaften 261 (1984), Springer-Verlag, Heidelberg

[3] *R. Coleman*, p-adic Banach spaces and families of modular forms, Invent. Math. 127 (1997), no. 3, 417-479

[4] *R. Coleman*, Notes on Wan's proof of Dwork's conjecture, private notes

[5] *R. Coleman and B. Mazur*, The eigencurve, in: Galois representations in arithmetic algebraic geometry (Durham, 1996), 1-113, London Math. Soc. Lecture Note Ser. 254, Cambridge Univ. Press, Cambridge 1998

[6] *R. Elkik*, Solutions d'equations à coefficients dans un anneau henselién, Ann. Sci. Ec. No. Sup. 6, no.4 (1973), 553-604

[7] *P. Monsky and G. Washnitzer*, Formal cohomology: I, Ann. of Math. 88 (1968), 181-217

[8] *P. Monsky*, Formal cohomology III. Fixed point theorems, Ann. of Math. (2) 93 (1971), 315-343

[9] *S. Lang*, Cyclotomic fields I+II, Graduate Texts in Mathematics 121, Springer-Verlag, New York 1990

[10] *M. van der Put*, The cohomology of Monsky and Washnitzer, Mém. Soc. Math. France (N.S.) 23 (1986), 4, 33-59

[11] *A. Robert*, A Course in p-adic analysis, Graduate Texts in Mathematics 198, Springer-Verlag, New York 2000

[12] *J.P. Serre*, Endomorphismes complétement continues des espaces de Banach p-adiques, Publ. Math. I.H.E.S. 12 (1962), 69-85

[13] *P. Schneider and J. Teitelbaum*, p-adic Fourier theory, Documenta Math. 6 (2001), 447-481

[14] *D. Wan* Dwork's conjecture on unit root zeta functions, Ann. of Math. (2) 150 (1999), no.3, 876-927

[15] *D. Wan*, Rank one case of Dwork's conjecture, J. Amer. Math. Soc. 13 (2000), 853-908

[16] *D. Wan*, Higher rank case of Dwork's conjecture, J. Amer. Math. Soc. 13 (2000), 807-852

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Hasse Invariant and Group Cohomology

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Abstract. Let $p \geq 5$ be a prime number. The Hasse invariant is a modular form modulo p that is often used to produce congruences between modular forms of different weights. We show how to produce such congruences between eigenforms of weights 2 and $p+1$, in terms of group cohomology. We also illustrate how our method works for inert primes $p \geq 5$ in the contexts of quadratic imaginary fields (where there is no Hasse invariant available) and Hilbert modular forms over totally real fields, cyclic and of even degree over the rationals.

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1 The puzzle

Let $p \geq 5$ be a prime number. In the theory of modular forms mod p (see [S] and [SwD]) a special role is played by the *Hasse invariant* and the Θ operator. We fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p}$, and denote the corresponding place of $\overline{\mathbf{Q}}$ by $\wp$. We have a modular form E_{p-1} of weight $p-1$ in $M_{p-1}(SL_2(\mathbf{Z}), \mathbf{Z}_p)$, that is congruent to 1 mod $\wp$ (see [S] and [SwD]). By congruence we will mean a congruence of Fourier coefficients at almost all primes. The modular form E_{p-1} is the normalised form of the classical Eisenstein series, and has q-expansion

$$1 - 2(p-1)/B_{p-1} \sum \sigma_{p-2}(n) q^n,$$

and the congruence property of E_{p-1} is then a consequence of the divisibility of the denominator of B_{p-1} by p (the theorem of Clausen-von Staudt: see [S, §1.1]).
Multiplying by E_{p-1} gives the fact that for any positive integer N prime to p a weight 2 form in $S_2(\Gamma_1(N), \mathbf{Z}_p)$ is congruent mod p to a weight $p+1$ form in $S_{p+1}(\Gamma_1(N), \mathbf{Z}_p)$. It follows that a weight 2 normalized eigenform in $S_2(\Gamma_1(N), \overline{\mathbf{Z}_p})$ is congruent mod $\wp$ to a weight $p+1$ eigenform in $S_{p+1}(\Gamma_1(N), \overline{\mathbf{Z}_p})$ (see [DS, §6.10]).

For $N \geq 5$ prime to p, the Hasse invariant constructed geometrically (see [Kz]) is a global section of the coherent sheaf $\omega^{\otimes p-1}_{X_1(N)_{\mathbf{F}_p}}$ where $\omega_{X_1(N)_{\mathbf{F}_p}}$ is the pull back of the canonical sheaf $\Omega_{\mathcal{E}/X_1(N)_{\mathbf{F}_p}}$ by the zero section of the map $\mathcal{E}_{\mathbf{F}_p} \to X_1(N)_{\mathbf{F}_p}$, and with $\mathcal{E}_{\mathbf{F}_p}$ the universal generalised elliptic curve over $X_1(N)_{\mathbf{F}_p}$: E_{p-1} can be interpreted as a characteristic zero lift of the Hasse invariant (Deligne).
The Θ operator on modular forms mod p is defined by:

$$\Theta(\sum a_n q^n) = \sum n a_n q^n$$

where $a_n \in \overline{\mathbf{F}_p}$. It preserves levels, and increases weights by $p+1$, i.e., it gives maps:

$$M_k(\Gamma_1(N), \mathbf{F}_p) \longrightarrow M_{k+p+1}(\Gamma_1(N), \mathbf{F}_p),$$

preserving cusp forms. The analog in group cohomology of the Θ operator on mod p modular forms, can be found in [AS]. The aim of this note is to find a group theoretic substitute for the Hasse invariant.
Unlike as is done in [AS] in the case of Θ, for good reasons we cannot find an *element* in group cohomology that is an analog of the Hasse invariant. What we do find instead is a procedure for raising weights by $p-1$ of mod p Hecke eigenforms of weight two (preserving the level) that is one of the principal uses of the Hasse invariant.
Using the Eichler-Shimura isomorphism the relevant Hecke modules are $H^1(\Gamma_1(N), \mathbf{F}_p)$ and $H^1(\Gamma_1(N), \mathsf{Symm}^{p-1}(\mathbf{F}_p^2))$. From the viewpoint of group cohomology the above considerations give that a Hecke system of eigenvalues $(a_l)_{l \neq p}$ in the former also arises from the latter. This at first sight is puzzling as indeed the $p-1^{\mathsf{st}}$ symmetric power of the standard 2-dimensional representation of $SL_2(\mathbf{F}_p)$ is irreducible. In this short note we "resolve" this puzzle.
Indeed the solution to the puzzle is implicit in an earlier paper of one of us (cf. Remark 4 at the end of Section 3 of [K]) where the issue arose in trying to understand why the methods for studying Steinberg lifts of an irreducible modular Galois representation $\rho: G_{\mathbf{Q}} \to GL_2(\overline{\mathbf{F}_p})$ were qualitatively different from those for studying principal series and supercuspidal lifts. The buzzwords there were that the p-dimensional minimal K-type of a Steinberg representation of $GL_2(\mathbf{Q}_p)$ also arises in the restriction to $GL_2(\mathbf{Z}_p)$ of any unramified principal series representation of $GL_2(\mathbf{Q}_p)$.
The key to the solution of this puzzle (again) is a study of the degeneracy map $H^1(\Gamma_1(N), \mathbf{F}_p)^2 \to H^1(\Gamma_1(N) \cap \Gamma_0(p), \mathbf{F}_p)$.
In Section 3 we will give applications of our method in the situtation of imaginary quadratic fields, where the "geometric Hasse invariant" perforce is not available. Furthermore modular forms in this setting do not have a multiplicative structure. We owe this observation, and indeed the suggestion that our methods should work in this case, to Ian Kiming. Our cohomological methods do work in this situation under the hypothesis that $p \geq 5$ is inert, but have the (inherent) defect that results are about characteristic p modular forms,

and may not be used directly to produce congruences between characteristic 0 eigenforms of different weights. This comes from the fact that in this situation torsion in cohomology can possibly occur even after localisation at "interesting" maximal ideals of the Hecke algebra (see the concluding remark).
In Section 4 we deal with the case of $p \geq 5$ inert in a totally real field that is cyclic and of even degree over $\mathbf{Q}$, and in Section 5 we spell out some consequences of [K] for raising of levels in higher weights.
It would be interesting to see if the other cases cases can be treated by extending these methods.

2 The solution to the puzzle

Let us recall the hypotheses: $p \geq 5$ is prime, and $N \geq 1$ is prime to p. Consider the cohomology groups $H^1(\Gamma_1(N), \mathbf{F}_p)$ and $H^1(\Gamma_1(N) \cap \Gamma_0(p), \mathbf{F}_p)$. We have the standard action of Hecke operators T_r on these cohomology groups. We recall that we only consider the action for $(r,p) = 1$. We have the degeneracy map

$$\alpha : H^1(\Gamma_1(N), \mathbf{F}_p)^2 \to H^1(\Gamma_1(N) \cap \Gamma_0(p), \mathbf{F}_p)$$

that is defined to be the sum $\alpha_1 + \alpha_2$ where α_1 is the restiction map, and α_2 the "twisted" restriction map, given by conjugation by

$$g := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

followed by restriction. The map α is equivariant for the T_r's that we consider. We have the following variant of a lemma of Ihara and Ribet (see [R], and also [CDT, 6.3.1]).

Lemma 1 *The map* $\alpha\colon H^1(\Gamma_1(N), \mathbf{F}_p)^2 \to H^1(\Gamma_1(N) \cap \Gamma_0(p), \mathbf{F}_p)$ *is injective.*

Proof. Let Δ be the subgroup of $SL_2(\mathbf{Z}[1/p])$ of elements congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ modulo N. The arguments of [S2, II, §1.4] show that Δ is the amalgam of $\Gamma_1(N)$ and $g\Gamma_1(N)g^{-1}$ along their intersection $\Gamma_1(N) \cap \Gamma_0(p)$. The universal property of amalgams then implies that the kernel of α is $H^1(\Delta, \mathbf{F}_p)$ i.e., $Hom(\Delta, \mathbf{F}_p)$. By [S1], each subgroup of finite index of $SL_2(\mathbf{Z}[1/p])$ is a congruence subgroup, hence each morphism from Δ to $\mathbf{F}_p$ factors through the image Δ_n of Δ in some $SL_2(\mathbf{Z}/n\mathbf{Z})$ with n prime to p. The result follows, as p is at least 5 and does not divide N. (We use that $SL_2(\mathbf{Z})$ maps surjectively to $SL_2(\mathbf{Z}/n\mathbf{Z})$.)

By Shapiro's lemma we see that $H^1(\Gamma_1(N) \cap \Gamma_0(p), \mathbf{F}_p)$ is isomorphic (as a Hecke module) to $H^1(\Gamma_1(N), \mathbf{F}_p[\mathbf{P}^1(\mathbf{F}_p)])$. Using an easy computation of Brauer characters we deduce that the semisimplification of $\mathbf{F}_p[\mathbf{P}^1(\mathbf{F}_p)]$ under the natural action of $\Gamma_1(N)$ (that factors through $\Gamma_1(N)/\Gamma_1(N) \cap \Gamma(p)$) is $\mathsf{id} \oplus \mathsf{Symm}^{p-1}(\mathbf{F}_p^2)$. In fact as the cardinality of $\mathbf{P}^1(\mathbf{F}_p)$ is prime to p we deduce that this is indeed true even before semisimplification, i.e., $\mathbf{F}_p[\mathbf{P}^1(\mathbf{F}_p)]$ is semisimple as a $SL_2(\mathbf{F}_p)$-module. The submodule id is identified with the constant functions, with complement the functions with zero average. Thus we identify $H^1(\Gamma_1(N)\cap\Gamma_0(p),\mathbf{F}_p)$ with $H^1(\Gamma_1(N),\mathbf{F}_p)\oplus H^1(\Gamma_1(N),\mathsf{Symm}^{p-1}(\mathbf{F}_p^2))$. The degeneracy map α takes the form:

$$H^1(\Gamma_1(N),\mathbf{F}_p)^2 \to H^1(\Gamma_1(N),\mathbf{F}_p)\oplus H^1(\Gamma_1(N),\mathsf{Symm}^{p-1}(\mathbf{F}_p^2)).$$

Lemma 2 *The map:*

$$\beta\colon H^1(\Gamma_1(N),\mathbf{F}_p) \longrightarrow H^1(\Gamma_1(N),\mathsf{Symm}^{p-1}(\mathbf{F}_p^2)),$$

that is the composition of α_2 with the projection of $H^1(\Gamma_1(N)\cap\Gamma_0(p),\mathbf{F}_p)$ to $H^1(\Gamma_1(N),\mathsf{Symm}^{p-1}(\mathbf{F}_p^2))$, is injective.

Proof. This is an immediate consequence of Lemma 1, and the fact that $\alpha_1\colon H^1(\Gamma_1(N),\mathbf{F}_p)\to H^1(\Gamma_1(N)\cap\Gamma_0(p),\mathbf{F}_p)$ has image exactly the first summand of $H^1(\Gamma_1(N),\mathbf{F}_p)\oplus H^1(\Gamma_1(N),\mathsf{Symm}^{p-1}(\mathbf{F}_p^2))$.

In view of the Eichler-Shimura isomorphism (see [DI, §12]), we have a new proof by a purely group cohomological method of the following well-known result.

Corollary 1 *Suppose moreover that $N\geq 5$. A semi-simple representation $\rho\colon G_{\mathbf{Q}}\to GL_2(\overline{\mathbf{F}_p})$ that arises from $S_2(\Gamma_1(N),\overline{\mathbf{Q}_p})$ also arises from $S_{p+1}(\Gamma_1(N),\overline{\mathbf{Q}_p})$.*

Proof. This follows from the above lemma, together with the following facts.

1. The degeneracy map β is Hecke equivariant for the T_r's that we consider, and it sends $H^1_{\mathrm{par}}(\Gamma_1(N),\mathbf{F}_p)$ to $H^1_{\mathrm{par}}(\Gamma_1(N),\mathsf{Symm}^{p-1}(\mathbf{F}_p^2))$. The subscript "par" denotes parabolic cohomology, i.e., the intersection of the kernels of the restriction maps to the cohomology of the unipotent subgroups of $\Gamma_1(N)$.

2. For V any $\Gamma_1(N)$-module that is free of finite rank over $\mathbf{Z}$, and such that $H^0(\Gamma_1(N),\mathbf{F}_p\otimes V^\vee)=0$, the map:
$$H^1_{\mathrm{par}}(\Gamma_1(N),V)\to H^1_{\mathrm{par}}(\Gamma_1(N),\mathbf{F}_p\otimes V)$$
is surjective (one uses that $H^1_{\mathrm{par}}(\Gamma_1(N),\mathbf{F}_p\otimes V)$ is a quotient of $H^1_c(Y_1(N),\mathbf{F}_p\otimes\mathcal{F}_V)$, with $\mathcal{F}_V$ the sheaf given corresponding to V, and that $H^2_c(Y_1(N),\mathbf{F}_p\otimes\mathcal{F}_V)=0$ by Poincaré duality).

Remarks.

1. One can ask the converse question as to which maximal ideals m of the Hecke algebra acting on $H^1(\Gamma_1(N),\mathsf{Symm}^{p-1}(\mathbf{F}_p^2))$ are pull backs of maximal ideals of the Hecke algebra acting on $H^1(\Gamma_1(N),\mathbf{F}_p)$. Then, for non-Eisenstein m, the answer is in terms of the Galois representation ρ_{m}: a necessary and sufficient condition is that ρ_{m} be finite flat at p (see [R1, Thm. 3.1]). Perhaps one does not expect to have a group cohomological approach to such a subtle phenomenon.

2. Let ρ be an irreducible 2-dimensional mod p representation of $G_{\mathbf{Q}}$. We just saw that if ρ arises from $S_2(\Gamma_1(N))$, then it also does so from $S_{p+1}(\Gamma_1(N))$, using only group cohomology. See [He] for the case of higher weights, when the raising of weights by $p-1$ is explained more directly in terms of the Jordan-Hölder series of the $\mathsf{Symm}^n(\mathbf{F}_p^2)$. On the other hand the method here being cohomological cannot be used to raise weights from 1 to p, while multiplication by the Hasse invariant can be used for this (see [DS]).

3 Imaginary quadratic fields

Let $p \geq 5$ be a prime number, K an imaginary quadratic field in which p is inert, and $\mathcal{N}$ a non-zero ideal in the ring of integers $\mathcal{O}_K$ not containing p.
Let $\Gamma_1(\mathcal{N})$ be the congruence subgroup of $SL_2(\mathcal{O}_K)$ of level $\mathcal{N}$. As before we have the degeneracy map:

$$\alpha\colon H^1(\Gamma_1(\mathcal{N}), \mathbf{F}_p)^2 \to H^1(\Gamma_1(\mathcal{N}) \cap \Gamma_0(p), \mathbf{F}_p)$$

that is defined to be the sum $\alpha_1 \oplus \alpha_2$ where α_1 is the restriction map, and α_2 the "twisted" restriction map, given by "conjugation" by

$$g := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

followed by restriction. Then we again have:

Lemma 3 *The map: $\alpha\colon H^1(\Gamma_1(\mathcal{N}), \mathbf{F}_p)^2 \to H^1(\Gamma_1(\mathcal{N}) \cap \Gamma_0(p), \mathbf{F}_p)$ is injective.*

Proof. One replaces $\mathbf{Z}$ by $\mathcal{O}_K$ and N by $\mathcal{N}$ in the proof of Lemma 1. Strong approximation (see [PR]) guarantees that the reduction map from $SL_2(\mathcal{O}_K)$ to $SL_2(\mathcal{O}_K/n\mathcal{O}_K)$ is surjective for all $n \geq 1$.

By Shapiro's lemma we see that $H^1(\Gamma_1(\mathcal{N}) \cap \Gamma_0(p), \mathbf{F}_p)$ is isomorphic to $H^1(\Gamma_1(\mathcal{N}), \mathbf{F}_p[\mathbf{P}^1(\mathbf{F}_\wp)])$, where $\mathbf{F}_\wp = \mathcal{O}_K/\wp$. Using an easy computation of Brauer characters we deduce that the semisimplification of $\mathbf{F}_\wp[\mathbf{P}^1(\mathbf{F}_\wp)]$ under the natural action of $\Gamma_1(\mathcal{N})$ (that factors through $\Gamma_1(\mathcal{N})/\Gamma_1(\mathcal{N}) \cap \Gamma(\wp)$) is $\mathsf{id} \oplus \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2) \otimes \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2)^\sigma$, with σ the non-trivial automorphism of $\mathbf{F}_\wp$, and the superscript denotes that the action has been twisted by σ. Note that $\mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2) \otimes \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2)^\sigma$ is irreducible as a $SL_2(\mathbf{F}_\wp)$-module: this is a particular case of the well-known tensor product theorem of Steinberg (see [St]). In fact as the cardinality of $\mathbf{P}^1(\mathbf{F}_\wp)$ is prime to p we deduce as before that this is indeed true even before semisimplification, i.e., $\mathbf{F}_p[\mathbf{P}^1(\mathbf{F}_\wp)]$ is semisimple as a $SL_2(\mathbf{F}_\wp)$-module.
Thus α maps $H^1(\Gamma_1(\mathcal{N}), \mathbf{F}_\wp)^2$ into the direct sum of $H^1(\Gamma_1(\mathcal{N}), \mathbf{F}_\wp)$ and $H^1(\Gamma_1(\mathcal{N}), \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2) \otimes \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2)^\sigma)$, and composing with the projection to the second term gives a map:

$$\beta\colon H^1(\Gamma_1(\mathcal{N}), \mathbf{F}_\wp) \to H^1(\Gamma_1(\mathcal{N}), \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2) \otimes \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2)^\sigma)$$

LEMMA 4 *The map β is injective.*

PROOF. After the above discussion, this is an immediate consequence of Lemma 3 as before.

The map β is equivariant for the the action of all Hecke operators outside p (i.e., induced by elements of $SL_2(\mathbf{Q}_l \otimes K)$ for $l \neq p$). Thus we have proved:

COROLLARY 2 *Each system of Hecke eigenvalues in $\overline{\mathbf{F}_p}$ that arises from $H^1(\Gamma_1(\mathcal{N}), \mathbf{F}_p)$ also arises from $H^1(\Gamma_1(\mathcal{N}), \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2) \otimes \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2)^\sigma)$.*

REMARK. This result as it stands does not yield any information about congruences of systems of Hecke eigenvalues occurring in characteristic zero, as there is a problem with lifting. More precisely, the obstruction is in the p-torsion of $H^2(\Gamma_1(\mathcal{N}), \mathsf{Symm}^{p-1}(\mathcal{O}^2) \otimes \mathsf{Symm}^{p-1}(\mathcal{O}^2)^\sigma)$.

4 TOTALLY REAL FIELDS

The method is also applicable at inert primes $p \geq 5$ in the case of Hilbert modular forms over cyclic totally real fields of even degree. We quickly sketch the approach which is similar to that of the previous two sections. Let $F/\mathbf{Q}$ be a totally real, cyclic extension of even degree, $\mathrm{Gal}(F/\mathbf{Q}) = \langle \sigma \rangle$, $p \geq 5$ an inert prime, with $\wp$ the unique prime of F above it, $\mathbf{F}_\wp$ the residue field at $\wp$, and $\mathcal{N}$ an ideal of the ring of integers of F that is prime to p.
Consider the quaternion algebra D over F ramified at all infinite places and unramified at all finite places, and for any F-algebra R, set $B(R) = (D \otimes_F R)^*$. Let $\mathbf{A}$ be the adeles of F, and $U_1(\mathcal{N})$ the standard open compact (mod centre) subgroup of $B(\mathbf{A})$. The space of mod p weight 2 modular forms $S(\mathcal{N})$ (resp., $S(\mathcal{N}, \wp)$) for $U_1(\mathcal{N})$ (resp., $U_1(\mathcal{N}) \cap U_0(\wp)$) in this case consists of functions $B(\mathbf{A}) \to \mathbf{F}_p$ that are left and right invariant under $B(F)$ and $U_1(\mathcal{N})$ (resp., $U_1(\mathcal{N}) \cap U_0(\wp)$) respectively, modulo the space of functions that factor through the norm. These spaces come equipped with Hecke actions. This time controlling the kernel of the degeneracy map $S(\mathcal{N})^2 \to S(\mathcal{N}, \wp)$, i.e., analog of Lemmas 1 and 3, is easier and follows from strong approximation. Note again that the representation $\mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2) \otimes \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2)^\sigma \otimes \cdots \otimes \mathsf{Symm}^{p-1}(\mathbf{F}_\wp^2)^{\sigma^{[F:\mathbf{Q}]-1}}$ of $GL_2(\mathbf{F}_\wp)$ (which again is a direct summand, with complement the trivial representation, of the induction of the trivial representation from the Borel subgroup of $GL_2(\mathbf{F}_\wp)$ to $GL_2(\mathbf{F}_\wp)$) is irreducible as a consequence of Steinberg's tensor product theorem.
Now following the method of the previous section, and invoking the Jacquet-Langlands correspondence yields the following result.

PROPOSITION 1 *With notation as above, suppose that an irreducible representation $\rho: G_F \to GL_2(\overline{\mathbf{F}_p})$ arises from a Hilbert modular form on $\Gamma_1(\mathcal{N})$ of weight $(2, \ldots, 2)$. Then it also arises from a a Hilbert modular form on $\Gamma_1(\mathcal{N})$ of weight $(p+1, \ldots, p+1)$.*

Remarks: It will be of interest to work out some of the Hasse invariants with non-parallel weights obtained by E. Goren from the viewpoint of this paper (see [G]).

5 Congruences between forms of level N and Np for weights ≥ 2

We take this opportunity to write down a level raising criterion from level N to level Np for all weights that is easily deduced from Corollary 9 of [K], and list some errata to [K].

Proposition 2 *Let f be a newform in $S_k(\Gamma_1(N))$ for an integer $k \geq 2$, such that the mod $\wp$ representation corresponding to it is irreducible. Then:*

- *If $k = 2$, f is congruent to a p-new form in $S_k(\Gamma_1(N) \cap \Gamma_0(p))$ if and only if $a_p(f)^2 = \varepsilon_f(p)$ mod $\wp$ where ε_f is the nebentypus of f.*
- *If $2 < k \leq p+1$, f is congruent to a p-new form in $S_k(\Gamma_1(N) \cap \Gamma_0(p))$ if and only if $a_p(f)$ is 0 mod $\wp$.*
- *If $k > p+1$, f is always congruent to a p-new form in $S_k(\Gamma_1(N) \cap \Gamma_0(p))$.*

Errata to [K]

One of us (C.K.) would like to point out some typos in [K]:

1. Lines 12 and 19 of Definition 10 page 143 of [K] replace
$$f\colon D(\mathbf{Q})\backslash D(\mathbf{A}^\infty)/V \longrightarrow Hom_{\mathcal{O}}(M, Symm^{k-2}(\mathcal{O})).$$
by
$$f\colon D(\mathbf{Q})\backslash D(\mathbf{A}^\infty) \longrightarrow Hom_{\mathcal{O}}(M, Symm^{k-2}(\mathcal{O})).$$
2. On line 12, page 146 of [K] replace $V_1(N)^p$ by $V_1(N)^p \times V_p$.

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References

[AS] A. Ash, G. Stevens, *Modular forms in characteristic l and special values of their L-functions*, Duke Math. J. 53 (1986), no. 3, 849–86

[CDT] B. Conrad, F. Diamond, R. Taylor. *Modularity of certain potentially Barsotti-Tate Galois representations.* J.A.M.S. 12 (1999), 521–567.

[DI] F. Diamond, J. Im *Modular forms and modular curves.* In "Seminar on Fermat's last theorem", Canadian Mathematical Society Conference Proceedings 17, 1995 (V. Kumar Murty, editor).

[DS] P. Deligne and J-P. Serre. *Formes modulaires de poids 1.* Ann. Sci. Ecole Norm. Sup. (4) 7, 507–530 (1974).

[G] E.Z. Goren. *Hasse invariants for Hilbert modular varieties.* Israel J. Math. 122 (2001), 157–174.

[He] A. Herremans. *A combinatorial interpretation of Serre's conjecture on modular Galois representations*, Preprint 2002-03, Orsay.

[K] C. Khare, *A local analysis of congruences in the (p,p) case: Part II*, Invent. Math. 143 (2001), 129–155.

[Kz] N. Katz, *A result on modular forms in characteristic p*, Modular functions of one variable V, pp. 53–61, Lecture Notes in Math., Vol. 601, Springer, Berlin, 1977.

[PR] V. Platonov and A. Rapinchuk. *Algebraic groups and number theory.* Academic Press, 1994.

[R] K.A. Ribet, *Congruence relations between modular forms*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 503–514, PWN, Warsaw, 1984.

[R1] K.A. Ribet, *Report on mod l representations of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Proceedings of Symposia in Pure Mathematics, 55 (1994), Part 2.

[S] J-P. Serre, *Formes modulaires et fonctions zeta p-adiques*, Modular functions of one variable III, pp. 191–268, Lecture Notes in Math., Vol. 350, Springer, 1973.

[S1] J-P. Serre, *Le probleme des groupes de congruence pour SL_2*, Annals of Math. 92 (1970), 489–527.

[S2] J-P. Serre, *Trees.* Springer-Verlag, 1980.

[St] R. Steinberg, *Tensor product theorems.* Proceedings of Symposia in Pure Mathematics Volume 47 (1987), 331–338.

[SwD] H.P.F. Swinnerton-Dyer, *On ℓ-adic representations and congruences for coefficients of modular forms*, Modular functions of one variable III, pp. 1–55, Lecture Notes in Math., Vol. 350, Springer, 1973.

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Variations on the Bloch-Ogus Theorem

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Abstract. Let R be a semi-local regular ring of geometric type over a field k. Let $\mathcal{U} = \operatorname{Spec} R$ be the semi-local scheme. Consider a smooth proper morphism $p : Y \to \mathcal{U}$. Let $Y_{k(u)}$ be the fiber over the generic point of a subvariety u of $\mathcal{U}$. We prove that the Gersten-type complex for étale cohomology

$$0 \to H^q_{\text{ét}}(Y, C) \to H^q_{\text{ét}}(Y_{k(\mathcal{U})}, C) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q-1}_{\text{ét}}(Y_{k(u)}, C(-1)) \to \dots$$

is exact, where C is a locally constant sheaf with finite stalks of $\mathbb{Z}/n\mathbb{Z}$-modules on Y_{et} and n is an integer prime to $\operatorname{char}(k)$.

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1 Introduction

The history of the subject of the present paper starts with the famous paper of D. Quillen [14] where he proves the geometric case of the Gersten's conjecture for K-functor. One may ask whether the similar result holds for étale cohomology.
The first answer on this question was given by S. Bloch and A. Ogus in [2]. They proved the analog of Gersten's conjecture for étale cohomology with coefficients in the twisted sheaf $\mu_n^{\otimes i}$ of n-th roots of unity. More precisely, let X be a smooth quasi-projective variety over a field k and let $x = \{x_1, \dots, x_m\} \subset X$ be a finite subset of points. We denote by $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ the semi-local scheme at x. Consider the sheaf μ_n of n-th roots of unity on the small étale site X_{et}, with n prime to $\operatorname{char}(k)$. Then the main result of [2] (see Theorem 4.2 and

Example 2.1) implies that the Gersten-type complex for étale cohomology with supports

$$0 \to H^q(\mathcal{U}, \mu_n^{\otimes i}) \to \coprod_{u \in \mathcal{U}^{(0)}} H^q_u(\mathcal{U}, \mu_n^{\otimes i}) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q+1}_u(\mathcal{U}, \mu_n^{\otimes i}) \to \cdots \qquad (\dagger)$$

is exact for all $i \in \mathbb{Z}$ and $q \geq 0$, where $\mathcal{U}^{(p)}$ denotes the set of all points of codimension p in $\mathcal{U}$.

The next step was done by O. Gabber in [8]. He proved that the complex (†) is exact for cohomology with coefficients in any torsion sheaf C on X_{et} that comes from the base field k, i.e., $C = p^* C'$ for some sheaf C' on $(\operatorname{Spec} k)_{et}$ and the structural morphism $p : X \to \operatorname{Spec} k$.

It turned out that the proof of Gabber can be applied to any cohomology theory with supports that satisfies the same formalism as étale cohomology do. This idea was realized in the paper [4] by J.-L. Colliot-Thélène, R. Hoobler and B. Kahn. Namely, they proved that a cohomology theory with support h^* which satisfies some set of axioms [4, Section 5.1] is effaceable [4, Definition 2.1.1]. Then the exactness of (†) follows immediately by trivial reasons [4, Proposition 2.1.2]. In particular, one gets the exactness of (†) for the case when $\mathcal{U}$ is replaced by the product $\mathcal{U} \times_k T$, where T is a smooth variety over k [4, Theorem 8.1.1]. It was also proven [4, Remark 8.1.2.(3), Corollary B.3.3] the complex (†) is exact for the case when $\dim \mathcal{U} = 1$ and the sheaf of coefficients $\mu_n^{\otimes i}$ is replaced by a bounded below complex of sheaves, whose cohomology sheaves are locally constant constructible, torsion prime to $\operatorname{char}(k)$. The goal of the present paper is to prove the latter case for any dimension of the scheme $\mathcal{U}$. Namely, we want to prove the following

1.1 Theorem. *Let X be a smooth quasi-projective variety over a field k. Let $x = \{x_1, \ldots, x_m\} \subset X$ be a finite subset of points and $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ be the semi-local scheme at x. Let $\mathcal{C}$ be a bounded below complex of locally constant constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et} with n prime to* $\operatorname{char}(k)$*, Then the E^1-terms of the coniveau spectral sequence yield an exact complex*

$$0 \to H^q(\mathcal{U}, \mathcal{C}) \to \coprod_{u \in \mathcal{U}^{(0)}} H^q_u(\mathcal{U}, \mathcal{C}) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q+1}_u(\mathcal{U}, \mathcal{C}) \to \cdots$$

of étale hypercohomology with supports.

1.2 Remark. For the definition of a constructible sheaf we refer to [1, IX] or [9, V.1.8]. Observe that a locally constant sheaf with finite stalks provides an example of a locally constant constructible sheaf (see [1, IX.2.13]).

1.3 Corollary. *Let R be a semi-local regular ring of geometric type over a field k. We denote by $\mathcal{U} = \operatorname{Spec} R$ the respective semi-local affine scheme. Let $\mathcal{C}$ be a bounded below complex of locally constant constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on U_{et} with n prime to* $\operatorname{char}(k)$*, Then the complex*

$$0 \to H^q_{ét}(\mathcal{U}, \mathcal{C}) \to H^q_{ét}(\operatorname{Spec} k(\mathcal{U}), \mathcal{C}) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q-1}_{ét}(\operatorname{Spec} k(u), \mathcal{C}(-1)) \to \cdots$$

is exact, where $k(\mathcal{U})$ is the function field of $\mathcal{U}$, $k(u)$ is the residue field of u and $\mathcal{C}(i) = \mathcal{C} \otimes \mu_n^{\otimes i}$.

Proof. Follows by purity for étale cohomology (see the proof of [4, 1.4]). □

1.4 Corollary. *Let R be a semi-local regular ring of geometric type over a field k. Let $p : Y \to \mathcal{U}$ be a smooth proper morphism, where $\mathcal{U} = \operatorname{Spec} R$. Let C be a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on Y_{et} with n prime to* $\operatorname{char}(k)$. *Then the complex*

$$0 \to H^q_{ét}(Y, C) \to H^q_{ét}(Y_{k(\mathcal{U})}, C) \to \coprod_{u \in \mathcal{U}^{(1)}} H^{q-1}_{ét}(Y_{k(u)}, C(-1)) \to \cdots$$

is exact, where $Y_{k(u)} = \operatorname{Spec} k(u) \times_{\mathcal{U}} Y$.

Proof. The cohomology of Y with coefficients in C coincide with the hypercohomology of $\mathcal{U}$ with coefficients in the total direct image Rp_*C [9, VI.4.2]. Observe that the bounded complex Rp_*C has locally constant constructible cohomology sheaves. Now by the main result of paper [12], a bounded complex of sheaves on $\mathcal{U}_{et}$ with locally constant constructible cohomology sheaves is in the derived category isomorphic to a bounded complex of locally constant constructible sheaves. Hence, there exists a bounded complex $\mathcal{C}$ of locally constant constructible sheaves that is quasi-isomorphic to the complex Rp_*C. Replace Rp_*C by $\mathcal{C}$ and apply the previous corollary. □

1.5 Remark. The assumptions on the sheaf C are essential. As it was shown in [7] for the case $k = \mathbb{C}$ and $C = \mathbb{Z}$ there are examples of extensions $Y/\mathcal{U}$ for which the map $H^q_{\mathrm{DR}}(Y) \to H^q_{\mathrm{DR}}(Y_{k(\mathcal{U})})$ is not injective.

1.6 Remark. The injectivity part of Theorem 1.1 (i.e., the exactness at the first term) has been proven recently in [15] by extending the arguments of Voevodsky [16].

The structure of the proof of Theorem 1.1 is the following. First, we give some general formalism (sections 2, 3 and 4). Namely, we prove that any functor F that satisfies some set of axioms (homotopy invariance, transfers, finite monodromy) is effaceable (Theorem 4.7). Then we apply this formalism to étale cohomology (section 5). More precisely, we check that the étale cohomology functor $F(X, Z) = H^*_Z(X, \mathcal{C})$ satisfies all the axioms and, hence, is effaceable. It implies Theorem 1.1 immediately.
We would like to stress that our axioms for the functor F are different from those in [4]. The key point of the proof is that we use Geometric Presentation Lemma of Ojanguren and Panin (see Lemma 3.5) instead of Gabber's. This fact together with the notion of a functor with finite monodromy allows us to apply the techniques developed in [10], [11] and [17].

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2 Definitions and Notations

2.1 Notation. In the present paper all schemes are assumed to be Noetherian and separated. By k we denote a fixed ground field. A variety over k is an integral scheme of finite type over k. To simplify the notation sometimes we will write k instead of the scheme $\operatorname{Spec} k$. We will write $X_1 \times X_2$ for the fibered product $X_1 \times_k X_2$ of two k-schemes. By $\mathcal{U}$ we denote a regular semi-local scheme of geometric type over k, i.e., $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ for a smooth affine variety X over k and a finite set of points $x = \{x_1, \ldots, x_n\}$ of X. By $\mathcal{X}$ we denote a relative curve over $\mathcal{U}$ (see 3.1.(i)). By $\mathcal{Z}$ and $\mathcal{Y}$ we denote closed subsets of $\mathcal{X}$. By Z and Z' we denote closed subsets of $\mathcal{U}$. Observe that $\mathcal{X}$ and $\mathcal{U}$ are essentially smooth over k and all schemes $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{Y}$, Z, Z' are of finite type over $\mathcal{U}$.

2.2 Notation. Let U be a k-scheme. Denote by $Cp(U)$ a category whose objects are couples (X, Z) consisting of an U-scheme X of finite type over U and a closed subset Z of the scheme X (we assume the empty set is a closed subset of X). Morphisms from (X, Z) to (X', Z') are those morphisms $f : X \to X'$ of U-schemes that satisfy the property $f^{-1}(Z') \subset Z$. The composite of f and g is $g \circ f$.

2.3 Notation. Denote by $F : Cp(U) \to Ab$ a contravariant additive functor from the category of couples $Cp(U)$ to the category of (graded) abelian groups. Recall that F is additive if one has an isomorphism

$$F(X_1 \amalg X_2, Z_1 \amalg Z_2) \cong F(X_1, Z_1) \oplus F(X_2, Z_2).$$

Sometimes we shall write $F_Z(X)$ for $F(X, Z)$ having in mind the notation used for cohomology with supports.

Now notions of a homotopy invariant functor, a functor with transfers and a functor that satisfies vanishing property will be given.

2.4 Definition. A contravariant functor $F : Cp(U) \to Ab$ is said to be homotopy invariant if for each U-scheme X smooth or essentially smooth over k and for each closed subset Z of X the map $F_Z(X) \to F_{Z \times \mathbb{A}^1}(X \times \mathbb{A}^1)$ induced by the projection $X \times \mathbb{A}^1 \to X$ is an isomorphism.

2.5 Definition. One says a contravariant functor $F : Cp(U) \to Ab$ satisfies vanishing property if for each U-scheme X one has $F(X, \emptyset) = 0$.

2.6 Definition. A contravariant functor $F : Cp(U) \to Ab$ is said to be endowed with transfers if for each finite flat morphism $\pi : X' \to X$ of U-schemes and for each closed subset $Z \subset X$ it is given a homomorphism of abelian groups $\operatorname{Tr}_X^{X'} : F_{\pi^{-1}(Z)}(X') \to F_Z(X)$ and the family $\{\operatorname{Tr}_X^{X'}\}$ satisfies the following properties:

(i) for each fibered product diagram of U-schemes with a finite flat morphism π

$$\begin{tikzcd} X' \arrow[d, "\pi"'] & X'_1 \arrow[l, "f'"'] \arrow[d, "\pi_1"] \\ X & X_1 \arrow[l, "f"'] \end{tikzcd}$$

and for each closed subset $Z \subset X$ the diagram

$$\begin{tikzcd} F_{Z'}(X') \arrow[r, "F(f_1)"] \arrow[d, "\mathrm{Tr}_X^{X'}"'] & F_{Z'_1}(X'_1) \arrow[d, "\mathrm{Tr}_{X_1}^{X'_1}"] \\ F_Z(X) \arrow[r, "F(f)"] & F_{Z_1}(X_1) \end{tikzcd}$$

is commutative, where $Z' = \pi^{-1}(Z)$, $Z_1 = f^{-1}(Z)$ and $Z'_1 = \pi_1^{-1}(Z_1)$;

(ii) if $\pi : X'_1 \amalg X'_2 \to X$ is a finite flat morphism of U-schemes, then for each closed subset $Z \subset X$ the diagram

$$\begin{tikzcd} F_{Z'}(X'_1 \amalg X'_2) \arrow[r, "\text{“+”}"] \arrow[dr, "\mathrm{Tr}_X^{X'_1 \amalg X'_2}"'] & F_{Z'_1}(X'_1) \oplus F_{Z'_2}(X'_2) \arrow[d, "\mathrm{Tr}_{X_1}^{X'_1} + \mathrm{Tr}_{X_2}^{X'_2}"] \\ & F_Z(X) \end{tikzcd}$$

is commutative, where $Z' = \pi^{-1}(Z)$, $Z'_1 = X'_1 \cap Z'$ and $Z'_2 = X'_2 \cap Z'$;

(iii) if $\pi : (X', Z') \to (X, Z)$ is an isomorphism in $Cp(U)$, then two maps $\mathrm{Tr}_X^{X'}$ and $F(\pi)$ are inverses of each other, i.e.

$$F(\pi) \circ \mathrm{Tr}_X^{X'} = \mathrm{Tr}_X^{X'} \circ F(\pi) = \mathrm{id}.$$

3 THE SPECIALIZATION LEMMA

The following definition is inspired by the notion of a good triple used by Voevodsky in [16].

3.1 DEFINITION. Let $\mathcal{U}$ be a regular semi-local scheme of geometric type over the field k. A triple $(\mathcal{X}, \delta, \mathfrak{f})$ consisting of an $\mathcal{U}$-scheme $p : \mathcal{X} \to \mathcal{U}$, a section $\delta : \mathcal{U} \to \mathcal{X}$ of the morphism p and a regular function $\mathfrak{f} \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is called a perfect triple over $\mathcal{U}$ if $\mathcal{X}$, δ and $\mathfrak{f}$ satisfy the following conditions:

(i) the morphism p can be factorized as $p : \mathcal{X} \xrightarrow{\pi} \mathbb{A}^1 \times \mathcal{U} \xrightarrow{pr} \mathcal{U}$, where π is a finite surjective morphism and pr is the canonical projection on the second factor;

(ii) the vanishing locus of the function $\mathfrak{f}$ is finite over $\mathcal{U}$;

(iii) the scheme $\mathcal{X}$ is essentially smooth over k and the morphism p is smooth along $\delta(\mathcal{U})$;

(iv) the scheme $\mathcal{X}$ is irreducible.

3.2 Remark. The property (i) says that $\mathcal{X}$ is an affine curve over $\mathcal{U}$. The property (iii) implies that $\mathcal{X}$ is a regular scheme. Since $\mathcal{X}$ and $\mathbb{A}^1 \times \mathcal{U}$ are regular schemes by [6, 18.17] the morphism $\pi : \mathcal{X} \to \mathbb{A}^1 \times \mathcal{U}$ from (i) is a finite flat morphism.

The following lemma will be used in the proof of Theorems 4.2 and 4.7.

3.3 Lemma. *Let $\mathcal{U}$ be a regular semi-local scheme of geometric type over an infinite field k. Let $(p : \mathcal{X} \to \mathcal{U}, \delta : \mathcal{U} \to \mathcal{X}, \mathfrak{f} \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$ be a perfect triple over $\mathcal{U}$. Let $F : Cp(\mathcal{U}) \to Ab$ be a homotopy invariant functor endowed with transfers which satisfies vanishing property (see 2.4, 2.6 and 2.5). Then for each closed subset $\mathcal{Z}$ of the vanishing locus of $\mathfrak{f}$ the following composite vanishes*

$$F_{\mathcal{Z}}(\mathcal{X}) \xrightarrow{F(\delta)} F_{\delta^{-1}(\mathcal{Z})}(\mathcal{U}) \xrightarrow{F(\mathrm{id}_{\mathcal{U}})} F_{p(\mathcal{Z})}(\mathcal{U})$$

3.4 Remark. The mentioned composite is the map induced by the morphism $\delta : (\mathcal{U}, p(\mathcal{Z})) \to (\mathcal{X}, \mathcal{Z})$ in the category $Cp(\mathcal{U})$. Observe that we have $\delta^{-1}(\mathcal{Z}) \subset p(\mathcal{Z})$, where $p(\mathcal{Z})$ is closed by (ii) of 3.1.

Proof. Consider the commutative diagram in the category $Cp(\mathcal{U})$

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{Y}) & \xrightarrow{\mathrm{id}_{\mathcal{X}}} & (\mathcal{X}, \mathcal{Z}) \\ {\scriptstyle\delta}\uparrow & & \uparrow{\scriptstyle\delta} \\ (\mathcal{U}, Z') & \xrightarrow{\mathrm{id}_{\mathcal{U}}} & (\mathcal{U}, Z) \end{array}$$

where $Z = \delta^{-1}(\mathcal{Z})$, $Z' = p(\mathcal{Z})$ and $\mathcal{Y} = p^{-1}(Z')$. It gives the relation $F(\mathrm{id}_{\mathcal{U}}) \circ F(\delta) = F(\delta) \circ F(\mathrm{id}_{\mathcal{X}})$. Thus to prove the theorem it suffices to check that the following composite vanishes

$$F_{\mathcal{Z}}(\mathcal{X}) \xrightarrow{F(\mathrm{id}_{\mathcal{X}})} F_{\mathcal{Y}}(\mathcal{X}) \xrightarrow{F(\delta)} F_{Z'}(\mathcal{U}) \qquad (\dagger)$$

By Lemma 3.5 below applied to the perfect triple $(\mathcal{X}, \delta, \mathfrak{f})$ we can choose the finite surjective morphism $\pi : \mathcal{X} \to \mathbb{A}^1 \times \mathcal{U}$ from (i) of 3.1 in such a way that it's fibers at the points 0 and 1 of $\mathbb{A}^1$ look as follows:

(a) $\pi^{-1}(\{0\} \times \mathcal{U}) = \delta(\mathcal{U}) \amalg \mathcal{D}_0$ (scheme-theoretically) and $\mathcal{D}_0 \subset \mathcal{X}_{\mathfrak{f}}$;

(b) $\pi^{-1}(\{1\} \times \mathcal{U}) = \mathcal{D}_1$ and $\mathcal{D}_1 \subset \mathcal{X}_{\mathfrak{f}}$.

Observe that $\mathcal{Y} = \pi^{-1}(\mathbb{A}^1 \times Z')$. Let $\mathcal{Z}'_0 = \pi^{-1}(\{0\} \times Z') \cap \mathcal{D}_0$ and $\mathcal{Z}'_1 = \pi^{-1}(\{1\} \times Z')$ be the closed subsets of $\mathcal{Y}$. By definition $\mathcal{Z}'_0$, $\mathcal{Z}'_1$ are the closed subsets of $\mathcal{D}_0$ and $\mathcal{D}_1$ respectively. Since $\mathcal{Z}$ is contained in the vanishing locus of $\mathfrak{f}$ and $\mathcal{D}_0$, $\mathcal{D}_1 \subset \mathcal{X}_{\mathfrak{f}}$ we have $\mathcal{Z} \cap \mathcal{D}_0 = \mathcal{Z} \cap \mathcal{D}_1 = \emptyset$. The latter means that there are two commutative diagrams in the category $Cp(\mathcal{U})$

$$\begin{array}{ccc} (\mathcal{X},\mathcal{Y}) & \xrightarrow{\mathrm{id}_{\mathcal{X}}} & (\mathcal{X},\mathcal{Z}) \\ {\scriptstyle I_0}\uparrow & & \uparrow{\scriptstyle I_0} \\ (\mathcal{D}_0,\mathcal{Z}'_0) & \xrightarrow{\mathrm{id}_{\mathcal{D}_0}} & (\mathcal{D}_0,\emptyset) \end{array} \quad \text{and} \quad \begin{array}{ccc} (\mathcal{X},\mathcal{Y}) & \xrightarrow{\mathrm{id}_{\mathcal{X}}} & (\mathcal{X},\mathcal{Z}) \\ {\scriptstyle I_1}\uparrow & & \uparrow{\scriptstyle I_1} \\ (\mathcal{D}_1,\mathcal{Z}'_1) & \xrightarrow{\mathrm{id}_{\mathcal{D}_1}} & (\mathcal{D}_1,\emptyset) \end{array}$$

where I_0, I_1 are the closed embeddings $\mathcal{D}_0 \hookrightarrow \mathcal{X}$ and $\mathcal{D}_1 \hookrightarrow \mathcal{X}$ respectively. By vanishing property 2.5 we have $F(\mathcal{D}_0,\emptyset) = F(\mathcal{D}_1,\emptyset) = 0$. Then applying F to the diagrams we immediately get

$$F(I_0) \circ F(\mathrm{id}_{\mathcal{X}}) = 0 \text{ and } F(I_1) \circ F(\mathrm{id}_{\mathcal{X}}) = 0. \tag{1}$$

Let i_0, $i_1 : \mathcal{U} \hookrightarrow \mathbb{A}^1 \times \mathcal{U}$ be the closed embeddings which correspond to the points 0 and 1 of $\mathbb{A}^1$ respectively. The homotopy invariance property 2.4 implies that

$$F(i_0) = F(i_1) : F_{\mathbb{A}^1 \times Z'}(\mathbb{A}^1 \times \mathcal{U}) \to F_{Z'}(\mathcal{U}). \tag{2}$$

The base change property 2.6.(i) applied to the fibered product diagram

$$\begin{array}{ccc} (\mathcal{X},\mathcal{Y}) & \xleftarrow{I_1} & (\mathcal{D}_1,\mathcal{Z}'_1) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ (\mathbb{A}^1 \times \mathcal{U}, \mathbb{A}^1 \times Z') & \xleftarrow{i_1} & (\mathcal{U}, Z') \end{array}$$

gives the relation

$$F(i_1) \circ \mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}} = \mathrm{Tr}^{\mathcal{D}_1}_{\mathcal{U}} \circ F(I_1), \tag{3}$$

where $\mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}} : F_{\mathcal{Y}}(\mathcal{X}) \to F_{\mathbb{A}^1 \times Z'}(\mathbb{A}^1 \times \mathcal{U})$ and $\mathrm{Tr}^{\mathcal{D}_1}_{\mathcal{U}} : F_{\mathcal{Z}'_1}(\mathcal{D}_1) \to F_{Z'}(\mathcal{U})$ are the transfer maps for the finite flat morphism π and $\pi|_{\mathcal{D}_1}$ respectively.
Consider the commutative diagram

$$\begin{tikzcd} F_{\mathcal{Z}}(\mathcal{X}) \arrow[r, "F(\mathrm{id}_{\mathcal{X}})"] & F_{\mathcal{Y}}(\mathcal{X}) \arrow[r, "(F(\delta){,}F(I_0))"] \arrow[d, "\mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}}"'] & F_{Z' \amalg \mathcal{Z}'_0}(\mathcal{U} \amalg \mathcal{D}_0) \arrow[r, "\text{“+”}"] \arrow[d, "\mathrm{Tr}"'] & F_{Z'}(\mathcal{U}) \oplus F_{\mathcal{Z}'_0}(\mathcal{D}_0) \arrow[dl, "\mathrm{id}+\mathrm{Tr}^{\mathcal{D}_0}_{\mathcal{U}}"] \\ & F_{\mathbb{A}^1 \times Z'}(\mathbb{A}^1 \times \mathcal{U}) \arrow[r, "F(i_0)"] & F_{Z'}(\mathcal{U}) & \end{tikzcd} \tag{4}$$

where the central square commutes by 2.6.(i) and the right triangle commutes by 2.6.(ii). In the diagram we identify $\mathcal{U}$ with $\delta(\mathcal{U})$ by means of the isomorphism $\delta : \mathcal{U} \to \delta(U)$ and use the property 2.6.(iii) to identify $\mathrm{Tr}^{\delta(\mathcal{U})}_{\mathcal{U}}$ with $F(\delta)$.
The following chain of relations shows that the composite (†) vanishes and we finish the proof of the lemma.

$$F(\delta) \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(1)}{=} (\mathrm{id}+\mathrm{Tr}^{\mathcal{D}_0}_{\mathcal{U}}) \circ (F(\delta), F(I_0)) \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(4)}{=} F(i_0) \circ \mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}} \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(2)}{=}$$

$$F(i_1) \circ \mathrm{Tr}^{\mathcal{X}}_{\mathbb{A}^1 \times \mathcal{U}} \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(3)}{=} \mathrm{Tr}^{\mathcal{D}_1}_{\mathcal{U}} \circ F(I_1) \circ F(\mathrm{id}_{\mathcal{X}}) \overset{(1)}{=} 0$$

□

The following lemma is the semi-local version of Geometric Presentation Lemma [10, 10.1]

3.5 Lemma. *Let R be a semi-local essentially smooth algebra over an infinite field k and A an essentially smooth k-algebra, which is finite over the polynomial algebra $R[t]$. Suppose that $e : A \to R$ is an R-augmentation and let $I = \ker e$. Assume that A is smooth over R at every prime containing I. Given $f \in A$ such that A/Af is finite over R we can find an $s \in A$ such that*

1. *A is finite over $R[s]$.*
2. *$A/As = A/I \times A/J$ for some ideal J of A.*
3. *$J + Af = A$.*
4. *A(s-1)+Af=A.*

Proof. In the proof of [10, 10.1] replace the reduction modulo maximal ideal by the reduction modulo radical of the semi-local ring. □

4 The Effacement Theorem

We start with the following definition which is a slightly modified version of [4, 2.1.1].

4.1 Definition. Let X be a smooth affine variety over a field k. Let $x = \{x_1, \dots, x_n\}$ be a finite set of points of X and let $\mathcal{U} = \mathrm{Spec}\, \mathcal{O}_{X,x}$ be the semi-local scheme at x. A contravariant functor $F : Cp(X) \to Ab$ is effaceable at x if the following condition satisfied:
Given $m \geq 1$, for any closed subset $Z \subset X$ of codimension m, there exist a closed subset $Z' \subset \mathcal{U}$ such that

(1) $Z' \supset Z \cap \mathcal{U}$ and $\mathrm{codim}_{\mathcal{U}}(Z') \geq m - 1$;

(2) the composite $F_Z(X) \xrightarrow{F(j)} F_{Z\cap\mathcal{U}}(\mathcal{U}) \xrightarrow{F(\mathrm{id}_{\mathcal{U}})} F_{Z'}(\mathcal{U})$ vanishes, where $j : \mathcal{U} \to X$ is the canonical embedding and $Z \cap \mathcal{U} = j^{-1}(Z)$.

4.2 Theorem. *Let X be a smooth affine variety over an infinite field k and $x \subset X$ be a finite set of points. Let $G : Cp(k) \to Ab$ be a homotopy invariant functor endowed with transfers which satisfies vanishing property (see 2.4, 2.6 and 2.5). Let $F = p^*G$ denote the restriction of G to $Cp(X)$ by means of the structural morphisms $p : X \to \mathrm{Spec}\, k$. Then F is effaceable at x.*

Proof. We may assume $x \cap Z$ in non-empty. Indeed, if $x \cap Z = \emptyset$ then the theorem follows by the vanishing property of F (see 2.5).
Let $f \neq 0$ be a regular function on X such that Z is a closed subset of the vanishing locus of f. By Quillen's trick [14, 5.12], [13, 1.2] we can find a morphism $q : X \to \mathbb{A}^{n-1}$, where $n = \dim X$, such that

(a) $q|_{f=0} : \{f = 0\} \to \mathbb{A}^{n-1}$ is a finite morphism;

(b) q is smooth at the points x;

(c) q can be factorized as $q = pr \circ \Pi$, where $\Pi : X \to \mathbb{A}^n$ is a finite surjective morphism and $pr : \mathbb{A}^n \to \mathbb{A}^{n-1}$ is a linear projection.

Consider the base change diagram for the morphism q by means of the composite $r : \mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x} \xrightarrow{j} X \xrightarrow{q} \mathbb{A}^{n-1}$.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "r_X"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ \mathcal{U} \arrow[r, "r"] & \mathbb{A}^{n-1} \end{tikzcd}$$

So we have $\mathcal{X} = \mathcal{U} \times_{\mathbb{A}^{n-1}} X$ and p, r_X denote the canonical projections on $\mathcal{U}$, X respectively. Let $\delta : \mathcal{U} \to \mathcal{X} = \mathcal{U} \times_{\mathbb{A}} X$ be the diagonal embedding. Clearly δ is a section of p. Set $\mathfrak{f} = r_X^*(f)$. Take instead of $\mathcal{X}$ it's irreducible component containing $\delta(\mathcal{U})$ and instead of $\mathfrak{f}$ it's restriction to this irreducible component (since $x \cap Z$ is non-empty the vanishing locus of $\mathfrak{f}$ on the component containing $\delta(\mathcal{U})$ is non-empty as well).
Now assuming the triple $(p : \mathcal{X} \to \mathcal{U}, \delta, \mathfrak{f})$ is a perfect triple over $\mathcal{U}$ (see 3.1 for the definition) we complete the proof as follows:
Let $\mathcal{Z} = r_X^{-1}(Z)$ be the closed subset of the vanishing locus of $\mathfrak{f}$. Let $Z' = p(\mathcal{Z})$ be a closed subset of $\mathcal{U}$. Since $r_X \circ \delta = j$ we have $\delta^{-1}(\mathcal{Z}) = j^{-1}(Z) = Z \cap \mathcal{U}$. By Specialization Lemma 3.3 applied to the perfect triple $(\mathcal{X}, \delta, \mathfrak{f})$ and the functor $j^*F : Cp(\mathcal{U}) \to Ab$ the composite

$$F_{\mathcal{Z}}(\mathcal{X}) \xrightarrow{F(\delta)} F_{Z \cap \mathcal{U}}(\mathcal{U}) \xrightarrow{F(\mathrm{id}_{\mathcal{U}})} F_{Z'}(\mathcal{U})$$

vanishes. In particular, the composite

$$F_Z(X) \xrightarrow{F(j)} F_{Z \cap \mathcal{U}}(\mathcal{U}) \xrightarrow{F(\mathrm{id}_{\mathcal{U}})} F_{Z'}(\mathcal{U}) \qquad (*)$$

vanishes as well. Clearly $Z' \supset Z \cap \mathcal{U}$. By 3.1.(i) we have $\dim \mathcal{X} = \dim \mathcal{U} + 1$. On the other hand the morphism $r_{\mathcal{X}} : \mathcal{X} \to X$ is flat (even essentially smooth) and, thus, $\operatorname{codim}_{\mathcal{X}}(\mathcal{Z}) = \operatorname{codim}_X Z$. Therefore, we have $\operatorname{codim}_{\mathcal{U}}(Z') = \operatorname{codim}_{\mathcal{X}}(\mathcal{Z}) - 1 = m - 1$. $\square$

Hence, it remains to prove the following:

4.3 Lemma. *The triple* $(p : \mathcal{X} \to \mathcal{U}, \delta, \mathfrak{f})$ *is perfect over* $\mathcal{U}$.

Proof. By the property (c) one has $q = pr \circ \Pi$ with a finite surjective morphism $\Pi : X \to \mathbb{A}^n$ and a linear projection $pr : \mathbb{A}^n \to \mathbb{A}^{n-1}$. Taking the base change of Π by means of $r : \mathcal{U} \to \mathbb{A}^{n-1}$ one gets a finite surjective $\mathcal{U}$-morphism $\pi : \mathcal{X} \to \mathbb{A}^1 \times \mathcal{U}$. This checks (i) of 3.1. Since the closed subset $\{f = 0\}$ of X is finite over $\mathbb{A}^{n-1}$ the closed subset $\{\mathfrak{f} = 0\}$ of $\mathcal{X}$ is finite over $\mathcal{U}$ and we get 3.1.(ii). Since q is smooth at x the morphism $r : \mathcal{U} \to \mathbb{A}^{n-1}$ is essentially smooth. Thus the morphism $r_X : \mathcal{X} \to X$ is essentially smooth as the base change of the morphism r. The variety X is smooth over k implies that $\mathcal{X}$ is essentially smooth over k as well. Since q is smooth at x the morphism p is smooth at each point $y \in \mathcal{X}$ with $r_X(y) \in x$. In particular p is smooth at the points $\delta(x_i)$ $(x_i \in \mathcal{U})$. Since $\mathcal{U}$ is semi-local $\delta(\mathcal{U})$ is semi-local and p is smooth along $\delta(\mathcal{U})$. This checks (iii) of 3.1. Since $\mathcal{X}$ is irreducible 3.1.(iv) holds. And we have proved the lemma and the theorem. □

To prove Theorem 4.2 in the case when F is defined over some smooth affine variety we have to put an additional condition on F. In order to formulate this condition we introduce some notations.

4.4 Notation. Let $\rho : Y \to X \times X$ be a finite étale morphism together with a section $s : X \to Y$ over the diagonal embedding $\Delta : X \to X \times X$, i.e., $\rho \circ s = \Delta$. Let $pr_1, pr_2 : X \times X \to X$ be the canonical projections. We denote $p_1, p_2 : Y \to X$ to be the composite $pr_1 \circ \rho$, $pr_2 \circ \rho$ respectively.
For a contravariant functor $F : Cp(X) \to Ab$ consider it's pull-backs p_1^*F and $p_2^*F : Cp(Y) \to Ab$ by means of p_1 and p_2 respectively. From this point on we denote $F_1 = p_1^*F$ and $F_2 = p_2^*F$. By definition we have

$$F_i(Y' \to Y, Z) = F(Y' \to Y \xrightarrow{p_i} X, Z).$$

4.5 Remark. In general case the functors F_1 and F_2 are not equivalent. Moreover, the functors pr_1^*F and pr_2^*F are different. But in the case when F comes from the base field k, i.e., $F = p^*G$ where $p : X \to \operatorname{Spec} k$ is the structural morphism and $G : Cp(k) \to Ab$ is a contravariant functor, these functors coincide with each other.

4.6 Definition. We say a contravariant functor $F : Cp(X) \to Ab$ has a finite monodromy of the type $(\rho : Y \to X \times X, s : X \to Y)$, where ρ is a finite étale morphism and s is a section of ρ over the diagonal, if there exists an isomorphism $\Phi : F_1 \to F_2$ of functors on $Cp(Y)$. A functor $F : Cp(X) \to Ab$ is said to be a functor with finite monodromy if F has a finite monodromy of some type.

4.7 Theorem. *Let X be a smooth affine variety over an infinite field k and $x \subset X$ be a finite set of points. Let $F : Cp(X) \to Ab$ be a homotopy invariant functor endowed with transfers which satisfies vanishing property (see 2.4, 2.6 and 2.5). If F is a functor with finite monodromy then F is effaceable at x.*

Proof. Similar to the proof of Theorem 4.2 let $f \neq 0$ be a regular function on X such that Z is a closed subset of the vanishing locus of f. We may assume $x \cap Z$ is non-empty. Consider the fibered product diagram from the proof of Theorem 4.2

$$\begin{tikzcd} \mathcal{X} \arrow[r, "r_X"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ \mathcal{U} \arrow[r, "r"] & \mathbb{A}^{n-1} \end{tikzcd}$$

We have the projection $p : \mathcal{X} = \mathcal{U} \times_{\mathbb{A}^{n-1}} X \to \mathcal{U}$, the section $\delta : \mathcal{U} \to \mathcal{X}$ of p and the regular function $\mathfrak{f} = r_X^*(f)$.

Since F is the functor with finite monodromy there is a finite étale morphism $\rho : Y \to X \times X$, a section $s : X \to Y$ of ρ over the diagonal embedding and a functor isomorphism $\Phi : F_1 \to F_2$ as in 4.4 and 4.6. Consider the base change diagram for the morphism $\rho : Y \to X \times X$ by means of the composite $g : \mathcal{X} \xrightarrow{(p,r_X)} \mathcal{U} \times X \xrightarrow{(j,\mathrm{id})} X \times X$

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{\rho}"'] & Y \arrow[d, "\rho"] \\ \mathcal{X} \arrow[r, "g"] & X \times X \end{tikzcd}$$

Then $\tilde{\rho}$ is a finite étale morphism and there is the section $\tilde{\delta} : \mathcal{U} \to \tilde{\mathcal{X}}$ of the composite $\tilde{p} = p \circ \tilde{\rho} : \tilde{\mathcal{X}} \to \mathcal{U}$ such that $\tilde{\rho} \circ \tilde{\delta} = \delta$ ($\tilde{\delta}$ is the base change of the morphism $s : X \to Y$ by means of $\tilde{g} : \tilde{\mathcal{X}} \to Y$). Set $\tilde{\mathfrak{f}} = \tilde{\rho}^{-1}(\mathfrak{f})$. As in the proof of 4.2 we replace $\tilde{\mathcal{X}}$ by it's irreducible component containing $\tilde{\delta}(\mathcal{U})$ and $\tilde{\mathfrak{f}}$ by it's restriction to this component. By Lemma 4.8 below the triple $(\tilde{p} : \tilde{\mathcal{X}} \to \mathcal{U}, \tilde{\delta}, \tilde{\mathfrak{f}})$ is perfect.

Let $\mathcal{Z} = r_X^{-1}(Z)$ and $\tilde{\mathcal{Z}} = \tilde{\rho}^{-1}(\mathcal{Z})$. Set $Z' = p(\mathcal{Z})$. Observe that $\tilde{p}(\tilde{\mathcal{Z}}) = Z'$. The commutative diagram

$$\begin{tikzcd} F_Z(X) \arrow[r, "F(j)"] \arrow[d, "F(r_X)"'] & F_{Z \cap \mathcal{U}}(\mathcal{U}) \arrow[r, "F(\mathrm{id}_{\mathcal{U}})"] & F_{Z'}(\mathcal{U}) \\ F_{\mathcal{Z}}(\mathcal{X}) \arrow[r, "F(\tilde{\rho})"] \arrow[ur, "F(\delta)"] & F_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{X}}) \arrow[u, "F(\tilde{\delta})"'] & \end{tikzcd}$$

shows that to prove the relation $F(\mathrm{id}_{\mathcal{U}}) \circ F(j) = 0$ (compare with $(*)$ of the proof of 4.2) it suffices to check the relation $F(\mathrm{id}_{\mathcal{U}}) \circ F(\tilde{\delta}) = 0$.

Consider the pull-backs of the functors F_1, F_2 and the functor isomorphism Φ by means of the morphism $\tilde{g} : \tilde{\mathcal{X}} \to Y$. We shall use the same notation F_1, F_2 and Φ for these pull-backs till the end of this proof. So we have $F_1 = \tilde{g}^*(p_1^* F)$ and $F_2 = \tilde{g}^*(p_2^* F)$. The isomorphism $\Phi : F_1 \xrightarrow{\cong} F_2$ of functors over $Cp(\tilde{\mathcal{X}})$

provides us with the following commutative diagram

$$\begin{tikzcd}
(F_1)_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{X}}) \arrow[r, "F_1(\tilde{\delta})"] \arrow[d, "\Phi"', "\cong"] & (F_1)_{Z\cap\mathcal{U}}(\mathcal{U}) \arrow[r, "F_1(\mathrm{id}_{\mathcal{U}})"] \arrow[d, "\Phi"', "\cong"] & (F_1)_{Z'}(\mathcal{U}) \arrow[d, "\cong"', "\Phi"] \\
(F_2)_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{X}}) \arrow[r, "F_2(\tilde{\delta})"] & (F_2)_{Z\cap\mathcal{U}}(\mathcal{U}) \arrow[r, "F_2(\mathrm{id}_{\mathcal{U}})"] & (F_2)_{Z'}(\mathcal{U})
\end{tikzcd}$$

where the structure of an $\tilde{\mathcal{X}}$-scheme on $\mathcal{U}$ is given by $\tilde{\delta}$.
Since $r_X \circ \tilde{\rho} = p_2 \circ \tilde{g}$ we have $\tilde{\rho}^*(r_X^* F) = F_2$. Thus to check the relation $F(\mathrm{id}_{\mathcal{U}}) \circ F(\tilde{\delta}) = 0$ for the functor F we have to verify the same relation $F_2(\mathrm{id}_{\mathcal{U}}) \circ F_2(\tilde{\delta}) = 0$ for the functor F_2. Then by commutativity of the diagram it suffices to prove the relation $F_1(\mathrm{id}_{\mathcal{U}}) \circ F_1(\tilde{\delta}) = 0$ for the functor F_1.
Since $j \circ \tilde{p} = p_1 \circ \tilde{g}$ we have $\tilde{p}^*(j^*F) = F_1 : Cp(\tilde{\mathcal{X}}) \to Ab$. Thereby it suffices to prove the relation $G(\mathrm{id}_{\mathcal{U}}) \circ G(\tilde{\delta}) = 0$ for the functor $G = j^*F : Cp(\mathcal{U}) \to Ab$. This relation follows immediately from Theorem 3.3 applied to the functor G, the triple $(\tilde{p} : \tilde{\mathcal{X}} \to \mathcal{U}, \tilde{\delta}, \tilde{\mathfrak{f}})$ and the closed subset $\tilde{\mathcal{Z}} \subset \tilde{\mathcal{X}}$. □

4.8 Lemma. *The triple $(\tilde{\mathcal{X}}, \tilde{\delta}, \tilde{\mathfrak{f}})$ is perfect over $\mathcal{U}$.*

Proof. Observe that the triple $(p : \mathcal{X} \to \mathcal{U}, \delta, \mathfrak{f})$ is perfect by Lemma 4.3, the morphism $\tilde{\rho} : \tilde{\mathcal{X}} \to \mathcal{X}$ is finite étale and $\tilde{\rho} \circ \tilde{\delta} = \delta$ for the section $\tilde{\delta} : \mathcal{U} \to \tilde{\mathcal{X}}$ of the morphism $\tilde{p} : \tilde{\mathcal{X}} \to \mathcal{U}$. For the finite surjective morphism of $\mathcal{U}$-schemes $\pi : \mathcal{X} \to \mathbb{A}^1 \times \mathcal{U}$ the composite $\tilde{\mathcal{X}} \xrightarrow{\tilde{\rho}} \mathcal{X} \xrightarrow{\pi} \mathbb{A}^1 \times \mathcal{U}$ is a finite surjective morphism of $\mathcal{U}$-schemes as well. This proves 3.1.(i). Since $\tilde{\rho}$ is finite and the vanishing locus of $\mathfrak{f}$ is finite over $\mathcal{U}$ the vanishing locus of the function $\tilde{\mathfrak{f}}$ is finite over $\mathcal{U}$ as well. This proves 3.1.(ii). Since $\tilde{\rho} \circ \tilde{\delta} = \delta$, $\tilde{\rho}$ is étale and p is smooth along $\delta(\mathcal{U})$ the morphism $\tilde{p}$ is smooth along $\tilde{\delta}(\mathcal{U})$. Since the scheme $\mathcal{X}$ is essentially smooth over k and $\tilde{\rho}$ is étale the scheme $\tilde{\mathcal{X}}$ is essentially smooth over k. This proves 3.1.(iii). Since $\tilde{\mathcal{X}}$ is irreducible we have 3.1.(iv). And the lemma is proven. □

5 Applications to Étale Cohomology

5.1 Definition. Let X be a smooth affine variety over a field k, n an integer prime to $char(k)$ and $\mathcal{C}$ a bounded below complex of locally constant constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et}. Since étale cohomology commute with inductive limits, we may suppose the complex $\mathcal{C}$ is also bounded from above. We consider the functor $F : Cp(X) \to Ab$ which is given by

$$F(Y \xrightarrow{f} X, Z) = H^*_Z(Y, f^*\mathcal{C}),$$

where the object on the right hand side are étale hypercohomology. Since a locally constant constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et} is representable by a finite étale scheme over X, we may assume that the hypercohomology are taken on the big étale site of X [9, V.1]. Hence, we have a well-defined functor.

5.2 Lemma. *The étale cohomology functor $F(X,Z) = H^*_Z(X,\mathcal{C})$ is a functor endowed with transfers (2.6).*

Proof. Let $\pi : Y \to X$ be a finite flat morphism of schemes. For an X-scheme X' set $Y' = X' \times_X Y$ and denote the projection $Y' \to X'$ by π'. If $X'' \xrightarrow{g} X'$ is an X-scheme morphism then set $Y'' = X'' \times_X Y$ and denote by $\pi'' : Y'' \to X''$ the projection on X'' and by $g_Y : Y'' \to Y'$ the morphism $g \times \mathrm{id}_Y$. If $Z \subset X$ is a closed subset then we set $S = \pi^{-1}(Z)$, $Z' = X' \times_X Z$, $Z'' = X'' \times_X Z$, $S' = (\pi')^{-1}(Z')$, $S'' = (\pi'')^{-1}(Z'')$.
If $Y' = Y_1' \amalg Y_2'$ (disjoint union) then set $\pi_i' = \pi'|_{Y_i'}$, $Y_i'' = g_Y^{-1}(Y_i')$, $S_i' = Y_i' \cap S'$, $S_i'' = Y_i'' \cap S''$ and define $g_{Y,i} : Y_i'' \to Y_i'$ to be the restriction of g_Y.
Let C be a sheaf on the big étale site Et/X. If $Z \subset X$ is a closed subset then for an X-scheme X' we denote $\Gamma_{Z'}(X',C) = \ker(\Gamma(X',C) \to \Gamma(X'-Z',C))$ and if $Y' = Y_1' \amalg Y_2'$ we denote $\Gamma_{S_i'}(Y_i',C) = \ker(\Gamma(Y_i',C) \to \Gamma(Y_i' - S_i', C))$.
Deligne in [5] constructed trace maps for finite flat morphisms. In particular, for a X-scheme X' and for every presentation of the scheme Y' in the form $Y' = Y_1' \amalg Y_2'$ there are certain trace maps $\mathrm{Tr}_{\pi_i'} : \Gamma_{S_i'}(Y_i',C) \to \Gamma_{Z'}(X',C)$. These maps satisfy the following properties

(i) (base change) the diagram

$$\begin{tikzcd}
\Gamma_{S_i''}(Y_i'',C) \arrow[d, "\mathrm{Tr}_{\pi_i''}"'] & \Gamma_{S_i'}(Y_i',C) \arrow[l, "g^*_{Y,i}"'] \arrow[d, "\mathrm{Tr}_{\pi_i'}"] \\
\Gamma_{Z''}(X'',C) & \Gamma_{Z'}(X',C) \arrow[l, "g^*_Y"']
\end{tikzcd}$$

commutes;

(ii) (additivity) the diagram

$$\begin{tikzcd}
\Gamma_{S'}(Y',C) \arrow[r, "\text{“+''}"] \arrow[d, "\mathrm{Tr}_{\pi'}"'] & \Gamma_{S_1'}(Y_1',C) \oplus \Gamma_{S_2'}(Y_2',C) \arrow[d, "\mathrm{Tr}_{\pi_1'} + \mathrm{Tr}_{\pi_2'}"] \\
\Gamma_{Z'}(X',C) \arrow[r, "\mathrm{id}"] & \Gamma_{Z'}(X',C)
\end{tikzcd}$$

commutes;

(iii) (normalization) if $\pi_1' : Y_1' \to X'$ is an isomorphism then the composite map

$$\Gamma_{Z'}(X',C) \xrightarrow{(\pi_1')^*} \Gamma_{S_1'}(Y_1',C) \xrightarrow{\mathrm{Tr}_{\pi_1'}} \Gamma_{Z'}(X',C)$$

is the identity;

(iv) maps $\mathrm{Tr}_{\pi_i'}$ are functorial with respect to sheaves C on Et/X.

Now let $0 \to C \to \mathfrak{I}^\bullet$ be an injective resolution of the sheaf C on Et/X. Then for a closed subset $Z \subset X$ and for a presentation $Y' = Y_1' \amalg Y_2'$ one has

$$H^p_{S_i'}(Y_i', C) := H^p(\Gamma_{S_i'}(Y_i', \mathfrak{I}^\bullet)) \ , \ H^p_{Z'}(X', C) := H^p(\Gamma_{Z'}(X', \mathfrak{I}^\bullet)).$$

Thereby the property (iv) shows that the trace maps $\mathrm{Tr}_{\pi_i'} : \Gamma_{S_i'}(Y_i', \mathfrak{I}^r) \to \Gamma_{Z'}(X', \mathfrak{I}^r)$ determine a morphism of complexes $\Gamma_{S_i'}(Y_i', \mathfrak{I}^\bullet) \to \Gamma_{Z'}(X', \mathfrak{I}^\bullet)$. Thus one gets the induced map which we will denote by $H^p(\mathrm{Tr}_{\pi_i'})$: $H^p_{S_i'}(Y_i', C) \to H^p_{Z'}(X', C)$. And these trace maps satisfy the following properties (the same as in Definition 2.6):

(i) the base changing property;

(ii) the additivity property;

(iii) the normalization property;

(iv) the functorality with respect to sheaves on Et/X.

□

5.3 Lemma. *The étale cohomology functor $F(X, Z) = H^*_Z(X, \mathcal{C})$ is a functor with finite monodromy (4.6).*

Proof. According to Definition 4.6 we have to show that there exist a finite étale morphism $\rho : Y \to X \times X$ together with a section $s : X \to Y$ of ρ over the diagonal and a functor isomorphism $\Phi : F_1 \to F_2$ on $Cp(Y)$, where $F_1 = \rho^* \circ pr_1^* F$ and $F_2 = \rho^* \circ pr_2^* F$.
To produce Y we use the following explicit construction suggested by H. Esnault (we follow [15]):
Let $\tilde{X}$ be a finite Galois covering with Galois group G such that the pullback of $\mathcal{C}$ to $\tilde{X}$ is a complex of constant sheaves. Consider the étale covering $\tilde{X} \times_k \tilde{X} \to X \times_k X$ with Galois group $G \times G$. Let $\widetilde{X \times_k X} = (\tilde{X} \times_k \tilde{X})_G$ be the unique intermediate covering associated with the diagonal subgroup $G = (g, g) \in G \times G$. The diagonal map $\tilde{X} \to \tilde{X} \times_k \tilde{X}$ induces a map s : $X \to \widetilde{X \times_k X}$ which is a section to the projection $\widetilde{X \times_k X} \to X \times_k X$ over the diagonal $X \cong \Delta_X \subset X \times_k X$. Let Y be the connected component of $X = im(s)$ in $\widetilde{X \times_k X}$. Then $Y \xrightarrow{\rho} X \times_k X$ is a connected Galois covering having a section s over Δ_X.
To check that there is the functor isomorphism Φ we refer to the end of section 4 of [15]. □

We also need the following technical lemma that is a slightly modified version of Proposition 2.1.2, [4]

5.4 Lemma. *Let $\mathcal{U}$ be a semi-local regular scheme of geometric type over a field k, i.e., $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{X,x}$ for some smooth affine variety X and a finite set of points $x = \{x_1, \ldots, x_n\}$ of X. Suppose the étale cohomology functor*

$F(Y,Z) = H^*_Z(Y,\mathcal{C})$ from Theorem 1.1 is effaceable at x. Then, in the exact couple [4, 1.1] defining the coniveau spectral sequence for $(\mathcal{U},\mathcal{C})$, the map $i^{p,q}$ is identically 0 for all $p>0$. In particular, we have $E_2^{p,q} = H^q(\mathcal{U},\mathcal{C})$ if $p=0$ and $E_2^{p,q}=0$ if $p>0$. And the Cousin complex [4, 1.3] yields the exact complex from Theorem 1.1.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H^n_Z(X,\mathcal{C}) \arrow[r] \arrow[d] & H^n_{Z\cap\mathcal{U}}(\mathcal{U},\mathcal{C}) \arrow[r] \arrow[d] & H^n_{Z'}(\mathcal{U},\mathcal{C}) \arrow[d] \\
H^n_{X^{(m)}}(X,\mathcal{C}) \arrow[r] & H^n_{\mathcal{U}^{(m)}}(\mathcal{U},\mathcal{C}) \arrow[r] & H^n_{\mathcal{U}^{(m-1)}}(\mathcal{U},\mathcal{C})
\end{tikzcd}$$

The composition of arrows in the first row is identically 0 for any n. Therefore the compositions $H^n_Z(X,\mathcal{C}) \to H^n_{\mathcal{U}^{(m)}}(\mathcal{U},\mathcal{C}) \to H^n_{\mathcal{U}^{(m-1)}}(\mathcal{U},\mathcal{C})$ are 0. Passing to the limit over Z, this gives that the compositions $H^n_{X^{(m)}}(X,\mathcal{C}) \to H^n_{\mathcal{U}^{(m)}}(\mathcal{U},\mathcal{C}) \to H^n_{\mathcal{U}^{(m-1)}}(\mathcal{U},\mathcal{C})$ are 0. Passing to the limit over open neighborhoods of x, we get that the map $i^{m,n-m} : H^n_{\mathcal{U}^{(m)}}(\mathcal{U},\mathcal{C}) \to H^n_{\mathcal{U}^{(m-1)}}(\mathcal{U},\mathcal{C})$ is itself 0 for any $m \geq 1$. □

Now we are ready to prove the main result of this paper stated in the Introduction.

Proof of Theorem 1.1. Assume the ground field k is infinite. In this case, observe that the étale cohomology functor $F(X,Z) = H^*_Z(X,\mathcal{C})$ satisfies all the hypotheses of Theorem 4.7. Indeed, it is homotopy invariant according to [4, 7.3.(1)]. It satisfies vanishing property by the very definition. It has transfer maps by Lemma 5.2 and it is a functor with finite monodromy by Lemma 5.3. So that by Theorem 4.7 the functor F is effaceable. Now Theorem 1.1 follows immediately from Lemma 5.4.

To finish the proof, i.e., to treat the case of a finite ground field, we apply the standard arguments with transfers for finite field extensions (see the proof of [4, 6.2.5]). □

5.5 Remark. If the complex $\mathcal{C}$ comes from the base field k, i.e., each sheaf in $\mathcal{C}$ can be represented as p^*C' for some sheaf C' on $(\operatorname{Spec} k)_{et}$, where $p : X \to \operatorname{Spec} k$ is the structural morphism. Then the étale cohomology functor F satisfies all the hypotheses of Theorem 4.2. Hence, Theorem 1.1 holds without assuming that F is a functor with finite monodromy.

REFERENCES

[1] M. Artin, A. Grothendieck et J. L. Verdier, Théorie des topos et cohomologie étale des schémas, SGA 4, Tome 3, Lecture Notes in Math. 305, Springer-Verlag, 1973. vi+640 pp.

[2] S. Bloch, A. Ogus, Gersten's Conjecture and the homology of schemes, *Ann. Sci. Ec. Norm. Super.*, 4(7): 181-202, 1974.

[3] J.-L. Colliot-Thélène, Birational invariants, purity and the Gersten conjecture, *in* "K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)", 1-64, Proc. Sympos. Pure Math. 58, Part 1, Amer. Math. Soc., Providence, RI, 1995.

[4] J.-L. Colliot-Thélène, R.T. Hoobler, B. Kahn, The Bloch-Ogus-Gabber Theorem, *in* "Algebraic K-theory", 31-94, Fields Inst. Commun. 16, Amer. Math. Soc., Providence, RI, 1997.

[5] P. Deligne, "SGA $4^{1/2}$, Cohomologie étale, Lecture Notes in Math. 569, Springer-Verlag, 1977

[6] D. Eisenbud, "Commutative Algebra", Graduate Texts in Mathematics 150, Springer, 1994.

[7] H. Esnault, E. Viehweg, Germs of de Rham cohomology classes which vanish at the generic point, C. R. Acad. Sci. Paris Sér. I Math., 326(7): 857-862, 1998.

[8] O. Gabber, Gersten's conjecture for some complexes of vanishing cycles. *Manuscripta Math.*, 85(3-4): 323-343, 1994.

[9] J.S. Milne, "Étale Cohomology", Princeton Mathematical Series 33, Princeton, New Jersey: Princeton University Press. XIII, 1980.

[10] M. Ojanguren, I. Panin, Rationally trivial hermitian spaces are locally trivial, *Math. Z.*, 237: 181-198, 2001.

[11] M. Ojanguren, I. Panin, A purity theorem for the Witt group, *Ann. Sci. Ec. Norm. Super.*, 32(1): 71-86, 1999.

[12] I. Panin, A. Smirnov, On a theorem of Grothendieck. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 272(7): 286–293, 2000.

[13] I. Panin, A. Suslin, On a conjecture of Grothendieck concerning Azumaya algebras, *St.Petersburg Math. J.*, 9(4): 851-858, 1998.

[14] D. Quillen, Higher algebraic K-theory I, *in* "Algebraic K-theory I", 85-147, Lecture Notes in Math. Vol. 341. Springer-Verlag, Berlin, 1973.

[15] A. Schmidt, K. Zainoulline, Generic Injectivity for Etale Cohomology and Pretheories, to appear in *J. of Algebra*, 2003. (also available on K-Theory preprint server N 556)

[16] V. Voevodsky, Cohomological theory of presheaves with transfers, *in* "Cycles, Transfers and Motivic Homology Theories", An. Math. Studies 143, Princeton University Press, 1998.

[17] K. Zainoulline, The Purity Problem for Functors with Transfers, *K-Theory J.*, 22(4): 303-333, 2001.

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A Short Proof of Rost Nilpotence via Refined Correspondences

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Abstract. I generalize the notion of composition of algebraic correspondences using the refined Gysin homorphism of Fulton-MacPherson intersection theory. Using this notion, I give a short self-contained proof of Rost's "nilpotence theorem" and a generalization of one important proposition used by Rost in his proof of the theorem.

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1. Introduction

In an elegant four page preprint "A shortened construction of the Rost motive" N. Karpenko (see also [4]) gives a construction of Rost's motive M_a assuming the following result of Rost widely known as the "nilpotence theorem."

Theorem 1.1. *Let Q be a smooth quadric over a field k with algebraic closure $\overline{k}$ and let $f \in \operatorname{End} M(Q)$ be an endomorphism of its integral Chow motive. Then, if $f \otimes \overline{k} = 0$ in $\operatorname{End} M(Q \otimes \overline{k})$, f is nilpotent.*

For the proof, Karpenko refers the reader to a paper of Rost which proves the theorem by invoking the fibration spectral sequence of the cycle module of a product (also due to Rost [6]). (In [7], A. Vishik gives another proof of Theorem 1.1 based on V. Voevodsky's theory of motives.)
The existence of the Rost motive and the nilpotence theorem itself are both essential to Voevodsky's proof of the Milnor conjecture. It is, therefore, desirable to have direct proofs of these fundamental results. The main goal of this paper is to provide such a proof in the spirit of Karpenko's preprint. To accomplish this, I use a generalization to singular schemes of the notion of composition of correspondences to obtain a proof of the theorem which avoids the use of cycle modules.

Both Rost's proof of Theorem 1.1 and the proof presented here involve two principal ingredients: (1) a theorem concerning nilpotent operators on $Hom(M(B), M(X))$ for B and X smooth projective varieties, (2) a decomposition theorem for the motive $M(Q)$ of a quadric Q with a k-rational point. For (1), we obtain an extension of Rost's results (Theorem 3.1) allowing the motive of B to be Tate twisted. Moreover, the method of proof can be used to extend the result to arbitrary varieties B. For (2), the theorem stated here (Theorem 4.1) is identical to Rost's, but the proof is somewhat simpler as we are able to perform computations with correspondences involving possibly singular varieties.

V. Chernousov, S. Gille and A. Merkurjev have recently generalized Theorem 1.1 to arbitrary homogeneous varieties [1]. Their approach is to write down a decomposition as in (2) for homogeneous varieties in terms of group theory and then to use the extension to (1) given here to prove a nilpotence result. I would like to thank Merkurjev for pointing out to me the usefulness of this extension.

1.1. Notation. As the main tool used in this paper is the intersection theory of Fulton-MacPherson, we use the notation of [2]. In particular, a *scheme* will be a scheme of finite type over a field and a *variety* will be an irreducible and reduced scheme. We use the notation Chow_k for k a field to denote the category of Chow motives whose definition is recalled below in Section 2. For a scheme X, $A_j X$ will denote the Chow group of dimension j cycles on X.

In section 3, we will use the notation $\mathbb{H}$ to denote the hyperbolic plane. That is, $\mathbb{H}$ is the quadratic space consisting of k^2 with quadratic form given by $q(x, y) = xy$.

2. Refined Intersections

Let V and W be schemes over a field k, let $\{V_i\}_{i=1}^m$ be the irreducible components of V and write $d_i = \dim V_i$. The group of degree r Chow correspondences is defined as

$$\mathrm{Corr}_r(V, W) = \oplus A_{d_i - r}(V_i \times W). \tag{1}$$

If X_1, X_2, X_3 are smooth proper schemes, then it is well-known that there is a composition

$$\begin{aligned} \mathrm{Corr}_r(X_1, X_2) \otimes \mathrm{Corr}_s(X_2, X_3) &\to \mathrm{Corr}_{r+s}(X_1, X_3) \\ g \otimes f &\mapsto f \circ g \end{aligned} \tag{2}$$

given by the formula

$$f \circ g = p_{13*}(p_{12}^* g \cdot p_{23}^* f) \tag{3}$$

where the $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the obvious projection maps. Using this formula, the category Chow_k of Chow motives can be defined as follows ([4], see also [3]): The objects are the triples (X, p, n) where X is a smooth projective scheme over k, $p \in \mathrm{Corr}_0(X,X)$ is a projector (that is, $p^2 = p$) and n is an integer. The morphisms are defined by the formula

$$\mathrm{Hom}((X,p,n),(Y,q,m)) = q\,\mathrm{Corr}_{m-n}(X,Y)p. \tag{4}$$

To fix notation, we remind the reader that the *Tate twist* of an object $M = (X,p,n)$ is the object $M(k) = (X,p,n+k)$, and the objects $\mathbb{Z}(k) = (\mathrm{Spec}\,k, \mathrm{id}, k)$ are customarily called the *Tate objects*. It is clear from (4) that

$$\mathrm{Hom}(\mathbb{Z}(k), M(X)) = A_k X, \quad \mathrm{Hom}(M(X), \mathbb{Z}(k)) = A^k X \tag{5}$$

where $M(X)$ is the motive $(X, \mathrm{id}, 0)$ associated to the scheme X.

2.1. Refined correspondences. The main observation behind this paper is that a composition generalizing that of (2) holds for arbitrary varieties X_1 and X_3 provided that X_2 is smooth and proper. To define this composition we use the the Gysin pullback through the regular embedding

$$X_1 \times X_2 \times X_3 \overset{\mathrm{id}\times\Delta\times\mathrm{id}}{\to} X_1 \times X_2 \times X_2 \times X_3.$$

We can then define the composition by the formula

$$f \circ g = p_{13*}((\mathrm{id} \times \Delta \times \mathrm{id})^!(g \otimes f)). \tag{6}$$

We need to verify that the definition given in (6) agrees with that of (3) and satisfies various functoriality properties needed to make it a useful extension. To state these properties in their natural generality, it is helpful to also consider (6) in a slightly different situation from that of (2). For X_2 a smooth scheme and X_1, X_3 arbitrary schemes, we define a composition

$$\begin{array}{rcl} A_r(X_1 \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) & \to & A_{r-s}(X_1 \times X_3) \\ g \otimes f & \mapsto & f \circ g \end{array} \tag{7}$$

where $f \circ g$ is defined as in (6). We consider (7) because $\oplus_i \mathrm{Corr}_i(X_1, X_2)$ is not necessarily equal to $\oplus_i A_i(X_1 \times X_2)$ unless X_1 is scheme with irreducible connected components. Therefore, in the case that X_1 does not have irreducible connected components, $\mathrm{Corr}_*(X_1, X_2)$ is not a reindexing of the Chow groups of $X_1 \times X_2$.

Proposition 2.1. *Let $X_i, i \in \{1,2,3\}$ be schemes with X_2 smooth and proper.*

(a) If all X_i are smooth and X_2 is proper, then the definition of $f \circ g$ for $g \in \mathrm{Corr}_r(X_1, X_2), f \in \mathrm{Corr}_s(X_2, X_3)$ given in (6) agrees with that of (3).

(b) If $\pi : X_1' \to X_1$ is a proper morphism, then the diagram

$$\begin{tikzcd} A_r(X_1' \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) \arrow[r] \arrow[d, "\pi_*"'] & A_{r-s}(X_1' \times X_3) \arrow[d, "\pi_*"] \\ A_r(X_1 \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) \arrow[r] & A_{r-s}(X_1 \times X_3) \end{tikzcd}$$

commutes. Here, for the vertical arrows, by π_* we mean the morphism induced by π_* on the first factor and the identity on the other factors.

(c) If $\phi : X_1' \to X_1$ is flat of constant relative dimension e, then

$$\begin{tikzcd}
A_r(X_1 \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) \arrow[r] \arrow[d, "\phi^*"'] & A_{r-s}(X_1 \times X_3) \arrow[d, "\phi^*"] \\
A_{r+e}(X_1' \times X_2) \otimes \mathrm{Corr}_s(X_2, X_3) \arrow[r] & A_{r+e-s}(X_1' \times X_3)
\end{tikzcd}$$

commutes.

Proof. First note that it suffices to prove the proposition for X_2 irreducible of dimension d_2. This is because $\mathrm{Corr}_s(X_2, X_3)$ and $A_r(X_1 \times X_2)$ are both direct sums over the irreducible components of X_2 and all of the maps in the theorem commute with these direct sum decompositions.
(a): Another formulation of (3) is that $f \circ g$ is given by

$$p_{13*}\Delta_{123}^!(p_{12}^* g \otimes p_{23}^* f)$$

where

$$\Delta_{123} : X_1 \times X_2 \times X_3 \to (X_1 \times X_2 \times X_3) \times (X_1 \times X_2 \times X_3)$$

is the obvious diagonal.
Consider the sequence of maps

$$X_1 \times X_2 \times X_3 \xrightarrow{\Delta_{123}} (X_1 \times X_2 \times X_3) \times (X_1 \times X_2 \times X_3) \xrightarrow{p_{12} \times p_{23}} X_1 \times X_2 \times X_2 \times X_3 \tag{8}$$

with composition $\Delta_2 : X_1 \times X_2 \times X_3 \to X_1 \times X_2 \times X_2 \times X_3$.
Since all X_i are smooth, $p_{12} \times p_{23}$ is a smooth morphism. It follows from ([2] Proposition 6.5.b) that

$$\begin{aligned} \Delta_2^!(g \otimes f) &= \Delta_{123}^!(p_{12} \times p_{23})^*(g \otimes f) \\ &= \Delta_{123}^!(p_{12}^* g \otimes p_{23}^* f). \end{aligned}$$

(a) now follows by taking push-forwards.
(b): We have a fiber diagram

$$\begin{tikzcd}
X_1' \times X_2 \times X_3 \arrow[r, "\Delta_2'"] \arrow[d] & X_1' \times X_2 \times X_2 \times X_3 \arrow[d] \\
X_1 \times X_2 \times X_3 \arrow[r, "\Delta_2"'] & X_1 \times X_2 \times X_2 \times X_3
\end{tikzcd} \tag{9}$$

where Δ_2' and Δ_2 are both induced by the diagonal

$$X_2 \to X_2 \times X_2.$$

Since Δ_2 and Δ_2' are both regular of codimension d_2, it follows from ([2], Proposition 6.2.c) that both morphisms induce the same Gysin pullback on the top row of the diagram.

By ([2], 6.2 (a)), proper push-forward and Gysin pull-back through a regular embedding commute. Applying this fact to (9), we have

$$\begin{aligned}
\pi_*(f \circ g) &= (\pi \times \mathrm{id}_3)_* p_{13*} \Delta_2^!(g \otimes f) \\
&= p_{13*}(\pi \times \mathrm{id}_2 \times \mathrm{id}_3)_* \Delta_2^!(g \otimes f) \\
&= p_{13*} \Delta_2^!(\pi \times \mathrm{id}_2 \times \mathrm{id}_2 \times \mathrm{id}_3)_*(g \otimes f) \\
&= p_{13*} \Delta_2^!((\pi \times \mathrm{id})_* g \otimes f)) \\
&= f \circ (\pi_* g).
\end{aligned}$$

(c) Here the argument is very similar to the one for (b): We have a pullback diagram

$$\begin{tikzcd}
X_1' \times X_2 \times X_3 \arrow[r, "\phi \times \mathrm{id}_2 \times \mathrm{id}_3"] \arrow[d, "p_{13}'"'] & X_1 \times X_2 \times X_3 \arrow[d, "p_{13}"] \\
X_1' \times X_3 \arrow[r, "\phi \times \mathrm{id}_3"'] & X_1 \times X_3.
\end{tikzcd}$$

Since the vertical arrows are proper and the horizontal arrows are flat, it follows from ([2], 1.7) that

$$p'_{13*}(\phi \times \mathrm{id}_2 \times \mathrm{id}_3)^* = (\phi \times \mathrm{id}_3)^* p_{13*} \tag{10}$$

We then consider the pullback

$$\begin{tikzcd}
X_1' \times X_2 \times X_3 \arrow[r, "\Delta_2'"] \arrow[d, "\phi"'] & X_1' \times X_2 \times X_2 \times X_3 \arrow[d, "\phi"'] \\
X_1 \times X_2 \times X_3 \arrow[r, "\Delta_2"'] & X_1 \times X_2 \times X_2 \times X_3
\end{tikzcd}$$

in which the vertical arrow are flat and the horizontal arrow are regular embeddings both of codimension d_2. By ([2], 6.2 (c)) it follows that the flat pullbacks commutes with the Gysin pullbacks; thus,

$$\begin{aligned}
\phi^*(f \circ g) &= (\phi \times \mathrm{id}_3)^* p'_{13*} \Delta_2^!(g \otimes f) \\
&= p_{13*}(\phi \times \mathrm{id}_2 \times \mathrm{id}_3)^* \Delta_2^!(g \times f) \\
&= p_{13*} \Delta_2^!((\phi^* g) \otimes f)) \\
&= f \circ (\phi^* g).
\end{aligned}$$

$\square$

Remark 2.2. If X_1 and X_3 are taken to be schemes with irreducible connected components, then in (b) and (c), we can replace $A_*(X_1 \times X_2)$ with $\mathrm{Corr}_*(X_1, X_2)$ after a shift in the indices. Then the roles of X_1 and X_3 in the theorem can also be interchanged by the symmetry of $\mathrm{Corr}_*(X, Y)$.

The fact that morphisms in $\mathrm{Corr}_*(\_, \_)$ are not in general composable is mitigated somewhat by the following result.

PROPOSITION 2.3. *Let $\{X_i\}_{i=1}^4$ be schemes with X_2 and X_3 smooth and proper.*

(a) If $\Delta \in \mathrm{Corr}_0(X_2 \times X_2)$ is the class of the diagonal then, the morphism $A_r(X_1 \times X_2) \to A_r(X_1 \times X_2)$ given by $f \mapsto \Delta \circ f$ is the identity.
(b) If $f_1 \in A_r(X_1 \times X_2)$ and $f_i \in \mathrm{Corr}_{r_i}(X_i, X_{i+1})$ for $i = 2, 3$, then

$$(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1).$$

In other words, composition is associative.

Proof. (a) can be easily checked on the level of cycles in $Z_r(X_1 \times X_2)$. For (b), the important point is the commutativity of the diagram

$$\begin{tikzcd} X_1 \times X_2 \times X_3 \times X_4 \arrow[r] \arrow[d] \arrow[dr, "\Delta_{23}"] & X_1 \times X_2 \times X_2 \times X_3 \times X_4 \arrow[d] \\ X_1 \times X_2 \times X_3 \times X_3 \times X_4 \arrow[r] & X_1 \times X_2 \times X_2 \times X_3 \times X_3 \times X_4 \end{tikzcd} \tag{11}$$

where the arrows are the obvious diagonal morphisms. Both compositions in (b) can be computed as $p_{14*}\Delta_{23}^!(f_3 \otimes f_2 \otimes f_1)$. □

3. Rost's Correspondence Theorem

If X and Y are smooth projective varieties and $f : M(X) \to M(Y)$ is a morphism, we obtain a morphism $f_* : A_r(X) \to A_r(Y)$ induced by the composition

$$\mathbb{Z}(r) \to M(X) \xrightarrow{f} M(Y)$$

using (5). Similarly, for a smooth projective variety B and an integer a, we obtain a morphism $f_* : \mathrm{Hom}(M(B)(a), M(X)) \to \mathrm{Hom}(M(B)(a), M(Y))$ given by

$$g \mapsto f \circ g \tag{12}$$

with $g \in \mathrm{Hom}(M(B)(a), M(X)) = \mathrm{Corr}_{-a}(B, X)$.
Rost's nilpotence theorem is a consequence of the following more general theorem concerning correspondences between smooth varieties.

Theorem 3.1. *Let B and X be smooth projective varieties over a field k with $\dim B = d$. For any $b \in B$, let X_b denote the fiber of the projection $\pi : B \times X \to B$. If $f \in \mathrm{End}(M(X))$ is a morphism such that $f_* A_r(X_b) = 0$ for all b and all $0 \le r \le d + a$, then*

$$f_*^{d+1} \mathrm{Hom}(M(B)(a), M(X)) = 0. \tag{13}$$

In the case $a = 0$, the theorem is due to Rost ([5], Proposition 1). Our proof of the theorem is based on Rost's proof, but uses the results of Section 2 in place of Rost's cycle module spectral sequence. Note that, while the hypotheses of the theorem assume that B is smooth, the the proof is essentially an induction on all subvarieties (smooth or otherwise) of B. Moreover, the result holds with a slight change of notation (which we describe after the proof) for arbitrary B.

Proof of Theorem 3.1. Let $Z_k(B\times X)$ denote the group of k-dimensional cycles on $B\times X$, and let $F_pZ_k(B\times X)$ denote the subgroup of $Z_k(B\times X)$ generated by subvarieties V of dimension k such that $\dim \pi(V)\le p$. Let $F_pA_k(B\times X)$ denote the image of $F_pZ_k(B\times X)$ under the rational equivalence quotient map. To prove the theorem, it is clearly sufficient to show that

$$f_* F_p A_{d+a}(B\times X)\subset F_{p-1}A_{d+a}(B\times X) \tag{14}$$

since $F_{-1}A_{d+a}(B\times X)=0$. Therefore, it is sufficient to show that, for V a $(d+a)$-dimensional subvariety of $B\times X$ such that $\dim\pi(V)=p$, $f_*[V]\in F_{p-1}A_{d+a}(B\times X)$.

Let $Y=\pi(V)$. By the hypotheses of the theorem, there is a nonempty open set $U\subset Y$ such that $f_*[V_U]=0$. (Here we write V_U for the fiber product $V\times_Y U$.) Let $W=Y-U$, and consider the short exact sequence of Chow groups

$$A_{d+a}W\times X\xrightarrow{i_*}A_{d+a}Y\times X\xrightarrow{j^*}A_{d+a}U\times X\to 0. \tag{15}$$

By the results of Section 2, $f_*[V_U]=j^*f_*[V]$ where $f_*[V]$ is the composition $f\circ[V]$ of f with $[V]$ viewed as an element of $\mathrm{Corr}_{p-d-a}Y\times X$. It follows that $f_*[V]$ lies in the image of the first morphism in (15). Thus $f_*[V]\in F_{p-1}A_{d+a}B\times X$. □

Remark 3.2. Using the associativity of composition (Proposition 2.3), it is easy to see that the above proof generalizes to the case where B is arbitrary. The statement of the theorem remains the same, except that $Hom(M(B)(a),M(X))$ is replaced with $\mathrm{Corr}_{-a}(B,X)$.

4. Rost Nilpotence

If $M=(Y,p,n)$ is a motive in Chow_k and X is an arbitrary scheme, we define

$$\mathrm{Corr}(X,M)=p\,\mathrm{Corr}_n(X,Y).$$

Since Y is smooth and projective, this definition makes sense by what we have seen in Section 2. If $j:U\to X$ is flat we obtain a pullback $\mathrm{Corr}(X,M)\to\mathrm{Corr}(U,M)$ and, if $p:X'\to X$ is proper, we obtain a pushforward $\mathrm{Corr}(X',M)\to\mathrm{Corr}(X,M)$. This follows from Proposition 2.1. Similarly, by Remark 2.2 we can define $\mathrm{Corr}(M,X)$.

Using this observation, we can easily obtain a result of Rost's on the decomposition of the motive of a quadric. To state the theorem, we must first recall a fact about quadrics with points.

Suppose Q is the projective quadric corresponding to a non-degenerate quadratic form q; that is, $Q=V(q)$. As we are discussing quadrics and quadratic forms, we will assume for the remainder of the paper that the field k over which Q and q are defined has characteristic not equal to 2. Suppose further that Q has a point over k. Then the quadratic form q splits as an orthogonal direct sum $q=\mathbb{H}\perp q'$. (This is a standard fact about quadratic forms which is also an easy exercise). Let Q' denote the (clearly smooth) quadric associated to q'.

THEOREM 4.1 (Rost decomposition). *If Q has a point over k then $M(Q) = \mathbb{Z} \oplus M(Q')(1) \oplus \mathbb{Z}(d)$ where $d = \dim Q$ and Q' is the smooth quadric of the proceeding paragraph.*

Proof. For the proof, we use Rost's methods and notation ([5], Proposition 2) with some simplifications coming from our results in the previous sections. ‘
We can write $q = xy + q'(z)$ where z denotes a d-dimensional variable. Let Q_1 denote the closed subvariety $V(x)$ and let p denote the closed point on Q_1 corresponding to the locus $x = z = 0, y = 1$. Note that $U_1 := Q - Q_1$ is isomorphic to $\mathbb{A}^d$. Moreover, $Q_1 - \{p\}$ is an $\mathbb{A}^1$-bundle over Q' via the morphism $(y, z) \mapsto z$. For any motive M, we thus obtain short exact sequences

$$\operatorname{Corr}(M, Q_1) \to \operatorname{Corr}(M, Q) \twoheadrightarrow \operatorname{Corr}(M, \mathbb{A}^d), \tag{16}$$
$$\operatorname{Corr}(M, p) \to \operatorname{Corr}(M, Q_1) \twoheadrightarrow \operatorname{Corr}(M(-1), Q'). \tag{17}$$

Here Q_1 is, in general, a *singular* quadric. However, by Theorem 2.3, each of the entries of (16) and (17) can each be interpreted as presheaves on the category of Chow motives given, for example, by the association $M \rightsquigarrow \operatorname{Corr}(M, Q_1)$. Moreover, by Proposition 2.1, the morphisms in (16) and (17) induce maps of presheaves, i.e., they are functorial in M.
In fact, in both sequences the first morphism is an injection and the second morphism is a split surjection. To see this we construct splittings for the first morphism in each sequence.
For (17), let $\pi : Q_1 \to p$ denote the projection to a point. Then $\pi_* : \operatorname{Corr}(M, Q_1) \to \operatorname{Corr}(M, p)$ induces a splitting. Again, by Proposition 2.1, this map is functorial in M.
For (16), let r denote the point corresponding to $x = 1, y = z = 0$, and let U denote the open subset $Q - \{r\}$ in Q. Then there is a morphism $\phi^\circ : U \to Q_1$ given by $(x, y, z) \mapsto (y, z)$. Let ϕ denote the closure of the graph of ϕ° in $\operatorname{Corr}(Q, Q_1)$. By the results of section 2, ϕ induces a morphism $\phi_* : \operatorname{Corr}(M, Q) \to \operatorname{Corr}(M, Q_1)$. We claim that ϕ_* splits (16) and is functorial in M. (This is not hard to check on the level of cycles.)
Since the push-forward on the second factor induces an isomorphism

$$\operatorname{Corr}(M, \mathbb{A}^d) \xrightarrow{\cong} \operatorname{Hom}(M, \mathbb{Z}(d)),$$

we have a decomposition

$$\operatorname{Hom}(M, M(Q)) = \operatorname{Hom}(M, \mathbb{Z}(d)) \oplus \operatorname{Hom}(M, \mathbb{Z}) \oplus \operatorname{Hom}(M, M(Q')(-1)). \tag{18}$$

The decomposition of the theorem the follows from Yoneda's lemma which applies in this case because of the functoriality of the decomposition with respect to M.

□

We are now prepared to prove Rost nilpotence, Theorem 1.1. The proof is essentially identical to Rost's, but I include it for the convenience of the reader.

We first note that, due to the inductive structure of the proof, it is actually helpful to strengthen the conclusion of the theorem slightly. We therefore restate the theorem with the stronger conclusion.

THEOREM 4.2 (Rost [5], Proposition 2). *For each $d \in \mathbb{N}$, there is a number $N(d)$ such that, if Q is a smooth quadric of dimension d over a field k and $f \in \operatorname{End}(M(Q))$ such that $f \otimes \overline{k} = 0$, then $f^{N(d)} = 0$.*

Proof. If $d = 0$, Q either consists either of two points defined over k or of one point defined over a quadratic extension of k. In the first case, $\operatorname{End}(M(Q)) = \operatorname{End}(\mathbb{Z} \oplus \mathbb{Z})$ and in the second $\operatorname{End}(M(Q))$ is isomorphic to the rank 2 subring of $\operatorname{End}(M(Q \otimes \overline{k}))$ consisting of matrices invariant under conjugation by $\binom{0\,1}{1\,0}$. The theorem, therefore, holds trivially with $N(0) = 1$.
We then induct on d. Suppose Q is a rank $d > 0$ quadric with a point over k. Then $M(Q)$ splits as in Rost's decomposition theorem. In fact, we also have a splitting

$$\operatorname{End}(M(Q)) = \operatorname{End}(\mathbb{Z}(d)) \oplus \operatorname{End}(\mathbb{Z}) \oplus \operatorname{End}(M(Q')). \tag{19}$$

This follows from the fact that the six cross terms (e.g. $Hom(\mathbb{Z}, \mathbb{Z}(d))$, $\operatorname{Hom}(\mathbb{Z}, M(Q'))$ and $\operatorname{Hom}(M(Q'), \mathbb{Z})$) are all zero for dimension reasons. As $\operatorname{End}(\mathbb{Z}(j)) = \mathbb{Z}$, we have

$$\operatorname{End}(M(Q)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \operatorname{End}(M(Q'))$$

and $f^{N(d-2)} = 0$ by the induction hypothesis applied to Q'.
If Q does not have a point over k, then $Q \otimes k(x)$ does have a point (trivially) over the residue field of any point x. Therefore $f^{N(d-2)} \otimes k(x) = 0$ for every such point x by the induction hypothesis. (For this to hold for $\dim Q = 1$, we have to set $N(-1) = 1$.) Now apply Theorem 3.1 to $f^{N(d-2)}$. We obtain the conclusion that $f^{(d+1)N(d-2)} = 0$. Thus we can take $N(d) = (d+1)N(d-2)$ and the theorem is proved. $\square$

Remark 4.3. The proof shows that we can take $N(d) = (d+1)!!$ in the theorem.

REFERENCES

[1] Vladimir Chernousov, Stefan Gille, and Alexander Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties http://www.math.ucla.edu/~merkurev/papers/nilpotence9.dvi.

[2] William Fulton. *Intersection theory*. Springer-Verlag, Berlin, second edition, 1998.

[3] Uwe Jannsen. Motives, numerical equivalence, and semi-simplicity. *Invent. Math.*, 107(3):447–452, 1992.

[4] Nikita A. Karpenko. Characterization of minimal Pfister neighbors via Rost projectors. *J. Pure Appl. Algebra*, 160(2-3):195–227, 2001.

[5] Markus Rost. The motive of a Pfister form. http://www.math.ohio-state.edu/~rost/motive.html.

[6] Markus Rost. Chow groups with coefficients. *Documenta Math.*, Vol 1., (1996) 319–393.

[7] Alexander Vishik. Integral motives of quadrics. http://www.mpim-bonn.mpg.de/html/preprints/preprints.html.

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A Lambda-Graph System for the Dyck Shift and Its K-Groups

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Abstract. A property of subshifts is described that allows to associate to the subshift a distinguishied presentation by a compact Shannon graph. For subshifts with this property and for the resulting invariantly associated compact Shannon graphs and their λ-graph systems the term 'Cantor horizon' is proposed. The Dyck shifts are Cantor horizon. The C^*-algebras that are obtained from the Cantor horizon λ-graph systems of the Dyck shifts are separable, unital, nuclear, purely infinite and simple with UCT. The K-groups and Bowen-Franks groups of the Cantor horizon λ-graph systems of the Dyck shifts are computed and it is found that the K_0-groups are not finitely generated.

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0. Introduction

Let Σ be a finite alphabet. On the shift space $\Sigma^{\mathbb{Z}}$ one has the left-shift that sends a point $(\sigma_i)_{i\in\mathbb{Z}}$ into the point $(\sigma_{i+1})_{i\in\mathbb{Z}}$. In symbolic dynamics one studies the dynamical systems, called subshifts, that are obtained by restricting the shift to a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$. For an introduction to symbolic dynamics see [Ki] or [LM]. A finite word in the symbols of Σ is said to be admissible for the subshift $X \subset \Sigma^{\mathbb{Z}}$ if it appears somewhere in a point of X. A subshift is uniquely determined by its set of admissible words. Throughout this paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of all nonnegative integers and the set of all positive integers respectively.

A directed graph G whose edges are labeled by symbols in the finite alphabet Σ is called a Shannon graph if for every vertex u of G and for every $\alpha \in \Sigma$, G has at most one edge with initial vertex u and label α. We say that a Shannon

graph G presents a subshift X if every vertex of G has a predecessor and a successor and if the set of admissible words of X coincides with the set of label sequences of finite paths on G. To a Shannon graph G there is associated a topological Markov chain $M(G)$. The state space of $M(G)$ is the set of pairs (u, α), where u is a vertex of G and α is the label of an edge of G with initial vertex u. Here a transition from state (u, α) to state (v, β) is allowed if and only if v is the final vertex of the edge with initial vertex u and label α. For a vertex u of Shannon graph G we denote the forward context of u by $\Gamma^+(u)$. $\Gamma^+(u)$ is the set of sequences in $\Sigma^{\mathbb{N}}$ that are label sequences of infinite paths in G that start at the vertex u. We say that a Shannon graph G is forward separated if vertices of G, that have the same forward context, are identical. The Shannon graphs that we consider in this paper are forward separated. We always identify the vertices of a forward separated Shannon graph G with their forward contexts, and then use on the vertex set of G the topology that is given by the Hausdorff metric on the set of nonempty compact subsets of $\Sigma^{\mathbb{N}}$.
There is a one-to-one correspondence between forward-separated compact Shannon graphs G such that every vertex has a predecessor and a class of λ-graph systems [KM]. We recall that a λ-graph system is a directed labelled Bratteli diagram with an additional structure. We write the vertex set of a λ-graph system as

$$V = \bigcup_{n \in \mathbb{Z}_+} V_{-n}.$$

Every edge with initial vertex in V_{-n}, has its final vertex in $V_{-n+1}, n \in \mathbb{N}$. It is required that every vertex has a predecessor and every vertex except the vertex in V_0 has a successor. In this paper we consider λ-graph-systems that are forward separated Shannon graphs. Their additional structure is given by a mapping

$$\iota : \bigcup_{n \in \mathbb{N}} V_{-n} \to V$$

such that

$$\iota(V_{-n}) = V_{-n+1}, \qquad n \in \mathbb{N}$$

that is compatible with the labeling, that is, if u is the initial vertex of an edge with label α and final vertex v, then $\iota(u)$ is the intial vertex of an edge with label α and final vertex $\iota(v)$.
Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ there is a one-to-one correspondence between the compact forward separated Shannon graphs that present X, and the forward separated Shannon λ-graph systems that present X. To describe this one-to-one correspondence denote for a vertex v of a Shannon graph by v_n the set of initial segments of length n of the sequences in $V, n \in \mathbb{N}$. The λ-graph system that corresponds to the forward separated Shannon graph has as its set V_{-n} the set of $v_n, n \in \mathbb{Z}_+$, v a vertex of G, and if in G there is an edge with initial vertex u and final vertex v and label α then in the corresponding λ-graph system there is an edge with initial vertex u_n, final vertex $v_{n-1}, n \in \mathbb{N}$,

and label α, the mapping ι of the corresponding λ-graph system deleting last symbols.
λ-graph systems can be described by their symbolic matrix systems [Ma]

$$(\mathcal{M}_{-n,-n-1}, I_{-n,-n-1})_{n\in\mathbb{Z}_+}.$$

Here $\mathcal{M}_{-n,-n-1}$ is the symbolic matrix

$$[\mathcal{M}_{-n,-n-1}(u,v)]_{u\in V_{-n}, v\in V_{-n-1}}$$

that is given by setting $\mathcal{M}_{-n,-n-1}(u,v)$ equal to $\alpha_1 + \cdots + \alpha_k$ if in V there is an edge with initial vertex v and final vertex u with label $\alpha_i, i = 1,\dots,k$ and by setting $\mathcal{M}_{-n,-n-1}(u,v)$ equal to zero otherwise. $I_{-n,-n-1}$ is the zero-one matrix

$$[I_{-n,-n-1}(u,v)]_{u\in V_{-n}, v\in V_{-n-1}}$$

that is given by setting $I_{-n,-n-1}(u,v)$ equal to one if $\iota(v) = u$ and by setting $I_{-n,-n-1}(u,v)$ equal to zero otherwise. We remark that the time direction considered here is opposite to the time direction in [Ma]. For symbolic matrix systems there is a notion of strong shift equivalence [Ma] that extends the notion of strong shift equivalence for transition matrices of topological Markov shifts [Wi] and of the symbolic matrices of sofic systems [BK,N].
To a symbolic matrix system there are invariantly associated K-groups and Bowen-Franks groups [Ma]. To describe them, let

$$M_{n,n+1} = [M_{n,n+1}(u,v)]_{u\in V_{-n}, v\in V_{-n-1}}$$

be the nonnegative matrix that is given by setting $M_{n,n+1}(u,v)$ equal to zero if $\mathcal{M}_{-n,-n-1}(u,v)$ is zero, and by setting it equal to the number of the symbols whose sum is $\mathcal{M}_{-n,-n-1}(u,v)$ otherwise. We let $I_{n,n+1}, n\in\mathbb{Z}_+$ be $I_{-n,-n-1}$. Let $m(n)$ be the cardinal number of the vertex set V_{-n}. Also denote by $\bar{I}^t_{n,n+1}, n\in\mathbb{Z}_+$ the homomorphism from $\mathbb{Z}^{m(n)}/(M^t_{n-1,n} - I^t_{n-1,n})\mathbb{Z}^{m(n-1)}$ to $\mathbb{Z}^{m(n+1)}/(M^t_{n,n+1} - I^t_{n,n+1})\mathbb{Z}^{m(n)}$ that is induced by $I^t_{n,n+1}$. Then

$$K_0(M,I) = \varinjlim_n \{\mathbb{Z}^{m(n+1)}/(M^t_{n,n+1} - I^t_{n,n+1})\mathbb{Z}^{m(n)}, \bar{I}^t_{n,n+1}\},$$

$$K_1(M,I) = \varinjlim_n \{\mathrm{Ker}(M^t_{n,n+1} - I^t_{n,n+1}) \text{ in } \mathbb{Z}^{m(n)}, I^t_{n,n+1}\}.$$

Let $\mathbb{Z}_I$ be the group of the projective limit $\varprojlim_n \{\mathbb{Z}^{m(n)}, I_{n,n+1}\}$. The sequence $M_{n,n+1} - I_{n,n+1}, n\in\mathbb{Z}_+$ acts on it as an endomorphism, denoted by $M - I$. The Bowen-Franks groups $BF^I(M,I), i = 0,1$ are defined by

$$BF^0(M,I) = \mathbb{Z}_I/(M-I)\mathbb{Z}_I, \qquad BF^1(M,I) = \mathrm{Ker}(M-I) \quad \text{in} \quad \mathbb{Z}_I.$$

Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we use for $x = (x_i)_{i\in\mathbb{Z}} \in X$ notation like

$$x_{[j,k]} = (x_i)_{j\le i\le k},$$

and we set

$$X_{[j,k]} = \{x_{[j,k]} \mid x \in X\}, \qquad j,k \in \mathbb{Z}, \quad j < k,$$

using similar notations when indices range is infinite intervals. We denote the forward context of a point x^- in $X_{(-\infty,0]}$ by $\Gamma^+(x^-)$,

$$\Gamma^+(x^-) = \{x^+ \in X_{[1,\infty)} \mid (x^-, x^+) \in X\}.$$

The set $G(X) = \{\Gamma^+(x^-) \mid x^- \in X_{(-\infty,0]}\}$ is the vertex set of a forward separated Shannon graph that presents X. The λ-graph system of its closure was introduced in [KM] as the canonical λ-graph system of the subshift X. It is canonically associated to the subshift in the sense that a topological conjugacy of subshifts induces a strong shift equivalence of their canonical λ-graph systems.

For a subshift $X \subset \Sigma^{\mathbb{Z}}$ that is synchronizing [Kr] (or semisynchronizing [Kr]) one has an intrinsically defined shift invariant dense subset $P_s(X)$ of periodic points of X, and one has associated to X the presenting forward separated Shannon graph whose vertex set is the set of forward contexts $\Gamma^+(x)$, where x is left asymptotic to a point in $P_s(X)$. These Shannon graphs are canonically associated to the synchronizing (or semisynchronizing) subshift, in the sense that a topological conjugacy of subshifts induces a block conjugacy of the topological Markov chains of the Shannon graphs and also a strong shift equivalence [Ma] of the λ-graph systems of their closures. Prototype examples of semisynchronizing subshifts are the Dyck shifts that can be defined via the Dyck inverse monoids. The Dyck inverse monoid is the inverse monoid (with zero) with generators $\alpha_n, \beta_n, 1 \le n \le N$, and relations

$$\begin{aligned} \alpha_n\beta_n &= 1, \qquad 1 \le n \le N, \\ \alpha_n\beta_m &= 0, \qquad 1 \le n, m \le N, \quad n \ne m \end{aligned}$$

and the Dyck shift D_N is defined as the subshift $D_N \subset \{\alpha_n, \beta_n \mid 1 \le n \le N\}^{\mathbb{Z}}$, whose admissible words $(\gamma_i)_{0\le i\le I}$ satisfy the condition

$$\prod_{0\le i\le I} \gamma_i \ne 0.$$

In section 1 we introduce another class of subshifts $X \subset \Sigma^{\mathbb{Z}}$ with an intrinsically defined shift invariant dense set $P_{Ch}(X)$ of periodic points. Again the Dyck shifts serve here as prototypes. In the Dyck shift D_N the points in $P_{Ch}(D_N)$ are such that during a period there appears an event that has the potential to influence even the most distant future. In other words, a point $(x_i)_{i\in\mathbb{Z}}$ in

D_N with period p is in $P_{Ch}(D_N)$ if the normal form of the word $(x_i)_{0\le i<p}$ is a word in the symbols $\alpha_n, 1 \le n \le N$. One can view here the record of an infinite sequence of events as a point in a Cantor discontinuum. With this in mind, we call the subshifts in this class Cantor horizon subshifts. The presenting Shannon graph with vertex set the set of forward contexts $\Gamma^+(y^-)$, where y^- is negatively asymptotic to a point in $P_{Ch}(X)$, is canonically associated to the Cantor horizon subshift $X \subset \Sigma^{\mathbb{Z}}$, and so is the λ-graph system of its closure, that we call the Cantor horizon λ-graph system of X. The Cantor horizon λ-graph system of a Cantor horizon subshift is a sub λ-graph system of its canonical λ-graph system.

The K-groups and Bowen-Franks groups of the symbolic matrix system $(\mathcal{M}^{D_N}, I^{D_N})$ for the canonical λ-graph systems of the Dyck shifts $D_N, N \ge 2$ were computed in [Ma2]. These are

$$K_0(\mathcal{M}^{D_N}, I^{D_N}) \cong \sum_{n\in\mathbb{N}} \mathbb{Z}, \qquad K_1(\mathcal{M}^{D_N}, I^{D_N}) \cong 0,$$

$$BF^0(\mathcal{M}^{D_N}, I^{D_N}) \cong 0, \qquad BF^1(\mathcal{M}^{D_N}, I^{D_N}) \cong \prod_{n\in\mathbb{N}} \mathbb{Z}.$$

In section 3 we determine the symbolic matrix system $(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)})$ of the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of the Dyck shift D_2, and we compute its K-groups. Denoting the group of all $\mathbb{Z}$-valued continuous functions on the Cantor discontinuum $\mathfrak{C}$ by $C(\mathfrak{C}, \mathbb{Z})$ one has

$$K_0(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)}) \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}),$$

and one has

$$K_1(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)}) \cong 0.$$

One can construct simple C^*-algebras from irreducible λ-graph systems [Ma3]. A λ-graph system is said to be irreducible if for a sequence $v_{-n} \in V_{-n}, n \in \mathbb{Z}_+$ of vertices with $\iota(v_{-n}) = v_{-n+1}$ and for a vertex u, there exists an $N \in \mathbb{Z}_+$ such that there is a path from v_{-N} to u. It is said to be aperiodic if for a vertex u, there exists an $N \in \mathbb{Z}_+$ such that for all $v \in V_{-N}$ there exist paths from v to u. The Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of the Dyck shift D_2 is irreducible and moreover aperiodic. Hence the resulting C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ is simple and purely infinite whose K_0-group and K_1-group are the above groups $K_0(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)})$ and $K_1(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)})$ respectively (cf. [Ma3]). In section 4, we compute the Bowen-Franks groups of the symbolic matrix system $(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)})$.

In section 5, we consider the K-groups and Bowen-Franks groups of the Dyck shifts $D_N, N \ge 2$. Here one has

$$K_0(\mathcal{M}^{Ch(D_N)}, I^{Ch(D_N)}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}),$$
$$K_1(\mathcal{M}^{Ch(D_N)}, I^{Ch(D_N)}) \cong 0.$$

1. Subshifts with Cantor horizon lambda-graph systems

Denoting for a given subshift $X \subset \Sigma^{\mathbb{Z}}$, the right context of an admissible block $x_{[i,j]}, x \in X, i, j \in \mathbb{Z}, i \leq j$, by $\Gamma^+(x_{[i,j]})$,

$$\Gamma^+(x_{[i,j]}) = \{y^+ \in X_{(j,\infty)} \mid (x_{[i,j]}, y^+) \in X_{[i,\infty)}\}$$

and its left context by $\Gamma^-(x_{[i,j]})$,

$$\Gamma^-(x_{[i,j]}) = \{y^- \in X_{(-\infty,i)} \mid (y^-, x_{[i,j]}) \in X_{(-\infty,j]}\},$$

we set

$$\omega^+(x_{[i,j]}) = \bigcap_{y^- \in \Gamma^-(x_{[i,j]})} \{y^+ \in X_{(j,\infty)} \mid (y^-, x_{[i,j]}, y^+) \in X\}.$$

Lemma 1.1. *Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}, X \subset \Sigma^{\mathbb{Z}}$ be subshifts and let $\psi : \widetilde{X} \to X$ be a topological conjugacy. Let for some $L \in \mathbb{Z}_+$ ψ be given by a $(2L+1)$-block map Ψ and ψ^{-1} be given by a $(2L+1)$-block map $\widetilde{\Psi}$. Let $\widetilde{N} \in \mathbb{N}$, and let $\tilde{x} \in \widetilde{X}$ be such that*

$$\omega^+(\tilde{x}_{(-L-\widetilde{N},-L]}) = \omega^+(\tilde{x}_{(-L-\tilde{n},-L]}), \qquad \tilde{n} \geq \widetilde{N}. \tag{1.1}$$

Then for $x = \psi(\widetilde{x})$ and $N = \widetilde{N} + 2L$,

$$\omega^+(x_{(-N,0]}) = \omega^+(x_{(-n,0]}), \qquad n \geq N. \tag{1.2}$$

Proof. Let $n \geq N$, and let

$$y^+ \in \omega^+(x_{(-n,0]}). \tag{1.3}$$

Let

$$\tilde{y}^+ = \widetilde{\Psi}(x_{[-L,0]}, y^+).$$

One has

$$\tilde{y}^+ \in \omega^+(\tilde{x}_{(-\widetilde{N}-L,-L]}), \tag{1.4}$$

which implies that

$$y^+ \in \omega^+(x_{(-N,0]}),$$

confirming (1.2). We note that by (1.1) one has that (1.4) follows from

$$\tilde{y}^+ \in \omega^+(\tilde{x}_{(-\tilde{n}-L,-L]}),$$

which in turn follows from (1.3). $\square$

Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift and $P(X)$ be its set of periodic points. Denote by $P_a(X)$ the set of $x \in P(X)$ such that there is an $N \in \mathbb{N}$ such that

$$\omega^+(x_{(-N,0]}) = \omega^+(x_{(-n,0]}), \qquad n \geq N.$$

Lemma 1.2. *Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}, X \subset \Sigma^{\mathbb{Z}}$ be subshifts and let $\psi : \widetilde{X} \to X$ be a topological conjugacy. Then*

$$\psi(P_a(\widetilde{X})) = P_a(X).$$

Proof. Apply Lemma 1.1. □

By Lemma 1.1 the following property of a subshift $X \subset \Sigma^{\mathbb{Z}}$ is invariant under topological conjugacy : For $x \in X$ and $N \in \mathbb{N}$ such that

$$\omega^+(x_{(-N,0]}) = \omega^+(x_{(-n,0]}), \qquad n \geq N$$

there exists an $M \in \mathbb{N}$ such that

$$\omega^+(x_{[-M,0)}) = \omega^+(x_{[-m,0)}), \qquad m \geq M.$$

For a subshift $X \subset \Sigma^{\mathbb{Z}}$ with this property we consider the subgraph $G_{Ch}(X)$ of $G(X)$ with vertices $\Gamma^+(u^-)$ where $u^- \in X_{(-\infty,0]}$ is negatively asymptotic to a point in $P_{Ch}(X) = P(X)\backslash P_a(X)$. If here $G_{Ch}(X)$ presents X then we say that X is a Cantor horizon subshift, and we call the λ-graph system of the closure of $G_{Ch}(X)$ the Cantor horizon λ-graph system of X. By Lemma 1.2 the Cantor horizon property is an invariant of topological conjugacy and the Cantor horizon λ-graph system is invariantly associated to the Cantor horizon subshift.

2. The Dyck shift

We consider the Dyck shift D_2 with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_0, \alpha_1\}, \Sigma^+ = \{\beta_0, \beta_1\}$. A periodic point x of D_2 with period p is not in $P_a(D_2)$ precisely if for some $i \in \mathbb{Z}$ the normal form of the word $(x_{i+q})_{0 \leq q < p}$ is a word in the symbols of Σ^-, in other words, if the multiplier of x in the sense of [HI] is negative. We also note that periodic points with negative multipliers give rise to the same irreducible component of $G_{Ch}(D_2)$ precisely if they have the same multiplier.

We describe the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of D_2: The vertices at level l are given by the words of length l in the symbols of Σ^-. The mapping ι deletes the first symbol of a word. A word $(\alpha_{i(n)})_{1 \leq n \leq l}$ accepts β_i precisely if $i(l) = i, i = 0, 1$, effecting a transition to the word $(\alpha_{i(n)})_{1 \leq n < l}$, and it accepts α_i, effecting a transition to the word $(\alpha_{i(n)})_{2 \leq n \leq l}$. The forward context of the word $a = (\alpha_{i(n)})_{1 \leq n \leq l}$ contains precisely all words $c = (\gamma_n)_{1 \leq n \leq l}$ in symbols of Σ such that (a, c) is admissible for D_2. In describing the Cantor horizon symbolic matrix system $(\mathcal{M}, I)$ of the Dyck shift and the resulting nonnegative matrix system (M, I) we use the reverse lexcographic order on the words in the symbols in Σ^-, that is, we assign to a word $(\alpha_{i(n)})_{1 \leq n \leq l} \in \Sigma^{-[1,l]}$ the number

$$\sum_{1 \leq n \leq l} i(n) 2^{n-1}.$$

One has then

$$\mathcal{M}_{0,-1} = [\beta_0 + \alpha_0 + \alpha_1,\ \ \beta_1 + \alpha_0 + \alpha_1] = [\alpha_0 + \alpha_1 + \beta_0,\ \ \alpha_0 + \alpha_1 + \beta_1],$$
$$I_{0,-1} = [1,\ \ 1].$$

For $l \in \mathbb{Z}_+$ and $a \in \{\alpha_0, \alpha_1, \beta_0, \beta_1\}$, let $I_l(a)$ be the $2^l \times 2^l$ diagonal matrix with diagonal entries a, and $\mathcal{S}_l(a)$ be the $2^{l-1} \times 2^{l+1}$ matrix $[\mathcal{S}_l(a)(i,j)]_{1 \le i \le 2^{l-1}, 1 \le j \le 2^{l+1}}$ where $\mathcal{S}_l(a)(i,j)$ is a for $j = 4i, 4i-1, 4i-2, 4i-3$, and is otherwise zero.

PROPOSITION 2.1. *For $l = 1, 2, \dots,$ the matrix $\mathcal{M}_{-l,-l-1}$ is a $2^l \times 2^{l+1}$ rectangular matrix that is given as the block matrix:*

$$\mathcal{M}_{-l,-l-1} = \begin{bmatrix} \mathcal{S}_l(\alpha_0) \\ \mathcal{S}_l(\alpha_1) \end{bmatrix} + [\, I_l(\beta_0) \ \mid \ I_l(\beta_1) \,] \tag{2.1}$$

and

$$I_{-l,-l-1}(i,j) = \begin{cases} 1 & (j = 2i-1, 2i), \\ 0 & \textit{elsewhere}. \end{cases} \tag{2.2}$$

Proof. The first summand in (2.1) describes the transitions that arise when a vertex accepts a symbol in Σ^+. The second summand arises from the transitions that arise when a vertex accepts a symbol in Σ^-, the arrangement of the components of the matrix as well as (2.2) being a component of the ordering of the vertices at level l and $l-1$. $\square$

We note that the λ-graph systems of the closures of the irreducible components of $G_{Ch}(X)$ are identical.

PROPOSITION 2.2. *The λ-graph system $\mathfrak{L}^{Ch(D_2)}$ for $(\mathcal{M}, I)$ is irreducible and aperiodic.*

Proof. Let $V_{-l}, l \in \mathbb{Z}_+$ be the vertex set of the λ-graph system $\mathfrak{L}^{Ch(D_2)}$. For any vertex u of V_{-l}, there are labeled edges from each of the vertices in V_{-2l} to the vertex u. This implies that $(\mathcal{M}, I)$ is aperiodic. $\square$

3. COMPUTATION OF THE K-GROUPS

Let $M_{l,l+1}$ for $l \in \mathbb{Z}_+$ be the nonnegative matrix obtained from $\mathcal{M}_{-l,-l-1}$ by setting all the symbols of the components of $\mathcal{M}_{-l,-l-1}$ equal to 1. The matrix $I_{l,l+1}$ for $l \in \mathbb{Z}_+$ is defined to be $I_{-l,-l-1}$. For $l > 1$ and $1 \le i \le 2^{l-2}$ let $a_i(l) = [a_i(l)_n]_{n=1}^{2^l}$ be the vector that is given by

$$a_i(l)_n = \begin{cases} 1 & (n = 4i-3, 4i-2, 4i-1, 4i), \\ 0 & \text{elsewhere}. \end{cases}$$

Define for $1 < l \in \mathbb{N}$ E_l as the $2^l \times 2^l$-matrix whose $i-$th column vector and whose $(2^{l-1}+i)-$th column vector are both equal to $a_i(l), 1 \le i \le 2^{l-2}$, the other column vectors being equal to zero vectors.
For $l > 1$ and $1 \le i \le 2^{l-1}$ let $b_i(l) = [b_i(l)_n]_{n=1}^{2^l}$ be the vector that is given by

$$b_i(l)_n = \begin{cases} 1 & (n = 2i-1, 2i), \\ 0 & \text{elsewhere.} \end{cases}$$

Define for $1 < l \in \mathbb{N}$ F_l as the $2^l \times 2^l$-matrix whose $i-$th column vector is equal to $b_i(l), 1 \le i \le 2^{l-1}$, the other column vectors being equal to zero vectors.
One has

$$I_{l,l+1}^t E_l = E_{l+1} I_{l,l+1}^t, \qquad I_{l,l+1}^t F_l = F_{l+1} I_{l,l+1}^t, \qquad l > 1. \tag{3.1}$$

Let I_l denote the unit matrix of size 2^l. Define a $2^l \times 2^{l-1}$ matrix $H_{l,l-1}$ by setting

$$H_{l,l-1} = \begin{bmatrix} I_{l-1} \\ -I_{l-1} \end{bmatrix}, \qquad l > 1.$$

Lemma 3.1. *For $l > 1$ one has*

$$(E_l - F_l) H_{l,l-1} = -I_{l-1,l}^t.$$

Proof. One has

$$E_l H_{l,l-1} = 0, \qquad F_l H_{l,l-1} = I_{l-1,l}^t.$$

□

Set $y_1(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $y_2(2) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $y_3(2) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $y_4(2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and define inductively for $l > 2$ vectors $y_i(l)$, $1 \le i \le 2^l$, where

$$y_i(l) = I_{l-1,l}^t y_i(l-1), \qquad 1 \le i \le 2^{l-1} \tag{3.2}$$

and

$$y_i(l) = H_{l,l-1} y_{i-2^{l-2}}(l-1), \qquad 2^{l-1} < i \le 2^{l-1} + 2^{l-2} \tag{3.3}$$

where one defines the vectors

$$y_i(l) = [y_i(l)_n]_{n=1}^{2^l}, \qquad 2^{l-1} + 2^{l-2} < i \le 2^l$$

by setting
(3.4)

$$y_{2^{l-1}+2^{l-2}+i}(l)_n = \begin{cases} 1 & (n = 4i-1, 4i, 2^{l-1}+2i), \\ -1 & (n = 2^{l-1}+4i-1, 2^{l-1}+4i), \\ 0 & \text{elsewhere,} \end{cases} \qquad 1 \le i \le 2^{l-3}$$

and by setting

$$(3.5)\quad y_{2^{l-1}+2^{l-2}+2^{l-3}+i}(l)_n = \begin{cases} 1 & (n = 2^{l-1} + 2^{l-2} + 2i), \\ 0 & elsewhere, \end{cases} \qquad 1 \le i \le 2^{l-3}.$$

Define for $l > 2$, T_l as the $2^l \times 2^l$-matrix whose column vectors $y_i(l), 1 \le i \le 2^l$. Here $y_1(l)$ has all components equal to 1, that is, $y_1(l)$ is the eigenvector of $E_l - F_l$ for the eigenvalue 1. Also the vectors

$$y_i(l), \qquad 2^{l-1} + 2^{l-2} < i \le 2^l$$

are linearly independent vectors in the kernel of $E_l - F_l$ and one sees from (3.1), (3.2), (3.3), (3.4) and (3.5) that T_l is invertible and that $T_l^{-1}(E_l - F_l)T_l$ is a matrix that is in a normal form. This normal form is a Jordan form in the sense that by conjugation with a suitable permutation matrix, followed by a conjugation with a suitable diagonal matrix whose entries are 1 or -1, the matrix assumes a Jordan form with Jordan blocks arranged along the diagonal. There will be one Jordan block of length 1 for the eigenvalue 1, and there will be 2^{l-1} Jordan blocks for the eigenvalue 0, and if one lists these by decreasing length then the $k-$th Jordan block for the eigenvalue 0 has length $l - \mu(k)$ where $\mu(k)$ be given by $2^{\mu(k)-1} < k \le 2^{\mu(k)}, 1 \le k \le 2^{l-2}$.
By an elementary column operation we will mean the addition or subtraction of one column vector from another or the exchange of two column vectors.

Lemma 3.2. *Let $l \in \mathbb{N}$ and let K be a $2^l \times 2^l$-matrix with column vectors $z_i, 1 \le i \le 2^l$,*

$$z_i = b_i(l), \qquad 1 \le i \le 2^{l-1},$$

and column vectors

$$z_{2^{l-1}+i} = [z_{2^{l-1}+i,n}]_{n=1}^{2^l}, \qquad 1 \le i \le 2^{l-1}$$

such that

$$\begin{aligned} &z_{2^{l-1}+i,2j-1} = z_{2^{l-1}+i,2j} = 0, \qquad 1 \le j < i, \\ &z_{2^{l-1}+i,2i-1} = 0, \qquad z_{2^{l-1}+i,2i} = 1, \\ &z_{2^{l-1}+i,2j-1} = 0, \qquad z_{2^{l-1}+i,2j} \in \{-1, 0, 1\}, \qquad i < j \le 2^{l-1}. \end{aligned}$$

Then K can be converted into the unit matrix by a sequence of elementary column operations.

Proof. Let the vector $c_j = [c_{j,n}]_{n=1}^{2^l}, 1 \le j \le 2^{l-1}$, be given by

$$c_{j,n} = \begin{cases} 1 & (n = 2j), \\ 0 & \text{elsewhere.} \end{cases}$$

and denote by $K[j], 1 \leq j \leq 2^{l-1}$ the matrix that is obtained by replacing in the matrix K the last j column vectors by the vectors $c_{2^{l-1}-i}, j \geq i \geq 1$. $K[1]$ is equal to K and $K[j], 1 < j \leq 2^{l-1}$, can be obtained from $K[j-1]$ by subtracting from and adding to the $(2^l - j)$-th column appropriate selections of the the $(2^l - i)$-th columns, $1 \leq i \leq j$. $K[2^{l-1}]$ has as its first 2^{l-1} column vectors the vectors $b_i(l), 1 \leq i \leq 2^{l-1}$, and as its last 2^{l-1} column vectors the vectors $c_i(l), 1 \leq i \leq 2^{l-1}$, and can be converted into the unit matrix by elementary column operations. $\square$

LEMMA 3.3. *Let $l \in \mathbb{N}$ and let K be a $2^l \times 2^l$-matrix with column vectors $z_i, 1 \leq i \leq 2^l$,*

$$z_i = b_i(l), \qquad 1 \leq i \leq 2^{l-1},$$

and column vectors

$$z_{2^{l-1}+i} = [z_{2^{l-1}+i,n}]_{n=1}^{2^l}, \qquad 1 < i \leq 2^{l-1}$$

such that

$$(z_{2^{l-1}+i,2j-1}, z_{2^{l-1}+i,2j}) \in \{(0,0),(1,1)\}, \qquad 1 \leq j < i,$$
$$z_{2^{l-1}+i,2i-1} = 0, \qquad z_{2^{l-1}+i,2i} = 1,$$
$$(z_{2^{l-1}+i,2j-1}, z_{2^{l-1}+i,2j}) \in \{(-1,-1),(0,0),(1,1),(0,1),(0,-1)\}, \ i < j < 2^{l-1}.$$

Then K can be converted into the unit matrix by a sequence of elementary column operations.

Proof. For all $i, 2^{l-1} < i \leq 2^l$, one subtracts from the i-th column of K and adds to the i-th column of K appropriate selections of the first 2^{l-1} columns of K to obtain a matrix to which Lemma 3.2 applies. $\square$

PROPOSITION 3.4. *The matrix T_l is unimodular.*

Proof. The matrix T_2 can be converted into the unit matrix by elementary column operations. The proof is by induction on l. Assume that the matrix $T_{l-1}, l > 2$ can be converted into the unit matrix by a sequence of elementary column operations. Then by (3.2) the matrix T_l can be converted by a sequence of elementary column operations into a matrix whose first 2^{l-1} column vectors are the vectors $b_i(l), 1 \leq i \leq 2^{l-1}$ and whose last 2^{l-1} column vectors are those of the matrix T_l, and by (3.2),(3.3) and (3.4) Lemma 3.3 is applicable to this matrix. $\square$

Define a $2^l \times 2^l$ matrix L_l by setting

$$L_l = I_l + E_l - F_l, \qquad l > 1.$$

Denote by $0_{k,l}$ the $2^k \times 2^l$ matrix with entries 0's. Also define permutation matrices $P_l(i,j), 1 \leq i,j \leq 2^l, l > 1$, by

$$P_l(i, 2^l - i + 1) = 1, \qquad 1 \leq i \leq 2^l,$$

and set

$$B_{l+1} = \begin{bmatrix} L_l & 0_{l,l} \\ P_l L_l P_l & 0_{l,l} \end{bmatrix}.$$

Lemma 3.5. $B_{l+1} = [\, M^t_{l,l+1} - I^t_{l,l+1} \mid 0_{l+1,l} \,], \quad l > 1.$

Proof. This follows from Proposition 2.1. □

Define $2^{l+1} \times 2^{l+1}$ matrices $J(l+1)$ and U_{l+1} by setting

$$J(l+1) = \begin{bmatrix} T_l^{-1} L_l T_l & 0_{l,l} \\ 0_{l,l} & 0_{l,l} \end{bmatrix}, \qquad U_{l+1} = \begin{bmatrix} T_l & 0_{l,l} \\ P_l T_l & I_l \end{bmatrix}.$$

Lemma 3.6. U_{l+1} *is unimodular and*

$$U_{l+1}^{-1} B_{l+1} U_{l+1} = J(l+1), \qquad l > 1.$$

Proof. One has

$$U_{l+1}^{-1} = \begin{bmatrix} T_l^{-1} & 0_{l,l} \\ -P_l & I_l \end{bmatrix}$$

and further

$$\begin{bmatrix} T_l^{-1} & 0_{l,l} \\ -P_l & I_l \end{bmatrix} \begin{bmatrix} L_l & 0_{l,l} \\ P_l L_l P_l & 0_{l,l} \end{bmatrix} \begin{bmatrix} T_l & 0_{l,l} \\ P_l T_l & I_l \end{bmatrix} = \begin{bmatrix} T_l^{-1} L_l T_l & 0_{l,l} \\ 0_{l,l} & 0_{l,l} \end{bmatrix}.$$

□

Define a $2^{l+1} \times 2^l$ matrix $G_{l+1,l}$ by setting

$$G_{l+1,l} = \begin{bmatrix} I_l \\ 0_{l,l} \end{bmatrix}, \qquad l > 1.$$

One has

$$I^t_{l,l+1} T_l = T_{l+1} G_{l+1,l}, \tag{3.6}$$

$$I^t_{l,l+1} P_l T_l = P_{l+1} T_{l+1} G_{l+1,l}. \tag{3.7}$$

Define a $2^{l+1} \times 2^l$ matrix $J_{l+1,l}$ by setting

$$J_{l+1,l} = \begin{bmatrix} G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix}, \qquad l > 1.$$

Lemma 3.7. $I^t_{l,l+1} U_l = U_{l+1} J_{l+1,l}, \quad l > 1.$

Proof. From (3.6) and (3.7), it follows that

$$\begin{aligned} \begin{bmatrix} I^t_{l-1,l} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix} \begin{bmatrix} T_{l-l} & 0_{l,l-1} \\ P_{l-1} T_{l-1} & I_{l-1} \end{bmatrix} &= \begin{bmatrix} T_l G_{l,l-1} & 0_{l,l-1} \\ P_l T_l G_{l,l-1} & I^t_{l-1,l} \end{bmatrix} \\ &= \begin{bmatrix} T_l & 0_{l,l} \\ P_l T_l & I_l \end{bmatrix} \begin{bmatrix} G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix}. \end{aligned}$$

□

Lemma 3.8.

$$J_{l+1,l}J(l) = J(l+1)J_{l+1,l}, \qquad l > 1.$$

Proof. By (3.1)

$$I^t_{l-1,l}L_{l-1} = L_l I^t_{l-1,l}$$

and by (3.6)

$$T^{-1}_{l+1}I^t_{l,l+1}T_l = G_{l+1,l}T^{-1}_l.$$

Therefore by (3.6)

$$\begin{aligned} G_{l,l-1}T^{-1}_{l-1}L_{l-1}T_{l-1} &= T^{-1}_l I^t_{l-1,l}L_{l-1}T_{l-1} \\ &= T^{-1}_l L_l I^t_{l-1,l}T_{l-1} \\ &= T^{-1}_l L_l T_l G_{l,l-1} \end{aligned}$$

and therefore

$$\begin{aligned} &\begin{bmatrix} G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix} \begin{bmatrix} T^{-1}_{l-1}L_{l-1}T_{l-1} & 0_{l,l-1} \\ 0_{l,l-1} & 0_{l,l-1} \end{bmatrix} \\ &\qquad = \begin{bmatrix} G_{l,l-1}T^{-1}_{l-1}L_{l-1}T_{l-1} & 0_{l,l-1} \\ 0_{l,l-1} & 0_{l,l-1} \end{bmatrix} \\ &\qquad = \begin{bmatrix} T^{-1}_l L_l T_l G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & 0_{l,l-1} \end{bmatrix} \\ &\qquad = \begin{bmatrix} T^{-1}_l L_l T_l & 0_{l,l-1} \\ 0_{l,l-1} & 0_{l,l-1} \end{bmatrix} \begin{bmatrix} G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix}. \end{aligned}$$

□

By the preceding lemma, the matrix $J_{l+1,l}$ induces a homomorphism

$$\bar{J}_{l+1,l} : \mathbb{Z}^{2^l}/J(l)\mathbb{Z}^{2^l} \to \mathbb{Z}^{2^{l+1}}/J(l+1)\mathbb{Z}^{2^{l+1}}$$

and by Proposition 3.4 and Lemma 3.6 the matrix U_l, as $B_l\mathbb{Z}^{2^l} = (M^t_{l-1,l} - I^t_{l-1,l})\mathbb{Z}^{2^{l-1}}$, induces a homomorphism

$$\bar{U}_l : \mathbb{Z}^{2^l}/J(l)\mathbb{Z}^{2^l} \to \mathbb{Z}^{2^l}/B_l\mathbb{Z}^{2^l}.$$

Lemma 3.9. *The diagram :*

$$\begin{array}{ccc} \mathbb{Z}^{2^l}/(M^t_{l-1,l} - I^t_{l-1,l})\mathbb{Z}^{2^{l-1}} & \xrightarrow{\bar{I}^t_{l,l+1}} & \mathbb{Z}^{2^{l+1}}/(M^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2^l} \\ {\scriptstyle \bar{U}_l}\Big\uparrow & & {\scriptstyle \bar{U}_{l+1}}\Big\uparrow \\ \mathbb{Z}^{2^l}/J(l)\mathbb{Z}^{2^l} & \xrightarrow{\bar{J}_{l+1,l}} & \mathbb{Z}^{2^{l+1}}/J(l+1)\mathbb{Z}^{2^{l+1}} \end{array}$$

is commutative.

Proof. Apply Lemma 3.7. $\square$

Define a diagonal matrix $D(l)$ by setting

$$D(l) = \mathrm{diag}(2, \overbrace{1, 1, \dots, 1}^{2^l - 1}), \qquad l \in \mathbb{N}.$$

As $J(l+1)\mathbb{Z}^{2^{l+1}} = D(l)\mathbb{Z}^{2^l} \oplus (\overbrace{0, \dots, 0}^{2^l})$ and hence $\mathbb{Z}^{2^{l+1}}/J(l+1)\mathbb{Z}^{2^{l+1}} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2^l}$, through the map $\varphi_l \oplus \mathrm{id}$ where $\varphi_l : \mathbb{Z}^{2^l}/D(l)\mathbb{Z}^{2^l} \to \mathbb{Z}/2\mathbb{Z}$ is defined by $\varphi_l([(x_i)_{i=1}^{2^l}]) = [x_1] \pmod 2$, we have

Proposition 3.10. *The following diagram is commutative:*

$$\begin{array}{ccc}
\mathbb{Z}^{2^l}/J(l)\mathbb{Z}^{2^l} & \xrightarrow{\bar{J}_{l+1,l}} & \mathbb{Z}^{2^{l+1}}/J(l+1)\mathbb{Z}^{2^{l+1}} \\
{\scriptstyle \varphi_{l-1}\oplus \mathrm{id}}\big\downarrow & & {\scriptstyle \varphi_l \oplus \mathrm{id}}\big\downarrow \\
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2^{l-1}} & \xrightarrow{\mathrm{id}\oplus I^t_{l-1,l}} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2^l}.
\end{array}$$

Corollary 3.11.

$$\varinjlim_l \{\mathbb{Z}^{2^{l+1}}/(M^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2^l}, \bar{I}^t_{l,l+1}\} \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}).$$

Proof. As the group of the inductive limit: $\varinjlim_l \{I^t_{l,l+1} : \mathbb{Z}^{2^l} \to \mathbb{Z}^{2^{l+1}}\}$ is isomorphic to $C(\mathfrak{C}, \mathbb{Z})$, we get the assertion. $\square$

Theorem 3.12.

$$K_0(M, I) \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}), \qquad K_1(M, I) \cong 0.$$

By Proposition 2.2, the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of D_2 is aperiodic so that the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ associated with the λ-graph system $\mathfrak{L}^{Ch(D_2)}$ is simple and purely infinite ([Ma3;Proposition 4.9]). It satisfies the UCT by [Ma3;Proposition 5.6] (cf.[Bro],[RS]). By [Ma3;Theorem 5.5], the K-groups $K_i(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}})$ are isomorphic to the K-groups $K_i(M, I)$ so that we get

Corollary 3.13. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ associated with the λ-graph system $\mathfrak{L}^{Ch(D_2)}$ is separable, unital, nuclear, simple, purely infinite with UCT such that*

$$K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong 0.$$

4. Computation of the Bowen-Franks groups

We compute next the Bowen-Franks groups $BF^0(M,I)$ and $BF^0(M,I)$.

Lemma 4.1. $\mathrm{Ext}^1_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}) \cong 0$.

Proof. From [Ro] (cf.[Sch;Theorem 1.3]) one has that for an inductive sequence $\{G_i\}$ of abelian groups there exists a natural short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i,\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\varinjlim G_i,\mathbb{Z}) \to \varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i,\mathbb{Z}) \to 0.$$

The lemma follows therefore from $C(\mathfrak{C},\mathbb{Z}) = \varinjlim_l \{I^t_{l,l+1} : \mathbb{Z}^{2^l} \to \mathbb{Z}^{2^{l+1}}\}$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^{2^l},\mathbb{Z}) = \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{2^l},\mathbb{Z}) = 0$. $\square$

As in [Ma;Theorem 9.6], one has the following lemma that provides a universal coefficient type theorem.

Lemma 4.2. *For $i = 0,1$ there exists an exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_i(M,I),\mathbb{Z}) \to BF^i(M,I) \to \mathrm{Hom}_{\mathbb{Z}}(K_{i+1}(M,I),\mathbb{Z}) \to 0.$$

Theorem 4.3.

$$BF^0(M,I) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. By Theorem 3.12 and by Lemma 4.2 one has

$$BF^0(M,I) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

As $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, the theorem follows from Lemma 4.1. $\square$

Theorem 4.4.

$$BF^1(M,I) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

Proof. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})$ is trivial. Therefore by Theorem 3.12

$$\mathrm{Hom}_{\mathbb{Z}}(K_0(M,I)) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

Since the group $K_1(M,I)$ is trivial, by Lemma 4.2, one gets

$$BF^1(M,I) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

$\square$

As the Bowen-Franks groups $BF^0(M,I)$ and $BF^1(M,I)$ are isomorphic to the Ext-groups $\mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}})(= \mathrm{Ext}(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}))$ and $\mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}})(= \mathrm{Ext}(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}} \otimes C_0(\mathbb{R}))$ for the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ (cf. [Ma3]), we obtain

Corollary 4.5.

$$\mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

5. General Dyck shifts

One can extend the preceding results for the Dyck shift D_2 to the general Dyck shifts D_N with $2N$ symbols $\alpha_n, \beta_n, 1 \le n \le N$, for $N > 2$, generalizing the previous discussions for the case of $N = 2$. We will briefly explain this. We consider the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of D_N as in the previous case and write its symbolic matrix system $(\mathcal{M}^{Ch(D_N)}_{-l,-l-1}, I^{Ch(D_N)}_{-l,-l-1})$ as $(\mathcal{M}_{l,l+1}, I_{l,l+1})$. We define the nonnegative matrices $M_{l,l+1}, I_{l,l+1}, l \in \mathbb{Z}_+$ in a similar way. The size of the matrices $M_{l,l+1}, I_{l,l+1}$ is $N^l \times N^{l+1}$. Let $I_l^{(N)}$ be the unit matrix with size N^l. For $l > 1$ and $1 \le i \le N^{l-2}$ let $a_i^{(N)}(l) = [a_i^{(N)}(l)_n]_{n=1}^{N^l}$ be the vector that is given by

$$a_i^{(N)}(l)_n = \begin{cases} 1 & (N^2(i-1)+1 \le n \le N^2 i), \\ 0 & \text{elsewhere.} \end{cases}$$

Define for $1 < l \in \mathbb{N}$ $E_l^{(N)}$ as the $N^l \times N^l$-matrix whose $i-$th, $(N^{l-1}+i)-$th, $(2N^{l-1}+i)-$th, $\dots$, $((N-1)N^{l-1}+i)-$th column vectors are all equal to $a_i^{(N)}(l), 1 \le i \le N^{l-2}$, the other column vectors being equal to zero vectors. Let $b_i^{(N)}(l) = [b_i^{(N)}(l)_n]_{n=1}^{N^l}$ be the vector that is given by

$$b_i^{(N)}(l)_n = \begin{cases} 1 & (N(i-1)+1 \le n \le N i), \\ 0 & \text{elsewhere.} \end{cases}$$

Define for $1 < l \in \mathbb{N}$ $F_l^{(N)}$ as the $N^l \times N^l$-matrix whose $i-$th column vector is equal to $b_i^{(N)}(l), 1 \le i \le N^{l-1}$, the other column vectors being equal to zero vectors.

Define a $N^l \times N^l$ matrix by

$$L_l^{(N)} = I_l^{(N)} + E_l^{(N)} - F_l^{(N)}, \qquad l \in \mathbb{N}.$$

Also define permutation matrices $P_l^{(N),k}, k = 1, 2, \dots, N-1$, by

$$P_l^{(N),k}(i, N^l - i + 1 - (k-1)) = 1, \qquad 1 \le i \le N^l,$$

where $1 \le N^l - i + 1 - (k-1) \le N^l$ is taken mod N^l. Denote by $0_{k,l}$ the $N^k \times N^l$ matrix with entries 0's.

We define an $N^{l+1} \times N^{l+1}$ matrix $B_{l+1}^{(N)}$ by

$$B_{l+1}^{(N)} = [M_{l,l+1}^t - I_{l,l+1}^t \mid 0_{l+1,l}],$$

that is written as the block matrix

$$\begin{bmatrix} L_l^{(N)} & 0_{l,l} \\ P_l^{(N),1} L_l^{(N)} P_l^{(N),1} & 0_{l,l} \\ \vdots & \vdots \\ P_l^{(N),N-1} L_l^{(N)} P_l^{(N),N-1} & 0_{l,l} \end{bmatrix}.$$

By an argument that is similar to the one of the previous sections, one can conclude then

Theorem 5.1.

$$K_0(M,I) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{C},\mathbb{Z}), \qquad K_1(M,I) \cong 0,$$
$$BF^0(M,I) \cong \mathbb{Z}/N\mathbb{Z}, \qquad BF^1(M,I) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

Corollary 5.2. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ associated with the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of D_N is separable, unital, nuclear, simple, purely infinite such that*

$$K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{C},\mathbb{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong 0,$$
$$\mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z}, \qquad \mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

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References

[BK] M. Boyle and W. Krieger, *Almost Markov and shift equivalent sofic systems*, Proceedings of Maryland Special Year in Dynamics 1986-87, Springer Verlag Lecture Notes in Math. **1342** (1988), 33–93.

[Bro] L. G. Brown, *The universal coefficient theorem for Ext and quasidiagonality*, Operator Algebras and Group Representation, Pitman Press **17** (1983), 60–64.

[HI] T. Hamachi and K. Inoue, *Embedding of shifts of finite type into the Dyck shift, preprint 2002.*

[Ki] B.P. Kitchens, *Symbolic dynamics*, Springer-Verlag, Berlin, Heidelberg and New York, 1998.

[Kr] W. Krieger, *On subshifts and topological Markov chains*, Numbers, information and complexity (Bielefeld 1998), Kluwer Acad. Publ. Boston MA (2000), 453–472.

[KM] W. Krieger and K. Matsumoto, *Shannon graphs, subshifts and lambda-graph systems*, J. Math. Soc. Japan **54** (2002), 877–899.

[LM] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[Ma] K. Matsumoto, *Presentations of subshifts and their topological conjugacy invariants*, Documenta Math. **4** (1999), 285-340.

[Ma2] K. Matsumoto, *K-theoretic invariants and conformal measures of the Dyck subshift*, preprint, 1999.

[Ma3] K. Matsumoto, *C^*-algebras associated with presentations of subshifts*, Documenta Math. **7** (2002), 1-30.

[N] M. Nasu, *Topological conjugacy for sofic shifts*, Ergodic Theory Dynam. Systems **6** (1986), 265–280.

[Ro] J. E. Roos, *Sur les dérivés de $\varprojlim$. Applications*, C. R. Acad. Sci. Paris **252** (1961), 3702–3704.

[RS] J. Rosenberg and C. Schochet, *The Künneth Theorem and the Universal Coefficient Theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431–474.

[Sch] C. Schochet, *The UCT, the Milnor sequence, and a canonical decomposition of the Kasparov group*, K-Theory **10** (1996), 49–72.

[Wi] R. F. Williams, *Classification of subshifts of finite type*, Ann. Math. **98** (1973), 120–153, erratum, Ann. Math. $99(1974), 380-381$.

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On the Height of Calabi-Yau Varieties in Positive Characteristic

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Abstract. We study invariants of Calabi-Yau varieties in positive characteristic, especially the height of the Artin-Mazur formal group. We illustrate these results by Calabi-Yau varieties of Fermat and Kummer type.

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1. Introduction

The large measure of attention that complex Calabi-Yau varieties drew in recent years stands in marked contrast to the limited attention for their counterparts in positive characteristic. Nevertheless, we think these varieties deserve a greater interest, especially since the special nature of these varieties lends itself well for excursions into the largely unexplored territory of varieties in positive characteristic. In this paper we mean by a Calabi-Yau variety a smooth complete variety of dimension n over a field with $\dim H^i(X,O_X)=0$ for $i=1,\ldots,n-1$ and with trivial canonical bundle. We study some invariants of Calabi-Yau varieties in characteristic $p>0$, especially the height h of the Artin-Mazur formal group for which we prove the estimate $h\leq h^{1,n-1}+1$ if $h\neq\infty$. We show how this invariant is related to the cohomology of sheaves of closed forms.

It is well-known that K3 surfaces do not possess non-zero global 1-forms. The analogous statement about the existence of global i-forms with $i=1$ and $i=n-1$ on a n-dimensional Calabi-Yau variety is not known and might well be false in positive characteristic. We show that for a Calabi-Yau variety of dimension ≥ 3 over an algebraically closed field k of characteristic $p>0$ with no non-zero global 1-forms there is no p-torsion in the Picard variety and $\mathrm{Pic}/p\mathrm{Pic}$ is isomorphic to NS/pNS with NS the Néron-Severi group of X. If in addition X does not have a non-zero global 2-form then $NS/pNS\otimes_{\mathbf{F}_p}k$ maps injectively into $H^1(X,\Omega^1_X)$. This yields the estimate $\rho\leq h^{1,1}$ for the Picard number. We also study Calabi-Yau varieties of Fermat type and of Kummer type to illustrate the results.

2. The Height of a Calabi-Yau Variety

The most conspicuous invariant of a Calabi-Yau variety X of dimension n in characteristic $p > 0$ is its *height*. There are several ways to define it, using crystalline cohomology or formal groups. In the latter setting one considers the functor $F_X^r : \mathrm{Art} \to \mathrm{Ab}$ defined on the category of local Artinian k-algebras with residue field k by

$$F_X^r(S) := \mathrm{Ker}\{H^r_{\mathrm{et}}(X \times S, \mathbb{G}_m) \longrightarrow H^r_{\mathrm{et}}(X, \mathbb{G}_m)\}.$$

According to a theorem of Artin and Mazur [2], for a Calabi-Yau variety X and $r = n$ this functor is representable by a smooth formal group Φ_X of dimension 1 with tangent space $H^n(X, O_X)$. Formal groups of dimension 1 in characteristic $p > 0$ are classified up to isomorphism by their height h which is a natural number ≥ 1 or ∞. In the former case ($h \neq \infty$) the formal group is p-divisible, while in the latter case the formal group is isomorphic to the additive formal group $\hat{\mathbb{G}}_a$.

For a non-singular complete variety X over an algebraically closed field k of characteristic $p > 0$ we let W_mO_X be the sheaf of Witt rings of length m, which is coherent as a sheaf of rings. It has three operators F, V and R given by $F(a_0, \ldots, a_m) = (a_0^p, \ldots, a_m^p)$, $V(a_0, \ldots, a_m) = (0, a_0, \ldots, a_m)$ and $R(a_0, \ldots, a_m) = (a_0, \ldots, a_{m-1})$ satisfying the relations $RVF = FRV = RFV = p$. The cohomology groups $H^i(X, W_mO_X)$ with the maps induced by R form a projective system of finitely generated $W_m(k)$-modules. The projective limit is the cohomology group $H^i(X, WO_X)$. Note that this need not be a finitely generated $W(k)$-module. It has semi-linear operators F and V.

Let X be a Calabi-Yau manifold of dimension n. The vanishing of the groups $H^i(X, O_X)$ for $i \neq 0, n$ and the exact sequence

$$0 \to W_{m-1}O_X \to W_mO_X \to O_X \to 0$$

imply that $H^i(X, W_mO_X)$ vanishes for $i = 1, \ldots, n-1$ and all $m > 0$, hence $H^i(X, WO_X) = 0$. We also see that restriction $R : W_mO_X \to W_{m-1}O_X$ induces a surjective map $H^n(X, W_mO_X) \to H^n(X, W_{m-1}O_X)$ with kernel $H^n(X, O_X)$. The fact that F and R commute implies that if the induced map $F : H^n(X, W_mO_X) \to H^n(X, W_mO_X)$ vanishes then $F : H^n(X, W_iO_X) \to H^n(X, W_iO_X)$ vanishes for $i < m$ too. It also follows that $H^n(X, W_iO_X)$ is a k-vector space for $i < m$.

It is known by Artin-Mazur [2] that the Dieudonné module of the formal group Φ_X is $H^n(X, WO_X)$ with WO_X the sheaf of Witt vectors of O_X. This implies the following result, cf. [3] where we proved this for K3-surfaces. We omit the proof which is similar to that for K3 surfaces.

Theorem 2.1. *For a Calabi-Yau manifold X of dimension n we have the following characterization of the height:*

$$h(\Phi_X) = \min\{i \geq 1\colon [F : H^n(W_i O_X) \to H^n(W_i O_X)] \neq 0\}.$$

We now connect this with de Rham cohomology. Serre introduced in [14] a map $D_i : W_i(O_X) \to \Omega^1_X$ of sheaves in the following way:

$$D_i(a_0, a_1, \ldots, a_{i-1}) = a_0^{p^{i-1}-1} da_0 + \ldots + a_{i-2}^{p-1} da_{i-2} + da_{i-1}.$$

It satisfies $D_{i+1}V = D_i$, and Serre showed that this induces an injective map of sheaves of additive groups

$$D_i : W_i O_X / F W_i O_X \to \Omega^1_X. \tag{1}$$

The exact sequence $0 \to W_i O_X \xrightarrow{F} W_i O_X \longrightarrow W_i O_X / F W_i O_X \to 0$ gives rise to an isomorphism

$$H^{n-1}(W_i O_X / F W_i O_X) \cong \operatorname{Ker}[F : H^n(W_i O_X) \to H^n(W_i O_X)]. \tag{2}$$

Proposition 2.2. *If $h \neq \infty$ then the induced map*

$$D_i : H^{n-1}(X, W_i O_X / F W_i O_X) \to H^{n-1}(X, \Omega^1_X)$$

is injective, and $\dim \operatorname{Im} D_i = \min\{i, h-1\}$.

Proof. We give a proof for the reader's convenience. Take an affine open covering $\{U_i\}$ of X. Assuming some D_ℓ is not injective, we let ℓ be the smallest natural number such that D_ℓ is not injective on $H^{n-1}(W_\ell O_X / F W_\ell O_X)$. Let $\alpha = \{f_I\}$ with $f_I = (f_I^{(0)}, \ldots, f_I^{(\ell-1)}) \in \Gamma(U_{i_0,\ldots,i_{n-1}}, W_\ell O_X)$ represent a non-zero element of $H^{n-1}(W_\ell O_X / F W_\ell O_X)$ such that $D_\ell(\alpha)$ is zero in $H^{n-1}(\Omega^1_X)$. Then there exists elements $\omega_J = \omega_{j_0 j_1 \cdots j_{n-2}}$ in $\Gamma(U_{j_0} \cap \ldots \cap U_{j_{n-2}}, \Omega^1_X)$ such that

$$\sum_{j=0}^{\ell-1} (f_I^{(j)})^{p^{\ell-j-1}} d \log f_I^{(j)} = \sum_j \omega_{I_j},$$

where the multi-index $I_j = \{i_0, \ldots, i_{n-1}\}$ is obtained from I by omitting i_j. By applying the inverse Cartier operator we get an equation

$$\sum_{j=0}^{\ell-1} (f_I^{(j)})^{p^{\ell-j}} d \log f_I^{(j)} + df_I^{(\ell)} = \sum_j \tilde{\omega}_{I_j}$$

for certain functions $f_I^{(\ell)}$ and differential forms $\tilde{\omega}_{I_j}$ with $C(\tilde{\omega}_{I_j}) = \omega_{I_j}$. Since α is an non-zero element of $H^{n-1}(W_\ell O_X / F W_\ell O_X)$, the element $\beta = (f_I^{(0)}, \ldots, f_I^{(\ell-1)}, f_I^{(\ell)})$ gives a non-zero element of $H^{n-1}(W_{\ell+1} O_X / F W_{\ell+1} O_X)$. In view of (2) for $i = \ell + 1$ the element β gives a non-zero element $\tilde{\beta}$ of $H^n(W_{\ell+1} O_X)$ such that $F(\tilde{\beta}) = 0$ in $H^n(W_{\ell+1} O_X)$. Take the element $\tilde{\alpha}$ in $H^n(W_\ell O_X)$ which corresponds to the element α under the isomorphism (2) for $i = \ell$. Then we have $F(\tilde{\alpha}) = 0$ in $H^n(W_\ell O_X)$, and $R^\ell(\tilde{\alpha}) \neq 0$ in $H^n(X, O_X)$ by the assumption on ℓ. Therefore, we have $R^\ell(\tilde{\beta}) \neq 0$ in $H^n(X, O_X)$, and the elements $V^jR^j(\tilde{\beta})$ for $j = 0, \dots, \ell$ generate $H^n(W_{\ell+1}O_X)$. Hence the Frobenius map is zero on $H^n(W_{\ell+1}O_X)$. Repeating this argument, we conclude that the Frobenius map is zero on $H^n(W_iO_X)$ for any $i > 0$ and this contradicts the assumption $h \neq \infty$.

Corollary 2.3. *If the height h of an n-dimensional Calabi-Yau variety X is not ∞ then $h \leq \dim H^{n-1}(\Omega_X^1) + 1$.*

Definition 2.4. A Calabi-Yau manifold X is called *rigid* if $\dim H^{n-1}(X, \Omega_X^1) = 0$.

Please note that the tangent sheaf Θ_X is the dual of Ω_X^1, hence by the triviality of the canonical bundle it is isomorphic to Ω_X^{n-1}. Therefore, by Serre duality the space of infinitesimal deformations $H^1(X, \Theta_X)$ is isomorphic to the dual of $H^{n-1}(X, \Omega_X^1)$.

Corollary 2.5. *The height of a rigid Calabi-Yau manifold X is either 1 or ∞.*

3. Cohomology Groups of Calabi-Yau Varieties

Let X be a Calabi-Yau variety of dimension n over k. The existence of Frobenius provides the de Rham cohomology with a very rich structure from which we can read off characteristic p properties. If $F : X \to X^{(p)}$ is the relative Frobenius operator then the Cartier operator C gives an isomorphism

$$\mathcal{H}^j(F_*\Omega^\bullet_{X/k}) = \Omega^j_{X,d\text{-closed}}/d\Omega_X^{j-1} \xrightarrow{\sim} \Omega^j_{X^{(p)}}$$

of sheaves on $X^{(p)}$. We generalize the sheaves $d\Omega_X^{j-1}$ and $\Omega^j_{X,d\text{-closed}}$ by setting (cf. [6])

$$B_0\Omega_X^j = (0), \quad B_1\Omega_X^j = d\Omega_X^{j-1}, \quad B_{m+1}\Omega_X^j = C^{-1}(B_m\Omega_X^j).$$

and

$$Z_0\Omega_X^j = \Omega_X^j, \quad Z_1\Omega_X^j = \Omega^j_{X,d\text{-closed}}, \quad Z_{m+1}\Omega_X^j = \mathrm{Ker}(dC^m).$$

Note that we have the inclusions

$$\begin{array}{c} 0 = B_0\Omega_X^j \subset B_1\Omega_X^j \subset \dots \subset B_m\Omega_X^j \subset \dots \\ \dots \subset Z_m\Omega_X^j \subset \dots \subset Z_1\Omega_X^j \subset Z_0\Omega_X^j = \Omega_X^j \end{array}$$

and that we have an exact sequence

$$0 \to Z_{m+1}\Omega_X^j \longrightarrow Z_m\Omega_X^j \xrightarrow{dC^m} d\Omega_X^j \to 0.$$

Alternatively, the sheaves $B_m\Omega_X^j$ and $Z_m\Omega_X^j$ can be viewed as locally free subsheaves of $(F^m)_*\Omega_X^j$ on $X^{(p^m)}$. Duality for the finite morphism F^m implies that for every $j \geq 0$ there is a perfect pairing of $O_{X^{(p^m)}}$-modules $F_*^m\Omega_X^j \otimes F_*^m\Omega_X^{n-j} \longrightarrow \Omega^n_{X^{(p^m)}}$ given by $(\alpha, \beta) \mapsto C^m(\alpha \wedge \beta)$. This induces perfect pairings of $O_{X^{(p^m)}}$-modules

$$B_m\Omega_X^j \otimes F_*^m\Omega_X^{n-j}/Z_m\Omega_X^{n-j} \to \Omega_{X^{(p^m)}}$$

and

$$Z_m\Omega_X^j \otimes F_*^m\Omega_X^{n-j}/B_m\Omega_X^{n-j} \to \Omega_{X^{(p^m)}}$$

Now we have an isomorphism $F_*^m\Omega_X^j/Z_m\Omega_X^j \cong B_m\Omega_X^{j+1}$ induced by the map d. Going back to the interpretation of the $B_m\Omega_X^j$ as sheaves on X we find in this way for $1 \leq j \leq n$ and $m > 0$ perfect pairings

$$B_m\Omega_X^j \otimes B_m\Omega_X^{n+1-j} \to \Omega_X^n \qquad (\omega_1 \otimes \omega_2) \mapsto C^m(\omega_1 \wedge \omega_2).$$

We first note another interpretation for $B_m\Omega_X^1$: the injective map of sheaves of additive groups $D_m : W_m(\mathcal{O}_X)/FW_m(\mathcal{O}_X) \to \Omega_X^1$ induces an isomorphism

$$D_m : W_m(\mathcal{O}_X)/FW_m(\mathcal{O}_X) \xrightarrow{\sim} B_m\Omega_X^1. \tag{3}$$

We write $h^i(X,-)$ for $\dim_k H^i(X,-)$. Note that duality implies $h^i(B_m\Omega_X^n) = h^{n-i}(B_m\Omega_X^1)$.

Proposition 3.1. *We have $h^i(X, B_m\Omega_X^1) = 0$ unless $i = n$ or $i = n-1$. If $i = n-1$ or $i = n$ we have*

$$h^i(B_m\Omega_X^1) = \begin{cases} \min\{m, h-1\} & \text{if } h \neq \infty \\ m & \text{if } h = \infty. \end{cases}$$

Proof. The statement about $h^{n-1}(B_m\Omega_X^1)$ follows from (3) and the characterization of the height given in Section 2. The other statements follow from the long exact sequence associated with the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \xrightarrow{d} d\mathcal{O}_X \longrightarrow 0$$

and the exact sequence

$$0 \to B_m \to B_{m+1} \xrightarrow{C^m} B_1 \to 0. \tag{4}$$

The details can safely be left to the reader. This concludes the proof.

The natural inclusions $B_i\Omega_X^j \hookrightarrow \Omega_X^j$ and $Z_i\Omega_X^j \hookrightarrow \Omega_X^j$ of sheaves of groups on X induce homomorphisms

$$H^1(B_i\Omega_X^j) \to H^1(\Omega_X^j) \quad \text{and} \quad H^1(Z_i\Omega_X^j) \to H^1(\Omega_X^j)$$

whose images are denoted by $\operatorname{Im} H^1(B_i\Omega_X^j)$ and $\operatorname{Im} H^1(Z_i\Omega_X^j)$. Note that we have a non-degenerate cup product pairing

$$\langle\, , \rangle : H^{n-1}(X, \Omega_X^1) \otimes H^1(X, \Omega_X^{n-1}) \to H^n(X, \Omega_X^n) \cong k.$$

Lemma 3.2. *The images $\operatorname{Im} H^{n-1}(B_i\Omega_X^1)$ and $\operatorname{Im} H^1(Z_i\Omega_X^{n-1})$ are orthogonal to each other for the pairing $\langle\, , \rangle$.*

Proof. From the definitions it follows that for elements $\alpha \in H^{n-1}(B_i\Omega_X^1)$ and $\beta \in H^1(Z_i\Omega_X^{n-1})$ we have $C^i(\alpha \wedge \beta) = 0$. The long exact sequence associated to

$$0 \to B_1\Omega_X^n \to Z_1\Omega_X^n \to \Omega_X^n \to 0 \tag{5}$$

together with the fact that $H^n(Z_i\Omega_X^n) = H^n(\Omega_X^n)$ for $i \geq 0$ implies that C acts without kernel on $H^n(\Omega_X^n)$. This proves the required orthogonality.

Lemma 3.3. *If $h \neq \infty$ we have* $\dim \mathrm{Im} H^1(X, Z_i\Omega_X^{n-1}) = \dim H^1(\Omega_X^{n-1}) - i$ *for* $0 \le i \le h-1$.

Proof. If the height $h = 1$ then we have $H^{n-1}(B_i\Omega_X^1) = 0$ by (4) and moreover the vanishing of $H^i(X, d\Omega_X^{n-1})$ and the exact sequence

$$0 \to Z_{i+1}\Omega_X^{n-1} \longrightarrow Z_i\Omega_X^{n-1} \xrightarrow{dC^i} d\Omega_X^{n-1} \to 0 \tag{6}$$

imply that $\mathrm{Im}\, H^1(X, Z_i\Omega_X^{n-1}) = H^1(X, \Omega_X^{n-1})$ for $i \geq 1$. For $2 \le h < \infty$, we know by Proposition 2.2 that $\mathrm{Im}\, H^{n-1}(X, B_i\Omega_X^1) \subset H^{n-1}(X, \Omega_X^1)$ is of dimension $\min\{i, h-1\}$. The exact sequence (6) gives an exact sequence

$$k \longrightarrow H^1(Z_{i+1}\Omega_X^{n-1}) \xrightarrow{\psi_{i+1}} H^1(Z_i\Omega_X^{n-1}) \longrightarrow k$$

from which we deduce that either $\dim \psi_{i+1}(H^1(Z_{i+1}\Omega_X^{n-1})) = \dim H^1(Z_i\Omega_X^{n-1}) + 1$ or $\dim \psi_{i+1}(H^1(Z_{i+1}\Omega_X^{n-1})) = \dim H^1(Z_i\Omega_X^{n-1})$. By induction $\dim \mathrm{Im} H^1(Z_i\Omega_X^{n-1})$ is at least $\dim H^1(\Omega_X^{n-1}) - i$. On the other hand, by Proposition 3.1 we have $\dim \mathrm{Im} H^1(Z_i\Omega_X^{n-1}) \le \dim H^1(\Omega_X^{n-1}) - i$ for $i \le h-1$.

Lemma 3.4. *If X is a Calabi-Yau manifold of dimension n with $h = \infty$ then*

$$(\mathrm{Im}\, H^{n-1}(X, B_i\Omega_X^1))^\perp = \mathrm{Im}\, H^1(Z_i\Omega_X^{n-1}).$$

Proof. We prove this by induction on i. By the exact sequence (5) we have $\dim H^i(X, d\Omega_X^{n-1}) = 1$ for $i = 0, 1$. Thus, by the exact sequence (6) we see that the difference $\dim \mathrm{Im} H^1(Z_i\Omega^{n-1}) - \dim \mathrm{Im} H^1(Z_{i+1}\Omega^{n-1})$ is equal to 0 or 1, and we have an exact sequence

$$H^1(Z_{i+1}\Omega^{n-1}) \xrightarrow{\phi} H^1(Z_i\Omega^{n-1}) \xrightarrow{dC^i} H^1(d\Omega_X^{n-1}).$$

Assume that $\mathrm{Im}\, H^{n-1}(B_{j-1}\Omega^1) \neq \mathrm{Im}\, H^{n-1}(B_j\Omega^1)$ for $j \le i$ and $\mathrm{Im}\, H^{n-1}(B_i\Omega^1) = \mathrm{Im}\, H^{n-1}(B_{i+1}\Omega^1)$. By Lemma 3.2,

$$\mathrm{Im} H^1(Z_{i-1}\Omega^{n-1}) \supset \mathrm{Im} H^1(Z_i\Omega^{n-1})$$

and $\mathrm{Im} H^1(Z_{i-1}\Omega^{n-1}) \neq \mathrm{Im} H^1(Z_i\Omega^{n-1})$ for $j \le i$. Suppose $\mathrm{Im} H^1(Z_i\Omega^{n-1}) \neq \mathrm{Im} H^1(Z_{i+1}\Omega^{n-1})$. The natural homomorphism $\phi : H^1(Z_{i+1}\Omega^{n-1}) \to H^1(Z_i\Omega^{n-1})$ is not surjective. Since $H^1(d\Omega_X^{n-1}) \cong k$, we see that $dC^i : H^1(Z_i\Omega^{n-1}) \to H^1(d\Omega_X^{n-1})$ is surjective and we factor it as

$$H^1(Z_i\Omega^{n-1}) \xrightarrow{C^i} H^1(\Omega^{n-1}) \xrightarrow{d} H^1(d\Omega_X^{n-1}).$$

Since dC^i is surjective, d is not the zero map on $C^i(H^1(Z_i\Omega^{n-1}))$. Therefore, we have

$$C^i(H^1(Z_i\Omega^{n-1})) \not\subset \mathrm{Im}\, H^1(Z_1\Omega^{n-1}).$$

Take an affine open covering of X, and take any Čech cocycle $C^i(\eta) = \{C^i(\eta_{jk})\}$ of $C^i(H^1(Z_i\Omega^{n-1}))$ with respect to this affine open covering. Take any element $\zeta \in H^{n-1}(B_i\Omega^1)$. Then there exists an element $\tilde{\zeta}$ such that $C^i(\tilde{\zeta}) = \zeta$. We consider the image of the element $C^i(\eta) \wedge \zeta$ in $H^n(X, \Omega_X^n)$. Then, we have

$$C^i(\tilde{\eta}) \wedge \zeta = C^i(\tilde{\eta}) \wedge C^i(\tilde{\zeta}) = C^i(\eta \wedge \tilde{\zeta})$$

Since $\mathrm{Im} H^{n-1}(B_{2i}\Omega^1) = \mathrm{Im} H^{n-1}(B_i\Omega^1)$, the image of $\tilde{\zeta}$ in $H^{n-1}(\Omega^1_X)$ is contained in $\mathrm{Im} H^{n-1}(B_i\Omega^1)$. As $\mathrm{Im} H^1(Z_i\Omega^{n-1}))$ is orthogonal to $\mathrm{Im} H^{n-1}(B_i\Omega^1)$, we see that $\eta\wedge\tilde{\zeta}$ is zero in $H^n(X,\Omega^n_X)$, and we have $C^i(\eta\wedge\tilde{\zeta}) = 0$ in $H^n(X,\Omega^n_X)$. Therefore, we see that the image of $C^i(H^1(Z_i\Omega^{n-1}))$ in $H^1(\Omega^{n-1}_X)$ is orthogonal to $\mathrm{Im} H^{n-1}(B_i\Omega^1)$ and we have

$$C^i(H^1(Z_i\Omega^{n-1})) \subset \ \mathrm{Im} H^{n-1}(B_i\Omega^1)^\perp \subset \ \mathrm{Im} H^1(Z_i\Omega^{n-1}) \subset \ \mathrm{Im} H^1(Z_1\Omega^{n-1}),$$

a contradiction. Hence, we have $\mathrm{Im} H^1(Z_i\Omega^{n-1}) = \mathrm{Im} H^1(Z_{i+1}\Omega^{n-1})$.

Collecting results we get the following theorem.

Theorem 3.5. *If X is a Calabi-Yau variety of dimension n and height h then for $i \le h-1$ we have*

$$\mathrm{Im} H^{n-1}(X, B_i\Omega^1_X)^\perp = \mathrm{Im} H^1(X, Z_i\Omega^{n-1}_X).$$

One reason for our interest in the spaces $\mathrm{Im} H^1(X, Z_i\Omega^{n-1}_X)$ comes from the fact that they play a role as tangent spaces to strata in the moduli space as in the analogous case of K3 surfaces, cf. [3]. We intend to come back to this in a later paper.

4. Picard groups

We suppose that X is a Calabi-Yau variety of dimension $n \ge 3$. We have the following result for the space of regular 1-forms.

Proposition 4.1. *All global 1-forms are indefinitely closed: for $i \ge 0$ we have $H^0(X, Z_i\Omega^1_X) = H^0(X,\Omega^1_X)$. The action of the Cartier operator on this space is semi-simple.*

Proof. Since the sheaves $B_i\Omega^1_X$ have non non-zero cohomology in degree 0 and 1 the exact sequence

$$0 \to B_i\Omega^1_X \longrightarrow Z_i\Omega^1_X \xrightarrow{C^i} \Omega^1_X \to 0.$$

implies $\dim H^0(Z_i\Omega^1_X) = \dim H^0(\Omega^1_X)$. Since the natural map $H^0(Z_i\Omega^1_X) \to H^0(\Omega^1_X)$ is injective, we have $H^0(Z_i\Omega^1_X) = H^0(\Omega^1_X)$. The second assertion follows from $H^0(B_i\Omega^1_X) = 0$.

It is well known that for a p^{-1}-linear semi-simple homomorphism λ on a finite-dimensional vector space V the map $\lambda - \mathrm{id}_V$ is surjective. This means that we have a basis of logarithmic differential forms $C\omega = \omega$.

Corollary 4.2. *If id denotes the identity homomorphism on $H^0(X,\Omega^1_X)$ the map $C - \mathrm{id} : H^0(X,\Omega^1_X) \to H^0(X,\Omega^1_X)$ is surjective.*

Proposition 4.3. *Suppose that X is a smooth complete variety for which all global 1-forms are closed and such that C gives a bijection $H^0(X, Z_1\Omega^1_X) \longrightarrow H^0(X,\Omega^1_X)$. Then we have an isomorphism*

$$H^0(X,\Omega^1_X) \cong \mathrm{Pic}(X)[p] \otimes_{\mathbf{Z}} k.$$

Proof. Let L be a line bundle representing an element $[L]$ of order p in $\mathrm{Pic}(X)$. Then there exists a rational function $g \in k(X)^*$ such that $(g) = pD$, where D is a divisor corresponding to L. One observes now by a local calculation that dg/g is a regular 1-form and thus defines an element of $H^0(X, \Omega_X^1)$. Conversely, if ω is a global regular 1-form with $C\omega = \omega$ then ω can be represented locally as df_i/f_i with respect to some open cover $\{U_i\}$. From the relation $df_i/f_i = df_j/f_j$ we see $d\log(f_i/f_j) = 0$ and this implies $d(f_i/f_j) = 0$. Hence we see that $f_i/f_j = \phi_{ij}^p$ form some 1-cocycle $\{\phi_{ij}\}$. This cocycle defines a torsion element of order p of $\mathrm{Pic}(X)$. These two maps are each others inverse and the result follows.

We are using the notation $\mathrm{Pic}(X)$ (resp. $NS(X)$) for the Picard group (resp. Néron-Severi group) of X. If L is a line bundle with transition functions $\{f_{ij}\}$ then $d\log f_{ij}$ represents the first Chern class of L. In this way we can define a homomorphism

$$\varphi_1 : \mathrm{Pic}(X) \longrightarrow H^1(Z_1\Omega_X^1), \quad [L] \mapsto c_1(L) = \{df_{ij}/f_{ij}\}$$

which obviously factors through $\mathrm{Pic}(X)/p\mathrm{Pic}(X)$.

Proposition 4.4. *The homomorphism* $\varphi_1 : \mathrm{Pic}(X)/p\mathrm{Pic}(X) \longrightarrow H^1(X, Z_1\Omega_X^1)$ *is injective.*

Proof. We take an affine open covering $\{U_i\}$. Suppose that there exists an element $[L]$ such that $\varphi_1([L]) = 0$. Then there exists a d-closed regular 1-form ω_i on an affine open set U_i such that $df_{ij}/f_{ij} = \omega_j - \omega_i$ on $U_i \cap U_j$ and we have $df_{ij}/f_{ij} = C(\omega_j) - C(\omega_i)$. Therefore, we have $\omega_j - C(\omega_j) = \omega_i - C(\omega_i)$ on $U_i \cap U_j$. This shows that there exists an regular 1-form ω on X such that $\omega = \omega_i - C(\omega_i)$ on U_i. By Corollary 4.2, there exists an element $\omega' \in H^0(\Omega_X^1)$ such that $(C - \mathrm{id})\omega' = \omega$. Replacing $\omega_i + \omega'$ by ω_i, we have

$$df_{ij}/f_{ij} = \omega_j - \omega_i$$

with $C(\omega_i) = \omega_i$. Then, there exists an regular function f_i on U_i such that $\omega_i = df_i/f_i$. So we have $d\log f_{ij} = d\log(f_j/f_i)$. Therefore, there exists a regular function φ_{ij} on $U_i \cap U_j$ such that $f_{ij} = (f_j/f_i)\varphi_{ij}^p$. Thus $[L]$ is a p-th power. We conclude that $\varphi_1 : \mathrm{Pic}(X)/p\mathrm{Pic}(X) \to H^1(Z_1\Omega_X^1)$ is injective.

Proposition 4.5. *The natural homomorphism* $H^1(Z_1\Omega_X^1) \to H^2_{DR}(X)$ *is injective.*

Proof. Let $\{U_i\}$ be an affine open covering of X. A Čech cocycle $\{\omega_{ij}\}$ in $H^1(Z_1\Omega_X^1)$ is mapped to $\{(0, \omega_{ij}, 0)\}$ in $H^2_{DR}(X)$. Suppose this element is zero in $H^2_{DR}(X)$. Then there exist elements $(\{f_{ij}\}, \{\omega_i\})$ with $f_{ij} \in \Gamma(U_i \cap U_j, O_X)$ and $\omega_i \in \Gamma(U_i, \Omega_X^1)$ such that

$$f_{jk} - f_{ik} + f_{ij} = 0 \quad \omega_{ij} = df_{ij} + \omega_j - \omega_i \quad d\omega_i = 0.$$

Since $\{f_{ij}\}$ gives an element of $H^1(O_X)$ and $H^1(O_X) = 0$, there exists an element $\{f_i\}$ such that $f_{ij} = f_j - f_i$ on $U_i \cap U_j$. Therefore, we have $\omega_{ij} = (df_j + \omega_j) - (df_i + \omega_i)$. Since $d(df_i + \omega_i) = 0$, we conclude that $\{\omega_{ij}\}$ is zero in $H^1(Z_1\Omega_X^1)$.

The results above imply the following theorem.

Theorem 4.6. *The natural homomorphism* $\mathrm{Pic}(X)/p\mathrm{Pic}(X) \longrightarrow H^2_{DR}(X)$ *is injective.*

Let us point out at this point that for a Calabi-Yau manifold X the Picard group $\mathrm{Pic}(X)$ is reduced and coincides with the Néron-Severi group $NS(X)$ because $NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(X)$ vanishes because of $H^1(X, O_X) = 0$.

Lemma 4.7. *For a Calabi-Yau manifold X of dimension $n \geq 3$ with non non-zero global 1-forms* $\mathrm{Pic}(X)$ *has no p-torsion.*

Proof. Take an affine open covering $\{U_i\}$ of X. Assume $\{f_{ij}\}$ represents an element $[L] \in \mathrm{Pic}(X)$ which is p-torsion. Then, there exist regular functions $f_i \in H^0(U_i, O_X^*)$ such that $f_{ij}^p = f_i/f_j$. The df_i/f_i on U_i glue together to yield a regular 1-form ω on X. Since $H^0(X, \Omega_X^1) = 0$, we see $\omega = 0$, i.e., $df_i = 0$. Therefore, there exist regular functions $g_i \in H^0(U_i, O_X^*)$ such that $f_i = g_i^p$. Hence, we have $\{f_{ij}\} \sim 0$ and we see that $\mathrm{Pic}(X)$ has no p-torsion.

Lemma 4.8. *Let X be a Calabi-Yau manifold X of dimension $n \geq 3$ with no non-zero global 2-forms. Then, the homomorphism*

$$\mathrm{Pic}(X)/p\mathrm{Pic}(X) \longrightarrow H^1(\Omega_X^1)$$

defined by $\{f_{ij}\} \mapsto \{df_{ij}/f_{ij}\}$ *is injective.*

Proof. By the assumption $H^0(X, \Omega_X^2) = 0$ we have $H^0(X, d\Omega_X^1) = 0$. Therefore, from the exact sequence

$$0 \to Z_1\Omega_X^1 \longrightarrow \Omega_X^1 \xrightarrow{d} d\Omega_X^1 \to 0,$$

we deduce a natural injection $H^1(Z_1\Omega_X^1) \longrightarrow H^1(\Omega_X^1)$. So the result follows from Lemma 4.4.

Theorem 4.9. *Let X be a Calabi-Yau manifold X of dimension $n \geq 3$ with $H^0(X, \Omega_X^i) = 0$ for $i = 1, 2$. Then the natural homomorphism*

$$NS(X)/pNS(X) \otimes_{\mathbf{F}_p} k \longrightarrow H^1(\Omega_X^1) = H^2_{dR}(X)$$

is injective and the Picard number satisfies $\rho \leq \dim_k H^1(\Omega_X^1)$.

Proof. Suppose that this homomorphism is not injective. Then with respect to a suitable affine open covering $\{U_i\}$ there exist elements $\{f_{ij}^{(\nu)}\}$ representing non-zero elements in $NS(X)/pNS(X)$, such that

$$\sum_{\nu=1}^{\ell} a_\nu df_{ij}^{(\nu)}/f_{ij}^{(\nu)} = 0 \quad \text{in } H^1(\Omega_X^1)$$

for suitable $a_\nu \in k$. We take such elements with the minimal ℓ. We may assume $a_1 = 1$ and we have $a_i/a_j \notin \mathbf{F}_p$ for $i \neq j$. By Lemma 4.8, we have $\ell \geq 2$. There exists $\omega_i \in H^1(U_i, \Omega_X^1)$ such that

$$\sum_{\nu=1}^{\ell} a_\nu df_{ij}^{(\nu)}/f_{ij}^{(\nu)} = \omega_j - \omega_i \tag{1}$$

on $U_i \cap U_j$. There exists an element $\tilde{\omega}_i \in H^1(U_i, \Omega^1_X)$ such that $C(\tilde{\omega}_i) = \omega_i$. Therefore, taking the Cartier inverse, we have

$$\sum_{\nu=1}^{\ell} \tilde{a}_\nu df^{(\nu)}_{ij}/f^{(\nu)}_{ij} + dg_{ij} = \tilde{\omega}_j - \tilde{\omega}_i$$

with $\tilde{a}_\nu \in k$, $\tilde{a}^p_\nu = a_\nu$, and suitable $dg_{ij} \in H^0(U_i \cap U_j, dO_X)$. Since $\{df^{(\nu)}_{ij}/f^{(\nu)}_{ij}\}$ is a cocycle, we see that $\{dg_{ij}\} \in H^1(X, dO_X)$. Since $H^1(X, dO_X) = 0$, there exists an element $dg_i \in H^0(U_i, dO_X)$ such that $dg_{ij} = dg_j - dg_i$. Therefore, we have

$$\sum_{\nu=1}^{\ell} \tilde{a}_\nu df^{(\nu)}_{ij}/f^{(\nu)}_{ij} = (\tilde{\omega}_j - dg_j) - (\tilde{\omega}_i - dg_i). \tag{2}$$

Subtracting (2) from (1) we get a non-trivial linear relation with a smaller ℓ in $H^1(\Omega^1_X)$, a contradiction.

Remark. In the case of a K3 surface X the natural homomorphism

$$NS(X)/pNS(X) \otimes_{\mathbf{F}_p} k \longrightarrow H^2_{DR}(X)$$

is not injective if X is supersingular in the sense of Shioda. Ogus showed that the kernel can be used for describing the moduli of supersingular K3 surfaces, cf. Ogus[10]. So the situation is completely different in dimension ≥ 3.

5. Fermat Calabi-Yau manifolds

Again p is a prime number and m a positive integer which is prime to p. Let f be a smallest power of p such that $p^f \equiv 1 \bmod m$ and put $q = p^f$. We denote by $\mathbf{F}_q$ a finite field of cardinality q. We consider the Fermat variety $X^r_m(p)$ over $\mathbf{F}_q$ defined by

$$X_0^m + X_1^m + \ldots + X_{r+1}^m = 0$$

in projective space $\mathbf{P}^{r+1}$ of dimension $r+1$. The zeta function of X^r_m over $\mathbf{F}_q$ was calculated by A. Weil (cf. [18]). The result is:

$$Z(X^r_m/\mathbf{F}_q, T) = \frac{P(T)^{(-1)^{r-1}}}{(1-T)(1-qT)\ldots(1-q^rT)},$$

where $P(T) = \prod_\alpha (1 - j(\alpha)T)$ with the product taken over a set of vectors α and $j(\alpha)$ is a Jacobi sum defined as follows. Consider the set

$$A_{m,r} = \{(a_0, a_1, \ldots, a_{r+1}) \in \mathbf{Z}^{r+2} \mid 0 < a_i < m, \textstyle\sum_{j=0}^{r+1} a_j \equiv 0 (\bmod m)\},$$

and choose a character $\chi : \mathbf{F}_q^* \to \mathbf{C}^*$ of order m. For $\alpha = (a_0, a_1, \ldots, a_{r+1}) \in A_{m,r}$ we define

$$j(\alpha) = (-1)^r \sum \chi(v_1^{a_1}) \ldots \chi(v_{r+1}^{a_{r+1}}),$$

where the summation runs over $v_i \in \mathbf{F}_q^*$ with $1 + v_1 + \ldots + v_{r+1} = 0$. Thus the $j(\alpha)$'s are eigenvalues of the Frobenius map over $\mathbf{F}_q$ on the ℓ-adic étale cohomology group $H^r_{et}(X^r_m, \mathbf{Q}_\ell)$.

Now, let $\zeta = \exp(2\pi i/m)$ be a primitive m-th root of unity, and $K = \mathbf{Q}(\zeta)$ the corresponding cyclotomic field with Galois group $G = \mathrm{Gal}(K/\mathbf{Q})$. For an

element $t \in (\mathbf{Z}/m\mathbf{Z})^*$ we let σ_t be the automorphism of K defined by $\zeta \mapsto \zeta^t$. The correspondence $t \leftrightarrow \sigma_t$ defines an isomorphism $(\mathbf{Z}/m\mathbf{Z})^* \cong G$ and we shall identify G with $(\mathbf{Z}/m\mathbf{Z})^*$ by this isomorphism. We define a subgroup H of order f of G by $H = \{p^j \bmod m \mid 0 \le j < f\}$
Let $\{t_1, \dots, t_g\}$ with $t_i \in \mathbf{Z}/m\mathbf{Z}^*$, be a complete system of representatives of G/H with $g = |G/H|$, and put

$$A_H(\alpha) = \sum_{t \in H} [\sum_{j=1}^{r+1} \langle ta_j/m \rangle],$$

where $[a]$ (resp. $\langle a \rangle$) means the integral part (resp. the fractional part) of a rational number a.
Choose a prime ideal $\mathcal{P}$ in K lying over p; it has norm $N(\mathcal{P}) = p^f = q$. If $\mathcal{P}_i$ denotes the prime ideal $\mathcal{P}^{\sigma^{-1}_{-t_i}}$ we have the prime decomposition $(p) = \mathcal{P}_1 \cdots \mathcal{P}_g$ in K and Stickelberger's theorem tells us that

$$(j(\alpha)) = \prod_{i=1}^{g} \mathcal{P}_i^{A_H(t_i\alpha)},$$

where $t_i\alpha = (t_i a_0, \dots, t_i a_{r+1})$. For the details we refer to Lang[7] or Shioda-Katsura[16].
Now we restrict our attention to Fermat Calabi-Yau manifolds $X^r_m(p)$ with $m = r + 2$.

Theorem 5.1. *Assume $r \ge 2$. Let Φ^r be the Artin-Mazur formal group of the r-dimensional Calabi-Yau variety $X = X^r_{r+2}(p)$. The height h of Φ^r is equal to either 1 or ∞. Moreover, $h = 1$ if and only if $p \equiv 1 \pmod{r+2}$.*

Before we give the proof of this theorem we state a technical lemma.

Lemma 5.2. *Under the notation above, assume $[\sum_{j=1}^{r+1} \langle ta_j/(r+2) \rangle] = 0$ with $t \in (\mathbf{Z}/(r+2)\mathbf{Z})^*$. Then $a_j = t^{-1}$ in $(\mathbf{Z}/(r+2)\mathbf{Z})^*$ for all $j = 0, 1, \dots, r+1$.*

Proof. Since $t \in (\mathbf{Z}/(r+2)\mathbf{Z})^*$, we have $\langle ta_j/(r+2) \rangle \ge 1/(r+2)$. Suppose there exists an index i such that $ta_i \not\equiv 1 \pmod{r+2}$. Then we have the inequality $\langle ta_i/(r+2) \rangle \ge 2/(r+2)$ and thus $\sum_{j=1}^{r+1} \langle ta_\rho/(r+2) \rangle \ge 1$, which contradicts the assumption. So we have $ta_i \equiv 1 \pmod{r+2}$ and $a_j \equiv t^{-1}$ for $j = 1, \dots, r+1$. Since $a_0 + a_1 + \ldots + a_{r+1} \equiv 0 \pmod{r+2}$, we conclude $a_0 \equiv t^{-1}$.
Proof of the theorem. The Dieudonné module $D(\Phi^r)$ of Φ^r is isomorphic to $H^r(X, WO_X)$. We denote by Q(W) the quotient field of the Witt ring $W(k)$ of k. Then, if $h < \infty$, we have

$$h = \dim_{Q(W)} H^r(X, WO_X) \otimes_{W(k)} Q(W).$$

and by Illusie [6] we know we have

$$H^r(X, WO_X) \otimes_{W(k)} Q(W) \cong H^r_{\mathrm{cris}}(X) \otimes Q(W)_{[0,1[}.$$

According to Artin-Mazur [2], the slopes of $H^r_{\mathrm{cris}}(X) \otimes Q(W)$ are given by $(\mathrm{ord}_{\mathcal{P}} q)/f$ and the $(\mathrm{ord}_{\mathcal{P}} j(\alpha))/f$. Hence, the height h is equal to the number of $j(\alpha)$ such that $A_H(\alpha) < f$.

First, assume $p \equiv 1 \pmod{r+2}$, i.e. $f = 1$. Then $H = \langle 1 \rangle$ and $A_H(\alpha) < f = 1$ implies $A_H(\alpha) = 0$. Therefore, by Lemma 5.2, we have $a_j = 1$ for all $j = 0, 1, \ldots, r+1$ and there is only one α, namely $\alpha = (1, 1, \ldots, 1)$, such that $\mathrm{ord}_{\mathcal{P}} j(\alpha) = 0$. So we conclude $h = 1$ in this case.

Secondly, assume $p \not\equiv 1 \pmod{r+2}$. By definition, we have $f \geq 2$. We now prove that there exists no α such that $A_H(\alpha) < f$. Suppose $A_H(\alpha) = \sum_{t \in H} [\sum_{j=1}^{r+1} \langle t a_j/(r+2) \rangle] < f$. Then there exists an element $t \in H$ such that $[\sum_{j=1}^{r+1} \langle t a_j/(r+2) \rangle] = 0$. By Lemma 5.2 we have $\alpha = (t^{-1}, t^{-1}, \ldots, t^{-1})$. For $t' \in H$ with $t' \neq t$ we have $[\sum_{\rho=1}^{r+1} \langle t' t^{-1}/(r+2) \rangle] \neq 0$. Therefore the inequality yields $[\sum_{j=1}^{r+1} \langle t' t^{-1}/(r+2) \rangle] = 1$ for $t' \in H$, $t' \neq t$. Since $A_H(t\alpha) = A_H(\alpha)$ for any $t \in H$, by a translation by t, we may assume $\alpha = (1, 1, \ldots, 1)$, i.e., $t = 1$. Moreover, we can take a representative of $t' \in H$ such that $0 < t' < r+2$. Then,

$$\begin{aligned} 1 &= [\sum_{j=1}^{r+1} \langle t' t^{-1}/(r+2) \rangle] = [\sum_{j=1}^{r+1} \langle t'/(r+2) \rangle] = [\sum_{j=1}^{r+1} t'/(r+2)] \\ &= [(r+1)t'/(r+2)] \end{aligned}$$

and we get $1 \leq (r+1)t'/(r+2) < 2$. By this inequality, we see $t' = 2$. Therefore, we have $H = \{1, 2\}$. Since H is a subgroup of $(\mathbf{Z}/(r+2)\mathbf{Z})^*$, we see that $2^2 \equiv 1 \bmod r+2$. Therefore, we have $r = 1$, which contradicts our assumption.

Hence there exists no α such that $\mathrm{ord}_{\mathcal{P}} j(\alpha) < 1$ and we conclude $h = \infty$ in this case. This completes the proof of the theorem.

For K3 surfaces we have two notions of supersingularity. We generalize these to higher dimensions.

Definition 5.3. A Calabi-Yau manifold X of dimension r is said to be of *additive Artin-Mazur type* ('supersingular in the sense of Artin') if the height of Artin-Mazur formal group associated with $H^r(X, O_X)$ is equal to ∞.

Definition 5.4. A non-singular complete algebraic variety X of dimension r is said to be *fully rigged* ('supersingular in the sense of Shioda') if all the even degree étale cohomology groups are spanned by algebraic cycles.

By the theorem above, we know that the Fermat Calabi-Yau manifolds are of additive Artin-Mazur type if and only if $p \not\equiv 1 \bmod m$ with $m = r+2$. As to being fully rigged we have the following theorem.

Theorem 5.5 (Shioda-Katsura [16]). *Assume $m \geq 4$, $(p, m) = 1$ and r is even. Then the Fermat variety $X_m^r(p)$ is fully rigged if and only if there exists a positive integer ν such that $p^\nu \equiv -1 \bmod m$.*

M. Artin conjectured that a K3 surface X is supersingular in the sense of Artin if and only if X is supersingular in the sense of Shioda. He also showed that "if part" holds. In the case of the Fermat K3 surface, i.e, $X_4^2(p)$, by the two theorems above, we see, as is well-known, that the Artin conjecture holds.

However, in the case of even $r \geq 4$, the above two theorems imply that this straightforward generalization of the Artin conjecture to higher dimension does not hold.

6. Kummer Calabi-Yau manifolds

Let A be an abelian variety of dimension $n \geq 2$ defined over an algebraically closed field of characteristic $p > 0$, and G be a finite group which acts on A faithfully. Assume that the order of G is prime to p, and that the quotient variety A/G has a resolution which is a Calabi-Yau manifold X. We call X a Kummer Calabi-Yau manifold. We denote by $\pi : A \longrightarrow A/G$ the projection, and by $\nu : X \longrightarrow A/G$ the resolution.

Theorem 6.1. *Under the assumptions above the Artin-Mazur formal group Φ^n_X is isomorphic to the Artin-Mazur formal group Φ^n_A.*

Proof. Since the order of G is prime to p, the singularities of A/G are rational, and we have $R^i\nu_* O_X = 0$ for $i \geq 1$. So by the Leray spectral sequence we have $H^n(A/G, O_{A/G}) \cong H^n(X, O_X) \cong k$ and $H^{n-1}(A/G, O_{A/G}) \cong H^{n-1}(X, O_X) \cong 0$. It follows that the Artin-Mazur formal group $\Phi^n_{A/G}$ is pro-representable by a formal Lie group of dimension 1 (cf. Artin-Mazur[2]). Since the tangent space $H^n(A/G, O_{A/G})$ of $\Phi^n_{A/G}$ is naturally isomorphic to the tangent space $H^n(X, O_X)$ of Φ^n_X as above, the natural homomorphism from $\Phi^n_{A/G}$ to Φ^n_X is non-trivial. One-dimensional formal groups are classified by their height and between formal groups of different height there are no non-trivial homomorphisms . So the height of $\Phi^n_{A/G}$ is equal to that of Φ^n_X and we thus see that $\Phi^n_{A/G}$ and Φ^n_X are isomorphic.
Since the order of G is prime to p, there is a non-trivial trace map from $H^n(A, O_A)$ to $H^n(A/G, O_{A/G})$. Therefore, $\pi^* : H^n(A/G, O_{A/G}) \longrightarrow H^n(A, O_A)$ is an isomorphism. Therefore, as above we see that the height of $\Phi^n_{A/G}$ is equal to the height of Φ^n_A, and that Φ^n_X is isomorphic to Φ^n_A. Q.e.d.

Though the following lemma might be well-known to specialists we give here a proof for the reader's convenience.

Lemma 6.2. *Let A be an abelian variety of dimension $n \geq 2$ and p-rank $f(A)$. The height h of the Artin-Mazur formal group Φ_A of A is as follows:*

(1) *$h = 1$ if A is ordinary, i.e., $f(A) = n$,*
(2) *$h = 2$ if $f(A) = n - 1$,*
(3) *$h = \infty$ if $f(A) \leq n - 2$.*

Proof. We denote by $H^i_{\mathrm{cris}}(A)$ the i-th cristalline cohomology of A and as usual by $H^i_{\mathrm{cris}}(A)_{[\ell,\ell+1[}$ the additive group of elements in $H^i_{\mathrm{cris}}(A)$ whose slopes are in the interval $[\ell, \ell+1[$. By the general theory in Illusie [6], we have

$$H^n(A, W(O_A)) \otimes_W Q(W) \cong (H^n_{\mathrm{cris}}(A) \otimes_W Q(W))_{[0,1[}$$

with $Q(W)$ the quotient field of W. The theory of Dieudonné modules implies

$$h = \dim_{Q(W)} D(\Phi_A) = \dim_{Q(W)} H^n(A, W(O_A)) \otimes_W Q(W) \quad \text{if } h < \infty,$$

and $\dim_{Q(W)} D(\Phi_A) = 0$ if $h = \infty$. We know the slopes of $H^1_{\rm cris}(A)$ for each case. Since we have

$$H^n_{\rm cris}(A) \cong \wedge^n H^1_{\rm cris}(A),$$

counting the number of slopes in $[0, 1[$ of $H^n_{\rm cris}(A)$ gives the result.

Corollary 6.3. *Let X be a Kummer Calabi-Yau manifold of dimension n obtained from an abelian variety A as above. Then the height of the Artin-Mazur formal group Φ^n_X is equal to either 1, 2 or ∞.*

Example 6.4. Assume $p \geq 3$. Let A be an abelian surface and ι the map $A \to A$ sending $a \in A$ to its inverse $-a \in A$. We denote by $Km(A)$ the Kummer surface of A, i.e., the minimal resolution of $A/\langle\iota\rangle$. Then $\Phi^2_{Km(A)}$ is isomorphic to Φ^2_A.

Example 6.5. Assume $p \geq 5$, and let ω be a primitive third root of unity. Let E be a non-singular complete model of the elliptic curve defined by $y^2 = x^3 + 1$, and let σ be an automorphism of E defined by $x \mapsto \omega x,\ y \mapsto y$. We set $A = E^3$ and put $\tilde{\sigma} = \sigma \times \sigma \times \sigma$. The minimal resolution X of $A/\langle\tilde{\sigma}\rangle$ is a Calabi-Yau manifold, and the Artin-Mazur formal group Φ^3_X is isomorphic to Φ^3_A.

Let ω be a complex number with positive imaginary part, and $L = \mathbf{Z} + \mathbf{Z}\omega$ be a lattice in the complex numbers $\mathbf{C}$. From here on, we consider an elliptic curve $E = \mathbf{C}/L$, and we assume that E has a model defined over an algebraic number field K. Then $A = E \times E \times E$ is an abelian threefold, and we let $G \subseteq \mathrm{Aut}_K(E)$ be a finite group which faithfully acts on A. We assume that G has only isolated fixed points on A and that the quotient variety A/G has a crepant resolution $\nu : X \to A/G$ defined over K, [12]. We denote by π the projection $A \to A/G$. For a prime p of K, we denote by $\bar{X}$ the reduction modulo p of X.

Example 6.6. [K. Ueno [17]] Assume that E is an elliptic curve defined over $\mathbf{Q}$ having complex multiplication $\sigma : E \to E$ by a primitive third root of unity. Then $G = \mathbf{Z}/3\mathbf{Z} = \langle\sigma\rangle$ acts diagonally on $A = E^3$. A crepant resolution of A/G gives a rigid Calabi-Yau manifold defined over $\mathbf{Q}$. For a prime number $p \geq 5$ the reduction modulo p of A is the abelian threefold given in Example 6.5.

Theorem 6.7. *Let X be a Calabi-Yau obtained as crepant resolution of A/G as above. Assume moreover that X is rigid. Then the elliptic curve E has complex multiplication and the intermediate Jacobian of X is isogenous to E.*

Corollary 6.8. *Under the assumptions as in the theorem, we take a prime p of good reduction for X and let $\bar{X}$ be the reduction of X modulo p. Then the height of the formal group $\Phi_{\bar{X}}$ is either* 1 *or ∞. It is ∞ if and only if the reduction of the intermediate Jacobian variety of X at p is a supersingular elliptic curve.*

Example 6.9. We consider the reduction $\bar{X}$ modulo p of the variety X in the Example 6.6. We assume the characteristic of the residue field of p is not equal to 2 and 3. Then the height $h(\Phi_{\bar{X}}) = \infty$ if and only if the reduction modulo p

of the intermediate Jacobian of X is a supersingular elliptic curve, and this is the case if and only if $p \equiv 2 \pmod 3$.

Before we prove the theorem we introduce some notation. We have a natural identification $H_1(E, \mathbf{Z}) = \mathbf{Z} + \mathbf{Z}\omega$. Fixing a non-zero regular differential form η on E determines a regular three form $p_1^*\eta \wedge p_2^*\eta \wedge p_3^*\eta = \Omega_A$ on A. We have a natural homomorphism

$$H^3(X, \mathbf{Z}) \to H^3_{dR}(X) \to H^3(X, O_X).$$

If X is rigid, the corresponding quotient $H^3(X, O_X)/H^3(X, \mathbf{Z})$ gives the intermediate Jacobian of X. Since $\dim_{\mathbf{C}} H^3(X, O_X) = 1$, the intermediate Jacobian of X is isomorphic to an elliptic curve.
We can define the period map with respect to Ω_A:

$$\pi_A : H_3(X, \mathbf{Z}) \longrightarrow \mathbf{C} \qquad \gamma \mapsto \int_\gamma \Omega_A.$$

By Poincaré duality we can identify π_A with the natural projection $H^3(X, \mathbf{Z}) \to H^3(X, \mathbf{C}) \to H^3(X, O_X) = \mathbf{C}$ (cf. Shioda [15], for instance.) There exists a regular 3-form Ω_X on X such that $\Omega_A = (\nu^{-1} \circ \pi)^* \Omega_X$. We can define π_X with respect to Ω_X for the Calabi-Yau manifold X as well.
In order to describe the structure of the intermediate Jacobian of X we look at the period map of an abelian threefold, following the method in Shioda [15] (also see Mumford [8]). Choose a basis u_1, u_2 of $H_1(E, \mathbf{R})$. This determines a $\mathbf{C}$-basis $H_1(E, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = H_1(E, \mathbf{C})$. If e_i for $i = 1, 2, 3$ is the standard basis of $\mathbf{C}^3$ then $u_{2i-1} = e_i$ and $u_{2i} = \omega e_i$ for $i = 1, 2, 3$ form a basis of $H_1(A, \mathbf{Z})$ and $A = \mathbf{C}^3/M$ with M the lattice generated by $u_1, \ldots, u_6$. The dual basis is denoted by v_i. The basis of $H^1(A, \mathbf{Z})$ determines a canonical basis $v_i \wedge v_j \wedge v_k$ of $H^3(A, \mathbf{Z})$. The natural homomorphism

$$p_A : H^3(A, \mathbf{Z}) \longrightarrow H^3_{dR}(A) \longrightarrow H^3(A, O_A) \cong \mathbf{C}.$$

is an element of $\mathrm{Hom}_{\mathbf{C}}(H^3(A, \mathbf{C}), \mathbf{C})$ and can be considered as an element of $H^3(A, \mathbf{C})$ and is given by

$$p_A = \sum_{i<j<k} \det(u_i, u_j, u_k) v^i \wedge v^j \wedge v^k.$$

Therefore the image of p_A in $\mathbf{C}$ is spanned by the complex numbers 1, ω, ω^2 and ω^3 over $\mathbf{Z}$.
We now give the proof of Theorem 6.7. Let S be the set of non-free points of the action of G on A. Then the restriction of π to $A \setminus S$ is étale on $A/G \setminus \pi(S)$. Since S is of codimension 3 in A, we have the following diagram:

$$\begin{array}{ccccc}
H^3(A, \mathbf{Z}) & \cong & H^3(A \setminus S, \mathbf{Z}) & \xrightarrow{p_A} & H^3(A, O_A) \cong \mathbf{C} \\
\downarrow \pi_* & & \downarrow (\pi\,|_{A \setminus S})_* & & \downarrow \pi_* \\
H^3(A/G, \mathbf{Z}) & \cong & H^3(A/G \setminus \pi(S), \mathbf{Z}) & \xrightarrow{p_{A/G}} & H^3(A/G, O_{A/G}) \cong \mathbf{C} \\
\downarrow \cong & & & & \downarrow \nu^* \\
H^3(X, \mathbf{Z}) & & & \xrightarrow{p_X} & H^3(X, O_X) \cong \mathbf{C}.
\end{array}$$

The vertical arrows on the right hand side give an identification of $H^3(A, O_A)$ and $H^3(X, O_X)$. Because p does not divide the order of G we see that $H^3(A, \mathbb{Q})$ maps surjectively to $H^3(A/G, \mathbb{Q}) = H^3(A, \mathbb{Q})^G$, hence the image of $H^3(A, \mathbb{Z})$ is commensurable with $H^3(X, \mathbb{Z})$.
Now $\mathrm{Im}\, p_X$ is a lattice in $\mathbf{C}$, and $\mathrm{Im}\, p_A$ is a lattice in $\mathbf{C}$ as well. We know that $\mathrm{Im}\, p_A$ is generated by 1, ω, ω^2 and ω^3 and thus ω is a quadratic number and the intermediate Jacobian has complex multiplication by $\mathbf{Q}(\omega)$. Hence the intermediate Jacobian $\mathbf{C}/\mathrm{Im}\, p_X$ of X is isogenous to E.

7. Questions

We close with two natural basic questions that suggest themselves.
Is there a function $f(n)$ such that a Calabi-Yau variety in characteristic $p > 0$ of dimension n lifts to characteristic 0 if $p > f(n)$? Note that Hirokado constructed a non-liftable Calabi-Yau threefold in characteristic 3, see [5] (see also [13]).
Can a Calabi-Yau variety of dimension 3 in positive characteristic have non-zero regular 1-forms or regular 2-forms?

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References

[1] M. Artin, Supersingular K3 surfaces, Ann. Scient. Ec. Norm. Sup., 7 (1974), 543–568.
[2] M. Artin and B. Mazur, Formal groups arising from algebraic varieties, Ann. Scient. Ec. Norm. Sup., 10 (1977), 87–132.
[3] G. van der Geer and T. Katsura, On a stratification of the moduli of K3 surfaces, J. Eur. Math. Soc., 2 (2000), 259-290.
[4] A. Grothendieck, Fondements de la géométrie algébrique (FGA), Extraits du Sém. Bourbaki, 1957-1962, Secr. Math., Paris, 1962.
[5] M. Hirokado, A non-liftable Calabi-Yau threefold in characteristic 3, Tohoku Math. J. 51 (1999), 479–487.
[6] L. Illusie, Complexe de de Rham-Witt et cohomologie cristalline, Ann. Sci. ENS, 12 (1979), 501–661.
[7] S. Lang, Cyclotomic Fields, Springer-Verlag, 1978.
[8] D. Mumford, Abelian Varieties, Oxford Univ. Press, 1970.
[9] A. Ogus, Supersingular K3 crystals, Astérisque 64 (1979), 3–86.
[10] A. Ogus, Singularities of the height strata in the moduli of K3 surfaces, In Moduli of Abelian Varieties, (Texel 1999), Progress in Math. 195,(2001), 325–343.

[11] A. Ogus, On the Hasse locus of a Calabi-Yau family, Math. Res. Lett. 8 (2001), 35–41.

[12] M. Reid, Canonical 3-folds, Journées de Géometrie Algébrique d'Angers, Sijthoff & Noordhoff, Alphen aan den Rijn, Germantown Md (1980), 273–310.

[13] S. Schröer, Some Calabi-Yau threefolds with obstructed deformations over the Witt vectors, math arXiv: math.AG/0302064.

[14] J. -P. Serre, Sur la topologie des variétés algébriques en caractéristique p, Symposion Internacional de topologia algebraica 1958, 24–53.

[15] T. Shioda, The period map of abelian surfaces, J. Fac. Sci. Univ. Tokyo. 25 (1978), 47-59.

[16] T. Shioda and T. Katsura, On Fermat varieties, Tohoku J. Math., 31 (1979), 97-115.

[17] K. Ueno, Classification theory of algebraic varieties and compact complex manifolds, Lecture Notes in Math. 439 Springer Verlag, 1975.

[18] A. Weil, Number of solutions of equations in finite fields, Bull. Amer. Math. Soc., 55 (1949), 497-508.

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Stability of Arakelov Bundles and Tensor Products without Global Sections

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Abstract. This paper deals with Arakelov vector bundles over an arithmetic curve, i.e. over the set of places of a number field. The main result is that for each semistable bundle E, there is a bundle F such that $E \otimes F$ has at least a certain slope, but no global sections. It is motivated by an analogous theorem of Faltings for vector bundles over algebraic curves and contains the Minkowski-Hlawka theorem on sphere packings as a special case. The proof uses an adelic version of Siegel's mean value formula.

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Introduction

G. Faltings has proved that for each semistable vector bundle E over an algebraic curve of genus g, there is another vector bundle F such that $E \otimes F$ has slope $g - 1$ and no global sections. (Note that any vector bundle of slope $> g-1$ has global sections by Riemann–Roch.) See [3] and [4] where this result is interpreted in terms of theta functions and used for a new construction of moduli schemes of vector bundles.

In the present paper, an arithmetic analogue of that theorem is proposed. The algebraic curve is replaced by the set X of all places of a number field K; we call X an arithmetic curve. Vector bundles are replaced by so-called Arakelov bundles, cf. section 3. In the special case $K = \mathbb{Q}$, Arakelov bundles without

global sections are lattice sphere packings, and the slope μ measures the packing density.
We will see at the end of section 4 that the maximal slope of Arakelov bundles of rank n without global sections is $d(\log n + \mathrm{O}(1))/2 + (\log \mathfrak{d})/2$ where d is the degree and $\mathfrak{d}$ is the discriminant of K. Now the main result is:

Theorem 0.1 *Let $\mathcal{E}$ be a semistable Arakelov bundle over the arithmetic curve X. For each $n \gg 0$ there is an Arakelov bundle $\mathcal{F}$ of rank n satisfying*

$$\mu(\mathcal{E} \otimes \mathcal{F}) > \frac{d}{2}(\log n - \log \pi - 1 - \log 2) + \frac{\log \mathfrak{d}}{2}$$

such that $\mathcal{E} \otimes \mathcal{F}$ has no nonzero global sections.

The proof is inspired by (and generalises) the Minkowski-Hlawka existence theorem for sphere packings; in particular, it is not constructive. The principal ingredients are integration over a space of Arakelov bundles (with respect to some Tamagawa measure) and an adelic version of Siegel's mean value formula. Section 2 explains the latter, section 3 contains all we need about Arakelov bundles, and the main results are proved and discussed in section 4.
This paper is a condensed and slightly improved part of the author's Ph. D. thesis [6]. I would like to thank my adviser G. Faltings for his suggestions; the work is based on his ideas. It was supported by a grant of the Max-Planck-Institut in Bonn.

1 Notation

Let K be a number field of degree d over $\mathbb{Q}$ and with ring of integers $\mathcal{O}_K$. Let $X = \operatorname{Spec}(\mathcal{O}_K) \cup X_\infty$ be the set of places of K; this might be called an 'arithmetic curve' in the sense of Arakelov geometry. X_∞ consists of r_1 real and r_2 complex places with $r_1 + 2r_2 = d$. $w(K)$ is the number of roots of unity in K.
For every place $v \in X$, we endow the corresponding completion K_v of K with the map $|\,.\,|_v : K_v \to \mathbb{R}_{\geq 0}$ defined by $\mu(a \cdot S) = |a|_v \cdot \mu(S)$ for a Haar measure μ on K_v. This is the normalised valuation if v is finite, the usual absolute value if v is real and its square if v is complex. The well known product formula $\prod_{v \in X} |a|_v = 1$ holds for every $0 \neq a \in K$. On the adele ring $\mathbb{A}$, we have the divisor map $\operatorname{div} : \mathbb{A} \to \mathbb{R}^X_{\geq 0}$ that maps each adele $a = (a_v)_{v \in X}$ to the collection $(|a_v|_v)_{v \in X}$ of its valuations.
Let $\mathcal{O}_v$ be the set of those $a \in K_v$ which satisfy $|a|_v \leq 1$; this is the ring of integers in K_v for finite v and the unit disc for infinite v. Let $\mathcal{O}_{\mathbb{A}}$ denote the product $\prod_{v \in X} \mathcal{O}_v$; this is the set of all adeles a with $\operatorname{div}(a) \leq 1$. By $D \leq 1$ for an element $D = (D_v)_{v \in X}$ of $\mathbb{R}^X_{\geq 0}$, we always mean $D_v \leq 1$ for all v.
We fix a canonical Haar measure λ_v on K_v as follows:

- If v is finite, we normalise by $\lambda_v(\mathcal{O}_v) = 1$.
- If v is real, we take for λ_v the usual Lebesgue measure on $\mathbb{R}$.

- If v is complex, we let λ_v come from the real volume form $idz \wedge d\bar{z}$ on $\mathbb{C}$. In other words, we take twice the usual Lebesgue measure.

This gives us a canonical Haar measure $\lambda := \prod_{v \in X} \lambda_v$ on $\mathbb{A}$. We have $\lambda(\mathbb{A}/K) = \sqrt{\mathfrak{d}}$ where $\mathfrak{d} = \mathfrak{d}_{K/\mathbb{Q}}$ denotes (the absolute value of) the discriminant. More details on this measure can be found in [12], section 2.1.

Let $V_n = \frac{\pi^{n/2}}{(n/2)!}$ be the volume of the unit ball in $\mathbb{R}^n$. For $v \in X_\infty$, we denote by $\mathcal{O}_v^n$ the unit ball with respect to the standard scalar product on K_v^n. Observe that this is *not* the n-fold Cartesian product of $\mathcal{O}_v \subseteq K_v$. Similarly, $\mathcal{O}_{\mathbb{A}}^n := \prod_{v \in X} \mathcal{O}_v^n$ is not the n-fold product of $\mathcal{O}_{\mathbb{A}} \subseteq \mathbb{A}$. Its volume $\lambda^n(\mathcal{O}_{\mathbb{A}}^n)$ is $V_n^{r_1}(2^n V_{2n})^{r_2}$.

2 A mean value formula

The following proposition is a generalisation of Siegel's mean value formula to an adelic setting: With real numbers and integers instead of adeles and elements of K, Siegel has already stated it in [10], and an elementary proof is given in [7]. (In the special case $l = 1$, a similar question is studied in [11].)

Proposition 2.1 *Let $1 \leq l < n$, and let f be a nonnegative measurable function on the space $\mathrm{Mat}_{n \times l}(\mathbb{A})$ of $n \times l$ adele matrices. Then*

$$\int\limits_{\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)} \sum_{\substack{M \in \mathrm{Mat}_{n \times l}(K) \\ \mathrm{rk}(M) = l}} f(g \cdot M)\, d\tau(g) = \mathfrak{d}^{-nl/2} \int\limits_{\mathrm{Mat}_{n \times l}(\mathbb{A})} f\, d\lambda^{n \times l} \tag{1}$$

where τ is the unique $\mathrm{Sl}_n(\mathbb{A})$-invariant probability measure on $\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)$.

Proof: The case $l = 1$ is done in section 3.4 of [12], and the general case can be deduced along the same lines from earlier sections of this book. We sketch the main arguments here; more details are given in [6], section 3.2.

Let G be the algebraic group Sl_n over the ground field K, and denote by τ_G the Tamagawa measure on $G(\mathbb{A})$ or any quotient by a discrete subgroup. The two measures τ and τ_G on $\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)$ coincide because the Tamagawa number of G is one.

G acts on the affine space $\mathrm{Mat}_{n \times l}$ by left multiplication. Denote the first l columns of the $n \times n$ identity matrix by $E \in \mathrm{Mat}_{n \times l}(K)$, and let $H \subseteq G$ be the stabiliser of E. This algebraic group H is a semi-direct product of Sl_{n-l} and $\mathrm{Mat}_{l \times (n-l)}$. Hence section 2.4 of [12] gives us a Tamagawa measure τ_H on $H(\mathbb{A})$, and the Tamagawa number of H is also one.

Again by section 2.4 of [12], we have a Tamagawa measure $\tau_{G/H}$ on $G(\mathbb{A})/H(\mathbb{A})$ as well, and it satisfies $\tau_G = \tau_{G/H} \cdot \tau_H$ in the sense defined there. In particular, this implies

$$\int\limits_{G(\mathbb{A})/H(K)} f(g \cdot E)\, d\tau_G(g) = \int\limits_{G(\mathbb{A})/H(\mathbb{A})} f(g \cdot E)\, d\tau_{G/H}(g).$$

It is easy to see that the left hand sides of this equation and of (1) coincide. According to lemma 3.4.1 of [12], the right hand sides coincide, too. □

3 Arakelov vector bundles

Recall that a (Euclidean) lattice is a free $\mathbb{Z}$-module Λ of finite rank together with a scalar product on $\Lambda \otimes \mathbb{R}$. This is the special case $K = \mathbb{Q}$ of the following notion:

Definition 3.1 An *Arakelov (vector) bundle* $\mathcal{E}$ over our arithmetic curve $X = \operatorname{Spec}(\mathcal{O}_K) \cup X_\infty$ is a finitely generated projective $\mathcal{O}_K$-module $\mathcal{E}_{\mathcal{O}_K}$ endowed with

- a Euclidean scalar product $\langle \_, \_ \rangle_{\mathcal{E},v}$ on the real vector space $\mathcal{E}_{K_v}$ for every real place $v \in X_\infty$ and
- a Hermitian scalar product $\langle \_, \_ \rangle_{\mathcal{E},v}$ on the complex vector space $\mathcal{E}_{K_v}$ for every complex place $v \in X_\infty$

where $\mathcal{E}_A := \mathcal{E}_{O_K} \otimes A$ for every $\mathcal{O}_K$-algebra A.

A first example is the trivial Arakelov line bundle $\mathcal{O}$. More generally, the trivial Arakelov vector bundle $\mathcal{O}^n$ consists of the free module $\mathcal{O}_K^n$ together with the standard scalar products at the infinite places.
We say that $\mathcal{E}'$ is a subbundle of $\mathcal{E}$ and write $\mathcal{E}' \subseteq \mathcal{E}$ if $\mathcal{E}'_{\mathcal{O}_K}$ is a direct summand in $\mathcal{E}_{\mathcal{O}_K}$ and the scalar product on $\mathcal{E}'_{K_v}$ is the restriction of the one on $\mathcal{E}_{K_v}$ for every infinite place v. Hence every vector subspace of $\mathcal{E}_K$ is the generic fibre of one and only one subbundle of $\mathcal{E}$.
From the data belonging to an Arakelov bundle $\mathcal{E}$, we can define a map

$$\| \,.\, \|_{\mathcal{E},v} : \mathcal{E}_{K_v} \longrightarrow \mathbb{R}_{\geq 0}$$

for every place $v \in X$:

- If v is finite, let $\|e\|_{\mathcal{E},v}$ be the minimum of the valuations $|a|_v$ of those elements $a \in K_v$ for which e lies in the subset $a \cdot \mathcal{E}_{\mathcal{O}_v}$ of $\mathcal{E}_{K_v}$. This is the nonarchimedean norm corresponding to $\mathcal{E}_{\mathcal{O}_v}$.
- If v is real, we put $\|e\|_{\mathcal{E},v} := \sqrt{\langle e, e \rangle_v}$, so we just take the norm coming from the given scalar product.
- If v is complex, we put $\|e\|_{\mathcal{E},v} := \langle e, e \rangle_v$ which is the square of the norm coming from our Hermitian scalar product.

Taken together, they yield a divisor map

$$\operatorname{div}_{\mathcal{E}} : \mathcal{E}_{\mathbb{A}} \to \mathbb{R}_{\geq 0}^X \qquad e = (e_v) \mapsto (\|e_v\|_{\mathcal{E},v}).$$

Although $\mathcal{O}_{\mathbb{A}}$ is not an $\mathcal{O}_K$-algebra, we will use the notation $\mathcal{E}_{\mathcal{O}_{\mathbb{A}}}$, namely for the compact set defined by

$$\mathcal{E}_{\mathcal{O}_{\mathbb{A}}} := \{e \in \mathcal{E}_{\mathbb{A}} : \operatorname{div}_{\mathcal{E}}(e) \leq 1\}.$$

Recall that these norms are used in the definition of the Arakelov degree: If $\mathcal{L}$ is an Arakelov line bundle and $0 \neq l \in \mathcal{L}_K$ a nonzero generic section, then

$$\deg(\mathcal{L}) := -\log \prod_{v \in X} \|l\|_{\mathcal{L},v}$$

and the degree of an Arakelov vector bundle $\mathcal{E}$ is by definition the degree of the Arakelov line bundle $\det(\mathcal{E})$. $\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$ is called the slope of $\mathcal{E}$. One can form the tensor product of two Arakelov bundles in a natural manner, and it has the property $\mu(\mathcal{E} \otimes \mathcal{F}) = \mu(\mathcal{E}) + \mu(\mathcal{F})$.
Moreover, the notion of stability is based on slopes: For $1 \leq l \leq \mathrm{rk}(\mathcal{E})$, denote by $\mu_{\max}^{(l)}$ the supremum (in fact it is the maximum) of the slopes $\mu(\mathcal{E}')$ of subbundles $\mathcal{E}' \subseteq \mathcal{E}$ of rank l. $\mathcal{E}$ is said to be stable if $\mu_{\max}^{(l)} < \mu(\mathcal{E})$ holds for all $l < \mathrm{rk}(\mathcal{E})$, and semistable if $\mu_{\max}^{(l)} \leq \mu(\mathcal{E})$ for all l.
To each projective variety over K endowed with a metrized line bundle, one can associate a zeta function as in [5] or [1]. We recall its definition in the special case of Grassmannians associated to Arakelov bundles:

Definition 3.2 If $\mathcal{E}$ is an Arakelov bundle over X and $l \leq \mathrm{rk}(\mathcal{E})$ is a positive integer, then we define

$$\zeta_{\mathcal{E}}^{(l)}(s) := \sum_{\substack{\mathcal{E}' \subseteq \mathcal{E} \\ \mathrm{rk}(\mathcal{E}') = l}} \exp(s \cdot \deg(\mathcal{E}')).$$

The growth of these zeta functions is related to the stability of $\mathcal{E}$. More precisely, we have the following asymptotic bound:

Lemma 3.3 *There is a constant $C = C(\mathcal{E})$ such that*

$$\zeta_{\mathcal{E}}^{(l)}(s) \leq C \cdot \exp(s \cdot l\mu_{\max}^{(l)}(\mathcal{E}))$$

for all sufficiently large real numbers s.

Proof: Fix $\mathcal{E}$ and l. Denote by $N(T)$ the number of subbundles $\mathcal{E}' \subseteq \mathcal{E}$ of rank l and degree at least $-T$. There are $C_1, C_2 \in \mathbb{R}$ such that

$$N(T) \leq \exp(C_1 T + C_2)$$

holds for all $T \in \mathbb{R}$. (Embedding the Grassmannian into a projective space, this follows easily from [9]. See [6], lemma 3.4.8 for more details.)
If we order the summands of $\zeta_{\mathcal{E}}^{(l)}$ according to their magnitude, we thus get

$$\begin{aligned} \zeta_{\mathcal{E}}^{(l)}(s) &\leq \sum_{\nu=0}^{\infty} N\left(-l\mu_{\max}^{(l)}(\mathcal{E}) + \nu + 1\right) \cdot \exp\left(s \cdot (l\mu_{\max}^{(l)}(\mathcal{E}) - \nu)\right) \\ &\leq \exp(s \cdot l\mu_{\max}^{(l)}(\mathcal{E})) \cdot \sum_{\nu=0}^{\infty} \frac{C_3}{\exp((s - C_1)\nu)}. \end{aligned}$$

But the last sum is a convergent geometric series for all $s > C_1$ and decreases as s grows, so it is bounded for $s \geq C_1 + 1$. □

4 The main theorem

The global sections of an Arakelov bundle $\mathcal{E}$ over $X = \operatorname{Spec}(\mathcal{O}_K) \cup X_\infty$ are by definition the elements of the finite set

$$\Gamma(\mathcal{E}) := \mathcal{E}_K \cap \mathcal{E}_{\mathcal{O}_\mathbb{A}} \subseteq \mathcal{E}_\mathbb{A}.$$

Note that in the special case $K = \mathbb{Q}$, an Arakelov bundle without nonzero global sections is nothing but a (lattice) sphere packing: $\Gamma(\mathcal{E}) = 0$ means that the (closed) balls of radius $1/2$ centered at the points of the lattice $\mathcal{E}_\mathbb{Z}$ are disjoint. Here larger degree corresponds to denser packings.

Theorem 4.1 *Let $\mathcal{E}$ be an Arakelov bundle over the arithmetic curve X. If an integer $n > \operatorname{rk}(\mathcal{E})$ and an Arakelov line bundle $\mathcal{L}$ satisfy*

$$1 > \sum_{l=1}^{\operatorname{rk}(\mathcal{E})} \mathfrak{d}^{-nl/2} \cdot \lambda^{nl}\left(\frac{K^*\mathcal{O}_\mathbb{A}^{nl}}{K^*}\right) \cdot \zeta_\mathcal{E}^{(l)}(n) \exp(l \deg(\mathcal{L})),$$

then there is an Arakelov bundle $\mathcal{F}$ of rank n and determinant $\mathcal{L}$ such that

$$\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0.$$

Proof: Note that any global section of $\mathcal{E} \otimes \mathcal{F}$ is already a global section of $\mathcal{E}' \otimes \mathcal{F}$ for a unique minimal subbundle $\mathcal{E}' \subseteq \mathcal{E}$, namely the subbundle whose generic fibre is the image of the induced map $(\mathcal{F}_K)^{\text{dual}} \to \mathcal{E}_K$. We are going to average the number of these sections (up to K^*) for a fixed subbundle $\mathcal{E}'$ of rank l.

Fix one particular Arakelov bundle $\mathcal{F}$ of rank n and determinant $\mathcal{L}$. Choose linear isomorphisms $\phi_{\mathcal{E}'} : K^l \to \mathcal{E}'_K$ and $\phi_\mathcal{F} : K^n \to \mathcal{F}_K$ and let

$$\phi : \operatorname{Mat}_{n \times l}(K) \xrightarrow{\sim} (\mathcal{E}' \otimes \mathcal{F})_K$$

be their tensor product. Our notation will not distinguish these maps from their canonical extensions to completions or adeles.

For each $g \in \operatorname{Sl}_n(\mathbb{A})$, we denote by $g\mathcal{F}$ the Arakelov bundle corresponding to the K-lattice $\phi_\mathcal{F}(gK^n) \subseteq \mathcal{F}_\mathbb{A}$. More precisely, $g\mathcal{F}$ is the unique Arakelov bundle satisfying $(g\mathcal{F})_\mathbb{A} = \mathcal{F}_\mathbb{A}$, $(g\mathcal{F})_{\mathcal{O}_\mathbb{A}} = \mathcal{F}_{\mathcal{O}_\mathbb{A}}$ and $(g\mathcal{F})_K = \phi_\mathcal{F}(gK^n)$. This gives the usual identification between $\operatorname{Sl}_n(\mathbb{A})/\operatorname{Sl}_n(K)$ and the space of Arakelov bundles of rank n and fixed determinant together with local trivialisations.

Observe that the generic fibre of $\mathcal{E}' \otimes g\mathcal{F}$ is $\phi(g\operatorname{Mat}_{n \times l}(K))$. A generic section is not in $\mathcal{E}'' \otimes g\mathcal{F}$ for any $\mathcal{E}'' \subsetneq \mathcal{E}'$ if and only if the corresponding matrix has rank l. So according to the mean value formula of section 2, the average number of global sections

$$\int\limits_{\operatorname{Sl}_n(\mathbb{A})/\operatorname{Sl}_n(K)} \operatorname{card}\left(\frac{K^*\Gamma(\mathcal{E}' \otimes g\mathcal{F})}{K^*} \setminus \bigcup_{\mathcal{E}'' \subsetneq \mathcal{E}'} \frac{K^*\Gamma(\mathcal{E}'' \otimes g\mathcal{F})}{K^*}\right) d\tau(g)$$

is equal to the integral

$$\mathfrak{d}^{-nl/2} \int\limits_{\mathrm{Mat}_{n\times l}(\mathbb{A})} (f_K \circ \mathrm{div}_{\mathcal{E}'\otimes\mathcal{F}} \circ \phi)\, d\lambda^{n\times l}. \tag{2}$$

Here the function $f_K : \mathbb{R}_{\geq 0}^X \to \mathbb{R}_{\geq 0}$ is defined by

$$f_K(D) := \begin{cases} 1/\mathrm{card}\{a \in K^* : \mathrm{div}(a) \cdot D \leq 1\} & \text{if } D \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

with the convention $1/\infty = 0$.
In order to compute (2), we start with the local transformation formula

$$\begin{aligned} &\lambda_v^{n\times l}\ (\{M \in \mathrm{Mat}_{n\times l}(K_v) : c_1 \leq \|\phi(M)\|_{\mathcal{E}'\otimes\mathcal{F},v} \leq c_2\}) = \\ &\lambda_v^{nl}\ (\{M \in K_v^{nl} : c_1 \leq \|M\| \leq c_2\}) \cdot \|\det(\phi)\|^{-1}_{\det(\mathcal{E}'\otimes\mathcal{F}),v} \end{aligned}$$

for all $c_1, c_2 \in \mathbb{R}_{\geq 0}$. Regarding this as a relation between measures on $\mathbb{R}_{\geq 0}$ and taking the product over all places $v \in X$, we get the equation

$$(\mathrm{div}_{\mathcal{E}'\otimes\mathcal{F}} \circ \phi)_* \lambda^{n\times l} = \exp\deg(\mathcal{E}' \otimes \mathcal{F}) \cdot (\mathrm{div}_{\mathcal{O}^{nl}})_* \lambda^{nl} \tag{3}$$

of measures on $\mathbb{R}_{\geq 0}^X$. Hence the integrals of f_K with respect to these measures also coincide:

$$\int\limits_{\mathrm{Mat}_{n\times l}(\mathbb{A})} (f_K \circ \mathrm{div}_{\mathcal{E}'\otimes\mathcal{F}} \circ \phi)\, d\lambda^{n\times l} = \exp(n\deg(\mathcal{E}') + l\deg(\mathcal{F})) \cdot \lambda^{nl}\left(\frac{K^*\mathcal{O}_{\mathbb{A}}^{nl}}{K^*}\right).$$

We substitute this for the integral in (2). A summation over all nonzero subbundles $\mathcal{E}' \subseteq \mathcal{E}$ yields

$$\begin{aligned} \int\limits_{\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)} &\mathrm{card}\left(\frac{K^*\Gamma(\mathcal{E}\otimes g\mathcal{F}) \setminus 0}{K^*}\right) d\tau(g) = \\ &= \sum_{l=1}^{\mathrm{rk}(\mathcal{E})} \mathfrak{d}^{-nl/2} \cdot \zeta_{\mathcal{E}}^{(l)}(n) \exp(l\deg(\mathcal{F})) \cdot \lambda^{nl}\left(\frac{K^*\mathcal{O}_{\mathbb{A}}^{nl}}{K^*}\right). \end{aligned}$$

But the right hand side was assumed to be less than one, so there there has to be a $g \in \mathrm{Sl}_n(\mathbb{A})$ with $\Gamma(\mathcal{E} \otimes g\mathcal{F}) = 0$. □

In order to apply this theorem, one needs to compute $\lambda^N(K^*\mathcal{O}_{\mathbb{A}}^N/K^*)$ for $N \geq 2$. We start with the special case $K = \mathbb{Q}$. Here each adele $a \in \mathcal{O}_{\mathbb{A}}^N$ outside a set of measure zero has a rational multiple in $\mathcal{O}_{\mathbb{A}}^N$ with valuation one at all finite places, and this multiple is unique up to sign. Hence we conclude

$$\lambda^N\left(\frac{\mathbb{Q}^*\mathcal{O}_{\mathbb{A}}^N}{\mathbb{Q}^*}\right) = \frac{V_N}{2} \cdot \prod_{p\ \mathrm{prime}} \lambda_p^N(\mathbb{Z}_p^N \setminus p\mathbb{Z}_p^N) = \frac{V_N}{2\zeta(N)}.$$

In particular, the special case $K = \mathbb{Q}$ and $\mathcal{E} = \mathcal{O}$ of the theorem above is precisely the Minkowski-Hlawka existence theorem for sphere packings [8], §15. For a general number field K, we note that the roots of unity preserve $\mathcal{O}_{\mathbb{A}}^N$. Then we apply Stirling's formula to the factorials occurring via the unit ball volumes and get

$$\lambda^N\left(\frac{K^*\mathcal{O}_{\mathbb{A}}^N}{K^*}\right) \le \frac{\lambda^N(\mathcal{O}_{\mathbb{A}}^N)}{w(K)} \le \left(\frac{2\pi e}{N}\right)^{dN/2} \cdot \left(\frac{1}{\pi N}\right)^{(r_1+r_2)/2} \cdot \frac{1}{2^{r_2/2}w(K)}.$$

Using such a bound and the asymptotic statement 3.3 about $\zeta_{\mathcal{E}}^{(l)}$, one can deduce the following corollary of theorem 4.1.

Corollary 4.2 *Let the Arakelov bundle $\mathcal{E}$ over X be given. If n is a sufficiently large integer and μ is a real number satisfying*

$$\mu_{\max}^{(l)}(\mathcal{E}) + \mu \le \frac{d}{2}(\log n + \log l - \log \pi - 1 - \log 2) + \frac{\log \mathfrak{d}}{2}$$

for all $1 \le l \le \mathrm{rk}(\mathcal{E})$, then there is an Arakelov bundle $\mathcal{F}$ of rank n and slope larger than μ such that $\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0$.

If $\mathcal{E}$ is semistable, this gives the theorem 0.1 stated in the introduction. Here is some evidence that these bounds are not too far from being optimal:

Proposition 4.3 *Assume given $\epsilon > 0$ and a nonzero Arakelov bundle $\mathcal{E}$. Let $n > n(\epsilon)$ be a sufficiently large integer, and let μ be a real number such that*

$$\mu_{\max}^{(l)}(\mathcal{E}) + \mu \ge \frac{d}{2}(\log n + \log l - \log \pi - 1 + \log 2 + \epsilon) + \frac{\log \mathfrak{d}}{2}$$

holds for at least one integer $1 \le l \le \mathrm{rk}(\mathcal{E})$. Then there is no Arakelov bundle $\mathcal{F}$ of rank n and slope μ with $\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0$.

Proof: Fix such an l and a subbundle $\mathcal{E}' \subseteq \mathcal{E}$ of rank l and slope $\mu_{\max}^{(l)}(\mathcal{E})$. For each $\mathcal{F}$ of rank n and slope μ, we consider the Arakelov bundle $\mathcal{F}' := \mathcal{E}' \otimes \mathcal{F}$ of rank nl. By Stirling's formula, the hypotheses on n and μ imply

$$\exp\deg(\mathcal{F}') \cdot \lambda^{nl}(\mathcal{O}_{\mathbb{A}}^{nl}) > 2^{nld} \cdot \mathfrak{d}^{nl/2}.$$

Now choose a K-linear isomorphism $\phi : K^{nl} \xrightarrow{\sim} \mathcal{F}'_K$ and extend it to adeles. Applying the global transformation formula (3), we get

$$\lambda^{nl}(\phi^{-1}\mathcal{F}'_{\mathcal{O}_{\mathbb{A}}}) > 2^{nld} \cdot \lambda^{nl}(\mathbb{A}^{nl}/K^{nl}).$$

According to Minkowski's theorem on lattice points in convex sets (in an adelic version like [11], theorem 3), $\phi^{-1}(\mathcal{F}'_{\mathcal{O}_{\mathbb{A}}}) \cap K^{nl} \ne \{0\}$ follows. This means that $\mathcal{F}'$ — and hence $\mathcal{E} \otimes \mathcal{F}$ — must have a nonzero global section. □

Observe that the lower bound 4.2 and the upper bound 4.3 differ only by the constant $d \log 2$. So up to this constant, the maximal slope of such tensor

products without global sections is determined by the stability of $\mathcal{E}$, more precisely by the $\mu^{(l)}_{\max}(\mathcal{E})$.
Taking $E = \mathcal{O}$, we get lower and upper bounds for the maximal slope of Arakelov bundles without global sections, as mentioned in the introduction. In the special case $\mathcal{E} = \mathcal{O}$ and $K = \mathbb{Q}$ of lattice sphere packings, [2] states that no essential improvement of corollary 4.2 is known whereas several people have improved the other bound 4.3 by constants.

References

[1] *V. V. Batyrev and Yu. I. Manin,* Sur le nombre des points rationnels de hauteur borné des variétés algébriques. Math. Ann. 286 (1990), 27–43.

[2] *J. H. Conway and N. J. A. Sloane,* Sphere packings, lattices and groups. 3rd edition. Springer-Verlag, New York 1999.

[3] *G. Faltings,* Stable G-bundles and projective connections. J. Algebraic Geom. 2 (1993), 507–568.

[4] *G. Faltings,* Moduli-stacks for bundles on semistable curves. Math. Ann. 304 (1996), 489–515.

[5] *J. Franke, Yu. I. Manin, and Yu. Tschinkel,* Rational points of bounded height on Fano varieties. Invent. Math. 95 (1989), 421–435.

[6] *N. Hoffmann,* On vector bundles over algebraic and arithmetic curves. PhD thesis, University of Bonn, 2002. Bonner Mathematische Schriften 351.

[7] *A. M. Macbeath and C. A. Rogers,* Siegel's mean value theorem in the geometry of numbers. Proc. Cambridge Philos. Soc. 54 (1958), 139–151.

[8] *H. Minkowski,* Diskontinuitätsbereich für arithmetische Äquivalenz. J. Reine Angew. Math. 129 (1905), 220–274.

[9] *S. H. Schanuel,* Heights in number fields. Bull. Soc. Math. France 107 (1979), 433–449.

[10] *C. L. Siegel,* A mean value theorem in geometry of numbers. Ann. of Math. 46 (1945), 340–347.

[11] *J. L. Thunder,* An adelic Minkowski-Hlawka theorem and an application to Siegel's lemma. J. Reine Angew. Math. 475 (1996), 167–185.

[12] *A. Weil,* Adeles and algebraic groups. Birkhäuser, Boston 1982.

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The Reciprocity Obstruction for Rational Points on Compactifications of Torsors under Tori over Fields with Global Duality

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Abstract. This paper studies the reciprocity obstruction to the local–global principle for compactifications of torsors under tori over a generalised global field of characteristic zero. Under a non-divisibility condition on the second Tate–Shafarevich group for tori, it is shown that the reciprocity obstruction is the only obstruction to the local–global principle. This gives in particular an elegant unified proof of Sansuc's result on the Brauer–Manin obstruction for compactifications of torsors under tori over number fields, and Scheiderer's result on the reciprocity obstruction for compactifications of torsors under tori over p-adic function fields.

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Let K be a field of characteristic zero that has $(n{+}2)$-dimensional global duality in étale cohomology with respect to a collection of n-local fields $K \subset K_v \subset \overline{K}$ indexed by $v \in \Omega_K$. Examples of such fields are totally imaginary number fields (then $n = 1$) and function fields over n-local fields. See Section 1 for details. Let X be a nonsingular complete variety over K. Writing $X(\mathbf{A}_K) := \prod_{v\in\Omega} X(K_v)$, we have a *reciprocity pairing*

$$X(\mathbf{A}_K) \times H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n)) \to \mathbf{Q}/\mathbf{Z}.$$

Writing $X(\mathbf{A}_K)^{\mathrm{rcpr}}$ for the collections of points that pair to zero with every $\omega \in H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))$, we have that $X(K) \hookrightarrow X(\mathbf{A}_K)^{\mathrm{rcpr}}$. In particular, when $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ then $X(K) = \emptyset$.

Hence the reciprocity pairing gives an obstruction to the local–global principle. When K is a number field, this obstruction is easily seen to be equivalent to the obstruction coming from the well-known *Brauer–Manin pairing*

$$X(\mathbf{A}_K) \times H^2(X, \mathbf{G}_{\mathrm{m}}) \to \mathbf{Q}/\mathbf{Z}$$

introduced by Yu. Manin in [Ma].

Main result

In this paper we will show that under a technical assumption on Galois cohomology of tori the reciprocity obstruction is the only obstruction to the local–global principle for any smooth compactification of a torsor under a torus over K (i.e., any nonsingular complete variety containing a principal homogeneous space under a torus over K as a Zariski-dense open subvariety).

THEOREM 1. *Let K be a field of characteristic zero with global duality. Assume that $Ш^2(K, T)$ is of finite exponent for every torus T over K.*
Then for any smooth compactification X of a torsor under a torus over K we have that $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ if and only if $X(K) = \emptyset$.

Proof. This follows immediately from Corollary 3.3 and Corollary 4.3. □

This generalises (and simplifies the proof of) the original result of Sansuc that for a smooth compactification of a torsor under a torus over a number field the Brauer–Manin obstruction is the only obstruction against the Hasse principle (see [San] and also [Sk]).
The condition on $Ш^2(K, T)$ is not only known to hold for number fields, but also for p-adic function fields (this follows from the duality theorems in [SvH]). In particular, we get a proof of the following unconditional result, due to Scheiderer (private communication), that has not appeared in the literature before.

COROLLARY. *Let p be a prime and let K be a p-adic function field (i.e., a finite extension of $\mathbf{Q}_p(t)$). Then for any smooth compactification X of a torsor under a torus over K we have that $X(\mathbf{A}_K)^{\mathrm{rcpr}} = \emptyset$ if and only if $X(K) = \emptyset$.*

I do not know any other examples of fields of characteristic zero with global duality and finite exponent for $Ш^2$ of tori, nor do I know any examples of tori over fields of characteristic zero with global duality where $Ш^2$ has infinite exponent.

Method of proof

The proof uses *pseudo-motivic* homology

$${}^1H_*(X, \mathbf{Z}) := \mathrm{Ext}^{-*}_{k_{\mathrm{sm}}}(R\Gamma(X/k, \mathbf{G}_{\mathrm{m}}), \mathbf{G}_{\mathrm{m}})$$

as defined in [vH1] for nonsingular complete varieties over a field k of characteristic zero (see Section 2 for some more information).

This homology theory (covariant in X) can be considered to be in between motivic homology and étale homology with coefficients in $\hat{\mathbf{Z}}$ (see [vH1], [vH2]). It is more tractable than motivic homology, but it still contains some important geometric/arithmetic data. In particular, in certain cases ${}^1H_0(X, \mathbf{Z})$ can decide whether X has k-rational points.

Theorem 2. *Let X be a smooth compactification of a torsor under a torus over a field k of characteristic zero. Then the degree map*

$$ {}^1H_0(X, \mathbf{Z}) \to {}^1H_0(\operatorname{Spec} k, \mathbf{Z}) = \mathbf{Z} $$

is surjective if and only if $X(k) \neq \emptyset$.

Proof. If $X(k) \neq \emptyset$, then functoriality of ${}^1H_0(-, \mathbf{Z})$ implies the surjectivity of the degree map. The converse follows from Corollary 4.3. □

This is the key result in the paper and in fact an easy consequence of Hilbert's Theorem 90 and Rosenlicht's result that the invertible functions on a torus are characters up to translation. Theorem 1 is then essentially a purely formal consequence of global duality. However, to avoid any unnecessary technical subtleties we will actually derive Theorem 1 from the slightly stronger Corollary 4.3.

As we will see in Section 5, the approach taken here is strongly related to the approach of Colliot-Thélène and Sansuc in the case of number fields: Corollary 4.3 is equivalent to their result that a smooth compactification of a torsor under a torus has rational points if and only if the so-called *elementary obstruction* vanishes. However, the proofs in the present paper are simpler, and extend easily to higher cohomological dimension. This can be explained by the fact that for the varieties under consideration the homological formalism of pseudo-motivic homology happens to be more natural than the dual *co*homological formalism of descent.

Structure of the paper

Most of this paper is devoted to setting up the conceptual framework and establishing its formal properties. In Section 1 we recall the concept of an n-local field, originally due to Parshin, and we will introduce a cohomological global analogue: $(n+2)$-dimensional global duality in étale cohomology. We will introduce the reciprocity pairing in this framework and establish some basic properties. In Section 2 we will recall the definition and basic properties of pseudo-motivic homology. In Section 3 we define a cap-product between pseudo-motivic homology and étale cohomology and we establish a partial duality.

After setting up the proper framework in the first three sections, we show in Section 4 that a principal homogeneous space under a torus actually coincides with the degree 1 part of its zero-dimensional homology. This is essentially a

rephrasing of Rosenlicht's result on the invertible functions on a torus. The main results then follow immediately.
Finally, in Section 5 we will compare the methods used here to other methods in the literature.

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1 HIGHER DIMENSIONAL LOCAL AND GLOBAL DUALITY

1.1 HIGHER DIMENSIONAL LOCAL DUALITY

In this paper, an *n-local field* (for $n \geq 1$) will be a field k that admits a sequence of fields

$$k_0, k_1, \ldots, k_n = k$$

such that:

- k_0 is a finite field
- For each $i > 0$ the field k_i is the quotient field of an excellent henselian discrete valuation ring $\mathscr{O}_{k_i}$ with residue field k_{i-1}.

A *generalised n-local field* will be a field satisfying the same hypotheses, except that k_0 is only required to be *quasi-finite*, i.e., a perfect field with absolute Galois group isomorphic to $\hat{\mathbf{Z}}$.
A generalised n-local field k with k_1 of characteristic zero satisfies *n-dimensional local duality in étale cohomology*:

- There is a canonical isomorphism $H^{n+1}_{\text{ét}}(k, \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$
- For any finite $\mathrm{Gal}(\bar{k}/k)$-module M and any $i \in \mathbf{Z}$ the Yoneda pairing

  $$H^i_{\text{ét}}(k, M) \times \mathrm{Ext}^{n+1-i}_{\text{ét}}(M, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+1}_{\text{ét}}(k, \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$$

  is a perfect pairing of finite groups.
- For a finite unramified $\mathrm{Gal}(\bar{k}/k)$-module M of order prime to the characteristic of k_{n-1}, the unramified cohomology of M is precisely the annihilator of the unramified cohomology of $\mathscr{H}om(M, \mathbf{Q}/\mathbf{Z}(n))$ in the duality pairing.

Here an *unramified* $\mathrm{Gal}(\bar{k}/k)$-module M is a Galois module on which the inertia group I acts trivially, and the unramified cohomology of M is the image in $H^i_{\text{ét}}(k, M) = H^i(\mathrm{Gal}(\bar{k}/k), M)$ of the Galois cohomology group $H^i(\mathrm{Gal}(\bar{k}/k)/I, M)$ under the restriction map (see for example [Mi, I, p.36]). In terms of étale cohomology it is the image of the restriction map $H^i_{\text{ét}}(\mathscr{O}_k, M) \to H^i_{\text{ét}}(k, M)$.

For ordinary local fields (the case $n = 1$ and finite k_0), the duality in étale cohomology is due to Tate (see for example [Mi, §I.4]). For higher dimensional local fields (with finite k_0) this is [DW, Th. 1.1, Prop. 1.2]. Since the proofs only rely on the cohomological properties of k_0, they easily generalise to the case of quasi-finite k_0.

1.2 Higher dimensional global duality

Let K be a field of characteristic zero and suppose we have

- A collection Ω of discrete valuations $v\colon K \to \mathbf{Z}$.
- An $n \geq 1$ such that for every $v \in \Omega$ the quotient field K_v^{h} of the henselisation $\mathscr{O}_v^{\mathrm{h}}$ of the discrete valuation ring $\mathscr{O}_v := \{x \in K\colon v(k) \geq 0\}$ is an n-local field.
- A noetherian ring $\mathscr{O}_K \subset K$ such that K is the quotient field of $\mathscr{O}_K$ and such that for all but finitely many $v \in \mathscr{O}$ we have that $\mathscr{O}_K \subset \mathscr{O}_v$.

We will use the notation $\mathbf{A}_K$ (or simply $\mathbf{A}$) for the ring of (henselian) *adèles* corresponding to (K, Ω), i.e., the subring of $\prod_{v \in \Omega} K_v^{\mathrm{h}}$ consisting of the $\{x_v\}_{v \in \Omega}$ with $x_v \in \mathscr{O}_v$ for all but finitely many $v \in \Omega$. Since for every finite $\mathrm{Gal}(\overline{K}/K)$-module M we have that M extends to a locally constant étale sheaf over an affine open subscheme $U \subset \operatorname{Spec} \mathscr{O}_K$, we may define the *adèlic étale cohomology group*

$$H^*_{\text{ét}}(\mathbf{A}_K, M) := \varinjlim_{\substack{V \subset U \\ \text{open affine}}} \left(\prod_{\substack{v \in \Omega \\ v \in V}} H^*_{\text{ét}}(\mathscr{O}_v^{\mathrm{h}}, M) \times \prod_{\substack{v \in \Omega \\ v \notin V}} H^*_{\text{ét}}(K_v^{\mathrm{h}}, M) \right).$$

By abuse of notation we write $v \in \operatorname{Spec} \mathscr{O}_K$ if $\mathscr{O}_K \subset \mathscr{O}_v$ and similarly for every affine open subscheme $U \subset \operatorname{Spec} \mathscr{O}_K$.

As an example, observe that the canonical isomorphisms $H^{n+1}(K_v^{\mathrm{h}}, \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$ (with the unramified part being zero) induce an isomorphism

$$H^{n+1}(\mathbf{A}, \mathbf{Q}/\mathbf{Z}(n)) \simeq \bigoplus_{v \in \Omega} \mathbf{Q}/\mathbf{Z}.$$

We write $\text{Ш}^i(K, M)$ for the kernel of the map

$$H^i_{\text{ét}}(K, M) \to H^i_{\text{ét}}(\mathbf{A}, M).$$

Similarly we define the complex of abelian groups $R\Gamma_{\text{ét}}(\mathbf{A}, M)$ for any étale sheaf (or complex of étale sheaves) M over some open subscheme $U \subset \operatorname{Spec} \mathscr{O}_K$. We have a map

$$R\Gamma_{\text{ét}}(K, M) \to R\Gamma_{\text{ét}}(\mathbf{A}, M)$$

and we define the complex $R\Gamma(K, \mathbf{A}; M)$ to be the complex of abelian groups that makes a triangle

$$R\Gamma_{\text{ét}}(K, \mathbf{A}; M) \to R\Gamma_{\text{ét}}(K, M) \to R\Gamma_{\text{ét}}(\mathbf{A}, M).$$

As the notation indicates, the corresponding cohomology groups $H^i_{\text{ét}}(K, \mathbf{A}; M) := H^i(R\Gamma(K, \mathbf{A}; M))$ should be thought of as relative cohomology groups. By definition we have a long exact sequence

$$\cdots \to H^i_{\text{ét}}(K, \mathbf{A}; M) \to H^i_{\text{ét}}(K, M) \to H^i_{\text{ét}}(\mathbf{A}, M) \to H^{i+1}_{\text{ét}}(K, \mathbf{A}; M) \to \ldots$$

Observe that, even if the henselian adèles used here are different from the usual adèles (defined using completions), their cohomology with finite coefficients is the same, since their Galois groups are isomorphic (see for example [Mi, App. I.A]).

Remark 1.1. The relative cohomology groups $H^*_{\text{ét}}(K, \mathbf{A}; -)$ can be thought of as the cohomology with compact supports of $\operatorname{Spec} K$ regarded as something very open in a compactification of $\operatorname{Spec} \mathscr{O}_K$ (compare [Mi, §II.2]). This way of seeing it is more in line with the Grothendieck–Verdier approach to cohomology and duality. However, a notation H^*_c can lead to confusion when studying the cohomology of varieties over K, so the 'Eilenberg–MacLane'-style of notation as relative cohomology seems more convenient.

For any finite $\operatorname{Gal}(\overline{K}/K)$-module M, any i, $j \in \mathbf{Z}$ we have that an $\omega \in \operatorname{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j))$ induces maps

$$\begin{gathered} H^q_{\text{ét}}(K, \mathbf{A}; M) \to H^{q+i}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)) \\ H^q_{\text{ét}}(K, M) \to H^{q+i}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(j)) \\ H^q_{\text{ét}}(\mathbf{A}, M) \to H^{q+i}_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)) \end{gathered}$$

which are compatible with the long exact sequences of the pair $(K, \mathbf{A})$. Allowing ω to vary we get the *Yoneda pairings*

$$\begin{gathered} H^q_{\text{ét}}(K, \mathbf{A}; M) \otimes \operatorname{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)), \\ H^q_{\text{ét}}(K, M) \otimes \operatorname{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(j)), \\ H^q_{\text{ét}}(\mathbf{A}, M) \otimes \operatorname{Ext}^i_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(j)) \to H^{q+i}_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)). \end{gathered}$$

We say that K has $(n+2)$-dimensional global duality in étale cohomology if:

- We have an isomorphism $H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$ such that the boundary map $H^{n+1}_{\text{ét}}(\mathbf{A}, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n))$ corresponds to the summation map $\bigoplus_{v \in \Omega} \mathbf{Q}/\mathbf{Z} \xrightarrow{\sum} \mathbf{Q}/\mathbf{Z}$.
- For every finite $\operatorname{Gal}(\overline{K}/K)$-module M and any $i \in \mathbf{Z}$ the Yoneda pairing
$$H^i_{\text{ét}}(K, \mathbf{A}; M) \times \operatorname{Ext}^{n+2-i}_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$$
is a nondegenerate pairing of abelian groups inducing an isomorphism $H^i_{\text{ét}}(K, \mathbf{A}; M) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{Ext}^{n+2-i}_{K_{\text{ét}}}(M, \mathbf{Q}/\mathbf{Z}(n)), \mathbf{Q}/\mathbf{Z})$.

As a purely formal consequence we get duality for any bounded complex $\mathscr{C}$ of constructible étale sheaves defined over an open subset $U \subset \operatorname{Spec} \mathscr{O}_K$: we have for any $i \in \mathbf{Z}$ that

$$H^i_{\text{ét}}(K, \mathbf{A}; \mathscr{C}) \times \operatorname{Ext}^{n+2-i}_{K_{\text{ét}}}(\mathscr{C}, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) \simeq \mathbf{Q}/\mathbf{Z}$$

is a nondegenerate pairing of abelian groups inducing an isomorphism $H^i_{\text{ét}}(K, \mathbf{A}; \mathscr{C}) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{Ext}^{n+2-i}_{K_{\text{ét}}}(\mathscr{C}, \mathbf{Q}/\mathbf{Z}(n)), \mathbf{Q}/\mathbf{Z})$.
Examples of fields that satisfy $(n+2)$-dimensional global duality in étale cohomology are

- Totally imaginary number fields (with $n = 1$).
- Function fields of curves over generalised $(n-1)$-local fields with k_1 of characteristic zero.

Remark 1.2. To get 3-dimensional global duality for number fields that admit real embeddings, one needs to take care of the real places separately (as in [Mi, §II.2]). Having done that, the methods of this paper still apply.

1.3 The reciprocity pairing

Let X be a nonsingular complete variety over a field K having $(n+2)$-dimensional global duality in étale cohomology.
For any i, $j \in \mathbf{Z}$ the restriction map gives pairings of sets

$$X(K) \times H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(K; \mathbf{Q}/\mathbf{Z}(j))$$
$$X(\mathbf{A}) \times H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))$$

where we use the notation

$$H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j)) := H^i_{\text{ét}}(\mathbf{A}, R\Gamma(X/K, \mathbf{Q}/\mathbf{Z}(j))).$$

When we compare these two pairings, we see that composition with the restriction map $H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j)) \to H^i_{\text{ét}}(X_{\mathbf{A}}, \mathbf{Q}/\mathbf{Z}(j))$ and the boundary map $H^i_{\text{ét}}(\mathbf{A}; \mathbf{Q}/\mathbf{Z}(j)) \to H^{i+1}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))$ transforms the second pairing into a pairing

$$X(\mathbf{A}) \times H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j)) \to H^{i+1}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(j))$$

with the property that the image of the map $X(K) \to X(\mathbf{A})$ lands into the subset

$$X(\mathbf{A})^{\perp H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j))} := \big\{\{x_v\} \in X(\mathbf{A})\colon \langle \{x_v\}, \omega\rangle = 0 \text{ for any } \omega \in H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j))\big\}$$

Taking $i = n+1$, $j = n$, we get the *reciprocity pairing*

$$X(\mathbf{A}) \times H^{n+1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$$

mentioned in the introduction, and we have that

$$X(K) \hookrightarrow X(\mathbf{A})^{\text{rcpr}} = X(\mathbf{A})^{\perp H^{n+1}_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(n))}.$$

1.4 Generalised global duality beyond finite coefficients

Later in this paper we will use $(n+2)$-dimensional global duality to detect elements in $H^2_{\text{ét}}(K,\mathbf{A};\mathbf{X}(M))$ for a finitely generated group scheme over K. Here a *finitely generated group scheme* over a perfect field k is a group scheme G such that $G(\bar{k})$ is a finitely generated group. By

$$\mathbf{X}(M) := \mathscr{H}om(M,\mathbf{G}_{\mathrm{m}})$$

we denote the Cartier dual of M.
Morally speaking, one would expect a nondegenerate pairing

$$H^n_{\text{ét}}(K, M \overset{L}{\otimes} \mathbf{Z}(n-1))) \times H^2_{\text{ét}}(K,\mathbf{A};\mathbf{X}(M)) \to H^{n+3}_{\text{ét}}(K,\mathbf{A};\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z}$$

for suitable 'motivic' complexes of sheaves $\mathbf{Z}(n-1)$ and $\mathbf{Z}(n)$ in the sense of Beilinson and Lichtenbaum (see [BMS], [L]; recall that we have $\mathbf{Z}(0)=\mathbf{Z}$ and $\mathbf{G}_{\mathrm{m}}=\mathbf{Z}(1)[1]$). Such a duality for 'integral' coefficients is known when K is a number field (cf. [Mi, §I.4]), but I do not know of such a full duality in any other case — even for K a p-adic function field, the results of [SvH] do give the required duality between $H^2_{\text{ét}}(K, M\otimes^L \mathbf{Z}(1))$ and $H^2_{\text{ét}}(K,\mathbf{A};\mathbf{X}(M))$, but this duality is obtained without introducing a complex $\mathbf{Z}(2)$.
To avoid these complications, we consider the torsion version

$$\begin{aligned}H^{n-1}_{\text{ét}}(K, M \overset{L}{\otimes} \mathbf{Q}/\mathbf{Z}(n-1))) \times H^2_{\text{ét}}(K,\mathbf{A};\mathbf{X}(M))&\\ \to H^{n+2}_{\text{ét}}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z}& \quad (1)\end{aligned}$$

which can be defined as the Yoneda pairing associated to the isomorphisms

$$M \overset{L}{\otimes} \mu_m^{\otimes n-1} \simeq R\mathscr{H}om(\mathbf{X}(M), \mathbf{G}_{\mathrm{m}} \overset{L}{\otimes} \mu_m^{\otimes n-1}) = R\mathscr{H}om(\mathbf{X}(M), \mu_m^{\otimes n}[1])$$

for all $m \in \mathbf{N}$.

Proposition 1.3. *Let K be a field that has $(n+2)$-dimensional global duality, and let M be a finitely generated group scheme over K. If $Ш^2(K,\mathbf{X}(M))$ is of finite exponent, then the pairing (1) is nondegenerate on the right.*

Proof. By hypothesis, we have an $N\in\mathbf{N}$ such that $Ш^2(K,\mathbf{X}(M))$ is N-torsion. By Hilbert's Theorem 90 and a restriction–corestriction argument we also have an $N'\in\mathbf{N}$ such that $H^1_{\text{ét}}(\mathbf{A},\mathbf{X}(M))$ is N'-torsion.
The long exact sequence of relative cohomology now implies that $H^2_{\text{ét}}(K,\mathbf{A};\mathbf{X}(M))$ is NN'-torsion, so this group embeds into $H^2_{\text{ét}}(K,\mathbf{A};\mathbf{X}(M)\otimes^L \mathbf{Z}/NN')$ by the Kummer sequence.
Global duality then implies that $H^2_{\text{ét}}(K,\mathbf{A};\mathbf{X}(M))$ embeds into the dual of $H^{n-1}_{\text{ét}}(K, M\otimes^L \mathbf{Z}/NN'(n-1)))$, hence into the dual of $H^{n-1}_{\text{ét}}(K, M\otimes^L \mathbf{Q}/\mathbf{Z}(n-1)))$. □

We will also use the following easy lemma.

Lemma 1.4. *Let K be a field that has $(n+2)$-dimensional global duality. The pairing*

$$H^{n+1}_{ét}(K, \mathbf{Q}/\mathbf{Z}(n)) \times H^1_{ét}(K, \mathbf{A}; \mathbf{Z}) \to H^{n+2}_{ét}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z} \qquad (2)$$

is nondegenerate on the right.

Proof. This follows easily from the fact that $H^1_{\text{ét}}(K, \mathbf{A}; \mathbf{Z}) = \left(\prod_{v\in\Omega} \mathbf{Z}\right)/\mathbf{Z}$, whereas $H^{n+1}_{\text{ét}}(K, \mathbf{Q}/\mathbf{Z}(n))$ surjects onto the kernel of the map $\bigoplus_{v\in\Omega} \mathbf{Q}/\mathbf{Z} = H^{n+1}_{\text{ét}}(\mathbf{A}, \mathbf{Q}/\mathbf{Z}(n)) \to H^{n+2}_{\text{ét}}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n))) = \mathbf{Q}/\mathbf{Z}$. $\square$

2 Pseudo-motivic homology

Let k be a field of characteristic zero. Let X be a nonsingular variety over k. We write X_{sm} for the smooth site over X (i.e., underlying category the smooth schemes of finite type over X and coverings the surjective smooth morphisms). For any sheaf $\mathscr{F}$ on X_{sm} we denote by $R\Gamma(X/K_{\text{sm}}, \mathscr{F})$ the total direct image in the derived category of sheaves on $(\operatorname{Spec} k)_{\text{sm}}$ of $\mathscr{F}$ under the structure morphism $X \to \operatorname{Spec} k$. With this notation we define

$$\begin{aligned}
\mathscr{C}^*(X, \mathbf{G}_{\text{m}}) &:= R\Gamma(X/k_{\text{sm}}, \mathbf{G}_{\text{m}}) \\
\mathscr{C}^{\text{c}}_*(X, \mathbf{Z}) &:= R\,\mathscr{H}\!om_{k_{\text{sm}}}(\mathscr{C}^*(X, \mathbf{G}_{\text{m}}), \mathbf{G}_{\text{m}}) \\
{}^1H^{\text{c}}_i(X, \mathbf{Z}) &:= H^{-i}(k_{\text{sm}}, \mathscr{C}^{\text{c}}_*(X, \mathbf{Z}))
\end{aligned}$$

When X is complete, we have that ${}^1H^{\text{c}}_i(X, \mathbf{Z}) = {}^1H_i(X, \mathbf{Z})$, *pseudo-motivic homology*, which was introduced and studied in [vH1] for nonsingular complete varieties. For a noncomplete X, we have that ${}^1H^{\text{c}}_i(X, \mathbf{Z})$ is *pseudo-motivic homology with compact supports*, which was studied in [vH2].

As in [vH2] we will work with a truncated version for technical reasons. We define

$$\begin{aligned}
\mathscr{C}^*(X, \mathbf{G}_{\text{m}})_\tau &:= \tau_{\leq 1} R\Gamma(X/k_{\text{sm}}, \mathbf{G}_{\text{m}}) \\
H^i(X, \mathbf{G}_{\text{m}})_\tau &:= H^i(k_{\text{sm}}, \mathscr{C}^*(X, \mathbf{G}_{\text{m}})_\tau) \\
\mathscr{C}^{\text{c}}_*(X, \mathbf{Z})_\tau &:= R\,\mathscr{H}\!om_{k_{\text{sm}}}(\mathscr{C}^*(X, \mathbf{G}_{\text{m}})_\tau, \mathbf{G}_{\text{m}}) \\
{}^1H^{\text{c}}_i(X, \mathbf{Z})_\tau &:= H^{-i}(k_{\text{sm}}, \mathscr{C}^{\text{c}}_*(X, \mathbf{Z})_\tau)
\end{aligned}$$

Finally, we will also need the associated 'truncated' cohomology theory with torsion coefficients, so we define

$$\begin{aligned}
\mathscr{C}^*(X, \mu_m^{\otimes j})_\tau &:= \mathscr{C}^*(X, \mathbf{G}_{\text{m}})_\tau \overset{L}{\otimes} \mu_m^{\otimes j} \\
H^i(X, \mu_m^{\otimes j})_\tau &:= H^i(k_{\text{sm}}, \mathscr{C}^*(X, \mu_m^{\otimes j})_\tau) \\
H^i(X, \mathbf{Q}/\mathbf{Z}(j))_\tau &:= \varinjlim_m H^i(X, \mu_m^{\otimes j})_\tau
\end{aligned}$$

Remark 2.1. We only need the smooth topology in the definition of the complexes $\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z})_{(\tau)}$. After that, the comparison between smooth cohomology and étale cohomology of the complexes of sheaves that we are using assures that we might as well compute everything on the étale site. In particular, there is no need to distinguish in notation between $H^*(k_{\mathrm{sm}},-)$ and $H^*(k_{\text{ét}},-)$, and we will normally just write $H^*(k,-)$

2.1 Some calculations

In the present paper we are interested in varieties with a finitely generated geometric Picard group. For these varieties the truncated pseudo-motivic homology has a very simple structure.

Lemma 2.2. *Assume X is a nonsingular complete geometrically irreducible variety over k such that the Picard scheme $\mathrm{Pic}_{X/k}$ is a finitely generated group scheme. Then we have a triangle*

$$\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}})[1]\to\mathscr{C}_*(X,\mathbf{Z})_\tau\to\mathbf{Z}.$$

Proof. By Cartier duality this follows from the fact that we have a triangle

$$\mathbf{G}_{\mathrm{m}}\to\mathscr{C}^*(X,\mathbf{G}_{\mathrm{m}})_\tau\to\mathrm{Pic}_{X/K}[-1]. \tag{3}$$

□

Corollary 2.3. *With X as above, we have a long exact sequence*

$$\cdots\to H^1(k,\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}}))\to{}^1H_0(X,\mathbf{Z})_\tau\to H^0(k,\mathbf{Z})$$
$$\to H^2(k,\mathscr{H}om(\mathrm{Pic}_{X/k},\mathbf{G}_{\mathrm{m}}))\to\cdots$$

Lemma 2.4. *Assume V is a nonsingular geometrically irreducible variety over k such that $\mathrm{Pic}_{V/k}=0$. Then we have a triangle*

$$\mathscr{H}om(\bar{k}[V]^*/\bar{k}^*,\mathbf{G}_{\mathrm{m}})\to\mathscr{C}_*(V,\mathbf{Z})_\tau\to\mathbf{Z}$$

Proof. Let $V\hookrightarrow X$ be a smooth compactification, let $Z\subset X$ be the closed complement, and let $\mathscr{Z}^1_Z(X/k)$ be the locally free abelian sheaf on k_{sm} associated to the Galois permutation module generated by the set of irreducible components of $\overline{Z}$ that are of codimension 1 in $\overline{X}$. Let M be the kernel of the surjective map of sheaves

$$\mathscr{Z}^1_Z(X/k)\to\mathrm{Pic}_{X/k}\,.$$

We have that M is a locally free abelian sheaf on k_{sm}, hence it is the sheaf associated to a torsion-free Galois module. It follows from the triangle (3) and the triangle [vH2, eq. (3)] that we have a triangle

$$\mathbf{G}_{\mathrm{m}}\to\mathscr{C}^*(V,\mathbf{Z})_\tau\to M.$$

Checking the global sections over $\bar{k}$ then gives that M is the sheaf corresponding to the finitely generated Galois module $\bar{k}[V]^*/\bar{k}^*$. □

Lemma 2.5. *Let X be a nonsingular projective variety over k and let $V \subset X$ be an open subvariety, then the natural map*

$$^1H_0^{\mathrm{c}}(V, \mathbf{Z})_\tau \to {}^1H_0(X, \mathbf{Z})_\tau$$

is surjective

Proof. This is part of [vH2, Cor. 1.5]. □

2.2 Homology classes of points

For any variety V over k we have that the covariantly functorial properties of pseudo-motivic homology give a natural map

$$V(k) \to {}^1H_0^{\mathrm{c}}(V, \mathbf{Z}).$$

We denote the homology class of a k-valued point $x \in V(k)$ by $[x]$, If x corresponds to a map $i\colon \operatorname{Spec} k \to V$ then $[x]$ corresponds to the morphism

$$R\Gamma(X/k, \mathbf{G}_{\mathrm{m}}) \to \mathbf{G}_{\mathrm{m}}$$

induced by the natural morphism

$$\mathbf{G}_{\mathrm{m}} \to i_*\mathbf{G}_{\mathrm{m}}$$

of sheaves on X. We will not make a distinction in notation between the class $[x] \in {}^1H_0^{\mathrm{c}}(V, \mathbf{Z})$ and its image under the truncation map ${}^1H_0^{\mathrm{c}}(V, \mathbf{Z}) \to {}^1H_0^{\mathrm{c}}(V, \mathbf{Z})_\tau$
The sheafified version of this map gives a morphism of sheaves (of sets) over k_{sm}

$$V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z})$$

with the image of V landing in the inverse image of 1 under the degree map

$$^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z}) \to \mathbf{Z}.$$

See [vH1] and [vH2] for more information.

Lemma 2.6. *Assume V is a nonsingular geometrically connected variety over k such that $\operatorname{Pic}_{V/k} = 0$. Then the morphism*

$$V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V, \mathbf{Z}) = R\,\mathscr{H}om_{k_{sm}}(\Gamma(V/k_{sm}, \mathbf{G}_{\mathrm{m}}), \mathbf{G}_{\mathrm{m}})$$

is given by locally sending a section $x \in V$ to the map that sends a local section f of $\Gamma(V/k_{sm}, \mathbf{G}_{\mathrm{m}})$ to $f(x)$.

Proof. This follows immediately from the definitions. □

3 The cap-product and partial generalised global duality for pseudo-motivic homology

3.1 Definition and basic properties of the cap product

Let X be a nonsingular variety over a field k of characteristic zero. Since $\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z}) = R\mathscr{H}om_{k_{\mathrm{sm}}}(\mathscr{C}^*(X,\mathbf{G}_{\mathrm{m}}),\mathbf{G}_{\mathrm{m}})$, we have well-defined Yoneda-products

$$ {}^1H^{\mathrm{c}}_i(X,\mathbf{Z}) \times H^j(X,\mathbf{G}_{\mathrm{m}}) \to H^{j-i}(k,\mathbf{G}_{\mathrm{m}}) $$
$$ {}^1H^{\mathrm{c}}_i(X,\mathbf{Z}) \times H^j(X,\mathbf{Q}/\mathbf{Z}(1)) \to H^{j-i}(k,\mathbf{Q}/\mathbf{Z}(1)). $$

Applying Tate twist to the torsion coefficients in the second pairing gives us

$$ {}^1H^{\mathrm{c}}_i(X,\mathbf{Z}) \times H^j(X,\mathbf{Q}/\mathbf{Z}(m)) \to H^{j-i}(k,\mathbf{Q}/\mathbf{Z}(m)). $$

for any $m \in \mathbf{Z}$. Similarly, we have the truncated versions

$$ {}^1H^{\mathrm{c}}_i(X,\mathbf{Z})_\tau \times H^j(X,\mathbf{G}_{\mathrm{m}})_\tau \to H^{j-i}(k,\mathbf{G}_{\mathrm{m}}) $$
$$ {}^1H^{\mathrm{c}}_i(X,\mathbf{Z})_\tau \times H^j(X,\mathbf{Q}/\mathbf{Z}(m))_\tau \to H^{j-i}(k,\mathbf{Q}/\mathbf{Z}(m)). $$

All these pairings can be called *cap-product pairings* and will be denoted by $-\cap-$.
For a k-valued point $x\colon \operatorname{Spec} k \hookrightarrow X$, and an $\omega \in H^j(X,\mathbf{Q}/\mathbf{Z}(m))_{(\tau)}$ we have that

$$ [x] \cap \omega = i^*\omega \in H^j(k,\mathbf{Q}/\mathbf{Z}(m)). \tag{4} $$

This follows easily from the definitions, in particular from the fact that the homology class $[x]$ is defined using the the natural maps $\mathbf{G}_{\mathrm{m}} \to i_*\mathbf{G}_{\mathrm{m}}$ and the pull-back homomorphism i^* is defined using the natural map $\mathbf{Q}/\mathbf{Z}(m) \to i_*\mathbf{Q}/\mathbf{Z}(m)$.

3.2 Partial generalised global duality for pseudo-motivic homology

Let K be a field of characteristic zero with global $(n+2)$-dimensional duality in étale cohomology, and let X be a nonsingular variety over K. We define:

$$ {}^1H^{\mathrm{c}}_i(X_{\mathbf{A}},\mathbf{Z})_{(\tau)} := H^{-i}(\mathbf{A},\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z})_{(\tau)}) $$
$$ {}^1H^{\mathrm{c}}_i(X,X_{\mathbf{A}};\mathbf{Z})_{(\tau)} := H^{-i}(K,\mathbf{A};\mathscr{C}^{\mathrm{c}}_*(X,\mathbf{Z})_{(\tau)}). $$

The cap product and the maps in cohomology for the pair $(K,\mathbf{A})$ give us pairings

$$ {}^1H^{\mathrm{c}}_i(X_{\mathbf{A}},\mathbf{Z})_{(\tau)} \times H^j(X,\mathbf{Q}/\mathbf{Z}(m))_{(\tau)} \to H^{j-i+1}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(m)) $$
$$ {}^1H^{\mathrm{c}}_i(X,X_{\mathbf{A}};\mathbf{Z})_{(\tau)} \times H^j(X,\mathbf{Q}/\mathbf{Z}(m))_{(\tau)} \to H^{j-i+1}(K,\mathbf{A};\mathbf{Q}/\mathbf{Z}(m)). $$

We will be interested in the case $i = 0$, $j = n + 1$, $m = n$. By equation (4) we get a commutative diagram of pairings

$$\begin{array}{ccccc} X(\mathbf{A}) & \times & H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_{(\tau)} & \xrightarrow{\mathrm{rcpr}} & \mathbf{Q}/\mathbf{Z} \\ \downarrow & & \| & & \| \\ {}^1H_0^{\mathrm{c}}(X_{\mathbf{A}}, \mathbf{Z})_{(\tau)} & \times & H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_{(\tau)} & \xrightarrow{\cap} & \mathbf{Q}/\mathbf{Z} \end{array} \tag{5}$$

The aim of this section is to identify the left kernel of the bottom pairing in the truncated version of the above diagram with the image of ${}^1H_0^{\mathrm{c}}(X, \mathbf{Z})_\tau$. Since the long exact sequences in the cohomology of the pair $(K, \mathbf{A})$ gives a long exact sequences in pseudo-motivic homology:

$$\cdots \to {}^1H_i^{\mathrm{c}}(X, X_{\mathbf{A}}, \mathbf{Z})_\tau \to {}^1H_i^{\mathrm{c}}(X, \mathbf{Z})_\tau \to {}^1H_i^{\mathrm{c}}(X_{\mathbf{A}}, \mathbf{Z})_\tau \to \\ {}^1H_{i-1}^{\mathrm{c}}(X, X_{\mathbf{A}}, \mathbf{Z})_\tau \to \cdots \tag{6}$$

it will be sufficient to prove that $H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau$ detects all elements in ${}^1H_{-1}^{\mathrm{c}}(X, X_{\mathbf{A}}, \mathbf{Z})_\tau$

Theorem 3.1. *Let X be a nonsingular complete variety over a field K having $(n+2)$-dimensional global duality in étale cohomology. Assume that $\mathrm{Pic}_{X/K}$ is a finitely generated group scheme and that $Ш^2(K, \mathbf{X}(\mathrm{Pic}_{X/K}))$ is of finite exponent. Then the cap-product pairing*

$${}^1H_{-1}(X, X_{\mathbf{A}}; \mathbf{Z})_\tau \times H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau \to H^{n+2}(K, \mathbf{A}; \mathbf{Q}/\mathbf{Z}(n)) = \mathbf{Q}/\mathbf{Z}$$

is nondegenerate on the left.

Proof. The triangle of Lemma 2.2 gives the following diagram of compatible pairings with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & H^2(K, \mathbf{A}; \mathbf{X}(\mathrm{Pic}_{X/K})) & \longrightarrow & {}^1H_{-1}(X, X_{\mathbf{A}}; \mathbf{Z})_\tau & \longrightarrow & H^1(K, \mathbf{A}; \mathbf{Z}) \\ & \times & & \times & & \times \\ 0 \leftarrow & H^{n-1}(K, \mathrm{Pic}_{X/K} \otimes^L \mathbf{Q}/\mathbf{Z}(n-1)) & \leftarrow & H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau & \leftarrow & H^{n+1}(K, \mathbf{Q}/\mathbf{Z}(n)) \\ & \downarrow & & \downarrow & & \downarrow \\ & \mathbf{Q}/\mathbf{Z} & = & \mathbf{Q}/\mathbf{Z} & = & \mathbf{Q}/\mathbf{Z} \end{array}$$

where the leftmost pairing is the pairing (1) and the rightmost pairing is the pairing of Lemma 1.4. Since those two pairings are nondegenerate on the $(K, \mathbf{A})$-side, it follows that the middle pairing is nondegenerate on the $(K, \mathbf{A})$-side as well. ☐

Corollary 3.2. *Let X be as in Theorem 3.1. Then the left kernel of the cap-product pairing*

$${}^1H_0(X_{\mathbf{A}}; \mathbf{Z})_\tau \times H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}$$

is precisely the image of the map

$${}^1H_0(X, \mathbf{Z})_\tau \to H_0(X_{\mathbf{A}}, \mathbf{Z})_\tau$$

Proof. This follows from Theorem 3.1, the exact sequence (6) for the pseudo-motivic homology of the pair $(X, X_{\mathbf{A}})$, and the fact that we have a compatible diagram of pairings

$$\begin{array}{ccccc} {}^1H_0(X_{\mathbf{A}};\mathbf{Z})_\tau & \times & H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau & \to & \mathbf{Q}/\mathbf{Z} \\ \downarrow & & \| & & \| \\ {}^1H_{-1}(X,X_{\mathbf{A}};\mathbf{Z})_\tau & \times & H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau & \to & \mathbf{Q}/\mathbf{Z} \end{array}$$

□

COROLLARY 3.3. *Let X be as in Theorem 3.1. If $X(\mathbf{A})^{\mathrm{rcpr}} \neq \emptyset$, then the degree map*

$$H_0(X,\mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective.

Proof. Take an adèlic point $\{x_v\} \in X(\mathbf{A})^{\mathrm{rcpr}}$. The compatibility between cap-product and the map $X(\mathbf{A}) \to {}^1H_0(X_{\mathbf{A}},\mathbf{Z})_\tau$ implies that its homology class

$$[\{x_v\}] \in {}^1H_0(X_{\mathbf{A}},\mathbf{Z})_\tau$$

is orthogonal to any $\omega \in H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))$, so certainly to any $\omega \in H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau$. Therefore, the homology class $[\{x_v\}]$ is the restriction of some $\gamma \in {}^1H_0(X,\mathbf{Z})$. Since each $[x_v] \in {}^1H_0(X_{K_v},\mathbf{Z})$ is of degree 1, the degree of γ is 1. □

4 PSEUDO-MOTIVIC HOMOLOGY OF COMPACTIFICATIONS OF TORSORS UNDER TORI

Throughout this section V will be a torsor under a torus T over a field k of characteristic zero.
Since $\mathrm{Pic}_{V/k} = 0$, Lemma 2.4 gives us that the complex $\mathscr{C}_*(V,\mathbf{Z})_\tau$ is in fact quasi-isomorphic to a sheaf which is represented by a group scheme locally of finite type (the extension of $\mathbf{Z}$ by a torus). We denote this group scheme by ${}^1\mathscr{H}_0^{\mathrm{c}}(V,\mathbf{Z})$.

PROPOSITION 4.1. (i) *The triangle of 2.4 is naturally isomorphic to the triangle of sheaves associated to the short exact sequence of group schemes*

$$0 \to T \to {}^1\mathscr{H}_0^{\mathrm{c}}(V,\mathbf{Z}) \to \mathbf{Z} \to 0$$

(ii) *The natural map $V \to {}^1\mathscr{H}_0^{\mathrm{c}}(V,\mathbf{Z})$ induces a T-equivariant isomorphism of V with the connected component of ${}^1\mathscr{H}_0^{\mathrm{c}}(V,\mathbf{Z})$ mapping to $1 \in \mathbf{Z}$.*

Proof. The first part of the proposition follows by Cartier duality from the sheafified version of Rosenlicht's result that we have a short exact sequence

$$0 \to \mathbf{G}_{\mathrm{m}} \to \Gamma(V/k_{\mathrm{sm}},\mathbf{G}_{\mathrm{m}}) \to \mathbf{X}(T) \to 0.$$

See [Ro], and also [Ray, Cor. VII.1.2.], [CTS, Prop. 1.4.2].
To get the second part of the proposition, we need the extra information that the map $\Gamma(V/k_{\rm sm}, \mathbf{G}_{\rm m}) \to \mathbf{X}(T)$ considered above is defined locally by sending a local section f of $\Gamma(V/k_{\rm sm}, \mathbf{G}_{\rm m})$ to the map that sends a local section t of T to $f(t\cdot x)/f(x)$ for any local section x of V. Comparing this with the description of the map $V \to {}^1\mathscr{H}_0^{\rm c}(V, \mathbf{Z})$ in Lemma 2.6 gives the desired result. □

Corollary 4.2. *For any field extension k'/k we have that the natural map*

$$V(k') \to {}^1H_0^{\rm c}(V_{k'}, \mathbf{Z})_\tau$$

gives a $T(k')$-equivariant isomorphism of $V(k')$ onto the subset of elements of ${}^1H_0^{\rm c}(V_{k'}, \mathbf{Z})_\tau$ of degree 1.

Proof. Immediate from the above, since ${}^1H_0^{\rm c}(V_{k'}, \mathbf{Z})_\tau = {}^1\mathscr{H}_0^{\rm c}(V, \mathbf{Z})(k')$ by Lemma 2.4. □

Corollary 4.3. *Let X be a nonsingular complete variety over k containing V as a Zariski-dense subvariety.*

(i) *The degree map*

$$^1H_0^{\rm c}(V, \mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective if and only if $V(k') \neq \emptyset$.

(ii) *The degree map*

$$^1H_0(X, \mathbf{Z})_\tau \to \mathbf{Z}$$

is surjective if and only if $X(k) \neq \emptyset$.

Proof. The first statement follows immediately from Corollary 4.2, whereas the second statement follows from the first combined with Lemma 2.5. □

Together with the results of Section 3 this implies the two theorems in the introduction.

Remark 4.4. It is clear from the above, that we can sharpen Theorem 1 by replacing the full group $H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))$ in the reciprocity pairing by the truncated group $H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau$, or its image under the map $H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))_\tau \to H^{n+1}(X, \mathbf{Q}/\mathbf{Z}(n))$.
In the case of a number field, this makes no difference, since for smooth compactifications of torsors under tori we have that our truncated cohomological Brauer group $H^2(X, \mathbf{G}_{\rm m})_\tau$ is equal to the full cohomological Brauer group $H^2(X, \mathbf{G}_{\rm m})$. (For an arbitrary variety X our truncated group $H^2(X, \mathbf{G}_{\rm m})_\tau$ is equal to the the so-called 'algebraic' cohomological Brauer group, i.e., the kernel of the map $H^2(X, \mathbf{G}_{\rm m}) \to H^2(\overline{X}, \mathbf{G}_{\rm m})$).

5 Comparison with the literature

Torsors under a torus T over a (generalised) global field K which are trivial everywhere locally are classified by $\text{Ш}^1(K,T)$. It follows from Rosenlicht's result and Hilbert Theorem 90 that $H^1(K,T)$ embeds into $H^2(K,\mathbf{X}(\mathrm{Pic}_{X/K}))$ for any smooth compactification X of a principal homogeneous space V under T. For a field K as in Theorem 1, duality gives that $\text{Ш}^1(K,T)$ embeds into the dual of $H^{n-1}(K,\mathrm{Pic}_{X/K}\otimes^L \mathbf{Q}/\mathbf{Z}(n-1))$, hence into the dual of $H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau$. Therefore, it is not very surprising that the reciprocity pairing detects a failure of the local–global principle.
The only problem is to relate the abstract 'arithmetic' pairing

$$\text{Ш}^1(K,T)\times H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}$$

to the 'geometric' reciprocity pairing

$$X(\mathbf{A})\times H^{n+1}(X,\mathbf{Q}/\mathbf{Z}(n))_\tau \to \mathbf{Q}/\mathbf{Z}.$$

We have seen that pseudo-motivic homology provides a nice conceptual intermediate to compare the two pairings, but there have been other approaches as well. The existing literature deals with number fields, so here we consider the Brauer group, rather than $H^2(X,\mathbf{Q}/\mathbf{Z}(1))_\tau$.
In [San] the comparison between the 'geometric' and the 'arithmetic' pairing is essentially done in Lemma 8.4, using explicit ways of representing classes in the Brauer group and explicit cochain calculations. If one would want to apply this approach to global duality fields of higher cohomological dimension, both the higher degree of the cochains and the fact that the coefficients would be in $\mathbf{Q}/\mathbf{Z}(n)$ should complicate things considerably.
A more conceptual approach, the *descent method*, due to Colliot-Thélène and Sansuc and described in [CTS] uses the concept of a *universal X-torsor* under groups of multiplicative type. The most streamlined version of this approach is probably presented in [Sk]. As in the present paper, the proof proceeds in two major steps. The first result is that for any nonsingular complete variety X over a number field K with $X(\mathbf{A}_K)^{\mathrm{Br}(X)_\tau}\neq\emptyset$ we have that the universal X-torsor exists. The second result is that for a smooth projective compactification of a torsor under a torus over K the universal X-torsor exists if and only if $X(K)\neq\emptyset$.
There is a very clear relation with the present paper: Colliot-Thélène and Sansuc show that the universal X-torsor exists if and only if the 2-fold extension of Galois modules

$$0\to\bar{k}^*\to\bar{k}(X)^*\to\mathrm{Div}(\overline{X})\to\mathrm{Pic}(\overline{X})\to 0$$

is trivial. This can be seen to be equivalent to the surjectivity of the degree map

$${}^1H_0(X,\mathbf{Z})_\tau\to\mathbf{Z}.$$

Therefore, the two steps of the proof in the present paper are equivalent to the two steps in the descent method, but in both steps the methods of proof are different. In particular in the first step the homology approach of the present paper seems much more efficient than the approach of Colliot-Thélène and Sansuc (or Skorobogatov's streamlined version in [Sk, Sec. 6.1]), where again the core of the proof is a comparison of the 'geometric' and the 'arithmetic' pairing using cocycle computations. Recently, Salberger has published a different proof of the first step in the descent method, which no longer needs explicit cocyle computations ([Sal, Prop. 1.26]). However, Salberger's proof does require a subtle cohomological construction that might not be easy to generalise to global fields of higher cohomological dimension.

References

[BMS] A. Beilinson, R. MacPherson, and V. Schechtman, *Notes on motivic cohomology*, Duke Math. J. 54 (1987), 679–710.

[CTS] J.-L. Colliot-Thélène and J.-J. Sansuc, *La descente sur les variétés rationnelles, II*, Duke Math. J. 54 (1987), 375–492.

[DW] C. Deninger and K. Wingberg, *Artin-Verdier duality for n-dimensional local fields involving higher algebraic K-sheaves*, J. Pure Appl. Algebra 43 (1986), no. 3, 243–255.

[L] S. Lichtenbaum, *Values of zeta-functions at nonnegative integers*, Number theory (Noordwijkerhout, 1983), Lect. Notes Math., vol. 1068, Springer-Verlag (1984), 127–138.

[Ma] Yu. I. Manin, *Le groupe de Brauer–Grothendieck en géométrie diophantienne*, Actes Congrès Intern. Math.(Nice 1970), vol. 1, Gauthiers-Villars (1971), 401–411.

[Mi] J.S. Milne, *Arithmetic duality theorems*, Perspectives in mathematics, no. 1, Academic Press (1986).

[Ray] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lect. Notes in Math., vol. 119, Springer-Verlag (1970).

[Ro] M. Rosenlicht, *Some rationality questions on algebraic groups*, Ann. Mat. Pura Appl. (4) 43 (1957), 25–50.

[Sal] P. Salberger, *On obstructions to the Hasse principle*, Arithmetic of higher dimensional algebraic varieties, (AIM Conference, Palo Alto 2002), Birkhäuser, to appear.

[San] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. 327 (1981), 12–80.

[SvH] C. Scheiderer and J. van Hamel, *Cohomology of tori over p-adic curves*, Math. Ann. 326 (2003), 155–183.

[Sk] A. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press (2001).

[vH1] J. van Hamel, *Lichtenbaum–Tate duality for varieties over p-adic fields*, preprint, http://www.maths.usyd.edu.au/u/vanhamel.

[vH2] J. van Hamel, *The Brauer–Manin obstruction for zero-cycles on Severi–Brauer fibrations over curves*, J. London Math. Soc. (2) 68 (2003), 317–337.

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Diffeotopy Functors of ind-Algebras and Local Cyclic Cohomology

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Abstract.

We introduce a new bivariant cyclic theory for topological algebras, called local cyclic cohomology. It is obtained from bivariant periodic cyclic cohomology by an appropriate modification, which turns it into a deformation invariant bifunctor on the stable diffeotopy category of topological ind-algebras. We set up homological tools which allow the explicit calculation of local cyclic cohomology. The theory turns out to be well behaved for Banach- and C^*-algebras and possesses many similarities with Kasparov's bivariant operator K-theory. In particular, there exists a multiplicative bivariant Chern-Connes character from bivariant K-theory to bivariant local cyclic cohomology.

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Introduction

A central topic of noncommutative geometry is the study of topological algebras by means of homology theories. The most important of these theories (and most elementary in terms of its definition) is topological K-theory. Various other homology theories have been studied subsequently. This has mainly been done to obtain a better understanding of K-theory itself by means of

theories which either generalize the K-functor or which provide explicitely calculable approximations of it. The latter is the case for cyclic homology, which was introduced by Connes [Co1], and independently by Tsygan [FT], in order to extend the classical theory of characteristic classes to operator K-theory, respectively to algebraic K-theory. Concerning operator K-theory, which is $\mathbb{Z}/2\mathbb{Z}$-graded by the Bott periodicity theorem, one is mainly interested in periodic cyclic theories. Periodic cyclic homology HP is defined as the homology of a natural $\mathbb{Z}/2\mathbb{Z}$-graded chain complex $\widehat{CC}_*$ associated to each complex algebra [Co1]. It can be expressed in terms of derived functors, which allows (in principle) its explicit calculation [Co]. There exists a natural transformation $ch: K_* \longrightarrow HP_*$, called the Chern-character, from K-theory to periodic cyclic homology [Co1]. If this Chern-character comes close to an isomorphism (after tensoring with $\mathbb{C}$), then periodic cyclic homology provides an explicitely calculable approximation of the K-groups one is interested in.
It turns out, however, that the Chern-character is often quite degenerate for Banach- and C^*-algebras. Unfortunately, this is the class of algebras, for which the knowledge of the K-groups would be most significant. The main reason for the degeneracy of Chern characters lies in the different functorial behavior of K-theory and periodic cyclic homology: due to its algebraic nature, cyclic homology possesses the continuity properties of K-theory only in a weak sense. The essential properties of K-theory are:

- Invariance with respect to (continuous) homotopies
- Invariance under topologically nilpotent extensions (infinitesimal deformations)
- Topological Morita invariance
- Excision
- Stability under passage to dense subalgebras which are closed under holomorphic functional calculus.
- Compatibility with topological direct limits

Periodic cyclic homology verifies only a list of considerably weaker conditions:

- Invariance with respect to diffeotopies (smooth homotopies) [Co1], [Go]
- Invariance under nilpotent extensions [Go]
- Algebraic Morita invariance [Co1]
- Excision [CQ2]

In the sequel we will ignore the excision property. The lists of remaining properties will be called strong respectively weak axioms.
To illustrate some of the differences between the two theories we discuss two well known examples.

Example 1: While for the algebra of smooth functions on a compact manifold M the periodic cyclic cohomology

$$HP^*(\mathcal{C}^\infty(M)) \simeq H_*^{dR}(M)$$

coincides with the de Rham homology of M [Co1], the periodic cyclic cohomology of the C^*-algebra of continuous functions on a compact space X is given by

$$HP^*(C(X)) \simeq C(X)'$$

the space of Radon measures on X [Ha]. Thus HP is not stable under passage to dense, holomorphically closed subalgebras. Taking $X = [0,1]$, one sees moreover that periodic cyclic (co)homology cannot be invariant under (continuous) homotopies.
Example 2: The inclusion $A \hookrightarrow M_n(A)$ of an algebra into its matrix algebra gives rise to a (co)homology equivalence by the Morita-invariance of HP. In contrast the inclusion $B \hookrightarrow \overline{\lim\limits_{n\to\infty} M_n(B)} = B \otimes_{C^*} \mathcal{K}$ ($\mathcal{K}$ the algebra of compact operators on a Hilbert space) of a C^*-algebra into its stable matrix algebra induces the zero map in periodic cyclic (co)homology [Wo]. Thus HP is not topologically Morita invariant. Moreover, it does not commute with topological direct limits. Finally it is known that periodic cyclic cohomology is not stable under topologically nilpotent extensions or infinitesimal deformations.
In order to obtain a good homological approximation of K-theory one therefore has to find a new cyclic homology theory which possesses a similar functorial behavior and is still calculable by means of homological algebra.
In this paper we introduce such a theory, called local cyclic cohomology. It is defined on the category of formal inductive limits of nice Fréchet algebras (ind-Fréchet algebras). A well behaved bivariant Chern-Connes character with values in bivariant local cyclic cohomology is constructed in [Pu2].
We proceed in two steps. In the first part of the paper we study diffeotopy functors of topological ind-algebras which satisfy the weak axioms. Our main result is a simple criterion, which guarantees that such a functor even satisfies the strong axioms. In the second part of the paper we modify periodic cyclic homology so that it satisfies this criterion and discuss the cyclic homology theory thus obtained.
A new basic object that emerges here is the stable diffeotopy category of ind-algebras (formal inductive limits of algebras). Its definition is in some sense similar to that of the stable homotopy category of spectra [Ad]. We construct first a triangulated prestable diffeotopy category, which possesses the usual Puppe exact sequences, by inverting the smooth suspension functor. Then we invert the morphisms with weakly contractible mapping cone to obtain the stable diffeotopy category. The criterion mentioned before can now be formulated as follows:

Theorem 0.1. *Let F be a functor on the category of ind-Fréchet algebras with approximation property [LT], which satisfies the weak axioms. Suppose that F is invariant under infinitesimal deformations and under stable diffeotopy, i.e. that it factors through the stable diffeotopy category. Then F also satisfies the strong axioms.*

In order to understand why this result holds we have to explain the significance of infinitesimal deformations. The approach of Cuntz and Quillen to periodic cyclic homology [CQ], [CQ1] emphasizes the invariance of the theory under quasinilpotent extensions. The corresponding notion for Fréchet algebras is that of a topologically nilpotent extension or infinitesimal deformation [Pu1], see also [Me], which is defined as an extension of Fréchet algebras with bounded linear section and topologically nilpotent kernel. Here a Fréchet algebra is called topologically nilpotent if the family of its relatively compact subsets is stable under taking multiplicative closures. Among the possible infinitesimal deformations of an algebra there is an initial or universal one [Pu], provided one works in the more general context of formal inductive limits of Fréchet algebras (or ind-Fréchet algebras). The universal infinitesimal deformation functor $\mathcal{T}$, which is left adjoint to the forgetful functor from ind-Fréchet algebras to a category with the same objects but with a more general kind of morphisms. These are the "almost multiplicative maps" which were introduced and studied in [Pu1]. By its very definition, every functor of ind-algebras, which is invariant under infinitesimal deformations, will be functorial with respect to almost multiplicative maps. This additional functoriality, which played already a fundamental role in [Pu], gives us the means to verify the strong axioms. For example, the inclusion of a dense, smooth subalgebra into a Banach algebra (with approximation property) possesses an almost multiplicative inverse up to stable diffeotopy. It is given by any family of linear regularization maps into the subalgebra, which converges pointwise to the identity. Thus this inclusion is turned by the given functor into an isomorphism.
It should be noted that there is an alternative way to introduce universal infinitesimal extensions, which is based on bornological algebras [Me]. This approach appears to be simpler, but does not seem to lead to homology theories which are accessible to calculation or which possess nice continuity properties. In order to obtain the results of this paper it is indispensable to work with ind-algebras (see section three). It allows to replace a large and complicated topological algebra by a large diagram of algebras of a very simple type. We split thus the information encoded in the initial data into a purely combinatorial and an algebro-analytic part of very particular type. This is reminiscent of algebraic topology where one replaces complicated spaces by model spaces given by simple building blocks and combinatorial gluing data.
In the second part of the paper we apply the results obtained so far to the functor given by bivariant periodic cyclic cohomology [CQ1]. For a pair of Fréchet algebras (A, B) it is given in terms of the natural cyclic bicomplex by

$$HP_*(A,B) := \quad Mor^*_{\mathfrak{Ho}}(\widehat{CC}(A), \widehat{CC}(B))$$

the group of chain homotopy classes of continuous chain maps of cyclic bicomplexes. Periodic cyclic (co)homology satisfies the weak axioms above, as mentioned at the beginning of the introduction. One can associate to it in a canonical way a homology theory which is invariant under infinitesimal deformations. This is analytic cyclic (co)homology

$$HC^{an}_*(A,B) := \quad Mor^*_{\mathfrak{Ho}}(\widehat{CC}(\mathcal{T}A), \widehat{CC}(\mathcal{T}B))$$

which was defined in [Pu] and developed in great generality in [Me]. Then we introduce the derived ind-category $\mathcal{D}$ which is obtained by localizing the chain homotopy category of $\mathbb{Z}/2\mathbb{Z}$-graded ind-chain complexes with respect to chain maps with weakly contractible mapping cone. Finally we define local cyclic cohomology as

$$HC^{loc}_*(A,B) := \quad Mor^*_{\mathcal{D}}(\widehat{CC}(\mathcal{T}A), \widehat{CC}(\mathcal{T}B))$$

Thus, by construction, local cyclic cohomology satisfies the assumptions of theorem (0.1). In particular, the first list of axioms holds for local cyclic cohomology, which behaves therefore very much like K-theory.

The second issue which distinguishes local cyclic cohomology among most other cyclic theories is its computability in terms of homological algebra.

There is a spectral sequence calculating morphism groups in the derived ind-category $\mathcal{D}$ which can be used to compute local cyclic cohomology groups. If $(\mathcal{C} = \text{“}\varinjlim_{i\in I}\text{”}\, C_i,\ \mathcal{C}' = \text{“}\varinjlim_{j\in J}\text{”}\, C'_j)$ is a pair of ind-chain complexes, then the E^2-term of the spectral sequence calculating $Mor^*_{\mathcal{D}}(\mathcal{C},\mathcal{C}')$ is given by

$$E^2_{pq} \;=\; R^p \varprojlim_{i\in I} \varinjlim_{j\in J} Mor^*_{\mathfrak{Ho}}(C_i, C'_j)$$

where $R^p \varprojlim_{i\in I}$ denotes the p-th right derived functor of the inverse limit over I.

If the cardinalities of the index set I is not too large, then the spectral sequence converges. A consequence of this result is the following theorem which is at the basis of most calculations of local cyclic cohomology groups.

Theorem 0.2. (Limit theorem)
Suppose that the Banach algebra A is the topological direct limit of the countable family of Banach algebras $(A_n)_{n\in\mathbb{N}}$ and suppose that A satisfies the approximation property (see (6.16)). Then there is a natural isomorphism

$$\lim_{n\to\infty} HC^{loc}_*(A_n) \quad \overset{\simeq}{\longrightarrow} \quad HC^{loc}_*(A)$$

of local cyclic homology groups and a natural short exact sequence

$$0 \longrightarrow \quad \varprojlim_n{}^1 HC^{*-1}_{loc}(A_n) \quad \longrightarrow \quad HC^*_{loc}(A) \quad \longrightarrow \quad \varprojlim_n HC^*_{loc}(A_n) \quad \longrightarrow 0$$

of local cyclic cohomology groups.

Although it might seem that nothing has been gained in this way, because one intractable cohomology group has been replaced by a limit of similarly intractable objects, the spectral sequence proves to be a surprisingly efficient tool for computations. The reason lies in the fact that, although the involved groups are mostly unknown, the transition maps in the corresponding limits often turn out to be quite accessible. We present a number of explicit calculations of local cyclic cohomology groups, which illustrate this principle.

The content of the different sections is as follows. In section one we introduce the notions of almost multiplicative morphism, topologically nilpotent extension, and universal infinitesimal deformation, which are used throughout the paper. This material is taken from [Pu1]. In section two the stable diffeotopy category of ind-algebras is introduced and section three presents various results about the stable diffeotopy type of universal infinitesimal deformations. The main theorem mentioned before is proved in section four. It is applied in section five to periodic cyclic homology. After a short review of the known cyclic homology theories we introduce local cyclic cohomology. In section six we develop the tools for computing local cyclic cohomology groups. Various natural transformations relating the different cyclic theories are discussed in section seven, see also [Me1], and in section eight we give examples of calculations of local cyclic cohomology. We present there also a partial solution of a problem posed by A. Connes in [Co3]. More detailed information can be found in the introductions to the various sections.

This paper is a completely revised and rewritten version of the preprint [Pu1]. A precursor of the theory presented here is asymptotic cyclic cohomology, which was introduced in [CM] and developed in [Pu]. While it shares the good functorial properties of local cyclic cohomology, there is no way to calculate asymptotic cyclic cohomology by homological means.

The excision property of local cyclic cohomology is a consequence of excision in analytic cyclic cohomology and is shown in [Pu2]. In that paper we construct a multiplicative bivariant Chern-Connes character

$$ch_{biv}: \quad KK_*(-,-) \quad \longrightarrow \quad HC_*^{loc}(-,-)$$

from Kasparov's bivariant K-theory [Ka] to bivariant local cyclic cohomology. This character provides a good approximation of (bivariant) K-theory. An equivariant version of the bivariant Chern-Connes character and the computational tools developed in this paper are used in [Pu4] and [Pu5] to verify the Kadison-Kaplansky idempotent conjecture in various cases. These applications show the potential power of local cyclic cohomology as a tool for solving problems in noncommutative geometry. The present paper and the articles [Pu2] and [Pu5] form the published version of the authors Habilitationsschrift presented at the Westfälische Wilhelms-Universität Münster.

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Contents

1 Topological ind-algebras and their universal infinitesimal deformations

1.1 Nice Fréchet algebras

A convenient category of algebras to work with for the purpose of this paper is the category of nice (or admissible) Fréchet algebras. These algebras are in many ways similar to Banach algebras. In addition, they are stable under a number of operations which cannot be performed in the category of Banach algebras, for example, the passage to a dense, holomorphically closed subalgebra. (We decided to replace the name "admissible Fréchet algebra" used in [Pu] and [Pu1] by that of a "nice Fréchet algebra" because the old terminology seemed us too ugly.)

Definition 1.1. [Pu] A Fréchet algebra A is called nice iff there exists an open neighborhood U of zero such that the multiplicative closure of any compact subset of U is precompact in A.

The open set U is called an "open unit ball" for A. It is by no means unique.
The class of nice Fréchet algebras contains all Banach algebras and derived subalgebras of Banach algebras [BC] and many Fréchet algebras which occur as dense, holomorphically closed subalgebras of Banach algebras.
Nice Fréchet algebras share a number of properties with Banach algebras: the spectrum of an element of a nice Fréchet-algebra is compact and nonempty and holomorphic functional calculus is valid in nice Fréchet-algebras. (This is most easily seen by noting that according to (1.5) a nice Fréchet algebra is the algebraic direct limit of Banach algebras. Another proof can be found in [Pu], section 1.)
The class of nice Fréchet algebras is closed under taking projective tensor products [Pu], (1.17). If A is nice with open unit ball U and if X is a compact space then the Fréchet algebra $C(X, A)$ is nice with open unit ball $C(X, U)$.

1.2 Formal inductive limits

In the sequel we will work with certain diagrams of algebras. An appropriate language to deal with such diagrams is provided by the notion of a formal inductive limit.

Definition 1.2. Let $\mathcal{C}$ be a category. The category ind-$\mathcal{C}$ of ind-objects or formal inductive limits over $\mathcal{C}$ is defined as follows.
The objects of ind-$\mathcal{C}$ are small directed diagrams over $\mathcal{C}$:

$$\begin{aligned}\mathrm{Ob}_{\text{ind-}\mathcal{C}} &= \{\text{“}\varinjlim_{i\in I}\text{”} A_i \,\Big|\, I \text{ a partially ordered directed set }\}\\ &= \{A_i, f_{ij} : A_i \to A_j,\ i \leq j \in I \,\Big|\, f_{jk}\circ f_{ij} = f_{ik}\}\end{aligned}$$

The morphisms between two ind-objects are given by

$$\mathrm{Mor}_{\text{ind-}\mathcal{C}}(\text{“}\varinjlim_{i\in I}\text{”}A_i,\ \text{“}\varinjlim_{j\in J}\text{”}B_j)\ :=\ \varprojlim_{i\in I}\varinjlim_{j\in J}\mathrm{Mor}_{\mathcal{C}}(A_i,B_j)$$

where the limits on the right hand side are taken in the category of sets.

There exists a fully faithful functor $\iota:\mathcal{C}\to\text{ind-}\mathcal{C}$ which identifies $\mathcal{C}$ with the full subcategory of constant ind-objects.

Lemma 1.3. *In ind-$\mathcal{C}$ there exist arbitrary inductive limits over directed index sets.*

This is [SGA], I, 8.5.1. Inductive limits in an ind-category will be denoted by $\underrightarrow{Lim}$. Even if direct limits exist in $\mathcal{C}$, they are usually different from the corresponding direct limit in ind-$\mathcal{C}$. If $(A_i)_{i\in I}$ is a small directed diagram in $\mathcal{C}$ which is viewed as diagram of constant ind-objects, then

$$\underset{i\in I}{\underrightarrow{Lim}}\,A_i\ \simeq\ \text{“}\varinjlim_{i\in I}\text{”}A_i$$

as objects of ind-$\mathcal{C}$. If $F:\mathcal{C}\to\mathcal{C}'$ is a functor to a category in which direct limits exist, then F possesses a unique extension $F':\text{ind-}\mathcal{C}\to\mathcal{C}'$ which commutes with direct limits. One has

$$F'(\text{“}\varinjlim_{i\in I}\text{”}A_i)\ =\ \varinjlim_{i\in I}F(A_i)$$

1.3 Diagrams of compactly generated algebras

It is our aim to construct and study continuous functors on categories of topological algebras. By this we mean either functors which are determined by their values on suitable families of dense subalgebras or more generally functors which commute with topological direct limits. The first step towards the construction of such functors will be the functorial replacement of "large" algebras by infinite diagrams of "small" algebras of a particular type. Part of the structure of a "large" algebra will be encoded in the combinatorics of the diagram and one is left with the study of "small" algebras with peculiar properties. The natural choice for these "small" algebras will be the Banach algebras generated by the compact subsets of the original algebra.

Definition and Lemma 1.4. *There exists a functor $\mathcal{B}$ from the category of nice Fréchet algebras to the category of ind-Banach algebras which assigns to a nice Fréchet algebra the diagram of minimal Banach completions of its compactly generated subalgebras.*

Proof: Let A be a nice Fréchet algebra and let U be an open unit ball for A. Fix a compact subset $S\subset U$ and denote by $A[S]$ the subalgebra of A generated by S. There exists a largest submultiplicative seminorm on $A[S]$ which satisfies $\| S \| \leq 1$. For $x \in A[S]$ it is given by

$$\| x \| = \inf_{x=\sum a_i s_i} \sum |a_i|$$

where the infimum is taken over the set of all presentations $x = \sum a_i s_i$ such that $a_i \in \mathbb{C}$ and $s_i \in S^\infty$, the multiplicative closure of S. The completion of $A[S]$ with respect to this seminorm is a Banach algebra denoted by A_S. Any inclusion $S \subset S' \subset U$ of compact subsets of U gives rise to a bounded homomorphism of Banach algebras $A_S \to A_{S'}$ so that one obtains an ind-Banach algebra

$$\mathcal{B}(A, U) := \text{“} \varinjlim_{\substack{S \subset U \\ S\, compact}} \text{”} A_S$$

Let $f : A \to A'$ be a bounded homomorphism of nice Fréchet algebras and fix open unit balls $U \subset A$ and $U' \subset A'$. For $S \subset U$ compact let $S' \subset U'$ be a compact set which absorbs $f(S^\infty)$ (S^∞ denotes the multiplicative closure of S). This is possible because S^∞ is precompact (A is nice) and f is bounded. The map f gives then rise to a bounded homomorphism $A_S \to A'_{S'}$. The collection of all these homomorphisms defines a morphism of ind-Banach algebras
$f_* : \mathcal{B}(A, U) \to \mathcal{B}(A', U')$. Applying this to the case $f = id$ shows that the ind-Banach algebra $\mathcal{B}(A, U)$ does not depend (up to unique isomorphism) on the choice of U. It will henceforth be denoted by $\mathcal{B}(A)$. The construction above shows furthermore that $\mathcal{B}(-)$ is a functor from the category of nice Fréchet algebras to the category of ind-Banach algebras. □

Lemma 1.5. *There exists a natural transformation of functors*

$$\phi : \mathcal{B} \to \iota$$

(see (1.2)). It is provided by the tautological homomorphism

$$\mathcal{B}(A) = \text{“} \varinjlim_{S \subset U} \text{”} A_S \to A$$

In fact $\varinjlim_{S \subset U} A_S = A$ *in the category of abstract algebras.*

Proof: The fact that the multiplicative closure of a compact subset S of a unit ball of A is precompact implies that the inclusion $A[S] \to A$ extends to a bounded homomorphism $A_S \to A$. These fit together to a bounded homomorphism

$$\phi_A : \mathcal{B}(A) \to \iota(A)$$

of ind-Fréchet algebras. It is clear that the homomorphisms ϕ_A define a natural transformation as claimed by the lemma. In fact

$$\varinjlim_S A[S] \xrightarrow{\simeq} \varinjlim_S A_S \xrightarrow{\simeq} A$$

where the limit is taken in the category of abstract algebras. This yields the second assertion. □

Lemma 1.6. *The functor $\mathcal{B}$ is fully faithful.*

Proof: Let $\psi : \mathcal{B}(A) \to \mathcal{B}(A')$ be a morphism of ind-Banach algebras. It gives rise to a homomorphism

$$\psi' : \quad A = \lim_{\substack{\longrightarrow \\ S}} A_S \quad \longrightarrow \quad \lim_{\substack{\longrightarrow \\ S'}} A'_{S'} = A'$$

of abstract algebras which maps precompact sets to bounded sets and is therefore bounded. This defines a map $mor_{ind-Alg}(\mathcal{B}(A), \mathcal{B}(A')) \to mor_{Alg}(A, A')$ which is clearly inverse to the map on morphism sets induced by $\mathcal{B}$. Therefore the functor $\mathcal{B}$ is fully faithful. □

The canonical extension of the functor $\mathcal{B}$ to the category of nice ind-Fréchet algebras (1.2) will be denoted by the same letter. To study it in further detail we introduce the following notion.

Definition 1.7. An ind-Banach algebra is called compact if it is isomorphic to an ind-Banach algebra "$\lim\limits_{\substack{\longrightarrow \\ i \in I}}$"$A_i$ satisfying the following condition: for every $i \in I$ there exists $i' \geq i$ such that the structure homomorphism $A_i \to A_{i'}$ is compact.

The proof of the following results is facilitated by the technical

Lemma 1.8. *Define a functor $\mathcal{B}'$ from the category of nice Fréchet algebras to the category of ind-Banach algebras by*

$$\mathcal{B}'(A) := \text{“} \lim_{\substack{\longrightarrow \\ S \subset U \\ S\, nullsequence}} \text{”} A_S$$

Then the canonical natural transformation $\mathcal{B}' \to \mathcal{B}$ is an isomorphism of functors.

Proof: This follows from the fact that every compact subset of a Fréchet space is contained in the convex hull of a nullsequence. (A proof is given for example in [Pu], (1.7).) The actual argument is however rather long and tedious and quite close to the one given in the proof of [Pu], (5.8). □

Lemma 1.9. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra. Then the ind-Banach algebra $\mathcal{B}(\mathcal{A})$ is compact.*

Proof: It suffices by lemma (1.8) to verify that $\mathcal{B}'(A)$ is compact for every nice Fréchet algebra A. Let S be a nullsequence in U. As A is nice the multiplicative closure of S is a nullsequence $S^\infty = (a_n)_{n \in \mathbb{N}}$. Choose a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of strictly positive real numbers tending to infinity such that $S' := (\lambda_n a_n)_{n \in \mathbb{N}}$ is still a nullsequence and let $S'' \subset U$ be a compact set absorbing S'. The induced homomorphism $A_S \to A_{S''}$ of the Banach algebras generated by S respectively. S'' is then compact. □

Proposition 1.10. *The functor $\mathcal{B}$ is right adjoint to the forgetful functor from the category of compact ind-Banach algebras to the category of nice ind-Fréchet algebras. In fact for every compact ind-Banach algebra $\mathcal{A}$ and every nice ind-Fréchet algebra $\mathcal{A}'$ the natural transformation ϕ (1.5) induces an isomorphism*

$$Mor_{ind-alg}(\mathcal{A},\, \mathcal{B}(\mathcal{A}')) \xrightarrow{\simeq} Mor_{ind-alg}(\mathcal{A},\, \mathcal{A}')$$

Proof: This is an immediate consequence of the definitions. □

Corollary 1.11. *For every compact ind-Banach algebra $\mathcal{A}$ the canonical homomorphism $\phi_{\mathcal{A}} : \mathcal{B}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{A}$ is an isomorphism.*

Proof: The corollary follows from (1.9) by applying twice the adjunction formula (1.10). □

Corollary 1.12. *The canonical natural transformation*

$$\phi_{\mathcal{B}} :\, \mathcal{B}^2 := \mathcal{B} \circ \mathcal{B} \xrightarrow{\simeq} \mathcal{B}$$

is an isomorphism of functors on the category of nice ind-Fréchet algebras.

Proof: This follows from (1.9) and (1.11). □

1.4 Almost multiplicative maps

A fundamental property of operator K-theory and other homology theories for topological algebras is the invariance under quasinilpotent extensions or infinitesimal deformations. The Cuntz-Quillen approach to periodic cyclic homology [CQ], [CQ1] for example is based on the deformation invariance of the theory.

Invariance under infinitesimal deformations is equivalent to the given theory being functorial not only under homomorphisms between algebras but also under homomorphisms between their quasinilpotent extensions. In other words, a deformation invariant theory extends to a functor on the category obtained by inverting epimorphisms with linear section and quasinilpotent kernel.

It is this "extended functoriality" which makes deformation invariance relevant for us and which will play a fundamental role in the present work.

The notions of (quasi)nilpotent extensions or infinitesimal deformations are well known for abstract and adically complete algebras. We develop the corresponding notions for complete locally convex algebras in close analogy. The existence of a universal quasinilpotent extension of adically complete algebras allows to describe explicitely the morphisms in the extended category. These correspond to linear maps $f : R \longrightarrow S$, for which products of a large number of curvature terms [CQ]

$$\omega_f(a,a') :=\, f(a\cdot a') \,-\, f(a)\cdot f(a'),\;\; a,a' \in R,$$

(measuring the deviation from multiplicativity) are small. It is straightforward to define the corresponding kind of morphisms for diagrams of Fréchet algebras

which leads to the notion of an almost multiplicative map. As such maps are stable under composition one obtains in this way a category. (In fact, we will introduce two different notions of almost multiplicativity depending on whether we are interested in uniform estimates or in estimates which are uniform on compact subsets only.)
We construct the universal infinitesimal deformation functors in the topological context as left adjoints of the corresponding forgetful functors to the almost multiplicative categories. This allows finally to introduce the notions of topological nilpotence and of a topologically quasifree algebra.
The basic motivation to study the class of almost multiplicative maps is that it contains a lot of interesting examples, in particular if one passes to diffeotopy categories. Essentially all the results of section three follow immediately from the existence of certain almost multiplicative morphisms in the stable diffeotopy category. We will comment on this fact in the introduction to section three.
An important class of almost multiplicative maps is provided by the (linear) asymptotic morphisms introduced by Connes and Higson [CH]. These are used by them to construct a universal bivariant K-functor for C^*-algebras. It will turn out that the stable diffeotopy category of universal infinitesimal deformations possesses a lot of similarities with the Connes-Higson category.
The reason for introducing topologically quasifree ind-algebras lies in their excellent homological behavior, which is similar to that of quasifree abstract (or adically complete) algebras exploited in [CQ]. The fact that universal infinitesimal deformations are topologically quasifree will allow us in section five to topologize the cyclic complexes of ind-Fréchet algebras in a straightforward way. The correct topologies on these complexes are not completely easy to find otherwise.
While for us the notion of topological nilpotence plays a minor role (compared to the notion of almost multiplicative morphisms), topological nilpotence is at the heart of the approach to analytic cyclic cohomology for bornological algebras presented by Meyer in his thesis [Me].
We begin by introducing a quite restrictive class of almost multiplicative maps. It will be used to facilitate the construction of universal infinitesimal deformations.

Definition 1.13. For a linear map $f : A \to B$ of algebras and a subset $T \subset A$ put

$$\omega(f,T) := \{\, f(aa') - f(a)f(a') \,|\, a, a' \in T \,\} \subset B$$

a) A bounded linear map $f : A \to B$ of Banach algebras is called strongly almost multiplicative if

$$\lim_{n\to\infty} \parallel \omega(f,T)^n \parallel^{\frac{1}{n}} = 0$$

for any bounded subset T of A.

b) A bounded linear morphism $\Phi = (\phi_{ij}) : \text{“}\varinjlim_{i\in I}\text{”} A_i \to \text{“}\varinjlim_{j\in J}\text{”} B_j$ of ind-Banach algebras is called strongly almost multiplicative if for all $i \in I$

and all bounded subsets $T_i \subset A_i$

$$\lim_{\overrightarrow{j\in J}} \overline{\lim_{n\to\infty}} \parallel \omega(\phi_{ij}, T_i)^n \parallel^{\frac{1}{n}} = 0$$

This is independent of the choice of the family of homomorphisms (ϕ_{ij}) representing the morphism Φ of ind-objects.

The more basic notion of almost multiplicativity is the following.

DEFINITION 1.14. a) A bounded linear map $f : A \to B$ of nice Fréchet algebras is called ALMOST MULTIPLICATIVE if for every compact subset $K \subset A$ the multiplicative closure

$$\omega(f, K)^\infty$$

of $\omega(f, K)$ is relatively compact in B.

b) A bounded linear morphism $\Psi = (\psi_{ij}) :$ "$\lim_{\overrightarrow{i\in I}}$"$A_i \to$ "$\lim_{\overrightarrow{j\in J}}$"$B_j$ of nice ind-Fréchet algebras is called almost multiplicative if for all $i \in I$ and all compact subsets $K_i \subset A_i$ the multiplicative closure

$$\omega(\psi_{ij}, K_i)^\infty$$

is relatively compact for sufficiently large $j \in J$.

It follows immediately from the definitions that a bounded linear morphism $\Psi : \mathcal{A} \to \mathcal{A}'$ of nice ind-Fréchet algebras is almost multiplicative if and only if $\mathcal{B}(\Psi) : \mathcal{B}(\mathcal{A}) \to \mathcal{B}(\mathcal{A}')$ is a strongly almost multiplicative morphism of ind-Banach algebras.

PROPOSITION 1.15. *The composition of strongly almost multiplicative morphisms of ind-Banach algebras is strongly almost multiplicative. Ind-Banach algebras therefore form a category under strongly almost multiplicative bounded linear morphisms. The same assertions hold for nice ind-Fréchet algebras and almost multiplicative maps.*

PROOF: It suffices by the previous remark to verify the proposition in the case of strongly almost multiplicative linear maps. For any linear map $\varphi : R \to S$ of algebras let $\omega_\varphi(r, r') := \varphi(rr') - \varphi(r)\varphi(r')$. If $f : A \to B$ and $g : B \to C$ are linear maps of algebras then the deviation from multiplicativity of $g \circ f$ is given by

$$\omega_{g\circ f}(a, a') = g(\omega_f(a, a')) + \omega_g(f(a), f(a'))$$

If f and g are bounded linear maps of Banach algebras and if $a_0, \ldots, a_{2n}$ are elements of the unit ball U of A then the previous equation and the "Bianchi identity" [CQ], (1.2)

$$\omega_f(a, a') \cdot f(a'') = \omega_f(a, a' \cdot a'') - \omega_f(a \cdot a', a'') + f(a) \cdot \omega(a', a'')$$

allow to express elements of $\omega(g\circ f)^n$ naturally in normal form as

$$\omega_{g\circ f}(a_1,a_2)\cdot\ldots\cdot\omega_{g\circ f}(a_{2n-1},a_{2n})=\sum_j g(\alpha_0^j)\omega_g(\alpha_1^j,\alpha_2^j)\cdot\ldots\cdot\omega_g(\alpha_{2k_j-1}^j,\alpha_{2k_j}^j)$$

where each element α_i^j is of the form

$$\alpha_i=f(a'_0)\omega_f(a'_1,a'_2)\cdot\ldots\cdot\omega_f(a'_{2l_i-1},a'_{2l_i}),\ \ a'_0,\ldots,a'_{2l_i}\in U$$

Moreover

$$\sharp\omega_f:=\sum_{i=0}^{2k_j}l_i\leq n\,,\quad \sharp\omega_g:=k_j\leq n\,,\quad \sharp\omega_f+\sharp\omega_g\geq n$$

for all j and the number $\sharp j$ of summands is bounded by $\sharp j\leq 9^n$. For all this see [Pu] (5.1). An easy calculation allows to deduce from these estimates that strongly almost multiplicative maps of Banach algebras are stable under composition. □

The following example provides a large number of almost multiplicative maps. Moreover it gives a hint why the stable diffeotopy category of universal infinitesimal deformations possesses similarities with the categories related to bivariant K-theories [CH],[Hi].

Example 1.16. *Let $f_t:A\to B$ be a linear asymptotic morphism of Banach algebras (or nice Fréchet-algebras) [CH], i.e. $(f_t)_{,t\geq 0}$ is a bounded continuous family of bounded linear maps such that*

$$\lim_{t\to\infty}f_t(aa')-f_t(a)f_t(a')=0\quad\forall a,a'\in A$$

Let $\tilde f:A\to C_b(\mathbb{R}_+,B)$ be the associated linear map satisfying $\mathrm{eval}_t\circ\tilde f=f_t$*. Then $\tilde f$ defines an almost multiplicative linear map*

$$\tilde f:A\to\ \text{“}\lim_{t\to\infty}\text{”}\,C_b([t,\infty[,B)$$

The class of almost multiplicative maps is considerably larger than the class of asymptotic morphisms. Whereas the curvature terms $\omega_f(a,a'),a,a'\in A$ of a linear asymptotic morphism become arbitrarily small in norm, almost multiplicativity means only that products of a large number of such terms become small in norm. So in particular the spectral radius of the curvature terms has to be arbitrarily small. This will explain an important difference between the homotopy category of asymptotic morphisms, used in E-theory, and the stable diffeotopy category of universal infinitesimal deformations: in the latter one the universal deformation of a Banach algebra is often isomorphic to the universal deformations of its dense and holomorphically closed subalgebras.

1.5 INFINITESIMAL DEFORMATIONS AND TOPOLOGICALLY NILPOTENT ALGEBRAS

With the notion of almost multiplicative morphism at hand one can introduce the topological analogs of nilpotence, infinitesimal deformation and formal smoothness.
As mentioned in the introduction of this section, Meyer has introduced a much more general notion of topological nilpotence which plays a crucial role in his approach to analytic cyclic cohomology for bornological algebras. We refer the reader to [Me].
We give here a slightly less general definition than the one in [Pu1], which suffices however for our purpose.

DEFINITION 1.17. a) A Banach algebra A is STRONGLY TOPOLOGICALLY NILPOTENT if the multiplicative closure of every norm bounded subset is norm bounded.

b) A Fréchet algebra A is TOPOLOGICALLY NILPOTENT if the multiplicative closure of every relatively compact subset is relatively compact.

c) An ind-Banach algebra "$\varinjlim\limits_{i\in I}$"$A_i$ is strongly topologically nilpotent if for each $i \in I$ and each bounded subset $U_i \subset A_i$ there exists $i' \geq i$ such that the image of the multiplicative closure U_i^∞ in $A_{i'}$ is bounded.

d) A nice ind-Fréchet algebra "$\varinjlim\limits_{j\in J}$"$B_j$ is topologically nilpotent if for each $j \in J$ and each compact subset $K_j \subset B_j$ there exists $j' \geq j$ such that the image of the multiplicative closure K_j^∞ in $B_{j'}$ is relatively compact.

Note that a topologically nilpotent Fréchet algebra is necessarily nice, the algebra itself being a possible open unit ball.

DEFINITION 1.18. Let

$$0 \;\to\; \mathcal{I} \;\to\; \mathcal{R} \;\xrightarrow{\pi}\; \mathcal{S} \;\to\; 0$$

be an extension of nice ind-Fréchet algebras (ind-Banach algebras) which possesses a bounded linear section. In particular, the ind-Fréchet space underlying $\mathcal{R}$ splits into the direct sum of $\mathcal{I}$ and $\mathcal{S}$. Then $\mathcal{R}$ is called a (STRONG) INFINITESIMAL DEFORMATION of $\mathcal{S}$ iff $\mathcal{I}$ is (strongly) topologically nilpotent.

The generic example of a (strongly) almost multiplicative map is given by

LEMMA 1.19. *Let*

$$0 \;\to\; \mathcal{I} \;\to\; \mathcal{R} \;\xrightarrow{\pi}\; \mathcal{S} \;\to\; 0$$

be a (strong) infinitesimal deformation of the nice ind-Fréchet algebra (ind-Banach algebra) $\mathcal{S}$. Then every bounded linear section of π is (strongly) almost multiplicative.

PROOF: Obvious from the definitions. □

We finally extend the notion of formal smoothness to the context of locally convex ind-algebras.

DEFINITION 1.20. A nice ind-Fréchet algebra (ind-Banach algebra) $\mathcal{A}$ is called (STRONGLY) TOPOLOGICALLY QUASIFREE if

$$Mor_{ind-alg}(\mathcal{A},\mathcal{R}) \xrightarrow{\pi_*} Mor_{ind-alg}(\mathcal{A},\mathcal{S})$$

is surjective for any (strong) infinitesimal deformation $\pi : \mathcal{R} \to \mathcal{S}$.

1.6 THE UNIVERSAL INFINITESIMAL DEFORMATION

In this section the universal infinitesimal deformation functor is introduced as the adjoint of the forgetful functor to the category of ind-algebras under almost multiplicative maps.

The construction of the universal infinitesimal deformation proceeds in two steps. The existence of a universal strong deformation of ind-Banach algebras is established first. Then the universal strong deformation of the diagram of compactly generated subalgebras of a nice ind-Fréchet algebra is identified as the universal deformation of the ind-algebra itself.

THEOREM 1.21. *The forgetful functor from the category of ind-Banach algebras to the category with the same objects and strongly almost multiplicative linear maps as morphisms possesses a left adjoint $\mathcal{T}'$, called the* STRONG UNIVERSAL INFINITESIMAL DEFORMATION *functor. This means that for all ind-Banach algebras $\mathcal{R}$, $\mathcal{S}$ there exists a natural and canonical isomorphism*

$$Mor_{\substack{ind\\alg}}(\mathcal{T}'\mathcal{R},\mathcal{S}) \xrightarrow{\simeq} Mor_{\substack{str\\alm\\mult}}(\mathcal{R},\mathcal{S})$$

PROOF: We proceed in several steps.

- We cite from [CQ], (1.2). Let R be an algebra and let $TR := \oplus_{k=1}^{\infty} R^{\otimes k}$ be the tensor algebra over R. Let $\rho : R \to TR$ be the canonical linear inclusion and let $\pi : TR \to R$ be the canonical algebra epimorphism satisfying $\pi \circ \rho = Id_R$. The associated extension of algebras

  $$0 \to IR \to TR \xrightarrow{\pi} R \to 0$$

  is the universal linear split extension of R. The kernel IR is a twosided ideal of TR and defines an adic filtration of TR. There is a canonical isomorphism of filtered vector spaces

  $$(TR, \text{IR-adic filtration}) \xleftarrow{\simeq} (\Omega^{ev}R, \frac{1}{2}\,\text{degree filtration})$$

between the tensor algebra over R and the module of algebraic differential forms of even degree over R. It is given by the formulas

$$\rho(a^0)\omega(a^1,a^2)\dots\omega(a^{2n-1},a^{2n}) \quad \longleftarrow \quad a^0da^1\dots da^{2n}$$
$$\omega(a^1,a^2)\dots\omega(a^{2n-1},a^{2n}) \quad \longleftarrow \quad da^1\dots da^{2n}$$

where $\omega(a,a') := \rho(aa') - \rho(a)\rho(a') \in IR$ is the curvature of ρ.

- Let R be a Banach algebra. For $\epsilon > 0$ let $\| - \|_\epsilon$ be the largest submultiplicative seminorm on TR satisfying $\| \rho(a) \|_\epsilon \leq 2 \| a \|_R$ and $\| \omega(a,a') \|_\epsilon \leq \epsilon \| a \|_R \cdot \| a' \|_R$. Denote the completion of TR with respect to this seminorm by TR_ϵ. It is a Banach algebra. By construction $\| - \|_\epsilon \leq \| - \|_{\epsilon'}$ for $\epsilon < \epsilon'$ so that the identity on TR extends to a bounded homomorphism $TR_{\epsilon'} \to TR_\epsilon$ of Banach algebras. Put $\mathcal{T}'R := \text{"}\lim\limits_{\epsilon\to 0}\text{"}\, TR_\epsilon$. It is called the strong universal infinitesimal deformation of R.

- Let $\| - \|_{\epsilon,0}$ respectively $\| - \|_{\epsilon,1}$ be the largest seminorms on TR satisfying

$$\| \rho(a^0)\omega(a^1,a^2)\dots\omega(a^{2n-1},a^{2n}) \|_{\epsilon,0} \leq \epsilon^n \ \| a^0 \|_R \cdots \| a^{2n} \|_R$$

respectively

$$\| \rho(a^0)\omega(a^1,a^2)\dots\omega(a^{2n-1},a^{2n}) \|_{\epsilon,1} \leq (2+2n)\,\epsilon^n \ \| a^0 \|_R \cdots \| a^{2n} \|_R$$

It follows from the Bianchi identity

$$\omega(a,a')\,\rho(a'') = \omega(a,a'a'') - \omega(aa',a'') + \rho(a)\,\omega(a',a'')$$

that they satisfy

$$\| xy \|_{\epsilon,0} \leq \| x \|_{\epsilon,1} \| y \|_{\epsilon,0}$$

for all $x,y \in TR$. With this it is not difficult to verify the estimates

$$\| - \|_{\epsilon,0} \leq \| - \|_{4\epsilon} \leq \| - \|_{4\epsilon,1}$$

on TR.

- Let $f : R \to S$ be a bounded homomorphism of Banach algebras and let $Tf : TR \to TS$ be the induced homomorphism of tensor algebras. It is immediate from the definitions that given $\epsilon > 0$ there exist $\epsilon' > 0$ and $C > 0$ such that

$$\| Tf(x) \|_{\epsilon',1} \leq C \ \| x \|_{\epsilon,0} \quad \forall x \in TR$$

With the previous estimate this implies that Tf extends to a bounded homomorphism

$$\mathcal{T}'f : \mathcal{T}'R \longrightarrow \mathcal{T}'S$$

of ind-Banach algebras. This allows to define the strong universal infinitesimal deformation of an ind-Banach algebra as

$$\mathcal{T}'(\text{“}\varinjlim_{i\in I}\text{”} R_i) \;:=\; \underset{\overrightarrow{i\in I}}{Lim}\, \mathcal{T}' R_i$$

- Let $\varphi : R \to \mathcal{S}$ be a strongly almost multiplicative linear morphism from a Banach algebra R to some ind-Banach algebra $\mathcal{S} :=$ “$\varinjlim_{j\in J}$”S_j. Let $\varphi_j : R \to S_j$ be a bounded linear map representing φ and satisfying

$$\overline{\lim_{n\to\infty}}\, \left(\| \,\omega_{\varphi_j}(U)^n \,\|_{S_j}\right)^{\frac{1}{n}} \;\leq\; \frac{\epsilon}{8}$$

where U denotes the unit ball of R. Fix n_0 such that $\| \,\omega_{\varphi_j}(U)^n \,\|_{S_j} \leq \left(\frac{\epsilon}{4}\right)^n$ for $n \geq n_0$. Let $T\varphi_j : TR \to S_j$ be the algebra homomorphism which is characterized by the condition $T\varphi_j \circ \rho = \varphi_j$. Then for arbitrary n one has the estimate

$$\| \,T\varphi_j\,(\rho(U)\omega(U,U)^n)\, \|_{S_j} \leq\, C(n,\epsilon)\;\| \,\rho(U)\omega(U,U)^n \,\|_{\frac{\epsilon}{4},0}$$

with $C(n,\epsilon) = \| \,\varphi_j\, \| \cdot (\| \,\varphi_j\, \| + \| \,\varphi_j\, \|^2)^{\,n} \cdot \left(\frac{\epsilon}{4}\right)^{-n}$ whereas for $n \geq n_0$ the stronger estimate

$$\| \,T\varphi_j\,(\rho(U)\omega(U,U)^n)\, \|_{S_j} \leq \| \,\varphi_j\, \| \cdot \| \,\rho(U)\omega(U,U)^n \,\|_{\frac{\epsilon}{4},0}$$

holds. This implies

$$\| \,T\varphi_j(x)\, \|_{S_j} \leq\, C\;\| \,x\, \|_{\frac{\epsilon}{4},0}$$

which shows with the previous estimates that $T\varphi_j$ extends to a bounded homomorphism $T'\varphi_j : TR_\epsilon \to S_j$. Therefore φ induces a homomorphism

$$\mathcal{T}'\varphi :\, \mathcal{T}'R \longrightarrow \mathcal{S}$$

of ind-Banach algebras. If $\phi : \mathcal{R} \longrightarrow \mathcal{S}$ is a strongly almost multiplicative morphism of ind-Banach algebras then one obtains similarly a homomorphism of ind-Banach algebras

$$\mathcal{T}'\varphi :\, \mathcal{T}'\mathcal{R} \longrightarrow \mathcal{S}$$

This construction is natural and defines a canonical and natural map

$$Mor_{\substack{str\\alm\\mult}}(\mathcal{R},\mathcal{S}) \quad\longrightarrow\quad Mor_{\substack{ind\\alg}}(\mathcal{T}'\mathcal{R},\mathcal{S})$$

- Let R be a Banach algebra and consider the canonical linear embedding $\rho : R \to \mathcal{T}'(R)$. It is bounded because $\| \,\rho(U)\, \|_\epsilon \leq 2$ and strongly almost multiplicative as

$$\| \,\omega_\rho(U,U)^n \,\|_\epsilon^{\frac{1}{n}} \leq \| \,\omega(U,U)^n \,\|_\epsilon^{\frac{1}{n}} \leq\, \epsilon$$

where U denotes the unit ball of R. Similarly the canonical linear morphism $\rho : \mathcal{R} \to \mathcal{T}'\mathcal{R}$ of ind-Banach algebras is strongly almost multiplicative. In particular, composition with homomorphisms of ind-Banach algebras defines a canonical map

$$\rho^* : \quad Mor_{\substack{ind\\alg}}(\mathcal{T}'\mathcal{R}, \mathcal{S}) \quad \longrightarrow \quad Mor_{\substack{str\\alm\\mult}}(\mathcal{R}, \mathcal{S})$$

that is obviously inverse to the map constructed above. Therefore ρ^* is an isomorphism.

- Finally it is easy to show that

$$\begin{array}{cccc} \mathcal{T}' : & Mor_{\substack{str\\alm\\mult}}(\mathcal{R}, \mathcal{S}) & \longrightarrow & Mor_{\substack{ind\\alg}}(\mathcal{T}'\mathcal{R}, \mathcal{T}'\mathcal{S}) \\ & \phi & \to & \mathcal{T}'\phi \end{array}$$

turns the strong universal infinitesimal deformation $\mathcal{T}'$ into a covariant functor from the category of ind-Banach algebras with strongly almost multiplicative morphisms to the category of ind-Banach algebras. The previous considerations show also that it is left adjoint to the forgetful functor.

□ The proof of the previous theorem gives no explicit description of the seminorms defining the strong universal infinitesimal deformation of a Banach algebra. If one works with formal inductive limits of Fréchet algebras instead of Banach algebras such an explicit description can be given.

DEFINITION AND LEMMA 1.22. *Let R be a Banach algebra and let TR be the tensor algebra over R. Denote by $\| - \|_{\epsilon,m}$ the largest seminorm on TR satisfying*

$$\| \rho(a^0)\omega(a^1, a^2) \ldots \omega(a^{2n-1}, a^{2n}) \|_{\epsilon,m} \leq (2 + 2n)^m \epsilon^n \| a^0 \| \ldots \| a^{2n} \|$$

a) *The seminorms $\| - \|_{\epsilon,m}$ are not submultiplicative but satisfy*

$$\| xy \|_{\epsilon,m} \leq \| x \|_{\epsilon,m+1} \cdot \| y \|_{\epsilon,m}$$

b) *The completion of the tensor algebra TR with respect to the seminorms $\| - \|_{\epsilon,m}$, $m \in \mathbb{N}$, is a nice Fréchet algebra TR^ϵ. An open unit ball of TR^ϵ is given by the open unit ball with respect to the seminorm $\| - \|_{\epsilon,1}$.*

c) *The formal inductive limit*

$$\mathfrak{T}'R := \text{“}\lim_{\epsilon \to 0}\text{”}\, TR^\epsilon$$

is isomorphic in the category of ind-Fréchet algebras to the strong universal infinitesimal deformation $\mathcal{T}'R$ of R.

PROOF: Assertions a) and b) are shown in [Pu], (5.6). Assertion c) is a consequence of the estimates

$$\| - \|_{\epsilon,0} \leq \| - \|_{4\epsilon} \leq \| - \|_{4\epsilon,1}$$

obtained in the proof of (1.21) and the fact that $\| - \|_{\epsilon',m} \leq C_m \ \| - \|_{\epsilon,0}$ for $\epsilon' < \epsilon$. □

Using the previous results the existence of a universal infinitesimal deformation functor can be established.

THEOREM 1.23. *The forgetful functor from the category of nice ind-Fréchet algebras to the category with the same objects and almost multiplicative linear maps as morphisms possesses a left adjoint $\mathcal{T}$, which is called the* UNIVERSAL INFINITESIMAL DEFORMATION *functor. This means that for all nice ind-Fréchet algebras $\mathcal{R}$, $\mathcal{S}$ there exists a natural and canonical isomorphism*

$$Mor_{\substack{ind\\alg}}(\mathcal{T}\mathcal{R}, \mathcal{S}) \quad \xrightarrow{\simeq} \quad Mor_{\substack{alm\\mult}}(\mathcal{R}, \mathcal{S})$$

The universal infinitesimal deformation functor is given by the composition

$$\mathcal{T} = \mathcal{T}' \circ \mathcal{B}$$

of the functor $\mathcal{B}$ (1.4), associating to an algebra the diagram of its compactly generated subalgebras, and the strong universal infinitesimal deformation functor $\mathcal{T}'$.

PROOF: Put $\mathcal{T} := \mathcal{T}' \circ \mathcal{B}$. For any nice ind-Fréchet algebras $\mathcal{R}$, $\mathcal{S}$ one has a sequence of natural isomorphisms

$$Mor_{\substack{alm\\mult}}(\mathcal{R}, \mathcal{S}) \xrightarrow{\simeq} Mor_{\substack{str\\alm\\mult}}(\mathcal{B}(\mathcal{R}), \mathcal{B}(\mathcal{S}))$$

by the remark following (1.14)

$$Mor_{\substack{str\\alm\\mult}}(\mathcal{B}(\mathcal{R}), \mathcal{B}(\mathcal{S})) \xrightarrow{\simeq} Mor_{\substack{ind\\alg}}(\mathcal{T}\mathcal{R}, \mathcal{B}(\mathcal{S}))$$

by the previous theorem and

$$Mor_{\substack{ind\\alg}}(\mathcal{T}\mathcal{R}, \mathcal{B}(\mathcal{S})) \xrightarrow{\simeq} Mor_{\substack{ind\\alg}}(\mathcal{T}\mathcal{R}, \mathcal{S})$$

by (1.10) and the following lemma. □

LEMMA 1.24. *For any nice ind-Fréchet algebra $\mathcal{R}$ the ind-Banach algebra $\mathcal{T}\mathcal{R}$ is compact.*

PROOF: The ind-Banach algebra $\mathcal{B}(\mathcal{R})$ is compact by (1.9). In order to show that $\mathcal{T}\mathcal{R} = \mathcal{T}'\mathcal{B}(\mathcal{R})$ is compact it suffices therefore to prove the following. The homomorphism $\mathcal{T}'f : \mathcal{T}'A \to \mathcal{T}'B$ induced by a compact homomorphism

$f : A \to B$ of Banach algebras is compact. As the notion of compactness is stable under isomorphism one can pass to the morphism $\mathfrak{T}'f : \mathfrak{T}'A \to \mathfrak{T}'B$ of ind-Fréchet algebras (1.22). The definition of the seminorms on this ind-algebra show immediately that for given $\epsilon > 0$ the homomorphisms $Tf : TA^\epsilon \to TB^{\epsilon'}$ are compact for $\epsilon' > 0$ small enough. This establishes the lemma. $\square$

It remains to verify that the functors constructed in the previous theorems merit their names and provide in fact (strong) infinitesimal deformations.

Lemma 1.25. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra. Then the canonical epimorphism $\pi : \mathcal{T}\mathcal{A} \to \mathcal{A}$ (resp. $\pi' : \mathcal{T}'\mathcal{A} \to \mathcal{A}$) adjoint to the identity of $\mathcal{A}$ via (1.21) (resp. (1.23)) is a (strong) infinitesimal deformation in the sense of (1.18).*

Proof: We show first that $0 \to \mathcal{I}'\mathcal{A} \to \mathcal{T}'\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$ is a strong infinitesimal deformation. By lemma (1.22) it suffices to verify the corresponding statement for the extension $0 \to \mathfrak{I}'\mathcal{A} \to \mathfrak{T}'\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$. Henceforth the notations of (1.22) are used. Let $S \subset IA^\epsilon := Ker(\pi : TA^\epsilon \to A)$ be a bounded set. Then for sufficiently small $\epsilon' < \epsilon$ the (bounded) image of S in $TA^{\epsilon'}$ satisfies $\| S \|_{\epsilon',1} \leq 1$. The estimate $\| xy \|_{\epsilon',0} \leq \| x \|_{\epsilon',1} \cdot \| y \|_{\epsilon',0}$ allows then to deduce that $\| S^\infty \|_{\epsilon',0} < \infty$. It follows that the image of S^∞ in $TA^{\epsilon''}$ is bounded for $\epsilon'' < \epsilon'$ (compare the proof of (1.22)). Thus $\mathcal{T}'\mathcal{A} \xrightarrow{\pi} \mathcal{A}$ is a strong infinitesimal deformation of $\mathcal{A}$. This result and (1.24) imply finally that $\mathcal{T}\mathcal{A} \xrightarrow{\pi} \mathcal{A}$ is an infinitesimal deformation in the sense of (1.18). $\square$

One can now make precise in which sense the almost multiplicative maps of (1.19) are generic.

Corollary 1.26. *Every almost multiplicative map $\phi : \mathcal{A} \to \mathcal{B}$ of nice ind-Fréchet algebras factorizes as $\phi = f \circ \psi$ where $\psi : \mathcal{A} \to \mathcal{A}'$ is a bounded linear section of an infinitesimal deformation $\pi : \mathcal{A}' \to \mathcal{A}$ and $f : \mathcal{A}' \to \mathcal{B}$ is a homomorphism of ind-Fréchet algebras.*

Finally the infinitesimal deformations given by the completed tensor algebras will be characterized by a universal property.

Theorem 1.27. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra. The extension*

$$0 \to \mathcal{I}\mathcal{A} \to \mathcal{T}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$$

with the canonical linear section $\rho : \mathcal{A} \to \mathcal{T}\mathcal{A}$ adjoint to the identity of $\mathcal{T}\mathcal{A}$ via (1.23) is the universal infinitesimal deformation of $\mathcal{A}$ in the following sense. Let

$$0 \to \mathcal{J} \to \mathcal{R} \xrightarrow{\pi'} \mathcal{S} \to 0$$

be an infinitesimal deformation of $\mathcal{S}$ with fixed bounded linear section and let $f : \mathcal{A} \to \mathcal{S}$ be a homomorphism of nice ind-Fréchet algebras. Then there exists a unique homomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{I}\mathcal{A} & \to & \mathcal{T}\mathcal{A} & \xrightarrow{\pi} & \mathcal{A} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \to & \mathcal{J} & \to & \mathcal{R} & \xrightarrow{\pi'} & \mathcal{S} & \to & 0 \end{array}$$

compatible with the given linear sections. In particular, $\mathcal{TA}$ is topologically quasifree (1.20).
In a similar sense for a given ind-Banach algebra $\mathcal{R}$ the extension

$$0 \rightarrow \mathcal{I}'\mathcal{R} \rightarrow \mathcal{T}'\mathcal{R} \xrightarrow{\pi} \mathcal{R} \rightarrow 0$$

is the universal strong infinitesimal deformation of $\mathcal{R}$. In particular, the ind-Banach algebra $\mathcal{T}'\mathcal{R}$ is strongly topologically quasifree.

2 The stable diffeotopy category of ind-algebras

A diffeotopy is a homotopy which depends smoothly on its parameter. Diffeotopy is a finer equivalence relation than homotopy. For example, the algebra of continuous functions on a closed interval vanishing at one endpoint is nullhomotopic but not nulldiffeotopic. Following closely some well known ideas of homotopy theory (see [Ad])we set up in this chapter a stable diffeotopy category of topological ind-algebras. Its construction proceeds in several steps. One defines first an unstable diffeotopy category in a straightforward way. Then the notions of suspension and mapping cone are introduced. The diffeotopy category is stabilized by inverting the suspension functor which gives rise to a prestable category which is already triangulated, i.e. which possesses long exact Puppe sequences. The stable diffeotopy category of ind-algebras is finally obtained from the prestable one by a category theoretic localization process which is necessary to get rid of some pathologies related to weakly contractible ind-algebras. We then present a criterion for detecting isomorphisms in the stable diffeotopy category which will be frequently used in the rest of the paper.
As mentioned before we begin by introducing the relation of diffeotopy between homomorphisms of topological ind-algebras.

Definition 2.1. (Diffeotopy category)

a) Let $\mathcal{C}^\infty([0,1])$ be the nice nuclear Fréchet algebra of smooth functions on the unit interval all of whose derivatives vanish at the endpoints. For an ind-Fréchet algebra “ $\underset{i\in I}{\varinjlim}$”$A_i$ let

$$\mathcal{C}^\infty([0,1],\,\mathcal{A}) := \text{“}\underset{i\in I}{\varinjlim}\text{”}\,\mathcal{C}^\infty([0,1],\,A_i) \;=\; \text{“}\underset{i\in I}{\varinjlim}\text{”}\,\mathcal{C}^\infty([0,1])\otimes_\pi A_i$$

It is again an ind-Fréchet algebra.

b) Two homomorphisms $\mathcal{A} \rightrightarrows \mathcal{A}'$ of ind-Fréchet algebras are called diffeotopic if they factorize as

$$\mathcal{A} \longrightarrow \mathcal{C}^\infty([0,1],\,\mathcal{A}') \rightrightarrows \mathcal{A}'$$

where the homomorphisms on the right hand side are given by evaluation at the endpoints. Diffeotopy is an equivalence relation. The equivalence classes are called diffeotopy classes of homomorphisms. The set of diffeotopy classes of homomorphisms between ind-Fréchet algebras $\mathcal{A}$ and $\mathcal{A}'$ is denoted by $[\mathcal{A}, \mathcal{A}']$.

c) The (unstable) diffeotopy category of ind-Fréchet algebras is the category with ind-Fréchet algebras as objects and with diffeotopy classes of ind-algebra homomorphisms as morphisms.

Now suspensions and mapping cones are defined which are necessary to triangulate diffeotopy categories, i.e. to establish Puppe sequences.

DEFINITION 2.2. (SUSPENSION AND MAPPING CONE)

a) Let $\mathcal{C}^\infty(]0,1[)$ be the nice nuclear Fréchet algebra of smooth functions on the unit interval which vanish together with all their derivatives at the endpoints. If $\mathcal{A} :=$ "$\varinjlim_{i\in I}$"A_i is an ind-Fréchet algebra then the ind-Fréchet algebra

$$\mathcal{SA} := \text{"}\varinjlim_{i\in I}\text{"}\,\mathcal{C}^\infty(]0,1[,\, A_i) \;=\; \text{"}\varinjlim_{i\in I}\text{"}\,\mathcal{C}^\infty(]0,1[) \otimes_\pi A_i$$

is called the suspension of $\mathcal{A}$. The suspension defines a functor of the category of ind-Fréchet algebras to itself.

b) Let $f :$ "$\varinjlim_{i\in I}$"$A_i \longrightarrow$ "$\varinjlim_{j\in J}$"A'_j be a homomorphism of ind-Fréchet algebras. Define a directed set K by

$$K := \{(i,j,f_{ij}) \,|\, i\in I, j\in J,\ f_{ij} : A_i \to A'_j \text{ represents } f\}$$

and by declaring $(i,j,f_{ij}) \leq (i',j',f_{i'j'})$ iff $i \leq i'$, $j \leq j'$ and the diagram

$$\begin{array}{ccc} A_{i'} & \xrightarrow{f_{i'j'}} & A'_{j'} \\ \uparrow & & \uparrow \\ A_i & \xrightarrow{f_{ij}} & A'_j \end{array}$$

commutes. The mapping cone $\mathbf{Cone}(f)$ of f is the ind-Fréchet algebra

$$\mathbf{Cone}(f) := \text{"}\varinjlim_{K}\text{"}\,Cone(f_{ij})$$

with

$$Cone(f_{ij}) := \{(a,\chi) \in A_i \times \mathcal{C}^\infty([0,1[,\, A'_j) \,|\, f_{ij}(a) = \chi(0)\}$$

Here $\mathcal{C}^\infty([0,1[)$ is the nice nuclear Fréchet subalgebra of $\mathcal{C}^\infty([0,1])$ consisting of the functions vanishing at the endpoint 1 of the unit interval.

Thus a morphism $\mathcal{B} \to \mathbf{Cone}(f)$ is given by a couple (φ, ν) consisting of a homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$ and a nulldiffeotopy ν of the composed map $\mathcal{B} \xrightarrow{\varphi} \mathcal{A} \xrightarrow{f} \mathcal{A}'$.
The suspension of an ind-algebra $\mathcal{A}$ is a special case of a mapping cone as $\mathcal{S}\mathcal{A} \simeq \mathbf{Cone}(p)$ where $p : \mathcal{C}^\infty([0,1[, \mathcal{A}) \to \mathcal{A}$ is the evaluation at 0. The mapping cone of a morphism $f : \mathcal{A} \to \mathcal{A}'$ fits into a natural sequence of homomorphisms

$$\xrightarrow{\mathcal{S}f} \mathcal{S}\mathcal{A}' \xrightarrow{s} \mathbf{Cone}(f) \xrightarrow{p} \mathcal{A} \xrightarrow{f} \mathcal{A}'$$

where s and p are defined on individual algebras of the formal inductive systems by $s : SA'_j \to Cone(f_{ij})$, $s(\chi) := (0, \chi)$ and $p : Cone(f_{ij}) \to A_i$, $p(a, \chi') := a$. The mapping cone functor commutes with suspensions, i.e. there exists a natural isomorphism $\mathbf{Cone}(\mathcal{S}f) \xrightarrow{\simeq} \mathcal{S}\mathbf{Cone}(f)$.
In the next step the diffeotopy category shall be stabilized so that it becomes a triangulated category with shift automorphism given by the inverse of the suspension. In order to do so the suspension has to be made an automorphism of the underlying category which leads to

Definition 2.3. (Prestable diffeotopy category) (See [Ma])
The prestable diffeotopy category of ind-Fréchet algebras is the additive category with objects given by pairs $(\mathcal{A}, n)$ consisting of an ind-Fréchet algebra $\mathcal{A}$ and an integer n and with the abelian groups

$$Mor^*((\mathcal{A}, n), (\mathcal{A}', n')) := \lim_{k \to \infty} [\mathcal{S}^{k-n}\mathcal{A}, \mathcal{S}^{k-n'}\mathcal{A}']$$

as morphisms. The transition maps in the limit are given by suspensions. The shift functor $T : T(\mathcal{A}, n) := (\mathcal{A}, n+1)$ is an automorphism of the prestable diffeotopy category and its inverse is canonically isomorphic to the suspension functor: $T^{-1} \simeq \mathcal{S}$.

The prestable diffeotopy category is in fact triangulated.

Lemma 2.4. *The prestable diffeotopy category is a triangulated category [KS] (1.5) in a natural way. The shift functor is given by the functor T of (2.3) and a triangle in the prestable diffeotopy category is distinguished iff it is isomorphic to a triangle of the form*

$$\mathcal{S}\mathcal{A}' \xrightarrow{s} \mathbf{Cone}(f) \xrightarrow{p} \mathcal{A} \xrightarrow{f} \mathcal{A}'$$

Proof: A classical result asserts that the homotopy category of pointed topological spaces becomes triangulated after inverting the suspension functor. Here one declares a triangle to be distinguished iff it is isomorphic to a cofibration sequence $X \xrightarrow{f} Y \longrightarrow Cone(f) \longrightarrow \Sigma X$. A proof of this can be found in [Ma], Chapter 1 and Appendix II. Section 1.4 of [KS] might also be helpful. The present lemma is obtained from this result by the following modifications.

One restricts to locally compact spaces, considers the dual function algebras and generalizes to arbitrary Fréchet algebras. Then one passes from algebras to ind-algebras. Finally one replaces the homotopy relation by the finer diffeotopy relation. The demonstration that the various prestable categories obtained along the way are triangulated carries over through each of these steps. This yields the assertion. □

As a consequence [KS], one obtains

Corollary 2.5. *Every homomorphism of ind-Fréchet algebras induces a covariant and a contravariant long exact Puppe sequence in the prestable diffeotopy category.*

The prestable diffeotopy category turns out to be too rigid for our purposes. In fact there is a class of ind-algebras, the weakly contractible ones, which one would like to be equivalent to zero in a reasonable stable diffeotopy category.

Definition 2.6. An ind-Fréchet algebra $\mathcal{A} = $ "$\varinjlim_{i\in I}$"A_i is called weakly contractible if for each $i \in I$ there exists $i' \geq i$ such that the structure homomorphism $A_i \to A_{i'}$ is nulldiffeotopic. It is called stably weakly contractible if $\mathcal{S}^k\mathcal{A}$ is weakly contractible for $k >> 0$.

Every direct limit (1.2) (in the category of ind-algebras) of weakly contractible ind-algebras is weakly contractible.

Lemma 2.7. *The family of stably weakly contractible ind-Fréchet algebras forms a null system, [KS], 1.6.6, in the prestable diffeotopy category.*

Proof: It is easily shown that the family $\mathcal{N}$ of stably weakly contractible ind-algebras is closed under isomorphism in the prestable diffeotopy category. If $f : \mathcal{A} \to \mathcal{A}'$ is a stable homomorphism of weakly contractible ind-algebras then it is almost immediate that $\mathbf{Cone}(f)$ is weakly contractible, too. □

Finally we arrive at

Definition 2.8. (Stable diffeotopy category) (See [Ad], [Ma])

The smooth stable diffeotopy category of ind-Fréchet algebras is the triangulated category obtained from the prestable diffeotopy category by inverting all morphisms with stably weakly contractible mapping cone. A triangle in the stable diffeotopy category is distinguished if it is isomorphic to the image of a distinguished triangle in the prestable diffeotopy category.

It follows in particular that exact covariant and contravariant Puppe sequences exist in the stable diffeotopy category.

In the sequel we will make use of the

Proposition 2.9. (Isomorphism criterion)

Let $f :$ "$\varinjlim_{i\in I}$"$A_i \longrightarrow$ "$\varinjlim_{j\in J}$"A'_j *be a homomorphism of ind-Fréchet algebras and suppose that the following conditions are satisfied: Every homomorphism* f_{ij} *representing the restriction of* f *to* A_i *fits into a diagram*

$$\begin{tikzcd}
A_{i'} \arrow[r, "f_{i'j'}"] & A'_{j'} \\
A_i \arrow[u] \arrow[r, "f_{ij}"] & A'_j \arrow[u] \arrow[ul, "\psi"]
\end{tikzcd}$$

such that

- *the vertical arrows are given by the structure homomorphisms of the corresponding ind-algebras.*
- *the horizontal arrows represent the restrictions of f to A_i respectively $A_{i'}$.*
- *the diagram commutes up to diffeotopy.*

Then the homomorphism f becomes an isomorphism in the stable diffeotopy category.

PROOF: A simple diagram chase shows that a morphism in the unstable diffeotopy category satisfies the criterion of the proposition iff its mapping cone is weakly contractible. Therefore the morphisms under considerations are exactly those belonging to the multiplicative system [KS], 1.6.7., associated to the null system of weakly contractible ind-algebras. In particular, these morphisms become isomorphisms in the stable diffeotopy category. □

There seems to be no reason for stable diffeotopy equivalences to be preserved under direct limits (in the category of ind-algebras). There is however a partial result in this direction.

PROPOSITION 2.10. *Let I be a directed set. Let $(\mathcal{A}_i)_{i\in I}$, $(\mathcal{B}_i)_{i\in I}$ be I-diagrams of ind-Fréchet algebras and let $f = (f_i : \mathcal{A}_i \longrightarrow \mathcal{B}_i)_{i\in I}$ be a morphism of I-diagrams. Suppose that the isomorphism criterion (2.9) applies to each of the morphisms f_i, $i \in I$. Then it applies also to the morphism*

$$\underset{\overrightarrow{i\in I}}{Lim}(f_i) : \underset{\overrightarrow{i\in I}}{Lim}\,\mathcal{A}_i \longrightarrow \underset{\overrightarrow{i\in I}}{Lim}\,\mathcal{B}_i$$

of direct limits which therefore is an isomorphism in the stable diffeotopy category.

PROOF: It is a tedious but straightforward exercise to show that $\underset{\overrightarrow{i\in I}}{Lim}\,\mathbf{Cone}(f_i) \overset{\simeq}{\longrightarrow} \mathbf{Cone}(\underset{\overrightarrow{i\in I}}{Lim}(f_i))$ is an isomorphism of ind-Fréchet algebras. As noted above, a morphism of ind-algebras satisfies the isomorphism criterion iff it has a weakly contractible mapping cone. Therefore the assertion follows from the fact that direct limits of weakly contractible ind-algebras are weakly contractible. □

3 The stable diffeotopy type of universal infinitesimal deformations

In this section some of the main results of this paper are presented. They describe the behavior of the universal infinitesimal deformations of nice ind-Fréchet algebras viewed as objects of the stable diffeotopy category. Among other things we show that under some mild technical assumptions the following assertions hold:

- The inclusion $\mathfrak{A} \hookrightarrow A$ of a dense and holomorphically closed subalgebra of a nice Fréchet algebra induces a stable diffeotopy equivalence of its universal infinitesimal deformations.
- The universal infinitesimal deformation of a topological direct limit of nice Fréchet algebras is stably diffeotopy equivalent to the inductive limit of the universal infinitesimal deformations of the individual algebras.

We will comment on these results in more detail in the introductions of the corresponding subsections. Instead we want to indicate why they hold.
Consider an inclusion $i : \mathfrak{A} \hookrightarrow A$ of a dense and holomorphically closed subalgebra of a nice Fréchet algebra. Suppose that a family of bounded linear "regularization" maps $\mathfrak{s} = (s_\alpha : A \to \mathfrak{A},\, \alpha \in \Lambda)$ is given which approximate the identity on $\mathfrak{A}$ uniformly on compact subsets. The norms of the "curvature" terms $\{\, \omega_\alpha(a, a') = s_\alpha(aa') - s_\alpha(a)s_\alpha(a'),\, a,\, a' \in A,\, \alpha \in \Lambda \,\}$, which measure the deviation of $\mathfrak{s}$ from multiplicativity, might be quite large in norm but their spectral radii will be very small (as they are the same if measured in $\mathfrak{A}$ or in A). Therefore large powers of curvature terms, or in many situations even any product of a large number of curvature terms, will be arbitrarily small in norm. Consequently the family $\mathfrak{s}$ of regularization maps is almost multiplicative and defines a morphism $\mathcal{T}\mathfrak{s} : \mathcal{T}A \longrightarrow \mathcal{T}\mathfrak{A}$ of universal infinitesimal deformations in the stable diffeotopy category. It turns out that the morphism $\mathcal{T}\mathfrak{s}$ provides a stable diffeotopy inverse of the morphism $\mathcal{T}i : \mathcal{T}\mathfrak{A} \longrightarrow \mathcal{T}A$ of universal deformations induced by the inclusion of $\mathfrak{A}$ into A. Thus the inclusion of a dense and holomorphically closed subalgebra induces a stable diffeotopy equivalence of universal infinitesimal deformations, provided that a sufficiently good family of linear regularizations exists. In order to guarantee this we will make some not too restrictive assumptions on the topological vector spaces underlying the algebras under consideration.

3.1 The Grothendieck approximation property

It turns out that the majority of the results presented in this section require that the topological vector spaces underlying the considered algebras verify a regularity condition. This condition is known as Grothendieck's approximation property [LT].

Definition 3.1. (Grothendieck approximation property) [LT]
Let E be a Fréchet space. Then E has the Grothendieck approximation property if the finite rank operators are dense in $\mathcal{L}(E)$ with respect to the topology of uniform convergence on compacta. Thus E possesses the approximation property iff for each seminorm $\| \;\|$ on E, for each $\epsilon > 0$, and each compact set $K \subset E$ there exists a bounded linear selfmap $\phi \in \mathcal{L}(E)$ of finite rank such that $\sup\limits_{x\in K}\|\phi(x) - x\| < \epsilon$.

Examples of Fréchet-algebras whose underlying topological vector spaces have the approximation property are

- nuclear Fréchet-algebras
- nuclear C^*-algebras
- l^p-spaces
- separable, symmetrically normed operator ideals
- the reduced group C^*-algebra of a finitely generated free group.

The algebra of all bounded operators on an infinite dimensional Hilbert space, on the contrary, does not have the approximation property.

3.2 Approximation by ind-algebras of countable type

In order to work with universal infinitesimal deformations of nice Fréchet algebras it turns out to be indispensable to dispose of small models of their stable diffeotopy type. In particular, one is interested in models which are given by a countable formal inductive limit. Under not too restrictive assumptions, their existence is guaranteed by

Theorem 3.2. (Approximation theorem)
Let A be a separable nice Fréchet algebra which possesses the Grothendieck approximation property. Let U be a convex open unit ball of A.
Let $0 \subset V_0 \subset V_1 \subset \ldots \subset V_n \subset \ldots$ be an increasing sequence of finite dimensional subspaces of A such that $\bigcup\limits_{n=0}^{\infty} V_n$ is a dense subalgebra of A, and let $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly monotone increasing sequence of positive real numbers such that $\lim\limits_{n\to\infty} \lambda_n = 1$. Put $S_n := V_n \cap \lambda_n\overline{U}$. Then the canonical morphism

$$\text{“}\lim_{n\to\infty}\text{”} A_{S_n} \hookrightarrow \mathcal{B}(A)$$

induces a stable diffeotopy equivalence of strong universal infinitesimal deformations. In particular, the universal infinitesimal deformation $\mathcal{T}A$ of A is stably diffeotopy equivalent to a countable formal inductive limit of Banach algebras.

Proof:
Recall that for a nice Fréchet algebra A one has $\mathcal{T}A = \text{“}\lim\limits_{\substack{S\subset U\\ \epsilon\to 0}}\text{”}\,(TA_S)_\epsilon$ in the notations of (1.4), (1.21) and (1.23), where S ranges over the family of compact subsets of some fixed open unit ball U of A. We want to apply the isomorphism criterion (2.9) to the morphism

$$\mathcal{T}'(\text{“}\lim_{n\to\infty}\text{”}\,A_{S_n}) = \mathop{Lim}_{n\to\infty}\mathcal{T}'(A_{S_n}) \longrightarrow \mathcal{T}'\mathcal{B}(A) = \mathcal{T}A$$

So it has to be shown that any structure morphism $i : (TA_{S_n})_{\epsilon_n} \longrightarrow (TA_S)_{\epsilon'}$ fits into a diagram

$$\begin{tikzcd}
(TA_{S_m})_{\epsilon_m} \arrow[r, "i'"] & (TA_{S'})_{\epsilon''} \\
(TA_{S_n})_{\epsilon_n} \arrow[u, "j"] \arrow[r, "i"'] & (TA_S)_{\epsilon'} \arrow[u, "j'"'] \arrow[ul, "T\phi"']
\end{tikzcd}$$

where the homomorphisms i, j, i', j' are given by the structure maps and which commutes up to diffeotopy.
By definition the identity homomorphism of TA_S gives rise to morphisms $(TA_S)_{\epsilon'} \longrightarrow \mathcal{T}'A_S \simeq \mathfrak{T}'A_S = \text{“}\lim\limits_{\epsilon_1\to 0}\text{”}\,(TA_S)^{\epsilon_1}$. In order to define the diagonal morphism in the desired diagram it suffices therefore to construct a bounded homomorphism $T\phi : (TA_S)^{\epsilon_1} \longrightarrow (TA_{S_m})_{\epsilon_m}$ for ϵ_1 given and suitable $m >> 0$ large and $\epsilon_m > 0$ small enough.
Fix ϵ with $0 < 4\epsilon < \epsilon_1$. As A possesses the Grothendieck approximation property there exists a bounded finite rank selfmap ϕ of A which is close to the identity on the (relatively compact) multiplicative closure S^∞ of S. We may suppose that $\phi_t := (1-t)\cdot\phi + t\cdot Id$ satisfies $\omega_{\phi_t}(S^\infty) \subset \epsilon U$ for $t \in [0,1]$ and that $\phi(A)$ is contained in the dense subspace $\bigcup_{n=0}^\infty V_n$ of A. In particular $\phi(A) \subset V_m$ for some $m >> 0$.
By definition the Banach algebra A_S is the completion of the subalgebra of A generated by S with respect to the seminorm $\|a\| := \inf \sum |\lambda_i|$ where the infimum is taken over all presentations $a = \sum \lambda_i\, s_i$ of a with $\lambda_i \in \mathbb{C}$, $s_i \in S^\infty$. It follows from this that ϕ induces a bounded linear map $\phi : A_S \longrightarrow A_{S_m}$ of Banach spaces for sufficiently large m such that $\phi(A) \bigcup \phi(A)^2 \subset V_m$. Fix such m and let C_0 be the norm of the linear map ϕ.

Let $a^0, \dots, a^{2n} \in A_S$. One finds in the notations of (1.21) and (1.22)

$$\| T\phi(\rho(a^0)\omega(a^1, a^2) \cdot \ldots \cdot \omega(a^{2n-1}, a^{2n})) \|_{\epsilon_m}$$

$$\leq \| T\phi(\rho(a^0)) \|_{\epsilon_m} \cdot \prod_{i=1}^{n} \| T\phi(\omega(a^{2i-1}, a^{2i})) \|_{\epsilon_m}$$

because $\| - \|_{\epsilon_m}$ is submultiplicative. The identity

$$T\phi(\omega(a, a')) = \varrho(\omega_\phi(a, a')) + \omega(\phi(a), \phi(a'))$$

shows then that

$$\| T\phi(\omega(a, a')) \|_{\epsilon_m} \leq 2 \| \omega_\phi(a, a') \|_{A_{S_m}} + \epsilon_m \| \phi(a) \|_{A_{S_m}} \cdot \| \phi(a') \|_{A_{S_m}}$$

$$\leq (2 \cdot \frac{m+1}{m} \cdot \epsilon + \epsilon_m \cdot C_0^2) \cdot \| a \|_{A_S} \cdot \| a' \|_{A_S}$$

So for ϵ_m sufficiently small

$$\| T\phi(\rho(a^0)\omega(a^1, a^2) \cdot \ldots \cdot \omega(a^{2n-1}, a^{2n})) \|_{\epsilon_m} \leq 2 \cdot C_0 \cdot \epsilon_1^n \cdot \| a^0 \|_{A_S} \cdot \ldots \cdot \| a^{2n} \|_{A_S}$$

from which the estimate

$$\| T\phi(\alpha) \|_{\epsilon_m} \leq C \cdot \| \alpha \|_{\epsilon_1, 0}, \ \ \forall \alpha \in (TA_S)^{\epsilon_1}$$

results. This establishes the existence of the diagonal morphism in the diagram. The same kind of estimate shows that, after possibly modifying the choice of m and ϵ_m, the one parameter family $T\phi_t = T((1-t) \cdot \phi + t \cdot Id)$ defines a diffeotopy connecting the homomorphisms $T\phi \circ i$ and j from $(TA_{S_n})_{\epsilon_n}$ to $(TA_{S_m})_{\epsilon_m}$. Similarly the same family $T\phi_t$ defines a diffeotopy between the homomorphisms $i' \circ T\phi$ and j' from $(TA_S)_{\epsilon'}$ to $(TA_{S'})_{\epsilon''}$ after choosing S' and ϵ'' appropriately. This completes the proof.

□

Corollary 3.3. *Let* "$\varinjlim_{i \in I}$"A^i *be a formal inductive limit of nice Fréchet algebras which possess the Grothendieck approximation property. Suppose that for each* $i \in I$ *a sequence* $(V_n^i)_{n \in \mathbb{N}}$ *of finite dimensional subspaces of* A^i *and a sequence* $(\lambda_n^i)_{n \in \mathbb{N}}$ *of real numbers has been chosen as in (3.2) and such that the structure maps* $A^i \to A^j$, $i \leq j$ *map* $\bigcup_n V_n^i$ *into* $\bigcup_n V_n^j$. *Then the countable ind-Banach algebras* "$\lim\limits_{n \to \infty}$"$A^i_{S_n}$ *form an inductive system, labeled by* I, *and the natural morphism*

$$\underset{i \in I}{\underrightarrow{Lim}} \left(\text{“} \lim_{n \to \infty} \text{”} A^i_{S_n} \right) \longrightarrow \underset{i \in I}{\underrightarrow{Lim}} \, \mathcal{B}(A^i) = \mathcal{B}\left(\text{“} \varinjlim_{i \in I} \text{”} A^i \right)$$

induces a stable diffeotopy equivalence of strong universal infinitesimal deformations.

Proof: The corollary follows from the proof of the previous theorem and proposition (2.10). □

3.3 Smooth subalgebras

A fundamental question in the study of functors of topological algebras is their compatibility with completions. Put differently, one asks how a functor behaves under passage to dense topological subalgebras. A prototype of such a stability phenomenon occurs in topological K-theory which is well known to be stable under passage to dense subalgebras which are closed under holomorphic functional calculus. Here we investigate stability properties of the universal infinitesimal deformation functor with values in the stable diffeotopy category. To this end we introduce a class of dense and holomorphically closed subalgebras of a nice Fréchet algebra, called smooth subalgebras. It contains in particular the domains of densely defined unbounded derivations. Under rather mild restrictions it is shown that the stable diffeotopy type of the universal infinitesimal deformation of a nice Fréchet algebra does not change under passage to smooth subalgebras. As a consequence, continuous homotopy equivalences of nice Fréchet algebras give rise to stable diffeotopy equivalences of their universal infinitesimal deformations.

Definition 3.4. Let $i : \mathfrak{A} \hookrightarrow A$ be an inclusion of Fréchet algebras with dense image and suppose that A is nice. Then $\mathfrak{A}$ is called a smooth subalgebra of A if there exists an open neighborhood U of 0 in A such that $i^{-1}(U)$ is an open unit ball of $\mathfrak{A}$ in the sense of (1.1).

In particular, smooth subalgebras of nice Fréchet algebras are nice. The condition of smoothness is quite restrictive. In fact, smooth subalgebras are closed under holomorphic functional calculus.

Lemma 3.5. *[Pu], (7.2). Let $\mathfrak{A} \subset A$ be a smooth subalgebra of the nice Fréchet algebra A. Then $\mathfrak{A}$ is closed under holomorphic functional calculus in A.*

The name "smooth subalgebra" is motivated by the following example.

Lemma 3.6. *[Pu], (7.4) Let A be a nice Fréchet algebra and let $\Delta := \{\delta_i,\, i \in I\}$ be an at most countable set of unbounded derivations on A. Suppose that there is a common dense domain dom(Δ) of all finite compositions of derivations in Δ. Then every at most countable set Σ of graph seminorms*

$$\| a \|_{k,f,m} := \sum_{J \subset \{1,\dots,k\}} \| \prod_{j \in J} \delta_{f(j)}\,(a) \,\|_m$$

defines a locally convex topology on dom(Δ), where $\| - \|_m$ ranges over a set of seminorms defining the topology of A, J runs over the ordered subsets of $\{1,\dots,k\}$ and f is a map from the finite set $\{1,\dots,k\}$ to the index set I. Denote by $\mathfrak{A}_\Sigma$ the Fréchet algebra obtained by completion of this locally convex algebra. Then $\mathfrak{A}_\Sigma$ is a smooth subalgebra of A.

Proof: We treat for simplicity the case $k = 1$, the reasoning in the general case being similar. Therefore the topology on $\mathfrak{A}$ is defined by the seminorms

$$\| a \|'_m := \| \partial a \|_m + \| a \|_m$$

Let $U \subset A$ be an open unit ball. We claim that $U' := U \cap \mathfrak{A}$ is an open unit ball in $\mathfrak{A}$. Let $K \subset U'$ be compact and choose $\lambda > 1$ such that $\lambda K \subset U'$ which is possible by the compactness of K. One finds for $a_j \in K$

$$\parallel \prod_1^n a_j \parallel'_m = \parallel \sum_{i=1}^n a_1 \cdot \ldots \cdot (\partial a_i) \cdot \ldots \cdot a_n \parallel_m + \parallel \prod_1^n a_j \parallel_m$$

$$\leq \sum_{i=1}^n \lambda^{1-n} \parallel \sum_{i=1}^n (\lambda a_1) \cdot \ldots \cdot (\partial a_i) \cdot \ldots \cdot (\lambda a_n) \parallel_m + \lambda^{-n} \parallel \prod_1^n (\lambda a_j) \parallel_m$$

By hypothesis $\lambda K \subset U$ has relatively compact multiplicative closure in A. Moreover $\partial(K) \subset A$ is compact. An estimation of the sum above yields therefore

$$\parallel \prod_1^n a_j \parallel'_m \leq (\lambda C_0 n + C_1)\lambda^{-n}$$

If one treats the case $k > 1$ one sees that the number of summands after differentiating a product of n factors k times equals n^k which is of subexponential growth in n so that the assertion holds then as well. $\square$

Another example of smooth subalgebras is provided by

Example 3.7. *[Pu], (7.9) Let A be a separable C^*-algebra and let τ be an (unbounded), densely defined, positive trace on A. Then its domain $\ell^1(A, \tau)$ is a smooth subalgebra of A.*

The basic result about smooth subalgebras is

Theorem 3.8. (Smooth subalgebra theorem)
Let A be a nice Fréchet algebra, let $\mathfrak{A}$ be a smooth subalgebra of A and suppose that at least one of the following conditions is satisfied

- *There exists a family $(\varphi_\lambda : A \to \mathfrak{A}, \lambda \in \Lambda)$ of bounded linear maps, labeled by a directed set Λ, such that $\{i \circ \varphi_\lambda(x), \lambda \in \Lambda\}$ is bounded and $\varinjlim_{\lambda \in \Lambda} i \circ \varphi_\lambda(x) = x$ for all $x \in A$.*
- *A possesses the Grothendieck approximation property.*

Then the inclusion

$$\mathfrak{A} \longrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: Let $i : \mathfrak{A} \hookrightarrow A$ be the inclusion and let $\mathcal{T}i : \mathcal{T}\mathfrak{A} \to \mathcal{T}A$ be the induced homomorphism of universal infinitesimal deformations. We will apply the isomorphism criterion (2.9) to show that $\mathcal{T}i$ is an isomorphism in the stable diffeotopy category.

Fix an open unit ball U of A such that $U' := i^{-1}(U)$ is an open unit ball of $\mathfrak{A}$. This is possible because $\mathfrak{A}$ is a smooth subalgebra of A. Let $S' \subset U'$ and

$S \subset U$ be compact and let $\epsilon' > 0$, $\epsilon > 0$ be such that $Ti : (T\mathfrak{A}_{S'})_{\epsilon'} \to (TA_S)_\epsilon$ represents the restriction of $\mathcal{T}i$ to $(T\mathfrak{A}_{S'})_{\epsilon'}$.
In order to verify the isomorphism criterion it suffices to show that this map fits into a diagram

$$\begin{tikzcd}
(T\mathfrak{A}_{S_1'})_{\epsilon_1'} \arrow[r, "Ti"] & (TA_{S_1})_{\epsilon_1} \\
(T\mathfrak{A}_{S'})_{\epsilon'} \arrow[u] \arrow[r, "Ti"'] & (TA_S)_\epsilon \arrow[u] \arrow[ul, "T\phi"']
\end{tikzcd}$$

with the vertical arrows given by structure maps and which commutes up to diffeotopy.
By our assumptions (and the theorem of Banach-Steinhaus in the first case mentioned above) there exist bounded linear maps $\varphi : A \to \mathfrak{A}$ such that the family $i \circ \varphi^t := (1-t) \cdot i \circ \varphi + t \cdot Id : A \to A,\ 0 \leq t \leq 1$ is arbitrarily close to the identity on S^∞ and $i(S^{'\infty})$. In particular, one can find for given $\epsilon_0 > 0$ a bounded linear map $\phi : A \to \mathfrak{A}$ such that $\omega_{\phi^t}(S^\infty) \subset \epsilon_0 U$ and $\omega_{\phi^t}(i(S^{'\infty})) \subset \epsilon_0 U$ for all $t \in [0,1]$. Consequently $\omega_\phi(S^\infty) \subset \epsilon_0 U'$ by our choice of open unit balls.
The arguments given in the proof of theorem (3.2) apply word for word and show that the homomorphism $T\phi : TA \longrightarrow T\mathfrak{A}$ defines (for a suitable choice of $S_1' \subset U'$ compact and $\epsilon_1' > 0$) a bounded algebra homomorphism $T\phi : (TA_S)_\epsilon \longrightarrow (T\mathfrak{A}_{S_1'})_{\epsilon_1'}$ which makes the left lower triangle of the diagram commute up to diffeotopy.
After choosing $S_1 \subset U$ and $\epsilon_1 > 0$ appropriately, the upper right triangle of the diagram will also commute up to diffeotopy by a similar reasoning.
This completes the proof of the theorem. □

COROLLARY 3.9. *Let* "$\varinjlim\limits_{i\in I}$"$A^i$ *be a nice ind-Fréchet algebra. For each* $i \in I$ *let* $\mathfrak{A}^i$ *be a smooth subalgebra of* A^i *satisfying the assumptions of (3.8). Suppose that the smooth subalgebras* $(\mathfrak{A}^i)_{i\in I}$ *form an inductive system* "$\varinjlim\limits_{i\in I}$"$\mathfrak{A}^i$ *under the structure maps of* "$\varinjlim\limits_{i\in I}$"$A^i$*. Then the canonical morphism*

$$\text{“}\varinjlim_{i\in I}\text{”}\mathfrak{A}^i \longrightarrow \text{“}\varinjlim_{i\in I}\text{”}A^i$$

induces a stable diffeotopy equivalence of universal deformations.

Proof: This follows from the proof of the previous theorem and proposition (2.10). □

3.3.1 Examples

Corollary 3.10. *Let A be a nice Fréchet algebra which possesses the Grothendieck approximation property. Let $\Delta := \{\delta_i,\, i \in I\}$ be an at most countable set of unbounded derivations on A and let $\mathfrak{A}_\Sigma$ be one of the completions of dom(Δ) introduced in (3.6). Then the inclusion*

$$\mathfrak{A}_\Sigma \hookrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: The corollary follows from (3.6) and (3.8). □

Corollary 3.11. *Let A be a nice Fréchet algebra and let $(\Phi^t)_{t\in\mathbb{R}}$ be a continuous one parameter group of automorphisms of A. Let Δ be the corresponding unbounded derivation with domain $\mathfrak{A}^\infty := \{a \in A,\, \Phi^t(a) \in \mathcal{C}^\infty(\mathbb{R}, A)\}$. Then the inclusion*

$$\mathfrak{A}^\infty \hookrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: Let u_λ, $\lambda \in \Lambda$, be a family of smooth functions with compact support on the real line which approach the delta distribution at 0. Then the family of regularization maps $\varphi_\lambda : A \longrightarrow \mathfrak{A}^\infty$, $\varphi_\lambda(a) := \int_{-\infty}^{\infty} u_\lambda(t)\, \Phi^t(a)\, dt$ satisfies the conditions of (3.8). The conclusion follows. □

Corollary 3.12. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra, let M be a smooth compact manifold without boundary, and let $k \geq 0$ be an integer. Then the canonical morphisms*

$$\mathcal{C}^\infty(M, \mathcal{A}) \hookrightarrow \mathcal{C}^k(M, \mathcal{A}) \hookrightarrow C(M, \mathcal{A})$$

of nice ind-Fréchet algebras induce stable diffeotopy equivalences of universal infinitesimal deformations.

Proof: The corollary follows as before from (3.6) and (3.8) by noting that $C(M, A)$ is nice (1.1) and by using convolution with a family of smooth kernels (k_λ) on $M \times M$, approaching the delta distribution along the diagonal, as family of regularization maps (φ_λ). There is also a version for manifolds with boundary. For the definition of the appropriate function spaces see [Pu], (7.7). □

Corollary 3.13. *Let $\mathcal{A}$ be a nice ind-Fréchet algebra and let $k \geq 0$ be an integer. Then the canonical inclusions*

$$\mathcal{C}^\infty([0,1],\mathcal{A}) \hookrightarrow \mathcal{C}^k([0,1],\mathcal{A}) \hookrightarrow C([0,1],\mathcal{A})$$

of nice ind-Fréchet algebras induce stable diffeotopy equivalences of universal infinitesimal deformations.

Proof: This is the case $M = [0,1]$ of (3.12) for manifolds with boundary. □

Corollary 3.14. *Let A be a separable C^*-algebra. Let τ be a densely defined, positive, unbounded trace on A and let $\ell^1(A,\tau)$ be its domain. Then the canonical inclusion*

$$\ell^1(\tau, A) \hookrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: By (3.7) the domain of τ is a smooth subalgebra of A. There exists a bounded approximate unit $(u_\lambda), \lambda \in \Lambda$, for A consisting of elements of the dense twosided ideal $\ell^1(\tau, A)$. Left-multiplication with u_λ provides the regularization maps φ_λ asked for in (3.8). For details see [Pu], (7.9). □

3.4 Topological direct limits

Another fundamental question in the study of functors of topological algebras is their behavior with respect to topological direct limits. As is well known topological K-theory commutes with arbitrary topological direct limits. Under rather mild restrictions the universal infinitesimal deformation functor with values in the stable diffeotopy category possesses a similar behavior.
It turns out that, under these restrictions, the universal infinitesimal deformation of a topological direct limit is stably diffeotopy equivalent to the direct limit (in the ind-category of algebras) of the individual universal deformations. This result provides an effective tool for calculations, as will be shown in a number of examples.

Theorem 3.15. (Limit theorem)
Let “$\underset{\lambda\in\Lambda}{\varinjlim}$”$A_\lambda$ be a directed family of nice Fréchet algebras and let

$$f = \underset{\leftarrow}{\lim} f_\lambda : \ “\underset{\lambda\in\Lambda}{\varinjlim}\text{”} A_\lambda \longrightarrow A$$

be a homomorphism to a nice Fréchet algebra A. Suppose that the following conditions hold:

- *A is separable and possesses the Grothendieck approximation property.*
- *The image $Im(f) := \underset{\lambda\in\Lambda}{\varinjlim} f_\lambda(A_\lambda)$ is dense in A.*

- *There exist seminorms $\| - \|_\lambda$ on A_λ, $\lambda \in \Lambda$, respectively $\| - \|$ on A, and a constant C such that*

  i) The set of elements of length less than 1 with respect to the seminorm is an open unit ball for A_λ, $\lambda \in \Lambda$, respectively A.

  ii)
$$\underset{\lambda' \geq \lambda}{Sup} \parallel i_{\lambda\lambda'}(a_\lambda) \parallel_{\lambda'} < \infty$$
  for all $a_\lambda \in A_\lambda$, $\lambda \in \Lambda$.

  iii)
$$\overline{\lim_{\overrightarrow{\lambda \in \Lambda}}} \parallel a_\lambda \parallel_\lambda \leq C \parallel f(a) \parallel$$
  for all
$$a = \lim_{\overrightarrow{\lambda \in \Lambda}} a_\lambda \in \lim_{\overrightarrow{\lambda \in \Lambda}} A_\lambda$$

Then
$$f : \text{“} \lim_{\overrightarrow{\lambda \in \Lambda}} \text{”} A_\lambda \longrightarrow A$$
induces a stable diffeotopy equivalence of universal infinitesimal deformations.

During the proof we will several times make use of the following

Lemma 3.16. *Let the assumptions of the previous theorem be valid. Then for given $K \subset A_\lambda$ compact and given $\epsilon > 0$ there exists $\lambda' \in \Lambda$ such that $\parallel i_{\lambda\lambda'}(a) \parallel_{\lambda'} \leq C \cdot \parallel f_\lambda(a) \parallel_A + \epsilon$ for all $a \in K$. Here C denotes the constant of the assumption of the previous theorem.*

Proof: By the theorem of Banach-Steinhaus and our assumptions the family $\{ i_{\lambda\lambda'} : A_\lambda \to (A_{\lambda'}, \| - \|_{\lambda'}), \lambda' \geq \lambda \} \bigcup \{ f_\lambda : A_\lambda \to (A, \| - \|_A) \}$ of bounded linear maps on A_λ is equicontinuous. Accordingly there exists a seminorm $\| - \|'$ on A_λ and a constant C' such that $\parallel i_{\lambda\lambda'}(a) \parallel_{\lambda'} \leq C' \cdot \parallel a \parallel'$ and $\parallel f_\lambda(a) \parallel_A \leq C' \cdot \parallel a \parallel'$ for all $a \in A_\lambda$ and $\lambda' > \lambda$. Choose a finite subset $\{y_1, \ldots, y_k\}$ of K such that the balls with respect to $\| - \|'$ around $y_1, \ldots, y_k$ with radius $\frac{\epsilon}{2C'(1+C)}$ cover K. Choose finally $\lambda' > \lambda$ so large that one has $\parallel i_{\lambda\lambda'}(y_l) \parallel_{\lambda'} \leq C \cdot \parallel f_\lambda(y_l) \parallel_A + \frac{\epsilon}{2}$ for all y_l, $1 \leq l \leq k$, which is possible by the assumptions of the theorem. With these choices the desired estimates hold. □

Proof of the theorem:
We want to apply the isomorphism criterion (2.9) to the morphism
$$\mathcal{T}f : \mathcal{T}(\text{“} \lim_{\overrightarrow{\lambda \in \Lambda}} \text{”} A_\lambda) = \underset{\overrightarrow{\lambda \in \Lambda}}{Lim} \mathcal{T}A_\lambda \longrightarrow \mathcal{T}A$$
of ind-Banach algebras. So let $(TA^\lambda_{S_\lambda})_{\epsilon_\lambda} \to (TA_S)_{\epsilon'}$ be a homomorphism representing $\mathcal{T}f_\lambda$ where $S_\lambda \subset A_\lambda$ and $S \subset A$ are compact sets satisfying $\parallel S_\lambda \parallel_{A_\lambda} < 1$ and $\parallel S \parallel_A < 1$ and $\epsilon_\lambda > 0$, $\epsilon' > 0$. It has to be shown that this map fits into a diagram of homomorphisms

$$\begin{tikzcd}
(TA^{\lambda'}_{S_{\lambda'}})_{\epsilon_{\lambda'}} \arrow[r] & (TA_{S'})_{\epsilon''} \\
(TA^{\lambda}_{S_\lambda})_{\epsilon_\lambda} \arrow[u] \arrow[r] & (TA_S)_{\epsilon'} \arrow[u] \arrow[ul, "T\psi"']
\end{tikzcd}$$

which commutes up to diffeotopy.
Denote by $S^\infty(S^\infty_\lambda)$ the relatively compact multiplicative closures of $S(S_\lambda)$.
Step 1:
Fix $\epsilon > 0$ such that $4\epsilon < \epsilon'$. As A possesses the Grothendieck approximation property there exists a bounded linear selfmap $\phi \in \mathcal{L}(A)$ of finite rank such that $\phi(A) \subset \varinjlim_\mu f(A_\mu)$ and $\|\omega(\phi^t,\, S^\infty \cup f_\lambda(S^\infty_\lambda))\|_A < \frac{\epsilon}{2C}$ for all $t \in [0,1]$ where $\phi^t := (1-t)\cdot Id + t\cdot\phi$ and where $C \geq 1$ is a constant as in the assumption of the theorem.
Step 2:
As ϕ is of finite rank one finds a finite dimensional subspace $V \subset A_\mu$ for some $\mu \geq \lambda$ such that $f_\mu : A_\mu \to A$ maps V onto $\phi(A)$. Let $s : \phi(A) \to V$ be any linear section of $f_\mu : V \to \phi(A)$. The set $K := \omega(s \circ \phi, S^\infty)$ is then a bounded and thus relatively compact subset of the finite dimensional space $W := V + V^2 \subset A_\mu$. Similarly $K' := \bigcup_{t=0}^1 \omega((1-t)\cdot i_{\lambda\mu} + t\cdot s\circ\phi\circ f_\lambda,\, S^\infty_\lambda)$ is a relatively compact subset of A_μ.
Step 3:
Choose according to the assumptions of the theorem and the previous lemma some $\lambda' \in \Lambda$ such that $\|i_{\mu\lambda'}(a)\|_{A_{\lambda'}} \leq C\cdot\|f_\mu(a)\|_A + \frac{\epsilon}{2}$ for all $a \in K \cup K' \subset A_\mu$. Put finally

$$\Psi := i_{\mu\lambda'} \circ s \circ \phi : A \to A_{\lambda'}$$

Step 4:
We estimate the deviation of Ψ from multiplicativity on S^∞.

$$\omega(\Psi,\, S^\infty) \,=\, i_{\mu\lambda'}(\omega(s\circ\phi,\, S^\infty)) \,=\, i_{\mu\lambda'}(K)$$

so that

$$\|\, \omega(\Psi,\, S^\infty)\, \|_{\lambda'} = \|\, i_{\mu\lambda'}(K)\, \|_{\lambda'}$$

$$\leq\, C\cdot \|\, f_\mu(K)\, \|_A \,+ \frac{\epsilon}{2} \,=\, C\cdot \|\, \omega(\phi,\, S^\infty)\, \|_A \,+ \frac{\epsilon}{2} \,<\, \epsilon$$

Step 5:
We estimate the deviation of $\chi_t := (1-t)\cdot i_{\lambda\lambda'} + t\cdot\Psi\circ f_\lambda$ from multiplicativity on S_λ^∞.

$$\omega(\chi_t,\, S_\lambda^\infty) \,=\, i_{\mu\lambda'}(\omega((1-t)\cdot i_{\lambda\mu} \,+\, t\cdot s\circ\phi\circ f_\lambda,\, S_\lambda^\infty)) \,\subset\, i_{\mu\lambda'}(K')$$

so that

$$\|\, \omega(\chi_t,\, S_\lambda^\infty) \,\|_{\lambda'} \leq \|\, i_{\mu\lambda'}(K') \,\|_{\lambda'}$$

$$\leq\, C\cdot \|\, f_\mu(K') \,\|_A \,+ \frac{\epsilon}{2} \,=\, C\cdot \|\, \omega(\phi^t,\, f_\lambda(S_\lambda^\infty)) \,\|_A \,+ \frac{\epsilon}{2} \,<\, \epsilon$$

Step 6:
We finally estimate the deviation of $\mu_t := (1-t)\cdot Id_A + t\cdot f_{\lambda'}\circ\Psi$ from multiplicativity on S^∞.

$$\omega(\mu_t,\, S^\infty) \,=\, \omega(\phi^t,\, S^\infty)$$

so that

$$\|\, \omega(\mu_t,\, S^\infty) \,\|_A < \,\epsilon$$

Step 7:
The arguments in the proof of (3.2) show then that for a suitable choice of $S_{\lambda'} \subset A_{\lambda'}$, $S' \subset A$ and $\epsilon_{\lambda'}$, $\epsilon'' > 0$:

- $T\psi : TA \to TA_{\lambda'}$ induces a bounded homomorphism $T\psi : (TA_S)_{\epsilon'} \longrightarrow (TA^{\lambda'}_{S_{\lambda'}})_{\epsilon_{\lambda'}}$ by the estimates of step 4.
- $T\chi_t$, $0 \leq t \leq 1$ defines a diffeotopy between $T\Psi\circ Tf_\lambda$ and $Ti_{\lambda\lambda'}$ by the estimates of step 5.
- $T\mu_t \,:\, (TA_S)_{\epsilon'} \longrightarrow (TA_{S'})_{\epsilon''}$, $0 \leq t \leq 1$, defines a diffeotopy between $Tf_{\lambda'}\circ T\Psi$ and the structure homomorphism by the estimates of step 6.

This establishes the desired diagram. The theorem is therefore proved. □
Note that for nice Fréchet algebras with Grothendieck approximation property the smooth subalgebra theorem (3.8) is a consequence of the previous direct limit theorem, applied to the constant inductive family given by the fixed smooth subalgebra.
As a special case of the limit theorem we obtain

Corollary 3.17. *Let* $``\underset{\overrightarrow{\lambda\in\Lambda}}{\lim}" A_\lambda$ *be a directed family of nice Fréchet algebras. Suppose that there exist seminorms* $\|-\|_\lambda$ *on* A_λ, $\lambda\in\Lambda$, *such that the set of elements of length less than 1 is an open unit ball for* A_λ, *and such that*

$$\underset{\overrightarrow{\lambda\in\Lambda}}{\lim}\ \|\, a \,\|_\lambda = \,0$$

for all $a \in \underset{\overrightarrow{\lambda\in\Lambda}}{\lim}\ A_\lambda$. *Then the universal infinitesimal deformation*

$$\mathcal{T}(\,``\underset{\overrightarrow{\lambda\in\Lambda}}{\lim}" A_\lambda) \,=\, \underset{\overrightarrow{\lambda\in\Lambda}}{Lim}\, \mathcal{T}A_\lambda$$

is a weakly contractible ind-Fréchet algebra.

Proof: The given conditions are equivalent to the assertion that the constant morphism “ $\varinjlim_{\lambda\in\Lambda}$” $A_\lambda \longrightarrow 0$ satisfies the assumptions of the limit theorem (3.15). The proof of this theorem shows therefore that the isomorphism criterion (2.9) applies to the constant morphism $\mathcal{T}($“ $\varinjlim_{\lambda\in\Lambda}$” $A_\lambda) \longrightarrow \mathcal{T}(0) = 0$. This implies our claim. □

3.4.1 Examples

The theorem allows to determine the stable diffeotopy type of universal infinitesimal deformations of numerous algebras which occur as topological direct limits. We present some examples.

Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. It is well known that every nontrivial twosided ideal $\mathcal{J}$ of $\mathcal{B}(\mathcal{H})$ satisfies $\mathcal{F} \subset \mathcal{J} \subset \mathcal{K}$, i.e. contains the smallest nonzero ideal $\mathcal{F}$ of finite rank operators and is contained in the largest ideal $\mathcal{K}$ of all compact operators. An ideal $\mathcal{J}$ is called symmetrically normed if it is complete with respect to a norm $\| \ \|_{\mathcal{J}}$ which satisfies the characteristic inequality $\|AXB\|_{\mathcal{J}} \leq \|A\|_{\mathcal{B}(\mathcal{H})} \cdot \|X\|_{\mathcal{J}} \cdot \|B\|_{\mathcal{B}(\mathcal{H})}$ for all $X \in \mathcal{J}, A, B \in \mathcal{B}(\mathcal{H})$, and $\|P\|_{\mathcal{J}} = \|P_{\mathcal{B}(\mathcal{H})}\| = 1$ for some (and therefore every) rank one projection $P \in \mathcal{B}(\mathcal{H})$. It follows easily from the definition that the inclusion $\mathcal{J} \hookrightarrow \mathcal{B}(\mathcal{H})$ is a bounded map of Banach spaces and that $\| \ \|_{\mathcal{B}(\mathcal{H})} \leq \| \ \|_{\mathcal{J}}$ on $\mathcal{J}$. This implies that $(\mathcal{J}, \| \ \|_{\mathcal{J}})$ is a nonunital Banach algebra. It is known that $\mathcal{J}$ is separable if and only if the ideal $\mathcal{F}$ of finite rank operators is dense in $\mathcal{J}$. (For all this consider [Co] and the references therein).

Corollary 3.18. *Let $\mathcal{J}$ be a separable, symmetrically normed operator ideal in $\mathcal{B}(\mathcal{H})$. Let i : “ $\lim\limits_{n\to\infty}$ ”$M_n(\mathbb{C}) \to \mathcal{J}$ be a homomorphism of ind-Banach algebras sending the matrix units (e_{kk}), $k \in \mathbb{N}$, to the orthogonal projections onto the lines spanned by the vectors of some orthonormal basis of $\mathcal{H}$. Then*

$$i : \text{“} \lim_{n\to\infty} \text{”} M_n(\mathbb{C}) \longrightarrow \mathcal{J}$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: First of all $\mathcal{J}$ possesses the Grothendieck approximation property. To see this consider the contraction with a finite rank projection P. It defines a linear selfmap of $\mathcal{J}$ of norm one because of the characteristic inequality $\|PXP\|_{\mathcal{J}} \leq \|P\|_{\mathcal{B}(\mathcal{H})}\|X\|_{\mathcal{J}}\|P\|_{\mathcal{B}(\mathcal{H})} = \|X\|_{\mathcal{J}}$. As the finite rank operators are dense in $\mathcal{J}$ ($\mathcal{J}$ is separable) it suffices to prove that for every finite set $\{A_1, .., A_k\}$ of finite rank operators and every $\epsilon > 0$ there exists a finite rank projection P satisfying $\|PA_iP - A_i\|_{\mathcal{J}} < \epsilon$ for $1 \leq i \leq k$. Every finite rank operator can be written as a product of three such operators: $A_i = B_iC_iD_i$. Therefore $\|PA_iP - A_i\|_{\mathcal{J}} \leq \|PB_i - B_i\|_{\mathcal{B}(\mathcal{H})}\|C_i\|_{\mathcal{J}}\|D_iP\|_{\mathcal{B}(\mathcal{H})}$ $+\|B_i\|_{\mathcal{B}(\mathcal{H})}\|C_i\|_{\mathcal{J}}\|D_iP - D_i\|_{\mathcal{B}(\mathcal{H})}$ so that the claim has only to be verified for

the operator norm for which it is obvious. Actually this argument shows also that the image of i is dense in $\mathcal{J}$.
As any two norms on the finite dimensional algebras $M_n(\mathbb{C})$ are equivalent the ind-Fréchet algebra "$\lim\limits_{n\to\infty}$" $M_n(\mathbb{C})$ does not depend (up to isomorphism) on the choice of the norms on the algebras $M_n(\mathbb{C})$, $n \in \mathbb{N}$. If we choose the norms obtained from the norm on $\mathcal{J}$ by restriction to $\lim\limits_{n\to\infty} i(M_n(\mathbb{C}))$ then the corollary follows immediately from theorem (3.15). $\square$

Corollary 3.19. *Let the notations of (3.18) be valid. Then for any nice ind-Fréchet algebra $\mathcal{A}$ the homomorphism*

$$Id \otimes i : \underset{n\to\infty}{Lim}\, M_n(\mathcal{A}) \;=\; \underset{n\to\infty}{Lim}\, \mathcal{A} \otimes_\pi M_n(\mathbb{C}) \;\longrightarrow\; \mathcal{A} \otimes_\pi \mathcal{J}$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: Let $\mathcal{A} =$ "$\underset{i\in I}{\varinjlim}$"$A_i$. A reasoning similar to the proof of (3.18) shows that the isomorphism criterion (2.9) applies to the morphisms
$\mathcal{T}(\underset{n\to\infty}{Lim}\, M_n(A_i)) \longrightarrow \mathcal{T}(A_i \otimes_\pi \mathcal{J})$ for all $i \in I$. Proposition (2.10) implies then that $\mathcal{T}(\underset{n\to\infty}{Lim}\, M_n(\mathcal{A})) \longrightarrow \mathcal{T}(\mathcal{A} \otimes_\pi \mathcal{J})$ is a stable diffeotopy equivalence as well. $\square$

Corollary 3.20. *Let A be a C^*-algebra. Let i : "$\lim\limits_{n\to\infty}$"$M_n(\mathbb{C}) \to \mathcal{K}(\mathcal{H})$ be an inclusion as defined in (3.18). Then the homomorphism*

$$Id \otimes i : \text{“}\lim_{n\to\infty}\text{”} M_n(A) \;=\; \text{“}\lim_{n\to\infty}\text{”} A \otimes_\pi M_n(\mathbb{C}) \;\longrightarrow\; A \otimes_{C^*} \mathcal{K}(\mathcal{H})$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

The proof is similar to that of (3.19).

Corollary 3.21. *Let "$\lim\limits_{n\to\infty}$"A_n be an inductive system of separable C^*-algebras and let A be the enveloping C^*-algebra of the algebraic direct limit $\lim\limits_{n\to\infty} A_n$. Suppose that A possesses the Grothendieck approximation property. Then the canonical homomorphism*

$$\text{“}\lim_{n\to\infty}\text{”} A_n \longrightarrow A$$

induces a stable diffeotopy equivalence of universal infinitesimal deformations.

Proof: Obvious from (3.15) and the fact that $\lim\limits_{n\to\infty} \parallel a_n \parallel_{A_n} = \parallel a \parallel_A$ for all $a = \lim\limits_{n\to\infty} a_n \in A$. $\square$

4 Diffeotopy functors on categories of ind-algebras

In this section we summarize what we obtained so far concerning our original goal of improving invariance and stability properties of functors of topological algebras. The results underline the crucial role played by the stable diffeotopy category in this question. We consider functors of nice ind-Fréchet algebras, which are invariant under diffeotopy, under infinitesimal deformations, and under passage to infinite matrix algebras. Suppose in addition that the given functor is not only invariant under diffeotopy, i.e. factors through the unstable diffeotopy category, but that it factors even through the stable diffeotopy category. Then it possesses in fact a number of remarkable properties:

- Continuous homotopy invariance
- Invariance under passage to dense smooth subalgebras (in the presence of the approximation property)
- Topological Morita invariance, i.e. invariance under passage to completions of the infinite matrix algebra over the given algebra

Thus the fact that a matrix stable and deformation invariant functor factors through the stable diffeotopy category ensures already that it behaves in many ways like K-theory. It turns out that among these three required properties the factorization property is the crucial one. Suppose that F is any functor on the ind-category of nice Fréchet algebras which factors through the stable diffeotopy category. Then there is a universal matrix stable and deformation invariant functor associated to it, given by the composition

$$F' := F \circ \mathcal{T} \circ M_\infty \simeq F \circ M_\infty \circ \mathcal{T}$$

with the universal infinitesimal deformation functor $\mathcal{T}$ and the infinite matrix functor M_∞. This functor will possess all the properties listed above.
The universal example of a stable diffeotopy functor is the tautological functor from the category of nice ind-Fréchet algebras to the stable diffeotopy category. The functor $\mathcal{T} \circ M_\infty$ with values in the stable diffeotopy category has therefore a lot of similarities with the (bivariant) K-functor.
It might be interesting to compare this functor to other functors and categories that have been constructed as models of bivariant K-theory such as Higson's category [Hi],[Cu], the category of asymptotic morphisms of Connes and Higson [CH], and the bivariant theories introduced by Cuntz in [Cu1]. There is however an important difference between all these theories and the one considered in the present paper because we completely ignore the excision problem. A closer study of these questions has to be undertaken elsewhere.

Definition 4.1. Let F be a functor on the ind-category of nice Fréchet algebras.

a) F is said to factor through the stable diffeotopy category if it is isomorphic to a functor of the form $F' \circ i$ where i is the canonical functor to the stable diffeotopy category.

b) F is called invariant under infinitesimal deformations, if for every topologically nilpotent extension of nice ind-Fréchet algebras

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

the induced morphism

$$F(\mathcal{A}) \xrightarrow{\simeq} F(\mathcal{B})$$

is an isomorphism.

c) For a nice ind-Fréchet algebra $\mathcal{A}$ let

$$M_\infty(\mathcal{A}) := \underset{n\to\infty}{\mathit{Lim}}\; M_n(\mathcal{A})$$

be its infinite matrix algebra with structure maps $M_n \hookrightarrow M_{n+1}$ given by the "inclusion of the upper left corner". The functor F is called matrix-stable if it turns the canonical morphism $\mathcal{A} \longrightarrow M_\infty(\mathcal{A})$ into an isomorphism.

Theorem 4.2. *Let F be a functor on the ind-category of nice Fréchet algebras which satisfies the following conditions:*

- *F factors through the stable diffeotopy category*
- *F is invariant under infinitesimal deformations*
- *F is matrix stable*

Then the following assertions hold

a) F is a homotopy functor, i.e. if $f, f' : \mathcal{A} \longrightarrow \mathcal{A}'$ are continuously homotopic homomorphisms of nice ind-Fréchet algebras then

$$F(f) = F(f')$$

b) If $A \hookrightarrow B$ is the inclusion of a smooth subalgebra into a nice Fréchet algebra possessing the Grothendieck approximation property, then

$$F(A) \xrightarrow{\simeq} F(B)$$

is an isomorphism.

b)' If

$$\mathcal{A} = \text{“}\varinjlim_{i\in I}\text{”} A_i \longrightarrow \text{“}\varinjlim_{i\in I}\text{”} B_i = \mathcal{B}$$

is a morphism of I-diagrams of nice Fréchet algebras such that $A_i \hookrightarrow B_i$ is the inclusion of a smooth subalgebra and such that B_i possesses the Grothendieck approximation property for all $i \in I$, then

$$F(\mathcal{A}) \xrightarrow{\simeq} F(\mathcal{B})$$

is an isomorphism.

c) Let $\mathcal{J}$ be a separable, symmetrically normed operator ideal and let $j : \mathbb{C} \to \mathcal{J}$ be a homomorphism which maps 1 to a projection of rank one. Then

$$F(Id \otimes_\pi j) : F(\mathcal{A}) \xrightarrow{\simeq} F(\mathcal{A} \otimes_\pi \mathcal{J})$$

is an isomorphism for every nice ind-Fréchet algebra $\mathcal{A}$.

c)' Let $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators on a separable Hilbert space and let $i : \mathbb{C} \to \mathcal{K}(\mathcal{H})$ be a homomorphism which maps 1 to a projection of rank one. Then

$$F(Id \otimes_{C^*} i) : F(\mathcal{B}) \xrightarrow{\simeq} F(\mathcal{B} \otimes_{C^*} \mathcal{K}(\mathcal{H}))$$

is an isomorphism for every ind-C^*-algebra $\mathcal{B}$.

Suppose in addition that F commutes with direct limits (Recall that direct limits exist in any ind-category). Then moreover the following is true.

d) If $(A_i)_{i\in I}$ is a directed family of nice Fréchet algebras such that the topological direct limit $\varinjlim_{i\in I} A_i$ exists in the category of nice Fréchet algebras and possesses the Grothendieck approximation property, then the natural morphism

$$\varinjlim_{i\in I} F(A_i) \xrightarrow{\simeq} F(\varinjlim_{i\in I} A_i)$$

is an isomorphism.

It should be noted that assertions a) b) and d) do not require the matrix stability of the functor under consideration.

Proof: For a nice ind-Fréchet algebra $\mathcal{A}$ denote by $\pi : \mathcal{TA} \longrightarrow \mathcal{A}$ the canonical epimorphism adjoint to the identity map of $\mathcal{A}$ (1.23). For any morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ of nice ind-Fréchet algebras there is the commutative diagram

$$\begin{array}{ccc} F(\mathcal{TA}) & \xrightarrow{F(\mathcal{T}f)} & F(\mathcal{TB}) \\ F(\pi)\downarrow & & \downarrow F(\pi) \\ F(\mathcal{A}) & \xrightarrow{F(f)} & F(\mathcal{B}) \end{array}$$

The invariance of F under infinitesimal deformations implies that the vertical arrows of the diagram are isomorphisms (1.25). Suppose now that the morphism f induces a stable diffeotopy equivalence of universal infinitesimal deformations, i.e. $\mathcal{T}f : \mathcal{T}\mathcal{A} \longrightarrow \mathcal{T}\mathcal{B}$ is a stable diffeotopy equivalence. Then the upper horizontal map in the diagram becomes an isomorphism, because F factors through the stable diffeotopy category. Thus one may conclude that $F(f) : F(\mathcal{A}) \longrightarrow F(\mathcal{B})$ is an isomorphism.

We now prove assertion a) of the theorem. By the smooth subalgebra theorem (3.8) the previous arguments apply to the inclusion of algebras $\mathcal{C}^\infty([0,1],\mathcal{A}) \longrightarrow C([0,1],\mathcal{A})$ so that $F(\mathcal{C}^\infty([0,1],\mathcal{A})) \xrightarrow{\simeq} F(C([0,1],\mathcal{A}))$ is an isomorphism. Any evaluation homomorphism $\mathcal{C}^\infty([0,1],\mathcal{A}) \longrightarrow \mathcal{A}$ is a diffeotopy equivalence and therefore turned by F into an isomorphism. Altogether this shows that any evaluation homomorphism $C([0,1],\mathcal{A}) \longrightarrow \mathcal{A}$ induces an isomorphism $F(C([0,1],\mathcal{A})) \xrightarrow{\simeq} F(\mathcal{A})$. This statement is equivalent to the homotopy invariance of F. Assertions b), c) and d) follow from the previous discussion and the smooth subalgebra theorem (3.8), respectively the direct limit theorem (3.15) and its corollaries (3.18) and (3.19). $\square$

We now make some observations concerning the problem of constructing functors which satisfy the conditions of the previous theorem. It turns out that one can associate in a universal way to any functor F on the ind-category of nice Fréchet algebras, which factors through the stable diffeotopy category, a functor F', which is matrix stable and invariant under infinitesimal deformations. The modified functor F' will be shown to satisfy the assertions of theorem (4.2). This result shows the crucial role played by the stable diffeotopy category in the search for functors of topological algebras with good homotopy, stability, and continuity properties.

Theorem 4.3. *Let F be a functor on the ind-category of nice Fréchet algebras and suppose that F factors through the stable diffeotopy category. Let*

$$F' := F \circ \mathcal{T} \circ M_\infty$$

be the functor obtained by composition with the universal infinitesimal deformation functor $\mathcal{T}$ and the infinite matrix functor M_∞. Then the functor F' is matrix stable, invariant under infinitesimal deformations, and satisfies all the assertions of Theorem (4.2).

Remark 4.4. *If one ignores Morita invariance there is a similar statement for the functor $F'' := F \circ \mathcal{T}$. It is universal among all functors which are invariant under infinitesimal deformations and equipped with a natural transformation to F. It satisfies assertions a), b) and d) of Theorem(4.2).*

Proof: We show first that F' is matrix stable. So let $\mathcal{A}$ be an ind-Fréchet algebra and let $\mathcal{A} \longrightarrow M_\infty(\mathcal{A})$ be the canonical inclusion. The induced homomorphism $M_\infty(\mathcal{A}) \longrightarrow M_\infty(M_\infty(\mathcal{A}))$ is known to be a diffeotopy equivalence. As the universal deformation functor preserves the relation of diffeotopy and as F is diffeotopy invariant, the conclusion follows. We verify next that F' is invariant under infinitesimal deformations. So let

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

be an infinitesimal deformation of $\mathcal{B}$. It follows that

$$0 \longrightarrow M_\infty(\mathcal{N}) \longrightarrow M_\infty(\mathcal{A}) \longrightarrow M_\infty(\mathcal{B}) \longrightarrow 0$$

is again an infinitesimal deformation. (Note that this has only to be verified for finite matrices of fixed size.) Applying the universal infinitesimal deformation functor $\mathcal{T}$ one obtains a morphism

$$\mathcal{T}(M_\infty(\mathcal{A})) \longrightarrow \mathcal{T}(M_\infty(\mathcal{B}))$$

which is a diffeotopy equivalence by the universal properties of $\mathcal{T}$ (1.27). Applying the diffeotopy invariant functor F one concludes that

$$F'(\mathcal{A}) \longrightarrow F'(\mathcal{B})$$

is an isomorphism. Let finally $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism of ind-algebras, such that for each $n \in \mathbb{N}$ the isomorphism criterion (2.9) applies to the morphism

$$\mathcal{T}(M_n(f)) : \mathcal{T}(M_n(\mathcal{A})) \longrightarrow \mathcal{T}(M_n(\mathcal{B}))$$

In particular $\mathcal{T}(M_n(f))$ is a stable diffeotopy equivalence. By (2.10) the direct limit

$$\mathcal{T}(M_\infty(f)) : \mathcal{T}(M_\infty(\mathcal{A})) \longrightarrow \mathcal{T}(M_\infty(\mathcal{B}))$$

of these morphisms is again a stable diffeotopy equivalence. Consequently the induced map

$$F'(f) : F'(\mathcal{A}) \longrightarrow F'(\mathcal{B})$$

is an isomorphism. The condition is satisfied in the following cases: For the inclusion of a smooth subalgebra (a diagram of smooth subalgebras) as in (3.8), for any of the morphisms $M_\infty(\mathcal{A}) \longrightarrow \mathcal{A} \otimes_\pi \mathcal{J}$ or $M_\infty(\mathcal{B}) \longrightarrow \mathcal{B} \otimes_{C^*} \mathcal{K}(\mathcal{H})$ considered in (3.18) and (3.19), and for the morphism $\underset{\overrightarrow{i \in I}}{Lim}\, A_i \longrightarrow A$ of a family $(A_i)_{i \in I}$ into a topological direct limit A, which possesses the Grothendieck approximation property (3.15). If, in the latter case, the functor F commutes in addition with direct limits, one deduces further from

$$F'(\underset{\overrightarrow{i \in I}}{Lim}\, A_i) = F(\mathcal{T}(M_\infty(\underset{\overrightarrow{i \in I}}{Lim}\, A_i))) = F(\underset{\overrightarrow{i \in I}}{Lim}\, \mathcal{T}(M_\infty(A_i))) = \underset{\overrightarrow{i \in I}}{Lim}\, F'(A_i)$$

that

$$\underset{\overrightarrow{i \in I}}{Lim}\, F'(A_i) \longrightarrow F'(A)$$

is an isomorphism. □

It should be noted that there is no reason for the functor F' to factor through the stable diffeotopy category in the way asked for in (4.1) although the original functor F does so. The reason lies in the fact that the suspension and universal infinitesimal deformation functors do not commute in any reasonable sense.

Corollary 4.5. *Consider the functor from the ind-category of nice Fréchet algebras to the stable diffeotopy category which associates to a nice ind-Fréchet algebra the universal infinitesimal deformation of its infinite matrix algebra. This functor is homotopy invariant, invariant under passage to smooth subalgebras (in the presence of the approximation property), and topologically Morita invariant (invariant under projective tensor products with separable symmetrically normed operator ideals or under the C^*-tensor product with the algebra of compact operators in the case of ind-C^*-algebras).*

In particular, this functor shows many similarities with a bivariant K-functor. The basic difference from a K-functor is that the present functor has no reason to satisfy excision.

5 Local cyclic cohomology

We apply now the ideas of the previous section in order to improve the homotopy, stability, and continuity properties of continuous periodic cyclic cohomology. Continuous periodic cyclic (co)homology has the following drawbacks which prevent it from being a good approximative model of K-theory. It is not invariant under continuous homotopies, it is not stable under tensorization with general operator ideals, it is not stable under passage to smooth subalgebras, and it is not compatible with topological direct limits. The considerations of the previous section suggest how to modify continuous periodic cyclic (co)homology in order to obtain a cyclic theory which does not have the mentioned disadvantages. The new cyclic theory should be invariant under infinitesimal deformations and should factor through the stable diffeotopy category of ind-algebras. There is indeed a canonical choice for a homology theory which satisfies these conditions. This is local cyclic (co)homology. The drawbacks mentioned before disappear under the passage from periodic to local cyclic cohomology. So it possesses in fact many properties which are typical for bivariant K-theory [Ka]. Besides this it will turn out to be accessible to direct computation. Local cyclic cohomology becomes thus a valuable tool for the study of problems in noncommutative geometry.

5.1 Cyclic cohomology theories

We recall some well known facts about various cyclic homology theories.

Periodic cyclic cohomology [Co1], [FT]

For a complex algebra A define the A-bimodule of algebraic differential forms by

$$\Omega^n A := \widetilde{A} \otimes A^{\otimes n}, \quad \Omega A := \bigoplus_n \Omega^n A$$

with $\widetilde{A} := A \oplus \mathbb{C}1$ the algebra obtained from A by adjoining a unit. The A-bimodule structure on ΩA is the obvious one. The Hochschild complex of A is given by

$$C_*(A) := (\Omega^* A,\, b)$$

with Hochschild differential

$$b(a_0\otimes\ldots\otimes a_n) := \sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_i a_{i+1}\otimes\ldots\otimes a_n + (-1)^n a_n a_0\otimes\ldots\otimes a_{n-1}$$

which equals

$$b(\omega da) = (-1)^{|\omega|}[\omega, a]$$

Its homology $HH_*(A,A) := H_*(C_*(A))$ is called the Hochschild homology of A. There is a canonical isomorphism

$$HH_*(A,A) \simeq Tor_*^{\widetilde{A}\otimes\widetilde{A}^{op}}(A,A)$$

Associated to the Hochschild complex there is the contractible $\mathbb{Z}/2$-graded cyclic bicomplex

$$CC_*(A) := \left(\bigoplus_{n\in\mathbb{Z}} \Omega^{*+2n}A,\, b+B\right)$$

where the Connes differential B is given by

$$B(a_0\otimes\ldots\otimes a_n) := \sum_{j=0}^{n}(-1)^{jn}\, 1\otimes a_j\otimes\ldots\otimes a_n\otimes a_0\otimes\ldots\otimes a_{j-1}$$

The Hodge-filtration of the cyclic bicomplex is the descending filtration defined by the subcomplexes

$$Fil^k_{Hodge}\, CC_*(A) := \left(b\Omega^k A \bigoplus \Omega^{\geq k}A, b+B\right)$$

generated by algebraic differential forms of degree at least k.
The periodic cyclic bicomplex $\widehat{CC}_*(A)$ of a complex algebra is the completion of the cyclic bicomplex $CC_*(A)$ with respect to the Hodge filtration:

$$\widehat{CC}_*(A) := \lim_{\overleftarrow{n}} CC_*(A)/Fil^n_{Hodge}CC_*(A)$$

Its homology $HP_*(A) := H_*(\widehat{CC}_*(A))$ is called the periodic cyclic homology of A. The cohomology $HP^*(A)$ of the dual chain complex is the periodic cyclic cohomology of A. The relation between periodic cyclic and Hochschild homology, which can be computed as a derived functor, allows the explicit calculation of periodic cyclic cohomology groups.
Cuntz and Quillen [CQ1] propose an approach to periodic cyclic (co)homology which emphasizes the $\mathbb{Z}/2\mathbb{Z}$-periodicity and the stability of the theory under nilpotent extensions. Moreover they work throughout in a bivariant setting.

For a complex algebra A consider the universal linear split extension

$$0 \to IA \to TA := \bigoplus_n A^{\otimes n} \to A \to 0$$

of A [CQ]. The completion $\widehat{TA} := \varprojlim TA/IA^n$ of the tensor algebra with respect to the corresponding adic topology is the universal topologically nilpotent extension of A in the category of adically complete algebras. It is quasifree [CQ] in the sense that every topologically nilpotent extension of it possesses a multiplicative linear section.
The X-complex [CQ1] of A is the $\mathbb{Z}/2\mathbb{Z}$-graded chain complex

$$X_*(A) := \quad \longrightarrow \quad A \quad \xrightarrow{\partial_0} \quad \Omega^1 A/[\Omega^1 A, A] \quad \xrightarrow{\partial_1} \quad A \quad \longrightarrow$$

$$\partial_0(a) \,=\, da, \quad \partial_1(a^0 da^1) \,=\, [a^0, a^1]$$

The X-complex of an adically complete algebra $\widehat{A} = \varprojlim A/I^n$ is defined as the adically complete chain complex $X_*(\widehat{A}) := \varprojlim X_*(A/I^n)$.
Cuntz and Quillen introduce the bivariant periodic cyclic cohomology of a pair of algebras (A, B) as [CQ1]

$$HP_*(A, B) := Mor_{\mathfrak{Ho}}\,(X_*(\widehat{TA}),\, X_*(\widehat{TB}))$$

where $\mathfrak{Ho}$ denotes the homotopy category of adically complete $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes. This functor coincides in the case $A = \mathbb{C}$ (resp. $B = \mathbb{C}$) with the periodic cyclic homology (resp. cohomology) as defined by Connes.
Bivariant periodic cyclic cohomology is a bifunctor on the category of abstract (and more generally of adically complete) complex algebras. Its fundamental properties are

- Homotopy invariance with respect to polynomial homotopies [Co1], [Go]
- Invariance under nilpotent extensions [Go]
- Morita invariance [Co1]
- Excision [CQ2]

The invariance under nilpotent extensions implies that the projection

$$\widehat{CC}_*(\widehat{TA}) := \varprojlim_n \widehat{CC}_*(TA/IA^n) \longrightarrow \widehat{CC}_*(A)$$

is a quasiisomorphism (in fact a chain homotopy equivalence). As mentioned before, the algebra $\widehat{TA}$ is quasifree [CQ], which is equivalent to the fact that it is of projective dimension at most one as bimodule over itself. It follows that the columns and therefore the total complex of $Fil^2_{Hodge}\widehat{CC}_*(\widehat{TA})$ are contractible, so that the projection

$$\widehat{CC}_*(\widehat{TA}) \longrightarrow \widehat{CC}_*(\widehat{TA})/Fil^2_{Hodge}\widehat{CC}_*(\widehat{TA}) \simeq X_*(\widehat{TA})$$

is a quasiisomorphism (in fact a chain homotopy equivalence), too. This establishes the equivalence of the different approaches to periodic cyclic cohomology. Any reasonable definition of a cyclic cohomology theory for topological algebras has to take the topologies of the underlying algebras into account. This is usually done by topologizing the cyclic complexes and by passing then to its completions. Several such theories have been proposed, in particular the following ones.

Continuous periodic cyclic cohomology [Co1]

Let A be a locally convex algebra with jointly continuous multiplication. The A-bimodule of continuous differential forms is given by

$$\Omega^n A := \widetilde{A} \otimes_\pi A^{\otimes_\pi^n} \quad \Omega A := \bigoplus_n \Omega^n A$$

The continuous Hochschild, cyclic and periodic cyclic complexes are defined similarly to the corresponding algebraic complexes by using continuous instead of algebraic differential forms. The homology HP_* of the continuous periodic cylic bicomplex $\widehat{CC}$ is called continuous periodic cyclic homology, the cohomology HP^* of the dual complex of bounded linear functionals on $\widehat{CC}$ is called the continuous periodic cyclic cohomology. It is calculated in the same way as the cyclic groups of abstract algebras with the noteworthy difference that topologically projective resolutions [Co1] have to be used for the computation of Hochschild groups.
The Cuntz-Quillen approach in the continuous case goes as follows. One considers the universal extension

$$0 \to IA \to TA := \oplus A^{\otimes_\pi n} \to A \to 0$$

of complete, locally convex algebras with bounded linear section and denotes by $\widehat{TA} := \varprojlim TA/IA^n$ its I-adic completion.
The X-complex is given in the continuous case by

$$X_*(A) := (A \oplus \Omega^1 A/\overline{[\Omega^1 A, A]}, \partial)$$

Bivariant periodic cyclic cohomology is then defined as

$$HP_*(A,B) := Mor_{\mathfrak{Ho}}(X_*(\widehat{TA}), X_*(\widehat{TB}))$$

the group of morphisms of the X-complexes in the homotopy category of complexes of complete locally convex vector spaces. As before these bivariant

groups coincide with the ones introduced by Connes if one of the variables equals $\mathbb{C}$.
The fundamental properties of bivariant continuous periodic cyclic cohomology are

- Diffeotopy invariance [Co1],
- Invariance under nilpotent extensions [Go],
- Morita invariance [Co1]
- Excision with respect to extensions with bounded linear section [Cu2]
- Existence of a Chern-character $K_* \longrightarrow HP_*$ on topological K-theory with values in continuous periodic cyclic homology [Co1]
- Existence of a Chern-Connes character for finitely summable Fredholm modules with values in continuous periodic cyclic cohomology [Co1]

Whereas continuous periodic cyclic (co)homology is rather well behaved for nuclear Fréchet algebras it has several serious drawbacks if one intends to use it as approximation to the K-functor for Banach- or C^*-algebras.

- The continuous periodic cyclic cohomology of a nuclear C^*-algebra A equals the space of bounded traces on A [Ha]. Thus for a compact Hausdorff space X the cohomology $HP^*(C(X))$ of the C^*-algebra $C(X)$ of continuous functions on X equals the space $C(X)'$ of Radon measures on X. Consequently continuous periodic cyclic cohomology is not invariant under continuous homotopies as $HP^*(C([0,1])) = C([0,1])'$ is infinite dimensional whereas $HP^*(\mathbb{C}) = \mathbb{C}$.
- The continuous periodic cyclic (co)homology of stable C^*-algebras $A \simeq A \otimes_{C^*} \mathcal{K}(\mathcal{H})$ vanishes altogether [Wo] while K-groups remain unaffected under stabilization.
- In the cases mentioned above the Chern-character with values in continuous periodic cyclic homology is obviously far from being rationally injective.

Entire cyclic cohomology [Co2]

The search for a Chern-character in K-homology for not necessarily finitely summable Fredholm modules led Connes [Co2] to the definition of entire cyclic cohomology. For a Banach algebra A let ΩA_ϵ be the completion of the space $\Omega A = \bigoplus_n \widetilde{A} \otimes_\pi A^{\otimes_\pi^n}$ with respect to the family of seminorms

$$\sum_n ([\frac{n}{2}]!)^{-1} \cdot R^{-n} \cdot \| - \|_A^{\otimes_\pi (n+1)}, \ \ R > 1$$

(For later use we introduce also the spaces $\Omega A_{\epsilon,r}$, $r < 1$, as the completions of ΩA with respect to the seminorms $\sum_n ([\frac{n}{2}]!)^{-1} \cdot (1+n)^m \cdot r^n \cdot \| - \|_A^{\otimes_\pi (n+1)}$ for $m \in \mathbb{N}$.) The entire cyclic bicomplex CC_*^ϵ is defined in the usual way using the space ΩA_ϵ instead of continuous differential forms. Its (co)homology is the entire cyclic (co)homology of A. The bivariant entire cyclic cohomology groups of a pair are defined as

$$HC_*^\epsilon(A,B) := Mor_{\mathfrak{Ho}}(CC_*^\epsilon(A), CC_*^\epsilon(B))$$

There are similar complexes $CC_*^{\epsilon,r}$, $r < 1$ based on the spaces $\Omega A_{\epsilon,r}$. Their cohomology will be denoted by $HC^*_{\epsilon,r}(B) := H^*(CC_*^{\epsilon,r}(B))$.
In the Cuntz-Quillen approach entire cyclic cohomology can be described in terms of the strong universal infinitesimal deformation functor $\mathcal{T}'$ (1.21) as

$$HC_*^\epsilon(A, B) = Mor_{\mathfrak{Ho}}(X_*(\mathcal{T}'A), X_*(\mathcal{T}'B))$$

where the morphisms are taken in the homotopy category $\mathfrak{Ho}$ of ind-complexes (5.2). This explains why Connes' definition of $CC_*^\epsilon(A)$ is natural.
The basic properties of entire cyclic (co)homology are

- Diffeotopy invariance [Co2]
- Invariance under strongly topologically nilpotent extensions [Pu1]
- Morita invariance [Co]
- Excision with respect to extensions with bounded linear section [Pu2]
- Existence of a Chern character $K_* \longrightarrow HC_*^\epsilon$ on topological K-theory with values in entire cyclic homology [Co2].
- Existence of a Chern-Connes character for Θ-summable Fredholm modules with values in entire cyclic cohomology [Co2].

Entire cyclic (co)homology can be characterized as the universal functor associated to periodic cyclic (co)homology which is invariant under strong infinitesimal deformations.
Considered as a functor on Banach- resp. C^*-algebras, entire cyclic cohomology has similar drawbacks as the continuous cyclic theory. Again the cohomology of a nuclear C^*-algebra coincides with the space of continuous traces [Kh]. Thus entire cyclic cohomology cannot be homotopy invariant. Moreover it vanishes identically on nuclear stable C^*-algebras. Finally it turns out to be very difficult to calculate entire cyclic cohomology groups directly in terms of their definition.
A basic problem, which actually motivated Connes to introduce cyclic cohomology [Co1] was the search for a Chern-character on K-homology and ultimately for a bivariant Chern-Connes character on Kasparov's bivariant K-theory [Ka]. The target theory of such a character must necessarily be invariant under continuous homotopies. Therefore there cannot exist a bivariant Chern-Connes character with values in any of the cyclic theories discussed so far.

Analytic cyclic cohomology [Pu], [Me]

Let $\mathcal{T}$ be the universal infinitesimal deformation functor (1.23) on the category of nice Fréchet algebras. The bivariant analytic cyclic cohomology of a pair of such algebras is defined [Pu], section 5, as

$$HC^*_{an}(A,\, B) \,:=\, Mor^*_{\mathfrak{Ho}}(X(\mathcal{T}A),\, X(\mathcal{T}B))$$

The basic properties of analytic cyclic (co)homology are

- Diffeotopy invariance
- Invariance under topologically nilpotent extensions
- Morita invariance
- Excision with respect to extensions with bounded linear section [Pu2], [Me]
- Existence of a Chern character $K_* \longrightarrow HC^\epsilon_*$ on topological K-theory with values in analytic cyclic homology.
- Existence of a Chern-Connes character for arbitrary Fredholm modules with values in analytic cyclic cohomology [Me].

Analytic cyclic (co)homology can be characterized by its invariance under topologically nilpotent extensions in a similar way as the entire theory is characterized by its invariance under strong infinitesimal deformations.
In [Me] Meyer develops analytic cyclic cohomology in much greater generality for bornological algebras and shows that most of its properties continue to hold in this broader context. In the case of the precompact bornology he recovers the theory introduced in [Pu].
Analytic cyclic cohomology is quite similar to the entire cyclic theory. The main difference is the existence of a Chern-Connes character for arbitrary Fredholm modules with values in analytic cyclic cohomology. Consequently it is possible to construct interesting analytic cyclic cocycles on general Banach- and C^*-algebras. Explicit calculations of cohomology groups turn out to be quite difficult, however. It remains an open problem, whether analytic cyclic cohomology is invariant under continuous homotopies [Me]. In particular, one does not know whether there exists a bivariant Chern-Connes character with values in analytic cyclic cohomology.

Asymptotic cyclic cohomology [CM], [Pu]

Asymptotic cyclic cohomology was introduced by Connes and Moscovici in [CM] and developed further in [Pu]. The novelty is the introduction of an asymptotic parameter space with one end at "infinity". An asymptotic cocycle should be thought of as a family of densely defined cyclic cocycles indexed by

the parameter space whose domain of definition grows larger and larger as the parameter tends to infinity.
Before we give the definition of the asymptotic theory we have to recall some notation from [Pu]. A DG-object (differential graded object) will be an integer-graded object equipped with a differential d of degree one satisfying $d^2 = 0$. Any morphism of DG objects is supposed to preserve gradings and to commute with the differentials. If A is a nice Fréchet algebra, then the ind-complex $X_*(\Omega\mathcal{T}A)$ is a DG-object in an obvious way. Let M be a smooth manifold and let $\mathfrak{U} = (U_\alpha)_{\alpha\in I}$ be a cover of M by relatively compact open sets. For an ind-Fréchet space $\mathcal{V} = $ "$\underset{\overrightarrow{i\in I}}{\lim}$"$V_i$ define an ind-DG-module by

$$\mathcal{E}(\mathfrak{U},\mathcal{V}) := Ker(\prod_\alpha \Omega_{dR}(U_\alpha)\otimes_\pi \mathcal{V} \longrightarrow \prod_{\alpha,\beta}\Omega_{dR}(U_\alpha\cap U_\beta)\otimes_\pi \mathcal{V})$$

Up to canonical isomorphism, this formal inductive limit does not depend on the choice of $\mathfrak{U}$ and will henceforth be denoted by $\mathcal{E}(M,\mathcal{V})$. If M itself is compact, then $\mathcal{E}(M,\mathcal{V})$ is isomorphic to the space $\Omega_{dR}(M)\otimes_\pi \mathcal{V}$ of differential forms on M with coefficients in $\mathcal{V}$, but for open M this is not the case.
Let $\mathcal{U}$ be a fundamental system of neighborhoods of ∞, ordered by inclusion, in the manifold $\mathbb{R}^n_+, n >> 0$, equipped with the topology of [Pu] (1.1). Put $\mathcal{E}(\mathcal{U},\mathcal{V}) := \underset{\overrightarrow{U\in\mathcal{U}}}{Lim}\,\mathcal{E}(U,\mathcal{V})$. If $\mathcal{V}$ is a ind-DG-complex, then so is $\mathcal{E}(\mathcal{U},\mathcal{V})$.
The asymptotic cyclic cohomology of a pair (A,B) of nice Fréchet algebras is defined as

$$HC^\alpha_*(A,B) := Mor^{DG}_{\mathfrak{Ho}}(X_*(\Omega\mathcal{T}A),\, \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{T}B)))$$

Asymptotic cyclic cohomology possesses the following properties.

- Continuous homotopy invariance [Pu] (6.15)
- Invariance under topologically nilpotent extensions [Pu1]
- Invariance under passage to certain smooth subalgebras [Pu] (7.1)
- Topological Morita invariance [Pu] (7.10)
- Excision with respect to extensions with bounded linear section [Pu2]
- Existence of a multiplicative bivariant Chern-Connes character $KK_* \longrightarrow HC^\alpha_*$ on bivariant K-theory with values in bivariant asymptotic cyclic cohomology. [Pu] (10.1)

So asymptotic cyclic cohomology possesses most of the properties one would like to have for a reasonable cyclic theory of topological algebras. The main drawback of the asymptotic theory lies in the fact that, just as for entire or analytic cyclic cohomology, there are no methods to calculate it directly by homological methods.

We will pass now from algebras to ind-algebras and will generalize various cyclic theories to this context. Cyclic homology theories for ind-algebras take their value in suitable homotopy categories of ind-complexes which we introduce next.

5.2 Homotopy categories of chain-complexes

We introduce some homotopy categories of chain complexes which are similar to the diffeotopy categories of algebras treated in section 2.
Let $\mathfrak{A}$ be a fixed additive category. We let $\mathfrak{C}$ be the category of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$, i.e. the category of formal inductive limits of $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes over $\mathfrak{A}$.

Definition 5.1. The homotopy category $\mathfrak{Ho}$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$ is the category with the same objects as $\mathfrak{C}$ and with homotopy classes of chain maps (of degree $k \in \mathbb{Z}/2\mathbb{Z}$) as morphisms (of degree $k \in \mathbb{Z}/2\mathbb{Z}$):

$$\begin{aligned}
&\mathrm{Mor}^k(\text{“}\varinjlim_{i\in I}\text{”}\, C_*^{(i)}, \text{“}\varinjlim_{j\in J}\text{”}\, D_*^{(j)}) \;:= \\
&= H^k(\mathrm{Hom}^*_{\mathfrak{C}}(\text{“}\varinjlim_{i\in I}\text{”}\, C^{(i)},\ \text{“}\varinjlim_{j\in J}\text{”}\, D^{(j)})) \\
&= H^k(\varprojlim_{i\in I}\varinjlim_{j\in J}\mathrm{Hom}^*_{\mathfrak{A}}(C^{(i)},\, D^{(j)})),
\end{aligned}$$

$$\mathrm{Hom}^*_{\mathfrak{A}}(C_\bullet, D_\bullet) := \left(\prod_l \mathrm{Hom}_{\mathfrak{A}}(C_l, D_{l+*}),\, \partial\right)$$

with

$$\partial(\phi) := \partial_D \circ \phi \,-\, (-1)^{deg(\phi)}\phi \circ \partial_C$$

Definition 5.2. Let $f\,:\,\text{“}\varinjlim_{i\in I}\text{”}\, C_*^i \longrightarrow \text{“}\varinjlim_{j\in J}\text{”}\, {C'}_*^j$ be a morphism of ind-complexes. Define a directed set K of triples (i,j,f_{ij}), $i\in I$, $j\in J$,
$f_{ij} : C_*^i \to {C'}_*^j$ in the same way as in (2.2) and define the mapping cone of f as the ind-complex

$$Cone(f) := \text{“}\varinjlim_{K}\text{”}\, Cone(f_{ij})$$

where

$$Cone(f_{ij})_* := (C_*^i[1] \oplus {C'}^j_*,\, \partial_{C_*^i[1]} \circ \pi_1 \oplus f_{ij}\circ\pi_1 + \partial_{{C'}_*^j}\circ\pi_2)$$

is the cone of the individual chain map f_{ij}. There are obvious morphisms of ind-complexes

$$\text{“}\varinjlim_{j\in J}\text{”}\, {C'}_*^j \longrightarrow Cone(f), \quad Cone(f) \longrightarrow \text{“}\varinjlim_{i\in I}\text{”}\, C_*^i[1]$$

LEMMA 5.3. *Call a triangle $X \to Y \to Z \to X[1]$ in $\mathfrak{Ho}$ distinguished if it is isomorphic to a triangle of the form $C \xrightarrow{f} C' \to Cone(f)$. Equipped with this family of distinguished triangles, the homotopy category $\mathfrak{Ho}$ of ind-complexes becomes a triangulated category.*

For a proof see [KS], 1.4.

DEFINITION 5.4. An ind-complex "$\varinjlim_{i\in I}$" C_*^i is called weakly contractible if for each $i \in I$ there exists $i' \geq i$, such that the structure map $C_*^i \to C_*^{i'}$ is nullhomotopic.

The family of weakly contractible ind-complexes defines a nullsystem in $\mathfrak{Ho}$.

DEFINITION 5.5. (DERIVED IND-CATEGORY)
The derived ind-category $\mathfrak{D}$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$ is the localization of the triangulated homotopy category $\mathfrak{Ho}$ of ind-complexes obtained by inverting the morphisms with weakly contractible mapping cone. It becomes a triangulated category by declaring a triangle in $\mathfrak{D}$ distinguished if it is isomorphic to the image of a distinguished triangle in $\mathfrak{Ho}$.

The isomorphism criteria (2.9) and (2.10) apply verbatim to morphisms in the derived ind-category.

5.3 CYCLIC COHOMOLOGY THEORIES OF IND-ALGEBRAS

CONTINUOUS PERIODIC CYCLIC COHOMOLOGY

The continuous periodic cyclic bicomplex defines a functor from the category of complete, locally convex algebras with jointly continuous multiplication to the category of complexes of complete, locally convex vector spaces. We still denote by $\widehat{CC}_*$ the unique extension of this functor to the corresponding ind-categories which commutes with direct limits. Thus one has

$$\widehat{CC}_*(\text{“}\varinjlim_{i\in I}\text{”} A_i) \,=\, \text{“}\varinjlim_{i\in I}\text{”}\, \widehat{CC}_*(A_i)$$

The bivariant continuous periodic cyclic cohomology of a pair

$$(\mathcal{A},\mathcal{B}) \,=\, (\text{“}\varinjlim_{i\in I}\text{”} A_i,\ \text{“}\varinjlim_{j\in J}\text{”} B_j)$$

of ind-algebras is then defined as

$$HP_*(\mathcal{A},\mathcal{B}) \,:=\, Mor^*_{\mathfrak{Ho}}(\widehat{CC}(\mathcal{A}),\, \widehat{CC}(\mathcal{B}))$$

the graded group of morphisms between the cyclic complexes in the homotopy category of ind-complexes. This group can be calculated as the cohomology of the single complex

$$\varprojlim_{i\in I}\varinjlim_{j\in J} Hom^*(\widehat{CC}(A_i),\, \widehat{CC}(B_j))$$

The Cuntz and Quillen approach generalizes similarly to ind-algebras and yields

$$HP_*(\mathcal{A},\,\mathcal{B})\,\simeq\,Mor^*_{\mathfrak{Ho}}(X(\widehat{T\mathcal{A}}),\,X(\widehat{T\mathcal{B}}))$$

the group of morphisms between the corresponding X-complexes of I-adic completions.
The Cartan homotopy formula [Go] shows that diffeotopic morphisms of ind-algebras induce chain homotopic maps of the corresponding cyclic ind-complexes. Therefore continuous periodic cyclic cohomology is invariant under diffeotopy, i.e. it descends to a functor from the unstable diffeotopy category of ind-algebras (2.1) to the chain homotopy category of ind-complexes.

Entire, analytic, and asymptotic cyclic cohomology

In a similar way the entire (analytic) cyclic bicomplex and entire (analytic, asymptotic) cyclic (co)homology can be naturally extended to ind-algebras. The Cuntz-Quillen approach provides a description of entire (analytic) cyclic cohomology of ind-Banach algebras (nice ind-Fréchet algebras) in terms of the (strong) universal infinitesimal deformation functor $\mathcal{T}'$ (1.21) (resp. $\mathcal{T}$ (1.23)) as follows:

$$HC^\epsilon_*(\mathcal{A},\,\mathcal{B})\,\simeq\,Mor^*_{\mathfrak{Ho}}(X(\mathcal{T}'\mathcal{A}),\,X(\mathcal{T}'\mathcal{B}))$$

$$HC^{an}_*(\mathcal{A},\,\mathcal{B})\,\simeq\,Mor^*_{\mathfrak{Ho}}(X(\mathcal{T}\mathcal{A}),\,X(\mathcal{T}\mathcal{B}))$$

$$HC^\alpha_*(\mathcal{A},\mathcal{B}) := Mor^{DG}_{\mathfrak{Ho}}(X_*(\Omega\mathcal{T}\mathcal{A}),\,\mathcal{E}(\mathcal{U},\,X_*(\Omega\mathcal{T}\mathcal{B})))$$

This shows the invariance of the entire (analytic) cyclic theory under strongly topologically nilpotent extensions (topologically nilpotent extensions). In fact there is the following characterization of entire (analytic) cyclic theory in terms of this invariance.

Lemma 5.6. *Let $F : ind-alg \longrightarrow \mathfrak{Ho}$ be a functor which is invariant under strongly topologically nilpotent extensions (topologically nilpotent extensions) and let $\Phi : F \to \widehat{CC}_*$ be a natural transformation to periodic cyclic homology. Then F factors uniquely through entire (analytic) cyclic homology.*

Proof: The canonical map $\widehat{CC}_*(\mathcal{T}'\mathcal{A}) \longrightarrow X(\mathcal{T}'\mathcal{A})$ (respectively $\widehat{CC}_*(\mathcal{T}\mathcal{A}) \longrightarrow X(\mathcal{T}\mathcal{A})$) is a deformation retraction, because $\mathcal{T}'\mathcal{A}$ (respectively $\mathcal{T}\mathcal{A}$) is strongly topologically quasifree (respectively topologically quasifree) (1.25), [CQ1]. The lemma follows then from a look at the natural commutative diagram

$$\begin{array}{ccccccc} & & & & F(\mathcal{T}'\mathcal{A}) & \overset{\simeq}{\longrightarrow} & F(\mathcal{A}) \\ & & & & \Phi\downarrow & & \downarrow\Phi \\ CC^\epsilon_*(\mathcal{A}) & \overset{\simeq}{\longrightarrow} & X_*(\mathcal{T}'\mathcal{A}) & \overset{\simeq}{\longleftarrow} & \widehat{CC}_*(\mathcal{T}'\mathcal{A}) & \underset{\pi_*}{\longrightarrow} & \widehat{CC}_*(\mathcal{A}) \end{array}$$

respectively

$$\begin{array}{ccccccc} & & & & F(\mathcal{T}\mathcal{A}) & \xrightarrow{\simeq} & F(\mathcal{A}) \\ & & & & \Phi\downarrow & & \downarrow\Phi \\ CC_*^{an}(\mathcal{A}) & \xrightarrow{\simeq} & X_*(\mathcal{T}\mathcal{A}) & \xleftarrow{\simeq} & \widehat{CC}_*(\mathcal{T}\mathcal{A}) & \xrightarrow[\pi_*]{} & \widehat{CC}_*(\mathcal{A}) \end{array}$$

and the fact that the transformations of the bottom lines coincide with the chain homotopy classes of the canonical morphisms of the cyclic complexes. □

5.4 Local cyclic cohomology

We are going to modify continuous periodic cyclic (co)homology in order to obtain a cyclic theory satisfying continuous homotopy invariance, topological Morita invariance, invariance under passage to smooth subalgebras, and compatibility with topological direct limits.
The results of section four tell us that a functor on the ind-category of nice Fréchet algebras possesses the desired properties provided

- It is invariant under infinitesimal deformations (topologically nilpotent extensions).
- It is matrix stable
- It factors through the stable diffeotopy category of ind-algebras.

Among the cyclic theories presented so far, analytic cyclic cohomology is characterized by its invariance under infinitesimal deformations. Moreover it is matrix stable. In order to obtain a cyclic theory which satisfies in addition the last condition we make the

Definition 5.7. (Local cyclic cohomology)
Let $\mathcal{T}$ be the universal infinitesimal deformation functor (1.23) on the category of nice ind-Fréchet algebras and let $\mathfrak{D}$ be the derived ind-category (5.5) of the category of complete, locally convex vector spaces. The bivariant local cyclic cohomology of a pair $(\mathcal{A}, \mathcal{B})$ of nice ind-Fréchet algebras is defined as

$$HC_*^{loc}(\mathcal{A}, \mathcal{B}) := Mor_{\mathfrak{D}}^*(X(\mathcal{T}\mathcal{A}), X(\mathcal{T}\mathcal{B}))$$

the group of morphisms in the derived ind-category between the X-complexes of the universal infinitesimal deformations of the given ind-algebras. The groups

$$HC_*^{loc}(\mathcal{A}) := HC_*^{loc}(\mathbb{C}, \mathcal{A})$$

respectively

$$HC^*_{loc}(\mathcal{A}) := HC_*^{loc}(\mathcal{A}, \mathbb{C})$$

are called the local cyclic homology, respectively local cyclic cohomology of $\mathcal{A}$.

An immediate consequence of the definition is the existence of a composition product.

Proposition 5.8. (Composition products)
Bivariant local cyclic cohomology is a bifunctor on the ind-category of nice Fréchet algebras. The composition of morphisms in the derived ind-category defines a natural associative composition product

$$\circ : HC_*^{loc}(\mathcal{A}, \mathcal{B}) \otimes HC_*^{loc}(\mathcal{B}, \mathcal{C}) \longrightarrow HC_*^{loc}(\mathcal{A}, \mathcal{C})$$

With this product the bivariant local cyclic cohomology $HC_*^{loc}(\mathcal{A}, \mathcal{A})$ becomes a unital ring, and the bivariant groups $HC_*^{loc}(\mathcal{A}, \mathcal{B})$ become $HC_*^{loc}(\mathcal{A}, \mathcal{A})$-$HC_*^{loc}(\mathcal{B}, \mathcal{B})$-bimodules. A bivariant local cyclic cohomology class is called a HC^{loc}-equivalence if the corresponding morphism of complexes in the derived ind-category is an isomorphism.

By its very definition, local cyclic cohomology satisfies the conditions mentioned before. This is shown in the following two propositions.

Proposition 5.9. *Consider the continuous periodic cyclic bicomplex as a functor on the ind-category of nice Fréchet algebras with values in the derived ind-category. Then this functor factors through the stable diffeotopy category of ind-algebras.*

Proof: The Cartan homotopy formula [Go], [CQ] shows that the functor $\widehat{CC}_*$ is invariant under diffeotopy. According to Cuntz [Cu2] continuous periodic cyclic cohomology satisfies excision for extensions with bounded linear section. In particular, a homomorphism $f : A \longrightarrow B$ of Fréchet algebras induces a natural chain homotopy equivalence $\widehat{CC}_*(Cone f) \longrightarrow Cone(\widehat{CC}_*(f))[1]$. Due to the naturality of its homotopy inverse this result carries over to ind-Fréchet algebras. This proves that the continuous periodic cyclic bicomplex $\widehat{CC}_*$ defines a homological functor on the prestable diffeotopy category (2.3). It remains to verify that this functor vanishes on weakly contractible ind-algebras which is evident from the definition of the derived ind-category and the Cartan homotopy formula in periodic cyclic homology. □

The proposition shows that theorem (4.3) applies to the functor $\widehat{CC}_*$. In particular, the functor $\widehat{CC}_* \circ \mathcal{T} \circ M_\infty$ with values in the derived ind-category possesses all the properties listed in theorem (4.2). It remains to identify it with local cyclic cohomology.

Lemma 5.10. *Let $\mathcal{T}$ be the universal infinitesimal deformation functor (1.23) and let M_∞ be the infinite matrix functor (4.1) on the ind-category of nice Fréchet algebras. There is an isomorphism of functors*

$$\widehat{CC}_* \circ \mathcal{T} \circ M_\infty \overset{\simeq}{\longrightarrow} X_* \circ \mathcal{T} \circ M_\infty \overset{\simeq}{\longrightarrow} X_* \circ \mathcal{T}$$

with values in the homotopy category of ind-complexes. Here the transformation on the right hand side is given by the contraction with the trace [Co1].

PROOF: We work in the homotopy category of ind-complexes. The universal infinitesimal deformation of any ind-algebra is topologically quasifree, which implies by [CQ1] that

$$\widehat{CC}_* \circ \mathcal{T} \circ M_\infty \quad \overset{\simeq}{\longrightarrow} \quad X_* \circ \mathcal{T} \circ M_\infty$$

is an isomorphism of functors. By making use of excision in analytic cyclic cohomology [Pu2], it suffices to verify that the contraction with the trace $\tau_* : X_*\mathcal{T}(M_\infty\mathcal{A}) \longrightarrow X_*\mathcal{T}\mathcal{A}$ is an isomorphism for unital $\mathcal{A}$. In this case τ_* factors as

$$X_*\mathcal{T}(M_\infty\mathcal{A}) \overset{\mu}{\longrightarrow} X_*\mathcal{T}(M_\infty\mathbb{C}) \otimes_\pi X_*\mathcal{T}\mathcal{A} \overset{\tau\otimes Id}{\longrightarrow} X_*\mathcal{T}\mathcal{A}$$

where the coproduct μ is an isomorphism by [Pu3]. It remains thus to verify that $\tau_* : X_*\mathcal{T}(M_\infty\mathbb{C}) \longrightarrow X_*\mathcal{T}\mathbb{C} \simeq \mathbb{C}$ is an isomorphism. It factorizes again as $X_*\mathcal{T}(M_\infty\mathbb{C}) \longrightarrow X_*(M_\infty\mathbb{C}) \longrightarrow \mathbb{C}$. The first map is an isomorphism because $M_\infty\mathbb{C}$ is topologically quasifree [CQ], and the second map is an isomorphism by the Morita invariance of cyclic homology. The lemma is proved. □

COROLLARY 5.11. *The functor $X_* \circ \mathcal{T} : ind - alg \longrightarrow \mathcal{D}$ from the ind-category of nice Fréchet algebras to the derived ind-category (5.5) is invariant under infinitesimal deformations, matrix stable, and factors through the stable diffeotopy category in the sense of (4.1).*

We are ready to list the basic properties of local cyclic (co)homology.

THEOREM 5.12. (HOMOTOPY INVARIANCE)
Bivariant local cyclic cohomology is invariant under continuous homotopies, i.e. for a nice ind-Fréchet algebra $\mathcal{A}$ any evaluation homomorphism

$$eval : C([0,1], \mathcal{A}) \longrightarrow \mathcal{A}$$

defines a HC_{loc}-equivalence

$$eval_* \in HC_*^{loc}(C([0,1], \mathcal{A}), \mathcal{A})$$

PROOF: This follows from (5.11), (5.7) and (4.2). □

THEOREM 5.13. (EXCISION) *[Pu2]*
Every extension

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

of nice ind-Fréchet algebras, which admits a local linear section ([Pu2],(5.12)), gives rise to natural long exact sequences

$$\begin{array}{ccccc} HC_*^{loc}(-,\mathcal{I}) & \longrightarrow & HC_*^{loc}(-,\mathcal{A}) & \longrightarrow & HC_*^{loc}(-,\mathcal{B}) \\ \partial\uparrow & & & & \downarrow\partial \\ HC_{*+1}^{loc}(-,\mathcal{B}) & \longleftarrow & HC_{*+1}^{loc}(-,\mathcal{A}) & \longleftarrow & HC_{*+1}^{loc}(-,\mathcal{I}) \end{array}$$

and

$$\begin{array}{ccccc} HC_*^{loc}(\mathcal{I},-) & \longleftarrow & HC_*^{loc}(\mathcal{A},-) & \longleftarrow & HC_*^{loc}(\mathcal{B},-) \\ \partial\downarrow & & & & \uparrow\partial \\ HC_{*+1}^{loc}(\mathcal{B},-) & \longrightarrow & HC_{*+1}^{loc}(\mathcal{A},-) & \longrightarrow & HC_{*+1}^{loc}(\mathcal{I},-) \end{array}$$

of local cyclic cohomology groups.

This is [Pu2], (5.12).

Theorem 5.14. (Topological Morita Invariance)
Let $\mathcal{A}$ be a nice ind-Fréchet algebra and let $\mathcal{B}$ be an ind-C^*-algebra.

a) Let

$$i : \mathcal{A} \longrightarrow M_n(\mathcal{A})$$

be a homomorphism which is given by exterior multiplication with a rank one projector in $M_n(\mathbb{C})$. Then

$$i_* \in HC_0^{loc}(\mathcal{A}, M_n(\mathcal{A}))$$

is a HC_{loc}-equivalence.

b) Let $\mathcal{J}$ be a separable, symmetrically normed operator ideal and let

$$i' : \mathcal{A} \longrightarrow \mathcal{A} \otimes_\pi \mathcal{J}$$

be a homomorphism which is given by exterior multiplication with a rank one projector in $\mathcal{J}$. Then

$$i'_* \in HC_0^{loc}(\mathcal{A}, \mathcal{A} \otimes_\pi \mathcal{J})$$

is a HC_{loc}-equivalence.

c) Let $\mathcal{K}$ be the C^*-algebra of compact operators and let

$$i'' : \mathcal{B} \longrightarrow \mathcal{B} \otimes_{C^*} \mathcal{K}$$

be a homomorphism which is given by exterior multiplication with a rank one projector in $\mathcal{K}$. Then

$$i''_* \in HC_0^{loc}(\mathcal{B}, \mathcal{B} \otimes_{C^*} \mathcal{K})$$

is a HC_{loc}-equivalence.

Proof: This follows from (5.11), (5.7) and (4.2). □

Theorem 5.15. (Invariance under passage to smooth subalgebras)
Let $\mathfrak{A}$ be a smooth subalgebra (3.4) of a nice Fréchet algebra A. If the inclusion

$$i : \mathfrak{A} \hookrightarrow A$$

satisfies the conditions of (3.8) then

$$i_* \in HC_0^{loc}(\mathfrak{A}, A)$$

is a HC^{loc}-equivalence. This is in particular the case if A possesses the Grothendieck approximation property. A similar assertion holds for the inclusion of I-diagrams of smooth subalgebras satisfying the conditions of (3.8).

Proof: This follows from (5.7), (5.11), and (4.2). □
In particular the algebra inclusions mentioned in (3.10) up to (3.14) induce HC^{loc}-equivalences.
According to [Pu3], there exists a natural exterior product on bivariant periodic, entire, analytic, and asymptotic cyclic cohomology. There is also a corresponding product in local cyclic cohomology.

Theorem 5.16. (Exterior products)
There exists a natural and associative exterior product

$$\times : \quad HC_*^{loc}(\mathcal{A}, \mathcal{C}) \otimes HC_*^{loc}(\mathcal{B}, \mathcal{D}) \quad \longrightarrow \quad HC_*^{loc}(\mathcal{A} \otimes_\pi \mathcal{B}, \mathcal{C} \otimes_\pi \mathcal{D})$$

on bivariant local cyclic cohomology of unital ind-algebras. The exterior product is compatible with the composition product in the sense that local cyclic cohomology classes α, β, α', β' satisfy

$$(\alpha \circ \beta) \times (\alpha' \circ \beta') \quad = \quad (\alpha \times \alpha') \circ (\beta \times \beta')$$

whenever these expressions are defined.

Proof: According to [Pu3] there exist natural continuous chain maps

$$\mu : \quad X\mathcal{T}(\mathcal{A} \otimes_\pi \mathcal{B}) \quad \longrightarrow \quad X(\mathcal{T}\mathcal{A}) \otimes_\pi X(\mathcal{T}\mathcal{B})$$

and

$$\nu : \quad X(\mathcal{T}\mathcal{A}) \otimes_\pi X(\mathcal{T}\mathcal{B}) \quad \longrightarrow \quad X\mathcal{T}(\mathcal{A} \otimes_\pi \mathcal{B})$$

which are naturally chain homotopy inverse to each other. On the level of chain maps of ind-complexes the exterior product is defined as

$$\alpha \times \beta := \nu_{\mathcal{A}', \mathcal{B}'} \circ (\alpha \otimes_\pi \beta) \circ \mu_{\mathcal{A}, \mathcal{B}}$$

The induced map on homology gives rise to the exterior product on analytic cyclic (co)homology. Its compatibility with the composition product follows immediately from the fact that the chain maps μ and ν are chain homotopy inverse to each other. Let $\varphi : X(\mathcal{T}\mathcal{A}) \longrightarrow X(\mathcal{T}\mathcal{B})$ be a chain map with weakly contractible mapping cone and let $\mathcal{C}$ be a nice unital ind-Fréchet algebra. The mapping cone of $\varphi \times Id_{\mathcal{C}} : X(\mathcal{T}(\mathcal{A} \otimes_\pi \mathcal{C})) \longrightarrow X(\mathcal{T}(\mathcal{B} \otimes_\pi \mathcal{C}))$ is then chain homotopy equivalent to the weakly contractible ind-complex $Cone(\varphi) \otimes_\pi X(\mathcal{T}\mathcal{C})$ and therefore weakly contractible itself. It follows that the transformation $\times$ descends to an exterior product on local cyclic (co)homology. $\square$

Proposition 5.17. (Chern character) *[Co1]*
The Chern character map of [Co1], [CQ1], defines a natural transformation

$$ch : \quad K_* \quad \longrightarrow \quad HC_*^{loc}$$

from topological K-theory to local cyclic homology.

Proof: This follows from [Co2] and [CQ1]. $\square$

Theorem 5.18. (Bivariant Chern-Connes character) *[Pu2]*

a) *There exists a natural transformation of bifunctors on the category of separable C^*-algebras*

$$ch_{biv} : \quad KK^*(-,-) \quad \longrightarrow \quad HC_*^{loc}(-,-)$$

from Kasparov's bivariant KK-theory to bivariant local cyclic cohomology called the bivariant Chern-Connes character.

b) *It is uniquely characterized by the following two properties:*

   – *If $f : A \to B$ is a homomorphism of C^*-algebras with associated class $[f] \in KK^0(A,B)$, then*

$$ch_{biv}([f]) = f_*$$

   – *If $\epsilon : 0 \to I \to A \to B \to 0$ is an extension of C^*-algebras with completely positive section and associated class $[\epsilon] \in KK^1(B,I)$, and if $[\delta] \in HP^1(B,I)$ denotes the boundary map in local cyclic homology then*

$$ch_{biv}([\epsilon]) = [\delta]$$

c) *The bivariant Chern-Connes character is multiplicative up to a period factor $2\pi i$.*
For any separable C^*-algebras A, B, C the diagram

$$\begin{array}{ccc}
KK^j(A,B) \otimes KK^l(B,C) & \xrightarrow{\circ} & KK^{j+l}(A,C) \\
\downarrow ch_{biv} \otimes ch_{biv} & & \downarrow ch_{biv} \\
HC_{lc}^j(A,B) \otimes HC_{lc}^l(B,C) & \xrightarrow[\frac{1}{(2\pi i)^{jl}} \circ]{} & HC_{lc}^{j+l}(A,C)
\end{array}$$

commutes, where the upper horizontal map is the Kasparov product and the lower horizontal map is given by $\frac{1}{(2\pi i)^{jl}}$ *times the composition product. (See [Pu] for an explanation of the factor* $2\pi i$*).*

d) (GROTHENDIECK-RIEMANN-ROCH THEOREM)
Let

$$ch' := \frac{1}{(2\pi i)^j}\, ch : K_j \longrightarrow HC_j^{loc}$$

be the normalized Chern character on K*-theory, and let* $\alpha \in KK^l(A,B)$*. Then the diagram*

$$\begin{array}{ccc} K_j(A) & \overset{-\otimes\alpha}{\longrightarrow} & K_{j+l}(B) \\ \downarrow ch' & & \downarrow ch' \\ HC_j^{loc}(A) & \underset{-\circ\frac{1}{(2\pi i)^{jl}}ch_{biv}(\alpha)}{\longrightarrow} & HC_{j+l}^{loc}(B) \end{array}$$

commutes.

e) Let $0 \to I \to A \to B \to 0$ *be an extension of separable* C^**-algebras with completely positive section. Then the bivariant Chern-Connes character is compatible with long exact sequences, i.e. the diagrams*

$$\begin{array}{cccccc} \longrightarrow & KK^j(-,B) & \overset{\delta}{\longrightarrow} & KK^{j-1}(-,I) & \longrightarrow \\ & \downarrow ch_{biv} & & \downarrow ch_{biv} & \\ \longrightarrow & HC^j_{loc}(-,B) & \underset{(2\pi i)^j\delta}{\longrightarrow} & HC^{j-1}_{loc}(-,I) & \longrightarrow \end{array}$$

and

$$\begin{array}{cccccc} \longleftarrow & KK^{j+1}(B,-) & \overset{\delta}{\longleftarrow} & KK^j(I,-) & \longleftarrow \\ & \downarrow ch_{biv} & & \downarrow ch_{biv} & \\ \longleftarrow & HC^{j+1}_{loc}(B,-) & \underset{(2\pi i)^j\delta}{\longleftarrow} & HC^j_{loc}(I,-) & \longleftarrow \end{array}$$

commute.

PROOF: This is [Pu2], (6.3). □

6 CALCULATION OF LOCAL CYCLIC COHOMOLOGY GROUPS

The other issue that distinguishes local cyclic cohomology from most cyclic theories is its computability in terms of homological algebra. Striking examples of such calculations are given in [Pu4] and [Pu5]. No computational tools similar to the ones presented here are available for entire, analytic, or asymptotic cyclic cohomology. Its nice functorial properties and its computability by homological methods make local cyclic cohomology a rather accessible invariant for a large class of algebras.

6.1 CALCULATION OF MORPHISM GROUPS IN THE DERIVED IND-CATEGORY

The aim of this section is the construction of a natural spectral sequence which calculates morphism groups in the derived ind-category. The strategy for obtaining such a spectral sequence is well known [Bo], and we just have to adapt it to the setting of this paper. The idea behind it is subsumed in

DEFINITION AND LEMMA 6.1. *Let $\mathfrak{C}$ be a triangulated category, let $\mathfrak{N}$ be a nullsystem in $\mathfrak{C}$ and denote by $\mathfrak{C}/\mathfrak{N}$ the corresponding quotient triangulated category.*

a) An object $X \in ob\ \mathfrak{C}$ is called $\mathfrak{N}$-colocal if $Mor_{\mathfrak{C}}(X, N) = 0,\ \forall N \in \mathfrak{N}$.

b) Suppose that X is $\mathfrak{N}$-colocal. Then the canonical map

$$Mor_{\mathfrak{C}}(X, Y) \xrightarrow{\simeq} Mor_{\mathfrak{C}/\mathfrak{N}}(X, Y)$$

is an isomorphism for all $Y \in ob\ \mathfrak{C} = ob\ \mathfrak{C}/\mathfrak{N}$.

c) Let $X \in ob\ \mathfrak{C}$ and suppose that there exists a morphism $f : P(X) \to X$ from an $\mathfrak{N}$-colocal object $P(X)$ to X such that $Cone(f) \in \mathfrak{N}$. Then

$$(f^*) : Mor_{\mathfrak{C}/\mathfrak{N}}(X, Y) \xrightarrow{\simeq} Mor_{\mathfrak{C}}(P(X), Y)$$

is an isomorphism for all $Y \in ob\ \mathfrak{C}$.

d) If the morphisms described in c) exist for every $X \in ob\ \mathfrak{C} = ob\ \mathfrak{C}/\mathfrak{N}$, then

$$P : \mathfrak{C}/\mathfrak{N} \longrightarrow \mathfrak{C},\ \ X \longrightarrow P(X)$$

becomes a functor which is left adjoint to the forgetful functor $\mathfrak{C} \longrightarrow \mathfrak{C}/\mathfrak{N}$.

In the sequel this lemma will be applied to the nullsystem of weakly contractible ind-complexes in the triangulated homotopy category of ind-complexes over an additive category.

EXAMPLE 6.2. *Let $\mathfrak{A}$ be a fixed additive category and let $\mathfrak{C}$ be the category of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$. The associated homotopy category is denoted by $\mathfrak{Ho}$ and its derived ind-category by $\mathfrak{D}$. Let finally $\mathfrak{N}$ be the nullsystem of weakly contractible ind-complexes in $\mathfrak{Ho}$. Then*

- *Every constant ind-complex (i.e. every ordinary chain complex over $\mathfrak{A}$) is $\mathfrak{N}$-colocal.*

- *The direct limit (in $\mathfrak{C}$) of $\mathfrak{N}$-colocal ind-complexes is not necessarily $\mathfrak{N}$-colocal.*

In fact, if limits of colocal ind-complexes were colocal, then all ind-complexes would be colocal as they are limits of constant ind-complexes. In particular, weakly contractible ind-complexes would be genuinely contractible, which is not the case.
We will construct now a canonical colocal model for the direct limit of a family of colocal ind-complexes. It will serve for calculations in the derived ind-category.
The notations of the previous example will be used throughout this section.

Definition 6.3. Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes

$$\mathbf{C}_i := \text{“}\lim_{\overrightarrow{J_i}}\text{”}\, C_*^{j_i}$$

over an additive category $\mathfrak{A}$. (In other words $\mathcal{C}$ is an ind-object over $\mathfrak{C}$.)

Let $\mathcal{F}$ be the set of triples (I', φ, f) such that

1) I' is a finite directed subset of I.
2) $\varphi : I' \to \coprod_{i\in I'} J_i$ is a map such that $\varphi(i) \in J_i$ for all $i \in I'$.
3) f is a collection of morphisms $f_{ii'} : C_*^{\varphi(i)} \longrightarrow C_*^{\varphi(i')}$, $i < i' \in I'$ representing the structure maps $\mathbf{C}_i \to \mathbf{C}_{i'}$ and such that $f_{i'i''} \circ f_{ii'} = f_{ii''}$ for $i < i' < i'' \in I'$.

The set $\mathcal{F}$ is partially ordered by putting $(I', \varphi, f) \leq (I'', \varphi', f')$ iff

1) $I' \subset I''$
2) $\varphi(i) \leq \varphi'(i)$ for all $i \in I'$
3) For all $i < i' \in I'$ the diagram $\begin{array}{ccc} C_*^{\varphi'(i)} & \xrightarrow{f'_{ii'}} & C_*^{\varphi'(i')} \\ \uparrow & & \uparrow \\ C_*^{\varphi(i)} & \xrightarrow{f_{ii'}} & C_{\varphi(i')} \end{array}$ commutes, where the vertical arrows are given by the structure maps of the ind-objects $\mathbf{C}_i$ and $\mathbf{C}_{i'}$, respectively. With this order $\mathcal{F}$ becomes in fact a directed set.

For $(I', \varphi, f) \in \mathcal{F}$ define a bicomplex $P_{**}^{(I',\varphi,f)}$ with underlying bigraded object

$$P_{pq}^{(I',\varphi,f)} := \bigoplus_{\substack{i_0>\ldots>i_p \\ i_0,\ldots,i_p\in I'}} C_q^{\varphi(i_p)}, \;\; p \in \mathbb{N},\, q \in \mathbb{Z}/2\mathbb{Z},$$

and with differentials ∂', ∂'' given as follows:

$$\partial' := \sum_{k=0}^{p} (-1)^k\, \partial_k : P_{pq}^{(I',\varphi,f)} \longrightarrow P_{(p-1)q}^{(I',\varphi,f)}$$

where the face maps ∂_k, $0 \leq k \leq p$, act on the indices $(i_0, \ldots, i_p)$ by deleting the k-th element of the string and on the corresponding objects by the identity if $k < p$ respectively by the morphism
$f_{i_p i_{p-1}} : C_q{}^{\varphi(i_p)} \to C_q{}^{\varphi(i_{p-1})}$ if $k = p$.

The second differential ∂'' is given by

$$\partial'' : P_{pq}^{(I',\varphi,f)} \longrightarrow P_{p(q-1)}^{(I',\varphi,f)}$$

$$\partial'' := (-1)^p \cdot d_q$$

where $d_q : C_q{}^{\varphi(i_p)} \to C_{q-1}{}^{\varphi(i_p)}$ is the differential in the complex $C_*^{\varphi(i_p)}$.

The total $\mathbb{Z}/2\mathbb{Z}$-graded chain complexes

$$P_*^{(I',\varphi,f)} := \left(\bigoplus_p P_{p,*-p}^{(I',\varphi,f)},\, d = \partial' + \partial'' \right)$$

form an ind-complex

$$\mathbf{P}(\mathcal{C}) := \text{“}\varinjlim_{\mathcal{F}}\text{”}\, P_*^{(I',\varphi,f)}$$

It is called the canonical resolution of $\mathcal{C}$.

LEMMA 6.4. *Let I be a directed set. The canonical resolution (6.3) defines a functor*

$$\mathbf{P} : \mathfrak{C}^I \longrightarrow \mathfrak{C}$$

from the category of I-diagrams over $\mathfrak{C}$ to $\mathfrak{C}$.

The canonical resolution provides a model for the direct limit of the family $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ in the following sense:

LEMMA 6.5. *Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed system of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$, let $\underset{i\in I}{\overrightarrow{Lim}}\, \mathbf{C}_i$ be its direct limit in the category of ind-complexes, and let $\mathbf{P}(\mathcal{C})$ be its canonical resolution. There exists a canonical morphism*

$$\mathbf{P}(\mathcal{C}) \longrightarrow \underset{i\in I}{\overrightarrow{Lim}}\, \mathbf{C}_i$$

of ind-complexes with weakly contractible mapping cone. If $\mathcal{C} \longrightarrow \mathcal{C}'$ is a morphism of I-diagrams of ind-complexes, then the corresponding diagram

$$\begin{array}{ccc} \mathbf{P}(\mathcal{C}) & \longrightarrow & \underset{i\in I}{\overrightarrow{Lim}}\, \mathbf{C}_i \\ \downarrow & & \downarrow \\ \mathbf{P}(\mathcal{C}') & \longrightarrow & \underset{i\in I}{\overrightarrow{Lim}}\, \mathbf{C}'_i \end{array}$$

commutes.

Proof: Consider the ind-complex

$$“\varinjlim_{\mathcal{F}}” E_*^{(I',\varphi,f)},\; E_*^{(I',\varphi,f)} := C_*^{\varphi(i')}$$

where i' is the largest element of the finite directed set I' (the transition morphism $E_*^{(I',\varphi,f)} \longrightarrow E_*^{(I'',\varphi',f')}$ equals $f'_{i'i''}$). An easy verification shows that this ind-complex is a direct limit of the family $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$:

$$“\varinjlim_{\mathcal{F}}” E_*^{(I',\varphi,f)} \simeq \underset{I}{\underrightarrow{Lim}}\, \mathbf{C}_i$$

Let

$$\pi : \mathbf{P}(\mathcal{C}) = “\varinjlim_{\mathcal{F}}” P_*^{(I',\varphi,f)} \longrightarrow “\varinjlim_{\mathcal{F}}” E_*^{(I',\varphi,f)}$$

be the morphism of ind-complexes which is given on the level of the individual complexes P_*^α, $\alpha = (I',\varphi,f) \in \mathcal{F}$, as follows:

$$\pi_\alpha : \bigoplus_p \bigoplus_{\substack{i_0 > \ldots > i_p \\ i_0,\ldots,i_p \in I'}} C_*^{\varphi(i_p)} \longrightarrow C_*^{\varphi(i')}$$

equals zero on direct summands corresponding to strings $i_0 > \ldots > i_p$ with $p > 0$ and is otherwise given by $f_{i_0 i'} : C_*^{\varphi(i_0)} \longrightarrow C_*^{\varphi(i')}$.
The cone of this morphism equals $Cone\,\pi \simeq “\varinjlim_{\mathcal{F}}” Cone\,\pi_\alpha$. We are going to show that $Cone\,\pi_\alpha$ is contractible for each $\alpha \in \mathcal{F}$ which implies that $Cone\,\pi$ is weakly contractible (it will not be genuinely contractible in general).
Let $s_\alpha : E_*^\alpha \longrightarrow P_*^\alpha$ be the morphism of $\mathbb{Z}/2\mathbb{Z}$-graded objects of $\mathfrak{A}$ which identifies $E_*^\alpha = C_*^{\varphi(i')}$ with the summand of P_*^α corresponding to the string $i_0 = i'$. Let furthermore $\chi_\alpha : P_*^\alpha \longrightarrow P_{*+1}^\alpha$ be the operator which vanishes on direct summands corresponding to strings $i_0 > \ldots > i_p$ with $i_0 = i'$ and identifies otherwise the direct summand corresponding to $i_0 > \ldots > i_p$ with the direct summand corresponding to $i' > i_0 > \ldots > i_p$. The morphism

$$h_\alpha : (Cone\,\pi_\alpha)_* = P_*^\alpha[1] \oplus E_*^\alpha \longrightarrow P_{*+1}^\alpha[1] \oplus E_{*+1}^\alpha = (Cone\,\pi_\alpha)_{*+1}$$

$$h_\alpha := \begin{pmatrix} -\chi_\alpha \circ (Id - s_\alpha \circ \pi_\alpha) & -\chi_\alpha \circ (\partial \circ s_\alpha - s_\alpha \circ \partial) + s_\alpha \\ 0 & 0 \end{pmatrix}$$

defines then a contracting homotopy of $Cone\,\pi_\alpha$. The naturality of the construction with respect to morphisms of I-diagrams is obvious. □

Lemma 6.6. *Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$ with canonical resolution $\mathbf{P}(\mathcal{C})$ and let $\mathbf{C}'$ be some other $\mathbb{Z}/2\mathbb{Z}$-graded ind-complex over $\mathfrak{A}$. Let $Q_*(\mathcal{C},\mathbf{C}') := \prod_p Q_{p,*-p}(\mathcal{C},\mathbf{C}')$ be the*

$\mathbb{Z}/2\mathbb{Z}$-graded total complex associated to the bicomplex of abelian groups

$$Q_{pq}(\mathcal{C},\mathbf{C}') := \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I}} Hom^q_{ind\text{-}\mathfrak{A}}(\mathbf{C}_{i_p},\mathbf{C}'),\ \ p\in\mathbb{N},\ q\in\mathbb{Z}/2\mathbb{Z},$$

where $Hom^q_{ind\text{-}\mathfrak{A}}(\mathbf{C}_{i_p},\mathbf{C}')$ denotes the morphisms of degree $q\in\mathbb{Z}/2\mathbb{Z}$ of the graded ind-objects over $\mathfrak{A}$ underlying the ind-complexes $\mathbf{C}_{i_p}$ and $\mathbf{C}'$. The differentials are given on the one hand by the simplicial differential $\partial' := \sum(-1)^k\partial_k$, deleting the appropriate index from the indexing strings and acting in the straightforward manner on the corresponding direct factor, and on the other hand by the differential $\partial''(\Phi) := \Phi\circ\partial_{\mathbf{C}_{i_p}} - (-1)^{|\Phi|}\partial_{\mathbf{C}'}\circ\Phi$ of the Hom-complex $Hom^*_{ind\text{-}\mathfrak{A}}(\mathbf{C}_{i_p},\mathbf{C}')$. Then there is a natural isomorphism

$$Mor^*_{\mathfrak{Ho}}(\mathbf{P}(\mathcal{C}),\mathbf{C}') \simeq H_*(Q_\bullet(\mathcal{C},\mathbf{C}'))$$

i.e. the graded group of morphisms from $\mathbf{P}(\mathcal{C})$ to $\mathbf{C}'$ in the homotopy category of ind-complexes is given by the homology of $Q_*(\mathcal{C},\mathbf{C}')$.

Proof: The graded group of morphisms between two objects $\mathbf{C}$, $\mathbf{C}'$ of the homotopy category $\mathfrak{Ho}$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$ can be calculated as the homology of the Hom-complex $Hom^*_{\text{ind-}\mathfrak{A}}(\mathbf{C},\,\mathbf{C}')$. Therefore one finds

$$Mor^n_{\mathfrak{Ho}}(\mathbf{P}(\mathcal{C}),\mathbf{C}')$$

$$= H^n(\,Hom^*_{\text{ind-}\mathfrak{A}}(\mathbf{P}(\mathcal{C}),\mathbf{C}')\,)$$

$$= H^n(\varprojlim_{\mathcal{F}}\varinjlim_{J} Hom^*_{\mathfrak{A}}\,(\,C^{(I',\varphi,f)},\,C'^{\,j}\,))$$

$$= H^n(\varprojlim_{\mathcal{F}}\varinjlim_{J} Hom^*_{\mathfrak{A}}\,(\bigoplus_p \bigoplus_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I'}} C^{\varphi(i_p)}\,[-p],\,C'^{\,j}\,))$$

$$= H^n(\varprojlim_{\mathcal{F}}\varinjlim_{J}\prod_p \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I'}} Hom^*_{\mathfrak{A}}\,(\,C^{\varphi(i_p)}\,[-p],\,C'^{\,j}\,))$$

$$= H^n(\varprojlim_{\mathcal{F}}\prod_p \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I'}} \varinjlim_{J} Hom^*_{\mathfrak{A}}\,(\,C^{\varphi(i_p)}\,[-p],\,C'^{\,j}\,))$$

because direct limits and finite products commute

$$= H^n(\prod_p \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I}} \varprojlim_{J_{i_p}}\varinjlim_{J} Hom^*_{\mathfrak{A}}\,(\,C^{(j_{i_p})}\,[-p],\,C'^{\,j}\,))$$

$$= H^n(\prod_p \prod_{\substack{i_0>\ldots>i_p\\ i_0,\cdots,i_p\in I}} Hom^*_{\text{ind-}\mathfrak{A}}\,(\,\mathbf{C}_{i_p}\,[-p],\,\mathbf{C}'\,))$$

$$= H^n(\prod_p Q_{p,*-p}(\mathcal{C}, \mathbf{C}'))$$

$$= H^n(Q_*(\mathcal{C}, \mathbf{C}'))$$

□

The following result justifies the introduction of the canonical resolution.

Proposition 6.7. *Let $\mathfrak{N}$ be the nullsystem of weakly contractible ind-complexes in the homotopy category $\mathfrak{Ho}$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$. If $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ is a directed family of $\mathfrak{N}$-colocal ind-complexes, then its canonical resolution $\mathbf{P}(\mathcal{C})$ is $\mathfrak{N}$-colocal as well.*

Proof: Let $\mathbf{C}'$ be a weakly contractible $\mathbb{Z}/2\mathbb{Z}$-graded ind-complex. According to lemma (6.6)

$$Mor^*_{\mathfrak{Ho}}(\mathbf{P}(\mathcal{C}), \mathbf{C}') \simeq H_*(Q_\bullet(\mathcal{C}, \mathbf{C}'))$$

The weak contractibility of $\mathbf{C}'$ implies that the columns of the bicomplex $Q_{**}(\mathcal{C}, \mathbf{C}')$ are acyclic. In fact their homology equals

$$\prod_{\substack{i_0 > \ldots > i_p \\ i_0, \ldots, i_p \in I}} Mor^q_{\mathfrak{Ho}}(\mathbf{C}_{i_p}, \mathbf{C}') = 0$$

The total complex $Q_*(\mathcal{C}, \mathbf{C}')$ is then acyclic as well which proves the claim. □
The following theorem provides the basis for most calculations in the derived ind-category.

Theorem 6.8. *Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes over $\mathfrak{A}$. Suppose that the ind-complexes $(\mathbf{C}_i)_{i\in I}$ are colocal with respect to the nullsystem of weakly contractible ind-complexes and let $\mathbf{C}'$ be some ind-complex.*

a) There exists a spectral sequence (E_r^{pq}, d_r) with E_2-term

$$E_2^{pq} = R^p \varprojlim_{i\in I} Mor^q_{\mathfrak{Ho}}(\mathbf{C}_i, \mathbf{C}')$$

which is natural in $\mathcal{C} \in \mathfrak{C}^I$ and $\mathbf{C}'_* \in \mathfrak{Ho}$. Here $R^p \varprojlim_{i\in I}$ denotes the p-th right derived functor of the inverse limit functor $\varprojlim_{i\in I}$.

b) Suppose that the higher derived limits $R^p \varprojlim_{i\in I}$ vanish for $p >> 0$. Then the spectral sequence converges to

$$E_\infty^{pq} = Gr^p\, Mor^{p+q}_{\mathfrak{D}}(\underset{\overrightarrow{i\in I}}{Lim}\, \mathbf{C}_i, \mathbf{C}')$$

c) *Suppose that the directed set I is countable. Then the spectral sequence collapses and gives rise to a natural short exact sequence*

$$\begin{array}{llllll} 0 & \to & \varprojlim_{i\in I}{}^{1}\, Mor_{\mathfrak{Ho}}^{n-1}(\mathbf{C}_i,\, \mathbf{C}') & \to & Mor_{\mathfrak{D}}^{n}(\,\underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i,\, \mathbf{C}'\,) & \to \\ & \to & \varprojlim_{i\in I} Mor_{\mathfrak{Ho}}^{n}(\,\mathbf{C}_i,\, \mathbf{C}'\,) & \to & 0 & \end{array}$$

Proof: Consider the chain complex $Q_*(\mathcal{C},\mathbf{C}')$ introduced in (6.6). We calculate its homology in two different ways. By lemma (6.6)

$$H_*(Q_\bullet(\mathcal{C},\mathbf{C}')) \simeq Mor_{\mathfrak{Ho}}^*(\mathbf{P}(\mathcal{C}),\mathbf{C}')$$

As the ind-complexes $\mathbf{C}_i$, $i\in I$, are $\mathcal{N}$-colocal by assumption, the ind-complex $\mathbf{P}(\mathcal{C})$ itself is $\mathcal{N}$-colocal by proposition (6.7). Therefore lemma (6.1) applies and shows that the canonical map

$$Mor_{\mathfrak{Ho}}^*(\mathbf{P}(\mathcal{C}),\mathbf{C}') \overset{\simeq}{\longrightarrow} Mor_{\mathfrak{D}}^*(\mathbf{P}(\mathcal{C}),\mathbf{C}')$$

is an isomorphism. By lemma (6.5) the canonical morphism

$$\pi:\ \mathbf{P}(\mathcal{C}) \longrightarrow \underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i$$

defines an isomorphism in the derived ind-category $\mathfrak{D}$ so that one obtains

$$Mor_{\mathfrak{D}}^*(\mathbf{P}(\mathcal{C}),\mathbf{C}') \overset{\simeq}{\longleftarrow} Mor_{\mathfrak{D}}^*(\underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i,\mathbf{C}')$$

This shows finally that

$$H_*(Q_\bullet(\mathcal{C},\mathbf{C}')) \simeq Mor_{\mathfrak{D}}^*(\underset{i\in I}{\underrightarrow{Lim}}\,\mathbf{C}_i,\, \mathbf{C}'\,)$$

We now exhibit a natural filtration of the complex $Q_*(\mathcal{C},\mathbf{C}')$ and calculate its homology by the associated spectral sequence.
The bicomplex $Q_{**}(\,\mathcal{C},\,\mathbf{C}'\,)$ possesses a natural descending filtration with associated graded modules given by the columns Q_{p*}, $p\geq 0$. We take $(\,E_r^{pq},\, d_r\,)$ to be the spectral sequence associated to the corresponding filtration of the total complex $Q_*(\,\mathcal{C},\mathbf{C}'\,)$. For the E_1-term one obtains

$$\begin{aligned} E_1^{pq} &= H^q(\,Q_{p*}(\,\mathcal{C},\,\mathbf{C}'\,),\,\partial''\,) \\ &= H^q\,(\prod_{\substack{i_0>\ldots>i_p\\ i_0,\ldots,i_p\in I}} Hom^*_{\text{ind-}\mathfrak{A}}(\mathbf{C}_{i_p},\,\mathbf{C}')\,) \\ &= \prod_{\substack{i_0>\ldots>i_p\\ i_0,\ldots,i_p\in I}} Mor_{\mathfrak{Ho}}^q(\,\mathbf{C}_{i_p},\,\mathbf{C}'\,) \end{aligned}$$

For the E_2-term one finds

$$E_2^{pq} \;=\; H^p\,(\prod_{\substack{i_0>\ldots>i_* \\ i_0,\cdots,i_*\in I}} Mor^q_{\mathfrak{Ho}}\,(\,\mathbf{C}_{i_*},\,\mathbf{C}'\,),\,\partial'\,)$$

This latter complex equals the standard complex calculating the higher inverse limits of the system

$$Mor^q_{\mathfrak{Ho}}\,(\,\mathbf{C}_i,\,\mathbf{C}'\,),\; i\in I$$

so that one obtains finally

$$E_2^{pq} \;=\; R^p \varprojlim_{i\in I} Mor^q_{\mathfrak{Ho}}\,(\,\mathbf{C}_i,\,\mathbf{C}'\,)$$

b) The vanishing of the higher inverse limits $R^p \varprojlim\limits_{i\in I}$ for $p >> 0$ implies that the projective system $H_*(Q\,(\,\mathcal{C},\mathbf{C}'\,)/Fil^k Q\,(\,\mathcal{C},\mathbf{C}'\,))$, $k\in\mathbb{N}$, satisfies the Mittag-Leffler condition. In particular

$$H_*(Q\,(\,\mathcal{C},\mathbf{C}'\,))\;\xrightarrow{\simeq}\;\varprojlim_{k} H_*(Q\,(\,\mathcal{C},\mathbf{C}'\,)/Fil^k\,(\,\mathcal{C},\mathbf{C}'\,)\,)$$

i.e. the spectral sequence converges.
c) Is an immediate consequence of a) and b) and the fact that for countable I the higher inverse limits $R^p \varprojlim\limits_{i\in I}$ vanish in degree $p>1$. □

In all applications we will deal exclusively with countable ind-complexes and therefore will only make use of part c) of the theorem.
Remark: In [Pu1] I erroneously claimed that the spectral sequence above converges in general. In fact there is no reason why that should be the case. I thank Ralf Meyer for pointing this out to me. However, in all situations where the spectral sequence can be calculated, the condition of b) is automatically satisfied so that no convergence problem arises.
The following consequences of the previous theorem will be particularly useful.

Theorem 6.9. *Let* $\mathbf{C} = $ "$\varinjlim\limits_{i\in I}$"$C_i$, $\mathbf{C}' = $ "$\varinjlim\limits_{j\in J}$"$C'_j$ *be* $\mathbb{Z}/2\mathbb{Z}$*-graded ind-complexes over* $\mathfrak{A}$*. Suppose that I is countable. Then there exists a short exact sequence*

$$\begin{array}{ccccccc} 0 & \to & \varprojlim\limits_{i\in I}{}^1 \varinjlim\limits_{j\in J} Mor^{n-1}_{\mathfrak{Ho}}\,(C_i,\,C'_j\,) & \to & Mor^n_{\mathfrak{D}}\,(\,\mathbf{C},\,\mathbf{C}'\,) & \to & \\ & \to & \varprojlim\limits_{i\in I} \varinjlim\limits_{j\in J} Mor^{n}_{\mathfrak{Ho}}\,(C_i,\,C'_j\,) & \to & 0 & & \end{array}$$

where $\mathfrak{Ho}$ *denotes the homotopy category of* $\mathbb{Z}/2\mathbb{Z}$*-graded chain complexes and* $\mathfrak{D}$ *denotes the derived ind-category over* $\mathfrak{A}$*.*

Proof: Identify each chain complex C_i, $i \in I$, with the associated constant ind-complex $\mathbf{C}_i$, which is $\mathfrak{N}$-colocal (6.2). The direct limit of the corresponding family of constant ind-complexes equals

$$\underset{\overrightarrow{i\in I}}{Lim}\,\mathbf{C}_i \simeq \mathbf{C}$$

Theorem (6.8) therefore applies and yields the assertion as there are natural isomorphisms

$$Mor^n_{\mathfrak{Ho}}(\mathbf{C}_i, \mathbf{C}') \simeq H^n(Hom^*_{\text{ind-}\mathfrak{A}}(\mathbf{C}_i, \mathbf{C}')) = H^n(\varinjlim_{j\in J} Hom^*_{\mathfrak{A}}(C_i, C'_j))$$

$$\simeq \varinjlim_{j\in J} H^n(Hom^*_{\mathfrak{A}}(C_i, C'_j)) \simeq \varinjlim_{j\in J} Mor^n_{\mathfrak{Ho}}(C_i, C'_j)$$

□

By a similar reasoning we obtain

Theorem 6.10. *Let* $\mathbf{C} = “\varinjlim_{i\in I}” C_i$ *be a* $\mathbb{Z}/2\mathbb{Z}$*-graded ind-complex and let* $(\mathbf{C}'_j)$, $j \in J$, *be a directed family of* $\mathbb{Z}/2\mathbb{Z}$*-graded ind-complexes over* $\mathfrak{A}$. *Suppose that* I *is countable. Then there exists a short exact sequence*

$$0 \to \varprojlim_{i\in I}{}^1 \varinjlim_{j\in J} Mor^{n-1}_{\mathfrak{Ho}}(C_i, \mathbf{C}'_j) \to Mor^n_{\mathfrak{D}}(\mathbf{C}, \underset{\overrightarrow{j\in J}}{Lim}\,\mathbf{C}'_j) \to$$

$$\to \varprojlim_{i\in I} \varinjlim_{j\in J} Mor^n_{\mathfrak{Ho}}(C_i, \mathbf{C}'_j) \to 0$$

where $\mathfrak{Ho}$ *denotes the homotopy category of* $\mathbb{Z}/2\mathbb{Z}$*-graded ind-complexes and* $\mathfrak{D}$ *denotes the derived ind-category over* $\mathfrak{A}$.

Whereas the previous result is needed for computations, the following one allows to treat direct limits.

Theorem 6.11. *Let* $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ *be a directed family of* $\mathbb{Z}/2\mathbb{Z}$*-graded ind-complexes and let* $\mathbf{C}'$ *be a* $\mathbb{Z}/2\mathbb{Z}$*-graded ind-complex over* $\mathfrak{A}$. *Suppose that* I *is countable. Then there exists a short exact sequence*

$$0 \to \varprojlim_{i\in I}{}^1 Mor^{n-1}_{\mathfrak{D}}(\mathbf{C}_i, \mathbf{C}') \to Mor^n_{\mathfrak{D}}(\underset{\overrightarrow{i\in I}}{Lim}\,\mathbf{C}_i, \mathbf{C}') \to$$

$$\to \varprojlim_{i\in I} Mor^n_{\mathfrak{D}}(\mathbf{C}_i, \mathbf{C}') \to 0$$

For the proof of the theorem we need

Lemma 6.12. *Every countable chain* $\mathcal{C}_0 \to \mathcal{C}_1 \to \dots$ *of ind-objects* $\mathcal{C}_i = “\varinjlim_{J_i}” C^{(i)}_{j_i}$ *is isomorphic to a chain* $\mathcal{C}'_0 \to \mathcal{C}'_1 \to \dots$ *of ind-objects with* *one and the same index set (equal to J) and morphisms $\mathcal{C}'_n \to \mathcal{C}'_{n+1}$ given by families $C'^{(n)}_j \to C'^{(n+1)}_j$, $j \in J$, such that the diagrams*

$$\begin{array}{ccc} C'^{(n)}_{j'} & \to & C'^{(n+1)}_{j'} \\ \uparrow & & \uparrow \\ C'^{(n)}_{j} & \to & C'^{(n+1)}_{j} \end{array}$$

commute for all $j < j' \in J$.

PROOF: Let J be the set of sequences $(f_k : C^{(k)}_{j_k} \to C^{(k+1)}_{j_{k+1}})_{k\in\mathbb{N}}$ of composable morphisms such that f_k is representing the restriction of $\mathcal{C}_k \to \mathcal{C}_{k+1}$ to $C^{(k)}_{j_k}$. The set J is partially ordered (and directed) in an obvious way. Define ind-objects $\mathcal{C}'_k$, $k \in \mathbb{N}$, with index set J, by putting $\mathcal{C}'_k :=$ " $\varinjlim_{J}$" $C^{(k)}_{j_k}$ and define morphisms $\mathcal{C}'_n \to \mathcal{C}'_{n+1}$, $n \in \mathbb{N}$, of ind-objects by the family $(\pi_n(\alpha))$, $\alpha \in J$, given by the n-th element $\pi_n(\alpha) = f_n : C^{(n)}_{j_n} \to C^{(n+1)}_{j_{n+1}}$ of the sequence α. There is a straightforward morphism of infinite chains of ind-objects from $(\mathcal{C}'_0 \to \mathcal{C}'_1 \to \dots)$ to $(\mathcal{C}_0 \to \mathcal{C}_1 \to \dots)$, which is easily seen to be an isomorphism. $\square$

PROOF OF THEOREM (6.11):
Let $\mathcal{C} = (\mathbf{C}_i)_{i\in I}$ be a directed family of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes, labeled by the countable index set I. After passage to a cofinal subset, which does not affect the statement of the theorem, we may assume that $I = \mathbb{N}$. By the previous lemma, we may further assume that $\mathcal{C}$ is given by a countable directed family of J-diagrams of complexes for some large directed set J. The canonical resolution of an ind-complex (6.3) is functorial on diagrams of complexes (6.4), so that we obtain a countable directed family $\mathbf{P}(\mathbf{C}_i)$ of $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes. Each of these ind-complexes is colocal with respect to the nullsystem of weakly contractible ind-complexes (6.7). Theorem (6.8) applies therefore and yields for any $\mathbb{Z}/2\mathbb{Z}$-graded ind-complex $\mathbf{C}'$ a short exact sequence

$$0 \to \varprojlim_{i\in I}{}^1 Mor^{n-1}_{\mathfrak{H}\mathfrak{o}}(\mathbf{P}(\mathbf{C}_i), \mathbf{C}') \to Mor^n_{\mathfrak{D}}(\underrightarrow{Lim}_{i\in I}\mathbf{P}(\mathbf{C}_i), \mathbf{C}') \to$$

$$\to \varprojlim_{i\in I} Mor^n_{\mathfrak{H}\mathfrak{o}}(\mathbf{P}(\mathbf{C}_i), \mathbf{C}') \to 0$$

The canonical projections $\pi_i : \mathbf{P}(\mathbf{C}_i) \to \mathbf{C}_i$ are natural in the sense that they give rise to a morphism of directed families (6.5). The cone of the induced morphism $\pi : \underrightarrow{Lim}_{i\in I}\mathbf{P}(\mathbf{C}_i) \longrightarrow \underrightarrow{Lim}_{i\in I}\mathbf{C}_i$ of direct limits equals the direct limit of the cones of the morphisms π_i. Because these cones are weakly contractible (6.5), the same holds for the cone of the morphism π. This morphism is therefore an isomorphism in the derived ind-category. As the groups $Mor^n_{\mathfrak{H}\mathfrak{o}}(\mathbf{P}(\mathbf{C}_i), \mathbf{C}')$ equal $Mor^n_{\mathfrak{D}}(\mathbf{C}_i, \mathbf{C}')$ by (6.7) and (6.1), the short exact sequence finally takes the form

$$0 \rightarrow \varprojlim_{i\in I}{}^1 Mor_{\mathfrak{D}}^{n-1}(\mathbf{C}_i, \mathbf{C}') \rightarrow Mor_{\mathfrak{D}}^{n}(\varinjlim_{i\in I} \mathbf{C}_i, \mathbf{C}') \rightarrow$$

$$\rightarrow \varprojlim_{i\in I} Mor_{\mathfrak{D}}^{n}(\mathbf{C}_i, \mathbf{C}') \rightarrow 0$$

□

6.2 Applications to local cyclic cohomology

The following results provide the basic tools for explicit calculations of local cyclic cohomology groups.

Theorem 6.13. (Approximation Theorem)
Let A be a nice separable Fréchet algebra which possesses the Grothendieck approximation property and let U be a convex open unit ball of A.
Let $V_0 \subset \ldots \subset V_n \subset \ldots$ be an increasing sequence of finite dimensional subspaces of A such that $\bigcup_{n=0}^{\infty} V_n$ is a dense subalgebra of A, and let $(\lambda_n)_{n\in\mathbb{N}}$, $(r_n)_{n\in\mathbb{N}}$, be monotone decreasing sequences of positive real numbers such that $\lim_{n\to\infty} \lambda_n = 1$, $\lim_{n\to\infty} r_n = 0$. Denote by A_n the Banach algebra obtained by completion of the subalgebra A generated by V_n with respect to the largest submultiplicative seminorm satisfying $\| \lambda_n V_n \cap U \| \leq 1$. Let $(TA)^r$, respectively $HC^*_{\epsilon,r}$, be the completions of the tensor algebra, respectively the cyclic bicomplex, introduced in (1.22), respectively in the section about entire cyclic cohomology.
Then there exists a natural isomorphism

$$\lim_{n\to\infty} HC_*^{\epsilon}(A_n) \stackrel{\simeq}{\longrightarrow} HC_*^{loc}(A)$$

of homology groups and a natural exact sequence

$$0 \longrightarrow \varprojlim_{n}{}^1 HC^{*-1}_{\epsilon,r_n}(A_n) \longrightarrow HC^*_{loc}(A) \longrightarrow \varprojlim_{n} HC^*_{\epsilon,r_n}(A_n) \longrightarrow 0$$

or

$$0 \longrightarrow \varprojlim_{n}{}^1 H^{*-1}(X((TA_n)^{r_n})) \longrightarrow HC^*_{loc}(A) \longrightarrow$$

$$\longrightarrow \varprojlim_{n} H^*(X((TA_n)^{r_n})) \longrightarrow 0$$

of cohomology groups. Thus the local cyclic (co)homology groups of a nice Fréchet algebra A with approximation property can be expressed in terms of suitable cyclic (co)homology groups of the approximating Banach algebras A_n, $n \in \mathbb{N}$. A similar statement holds if the modified entire cyclic complexes of the approximating algebras are replaced by the corresponding analytic cyclic complexes.

Corollary 6.14. *Let A be a Banach algebra which possesses the Grothendieck approximation property. Let $S \subset A$ be a finite set which generates a dense subalgebra A' of A and let for $\lambda > 1$ be A_λ the completion of A' with respect to the largest submultiplicative seminorm satisfying $\| S \| \leq \lambda$. Then there exists a natural isomorphism*

$$\lim_{\lambda \to 1} HC_*^\epsilon(A_\lambda) \quad \stackrel{\simeq}{\longrightarrow} \quad HC_*^{loc}(A)$$

of homology groups and a natural exact sequence

$$0 \longrightarrow \quad \varprojlim_{\lambda,\, r}{}^1 HC_{\epsilon,r}^{*-1}(A_\lambda) \longrightarrow \quad HC_{loc}^*(A) \quad \longrightarrow \varprojlim_{\lambda,\, r} HC_{\epsilon,r}^*(A_\lambda) \quad \longrightarrow 0$$

or

$$0 \quad \longrightarrow \quad \varprojlim_{\lambda,\, r}{}^1 H^{*-1}(X((TA_\lambda)^r)) \quad \longrightarrow \quad HC_{loc}^*(A) \quad \longrightarrow$$

$$\longrightarrow \quad \varprojlim_{\lambda,\, r} H^*(X((TA_\lambda)^r)) \quad \longrightarrow \qquad 0$$

of cohomology groups.

Remark 6.15. *It should be noted that although entire and analytic cyclic cohomology groups are usually very difficult to compute, a direct or inverse limit of such groups can be quite accessible to calculation.*

Proof: By the approximation theorem for ind-algebras (3.2) there are isomorphisms

$$\text{“}\varinjlim_n\text{”}\, (TA_n)^{r_n} \stackrel{\simeq}{\longrightarrow} \underset{\overrightarrow{n}}{Lim}\, \mathcal{T}'(A_n) \stackrel{\simeq}{\longrightarrow} \mathcal{T}'\mathcal{B}(A) \,=\, \mathcal{T}(A)$$

in the stable diffeotopy category. Passing to continuous cyclic bicomplexes and noting that the ind-algebra “ $\varinjlim_n$” $(TA_n)^{r_n}$ is strictly topologically quasifree, one obtains an isomorphism

$$\text{“}\varinjlim_n\text{”}\, X_*((TA_n)^{r_n}) \stackrel{\simeq}{\longrightarrow} X_*(\mathcal{T}A)$$

in the derived ind-category. As the ind-complex $X_*(\mathcal{T}(\mathbb{C}))$ is chain homotopy equivalent to the constant ind-complex $\mathbb{C}$, which is $\mathcal{N}$-colocal by (6.2), the first assertion follows from the identity $\varinjlim H_*(C_i) \stackrel{\simeq}{\longrightarrow} Mor_{\mathcal{D}}^*(\mathbb{C}, \text{"}\varinjlim\text{"}\, C_i)$. The second assertion is a consequence of (6.9). The equivalence of the two exact sequences follows from the comparison of the Connes and Cuntz-Quillen approach to cyclic homology [CQ1], [Pu] (5.27). □

Theorem 6.16. (Limit Theorem)
Let “ $\varinjlim_{\lambda \in \Lambda}$ ” A_λ be a countable directed family of nice Fréchet algebras and let

$$f = \varprojlim f_\lambda : \text{ “}\varinjlim_{\lambda \in \Lambda}\text{”}\, A_\lambda \longrightarrow A$$

be a homomorphism to a nice Fréchet algebra A. Suppose that the following conditions hold:

- *A is separable and possesses the Grothendieck approximation property.*
- *The image $Im(f) := \varinjlim_{\lambda\in\Lambda} f_\lambda(A_\lambda)$ is dense in A.*
- *There exist seminorms $\| - \|_\lambda$ on A_λ, $\lambda \in \Lambda$, respectively $\| - \|$ on A, and a constant C such that*
  - *i) The set of elements of length less than 1 with respect to the seminorm is an open unit ball for A_λ, $\lambda \in \Lambda$, respectively A.*
  - *ii)*
  $$\overline{\varinjlim_{\lambda\in\Lambda}} \parallel a_\lambda \parallel_\lambda \leq C \parallel f(a) \parallel$$
  for all
  $$a = \varinjlim_{\lambda\in\Lambda} a_\lambda \in \varinjlim_{\lambda\in\Lambda} A_\lambda$$

Then there exists a natural isomorphism

$$\varinjlim_{\lambda\in\Lambda} HC_*^{loc}(A_\lambda) \xrightarrow{\simeq} HC_*^{loc}(A)$$

of local cyclic homology groups and for any nice ind-Fréchet algebra $\mathcal{B}$ a natural exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varprojlim^1_{\lambda\in\Lambda} HC_{*-1}^{loc}(A_\lambda, \mathcal{B}) & \longrightarrow & HC_*^{loc}(A, \mathcal{B}) & \longrightarrow \\ & \longrightarrow & \varprojlim_{\lambda\in\Lambda} HC_*^{loc}(A_\lambda, \mathcal{B}) & \longrightarrow & 0 & \end{array}$$

of bivariant local cyclic cohomology groups.

Proof: This follows from the limit theorem for ind-algebras (3.15), theorem (6.11), and the remark about the colocality of $X_*(\mathcal{T}\mathbb{C})$ made in the proof of the previous theorem. □

7 Relations between cyclic cohomology theories

The various cyclic cohomology theories are related by a number of natural transformations. These fall into two groups: transformations of functors of one variable, i.e. of homology or cohomology, and transformations of bifunctors. All transformations preserve exterior products and in the bivariant case they preserve composition products as well. We will also comment on comparison results for the various cyclic theories.

For an ind-Banach algebra $\mathcal{R}$ the identity of its tensor algebra induces a natural bounded homomorphism $\mathcal{T}'\mathcal{R} \longrightarrow \widehat{T}\mathcal{R}$ of completed tensor algebras and thus a natural transformation $\mathcal{T}' \longrightarrow \widehat{T}$ of functors. Recall the functor $\mathcal{B}$ associating to a nice ind-Fréchet algebra the diagram of associated compactly generated Banach algebras and the transformation $\mathcal{B} \to \iota$ to the identity functor (1.5). Using these one obtains natural transformations

$$\mathcal{T} = \mathcal{T}' \circ \mathcal{B} \longrightarrow \mathcal{T}' \circ \iota = \mathcal{T}' \longrightarrow \widehat{T}$$

for ind-Banach algebras and

$$\mathcal{T} = \mathcal{T}' \circ \mathcal{B} \longrightarrow \widehat{T} \circ \mathcal{B} \longrightarrow \widehat{T}$$

for nice ind-Fréchet algebras. Passing to X-complexes and taking (co)homology groups we end up with the following

PROPOSITION 7.1. *There exist canonical natural transformations*

$$HC_*^{an}(-) \longrightarrow HC_*^{\epsilon}(-) \longrightarrow HP_*(-)$$

of cyclic homology theories for (ind-)Banach algebras, respectively

$$HC_*^{an}(-) \longrightarrow HP_*(-)$$

of cyclic homology theories for nice (ind-)Fréchet algebras. All these transformations are compatible with exterior products and with the Chern-character from topological K-theory.

PROPOSITION 7.2. *There exist canonical natural transformations*

$$HP^*(-) \longrightarrow HC_{\epsilon}^*(-) \longrightarrow HC_{an}^*(-)$$

of cyclic cohomology theories for (ind-)Banach algebras, respectively

$$HP^*(-) \longrightarrow HC_{an}^*(-)$$

of cyclic cohomology theories for nice (ind-)Fréchet algebras. All these transformations are compatible with exterior products.

There exist various Chern characters in K-homology [Co], which are defined for suitable classes of Fredholm modules and take values in the different cyclic cohomology theories. A detailed study of the relations between these characters will be the content of another paper.
The compatibility of the transformations with exterior products is clear because these are induced by explicit natural chain maps of cyclic complexes which are continuous with respect to all relevant topologies.
It arises the question to what extent these transformations are equivalences. The comparison problem turns out to be simpler for cohomology than for homology. In [Me1] Meyer obtains a number of results concerning this problem.

He presents examples of nice Fréchet algebras for which analytic and continuous periodic cyclic homology are different. The simplest example he provides is given by the algebra $\mathcal{S}(\mathbb{Z})$ of sequences of rapid decay. One should also expect that there exist Banach algebras for which analytic, entire, and continuous periodic cyclic homology are different from each other. However no such examples have been exhibited so far. In Meyer's example the continuous periodic and analytic cohomology groups are different as well.
In [Me] Meyer constructs a Chern character for arbitrary Fredholm modules with values in analytic cyclic cohomology. The character is compatible with the index pairing. This allows to exhibit nontrivial analytic cyclic cocycles for large classes of Banach and even C^*-algebras. However there seem to be no methods to determine the corresponding cohomology groups. On the one hand the foregoing discussion shows in particular that $HC^*_{an}(\mathcal{K}(\mathcal{H}))$ does not vanish. On the other hand the results of Haagerup [Ha] and Khalkhali [Kh] imply that the entire cyclic cohomology groups of a nuclear C^*-algebra are isomorphic to the space of continuous traces on that algebra. Therefore $HC^*_\epsilon(\mathcal{K}(\mathcal{H}))$ vanishes. Thus the natural transformation $HC^*_{an} \longrightarrow HC^*_\epsilon$ from analytic to entire cyclic cohomology cannot be an equivalence.
We come now to the transformations relating bivariant cyclic theories.

Proposition 7.3. *There exist canonical natural transformations*

$$HC^{an}_*(-,-) \quad\longrightarrow\quad HC^{\alpha}_*(-,-) \quad\longrightarrow\quad HC^{loc}_*(-,-)$$

of bivariant cyclic cohomology theories of nice (ind-)Fréchet algebras. All these transformations are compatible with composition and exterior products.

Proof: Let $\mathfrak{Ho} \longrightarrow \mathfrak{D}$ be the canonical functor from the homotopy category of ind-complexes to the derived ind-category. It induces a natural map

$$Mor_{\mathfrak{Ho}}(X_*\mathcal{T}(-), X_*\mathcal{T}(-)) \longrightarrow Mor_{\mathfrak{D}}(X_*\mathcal{T}(-), X_*\mathcal{T}(-))$$

of morphism groups which defines the desired transformation from bivariant analytic to bivariant local cyclic cohomology. The compatibility with composition products follows from the compatibility of functors with the composition of morphisms and the compatibility with exterior products is a consequence of the construction of the product in local cyclic cohomology. A bit more work is needed to construct the desired transformations of asymptotic cyclic cohomology. Let DG-$\mathfrak{Ho}$ be the homotopy category of ind-complexes of DG-modules and let DG-$\mathfrak{D}$ be the localization of this homotopy category with respect to the null system given by weakly contractible ind-complexes of DG-modules. Let

$$Mor_{\mathfrak{Ho}}(X_*(\mathcal{T}\mathcal{A}),\, X_*(\mathcal{T}\mathcal{B})) \longrightarrow Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{T}\mathcal{A}),\, \mathcal{E}(\mathcal{U},\, X_*(\mathcal{T}\mathcal{B})))$$

be the transformation which extends a given morphism of ind-complexes φ to the morphism of DG-ind-complexes that equals $\varphi\otimes 1$ in degree zero and vanishes in positive degrees. By [Pu], (4.14), the canonical projection of $X_*(\Omega\mathcal{T}\mathcal{B})$ onto its degree zero subspace induces a natural isomorphism

$$Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{T}\mathcal{A}),\, \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{T}\mathcal{B})))$$

$$\downarrow\simeq$$

$$Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{T}\mathcal{A}),\, \mathcal{E}(\mathcal{U},\, X_*(\mathcal{T}\mathcal{B})))$$

By composition one obtains a natural transformation

$$Mor_{\mathfrak{Ho}}(X_*\mathcal{T}(-),\, X_*\mathcal{T}(-)) \longrightarrow Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{T}(-)),\, \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{T}(-))))$$

from bivariant analytic to bivariant asymptotic cyclic cohomology.
The fact that the ordered family of neighborhoods of ∞ in $\mathbb{R}^n_+$ contains a cofinal family of convex open sets, the Cartan homotopy formula [Pu], (4.11), (4.12) for the asymptotic parameter space, and the isomorphism criterion (2.9) imply that

$$Id \otimes 1 :\, X_*(\Omega\mathcal{T}(-)) \longrightarrow \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{T}(-)))$$

is an isomorphism in DG-$\mathfrak{D}$. Then the canonical functor DG-$\mathfrak{Ho}$ $\longrightarrow$ DG-$\mathfrak{D}$ induces a natural map

$$HC^\alpha_*(A,B) = Mor_{\text{DG-}\mathfrak{Ho}}(X_*(\Omega\mathcal{T}\mathcal{A}), \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{T}\mathcal{B})))$$

$$\longrightarrow\; Mor_{\text{DG-}\mathfrak{D}}(X_*(\Omega\mathcal{T}\mathcal{A}), \mathcal{E}(\mathcal{U},\, X_*(\Omega\mathcal{T}\mathcal{B})))$$

$$\simeq\; Mor_{\text{DG-}\mathfrak{D}}(X_*(\Omega\mathcal{T}\mathcal{A}), X_*(\Omega\mathcal{T}\mathcal{B})) \;\simeq\; Mor_{\text{DG-}\mathfrak{D}}(X_*(\mathcal{T}\mathcal{A}), X_*(\mathcal{T}\mathcal{B}))$$

by [Pu], (6.9) and (4.14)

$$\simeq\; Mor_{\mathfrak{D}}(X_*(\mathcal{T}\mathcal{A}), X_*(\mathcal{T}\mathcal{B})) \;=\; HC^{loc}_*(A,B)$$

It is obvious that this map defines a natural transformation. The composition $HC^{an}_* \longrightarrow HC^\alpha_* \longrightarrow HC^{loc}_*$ clearly coincides with the transformation described at the beginning of the proof. □

COROLLARY 7.4. (FUNCTORIALITY UNDER LINEAR ASYMPTOTIC MORPHISMS) *Let $f_t : A \longrightarrow B, t > 0$, be a linear asymptotic morphism of nice Fréchet-algebras [CH]. Then f induces a natural element $f_* \in HC^{loc}_0(A,B)$ depending only on the continuous homotopy class of f. Moreover $(g \circ f)_* = g_* \circ f_*$ under the composition product. Consequently local cyclic cohomology of nice Fréchet-algebras is functorial under linear asymptotic morphisms.*

PROOF: This follows from the corresponding statement for asymptotic cyclic cohomology [Pu], (6.11), by applying the natural transformation to bivariant local cyclic cohomology. □

Lemma 7.5. *In all the previously mentioned cyclic theories there exist canonical natural equivalences*

$$H_*(-) \xrightarrow{\simeq} H_*(\mathbb{C}, -)$$

and

$$H^*(-) \xrightarrow{\simeq} H_*(-, \mathbb{C})$$

between homology resp. cohomology groups and suitable bivariant cohomology groups. These are compatible with exterior products and with the natural transformations between the various cyclic theories.

Proof: This follows from the fact that the canonical chain map

$$\mathbb{C} = X_*(\mathbb{C}) \longrightarrow X_*(\mathcal{T}\mathbb{C}), \, 1 \to ch(e)$$

is an isomorphism in the homotopy category of ind-complexes [CQ1]. □
In particular, one obtains from (7.3) canonical natural transformations

$$HC_*^{an}(-) \longrightarrow HC_*^{\alpha}(-) \longrightarrow HC_*^{loc}(-)$$

in homology and

$$HC^*_{an}(-) \longrightarrow HC^*_{\alpha}(-) \longrightarrow HC^*_{loc}(-)$$

in cohomology.
Concerning the transformations of homology groups one finds

Proposition 7.6. *The canonical natural transformations*

$$HC_*^{an}(-) \xrightarrow{\simeq} HC_*^{\alpha}(-) \quad \textit{and} \quad HC_*^{\alpha}(-) \xrightarrow{\simeq} HC_*^{loc}(-)$$

are natural equivalences.

Proof: The first assertion is shown in [Pu], (6.9). The ind-complex $X_*(\mathcal{T}\mathbb{C})$ is isomorphic to the constant ind-complex $\mathbb{C}$ ([CQ1]) and thus $\mathfrak{N}$-colocal (6.2). Therefore $HC_*^{an}(-) \xrightarrow{\simeq} HC_*^{loc}(-)$ is an isomorphism by (6.1) which implies the second assertion. □
Not much is known about the comparison between bivariant analytic and bivariant asymptotic or local cyclic cohomology. The basic unsolved question is whether analytic cyclic cohomology is invariant under continuous homotopies [Me] as it is the case for the asymptotic and local theories.
Finally a remark about the comparison between bivariant asymptotic and local cyclic cohomology. The functorial properties of both theories are identical (with the exception of the results depending on the approximation property). Recall that local cyclic cohomology was obtained from the analytic cyclic theory by turning it into a functor which factors through the stable diffeotopy category. This latter was obtained by inverting morphisms of ind-algebras with weakly contractible mapping cone. Asymptotic cyclic cohomology can be interpreted in a similar way. It is constructed by making analytic cyclic cohomology factor

through the category of ind-algebras obtained by inverting morphisms with weakly contractible mapping cone labeled by a countable index set. So there is some reason to believe that both theories coincide for certain sufficiently "small" algebras.
We finally summarize the natural transformations between the various cyclic homology and cohomology theories in the diagrams

$$HP_*(-) \longleftarrow HC^{\epsilon}_*(-) \longleftarrow HC^{an}_*(-)$$

$$HC^{an}_*(-) \xrightarrow{\simeq} HC^{\alpha}_*(-) \xrightarrow{\simeq} HC^{loc}_*(-)$$

and

$$HP^*(-) \longrightarrow HC^*_{\epsilon}(-) \longrightarrow HC^*_{an}(-)$$

$$HC^*_{an}(-) \longrightarrow HC^*_{\alpha}(-) \longrightarrow HC^*_{loc}(-)$$

All transformations are compatible with exterior products and the transformations in homology are compatible with the Chern character from K-theory.

8 Examples

In this last section we give some simple but characteristic examples of explicit calculations of local cyclic cohomology groups. They illustrate the abstract computation scheme developed in section 6. Examples of a similar but more involved nature can be found [Pu4] and [Pu5]. We finally apply local cyclic cohomology to obtain a partial solution of a problem on n-traces formulated in [Co3].
The general idea is to realize the local cyclic (co)homology $HC^{loc}(A)$ of a given algebra A as a limit of the (co)homology groups $HC^{loc}(A_n)$ of a countable directed family (A_n), $n \in \mathbb{N}$, of approximating algebras of a simpler type. Whereas it is usually not possible to compute these approximating (co)homology groups, the transition maps in this directed family often turn out to be amenable to study. And they are all one needs to determine the limit one is interested in.
In the presence of the approximation property one can try to proceed as follows.
1) One looks for a dense subalgebra $\mathfrak{A}$ of A with nice homological properties. By this we mean for example that $\mathfrak{A}$ is of finite Hochschild-homological dimension. Consequently $Fil^k_{Hodge}(\widehat{CC}_*(\mathfrak{A})$ will be contractible for $k >> 0$. If one is lucky the quotient complex $\widehat{CC}(\mathfrak{A})/Fil^k_{Hodge}(\widehat{CC}_*(\mathfrak{A})$ can be identified up to chain homotopy equivalence with a complex with known homology.
2) One chooses an increasing family $0 \subset V_1 \subset \ldots \subset V_n \subset \ldots$ of finite dimensional subspaces of $\mathfrak{A}$ such that $\bigcup V_n$ is a dense subalgebra of A and constructs the enveloping approximating Banach algebras $(A_n), n \in \mathbb{N}$, as in (3.15). By the approximation theorem (6.13) the canonical morphism

$$\text{“}\lim_{n\to\infty}\text{”}\, CC^{\epsilon}_*(A_n) \sim \text{“}\lim_{n\to\infty}\text{”}\, X_*(\mathcal{T}A_n) \longrightarrow X_*(\mathcal{T}A)$$

is then an isomorphism in the derived ind-category. If one is very lucky the vanishing result of step one carries over to the Banach completions $A_n, n \in \mathbb{N}$, so that

$$\text{“}\lim_{n\to\infty}\text{”}\, Fil^k_{Hodge} CC^\epsilon_*(A_n)$$

becomes contractible for $k >> 0$.
3) If one is able to get a good hold of the Banach algebras $A_n, n \in \mathbb{N}$, constructed in step two, one can identify the formal inductive limit

$$\text{“}\lim_{n\to\infty}\text{”}\, CC^\epsilon_*(A_n)/Fil^k_{Hodge} CC^\epsilon_*(A_n) \;\sim\; X_*(\mathcal{T}A)$$

(up to chain homotopy equivalence) with a well known small chain complex.
This is how we will proceed in our first example, the algebra of holomorphic functions on an annulus, where all three steps can be carried out without any difficulty. In the second example, the C^*-algebra of continuous functions on a compact metrizable space, there is no really good choice for a dense subalgebra of finite homological dimension. For special types of spaces like smooth manifolds or finite simplicial complexes there are many more or less natural choices of dense smooth subalgebras. But none of these possess topologically projective resolutions which allow to carry out the second step above. For a compact subset $X \subset \mathbb{R}^n$ the optimal choice seems to take the subalgebra of polynomial functions in $C(X)$ as dense subalgebra and to take the rings of bounded holomorphic functions on a sequence of smaller and smaller Grauert tubes around X as approximating Banach algebras. But the lack of a nice contracting homotopy of the acyclic Koszul complex on such a tube makes it impossible to follow the strategy outlined above. We approximate instead a given compact space by a sequence of smooth compact manifolds with boundary and use the limit theorem (6.16) to reduce to the case of the algebra of smooth functions on a manifold. The local cyclic cohomology of these algebras can be calculated by the diffeotopy invariance and excision property of the theory.
In the third example we finally treat a noncommutative algebra, the reduced group C^*-algebra $C^*_r(F_n)$ of a finitely generated free group. In this case it is easy to follow the first two steps outlined above, the dense subalgebra in question being obviously the group ring. The third step however cannot be carried out directly because one has no control of the approximating Banach algebras constructed in step two. We calculate instead the local cyclic cohomology of a smooth dense Banach subalgebra $\mathcal{A}(F_n)$ of $C^*_r(F_n)$, introduced by Haagerup [Ha1], which can be done by the strategy outlined above. We refer then to the smooth subalgebra theorem (3.8) to deduce the corresponding result for the group C^*-algebra.
We want to make a remark on the possibility of using the outlined strategy (and in particular the second step of it) in concrete calculations. Suppose that a dense subalgebra $\mathfrak{A}$ of finite homological dimension d of a Banach algebra A is given. If $d = 1$, i.e. if $\mathfrak{A}$ is quasifree [CQ], then $\text{“}\lim\limits_{n\to\infty}\text{”}\, Fil^{d+1}_{Hodge} CC^\epsilon_*(A_n)$ will be contractible for any approximating sequence A_n, $n \in \mathbb{N}$, as constructed

above (8.8). We want to emphasize however that this is an exceptional phenomenon and usually does not occur in homological dimension $d > 1$. Take for example $A = \ell^1(\Gamma)$, the Banach convolution algebra of a finitely generated discrete group Γ, and choose as dense subalgebra the group ring $\mathfrak{A} = \mathbb{C}[\Gamma]$. Then the Hochschild homological dimension d of $\mathfrak{A}$ equals the homological dimension of the group Γ. The assertion that "$\lim\limits_{n\to\infty}$" $Fil^{d+1}_{Hodge}CC^{\epsilon}_*(A_n)$ is contractible for some d and some approximating sequence $(A_n), n \in \mathbb{N}$, implies however that every cohomology class in $H^*(\Gamma,\mathbb{C})$ can be represented by group cocycles which are of subexponential growth with respect to any word metric on Γ, and this is rarely the case. A notable exception, where the described strategy in fact works, is the class of hyperbolic and nonpositively curved groups [Pu4], [Pu5].

It should be noted that in the presented examples the images of the approximating Banach algebras $A_n, n \in \mathbb{N}$, are not closed under holomorphic functional calculus in the ambient Banach algebra A. Dense and holomorphically closed subalgebras play a central role in K-theory but do not seem to be relevant in questions related to cyclic cohomology.

8.1 Rings of holomorphic functions on an annulus

Let $U_R := \{z \in \mathbb{C},\, R^{-1} < |z| < R\}$, $R > 1$, be the R-annulus in the complex plane. It is known that every domain in $\mathbb{C}$ with infinite cyclic fundamental group is biholomorphically equivalent to exactly one R-annulus. We consider the algebra

$$\mathcal{O}(\overline{U})_R := \mathcal{O}(U)_R \cap C(\overline{U_R})$$

of holomorphic functions on the annulus which extend continuously to its boundary. It is a unital Banach algebra with respect to the maximum norm.

We are going to determine the local cyclic cohomology of $\mathcal{O}(\overline{U})_R$. It is well known that the Banach algebras $\mathcal{O}(\overline{U}_R)$, $R > 1$, possess the Grothendieck approximation property and contain the ring of Laurent polynomials as a dense subalgebra. The algebra $\mathcal{O}(\overline{U}_R)$ is a topological direct limit of the family $\mathcal{O}(\overline{U}_{R'})$, $R' > R$, in the sense of (3.15). Therefore we deduce from the limit theorem (6.16) that the canonical chain map

$$\underset{R'\to R}{Lim}\, X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'})) \longrightarrow X_*\mathcal{T}(\mathcal{O}(\overline{U}_R))$$

is an isomorphism in the derived ind-category. One might have the impression that nothing has been gained by this because a complex which is quite hard to analyze has been replaced by a limit of similar complexes. It turns out however that the transition maps in the above limit are quite accessible to computation. So the limit $\underset{R'\to R}{Lim}\, X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'}))$ can be calculated although one has essentially no information about the individual complexes in the underlying directed family. Phenomena of this kind often arise in calculations of local cyclic cohomology groups and show the importance of the approximation and limit

theorems in explicit computations. In fact these theorems distinguish local cyclic cohomology among the known cyclic theories.
In order to carry out the computation we need to recall the reduced tensor algebra and reduced infinitesimal deformations. The reduced tensor algebra of a unital algebra A is $RA := \widetilde{TA}/(1-\rho(1_A))$. The functor $R(-)$ is characterized as the left adjoint of the forgetful functor to the category of unital algebras with unital linear maps as morphisms. The reduced universal infinitesimal deformation $\mathcal{RA} := \widetilde{\mathcal{TA}}/(1-\rho(1_{\mathcal{A}}))$ of a nice unital ind-Fréchet algebra $\mathcal{A}$ is characterized similarly by an obvious universal property. The natural map of X-complexes of universal deformations $X_*(\mathcal{TA}) \longrightarrow X_*(\mathcal{RA})$ is a chain homotopy equivalence.

Lemma 8.1. *Let $A = \mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials and consider the norms $\| \sum\limits_n a_n z^n \|_r := \sum\limits_n |a_n| \cdot r^{|n|}$, $r > 1$. Let $\| - \|^r_{N,m}$ be the largest seminorm on the reduced tensor algebra RA satisfying*

$$\| \rho(z^{k_0})\omega(z^{k_1}, z^{k_2}) \cdot \ldots \cdot \omega(z^{k_{2n-1}}, z^{k_{2n}}) \|^r_{N,m} \leq (2+2n)^m \cdot N^{-n} \cdot r^{k_1+\ldots+k_{2n}}$$

Let $\varphi : A \to RA$ be the algebra homomorphism which splits the canonical projection
$\pi : RA \to A$ and is characterized by

$$\varphi(z) = \rho(z), \quad \varphi(z^{-1}) = \varphi(z)^{-1} = \rho(z^{-1}) \sum_{n=0}^{\infty} \omega(z, z^{-1})^n$$

Then for given $r' > r > 1$ there exists $N_0 >> 0$ and constants $C_m, m \in \mathbb{N}$, such that

$$\| \varphi(f) \|^r_{N,m} \leq C_m \cdot \| f \|_{r'}$$

for all $f \in A$ and $N \geq N_0$.

Proof: The straightforward calculation based on the Bianchi-identity $\omega(a, a')\rho(a'') = \omega(a, a'a'') - \omega(aa', a'') + \rho(a)\omega(a', a'')$ is left to the reader. □

Lemma 8.2. *Denote by $\mathcal{R}$ the reduced infinitesimal deformation functor. The canonical projection*

$$\pi : \underset{R' \to R}{Lim}\ \mathcal{R}(\mathcal{O}(\overline{U}_{R'})) \longrightarrow \text{“}\lim_{R' \to R}\text{”}(\mathcal{O}(\overline{U}_{R'}))$$

is a diffeotopy equivalence of ind-algebras. Consequently the projection

$$\pi_* : \underset{R' \to R}{Lim}\ X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'})) \longrightarrow \text{“}\lim_{R' \to R}\text{”} X_*(\mathcal{O}(\overline{U}_{R'}))$$

is a chain homotopy equivalence of ind-complexes.

Proof: Denote by A_r the completion of the algebra A of Laurent polynomials with respect to the norm $\| - \|_r$ introduced in (8.1). It follows from Cauchy's integral formula that the ind-algebra "$\lim\limits_{R'\to R}$" $(\mathcal{O}(\overline{U}_{R'}))$ is canonically isomorphic to "$\lim\limits_{R'\to R}$" $A_{R'}$. By the estimates of the previous lemma the reduced universal infinitesimal deformation $\pi : \underset{R'\to R}{Lim}\, \mathcal{R}(A_{R'}) \longrightarrow$ "$\lim\limits_{R'\to R}$" $A_{R'}$ possesses a multiplicative section. Consequently the ind-algebras "$\lim\limits_{R'\to R}$" $A_{R'}$ and "$\lim\limits_{R'\to R}$" $\mathcal{O}(\overline{U}_{R'})$ are topologically quasifree. This implies the first assertion and the second assertion follows from the Cartan homotopy formula for the X-complexes of quasifree algebras [CQ1] and the fact that the canonical morphism $\underset{R'\to R}{Lim}\, X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'})) \longrightarrow \underset{R'\to R}{Lim}\, X_*\mathcal{R}(\mathcal{O}(\overline{U}_{R'}))$ is a chain homotopy equivalence. □

PROPOSITION 8.3. *Let $\mathcal{O}(\overline{U}_R)$, $R > 1$, be the Banach algebra of holomorphic functions on the annulus*

$$U_R = \{z \in \mathbb{C},\, R^{-1} < |z| < R\}$$

which extend continuously to its boundary. Then there is a canonical isomorphism

$$X_*\mathcal{T}(\mathcal{O}(\overline{U}_R)) \overset{\simeq}{\longrightarrow} \mathbb{C} \bigoplus \mathbb{C}[1]$$

in the derived ind-category. For any pair of nice ind-Fréchet algebras $(\mathcal{A}, \mathcal{B})$ there are canonical and natural isomorphisms

$$HC_*^{loc}(\mathcal{O}(\overline{U}_R) \otimes_\pi \mathcal{A}, \mathcal{B}) \simeq HC_*^{loc}(\mathcal{A}, \mathcal{B}) \oplus HC_{*+1}^{loc}(\mathcal{A}, \mathcal{B})$$

and

$$HC_*^{loc}(\mathcal{A}, \mathcal{O}(\overline{U}_R) \otimes_\pi \mathcal{B}) \simeq HC_*^{loc}(\mathcal{A}, \mathcal{B}) \oplus HC_{*+1}^{loc}(\mathcal{A}, \mathcal{B})$$

of bivariant local cyclic cohomology groups.

PROOF: It is not easy to determine the precise structure of the complexes $X_*(\mathcal{O}(\overline{U}_R))$, $R > 1$. Using Cauchy's integral formula and the fact that the inclusion maps $\mathcal{O}(\overline{U}_{R'}) \to \mathcal{O}(\overline{U}_{R''})$ for $R' > R'' > 1$ are nuclear, one may conclude at least that the identity map on the space of algebraic differential forms over the ring of Laurent polynomials induces an isomorphism of ind-complexes

$$\text{“}\lim_{R'\to R}\text{”}\, X_*(\mathcal{O}(\overline{U}_{R'})) \overset{\simeq}{\longrightarrow} \text{“}\lim_{R'\to R}\text{”}\, \Omega^*_{dR}(\mathcal{O}(U_{R'}))$$

where $\Omega^*_{dR}(\mathcal{O}(U_{R'}))$ denotes the analytic de Rham complex on the open annulus $\mathcal{O}(U_{R'})$. It is obvious from de Rham theory that the latter is chain homotopy equivalent to $\mathbb{C} \bigoplus \mathbb{C}[1]$. So in the end one obtains a chain of isomorphisms

$$X_*\mathcal{T}(\mathcal{O}(\overline{U}_R)) \overset{\simeq}{\longleftarrow} \underset{R'\to R}{Lim}\, X_*\mathcal{T}(\mathcal{O}(\overline{U}_{R'})) \overset{\simeq}{\longrightarrow} \text{“}\lim_{R'\to R}\text{”}\, X_*(\mathcal{O}(\overline{U}_{R'}))$$

$$\text{“}\lim_{R'\to R}\text{”}\, X_*(\mathcal{O}(\overline{U}_{R'})) \overset{\simeq}{\longrightarrow} \text{“}\lim_{R'\to R}\text{”}\, \Omega^*_{dR}(\mathcal{O}(U_{R'})) \overset{\simeq}{\longrightarrow} \mathbb{C} \bigoplus \mathbb{C}[1]$$

in the derived ind-category. □

We note that neither the periodic, nor the analytic or asymptotic cyclic cohomology groups of the algebras $\mathcal{O}(\overline{U}_R)$ seem to be known. The analytic and asymptotic cyclic homology groups on the other hand coincide of course with the local ones computed here.

Thus in the example considered above the existence of a dense subalgebra of A of finite (Hochschild)-homological dimension $d = 1$ implies the contractability of the limit $\underset{n\to\infty}{Lim}\, Fil^{d+1}_{Hodge} CC^{an}_*(A_n)$ in an approximating sequence A_n of Banach subalgebras of A. It should be noted that this phenomenon is rather exceptional and in some sense peculiar to subalgebras of homological dimension at most one (quasifree algebras).

8.2 Commutative C^*-algebras

As another example we calculate the bivariant local cyclic cohomology of separable commutative C^*-algebras (see also [Pu], Chapter 11). It would be nice to apply directly the computational methods developed in this paper. Despite serious efforts I was not able to do this and therefore we have to refer in addition to the excision property [Pu2] of local cyclic cohomology. Using excision we obtain

Proposition 8.4. *Let M be a smooth compact manifold with (possibly empty) boundary. Then there is a natural chain homotopy equivalence*

$$CC^{an}_*(\mathcal{C}^\infty(M)) \xrightarrow{\sim} H^*(M, \mathbb{C})$$

from the analytic cyclic bicomplex of $\mathcal{C}^\infty(M)$ to the $\mathbb{Z}/2\mathbb{Z}$-graded sheaf cohomology groups of M, viewed as complex with vanishing differentials.

Proof: We proceed in several steps.

- Let $(M, \partial M)$ be a smooth compact Riemannian manifold with boundary and let $\mathcal{C}^\infty(M, \partial M)$ respectively $\mathcal{C}^\infty_0(M, \partial M)$ be the algebras of smooth functions on M vanishing along ∂M, respectively vanishing of infinite order along ∂M. We claim that the inclusion

$$\mathcal{C}^\infty_0(M, \partial M) \hookrightarrow \mathcal{C}^\infty(M, \partial M)$$

  is a diffeotopy equivalence and that the induced morphism

$$\Omega^*_{dR}(M, \partial M) \longrightarrow \Omega_{0,dR}(M, \partial M)$$

  of the associated de Rham complexes is a chain homotopy equivalence. In fact let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly monotone increasing smooth homeomorphism of the real halfline which is a diffeomorphism outside the origin, equals the identity outside $[0, 1]$, and has vanishing Taylor series at the origin. An open tubular neighborhood W of ∂M in M can be identified with $\partial M \times \mathbb{R}_+$. One can extend the smooth homeomorphism $\mathrm{Id} \times \varphi$ of $\partial M \times \mathbb{R}_+$ to a smooth homeomorphism ψ of M by putting it equal to the identity outside W. The algebra homomorphism $\psi^* : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ maps $\mathcal{C}^\infty(M, \partial M)$ to $\mathcal{C}_0^\infty(M, \partial M)$ and is obviously an inverse to the inclusion $\mathcal{C}_0^\infty(M, \partial M) \hookrightarrow \mathcal{C}^\infty(M, \partial M)$ up to diffeotopy. By applying the Cartan homotopy formula one obtains the corresponding statement for the de Rham complexes.

- Let $(M, \partial M)$ be a smooth compact n-dimensional manifold with (possibly empty) boundary ∂M. We assume without loss of generality that M is connected.

  Every appropriate Morse function on M provides a filtration

  $$D^n = M_0 \subset M_1 \subset \ldots \subset M_j = M$$

  such that for $i = 0, \ldots, j-1$

  - M_i is a codimension 0 submanifold with corners of M which does not intersect the boundary ∂M.
  - The extension of nice nuclear Fréchet algebras

    $$0 \to \mathcal{C}^\infty(M_{i+1}, M_i) \to \mathcal{C}^\infty(M_{i+1}) \to \mathcal{C}^\infty(M_i) \to 0$$

    possesses a bounded linear section.
  - $$\begin{aligned} \mathcal{C}_0^\infty(M_{i+1}, M_i) &\simeq \mathcal{C}_0^\infty(D^k \times D^{n-k}, \partial(D^k) \times D^{n-k}) \\ &\simeq \mathcal{C}_0^\infty(D^k, \partial(D^k)) \otimes_\pi \mathcal{C}^\infty(D^{n-k}) \end{aligned}$$

- Suppose that a filtration of $(M, \partial M)$ as constructed before is given. We show by induction over i that the canonical chain map

  $$CC_*^{an}(\mathcal{C}^\infty(M)) \longrightarrow \Omega_{dR}^*(M)$$

  obtained by antisymmetrization of differential forms [Co] is a chain homotopy equivalence. Consider the commutative diagram

  $$\begin{array}{ccccc} CC_*^{an}(\mathcal{C}^\infty(M_{i+1}, M_i)) & \to & CC_*^{an}(\mathcal{C}^\infty(M_{i+1})) & \to & CC_*^{an}(\mathcal{C}^\infty(M_i)) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{dR}^*(M_{i+1}, M_i) & \to & \Omega_{dR}^*(M_{i+1}) & \to & \Omega_{dR}^*(M_i) \end{array}$$

  of complexes. By the excision theorem in analytic cyclic (co)homology [Pu2] the upper line is a distinguished triangle. The lower line is an exact sequence of complexes with bounded linear section and is thus a distinguished triangle as well. By the properties of the filtration of M, the induction hypothesis, and the five lemma it suffices to verify that

$$CC_*^{an}(\mathcal{C}^\infty(D^n)) \longrightarrow \Omega_{dR}^*(D^n)$$

and

$$\begin{gathered} CC_*^{an}(\mathcal{C}_0^\infty(D^k \times D^{n-k}, \partial(D^k) \times D^{n-k})) \\ \downarrow \\ \Omega_{0,dR}^*(D^k \times D^{n-k}, \partial(D^k) \times D^{n-k}) \end{gathered}$$

are chain homotopy equivalences. By the Cartan homotopy formulas for the analytic cyclic bicomplex and the de Rham complex the last statement is equivalent to the assertion that

$$CC_*^{an}(\mathcal{C}_0^\infty(D^k, \partial(D^k))) \longrightarrow \Omega_{0,dR}^*(D^k, \partial(D^k))$$

is a chain homotopy equivalence. This follows however from a simple induction over k making use of the Cartan homotopy formulas, excision in analytic cyclic cohomology, and the arguments in the first part of this demonstration.

- The classical theorems of de Rham and Hodge imply that for a smooth compact manifold without boundary there is a chain homotopy equivalence $\Omega_{dR}^*(M) \xrightarrow{\sim} H^*(M, \mathbb{C})$ where $H^*(M, \mathbb{C})$ is viewed as complex with zero differentials. We present here the proof of A. Weil which neatly covers the case of manifolds with boundary. Choose a Riemannian metric on M and let $\mathcal{U} = (U_0, \ldots, U_k)$ be a finite open cover of M by geodesically convex balls (semiballs) such that no ball with center in the interior meets the boundary of M. Consider the bicomplex

$$\check{C}^{pq}(\mathcal{U}, \Omega^*) := \prod_{i_0 < \ldots < i_p} \Omega_{dR}^q(U_{i_0} \cap \ldots \cap U_{i_p})$$

with differentials given by the Čech-differential in the horizontal and the de Rham differential in the vertical direction. On the one hand there is a canonical embedding $\Omega_{dR}^*(M) \hookrightarrow \check{C}^*(\mathcal{U}, \Omega^*)$ into the first column of $\check{C}^*(\mathcal{U}, \Omega^*)$ given by restriction of differential forms. The fact that sheaves of differential forms are fine allows to deduce that $\Omega_{dR}^*(M)$ becomes a retract of $\check{C}^*(\mathcal{U}, \Omega^*)$. On the other hand there is a canonical embedding $\check{C}^*(\mathcal{U}, \mathbb{C}) \hookrightarrow \check{C}^*(\mathcal{U}, \Omega^*)$ of the Čech complex of $\mathcal{U}$ with coefficients in the constant sheaf $\mathbb{C}$ into the first line of $\check{C}^*(\mathcal{U}, \Omega^*)$. The fact that any intersection of the balls $U_i, 0 \leq i \leq k$, is geodesically convex and the Cartan homotopy formula show, that $\check{C}^*(\mathcal{U}, \mathbb{C})$ is a retract of $\check{C}^*(\mathcal{U}, \Omega^*)$ as well. Therefore the de Rham complex $\Omega_{dR}^*(M)$ is chain homotopy equivalent to the finite dimensional complex $\check{C}^*(\mathcal{U}, \mathbb{C})$ and in particular to the complex with vanishing differentials given by the cohomology of the latter one. As $\mathcal{U}$ is a Leray cover the cohomology of $\check{C}^*(\mathcal{U},\mathbb{C})$ coincides with $H^*(M,\mathbb{C})$. Altogether we have shown that the analytic cyclic bicomplex $CC_*^{an}(\mathcal{C}^\infty(M))$ of $\mathcal{C}^\infty(M)$ is chain homotopy equivalent to the complex with vanishing differentials $H^*(M,\mathbb{C})$ given by the $\mathbb{Z}/2\mathbb{Z}$-graded sheaf cohomology of M with coefficients in $\mathbb{C}$. The naturality of the chain map is clear.

□

PROPOSITION 8.5. *Let X be a compact metrizable space and let $C(X)$ be the C^*-algebra of continuous functions on X.*

a) *There exists a projective system $(M_n, \partial M_n), n \in \mathbb{N}$, of smooth manifolds (with boundary) and smooth maps, and a continuous map*

$$\varprojlim f_n : X \longrightarrow \text{“}\varprojlim\text{”}(M_n,\, \partial M_n)$$

such that the family $\{ f_n^{-1}(U_n),\, U_n \subset M_n \text{ open},\, n \in \mathbb{N} \}$ forms a basis of the topology of X and such that the induced morphism

$$\text{“}\lim_{n\to\infty}\text{”}\, \mathcal{C}^\infty(M_n) \longrightarrow C(X)$$

satisfies the assumptions of the limit theorem (6.16).

b) *There is an isomorphism*

$$\text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n,\, \mathbb{C}) \xrightarrow{\simeq} X_*\mathcal{T}(C(X))$$

in the derived ind-category. Here $H^*(M_n,\, \mathbb{C})$ denotes the sheaf cohomology of M_n, viewed as $\mathbb{Z}/2\mathbb{Z}$-graded complex with zero differentials.

c) *If $\mathcal{A}$ is a nice ind-Fréchet algebra, then there is a similar isomorphism*

$$\text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n,\, \mathbb{C}) \otimes X_*\mathcal{T}(\mathcal{A}) \xrightarrow{\simeq} X_*\mathcal{T}(C(X,\, \mathcal{A}))$$

in the derived ind-category, which is natural in $\mathcal{A}$.

PROOF: It is well known that the Gelfand transform, which assigns to a commutative C^*-algebra its spectrum, defines an antiequivalence between the category of commutative C^*-algebras and the category of locally compact Hausdorff spaces. Under the Gelfand transform separable algebras correspond to metrizable spaces. Let X be a compact metrizable space, let $A = C(X)$ be the separable C^*-algebra of continuous functions on X, and let $(a_n), n \in \mathbb{N}, a_0 = 1$, be a countable system of selfadjoint elements generating a dense involutive subalgebra of A. For each $n \in \mathbb{N}$ let $A_n \subset A$ be the C^*-subalgebra generated by $\{a_0, \ldots, a_n\}$. Then $A = \lim\limits_{n\to\infty} A_n$ as C^*-algebras. The map $i_n : Sp(A_n) \hookrightarrow \mathbb{R}^n$, which associates to a character $\chi \in Sp(A_n)$

the n-tuple $(\chi(a_1), \ldots \chi(a_n))$ defines a faithful embedding of $Sp(A_n)$ into euclidean n-space. Denote by X_n its image. Then $\pi_{n+1}(X_{n+1}) = X_n$ where $\pi_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates. Let finally M_n be a family of smooth manifolds with boundary satisfying the following conditions for all $n \in \mathbb{N}$:

- M_n is a smooth codimension zero submanifold with boundary of $\mathbb{R}^n$.
- $\overset{\circ}{M}_n$ is an open neighborhood of X_n
- M_n is contained in a $\frac{1}{n}$-neighborhood of X_n.
- $\pi_{n+1}(M_{n+1}) \subset M_n$.

It is then clear that the family (M_n) satisfies the second assertion of part a) of the proposition. Let $(\mathcal{U}_n), n \in \mathbb{N}$, be a countable family of finite open covers of X such that $\bigcup_n \{U, U \in \mathcal{U}_n\}$ forms a basis of the topology of X and choose for each n a partition of unity subordinate to $\mathcal{U}_n$. If one takes as a countable generating system for $C(X)$ the family of functions occurring in these partitions of unity, then a corresponding family of manifolds will also satisfy the first claim. Assertion b) is just a special case of c) which we show now. Let $\mathcal{A} =$ " $\underset{\overrightarrow{i \in I}}{\lim}$" A_i be a nice ind-Fréchet algebra. Then for fixed $i \in I$ the morphism

$$\lim_{n\to\infty} \mathcal{C}^\infty(M_n, A_i) \longrightarrow C(X, A_i)$$

satisfies the conditions of theorem (6.16). Consequently, the isomorphism criterion (2.9) applies to the morphism

$$\underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n, A_i)) \longrightarrow X_*\mathcal{T}(C(X, A_i))$$

We deduce therefore from proposition (2.10) that the canonical morphism of ind-complexes

$$\underset{n\to\infty}{Lim} X_*\mathcal{T}(\mathcal{C}^\infty(M_n, \mathcal{A})) \simeq \underset{\overrightarrow{i\in I}}{Lim} \left(\underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n, A_i)) \right) \longrightarrow$$

$$\longrightarrow \underset{\overrightarrow{i\in I}}{Lim}\, (X_*\mathcal{T}(C(X, A_i))) \simeq X_*\mathcal{T}(C(X, \mathcal{A}))$$

is an isomorphism in the derived ind-category. By the Eilenberg-Zilber theorem for cyclic complexes, [Pu3] and (5.16), there is a chain homotopy equivalence of ind-complexes

$$\underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n, \mathcal{A})) \simeq \underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n)) \otimes_\pi X_*\mathcal{T}(\mathcal{A})$$

The previous proposition and (2.10) show then that in the derived ind-category there are isomorphisms

$$\underset{n\to\infty}{Lim}\, X_*\mathcal{T}(\mathcal{C}^\infty(M_n)) \otimes_\pi X_*\mathcal{T}(\mathcal{A}) \overset{\simeq}{\longrightarrow}$$

$$\underset{n\to\infty}{Lim}\,(H^*(M_n,\mathbb{C})\otimes_\pi X_*\mathcal{T}(\mathcal{A}))\simeq\left(\text{“}\lim_{n\to\infty}\text{”}\,H^*(M_n,\mathbb{C})\right)\otimes X_*\mathcal{T}(\mathcal{A})$$

Altogether one obtains the desired isomorphism

$$\left(\text{“}\lim_{n\to\infty}\text{”}\,H^*(M_n,\mathbb{C})\right)\otimes X_*\mathcal{T}(\mathcal{A})\xrightarrow{\simeq}X_*\mathcal{T}(C(X,\mathcal{A}))$$

Its naturality is obvious. □

THEOREM 8.6. *Let X, Y be locally compact metrizable spaces and let $C_0(X), C_0(Y)$ be the corresponding C^*-algebras of continuous functions vanishing at infinity. For a locally compact space denote by $H_c^*(-,\mathcal{F})$ its sheaf cohomology with compact supports and coefficients in the sheaf $\mathcal{F}$.*

a) *The even(odd) local cyclic homology groups of $C_0(X)$ are naturally isomorphic to the direct sum of the even(odd) sheaf cohomology groups of X with compact supports and complex coefficients*

$$HC_*^{loc}(C_0(X))\quad\xrightarrow{\simeq}\quad\bigoplus_{n\in\mathbb{Z}}H_c^{*+2n}(X,\mathbb{C})$$

b) *The even(odd) local cyclic cohomology groups of $C_0(X)$ are naturally isomorphic to the direct product of the even(odd) Borel-Moore homology groups of X with compact supports and complex coefficients [BM]*

$$HC^*_{loc}(C_0(X))\quad\xrightarrow{\simeq}\quad\prod_{n\in\mathbb{Z}}H^c_{*+2n}(X,\mathbb{C})$$

c) *The even(odd) bivariant local cyclic cohomology groups of the pair $(C_0(X), C_0(Y))$ are naturally isomorphic to the space of even(odd) linear maps from the direct sum of the sheaf cohomology groups of X with compact supports and complex coefficients to corresponding direct sum of the sheaf cohomology groups of Y*

$$HC_*^{loc}(C_0(X),\,C_0(Y))\simeq Hom^*\left(\bigoplus_{n\in\mathbb{Z}}H_c^{*+2n}(X,\mathbb{C}),\bigoplus_{m\in\mathbb{Z}}H_c^{*+2m}(Y,\mathbb{C})\right)$$

d) *Let $\mathcal{A}$ be a nice ind-Fréchet algebra. Then there is a natural isomorphism*

$$HC_*^{loc}(C_0(X,\mathcal{A}))\quad\xrightarrow{\simeq}\quad\bigoplus_{n\in\mathbb{Z}}H_c^{*+2n}(X,HC_*^{loc}(\mathcal{A}))$$

which identifies the local cyclic homology groups of the ind-algebra of $\mathcal{A}$-valued continuous functions on X vanishing at infinity with the direct sum of the sheaf cohomology groups of X with compact supports and coefficients in the constant sheaf $HC_*^{loc}(\mathcal{A})$.

e) Let $\mathcal{B}$ be a further nice ind-Fréchet algebra. Then there is a natural isomorphism

$$HC_*^{loc}(C_0(X,\mathcal{A}),\,\mathcal{B}) \;\overset{\simeq}{\longrightarrow}\; Hom^*\left(\bigoplus_{n\in\mathbb{Z}} H_c^{*+2n}(X,\mathbb{C}), HC_*^{loc}(\mathcal{A},\mathcal{B})\right)$$

There is a certain asymmetry in the statements concerning homology and cohomology which is due to the fact, that maps from but not into a direct limit are characterized by a universal property.
For the proof we will need the

Lemma 8.7. *Let $\mathcal{C},\mathcal{C}'$ be $\mathbb{Z}/2\mathbb{Z}$-graded ind-complexes of complete, locally convex vector spaces. Suppose that $\mathcal{C}$ = "$\varinjlim_{i\in I}$"$C_*^{(i)}$ is a formal inductive limit of finite dimensional complexes $C_*^{(i)}$ with vanishing differentials and that $\mathcal{C}'$ is colocal (see (6.1)) with respect to the nullsystem of weakly contractible ind-complexes. Then the following holds*

a) The ind-complex $\mathcal{C}\otimes_\pi\mathcal{C}'$ is $\mathcal{N}$-colocal.

b) For any ind-complex $\mathcal{C}''$ there is a natural isomorphism

$$Mor^*_{\mathfrak{Ho}}(\mathcal{C}\otimes_\pi\mathcal{C}',\,\mathcal{C}'') \,\simeq\, Mor^*_{Vect}(\varinjlim_{i\in I} C_*^{(i)},\, Mor^*_{\mathfrak{Ho}}(\mathcal{C}',\,\mathcal{C}''))$$

where on the right hand side the morphisms are taken in the category $Vect$ of abstract $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces.

Proof: Let $\mathcal{C}'$ = "$\varinjlim_{j\in J}$"$\overline{C}_*^{(j)}$, $\mathcal{C}''$ = "$\varinjlim_{k\in K}$"$\overline{\overline{C}}_*^{(k)}$ be ind-complexes. Then

$$Mor^*_{\mathfrak{Ho}}(\mathcal{C}\otimes_\pi\mathcal{C}',\,\mathcal{C}'') \,=\, H^*(\varprojlim_{I\times J}\,\varinjlim_{K}\,Hom_{cont}(C^{(i)}\otimes\overline{C}^{(j)},\,\overline{\overline{C}}^{(k)}))$$

$$=\, H^*(\varprojlim_{I\times J}\,\varinjlim_{K}\,Hom_{Vect}(C^{(i)},\,Hom_{cont}(\overline{C}^{(j)},\,\overline{\overline{C}}^{(k)}))$$

$$=\, H^*(\,Hom_{Vect}(\varinjlim_{I} C^{(i)},\,\varprojlim_{J}\,\varinjlim_{K}\,Hom_{cont}(\overline{C}^{(j)},\,\overline{\overline{C}}^{(k)})\,)$$

because the complexes $C^{(i)}, i\in I$, are finite dimensional

$$=\, Hom^*_{Vect}(\varinjlim_{I} C^{(i)},\,H^*(\varprojlim_{J}\,\varinjlim_{K}\,Hom_{cont}(\overline{C}^{(j)},\,\overline{\overline{C}}^{(k)})\,))$$

because the differentials of the complexes $C^{(i)}, i\in I$, vanish

$$=\, Hom^*_{Vect}(\varinjlim_{I} C^{(i)},\,Mor^*_{\mathfrak{Ho}}(\mathcal{C}',\,\mathcal{C}''))$$

which proves the second assertion. If $\mathcal{C}'$ happens to be $\mathcal{N}$-colocal, then for weakly contractible ind-complexes $\mathcal{C}''$ one has $Mor^*_{\mathfrak{Ho}}(\mathcal{C}', \mathcal{C}'') = 0$ so that $Mor^*_{\mathfrak{Ho}}(\mathcal{C} \otimes_\pi \mathcal{C}', \mathcal{C}'') = 0$ by the previous calculation. This implies the first assertion. $\square$

Proof of theorem (8.6):

Let X and Y be compact metrizable spaces and let $\mathcal{A}, \mathcal{B}$ be nice ind-Fréchet algebras. We begin by calculating the local cyclic cohomology of the pair $(C(X, \mathcal{A}), \mathcal{B})$. Let $(M_n)_{n\in\mathbb{N}}$ be an approximating family of manifolds for X as constructed in (8.5). Then the projection maps $f_n : X \to M_n$ give rise to an isomorphism

$$\lim_{n\to\infty} H^*(M_n, \mathbb{C}) \simeq \lim_{n\to\infty} H^*(\check{C}(M_n, \mathbb{C})) \simeq H^*(\lim_{n\to\infty} \check{C}(M_n, \mathbb{C}))$$

$$\simeq H^*(\check{C}(X, \mathbb{C})) \simeq H^*(X, \mathbb{C})$$

by (8.5) a) where $\check{C}(-, \mathbb{C})$ denotes the Čech-complex calculating the cohomology of the constant sheaf $\mathbb{C}$. According to proposition (8.5) there is an isomorphism $X_*\mathcal{T}(C(X, \mathcal{A})) \simeq$ "$\lim\limits_{n\to\infty}$" $H^*(M_n, \mathbb{C}) \otimes_\pi X_*(\mathcal{T}\mathcal{A})$ in the derived ind-category. Let $\mathbf{P}(X_*(\mathcal{T}\mathcal{A}))$ be an $\mathcal{N}$-colocal model of $X_*(\mathcal{T}\mathcal{A})$. Then "$\lim\limits_{n\to\infty}$" $H^*(M_n, \mathbb{C}) \otimes_\pi \mathbf{P}(X_*(\mathcal{T}\mathcal{A}))$ is an $\mathcal{N}$-colocal model of "$\lim\limits_{n\to\infty}$" $H^*(M_n, \mathbb{C}) \otimes_\pi X_*(\mathcal{T}\mathcal{A})$ by (8.7), (6.5) and (2.10). With these remarks in mind one finds

$$HC^{loc}_*(C(X, \mathcal{A}), \mathcal{B}) = Mor_{\mathfrak{D}}(X_*(\mathcal{T}C(X, \mathcal{A})), X_*(\mathcal{T}\mathcal{B}))$$

$$= Mor_{\mathfrak{D}}(\text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n, \mathbb{C}) \otimes_\pi X_*(\mathcal{T}\mathcal{A}), X_*(\mathcal{T}\mathcal{B}))$$

$$= Mor_{\mathfrak{Ho}}(\text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n, \mathbb{C}) \otimes_\pi \mathbf{P}(X_*(\mathcal{T}\mathcal{A})), X_*(\mathcal{T}\mathcal{B}))$$

$$= Hom^*_{Vect}(\lim_{n\to\infty} H^*(M_n, \mathbb{C}), Mor_{\mathfrak{Ho}}(\mathbf{P}(X_*(\mathcal{T}\mathcal{A})), X_*(\mathcal{T}\mathcal{B})))$$

by lemma (8.7)

$$= Hom^*_{Vect}(\lim_{n\to\infty} H^*(M_n, \mathbb{C}), Mor_{\mathfrak{D}}(X_*(\mathcal{T}\mathcal{A}), X_*(\mathcal{T}\mathcal{B})))$$

$$= Hom^*_{Vect}(\lim_{n\to\infty} H^*(M_n, \mathbb{C}), HC^{loc}_*(\mathcal{A}, \mathcal{B}))$$

$$= Hom^*_{Vect}(H^*(X, \mathbb{C}), HC^{loc}_*(\mathcal{A}, \mathcal{B}))$$

where $H^*(X, \mathbb{C}) = \bigoplus_n H^{*+2n}(X, \mathbb{C})$ is the $\mathbb{Z}/2\mathbb{Z}$-graded sheaf cohomology of X with coefficients in $\mathbb{C}$. For d) one finds similarly

$$HC^{loc}_*(C(X, \mathcal{A})) = Mor_{\mathfrak{D}}(\mathbb{C}, X_*(\mathcal{T}C(X, \mathcal{A})))$$

$$\simeq Mor_{\mathfrak{D}}(\mathbb{C}, \text{“}\lim_{n\to\infty}\text{”}\, H^*(M_n, \mathbb{C}) \otimes_\pi X_*(\mathcal{T}\mathcal{A}))$$

$$\simeq \lim_{n\to\infty} H^*(M_n, \mathbb{C}) \otimes Mor_{\mathfrak{D}}(\mathbb{C}, X_*(\mathcal{T}\mathcal{A}))$$

$$\simeq \lim_{n\to\infty} H^*(M_n,\mathbb{C}) \otimes HC_*^{loc}(\mathcal{A}) \simeq H^*(X, HC_*^{loc}(\mathcal{A}))$$

Finally assertion c) follows from d) and e) by taking $\mathcal{A} = \mathbb{C}$, $\mathcal{B} = C(Y)$ while a) and b) are special cases of c). This establishes the theorem for compact spaces. The locally compact case follows easily by using the excision property of local cyclic cohomology.

□

8.3 Reduced C^*-algebras of free groups

We will calculate the local cyclic cohomology of the reduced group C^*-algebra of a finitely generated free group. (See also [Pu], Chapter 11.)

Let F_n be a free group on n generators. The action by left translation induces a unitary action on the Hilbert space $\mathcal{H} = \ell^2(F_n)$ of square integrable functions. Consider the corresponding representation of the group algebra $\mathbb{C}[F_n]$ on $\mathcal{H}$. The enveloping C^*-algebra of its image is the reduced group-C^*-algebra $C_r^*(F_n)$. It is well known that $C_r^*(F_n)$ possesses the Grothendieck approximation property [Ha1]. So we can make use of the approximation theorem (6.13). Moreover the dense subalgebra $\mathbb{C}[F_n]$ is quasifree [CQ]. The formal part of our calculation is the content of

Lemma 8.8. *Let R be a dense, finitely generated, unital, and quasifree subalgebra of the nice Fréchet algebra A with open unit ball U. Suppose that A possesses the Grothendieck approximation property. Let V be a finite dimensional subspace containing 1 and generating R as an algebra and denote by A_n the completion of R with respect to the largest submultiplicative seminorm satisfying $\| V^n \cap U \| \leq 1 + \frac{1}{n}$. Then the canonical morphisms of ind-complexes*

$$\text{“}\lim_{n\to\infty}\text{”} X_*(A_n) \overset{\simeq}{\longleftarrow} \underset{n\to\infty}{Lim}\, X_*(\mathcal{T}A_n) \overset{\simeq}{\longrightarrow} X_*(\mathcal{T}A)$$

are isomorphisms in the derived ind-category.

Proof: Because R is quasifree, there exists a connection ∇ in the sense of Cuntz-Quillen [CQ] on $\Omega^1 R$. It extends to a connection on $\Omega^+ R$ by the formula $\nabla(a^0 da^1 \dots da^n) := a^0 \nabla(da^1) da^2 \dots da^n$. A connection gives rise to a contracting chain homotopy of the subcomplex $Fil^2_{Hodge}\widehat{CC}_*(R)$ of the periodic cyclic bicomplex of R, which is given by the formula $h = \sum_{k=0}^{\infty} (-\nabla \circ B)^k \circ \nabla$, as well as to an explicit linear section s of the quotient map $p : \widehat{CC}_*(R) \to X_*(R)$ satisfying $s \circ p = Id - b \circ \nabla$ on forms of degree one. Under the assumptions of the lemma these linear maps extend for given n to bounded linear operators $h : Fil^2_{Hodge} CC_*^{an}(A_n) \longrightarrow Fil^2_{Hodge} CC_{*+1}^{an}(A_{n'})$ respectively $s : X_*(A_n) \longrightarrow CC_*^{an}(A_{n'})$, $n' >> n$. In order to show this one writes every element of R as a linear combination of products of a finite generating set $S \subset V \cap U$ and makes use of the formula

$$\begin{aligned} \nabla(d(s_1 \dots s_N)) &= \sum s_1 \dots s_{k-1} \nabla(ds_k) s_{k+1} \dots s_N \\ &+ \sum s_1 \dots s_{k-1} ds_k d(s_{k+1} \dots s_N) \end{aligned}$$

Note that because the elements $s_1, \ldots, s_N$ belong to a finite subset of R, the differential forms $\nabla(ds_k) \in \Omega^2 R$ are finite in number. Details of the straightforward calculation can be found in [Pu], (11.22), (11.23). It follows from this result that $\underset{n\to\infty}{Lim}\, CC^{an}_*(A_n) \sim \underset{n\to\infty}{Lim}\, X_*(\mathcal{T}A_n) \longrightarrow$ “$\lim\limits_{n\to\infty}$” $X_*(A_n)$ is a chain homotopy equivalence of ind-complexes. This establishes the first part of the assertion. The second part follows from the limit theorem (6.16). $\square$

Unfortunately it is often difficult to apply this result directly because one has no information about the auxiliary algebras A_n used in the lemma. In the case $A = C^*_r(F_n)$, $R = \mathbb{C}[F_n]$, I do not see how to calculate the homology of the complexes $X_*(A_n)$ directly. It seems therefore to be preferable to pass first of all to a sufficiently large but well understood Banach subalgebra of $C^*_r(F_n)$ containing $\mathbb{C}[F_n]$, and to apply the previous lemma to the latter subalgebra. Such a good Banach subalgebra has been constructed by Haagerup [Ha1].

Proposition 8.9. *(Haagerup) Let F_n be a free group on n generators $s_1, \ldots s_n$ and let $|-|_S$ be the corresponding word length function. Let $\mathcal{A}(F_n)$ be the completion of the group ring $\mathbb{C}[F_n]$ with respect to the seminorms*

$$\| \sum a_g u_g \|_k^2 = \sum |a_g|^2 \cdot (1 + |g|_S)^{2k}, \;\; k \in \mathbb{N},$$

Then $\mathcal{A}(F_n)$ is a nice Fréchet subalgebra of the reduced group C^*-algebra $C^*_r(F_n)$. Moreover it coincides with the domain of an unbounded derivation on $C^*_r(F_n)$.

Applying the smooth subalgebra theorem (3.8) and lemma (8.8), we deduce from Haagerup's result

Proposition 8.10. *Let F_n be a free group on n generators $s_1, \ldots, s_n$. Let $\mathcal{A}(F_n)$ be the associated Haagerup algebra and let V be the linear span of $s_1^{\pm 1}, \ldots, s_n^{\pm 1}$ in $\mathbb{C}[\Gamma] \subset \mathcal{A}(F_n)$. Let $\mathcal{A}_k(F_n)$ be the Banach subalgebras of $\mathcal{A}(F_n)$ introduced in lemma (8.8). Then there is an isomorphism*

$$\text{“}\lim_{k\to\infty}\text{”}\, X_*(\mathcal{A}_k(F_n)) \quad \simeq \quad X_*(\mathcal{T}C^*_r(F_n))$$

in the derived ind-category.

Lemma 8.11. *In the notations of 8.10 the continuous linear map*

$$\begin{array}{ccc} \mathcal{A}_k(F_n)^n & \longrightarrow & X_1(\mathcal{A}_k(F_n)) \\ (a_1, \ldots, a_n) & \longrightarrow & a_1 ds_1 + \ldots + a_n ds_n \end{array}$$

induces an isomorphism

$$\text{“}\lim_{k\to\infty}\text{”}\, \mathcal{A}_k(F_n)^n \overset{\simeq}{\longrightarrow} \text{“}\lim_{k\to\infty}\text{”}\, X_1(\mathcal{A}_k(F_n))$$

of ind-Fréchet spaces.

Proof: We use the notations of (8.8) and (8.10). Let ∇ be the unique connection on $\Omega^1(\mathbb{C}[F_n])$ satisfying $\nabla(ds_i) = 0$ for $i = 1, \dots, n$. The image of the associated linear embedding $s : X_1(\mathbb{C}[F_n]) \longrightarrow \Omega^1(\mathbb{C}[F_n])$ coincides then with the subspace $(Id - b \circ \nabla)\Omega^1(\mathbb{C}[F_n]) = \mathbb{C}[F_n]ds_1 + \ldots + \mathbb{C}[F_n]ds_n$ of $\Omega^1(\mathbb{C}[F_n])$. The lemma follows from the fact (8.8) that s extends to a bounded morphism "$\lim\limits_{k\to\infty}$" $X_1(\mathcal{A}_k(F_n)) \longrightarrow$ "$\lim\limits_{k\to\infty}$" $\Omega^1\mathcal{A}_k(F_n)$ of ind-Fréchet spaces (8.8). $\Box$

From now on this identification of "$\lim\limits_{k\to\infty}$" $X_1(\mathcal{A}_k(F_n))$ will be understood. We determine the homotopy type of the ind-complex "$\lim\limits_{k\to\infty}$" $X_*(\mathcal{A}_k(F_n))$ in two steps.

Lemma 8.12. *Let F_n be a free group on n generators $s_1, \dots, s_n$.*
Let $h' : X_*(\mathbb{C}[F_n]) \to X_{*+1}(\mathbb{C}[F_n])$ be the linear operator which vanishes on X_1 and maps the element $g \in F_n \subset \mathbb{C}[F_n] = X_0(\mathbb{C}[F_n])$ to $uvd(u^{-1}) \in X_1(\mathbb{C}[F_n])$ if $g = uvu^{-1}$ is the unique reduced presentation of g in terms of the generators $s_1^{\pm1}, \dots, s_n^{\pm1}$ such that the first letter of v is different from the inverse of its last letter.

a) The operator $\pi' := Id - (h' \circ \partial + \partial \circ h')$ defines a deformation retraction of $X_*(\mathbb{C}[F_n])$ onto the direct sum $X'_*(\mathbb{C}[F_n])_{hom} \bigoplus X'_*(\mathbb{C}[F_n])_{inhom}$ of the following subcomplexes. The finite dimensional subcomplex $X'_*(\mathbb{C}[F_n])_{hom}$ which is given by the linear span of $1 \in X_0(\mathbb{C}[\Gamma])$ and the finite set $\{s_i^{-1}ds_i,\ i = 1, \dots, n\} \subset X^1(\mathbb{C}[\Gamma])$. It has vanishing differential and is thus isomorphic to the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $H_*(F_n, \mathbb{C})$, viewed as trivial chain complex. The subcomplex $X'_*(\mathbb{C}[F_n])_{inhom}$ which is given by the linear span of the nontrivial elements $g \in F_n \subset X_0(\mathbb{C}[F_n])$, for which the first letter of the reduced word representing g is different from the inverse of its last letter, and of the elements of the form $g'ds_i$, $g''d(s_i^{-1}) \in X^1(\mathbb{C}[F_n])$, $i = 1, \dots, n$ such that the first and last letter of the reduced word representing g' (respectively g'') is different from s_i^{-1} (respectively s_i).

b) The operator π' is continuous in the sense that it gives rise to a deformation retraction of completed complexes

$$\pi' : \quad \text{“}\lim_{k\to\infty}\text{”} X_*(\mathcal{A}_k(F_n)) \quad \longrightarrow \quad \begin{array}{c} \text{“}\lim\limits_{k\to\infty}\text{”} X'_*(\mathcal{A}_k(F_n))_{hom} \\ \oplus \\ \text{“}\lim\limits_{k\to\infty}\text{”} X'_*(\mathcal{A}_k(F_n))_{inhom} \end{array}$$

This follows from a straightforward calculation.

Lemma 8.13. *There is an isomorphism of ind-complexes*

$$\text{“}\lim_{k\to\infty}\text{”} X'_*(\mathcal{A}_k(F_n))_{hom} \simeq H_*(F_n, \mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}^n[1]$$

whereas the ind-complex "$\lim\limits_{k\to\infty}$" $X'_*(\mathcal{A}_k(F_n))_{inhom}$ is contractible.

Proof: We show the second assertion, the first being obvious from the definitions made in (8.12). Let g, $g'ds_i$, $g''ds_i^{-1}$, $(g, g', g'' \in F_n)$ be generating elements of the complex $X'_*(\mathbb{C}[F_n])_{inhom}$ and suppose that they are represented by reduced words. Then there occur no cancellations under multiplication of $(g', s_i) \to g's_i$ and $(g'', s_i^{-1}) \to g''s_i^{-1}$ and under cyclic permutations of the letters of g, $g's_i$ and $g''s_i^{-1}$. Due to this the underlying spaces of the complex $X'_*(\mathbb{C}[F_n])_{inhom}$ can be interpreted as subspaces of the tensor algebra over the vector space with basis $s_1^{\pm 1}, \ldots, s_n^{\pm 1}$ and the differentials can be described in terms of the action of the appropriate cyclic group on the tensor powers of the basis elements. Thus one finds that the differential $\partial_0 : X'_0(\mathbb{C}[F_n])_{inhom} \to X'_1(\mathbb{C}[F_n])_{inhom}$ corresponds to the cyclic averaging operator N and that the differential $\partial_1 : X'_1(\mathbb{C}[F_n])_{inhom} \to X'_0(\mathbb{C}[F_n])_{inhom}$ corresponds to the operator $1-T$ where T generates the cyclic action. This shows that $X'_*(\mathbb{C}[F_n])_{inhom}$ is acyclic, i.e. has vanishing homology. A contracting homotopy operator can be given on generating elements of length $n+1$ by the formulas $h''_0 = \frac{1}{n+1}(T^{n-1}+2T^{n-2}+\ldots+(n-1)T+1)$, respectively $h''_1 = \frac{1}{n+1}\cdot 1$. A simple calculation shows that this contracting homotopy operator is continuous with respect to the topology of the ind-complex “$\lim\limits_{k\to\infty}$” $X'_*(\mathcal{A}_k(F_n))_{inhom}$, whence the result. □

We can summarize now what we have obtained in the following

Theorem 8.14. *a) Let F_n be a free group on n generators and let $C_r^*(F_n)$ be its reduced group C^*-algebra. Then there is a canonical isomorphism*

$$X_*(\mathcal{T}C_r^*(F_n)) \quad \overset{\simeq}{\longleftarrow} \quad H_*(F_n, \mathbb{C}) \quad \simeq \quad \mathbb{C} \bigoplus \mathbb{C}^n[1]$$

in the derived ind-category.

b) Let F', F'' be finitely generated free groups and let $\mathcal{A}$, $\mathcal{B}$ be nice ind-Fréchet algebras. Then there is a canonical isomorphism between

$$HC_*^{loc}(C_r^*(F') \otimes_\pi \mathcal{A},\, C_r^*(F'') \otimes_\pi \mathcal{B})$$

and

$$Hom^*(H_*(F', \mathbb{C}),\, H_*(F'', \mathbb{C})) \otimes HC_*^{loc}(\mathcal{A}, \mathcal{B})$$

which is natural in $\mathcal{A}$ and $\mathcal{B}$.

Proof: The first assertion follows from (8.10), (8.12) and (8.13). The Eilenberg-Zilber theorem for cyclic bicomplexes provides a chain homotopy equivalence (5.16)

$$X_*(\mathcal{T}(C_r^*(F) \otimes_\pi \mathcal{A})) \overset{\simeq}{\longrightarrow} X_*(\mathcal{T}(C_r^*(F)) \otimes_\pi X_*(\mathcal{T}\mathcal{A})$$

A careful look at the morphism $X_*(\mathcal{T}C_r^*(F_n)) \overset{\simeq}{\longleftarrow} H_*(F_n, \mathbb{C})$ of the first assertion shows that it is the composition of a morphism with weakly contractible mapping cone and a series of chain homotopy equivalences. Therefore its mapping cone is weakly contractible. Thus the isomorphism criterion (2.10) applies to the chain map $H_*(F_n, \mathbb{C}) \otimes X_*(\mathcal{T}\mathcal{A}) \longrightarrow X_*(\mathcal{T}(C_r^*(F)) \otimes_\pi X_*(\mathcal{T}\mathcal{A})$ showing that the latter is an isomorphism in the derived ind-category. □

8.4 n-traces and analytic traces on Banach algebras

In his monumental paper [Co3] Alain Connes introduced a special type of densely defined unbounded cyclic cocycles on Banach algebras, called n-traces. Every n-trace defines an additive functional on the K-theory of the underlying algebra, and represents thus a sensitive tool to detect nontrivial elements in K-groups. In [Co3] it is asked whether n-traces can be viewed as cocycles of a suitable cohomology theory and in particular how to define a cohomology relation between n-traces. We give here a partial answer for algebras with approximation property. This was inspired by a remark of Alain Connes.
First we recall the

Definition 8.15. Let A be a Banach algebra.

a) (A. Connes) [Co3]
An n-trace on A is a cyclic n-cocycle $\tau : \Omega^n\mathfrak{A} \to \mathbb{C}$ on a dense subalgebra $\mathfrak{A}$ of A such that for any $a^1, \ldots, a^n \in \mathfrak{A}$ there exists $C(a^1, \ldots, a^n) < \infty$ such that

$$|\tau((x^1da^1)\cdot(x^2da^2)\cdot\ldots\cdot(x^nda^n))| \leq C(a^1,\ldots,a^n)\cdot\|x^1\|_A\cdot\ldots\cdot\|x^n\|_A$$

for all $x_i \in \mathfrak{A}$.

b) An analytic trace on A is a cocycle τ' on the cyclic bicomplex $CC_*(\mathfrak{A})$ of a dense subalgebra $\mathfrak{A}$ of A, such that for every finite subset $S \subset \mathfrak{A}$ there exist constants $C_n(S), n \in \mathbb{N}$, satisfying

$$|\tau'((x^1da^1)\cdot(x^2da^2)\cdot\ldots\cdot(x^nda^n))| \leq C_n(S)\cdot(\frac{n}{2})!\cdot\|x^1\|_A\cdot\ldots\cdot\|x^n\|_A$$

for all $x^1, \ldots, x^n \in \mathfrak{A}$, $a^1, \ldots, a^n \in S$ and

$$\lim_{n\to\infty} C_n(S)^{\frac{1}{n}} = 0$$

In particular every n-trace is analytic.

Now our result is

Theorem 8.16. *Let A be a separable Banach algebra with approximation property. Then every analytic trace τ_n on A defines a unique local cyclic cohomology class*

$$[\tau_n] \in HC^n_{loc}(A)$$

The linear functional on $K_n(A)$ associated to τ_n by [Co3] coincides with the Chern character pairing (5.17) with the class $[\tau_n]$ in local cyclic cohomology.

Proof: Let τ be an analytic trace on A. Denote by $\mathfrak{A}$ its dense domain of definition and let $0 \subset V_1 \subset V_2 \subset \ldots \subset V_m \subset \ldots$ be a chain of finite dimensional subspaces of $\mathfrak{A}$ whose union is a dense subalgebra of A. Following

(6.13) choose strictly monotone decreasing sequences (λ_n), (r_n), $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} \lambda_n = 1$ and $\lim\limits_{n\to\infty} r_n = 0$, and denote by A_n the completion of the subalgebra $A[V_n]$ of $\mathfrak{A} \subset A$ generated by V_n with respect to the largest submultiplicative seminorm satisfying $\| V_n \cap \overline{U} \| \leq \lambda_n$ (U the open unit ball of A). Denote by $(TA_n)^{(r_n)}$ the completed tensor algebras introduced in (1.22). We claim that the analytic trace τ defines a cocycle on the ind-complex

$$\text{“}\lim_{n\to\infty}\text{”}\, X_*((TA_n)^{r_n})$$

The natural inclusions of algebras $TA[V_1] \subset \ldots \subset TA[V_n] \subset \ldots \subset T\mathfrak{A}$ induce chain maps $\lim\limits_{n\to\infty} X_*(TA[V_n]) \longrightarrow \lim\limits_{n\to\infty} CC_*(A[V_n]) \longrightarrow CC_*(\mathfrak{A})$ where the first chain map is the normalized Cuntz-Quillen projection. The analytic trace yields therefore a cocycle $\tau' \in \lim\limits_{\overleftarrow{n}} X^*(TA[V_n])$ and it remains to check that τ' is continuous, i.e. extends to a functional on the completion $X_*((TA_n)^{r_n})$ of $X_*(TA_n)$. As the Cuntz-Quillen projection is continuous ([Pu], 5.25) it suffices to prove the estimates

$$|\tau(a^0da^1\ldots da^k)| \leq C(m)\cdot(\frac{k}{2})!\cdot(r_n)^{\frac{k}{2}}$$

for some constant $C(m)$ and all $k \in \mathbb{N}$, $a^0,\ldots,a^k \in K_n^\infty$, the multiplicative closure of $K_n := V_n \cap \lambda_n^{-1}\overline{U} \subset \mathfrak{A}$. The set $K_n \subset V_n$ being bounded, there exist finitely many elements $c_1,\ldots,c_l \in V_n$ such that K_n is contained in the circled convex hull of $S := \{c_1,\ldots,c_l\}$. The estimate $|\phi_k((x^1da^1)\cdot\ldots\cdot(x^kda^k))| \leq C_k(S)\cdot(\frac{k}{2})!\cdot\|x^1\|\cdot\ldots\cdot\|x^k\|$ for all $x^1,\ldots,x^k \in \mathfrak{A}$ and $a^1,\ldots,a^k \in K_n$ follows. Let now $a^1,\ldots,a^k \in K_n^\infty$. This means that these elements can be written as products $a^j = b_j^1\ldots b_j^{l_j}$ with $b_j^i \in K_n$. In particular $\mathrm{d}a^j = \Sigma_{i=1}^{l_j} b_j^1\ldots b_j^{i-1}db_j^ib_j^{i+1}\ldots b_j^{l_j}$. As by construction $\|K_n\|_A \leq \lambda_n^{-1}$ the continuity property i) of the analytic trace implies

$$|\tau(a^0da^1\ldots da^k)| \leq (\Pi_{j=1}^k l_j)\cdot\lambda_n^{-(\Sigma_{j=1}^k l_j - k)}\cdot(\frac{k}{2})!\cdot C_k(S)$$

$$\leq (\frac{k}{2})!\cdot C_k(S)\cdot(\lambda_n)^k\cdot\Pi_{j=1}^k l_j(\lambda_n)^{-l_j} \;\leq\; (\frac{k}{2})!\cdot C_k(S)\cdot C(\lambda_n)^k$$

for a suitable constant $C(\lambda_n)$ and all k.
Because $(C_k(S)^{\frac{1}{k}}\cdot r_n^{-1}\cdot C(\lambda_n))^k \leq C'(n)$ by condition ii), one has

$$|\tau(a^0da^1\ldots da^k)| \leq C'(n)\cdot(\frac{k}{2})!\cdot r_n^k$$

for all k, and the claim follows. Thus τ defines an element of

$$Mor^*_{\mathfrak{Ho}}(\text{“}\lim_{n\to\infty}\text{”}\, X_*((TA_n)^{r_n}),\mathbb{C})$$

As the canonical morphism

$$\text{“}\lim_{n\to\infty}\text{”}\, X_*((TA_n)^{r_n}) \longrightarrow X_*(\mathcal{T}A)$$

is an isomorphism in the derived ind-category by the approximation theorem (6.13), the analytic trace τ defines a cohomology class

$$[\tau] \in Mor^*_{\mathfrak{D}}(\text{“}\lim_{n\to\infty}\text{”}\, X_*((TA_n)^{r_n}), \mathbb{C}) \simeq Mor^*_{\mathfrak{D}}(X_*(\mathcal{T}A), \mathbb{C}) = HC^*_{loc}(A)$$

This establishes the theorem.

□

References

[Ad] J.F. Adams, Stable homotopy and generalized homology, University of Chicago Lecture Notes (1974), 373pp.

[BC] B. Blackadar, J. Cuntz, Differential Banach algebra norms and smooth subalgebras of C^*-algebras, J. Operator Theory 26 (1991), 255-282

[BM] A. Borel, J. Moore, Homology theory for locally compact spaces, Michigan Math. J. 7 (1960), 137-159

[Bo] A. Bousfield, The localization of spaces with respect to homology, Topology 14 (1975), 133-150

[Co] A. Connes, Noncommutative geometry, Academic Press (1994), 661 pp.

[Co1] A. Connes, Noncommutative differential geometry, Publ. Math. IHES 62 (1985), 41-144

[Co2] A. Connes, Entire cyclic cohomology of Banach algebras and characters of Theta-summable Fredholm modules, K-Theory 1 (1988), 519-548

[Co3] A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation, in "Geometric methods in operator algebras", Pitman Research Notes 123 (1986), 52-144

[CH] A. Connes, N. Higson, Déformations, morphismes asymptotiques et K-théorie bivariante, CRAS 311 (1990), 101-106

[CM] A. Connes, H. Moscovici, Cyclic Cohomology, the Novikov Conjecture and hyperbolic groups, Topology 29 (1990), 345-388

[Cu] J. Cuntz, A new look at KK-theory, K-theory 1 (1987), 31-51

[Cu1] J. Cuntz, Bivariante K-Theorie für lokalkonvexe Algebren und der Chern-Connes Charakter, Docum. Math. J. DMV 2 (1997), 139-182

[Cu2] J. Cuntz, Excision in periodic cyclic theory for topological algebras, Fields Inst. Comm. 17 (1997), 43-53

[CQ] J. Cuntz, D. Quillen, Algebra extensions and nonsingularity, Journal of the AMS 8(2) (1995), 251-289

[CQ1] J. Cuntz, D. Quillen, Cyclic homology and nonsingularity, Journal of the AMS 8(2) (1995), 373-442

[CQ2] J. Cuntz, D. Quillen, Excision in bivariant periodic cyclic cohomology, Invent. Math. 127 (1997), 67-98

[FT] B. Feigin, B. Tsygan, Additive K-theory, in "K-theory, arithmetic and geometry", Springer Lecture Notes 1289 (1987), 67-209

[Go] T. Goodwillie, Cyclic homology, derivations, and the free loopspace, Topology 24 (1985), 187-215

[Ha] U. Haagerup, All nuclear C^*-algebras are amenable, Invent. Math. 74 (1983), 305-319

[Ha1] U. Haagerup, An example of a nonnuclear C^*-algebra which has the metric approximation property, Invent. Math. 50 (1979), 279-293

[Hi] N. Higson, A characterization of KK-theory, Pacific J. Math. 126 (1987), 253-276

[Ka] G. Kasparov, Operator K-functor and extensions of C^*-algebras, Izv.Akad.Nauk.CCCP Ser.Math. 44 (1980), 571-636

[Kh] M. Khalkhali, Algebraic connections, universal bimodules and entire cyclic cohomology, Comm.Math.Phys.161 (1994), 433-446

[KS] M. Kashiwara, P. Shapira, Sheaves on manifolds, Springer Grundlehren 292 (1990), 512pp.

[LT] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces I, Springer Ergebnisse d. Math. 92 (1977), 188pp.

[Ma] H. Margolis, Spectra and the Steenrod algebra, North Holland (1983), 489pp.

[Me] R. Meyer, Analytic cyclic cohomology, Preprintreihe SFB 478 Münster 61 (1999), 131pp.

[Me1] R. Meyer, Comparisons between periodic, analytic, and local cyclic cohomology, Preprintreihe SFB 478 Münster (2002), 11pp.

[Pu] M. Puschnigg, Asymptotic cyclic cohomology, Springer Lecture Notes 1642 (1996), 238 pp.

[Pu1] M. Puschnigg, Cyclic homology theories for topological algebras, K-theory preprint archives 292 (1998), 47 pp.

[Pu2] M. Puschnigg, Excision in cyclic homology theories, Invent. Math. 143 (2001), 249-323

[Pu3] M. Puschnigg, Explicit product structures in cyclic homology theories, K-Theory 15 (1998), 323-345

[Pu4] M. Puschnigg, Local cyclic cohomology and the bivariant Chern-Connes character of the gamma-element, K-theory preprint archive 356 (1999), 65pp.

[Pu5] M. Puschnigg, The Kadison-Kaplansky conjecture for word-hyperbolic groups, Invent. Math. 149 (2002), 153-194

[SGA] M. Artin, A. Grothendieck, J.L. Verdier, Théorie des topos et cohomologie étale des schémas, SGA 4(1), Springer Lecture Notes 269 (1972), 525 pp.

[Wo] M. Wodzicki, Vanishing of cyclic homology of stable C^*-algebras, CRAS 307 (1988), 329-334

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Un Isomorphisme Motivique entre Deux Variétés Homogènes Projectives sous l'Action d'un Groupe de Type G_2

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Abstract. Dans tout cet article, on désigne par k un corps de caractéristique différente de 2 et on appelle variété tout k-schéma séparé et de type fini.

L'objet du présent article est d'étudier $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$, les variétés homogènes projectives associées à chacune des deux racines d'un groupe de type G_2. La première d'entre elles, $\mathcal{X}(\alpha_1)$, est une quadrique projective de dimension 5 associée à une voisine de Pfister et l'autre, $\mathcal{X}(\alpha_2)$, est une variété de Fano (de genre 10). Ces deux variétés ne sont pas isomorphes, pourtant elles le deviennent en tant qu'objets d'une catégorie plus large, à savoir la catégorie des correspondances (et par conséquent également dans la catégorie des motifs de Chow). Nous établissons que ce résultat est vrai que les variétés soient déployées ou non. En première partie, nous rappelons quelques résultats classiques sur les algèbres d'octonions et construisons un modèle d'algèbre d'octonions déployée. En seconde partie, étape importante de notre travail, nous construisons une structure cellulaire de $\mathcal{X}(\alpha_2)$ lorsqu'elle est déployée. C'est également dans cette partie que nous déterminons la structure de l'anneau de Chow déployé de la variété $\mathcal{X}(\alpha_2)$. Enfin, en troisième partie, après avoir introduit nos notations et rappelé les résultats nécessaires sur la catégorie des correspondances, nous établissons l'isomorphisme motivique en toute généralité.

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Première partie
Rappels sur les algèbres d'octonions

1 Généralités

Pour commencer, nous rappelons quelques propriétés sur les algèbres à composition en général et les octonions en particulier. Pour un exposé plus complet, on pourra consulter [SV] dont la plupart des résultats suivants sont tirés.

Définition 1. *Soit C une algèbre unitaire d'unité 1, non nécessairement associative et telle qu'il existe, sur C, une forme quadratique q non dégénérée qui permette la composition, i.e. telle que*

$$\forall x, y \in C \qquad q(xy) = q(x)q(y).$$

Une telle algèbre C est appelée algèbre à composition.

Remarque 1. *On verra plus loin (cf. corollaire 1) que la forme quadratique q est unique.*

On désigne par $B_q(\cdot,\cdot)$ la forme bilinéaire symétrique associée à la forme quadratique q, i.e. définie par

$$\forall x, y \in C, \quad B_q(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y)).$$

On définit également la forme linéaire suivante, appelée trace par

$$\forall x \in C, \quad T(x) = 2B_q(x,1).$$

Dans la suite, nous allons identifier le corps de base k avec son image dans C (et plus tard dans O) ainsi q, B_q et T seront considérées comme étant à valeurs dans C. Cet abus nous permet d'alléger significativement nos notations.

La donnée de la forme quadratique q induit sur C l'existence d'une involution.

Proposition 1. *Soit C une algèbre à composition. L'application*

$$\begin{array}{cccc} \overline{\cdot}: & C & \longrightarrow & C \\ & x & \mapsto & \overline{x} = 2B_q(x,1) - x \end{array}$$

est un anti-automorphisme[1] *involutif de C.*

Démonstration. Voir par exemple [Gar99, Lem. 2.3.6 p. 14]. □

Cette involution joue un rôle important car elle permet de retrouver la forme quadratique q et par conséquent elle caractérise également C.

[1] i.e. un automorphisme de l'espace vectoriel C vérifiant $\forall x, y \in C \quad \overline{xy} = \overline{y}\,\overline{x}$.

Lemme 1. *Soit $x \in C$ on a*

$$q(x) = x\overline{x}$$

et

$$T(x) = x + \overline{x}.$$

Démonstration. Voir [Gar99, Lem. 2.3.6 p. 6] pour le premier point ; le second est trivial. ☐

Ainsi, il existe dans la littérature des présentations des algèbres à composition à partir de la donnée d'une involution $\overline{\cdot}$ sur C telle que $x\overline{x}$ soit une forme quadratique non-dégénérée et $x + \overline{x}$ une forme linéaire.

Ces deux constructions sont parfaitement équivalentes. En outre, cette involution permet parfois de simplifier certains calculs dans C tout comme le résultat suivant :

Proposition 2. *Pour tout x d'une algèbre à composition C, on a*

$$x^2 - T(x)x + q(x) = 0 \tag{1}$$

et pour tous $x,\ y \in C$, on a également

$$xy + yx - T(x)y - T(y)x + 2B_q(x,\ y) = 0. \tag{2}$$

Si le sous-espace $k1 \oplus kx$ est de dimension 2 et non dégénéré, c'est une algèbre à composition.

Démonstration. Voir [SV, Prop. 1.2.3 p. 6]. ☐

La formule (2) implique en particulier que $xy = -yx$ dès que $x, y \in \ker T$ et que x et y sont des vecteurs orthogonaux. Ce dernier point est primordial pour la suite. D'autre part, la proposition 2 a pour conséquence (voir encore une fois [SV]) les deux corollaires ci-dessous.

Corollaire 1. *La forme quadratique q d'une algèbre à composition C est déterminée de façon unique par l'algèbre C.*

Corollaire 2. *Les algèbres à composition C sont puissances-associatives, i.e. pour tout $x \in C$, la sous-algèbre $k[x]$ est associative.*

Le premier de ces corollaires implique donc que la définition de C posée au départ de ce texte est correcte. Le second est quant à lui un moyen très pratique de simplifier un grand nombre de calculs lorsque l'on travaille dans une telle algèbre.

Autre curiosité des algèbres à composition, ces dernières n'existent pas en toute dimension. De surcroît, plus la dimension est grande, plus on perd de bonnes propriétés de l'algèbre. Le résultat précis concernant ces derniers points est énoncé sous la forme du théorème suivant :

Théorème 1. *Les dimensions possibles pour une algèbre à composition sont* 1, 2, 4 *et* 8. *Les algèbres à composition de dimension* 1 *ou* 2 *sont commutatives et associatives, celles de dimension* 4 *sont associatives mais non commutatives, et quant à celles de dimension* 8 *elles ne sont ni l'un ni l'autre.*

Remarque 2. *Les algèbres à composition de dimension* 4 *sont plus connues sous le nom d'algèbres de* quaternions.

Nous introduisons maintenant les algèbres d'octonions :

Définition 2. *Toute algèbre de composition C de dimension* 8 *est appelée* algèbre d'octonions *ou encore* algèbre de Cayley.

À partir de maintenant et jusqu'à la fin de ce texte, nous désignons les algèbres d'octonions par la lettre O.

Définition 3. *Soit O une algèbre d'octonions. Si la forme quadratique q est isotrope, on parlera d'algèbre d'octonions* déployée.

Remarque 3. *Sur tout corps de base, il existe une unique (à isomorphisme près) algèbre d'octonions déployées. En revanche, sur un corps de base fixé, il peut exister de nombreuses algèbres d'octonions non-déployée. Toute algèbre non déployée se déploie sur une extension quadratique du corps de base. Pour plus de détails, voir par exemple [S, Chap. 3 §4] ou [SV, Théo. 1.8.1 p. 19].*

2 Un modèle d'algèbre d'octonions déployée

Nous allons maintenant construire O une algèbre d'octonions déployée. De ce que nous avons dit en remarque 3, de telles algèbres sont uniques à isomorphisme près. Ainsi, sauf mention explicite du contraire, lorsque par la suite nous parlerons d'algèbre d'octonions déployée, nous désignerons le modèle d'algèbre que nous allons construire maintenant.

On se donne le k-espace vectoriel

$$O = \mathfrak{M}_2(k) \times \mathfrak{M}_2(k)$$

où $\mathfrak{M}_2(k)$ désigne l'algèbre des matrices 2×2 à coefficients dans le corps k et on munit O du produit

$$\forall x, y \in O \qquad xy = (x_1,\ x_2)(y_1,\ y_2) = (x_1 y_1 + \tilde{y}_2 x_2,\ y_2 x_1 + x_2 \tilde{y}_1)$$

où

$$\tilde{y}_i = {}^t\operatorname{co}(y_i),$$

$\operatorname{co}(y_i)$ désignant la matrice constituée des cofacteurs de y_i.

De cette façon et comme signalé dans le corollaire 1, la forme quadratique q sur O est uniquement déterminée et s'exprime[2] sous la forme suivante :

$$\forall x = (x_1, x_2) \in O, \qquad q(x) = \det(x_1) - \det(x_2).$$

[2]Les relations et le produit dans O étant connus, avec un peu d'algèbre linéaire le résultat se déduit facilement.

Il est clair que O est une algèbre unitaire d'unité $1 = (Id,\ 0)$ et que la forme quadratique q est isotrope. Par conséquent, pour prouver qu'il s'agit d'une algèbre d'octonions (déployée), il suffit de montrer que la forme quadratique q permet la composition. Ce dernier point peut s'établir à la main moyennant quelques calculs fastidieux que nous ne reproduisons pas ici. Une autre façon de le voir serait de constater que l'algèbre ainsi construite est le résultat du procédé de duplication de Cayley-Dickson appliqué à l'algèbre $\mathfrak{M}_2(k)$. Des détails sur ces derniers points se trouvent, par exemple, dans [SV, §1.5 et §1.8] ou encore dans [Gar99, §2.3].

L'anti-automorphisme et la trace associés à q sont, quant à eux, donnés par le lemme suivant :

Lemme 2. *Pour tout $x \in C$,*

$$\overline{x} = (\tilde{x}_1,\ -x_2),$$
$$T(x) = (\mathrm{trace}(x_1),\ 0).$$

Démonstration. Là encore un calcul direct est possible, sinon le lecteur peut toujours se reporter à [SV, §1.8]. □

Nous considérons maintenant les vecteurs

$$e_0 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\quad [0]\right); \qquad f_0 = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\quad [0]\right) \tag{3}$$
$$e_1 = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad [0]\right); \qquad f_1 = \left(\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},\quad [0]\right) \tag{4}$$
$$e_2 = \left([0],\quad \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right); \qquad f_2 = \left([0],\quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{5}$$
$$e_3 = \left([0],\quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right); \qquad f_3 = \left([0],\quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \tag{6}$$

où $[0]$ désigne la matrice nulle ; ces vecteurs forment une base de O qui vérifie de nombreuses propriétés dont les plus importantes sont résumées dans le lemme 3 ci-dessous.

Lemme 3. *Tout d'abord on a*

$$\begin{array}{lll} e_0 + f_0 = 1, & \overline{e_0} = f_0, & 2B_q(e_0, f_0) = 1, \\ e_0^2 = e_0, & f_0^2 = f_0, & e_0 f_0 = f_0 e_0 = 0 \end{array}$$

et pour tout $i \in \{1,\ 2,\ 3\}$,

$$\begin{array}{lll} q(e_i) = q(f_i) = 0, & 2B_q(e_i, f_j) = \delta_{ij}, & e_i^2 = 0, \\ f_i^2 = 0, & \overline{e_i} = -e_i, & \overline{f_i} = -f_i, \\ f_0 e_i = e_i e_0 = 0, & e_0 f_i = f_i f_0 = 0, & e_0 e_i = e_i f_0 = e_i, \\ f_0 f_i = f_i e_0 = f_i, & e_i f_j = -\delta_{ij} e_0, & f_i e_j = -\delta_{ij} f_0, \end{array}$$

$$e_i e_{i+1} = -e_{i+1} e_i = -f_{i+2}, \qquad f_i f_{i+1} = -f_{i+1} f_i = -e_{i+2}$$

ces deux dernières égalités étant vraies avec les indices $i+1$ *et* $i+2$ *pris modulo* 3 *à valeur dans l'ensemble* $\{1, 2, 3\}$.

Démonstration. Voir [Sch62, §1 p. 202]. □

En fait, il existe de nombreuses bases de O vérifiant ces propriétés et c'est avec de telles bases que nous allons travailler. On énonce donc :

Définition 4. *Toute base de O vérifiant les propriétés du lemme 3 est appelée* base normale *(cf. [Sch62, §1 p. 202]).*

Remarque 4. *La base utilisée dans [SV] n'est pas une base normale.*

Dans la suite de ce texte, nous travaillerons avec une base normale et comme certains résultats du lemme 3 nous serons très utiles, nous les avons résumés dans la figure 1 ci-dessous.

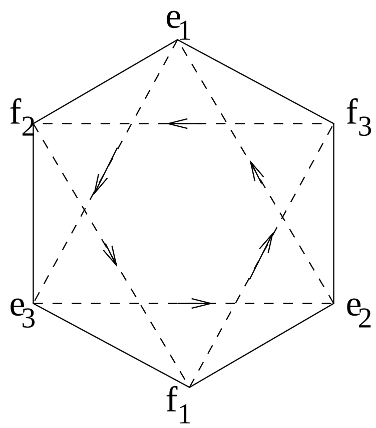

Fig. 1 – Diagramme illustrant la multiplication dans O.
Les traits pleins relient deux éléments dont le produit est nul. En pointillé le produit de ces deux éléments donne celui situé au-dessus du trait et la flèche indique la positivité. Par exemple $f_2 f_3 = -e_1$.

Deuxième partie
Décomposition cellulaire de $\mathcal{X}(\alpha_2)$

3 Définition des variétés

Rappelons, tout d'abord, quelques résultats classiques concernant les groupes algébriques et les variétés homogènes projectives sous jacentes. À l'aide de ces résultats, nous définissons les variétés $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ associées à un groupe de type G_2. Pour un exposé complet sur les groupes algébriques, nous vous conseillons la lecture de [B] et vous renvoyons à [MPW96] en ce qui concerne le choix de nos notations.

3.1 Variétés homogènes projectives

Soient G un groupe algébrique, T un tore maximal de G et B un sous-groupe de Borel le contenant. Le choix de ce groupe de Borel fixe un ensemble $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ de racines simples de G par rapport à T et pour tout α_i, il existe U_{α_i} un unique sous-groupe de racine de G. On associe également à toute racine α_i un autre sous-groupe de G, son sous-groupe de racine opposé, noté $U_{-\alpha_i}$, qui est en fait le sous-groupe de racine associé à la racine $-\alpha_i$. Dès lors, à tout sous-ensemble Θ de Δ, on associe P_Θ, un sous-groupe parabolique de G défini par

$$P_\Theta = \mathrm{gr}(T,\ \{U_\alpha |\ \alpha \in \Delta\},\ \{U_{-\alpha} |\ \alpha \notin \Theta\})$$

où $\mathrm{gr}(\mathcal{E})$ désigne le groupe engendré par l'ensemble $\mathcal{E}$. Par exemple, pour $\Theta = \Delta$, le sous-groupe parabolique associé est le sous-groupe de Borel B et pour $\Theta = \emptyset$, c'est G lui-même. On notera P_{α_i} le groupe $P_{\{\alpha_i\}}$ associé à une seule racine α_i. Enfin, on associe à Θ la variété projective G/P_Θ, homogène sous l'action de G, que nous notons[3] $\mathcal{X}(\Theta)$. Une telle variété $\mathcal{X}(\Theta)$ est évidemment lisse et est définie sur k, si et seulement si Θ est stable sous l'action du groupe de Galois absolu de k.

Parallèlement, on peut associer à un groupe algébrique un certain ensemble E, classiquement appelé immeuble sphérique. En détail, cet ensemble peut être un k-espace vectoriel, un k-espace vectoriel quadratique, hermitien ou encore une k-algèbre. Ainsi, lorsque les racines sont convenablement[4] numérotées, $\mathcal{X}(\alpha_i)$ est une variété constituée à partir de certains sous-espaces de E, de dimension i sur k. Ensuite, on définit une relation d'incidence sur ces i-espaces particuliers et on obtient les variétés associées aux autres sous-ensembles de Δ à partir des variétés $\mathcal{X}(\alpha_i)$. Concrètement, la variété $\mathcal{X}(\alpha_{i_1}, \ldots, \alpha_{i_l})$ associée au sous-ensemble $\{\alpha_{i_1}, \ldots, \alpha_{i_l}\}$ de Δ, a pour k-points l'ensemble :

$$\left\{ (V_1, \ldots, V_l) \in X(\alpha_{i_1})(k) \times \ldots \times X(\alpha_{i_l})(k) \mid \begin{array}{c} V_i \text{ est incident à } V_j \\ \text{pour tout } 1 \leqslant i, j \leqslant l \end{array} \right\}.$$

Dans le cadre de notre étude, G est un groupe de type G_2. Un tel groupe peut se définir à l'aide d'une algèbre d'octonions comme cela nous est appris par le résultat suivant :

Théorème 2. *Le groupe G des automorphismes d'une algèbre d'octonions O est un k-groupe algébrique simple adjoint de type G_2. Tout groupe adjoint de ce type est obtenu de cette façon.*

[3] Là encore, nous notons simplement $\mathcal{X}(\alpha_i)$ à la place de $\mathcal{X}(\{\alpha_i\})$.

[4] C'est-à-dire comme nous l'avons fait ici.

Démonstration. Voir [SV, Th. 2.3.5 p. 33] ou sans preuve [Car14, p. 298] et [Car52, p. 443]. □

Un groupe de type G_2 est un groupe de rang 2. Il possède par conséquent deux racines, notées α_1 et α_2 et son diagramme de Dynkin est donné par la figure 2 ci-dessous.

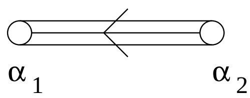

Fig. 2 – Diagramme de Dynkin de G.

En dehors du point $\mathcal{X}(\emptyset)$, nous avons donc trois variétés $\mathcal{X}(\alpha_1)$, $\mathcal{X}(\alpha_2)$ et $\mathcal{X}(\alpha_1, \alpha_2)$, cette dernière étant celle dont la dimension est la plus grande. L'ensemble que l'on associe à G pour décrire ses variétés homogènes projectives est une algèbre d'octonions O et la relation d'incidence est tout simplement l'inclusion. Les variétés $\mathcal{X}(\alpha_i)$ $(i \in \{1, 2\})$ sont constituées des sous-espaces V_i de dimension i de O dont les éléments sont de trace nulle et tels que si x et y sont deux éléments de V_i alors $xy = 0$.

Dans le cas présent, les variétés $\mathcal{X}(\alpha_i)$ $(i \in \{1, 2\})$ et $\mathcal{X}(\alpha_1, \alpha_2)$, sont définies sur k si et seulement si O est déployée. Par extension, nous les qualifions alors de variétés déployées.

Notre prochain objectif est d'exhiber la décomposition cellulaire de $\mathcal{X}(\alpha_2)$. Pour ce faire, le langage des foncteurs de points nous est apparu comme le plus indiqué. Toutefois, il nous semble, là encore, nécessaire de rappeler brièvement quelques notions essentielles avant de rentrer dans le vif du sujet.

3.2 Foncteurs de points

On désigne par foncteur de points, tout foncteur covariant de la catégorie $\mathfrak{Alg}_k$ des k-algèbres associatives, unitaires et commutatives, à valeurs dans la catégorie $\mathfrak{Ens}$ des ensembles.

Un tel foncteur $\mathcal{F}$ est dit représentable s'il existe une k-variété X telle que pour tout élément R de $\mathfrak{Alg}_k$ on ait :

$$\mathcal{F}(R) = \mathrm{Hom}_{k-\mathrm{Sch}}(\mathit{Spec}\ R, X)$$

On dit aussi que le foncteur $\mathcal{F}$ est représenté par X. On remarque au passage que l'application $X \mapsto \mathrm{Hom}_{k-\mathrm{Sch}}(-, X)$ définit une transformation naturelle allant de la catégorie des k-variétés dans celle des foncteurs de points représentables qui est une équivalence de catégorie (cf [DG, Th. de comparaison p. 18]).

Un exemple de foncteur de points est donné par le foncteur affine $\mathit{Spec}\ R$ associé à la k-algèbre R, défini pour tout élément S de $\mathfrak{Alg}_k$ par

$$Spec\ R(S) = \mathrm{Hom}_{\mathfrak{Alg}_k}(R, S).$$

Ce foncteur est clairement représenté par la k-variété $\mathrm{Spec}\, R$ et c'est pour cette raison qu'il est noté de la même manière.

On dit que $\mathcal{G}$ est un SOUS-FONCTEUR de $\mathcal{F}$, si pour toute k-algèbre R, $\mathcal{G}(R)$ est un sous-ensemble de $\mathcal{F}(R)$ et si pour tout homomorphisme $\varphi \in \mathrm{Hom}_{\mathfrak{Alg}_k}(R, S)$ (où $S \in \mathfrak{Alg}_k$), l'application $\mathcal{G}(\varphi)\colon \mathcal{G}(R) \to \mathcal{G}(S)$ est la restriction de $\mathcal{F}(\varphi)$.

On se donne maintenant un foncteur $\mathcal{F}$ représenté par une k-variété X. On dit alors qu'un sous-foncteur $\mathcal{G}$ de $\mathcal{F}$ est OUVERT (respectivement FERMÉ), s'il s'agit d'un sous-foncteur représentable de $\mathcal{F}$ qui est représenté par une sous-variété ouverte (respectivement fermée) de X.

À présent, nous allons définir les foncteurs de points GRASSMANNIENNES. Pour cela, on se donne un k-espace vectoriel V de dimension finie n et pour toute k-algèbre R, on pose $V_R = V \otimes_k R$.

DÉFINITION 5. *Soit $i \in \{1, \dots, n\}$, le foncteur grassmannienne $\Gamma_i(V)$ des sous-espaces de dimension i est défini par les données suivantes :*

- *pour toute k-algèbre R, l'ensemble $\Gamma_i(V)(R)$ est constitué des facteurs directs de rang i de V_R, en d'autres termes, il s'agit des sous-modules projectifs M de rang i de V_R tels que V_R/M est également projectif;*
- *pour tout homomorphisme $\varphi \in \mathrm{Hom}_{\mathfrak{Alg}_k}(R, S)$, l'application $\Gamma_i(V)(R) \to \Gamma_i(V)(S)$ est définie par la tensorisation par S sur R, i.e. par*

$$V_R \supset M \mapsto M \otimes_R S \subset V_S.$$

On pourra par exemple consulter [EH, Ex. VI-18 p. 261] pour vérifier qu'il s'agit bien du foncteur de points associé à la grassmannienne des i-sous-espaces de V.

Soit maintenant $f\colon V \times V \to W$ une application k-bilinéaire où W est aussi un k-espace vectoriel de dimension fini. On définit alors un sous-foncteur $\Gamma_i(V,\ f)$ de $\Gamma_i(V)$, i.e. celui des sous-espaces totalement isotropes de dimension i de V, en posant comme ensemble de ses R-points ($R \in \mathfrak{Alg}_k$) :

$$\Gamma_i(V, f)(R) = \{M \in \Gamma_i(V)(R) \mid f_R(M, M) = 0\}$$

où f_R est l'application R-bilinéaire induite par f. Ici nous utiliserons cette définition dans le cas où $V = W = O$ et où f est la multiplication de l'algèbre. Pour plus de détails concernant ce formalisme et ces définitions nous renvoyons le lecteur à [Kar01, §9 p. 23].

Nous sommes maintenant en mesure de donner les définitions explicites de $\mathcal{X}(\alpha_1)$, $\mathcal{X}(\alpha_2)$ et $\mathcal{X}(\alpha_1, \alpha_2)$ en terme de foncteurs de points. Ces définitions sont donc la version fonctorielle des définitions classiques que l'on peut par exemple consulter dans [Sch62, §6 p. 207].

3.3 Les foncteurs de points $\mathcal{X}(\alpha_1)$, $\mathcal{X}(\alpha_2)$ et $\mathcal{X}(\alpha_1, \alpha_2)$

Dans les définitions de $\mathcal{X}(\alpha_i)$ ($i \in \{1,2\}$) et $\mathcal{X}(\alpha_1, \alpha_2)$, les éléments mis en jeu sont tous de traces nulles, par conséquent nous pouvons nous restreindre à l'hyperplan $H = \ker T$ au lieu de travailler sur O tout entier et ce, même si lorsqu'il s'agit d'effectuer le produit de deux éléments, le résultat est toujours vu dans O. Nous obtenons ainsi pour tout élément R de $\mathfrak{Alg}_k$, les définitions allégées suivantes :

$$\mathcal{X}(\alpha_1)(R) = \{D \ \in \Gamma_1(H)(R) \ |\forall u, v \in D, \ uv = 0\},$$
$$\mathcal{X}(\alpha_2)(R) = \{P \in \Gamma_2(H)(R) \ |\forall u, v \in P, \ uv = 0\}$$

et

$$\mathcal{X}(\alpha_1, \alpha_2)(R) = \{(D, \ P) \in \mathcal{X}(\alpha_1)(R) \times \mathcal{X}(\alpha_2)(R) \mid D \subset P\}$$

où $\Gamma_1(H)$ et $\Gamma_2(H)$ désignent donc respectivement le foncteur grassmannienne des droites et des plans de H.

Regardons maintenant ces définitions d'un peu plus près. Nous avons déjà fait remarquer que tous les éléments x de O vérifient (1), i.e.

$$x^2 - T(x)x + q(x) = 0.$$

Par conséquent la définition de $\mathcal{X}(\alpha_1)$ devient

$$\mathcal{X}(\alpha_1)(R) = \{D \ \in \Gamma_1(H)(R) \ |\forall u \in D \ (q_{|_H})_R(u) = 0\}$$

où $(q_{|_H})_R$ désigne la forme quadratique q restreinte à H et étendue à R. Toujours pour alléger les notations, nous posons $q' = q_{|_H}$. En effet, un module projectif de rang 1 définit un faisceau localement libre de rang 1. Par conséquent, localement la condition $uv = 0$ est équivalente à $su^2 = 0$ (puisque le module est libre de rang 1, il existe un scalaire s tel que $v = su$) et donc équivalente à $q'(u) = 0$. Ce résultat étant vrai localement, il l'est également globalement. Ainsi, il devient clair que

$$\mathcal{X}(\alpha_1) = \Gamma_1(H, q')$$

i.e. que $\mathcal{X}(\alpha_1)$ est le foncteur des droites isotropes de H ou en d'autres termes, il s'agit d'une quadrique projective de dimension 5. Ce résultat a déjà été mis en évidence par M. Demazure (cf. [Dem77, §2 c)]), il s'agit d'un des quelques cas où des variétés homogènes projectives associées à deux groupes algébriques bien distincts sont isomorphes. En ce qui nous concerne, cet article présente aussi un autre intérêt ; si G désigne encore une fois un groupe adjoint de type G_2, l'article de M. Demazure nous apprend que $\mathrm{Aut}(\mathcal{X}(\alpha_1)) \simeq \mathrm{SO}(q')$ et que $\mathrm{Aut}(\mathcal{X}(\alpha_2)) \simeq G$. Dès lors, les variétés $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ ont des groupes d'automorphismes différents et ne sont donc pas isomorphes en tant que variétés projectives lisses.

Concernant maintenant $\mathcal{X}(\alpha_2)$, nous constatons là encore que tout ses points sont des plans totalement isotropes. En effet, ils sont de traces nulles et le produit de deux éléments quelconques est également nul. Ainsi, en vertu des équations (1) et (2) que nous rappelons :

$$x^2 - T(x)x + q(x) = 0$$

et

$$xy + yx - T(x)y - T(y)x + 2B_q(x,\ y) = 0,$$

nous constatons que $q(x) = B_q(x,\ y) = 0$, i.e. que tous les éléments sont isotropes et orthogonaux deux à deux. En revanche, un plan totalement isotrope constitué d'éléments de trace nulle n'appartient pas nécessairement à $\mathcal{X}(\alpha_2)$. En effet, dans ce cas là les formules (1) et (2) nous indiquent juste que $x^2 = 0$ et $xy + yx = 0$ ce qui n'implique pas que $xy = yx = 0$. Par exemple, dans la base normale (3), le module libre engendré par f_1 et f_2 est bien isotrope et vérifie les conditions requises car $f_1 f_2 = -f_2 f_1$ mais $f_1 f_2 = -e_3 \neq 0$. En conclusion, nous ne pouvons pas simplifier la définition de $\mathcal{X}(\alpha_2)$.

Nous allons maintenant donner la structure cellulaire de $\mathcal{X}(\alpha_2)$.

4 Structure cellulaire de $\mathcal{X}(\alpha_2)$

Dans toute cette sous-section, nous désignons par O une algèbre d'octonions déployée.

On rappelle que la structure cellulaire d'une variété alg"ébrique X (lisse et complète) est la donnée d'une filtration

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

constituée de sous-variétés fermées X_i telles que les différences $X_i \backslash X_{i-1}$, avec $i \in \{0, \ldots, n\}$, soient des espaces affines. Par la suite, on appelle ces différences des cellules.

D'après l'article de [Köc91], on sait que lorsque $\mathcal{X}(\alpha_2)$ est déployée, une telle filtration existe. Toutefois, nous voulons l'établir explicitement afin de prouver plus loin dans ce texte la rationnalité de certains cycles. Pour cela, nous allons partir d'une structure cellulaire de $\Gamma_2(H)$ et en prendre l'intersection avec $\mathcal{X}(\alpha_2)$ avant de raffiner la filtration en supprimant les termes redondants (i.e. ceux dont les cellules sont vides).

Remarque 5. *En général, même lorsqu'une variété X admet une structure cellulaire et que Y est une sous-variété fermée de X, l'intersection de la structure cellulaire de X avec Y ne donne pas nécessairement une structure cellulaire pour Y (considérez par exemple une quadrique projective anisotrope dans un espace projectif...). Le fait que cela fonctionne dans le cas présent est donc assez exceptionnel.*

La construction d'une telle filtration pour $\Gamma_2(H)$ se fait à partir de ses VARIÉTÉS DE SCHUBERT (cf. ci-dessous), lesquelles se définissent à partir d'un drapeau de H. Dans le cas d'une grassmannienne cette construction ne dépend pas du choix du drapeau de départ. En revanche, ce choix est primordial si l'on veut être capable d'écrire explicitement le résultat de l'intersection de la structure cellulaire de $\Gamma_2(H)$ avec $\mathcal{X}(\alpha_2)$. En l'occurence, pour construire ce drapeau nous partons d'une base normale (cf. (3)) de O, notre algèbre d'octonions déployée et nous munissons H d'une base qui s'en déduit. Concrètement nous avons

$$H = \mathrm{Vect}\{e_1,\ f_2,\ f_3,\ e = e_0 - f_0,\ e_3,\ e_2,\ f_1\}$$

et l'ordre dans lequel nous venons d'écrire les vecteurs qui engendrent H est important. En effet, pour définir notre drapeau, nous prenons comme premier espace vectoriel (le plus grand), $V_7 = H$. Nous déduisons ensuite V_i, nouvel espace vectoriel de la filtration comme étant égal à $\mathrm{Vect}\{v_{j_i}, \ldots, v_{j_1}\}$ où les v_{j_k} sont les i derniers vecteurs de l'ensemble $\{e_1,\ f_2,\ f_3,\ e = e_0 - f_0,\ e_3,\ e_2,\ f_1\}$ pris dans le même ordre. Par exemple, $V_6 = \mathrm{Vect}\{f_2,\ f_3,\ e,\ e_3,\ e_2,\ f_1\}$ et $V_3 = \mathrm{Vect}\{e_3,\ e_2,\ f_1\}$.

Un drapeau de H étant maintenant fixé, on peut définir de façon classique les VARIÉTÉS DE SCHUBERT (voir par exemple [M] ou [F]) de $\Gamma_2(H)$. Pour cela, à un couple $\lambda = (\lambda_1, \lambda_2)$ (avec $5 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant 0$) on associe la variété dont les R-points ($R \in \mathfrak{Alg}_k$) sont les suivants :

$$X_\lambda(R) = \{M \in \Gamma_2(V)(R) \mid \mathrm{rg}(M \cap (V_{5+i-\lambda_i})_R) \geqslant i,\ 1 \leqslant i \leqslant 2\}.$$

Ces variétés ne forment évidemment pas une filtration mais en prenant des réunions de variétés (et/ou de cellules) de Schubert, il est possible d'en fabriquer une. Nous n'en dirons pas plus ici sur les variétés de Schubert et nous ne reproduisons pas non plus la structure cellulaire de $\Gamma_2(H)$; nous nous contentons de donner la structure cellulaire qui en résulte pour $\mathcal{X}(\alpha_2)$ et dont les R-points ($R \in \mathfrak{Alg}_k$) sont les suivants :

$$\begin{aligned}
\mathcal{X}_5(R) &= \mathcal{X}(\alpha_2)(R)\\
\mathcal{X}_4(R) &= \{P \in \mathcal{X}(\alpha_2)(R) \mid \mathrm{rg}(P \cap (V_7)_R) \geqslant 2,\ \mathrm{rg}(P \cap (V_5)_R) \geqslant 1\}\\
\mathcal{X}_3(R) &= \{P \in \mathcal{X}(\alpha_2)(R) \mid \mathrm{rg}(P \cap (V_6)_R) \geqslant 2,\ \mathrm{rg}(P \cap (V_3)_R) \geqslant 1\}\\
\mathcal{X}_2(R) &= \{P \in \mathcal{X}(\alpha_2)(R) \mid \mathrm{rg}(P \cap (V_5)_R) \geqslant 2,\ \mathrm{rg}(P \cap (V_2)_R) \geqslant 1\}\\
\mathcal{X}_1(R) &= \{P \in \mathcal{X}(\alpha_2)(R) \mid \mathrm{rg}(P \cap (V_3)_R) \geqslant 2,\ \mathrm{rg}(P \cap (V_1)_R) \geqslant 1\}\\
\mathcal{X}_0(R) &= \{P \in \mathcal{X}(\alpha_2)(R) \mid \mathrm{rg}(P \cap (V_2)_R) \geqslant 2,\ \mathrm{rg}(P \cap (V_1)_R) \geqslant 1\}
\end{aligned}$$

Là encore, il n'est pas nécessaire de vérifier qu'il s'agit bien de sous-foncteurs fermés. Pour s'en convaincre on peut, par exemple, adapter les arguments donnés dans [Kar01, Lem. 9.7 p. 25].

Maintenant que nous avons une filtration, il nous faut vérifier que les cellules sont bien des espaces affines afin d'avoir effectivement une décomposition cellulaire. Tout d'abord, nous introduisons la notation suivante pour désigner les cellules :

$$\forall i \in \{0, \ldots, 5\} \qquad \mathcal{X}_{(i \backslash i-1)} = \mathcal{X}_i \backslash \mathcal{X}_{i-1}$$

avec la convention $\mathcal{X}_{-1} = \emptyset$. Il ne nous reste maintenant plus qu'à établir le résultat suivant :

Lemme 4. $\forall i \in \{0, \ldots, 5\}$,

$$\mathcal{X}_{(i \backslash i-1)} \simeq \mathbb{A}^i$$

où $\mathbb{A}^i$ désigne le foncteur espace affine de dimension i.

Démonstration. Dans sa démarche, la preuve est la même pour toutes les cellules. Nous allons par conséquent nous limiter à celle de $\mathcal{X}_{(4\backslash 3)} \simeq \mathbb{A}^4$.

On se donne donc R une k-algèbre et P un R-point de $\mathcal{X}_{(4\backslash 3)}(R)$. Nous allons établir qu'il existe une bijection entre $\mathcal{X}_{(4\backslash 3)}(R)$ et R^4.

Soit $j \colon V_7 \twoheadrightarrow V_7/V_6$ la projection canonique. Dire que $P \in \mathcal{X}_{(4\backslash 3)}(R)$ est équivalent à dire que d'une part, l'application j étendue à R et restreinte à P, $(j_R)_{|P}$, est surjective, d'autre part que $\ker(j_R)_{|P} \subset (V_5)_R$ et enfin que $(V_5)_R \to (V_5/V_4)_R$ restreinte à $\ker(j_R)_{|P}$ est un isomorphisme. En conséquence, nous posons $i \colon (V_5/V_4)_R \hookrightarrow P$ l'application induite par l'isomorphisme $(V_5/V_4)_R \xrightarrow{\sim} \ker(j_R)_{|P}$. La situation peut donc se résumer à la suite exacte courte suivante :

$$0 \to (V_5/V_4)_R \overset{i}{\hookrightarrow} P \overset{(j_R)_{|P}}{\twoheadrightarrow} (V_7/V_6)_R \to 0.$$

Il en découle que

$$P \simeq (V_5/V_4)_R \oplus (V_7/V_6)_R$$

où l'isomorphisme dépend de la donnée d'une section de $(j_R)_{|P}$. De cet isomorphisme nous déduisons que P est en fait un module libre et par conséquent nous allons pouvoir raisonner à partir des bases des V_i. Comme $(V_7/V_6)_R$ est engendré par $\overline{e_1}$, classe de e_1 modulo $(V_6)_R$ et $(V_5/V_4)_R$ par $\overline{f_3}$, classe de f_3 modulo $(V_4)_R$, une façon générique de remonter ces éléments est de prendre

$$m = m(a,\ b,\ c,\ d,\ g,\ h) = e_1 + a \cdot f_2 + b \cdot f_3 + c \cdot e + d \cdot e_3 + g \cdot e_2 + h \cdot f_1$$

et

$$n = n(w,\ x,\ y,\ z) = f_3 + w \cdot e + x \cdot e_3 + y \cdot e_2 + z \cdot f_1$$

où $a,\ b,\ c,\ d,\ g,\ h,\ w,\ x,\ y$ et z sont des scalaires. Ces deux éléments forment bien un module libre de rang 2. En fait, ce module est le même si nous remplaçons m par $m - b \cdot n$ et ainsi nous éliminons un scalaire et réduisons le problème. En conséquence et quitte à renommer les scalaires restants nous pouvons travailler avec

$$m = m(a,\ c,\ d,\ g,\ h) = e_1 + a \cdot f_2 + c \cdot e + d \cdot e_3 + g \cdot e_2 + h \cdot f_1$$

au lieu de la précédente définition de m. Dans l'ensemble de tous les modules qu'il est possible de générer à partir de m et n en faisant varier la valeur des scalaires, nous nous intéressons au sous-ensemble des modules qui appartiennent à $\mathcal{X}_{(4\backslash 3)}(R)$. Par définition même, m et n sont dans H, il ne nous reste donc plus qu'à vérifier que $m \cdot n = 0$ et $m^2 = n^2 = 0$.

En effectuant les calculs, nous obtenons

$$m \cdot n = 0 \iff \begin{cases} 0 = & hw - cz - xg + yd \\ 0 = & xc - dw + az \\ 0 = & yc + h - gw \\ 0 = & cw - z - d \\ 0 = & cw - ya \\ 0 = & x + aw \\ 0 = & -w - a \\ 0 = & -y - c \end{cases} \iff \begin{cases} h = & c^2 + gw \\ d = & cw - z \\ x = & w^2 \\ y = & -c \\ a = & -w \end{cases}$$

ce qui nous laisse déjà les variables c, g, w et z libres. Le calcul suivant nous donne

$$m^2 = 0 \iff 0 = c^2 - h - ag \iff h = c^2 - ag$$

qui est trivialement vérifié avec les précédentes relations. Enfin, le dernier calcul nous donne

$$n^2 = 0 \iff 0 = w^2 - x \iff x = w^2$$

qui est là encore une relation déjà présente dans le premier calcul.

Réciproquement, si nous considérons un module libre P engendré par $m = m(-w,\ c,\ cw - z,\ g,\ c^2 + gw)$ et $n = n(w,\ w^2,\ -c,\ z)$, les calculs précédents montrent directement que $m^2 = n^2 = m \cdot n = 0$ et par conséquent que $P \in \mathcal{X}_{(4\backslash 3)}(R)$. Nous avons donc mis en bijection $\mathcal{X}_{(4\backslash 3)}(R)$ et l'ensemble des modules engendrés par $m = m(-w,\ c,\ cw - z,\ g,\ c^2 + gw)$ et $n = n(w,\ w^2,\ -c,\ z)$ qui est en bijection avec R^4. Comme annoncé, nous avons ainsi établi que

$$\mathcal{X}_{(4\backslash 3)} \simeq \mathbb{A}^4.$$

Le calcul des autres cellules se fait en procédant de la même façon et finalement, pour tout $i \in \{0, \dots, 5\}$, nous trouvons que

$$\mathcal{X}_{(i\backslash i-1)} \simeq \mathbb{A}^i.$$

$\square$

5 Anneau de Chow de $\mathcal{X}(\alpha_2)$

On rappelle qu'à toute variété algébrique X qui est lisse et complète, on associe son anneau de Chow $\mathrm{CH}^*(X)$, engendré sur $\mathbb{Z}$ par les classes de cycles algébriques sur X modulo l'équivalence rationnelle et gradué par la codimension des classes de cycles. On peut également considérer $\mathrm{CH}_*(X)$, l'anneau gradué cette fois-ci par la dimension des classes de cycles et lorsque X est irréductible, on a $\mathrm{CH}^p(X) = \mathrm{CH}_{d-p}(X)$ où $d = \dim X$. Bien que cela soit un abus de langage, nous parlerons souvent de cycles plutôt que de classes de cycles.

Le but de cette section est de déterminer, dans le cas déployé, la structure générale de l'anneau de Chow de $\mathcal{X}(\alpha_2)$. Pour cela, nous allons exhiber les relations que vérifient les générateurs de $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$.

Lorsqu'une variété possède une structure cellulaire, nous savons (c.f. [F, Ex. 1.9.1 p. 23]) que son groupe de Chow est librement engendré par les classes pour l'équivalence rationnelle de l'adhérence des cellules. En conséquence, la construction de la structure cellulaire de $\mathcal{X}(\alpha_2)$ précédemment établie nous permet d'affirmer que $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$ possède un unique générateur libre par codimension. Ce qu'il nous faut maintenant comprendre, c'est comment ces générateurs se multiplient entre eux. Pour cela, nous allons utiliser le fait que $\mathcal{X}(\alpha_1, \alpha_2)$ peut être vue comme une fibration projective au-dessus de $\mathcal{X}(\alpha_2)$ au moyen de l'application suivante :

$$\begin{array}{rccc} pr\colon \mathcal{X}(\alpha_1, \alpha_2) & \longrightarrow & \mathcal{X}(\alpha_2) \\ (D, P) & \mapsto & P. \end{array}$$

Cette fibration induit alors par pull-back une application qui fait du groupe $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$ un $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$-module libre de rang 2 (c.f. [F, Ex. 8.3.4 p. 141 et Th. 3.3 b) p. 64]). Nous allons utiliser cette structure de $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$-module pour calculer de deux façons différentes des invariants de $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$. En comparant les résultats de ces deux calculs nous obtiendrons de façon presque complète la table de multiplication des générateurs de $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$. Nous verrons que ces relations sont suffisantes pour établir l'isomorphisme motivique. Pour commencer, il nous faut en apprendre d'avantage sur $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$.

5.1 Relations dans $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$

En premier lieu, il est connu (voir par exemple [Mar76, Lem. 5.1.1 p. 237]) que $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$ est un groupe libre engendré par un générateur libre en codimension 0 et 6 et deux générateurs libres dans les autres codimensions. Ainsi, nous désignons par g^i et h^i les générateurs de $\mathrm{CH}^i(\mathcal{X}(\alpha_1, \alpha_2))$ pour i dans $\{1, \ldots, 5\}$ et par g^0 et g^6 ceux de $\mathrm{CH}^0(\mathcal{X}(\alpha_1, \alpha_2))$ et $\mathrm{CH}^6(\mathcal{X}(\alpha_1, \alpha_2))$ respectivement. En second lieu, le calcul du produit de n'importe quel élément de $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$ avec un élément de codimension 1 peut être obtenu par une formule du type Pieri-Giambelli, la formule de Chevalley établie dans [Dem74, §4 Cor. 2 p. 78]. Grâce à cette formule, nous calculons la table de multiplication des générateurs de $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$.

$$
\begin{array}{llll}
(g^1)^2 = 3g^2 & (h^1)^2 = h^2 & h^1g^2 = g^3 + h^3 & g^1h^2 = g^3 + 3h^3 \\
(g^1)^3 = 6g^3 & (h^1)^3 = 2h^3 & h^1g^3 = 2g^4 + h^4 & g^1h^3 = g^4 + 2h^4 \\
(g^1)^4 = 18g^4 & (h^1)^4 = 2h^4 & h^1g^4 = g^5 + h^5 & g^1h^4 = g^5 + 3h^5 \\
(g^1)^5 = 18g^5 & (h^1)^5 = 2h^5 & h^1g^5 = g^6 & g^1h^5 = g^6 \\
(g^1)^6 = 0 & (h^1)^6 = 0 & h^1g^1 = h^2 + g^2 &
\end{array}
$$

Comme précédemment annoncé, nous allons maintenant calculer les invariants de $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$.

5.2 Calcul des invariants de $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$

Nous désignons par A le sous-anneau de $\mathrm{CH}^*(\mathcal{X}(\alpha_1, \alpha_2))$ engendré par les éléments du groupe $\mathrm{CH}^1(\mathcal{X}(\alpha_1, \alpha_2))$. Nous considèrons ensuite A^i le groupe abélien constitué des éléments de codimension i de A (i.e. engendré par les polynômes homogènes de degré i en g^1 et h^1). Par conséquent, après calculs nous obtenons que

$$
\begin{aligned}
A^1 &= \mathrm{gr}(g^1,\ h^1) \\
A^2 &= \mathrm{gr}((g^1)^2,\ g^1h^1,\ (h^1)^2) \\
&= \mathrm{gr}(3g^2,\ g^2 + h^2,\ h^2) \\
A^3 &= \mathrm{gr}((g^1)^3,\ (g^1)^2h^1,\ g^1(h^1)^2,\ (h^1)^3) \\
&= \mathrm{gr}(6g^3,\ 3(g^3 + h^3),\ g^3 + 3h^3,\ 2h^3) \\
A^4 &= \mathrm{gr}((g^1)^4,\ (g^1)^3h^1,\ (g^1)^2(h^1)^2,\ g^1(h^1)^3,\ (h^1)^4) \\
&= \mathrm{gr}(18g^4,\ 12g^4 + 6h^4,\ 6(g^4 + h^4),\ 2g^4 + 4h^4,\ 2h^4) \\
A^5 &= \mathrm{gr}((g^1)^5,\ (g^1)^4h^1,\ (g^1)^3(h^1)^2,\ (g^1)^2(h^1)^3,\ g^1(h^1)^4,\ (h^1)^5) \\
&= \mathrm{gr}(18g^5,\ 18(g^5 + h^5),\ 12g^5 + 18h^5,\ 6g^5 + 12h^5,\ 2g^5 + 6h^5,\ 2h^5) \\
A^6 &= \mathrm{gr}((g^1)^6,\ (g^1)^5h^1,\ (g^1)^4(h^1)^2,\ (g^1)^3(h^1)^3,\ (g^1)^2(h^1)^4,\ g^1(h^1)^5,\ (h^1)^6) \\
&= \mathrm{gr}(0, 2g^6,\ 6g^6,\ 12g^6,\ 18g^6,\ 18g^6, 0)
\end{aligned}
$$

où là encore, nous désignons par $\mathrm{gr}(\mathcal{E})$ le groupe abélien libre engendré par les éléments de $\mathcal{E}$ sur $\mathbb{Z}$.

Nous rappellons que pour tout i dans $\{1, \dots 5\}$, nous avons

$$\mathrm{CH}^i(\mathcal{X}(\alpha_1, \alpha_2)) = \mathrm{gr}(g^i, h^i)$$

et que

$$\mathrm{CH}^6(\mathcal{X}(\alpha_1, \alpha_2)) = \mathrm{gr}(g^6).$$

En calculant les différents quotients nous trouvons ainsi que :

$$\begin{aligned}
(\mathrm{CH}^1(\mathcal{X}(\alpha_1,\alpha_2)) : A^1) &= 1\\
(\mathrm{CH}^2(\mathcal{X}(\alpha_1,\alpha_2)) : A^2) &= 1\\
(\mathrm{CH}^3(\mathcal{X}(\alpha_1,\alpha_2)) : A^3) &= 2\\
(\mathrm{CH}^4(\mathcal{X}(\alpha_1,\alpha_2)) : A^4) &= 4\\
(\mathrm{CH}^5(\mathcal{X}(\alpha_1,\alpha_2)) : A^5) &= 4\\
(\mathrm{CH}^6(\mathcal{X}(\alpha_1,\alpha_2)) : A^6) &= 2
\end{aligned}$$

où nous désignons par $(\mathrm{CH}^i(\mathcal{X}(\alpha_1,\alpha_2)) : A^i)$, l'indice du sous-groupe A^i dans $\mathrm{CH}^i(\mathcal{X}(\alpha_1,\alpha_2))$.

Ces indices sont des invariants de l'anneau $\mathrm{CH}^*(\mathcal{X}(\alpha_1,\alpha_2))$ dont nous allons nous servir pour déterminer la structure d'anneau de $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$. Pour cela, nous allons calculer une nouvelle fois ces indices en utilisant la structure de $\mathrm{CH}^i(\mathcal{X}(\alpha_2))$-module de $\mathrm{CH}^i(\mathcal{X}(\alpha_1,\alpha_2))$.

5.3 Calcul de la structure de $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$

Pour commencer, nous rappelons que le groupe $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$ possède un seul générateur par codimension que nous notons h_2^i ($i \in \{0,\dots,5\}$). Ensuite, nous avons déjà signalé que $\mathrm{CH}^*(\mathcal{X}(\alpha_1,\alpha_2))$ est un $\mathrm{CH}(\mathcal{X}(\alpha_2))$-module libre de rang 2 . Plus précisement, il admet pour base l'ensemble $\{1,\ \zeta\}$ où 1 désigne l'élément neutre et ζ est un élément qui vérifie

$$\zeta^2 - c_1 h_2^1 \zeta + c_2 h_2^2 = 0$$

et où c_i désigne la $i^{ème}$ classe de Chern du fibré vectoriel associé au fibré projectif sur $\mathcal{X}(\alpha_2)$ (voir [H, App. A §3]). Ainsi, pour i à valeurs dans l'ensemble $\{1,\dots,5\}$, les groupes $\mathrm{CH}^i(\mathcal{X}(\alpha_1,\alpha_2))$ correspondent à $\mathrm{gr}(h_2^i,\ \zeta h_2^{i-1})$, et pour les indices 0 et 6, nous avons $\mathrm{CH}^0(\mathcal{X}(\alpha_1,\alpha_2)) \simeq \mathrm{CH}^0(\mathcal{X}(\alpha_2))$, et $\mathrm{CH}^6(\mathcal{X}(\alpha_1,\alpha_2)) \simeq \mathrm{CH}^5(\mathcal{X}(\alpha_2))\zeta$. Nous nous donnons ensuite quatre nombres entiers positifs l, m, n et p tels que

$$(h_2^1)^2 = l h_2^2, \qquad (h_2^1)^3 = lm h_2^3, \qquad (h_2^1)^4 = lmn h_2^4, \qquad (h_2^1)^5 = lmnp h_2^5.$$

Ceci étant posé, nous allons, là encore, considérer les sous-groupes A^i exprimés cette fois-ci en fonction de $(h_2^1)^i$ et $\zeta(h_2^1)^{i-1}$. Pour commencer,

$$A^1 = \mathrm{gr}(h_2^1,\ \zeta)$$

nous avons donc bien

$$(\mathrm{CH}^1(\mathcal{X}(\alpha_1,\alpha_2)) : A^1) = 1.$$

Ensuite,

$$\begin{aligned} A^2 &= \mathrm{gr}((h_2^1)^2,\ h_2^1\zeta,\ \zeta^2) \\ &= \mathrm{gr}(lh_2^2,\ h_2^1\zeta,\ c_1h_2^1\zeta - c_2h_2^2) \\ &= \mathrm{gr}(lh_2^2,\ c_2h_2^2,\ h_2^1\zeta) \end{aligned}$$

et comme

$$(\mathrm{CH}^2(\mathcal{X}(\alpha_1,\alpha_2)) : A^2) = 1$$

nous en déduisons que les nombres l et c_2 sont premiers entres eux. Ce résultat sera très largement exploité dans les calculs suivants. Concernant le groupe A^3, nous avons

$$\begin{aligned} A^3 &= \mathrm{gr}((h_2^1)^3,\ (h_2^1)^2\zeta,\ h_2^1\zeta^2,\ \zeta^3) \\ &= \mathrm{gr}((lmh_2^3,\ lh_2^2\zeta,\ lc_1h_2^2\zeta - mc_2h_2^3,\ (lc_1^2 - c_2)h_2^2\zeta - mc_1c_2h_2^3) \\ &= \mathrm{gr}(mh_2^3,\ h_2^2\zeta) \end{aligned}$$

d'où

$$(\mathrm{CH}^3(\mathcal{X}(\alpha_1,\alpha_2)) : A^3) = m = 2.$$

Nous calculons maintenant A^4,

$$\begin{aligned} A^4 =& \mathrm{gr}((h_2^1)^4,\ (h_2^1)^3\zeta,\ (h_2^1)^2\zeta^3,\ h_2^1\zeta^3,\ \zeta^4) \\ =& \mathrm{gr}(2lnh_2^4,\ 2lh_2^3\zeta,\ 2lc_1h_2^3\zeta - 2nc_2h_2^4,\ 2(lc_1^2 - c_2)h_2^3\zeta - 2nc_1c_2h_2^4, \\ & \qquad 2c_1(lc_1^2 - 2c_2)h_2^3\zeta - \frac{2n}{l}(lc_1^2 - c_2)c_2h_2^4) \\ =& \mathrm{gr}(2nh_2^4,\ 2h_2^3\zeta,\ \frac{2n}{l}c_2^2h_2^4) \end{aligned}$$

ainsi

$$(\mathrm{CH}^4(\mathcal{X}(\alpha_1,\alpha_2)) : A^4) = 2\,\mathrm{pgcd}(2n, \frac{2n}{l}c_2) = 4$$

où $\mathrm{pgcd}(2n, \frac{2n}{l}c_2)$ désigne le plus grand commun multiple de $\frac{2n}{l}c_2$ et $2n$. Par suite, comme $\mathrm{pgcd}(2n, \frac{2n}{l}c_2) = 2$, nécessairement $\mathrm{pgcd}(n, \frac{n}{l}c_2) = 1$ et ainsi $l = n$. Nous calculons maintenant

$$\begin{aligned} A^5 =& \mathrm{gr}((h_2^1)^5,\ (h_2^1)^4\zeta,\ (h_2^1)^3\zeta^2,\ (h_2^1)^2\zeta^3,\ h_2^1\zeta^4,\ \zeta^5) \\ =& \mathrm{gr}(2l^2ph_2^5,\ 2l^2h_2^4\zeta,\ 2l^2c_1h_2^4\zeta - 2lpc_2h_2^5, 2l(lc_1^2 - c_2)h_2^4\zeta - 2lpc_1c_2h_2^5, \\ & \quad 2lc_1(lc_1^2 - 2c_2)h_2^4\zeta - 2pc_2(lc_1^2 - c_2)h_2^5, \\ & \qquad (2l^2c_1^4 - 6lc_1^2c_2 + 2c_2^2)h_2^4\zeta - 2pc_1c_2(lc_1^2 - 2c_2)h_2^5) \\ =& \mathrm{gr}(2ph_2^5,\ 2h_2^4\zeta) \end{aligned}$$

ainsi

$$(\mathrm{CH}^5(\mathcal{X}(\alpha_1,\alpha_2)) : A^5) = 4p = 4$$

d'où $p = 1$. Enfin, nous calculons

$$\begin{aligned}
A^6 =\, & \mathrm{gr}((h_2^1)^6,\ (h_2^1)^5\zeta,\ (h_2^1)^4\zeta^2,\ (h_2^1)^3\zeta^3,\ (h_2^1)^2\zeta^4,\ h_2^1\zeta^5,\ \zeta^6) \\
=\, & \mathrm{gr}(0,\ 2l^2h_2^5\zeta,\ 2l^2c_1h_2^5\zeta,\ 2l(lc_1^2-c_2)h_2^5\zeta,\ 2lc_1(lc_1^2-2c_2)h_2^5\zeta, \\
& \quad (2l^2c_1^4-6lc_1^2c_2+2c_2^2)h_2^5\zeta,\ (2l^2c_1^5-8lc_1^3c_2+6c_1c_2^2)h_2^5\zeta) \\
=\, & \mathrm{gr}(2h_2^5\zeta)
\end{aligned}$$

et nous trouvons bien que

$$(\mathrm{CH}^6(\mathcal{X}(\alpha_1,\alpha_2)) : A^6) = 2.$$

Finalement, la table de multiplication partielle de $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$ est

$$\begin{aligned}
(h_2^1)^2 &= lh_2^2 \\
(h_2^1)^3 &= 2lh_2^3 \\
(h_2^1)^4 &= 2l^2h_2^4 \\
(h_2^1)^5 &= 2l^2h_2^5
\end{aligned}$$

Malgré tous nos efforts, l reste donc jusque là indéterminé. Toutefois, l'image de l'élément h_2^1 dans $\mathrm{CH}^*(\mathcal{X}(\alpha_1,\alpha_2)$ par l'application $pr^*\colon \mathrm{CH}^*(\mathcal{X}(\alpha_2)) \to \mathrm{CH}^*(\mathcal{X}(\alpha_1,\alpha_2))$ induite par la fibration projective, est une combinaison de g^1 et h^1, i.e. $pr^*(h_2^1) = ag^1 + bh^1$ avec a et b dans $\mathbb{Z}$, premiers entre eux (puisque $pr^*(h_2^1)$ peut être choisis comme un des générateurs de $\mathrm{CH}^1(\mathcal{X}(\alpha_1,\alpha_2))$). Or, nous avons

$$lpr^*(h_2^2) = (pr^*(h_2^1))^2 = (3a^2+2ab)g^1 + (b^2+2ab)h^2$$

ce qui impose donc à a et b d'être pairs si l l'est. Ceci est en contradiction avec le fait que a et b sont premiers entre eux. Par conséquent, l est nécessairement impair et ce fait est primordial pour établir l'isomorphisme motivique. En effet, bien que n'ayant pas totalement déterminé la structure de $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$, nous verrons que les informations dont nous disposons seront suffisantes pour établir l'isomorphisme motivique entre $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$.

Troisième partie
L'isomorphisme motivique

Nous allons maintenant commencer par introduire la catégorie des correspondances sur laquelle nous allons travailler ainsi que les résultats accompagnant cette théorie. Il s'agit de résultats classiques et la motivation principale n'est autre que l'introduction de nos notations.

6 Correspondances et motifs

Nous désignerons par $\mathfrak{Var}_k$ la catégorie des k-variétés lisses, complètes mais non nécessairement connexes (on inclut également $\emptyset$ dans $\mathfrak{Var}_k$). Nous avons déjà signalé que pour X dans $\mathfrak{Var}_k$, nous désignons par $\mathrm{CH}^*(X)$ l'anneau de Chow de X et que bien que cela soit un abus de langage, nous parlons de cycles plutôt que de classes de cycles.

Une CORRESPONDANCE de[5] X dans Y, où X, Y sont dans $\mathfrak{Var}_k$, est par définition un cycle dans $\mathrm{CH}^*(X \times Y)$. La composition des correspondances se fait de la façon classique (voir par exemple [F, Défi. 16.1.1 p. 305]) suivante :

$$\begin{array}{ccc} \mathrm{CH}^*(X \times Y) \times \mathrm{CH}^*(Y \times Z) & \longrightarrow & \mathrm{CH}^*(X \times Z) \\ (f, g) & \mapsto & g \circ f = (pr_{13})_*((f \times Z) \cdot (X \times g)) \end{array}$$

où $\cdot$ désigne la multiplication des cycles dans $\mathrm{CH}^*(X \times Y \times Z)$ et $(pr_{13})_*$ le push-forward par rapport à la projection

$$\begin{array}{cccc} pr_{13}\colon & X \times Y \times Z & \longrightarrow & X \times Z \\ & (x, y, z) & \mapsto & (x, z). \end{array}$$

Soient maintenant X et Y dans $\mathfrak{Var}_k$ avec Y supposée connexe (ou d'une façon plus générale équidimensionnelle). On définit dans un premier temps une CORRESPONDANCE DE X DANS Y DE DEGRÉ p comme étant un cycle homogène de $\mathrm{CH}^{\dim Y + p}(X \times Y)$. On étend ensuite cette définition au cas d'une variété Y de $\mathfrak{Var}_k$ quelconque en désignant par correspondance de degré p, les cycles homogènes de $\oplus_i \mathrm{CH}^{\dim Y_i + p}(X \times Y_i)$ où les Y_i désignent les composantes connexes de Y. Notez bien que l'on a

$$\oplus_i \mathrm{CH}^{\dim Y_i + p}(X \times Y_i) = \oplus_j \mathrm{CH}_{\dim X_j - p}(X_j \times Y)$$

où les X_j désignent les composantes connexes de X.

Lorsque l'on compose des correspondances, les degrés s'ajoutent ([F, Ex. 16.1.1 p. 308]), ainsi l'ensemble des correspondances de degré 0 est stable par composition et on définit par conséquent :

DÉFINITION 6. *Soit $\mathfrak{Corr}_k^0$ la catégorie additive dont les objets sont ceux de $\mathfrak{Var}_k$ et les groupes de morphismes sont les correspondances de degré* 0.

Une démonstration du fait que $\mathfrak{Corr}_k^0$ est bien une catégorie est consultable dans [F, §16.1 p. 305] . Nous signalons tout de même que l'application identité de X, id_X, est donnée par la classe de l'application diagonale sur $X \times X$. Par ailleurs, nous désignerons par $\mathrm{End}(X)$ le groupe $\mathrm{Hom}(X, X)$.

[5] Si $Y = X$ on parle de correspondance sur X tout court.

Dans ce texte, nous ne composons que des correspondances décomposées et homogènes. Dans ce cas de figure, la composition de deux correspondances se calcule explicitemment grâce au lemme suivant :

Lemme 5. *Soient* X, Y, $Z \in \mathfrak{Var}_k$, $f \in \mathrm{CH}^*(X)$,$g, g' \in \mathrm{CH}^*(Y)$ *et* $h \in \mathrm{CH}^*(Z)$. *On suppose que la variété* Y *est connexe (ou de façon plus générale équidimensionnelle) et que les cycles* g *et* g' *sont homogènes. Alors on a*

$$\begin{aligned}(g' \times h) \circ (f \times g) = &\ (pr_{13})_*(f \times (g \cdot g') \times h) \\ = &\begin{cases} \deg(g \cdot g')(f \times h) & \mathrm{si}\,\mathrm{codim}(g) + \mathrm{codim}(g') = \dim Y \\ 0 & \mathrm{sinon} \end{cases}\end{aligned}$$

où $\deg(-)$ *désigne le degré d'un* 0*-cycle (voir [F][Défi. 1.4 p. 13]).*

D'autre part, on définit également la transposé ${}^t f = \tau_*(f)$ dans $\mathrm{Hom}(Y, X)$ d'une correspondance f de $\mathrm{Hom}(X, Y)$ où $\tau\colon X \times Y \to Y \times X$ est la permutation des points, i.e. $\tau(x, y) = (y, x)$.

Enfin, si X est une variété définie sur le corps de base k et $\mathbb{L}/k$ est une extension de corps, on désigne par $X_{\mathbb{L}}$ la variété X définie sur $\mathbb{L}$. On dit qu'un cycle f de $\mathrm{CH}^*(X_{\mathbb{L}})$ est défini sur k, si f est dans l'image de l'homomorphisme de restriction $\mathrm{res}_{\mathbb{L}/k}\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(X_{\mathbb{L}})$. De même, on dit qu'une correspondance est définie sur k, si elle l'est en tant que cycle. De tels cycles ou correspondances sont qualifiés de rationnels.

Ces préliminaires terminés, nous allons maintenant prouver que les variétés $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ sont isomorphes dans la catégorie $\mathfrak{Corr}_k^0$ et qu'il s'agit par conséquent d'un isomorphisme motivique.

7 Isomorphisme

Nous allons maintenant nous employer à prouver l'isomorphisme. Pour cela nous allons procéder en trois étapes. Nous allons prouver qu'un tel isomorphisme existe lorsque les variétés $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ sont déployées. Ensuite, nous allons établir un théorème de nilpotence et enfin, grâce à ce théorème de nilpotence, nous prouverons l'isomorphisme en toute généralité.

7.1 Cas déployé

Nous nous plaçons dans le cas où les variétés $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ sont déployées. On rappelle que cela signifie que l'algèbre d'octonions sur laquelle le groupe de type G_2 agit l'est, ou encore de façon équivalente que la forme quadratique q définie sur l'algèbre des octonions est isotrope. Nous allons donc établir qu'il existe un cycle de degré 0 qui réalise un isomorphisme entre $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$. Pour prouver cet isomorphisme il nous faut trouver deux correspondances f et g, f dans $\mathrm{Hom}(\mathcal{X}(\alpha_1), \mathcal{X}(\alpha_2))$ et g dans $\mathrm{Hom}(\mathcal{X}(\alpha_2), \mathcal{X}(\alpha_1))$ telles que $g \circ f = \Delta_1$ et $f \circ g = \Delta_2$ où Δ_1 et Δ_2 désignent respectivement la classe de l'application diagonale de $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$. Si Δ_1 est déjà connu, il va nous falloir expliciter Δ_2.

Nous allons commencer par fixer quelques notations. Soit j un entier naturel de l'ensemble $\{0,\dots,5\}$, nous appelons h_1^j le générateur de $\mathrm{CH}^j(\mathcal{X}(\alpha_1))$ et h_2^j celui de $\mathrm{CH}^j(\mathcal{X}(\alpha_2))$. Les relations multiplicatives entre ces générateurs sont désormais (presque totalement) déterminées (cf. sous-section 5.3). En outre, comme $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ ont toutes les deux[6] une structure cellulaire , $\mathcal{X}(\alpha_1)\times\mathcal{X}(\alpha_1)$, $\mathcal{X}(\alpha_1)\times\mathcal{X}(\alpha_2)$, $\mathcal{X}(\alpha_2)\times\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)\times\mathcal{X}(\alpha_2)$ en ont également une que l'on déduit de celle de $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ (voir [Kar01, Défi. 7.2 p. 18]). Ceci nous permet d'établir le résultat suivant :

LEMME 6. *Les applications*

$$\begin{array}{ccc} \mathrm{CH}^*(\mathcal{X}(\alpha_i))\otimes\mathrm{CH}^*(\mathcal{X}(\alpha_j)) & \to & \mathrm{CH}^*(\mathcal{X}(\alpha_i)\times\mathcal{X}(\alpha_j)) \\ (f\otimes g) & \mapsto & f\times g \end{array}$$

où i et j parcourent $\{1,2\}$, sont des isomorphismes.

En fait, ce résultat est plus général ; il est vérifié dès que les deux variétés ont une structure cellulaire.

D'après le lemme 6, les générateurs de $\mathrm{CH}^*(\mathcal{X}(\alpha_i)\times\mathcal{X}(\alpha_j))$ s'expriment en fonction de ceux de $\mathrm{CH}^*(\mathcal{X}(\alpha_1))$ et $\mathrm{CH}^*(\mathcal{X}(\alpha_2))$. En particulier, en tant qu'anneaux, ils sont sans torsion.

REMARQUE 6. *On retrouve de cette façon un résultat déjà établi dans [Köc91, Cor. 1.5 p. 365].*

Nous sommes maintenant en mesure d'en déduire l'expression de l'application diagonale de $\mathcal{X}(\alpha_2)$:

LEMME 7.

$$\Delta_2=\sum_{i=0}^{5}h_2^i\times h_2^{5-i}$$

Démonstration. Tout ce qu'il y a faire ici, c'est de prouver que la correspondance $\sum_{i=0}^5 h_2^i\times h_2^{5-i}$ agit trivialement sur les générateurs de $\mathrm{End}(\mathcal{X}(\alpha_2)) = \mathrm{CH}^5(\mathcal{X}(\alpha_2)\times\mathcal{X}(\alpha_2))$. D'après le lemme 5 on a :

$$(h_2^i\times h_2^{5-i})\circ(h_2^j\times h_2^{5-j})=\begin{cases}\deg(h_2^{5-j}\cdot h_2^i)(h_2^j\times h_2^{5-i}) & \text{si } i=j\\ 0 & \text{sinon}\end{cases}$$

Si $i=j$, nous devons tout d'abord calculer $h_2^{5-i}\cdot h_2^i$ pour toutes les valeurs de j possibles, à savoir 0, 1 et 2. Il est tout d'abord clair en vertu de ce que nous avons calculé que

[6]La structure cellulaire de $\mathcal{X}(\alpha_1)$ bien que non décrite ici est en fait classique et bien connue.

$$h_2^0 \cdot h_2^5 = h_2^5 \cdot h_2^0 = h_2^5$$

et que

$$h_2^1 \cdot h_2^4 = h_2^4 \cdot h_2^1 = h_2^5.$$

Pour $j = 2$, nous calculons tout d'abord

$$\begin{aligned} 2h_2^2 \cdot h_2^3 &= (h_2^1)^2 \cdot h_2^3 \\ &= h_2^1 \cdot 2h_2^4 \\ &= 2h_2^5 \end{aligned}$$

et par conséquent nous avons

$$h_2^2 \cdot h_2^3 = h_2^3 \cdot h_2^2 = h_2^5,$$

en vertu de quoi nous concluons que

$$\deg(h_2^{5-j} \cdot h_2^j) = \deg(h_2^5) = 1$$

pour tout indice j de l'ensemble $\{0, \dots, 5\}$ et de là, nous en déduisons que pour tout indice j,

$$(\sum_{i=0}^{5} h_2^i \times h_2^{5-i}) \circ (h_2^j \times h_2^{5-j}) = h_2^j \times h_2^{5-j}.$$

De l'unicité de l'élément neutre, nous concluons que

$$\Delta_2 = \sum_{i=0}^{5} h_2^i \times h_2^{5-i}.$$

□

Nous allons maintenant exprimer la correspondance réalisant l'isomorphisme motivique dans le cas déployé.

Proposition 3. *Soit*

$$J = \sum_{i=0}^{5} h_1^i \times h_2^{5-i} \in \mathrm{CH}^5(\mathcal{X}(\alpha_1) \times \mathcal{X}(\alpha_2)),$$

J réalise un isomorphisme motivique entre $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ dont l'inverse est la correspondance transposée ${}^tJ$.

Démonstration. Nous pouvons nous contenter de prouver que $J \circ^t J = \Delta_2$. Pour cela nous utilisons une fois encore le lemme 5 pour calculer :

$$(h_1^i \times h_2^{5-i}) \circ (h_2^j \times h_1^{5-j}) = \begin{cases} \deg(h_1^{5-j} \cdot h_1^i)(h_2^j \times h_2^{5-i}) & \text{si } i = j \\ 0 & \text{sinon} \end{cases}$$

et là encore nous avons

$$\deg(h_1^{5-j} \cdot h_1^i) = \deg(h_1^5) = 1$$

puisque

$$\begin{aligned} (h_1^1)^2 &= h_1^2 \\ (h_1^1)^3 &= 2h_1^3 \\ (h_1^1)^4 &= 2h_1^4 \\ (h_1^1)^5 &= 2h_1^5 \end{aligned}$$

comme on peut s'en assurer en consultant [Kar90]. Par conséquent, il vient tout naturellement en développant que

$$J \circ^t J = \sum_{i=0}^{5} h_2^i \times h_2^{5-i} = \Delta_2.$$

Nous ne les reproduisons pas ici, mais le même type de calculs nous montre que ${}^tJ \circ J = \Delta_1$ et par conséquent, J est donc bien un isomorphisme motivique entre $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ comme annoncé. □

Nous prouvons maintenant un théorème de nilpotence pour $\mathcal{X}(\alpha_2)$ comme M. Rost l'a fait dans le cas des quadriques (voir [Bro03]).

7.2 Théorème de nilpotence et conséquence

En premier lieu, nous reproduisons ici quelques résultats établis par M. Rost. On désigne par X et B deux variétés algébriques lisses sur le corps de base k et $\pi \colon B \times X \to B$ la première projection. Pour tout élément b de B, $X_b = \mathit{Spec}\, k(b) \times_k X$ désigne la fibre au-dessus de b et $k(b)$ le corps résiduel en b. Pour toute correspondance f de $\mathrm{End}(X)$, on note $f_b \in \mathrm{End}_{\mathfrak{Corr}^0_{(k(b))}}(X_b)$ l'élément obtenu par changement de base.

Le résultat suivant est démontré dans [Bro03, Th. 3.1 p. 74].

Proposition 4. *Soit $f \in \mathrm{End}(X)$ et supposons que*

$$(f_b)_* \left(\mathrm{CH}_i(X_b)\right) = 0$$

pour tout $b \in B$ et tout $i \in \{0, \ldots, \dim B\}$. Alors

$$f^{(1+\dim B)} \circ \mathrm{Hom}(B, X) = 0.$$

Remarque 7. *La puissance de f est prise dans l'anneau* $\mathrm{End}(X)$.

Nous établissons maintenant le résultat suivant :

Lemme 8. *Si $\mathcal{X}(\alpha_2)$ est déployée, alors*

$$\mathrm{End}(\mathcal{X}(\alpha_2)) = \oplus^i \mathrm{End}_{\mathbb{Z}}\left(\mathrm{CH}^i(\mathcal{X}(\alpha_2))\right).$$

Démonstration. D'aprés [Ros90, Lem. 6 p. 7] ce résultat est vrai pour les quadriques déployées[7]. Il est donc vrai pour $\mathcal{X}(\alpha_1)$ lorsqu'elle l'est. Nous savons maintenant que dans ce cas $\mathcal{X}(\alpha_2)$ lui est isomorphe. Ce résultat est donc également vrai pour $\mathcal{X}(\alpha_2)$. □

Nous énonçons maintenant notre théorème de nilpotence.

Proposition 5 (Théorème de nilpotence). *Soit $f \in \mathrm{End}(\mathcal{X}(\alpha_2))$ et $\mathbb{L}/k$ une extension quelconque du corps de base k. Si $f_{\mathbb{L}} = 0 \in \mathrm{End}_{\mathfrak{Corr}^0_{\mathbb{L}}}(\mathcal{X}(\alpha_2)_{\mathbb{L}})$, alors il existe un entier n tel que $f^n = 0$.*

Démonstration. Dans le cas où $\mathcal{X}(\alpha_2)$ est déployée, comme

$$\mathrm{End}(\mathcal{X}(\alpha_2)) = \oplus_i \mathrm{End}_{\mathbb{Z}}\left(\mathrm{CH}^i(\mathcal{X}(\alpha_2))\right),$$

le fait que $f_{\mathbb{L}} = 0$ implique nécessairement que $f = 0$ car $\mathrm{End}(\mathcal{X}(\alpha_2))$ est invariant par extension.

Si maintenant nous sommes dans le cas où $\mathcal{X}(\alpha_2)$ n'est pas déployée, on applique alors la propositon prcédente avec $B = \mathcal{X}(\alpha_2)$. Nous allons également devoir utiliser, et donc prouver, le résultat suivant :

Lemme 9. *Pour tout $x \in \mathcal{X}(\alpha_2)$, la fibre $\mathcal{X}(\alpha_2)_x$ est déployée.*

Démonstration. Il est clair que $\mathcal{X}(\alpha_2)_x$ a un point rationnel et en conséquence, il existe un plan P de l'algèbre d'octonions O tel que la trace restreinte à P est identiquement nulle et que le produit de deux éléments quelconques de P est lui aussi nul. Nous avons déja fait remarquer (cf. sous-section 3.3) que dans ce cas là, P est totalement isotrope. Ainsi, la forme quadratique q sur O est également isotrope, d'où O est déployée d'après le théorème 1.8.1 de [SV]. Par conséquent, $\mathcal{X}(\alpha_2)_x$ est déployée. □

La variété $\mathcal{X}(\alpha_2)_x$ étant totalement déployée, en utilisant ce que l'on à déjà dit en début de preuve, $(f_x)_{\mathbb{L}} = 0$ implique que $f_x = 0$. Dès lors, on peut appliquer la proposition précédente (la condition $(f_x)_* \left(\mathrm{CH}_i(\mathcal{X}(\alpha_2)_x)\right) = 0$ est trivialement vérifiée) et en déduire que

[7] Dans l'article de M. Rost, le résultat est établi sur les dimensions mais comme nous travaillons avec des variétés irréductibles nous pouvons passer à la codimension sans problème.

$$f^6 = 0.$$

Le résultat est donc établi. □

De ce résultat nous déduisons un théorème d'isomorphisme :

Corollaire 3 (Théorème d'isomorphisme). *Sous les hypothèses de la proposition 5, si $f_{\mathbb{L}}$ est un isomorphisme alors f en est un.*

Démonstration. Si $\mathcal{X}(\alpha_2)_{\mathbb{L}}$ n'est pas déployée, elle l'est sur une extension plus grande. En conséquence, quitte à passer sur une autre extension, nous pouvons supposer que $\mathcal{X}(\alpha_2)_{\mathbb{L}}$ est déployée. En utilisant le lemme 8, nous constatons que

$$\mathrm{End}(\mathcal{X}(\alpha_2)) = \oplus_{i=0}^{5} \mathrm{End}_{\mathbb{Z}}(\mathrm{CH}^i(\mathcal{X}(\alpha_2)))$$

et ainsi la correspondance $f_{\mathbb{L}}$ est totalement déterminée par son action sur les groupes $\mathrm{CH}^i(\mathcal{X}(\alpha_2)) \simeq \mathbb{Z}$. D'où $f_{\mathbb{L}}$ satisfait l'équation $t^2 - 1 = 0$. Ainsi, $f_{\mathbb{L}}^2 = (\Delta_2)_{\mathbb{L}}$ et d'après la proposition 5, nous en déduisons que $f^2 = \Delta_2 + g$, où g est un élément nilpotent. En conclusion, f est un automorphisme. □

Remarque 8. *V. Chernousov, S. Gille et A. Merkurjev ont depuis généralisé ce résultat (et le théorème de nilpotence 5) dans leur preprint [CGM03, Th. 7.4 p. 13].*

Nous allons maintenant établir que le cycle J est rationnel et ainsi être en mesure de conclure que l'isomorphisme motivique est aussi réalisé dans le cas anisotrope.

7.3 Cas anisotrope

Nous supposons à présent que les variétés $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ sont anisotropes sur le corps de base k. Nous considérons alors $\mathbb{K}$ une clôture algébrique de k et les variétés $\mathcal{X}(\alpha_1)_{\mathbb{K}}$ et $\mathcal{X}(\alpha_2)_{\mathbb{K}}$ sont, elles, clairement déployées. Comme précédemment, nous désignons alors par, h_1^i et h_2^i les générateurs respectifs de $\mathrm{CH}^i(\mathcal{X}(\alpha_1)_{\mathbb{K}})$ et $\mathrm{CH}^i(\mathcal{X}(\alpha_2)_{\mathbb{K}})$. Nous annonçons maintenant :

Proposition 6. *La correspondance*

$$J = \sum_{i=0}^{5} h_1^i \times h_2^{5-i} \in \mathrm{CH}^5(\mathcal{X}(\alpha_1)_{\mathbb{K}} \times \mathcal{X}(\alpha_2)_{\mathbb{K}})$$

est rationnelle.

Démonstration. Sur $\mathbb{K}$, les variétés $\mathcal{X}(\alpha_1)_{\mathbb{K}}$ et $\mathcal{X}(\alpha_2)_{\mathbb{K}}$ sont déployées et admettent une structure cellulaire. Par conséquent, comme nous l'avons déjà fait remarquer

$$\mathrm{CH}^*(\mathcal{X}(\alpha_1)_{\mathbb{K}}) \otimes \mathrm{CH}^*(\mathcal{X}(\alpha_2)_{\mathbb{K}}) \simeq \mathrm{CH}^*(\mathcal{X}(\alpha_1)_{\mathbb{K}} \times \mathcal{X}(\alpha_2)_{\mathbb{K}})$$

et donc $\mathrm{CH}^*(\mathcal{X}(\alpha_1)_{\mathbb{K}} \times \mathcal{X}(\alpha_2)_{\mathbb{K}})$ est engendré par les $h_1^i \times h_2^j$. Pour prouver la rationnalité de J, nous allons procéder en plusieurs étapes et pour cela démontrer les deux lemmes suivants :

Lemme 10. *Le cycle* $h_2^1 \in \mathrm{CH}^1(\mathcal{X}(\alpha_2)_{\mathbb{K}})$ *est rationnel.*

Démonstration. Lorsque nous avons calculé la structure cellulaire de $\mathcal{X}(\alpha_2)_{\mathbb{K}}$ (cf. section 4) nous avons trouvé que le premier terme de la filtration admettait pour R-points ($R \in \mathfrak{Alg}_k$) :

$$\mathcal{X}_4(R) = \{P \in \mathcal{X}(\alpha_2)(R) \mid \mathrm{rg}(P \cap (V_7)_R) \geqslant 2,\ \mathrm{rg}(P \cap (V_5)_R) \geqslant 1\}$$

Par ailleurs, le groupe de Chow de la grassmannienne $\Gamma_2(H)_{\mathbb{K}}$ a également un seul générateur en codimension 1 et ce générateur est la classe de l'adhérence de la cellule de Schubert associée à la variété dont les R-points sont

$$(\Gamma_2(H)_{\mathbb{K}})_{(1,0)}(R) = \{P \in \Gamma_2(H)(R) \mid \mathrm{rg}(P \cap (V_7)_R) \geqslant 2,\ \mathrm{rg}(P \cap (V_5)_R) \geqslant 1\}.$$

Nous constatons ainsi que $\mathcal{X}_4(R) = \mathcal{X}(\alpha_2)_{\mathbb{K}}(R) \cap (\Gamma_2(H)_{\mathbb{K}})_{(1,0)}(R)$ et comme nous avons de plus

$$\mathrm{codim}_{\mathcal{X}(\alpha_2)_{\mathbb{K}}} \mathcal{X}_4 = \mathrm{codim}_{\Gamma_2(H)_{\mathbb{K}}} \mathcal{X}(\alpha_2)_{\mathbb{K}} \cap (\Gamma_2(H)_{\mathbb{K}})_{(0,1)},$$

l'image du générateur libre de $\mathrm{CH}^1(\Gamma_2(H)_K)$ par le pull-back de l'application $\mathcal{X}(\alpha_2)_{\mathbb{K}} \to \Gamma_2(H)_{\mathbb{K}}$ est exactement $h_2^1 = [\mathcal{X}_4]$. Nous considérons alors les pull-backs des extensions des scalaires

$$\mathcal{X}(\alpha_2)_{\mathbb{K}} \longrightarrow \mathcal{X}(\alpha_2) \qquad \text{et} \qquad \Gamma_2(H)_{\mathbb{K}} \longrightarrow \Gamma_2(H)$$

c'est-à-dire les applications

$$\mathrm{res}_{\mathbb{K}/k} \colon \mathrm{CH}^1(\mathcal{X}(\alpha_2)) \longrightarrow \mathrm{CH}^1(\mathcal{X}(\alpha_2)_{\mathbb{K}})$$

et

$$\mathrm{res}_{\mathbb{K}/k} \colon \mathrm{CH}^1(\Gamma_2(H)) \longrightarrow \mathrm{CH}^1(\Gamma_2(H)_{\mathbb{K}}).$$

Il est maintenant entendu qu'une base normale n'existe pas sur le corps de base k, toutefois comme nous l'avons déjà signalé, n'importe quelle base permet de définir toutes les variétés de Schubert d'une grassmannienne et les classes modulo équivalence rationnelle de ces variétés forment un système de générateurs libres de $\mathrm{CH}^*(\Gamma_2(H))$. Dès lors, il est bien clair que

$$\mathrm{CH}^*(\Gamma_2(H)) \simeq \mathrm{CH}^*(\Gamma_2(H)_{\mathbb{K}})$$

et ainsi en considérant le diagramme commutatif suivant

$$\begin{tikzcd}
\mathrm{CH}^1(\Gamma_2(H)_{\mathbb{K}}) \arrow[r] & \mathrm{CH}^1(\mathcal{X}(\alpha_2)_{\mathbb{K}}) \\
\mathrm{CH}^1(\Gamma_2(H)) \arrow[r] \arrow[u] & \mathrm{CH}^1(\mathcal{X}(\alpha_2)) \arrow[u, "\mathrm{res}_{\mathbb{K}/k}"']
\end{tikzcd}$$

nous en déduisons que le cycle h_2^1 est rationnel.

□

Lemme 11. *Le cycle* $c = h_1^0 \times h_2^3 + h_1^3 \times h_2^0 \in \mathrm{CH}^3(\mathcal{X}(\alpha_1)_{\mathbb{K}} \times \mathcal{X}(\alpha_2)_{\mathbb{K}})$ *est rationnel.*

Démonstration. Pour établir la rationalité de c, nous considérons le diagramme commutatif suivant :

$$\begin{tikzcd}
\mathrm{CH}^3(\mathcal{X}(\alpha_1)_{\mathbb{K}} \times \mathcal{X}(\alpha_2)_{\mathbb{K}}) \arrow[r, "(id_{\mathcal{X}(\alpha_1)_{\mathbb{K}}} \times p_{\mathbb{K}})^*"] & \mathrm{CH}^3(\mathcal{X}(\alpha_1)_{\mathbb{K}(\mathcal{X}(\alpha_2)_{\mathbb{K}})}) \\
\mathrm{CH}^3(\mathcal{X}(\alpha_1) \times \mathcal{X}(\alpha_2)) \arrow[r, "(id_{\mathcal{X}(\alpha_1)} \times p)^*"] \arrow[u, "\mathrm{res}_{\mathbb{K}/k}"] & \mathrm{CH}^3(\mathcal{X}(\alpha_1)_{k(\mathcal{X}(\alpha_2))}) \arrow[u, "\mathrm{res}_{\mathbb{K}(\mathcal{X}(\alpha_2)_{\mathbb{K}})/k(\mathcal{X}(\alpha_2))}"']
\end{tikzcd}$$

où les flèches horizontales sont les pull-backs par rapport aux morphismes plats

$$id_{\mathcal{X}(\alpha_1)} \times p \colon \mathcal{X}(\alpha_1)_{k(\mathcal{X}(\alpha_2))} \to \mathcal{X}(\alpha_1) \times \mathcal{X}(\alpha_2)$$

et

$$id_{\mathcal{X}(\alpha_1)_{\mathbb{K}}} \times p_{\mathbb{K}} \colon \mathcal{X}(\alpha_1)_{\mathbb{K}(\mathcal{X}(\alpha_2)_{\mathbb{K}})} \to \mathcal{X}(\alpha_1)_{\mathbb{K}} \times \mathcal{X}(\alpha_2)_{\mathbb{K}}$$

où p (resp. $p_{\mathbb{K}}$) est le morphisme point générique de $\mathcal{X}(\alpha_1)$ (resp. $\mathcal{X}(\alpha_1)_{\mathbb{K}}$).

Il est clair que la forme quadratique q est isotrope sur $k(\mathcal{X}(\alpha_2))$ (comme tout à l'heure dans le lemme 9 nous rajoutons un point rationnel à $\mathcal{X}(\alpha_1)$). Par conséquent, elle est totalement isotrope (par le même argument que dans la preuve du lemme 9) et $(h_1^3)_{\mathbb{K}(\mathcal{X}(\alpha_2)_{\mathbb{K}})}$ est défini sur $k(\mathcal{X}(\alpha_2))$ (puisque dans ce cas là, tous les cycles le sont). Comme $(id_{\mathcal{X}(\alpha_1)} \times p)^*$ est surjective (voir par exemple [IK00, §5, Prop. 5.1 p. 8]), il en découle, qu'il existe un cycle $d \in \mathrm{CH}^3(\mathcal{X}(\alpha_1)_{\mathbb{K}} \times \mathcal{X}(\alpha_2)_{\mathbb{K}})$ défini sur k tel que

$$(id_{\mathcal{X}(\alpha_1)_{\mathbb{K}}} \times p_{\mathbb{K}})^*(d) = (h_1^3)_{\mathbb{K}(\mathcal{X}(\alpha_2)_{\mathbb{K}})}.$$

Nous calculons maintenant l'action de $(id_{\mathcal{X}(\alpha_1)_{\mathbb{K}}} \times p_{\mathbb{K}})^*$ sur les éléments engendrant le groupe $\mathrm{CH}^3(\mathcal{X}(\alpha_1)_{\mathbb{K}} \times \mathcal{X}(\alpha_2)_{\mathbb{K}})$:

$$(id_{\mathcal{X}(\alpha_1)_{\mathbb{K}}} \times p_{\mathbb{K}})^*(h_1^i \times h_2^{3-i}) = \begin{cases} (h_1^3)_{\mathbb{K}(\mathcal{X}(\alpha_2)_{\mathbb{K}})} & \text{si } i = 3 \\ 0 & \text{sinon} \end{cases}$$

Et comme $(id_{\mathcal{X}(\alpha_1)_{\mathbb{K}}} \times p_{\mathbb{K}})^*(d) = (h_1^3)_{\mathbb{K}(\mathcal{X}(\alpha_2)_{\mathbb{K}})}$, il s'ensuit que

$$d = h_1^3 \times h_2^0 + \sum_{i=1}^{2} a_i h_1^i \times h_2^{3-i} + a h_1^0 \times h_2^3$$

pour a_1, a_2 et a dans $\mathbb{Z}$. Nous savons que h_2^1 est rationnel d'après le lemme 10 donc $(h_2^1)^2 = l h_2^2$ l'est. D'autre part, par un argument de transfert, $2h_2^2$ est aussi rationnel et comme l est impair (comme nous l'avons prouvé à la fin de la sous-section 5.3), h_2^2 est rationnel. D'autre part, h_1^1 est aussi un cycle rationnel, ce résultat classique relève du même argument que pour h_2^1 et enfin, comme $h_1^2 = (h_1^1)^2$, c'est aussi un cycle rationnel. De là, nous déduisons que $h_1^1 \times h_2^2$ et $h_1^2 \times h_2^1$ sont rationnels. De la même façon, $h_1^0 \times h_2^1$ et $h_1^0 \times h_2^2$ sont aussi rationnels et comme nous pouvons écrire que

$$2h_1^0 \times h_2^3 = (h_1^0 \times h_2^1) \cdot (h_1^0 \times h_2^2),$$

il s'ensuit que le cycle $2h_1^0 \times h_2^3$ est lui aussi rationnel. Dès lors en soustrayant des multiples de ces cycles à d, il en découle que selon la parité de a, soit $h_1^3 \times h_2^0$ est rationnel, soit $h_1^3 \times h_2^0 + h_1^0 \times h_2^3$ l'est. Dans ce dernier cas, la preuve est terminée. Dans l'autre cas, nous devons refaire exactement le même raisonnement avec $\mathcal{X}(\alpha_2)_{\mathbb{K}(\mathcal{X}(\alpha_1)_{\mathbb{K}})}$ au lieu de $\mathcal{X}(\alpha_1)_{\mathbb{K}(\mathcal{X}(\alpha_2)_{\mathbb{K}})}$, avec l'hypothèse que $h_1^3 \times h_2^0$ est rationnel, pour en déduire que $h_1^0 \times h_2^3$ l'est aussi. Dans tous les cas, nous pouvons conclure que

$$c = h_1^3 \times h_2^0 + h_1^0 \times h_2^3$$

est un cycle rationnel. □

Les arguments précédents, nous montrent que les cycles $h_1^1 \times h_2^1$ et $h_1^2 \times h_2^0$ sont rationnels. Par conséquent, les correspondances

$$\begin{aligned}
(h_1^0 \times h_2^2) \cdot c &= h_1^3 \times h_2^2 + h_1^0 \times h_2^5\\
(h_1^1 \times h_2^1) \cdot c &= h_1^4 \times h_2^1 + l h_1^1 \times h_2^4\\
(h_1^2 \times h_2^0) \cdot c &= h_1^5 \times h_2^0 + h_1^2 \times h_2^3
\end{aligned}$$

sont également rationnelles. D'autre part, par un argument de transfert, $2h_1^1 \times h_2^4 = h_1^1 \times 2h_2^4$ est rationnel et comme l est impair,

$$h_1^1 \times h_2^4 + h_1^4 \times h_2^1$$

est rationnel. Le cycle J étant égal à la somme des trois cycles, $h_1^3 \times h_2^2 + h_1^0 \times h_2^5$, $h_1^1 \times h_2^4 + h_1^4 \times h_2^1$ et $h_1^5 \times h_2^0 + h_1^2 \times h_2^3$, il est par voie de conséquence rationnel. □

Le fait que J soit une correspondance rationnelle signifie qu'il existe une correspondance f telle que $f_{\mathbb{K}} = J$. Nous avons établi dans la proposition 3 que J induit un isomorphisme entre $\mathcal{X}(\alpha_1)_{\mathbb{K}}$ et $\mathcal{X}(\alpha_2)_{\mathbb{K}}$. De même, ${}^tJ$ induit un isomorphisme entre $\mathcal{X}(\alpha_2)_{\mathbb{K}}$ et $\mathcal{X}(\alpha_1)_{\mathbb{K}}$ et est également rationnelle comme transposé d'une correspondance rationnelle. Ainsi, d'après le théorème d'isomorphisme (cf. corollaire 3), $f \circ {}^t f$ et ${}^t f \circ f$ sont des automorphismes motiviques de $\mathcal{X}(\alpha_2)$ et $\mathcal{X}(\alpha_1)$ respectivement. Par conséquent, f réalise un isomorphisme motivique de $\mathcal{X}(\alpha_1)$ dans $\mathcal{X}(\alpha_2)$.

Nous avons donc prouvé que les variétés $\mathcal{X}(\alpha_1)$ et $\mathcal{X}(\alpha_2)$ bien que non isomorphes en tant que variétés algébriques lisses sont motiviquement isomorphes.

Références

[B] A. Borel, *Linear Algebraic Groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[Bro03] P. Brosnan, *A short proof of Rost nilpotence via refined correspondences*, Documenta Math. 8 (2003) 69–78.

[Car14] É. Cartan, *Les groupes réels simples finis et continus*, Ann. Sci. École Norm. Sup. 31, 263–355, 1914.

[Car52] É. Cartan, *Œuvres Complètes. Partie I. Groupes de Lie*, Gauthier-Villars, Paris, 1952.

[CGM03] V. Chernousov, S. Gille and A. Merkurjev, *Motivic decomposition of isotropic projective homogeneous varieties*, Preprint, 2003. (voir `http ://www.mathematik.uni-bielefeld.de/lag/`).

[Dem74] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Scient. Éc. Norm. Sup. 7 (4), 53–88, 1974. Collection of Henri Cartan on the occasion of his 70th Birthday, I.

[Dem77] M. Demazure, *Automorphismes et déformations des variétés de Borel*, Invent. Math. 39 (2), 179–186, 1977.

[DG] M. Demazure et P. Gabriel, *Groupes Algébriques. Tome I : Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeur, Paris, 1970. Avec un appendice *Corps de classes local* par Michel Hazewinkel.

[EH] D. Eisenbud and J. Harris, *The Geometry of Schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[F] W. Fulton, *Intersection Theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Gar99] R. S. Garibaldi, *Exceptional algebraic groups*, ETH Zürich, Summer Semester 1999. (voir `http ://www.mathcs.emory.edu/ skip/classes/eag/eag.html`).

[H] R. Hartshorne,*Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[IK00] O. Izhboldin and N. A. Karpenko, *Some new examples in the theory of quadratic forms*. Math. Z. 234 (4), 647–695, 2000.

[Kar90] N. A. Karpenko, *Algebro-geometric invariants of quadratics forms*, Leningrad Math. J. 2 (1), 119–138, 1991.

[Kar01] N. A. Karpenko, *Cohomology of relative cellular spaces and isotropic flag varieties*, St-Petersburg Math. J. 12, 1–50, 2001.

[Köc91] B. Köck, *Chow motif and higher Chow theory of G/P*, Manuscripta math. 70 (4), 363–372, 1991.

[M] L. Manivel, *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence*, volume 3 des *Cours spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 1998.

[Mar76] R. Marlin, *Comparaison de l'anneau de Chow et de l'anneau de Grothendieck*. Dans le Séminaire de géométrie analytique (École Norm. Sup., Paris, 1974–75), 229–240. Astérisque, n. 36-37. Soc. Math. France, Paris, 1976.

[MPW96] A. S. Merkurjev, I. Panin, and A. R. Wadsworth, *Index reduction formulas for twisted flag varieties*, I, K-Theory 10 (6), 517–596, 1996.

[Ros90] M. Rost, *The motive of a Pfister form*. Prépublication, 1990 (voir `http ://www.math.ohio-state.edu/ rost/motive.html`).

[S] R. D. Schafer, *An introduction to nonassociative algebras*, volume 22 of Pure and Applied Math., Academic Press, New York–London, 1966.

[Sch62] G. J. Schellekens, *On a hexagonic structure. I*, Nederl. Akad. Wetensch. Proc. Ser. A 65 Indag. Math. 24, 201–217, 1962.

[SV] T. A. Springer, F.D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer Monographs in Mathematics, Springer-Verlag, 2000.

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Essential Dimension: a Functorial Point of View (After A. Merkurjev)

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Abstract. In these notes we develop a systematic study of the essential dimension of functors. This approach is due to A. Merkurjev and can be found in his unpublished notes [12]. The notion of essential dimension was earlier introduced for finite groups by J. Buhler and Z. Reichstein in [3] and for an arbitrary algebraic group over an algebraically closed field by Z. Reichstein in [14]. This is a numerical invariant depending on the group G and the field k. This number is denoted by $\mathrm{ed}_k(G)$. In this paper we insist on the behaviour of the essential dimension under field extension k'/k and try to compute $\mathrm{ed}_k(G)$ for *any* k. This will be done in particular for the group $\mathbb{Z}/n$ when $n \leq 5$ and for the circle group. Along the way we define the essential dimension of functor with versal pairs and prove that all the different notions of essential dimension agree in the case of algebraic groups. Applications to finite groups are given. Finally we give a proof of the so-called homotopy invariance, that is $\mathrm{ed}_k(G) = \mathrm{ed}_{k(t)}(G)$, for an algebraic group G defined over an infinite field k.

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Contents

Summary of the paper

In Section 1, we introduce the notion of essential dimension of a covariant functor from the category of field extensions over a base field k to the category of sets. This notion is due to A. Merkurjev and can be found in [12]. We then study the behaviour of this notion under products, coproducts and field extensions. Along the way, we define the notion of fibration of functors.

In Section 2, we introduce the essential dimension of an algebraic group G defined over an *arbitrary* field k. We then give some examples of computation of this essential dimension, including the case of the circle group.

In Section 3, we give an upper bound for the essential dimension of an algebraic group which acts linearly and generically freely on a finite-dimensional vector space. As an application, we show that the essential dimension of any algebraic group is finite. Compare this material with [14] where the essential dimension of G is defined taking the point of view of G-actions. Very sketchy proofs of these results can be found in [12]. For the convenience of the reader, we present complete proofs of them using the ideas of [12], filling in technical details. We then apply the previous results to estimate the essential dimension of finite abelian groups and dihedral groups when the base field is sufficiently large.

In Section 4, we introduce Merkurjev's notions of n-simple functors and non-constant morphisms (see [12]). We apply it to give lower bounds of essential dimension of some algebraic groups (e.g. symmetric groups) using non-trivial cohomological invariants always following [12].

In Section 5, inspired by Rost's definition of essential dimension for some sub-functors of Milnor's K-theory (see [16]), we define the notion of versal pair for functors from the category of commutative and unital k-algebras to the category of sets. We then define the (Rost's) essential dimension for functors having a versal pair, and compare it to Merkurjev's essential dimension.

In Section 6, we introduce the notion of generic torsor, following [9]. We then prove that the essential dimension of an algebraic group G is the essential dimension of a generic torsor. We also compare the essential dimension of an algebraic group G with that of any closed subgroup. Along the way the notion of compression of torsors is introduced following [14]. The present approach has the advantage that no hypothesis on the ground field is needed. Again, ideas of proofs of these results can be found in [12]. We use them, filling the details and reformulating them in terms of versal pairs.

In Section 7, we focus on essential dimension of finite constant group schemes. First of all, we prove that the essential dimension of such a group G is the minimum of the $\mathrm{trdeg}(E:k)$ for all the fields $E \subseteq k(V)$ on which G acts faithfully (see [3]). We then apply these results to compute essential dimension of cyclic and dihedral groups over the field of real numbers, and essential dimension of cyclic groups of order at most 5 over any base field.

Finally, in Section 8, we give the proof of the homotopy invariance for essential dimension of algebraic groups defined over an infinite field.

1. Introduction

Let k be a field. We denote by $\mathfrak{C}_k$ the category of field extensions of k, i.e. the category whose objects are field extensions K over k and whose morphisms are field homomorphisms which fix k. We write $\mathfrak{F}_k$ for the category of all *covariant* functors from $\mathfrak{C}_k$ to the category of sets. For such a functor $\mathbf{F}$ and for a field extension K/k we will write $\mathbf{F}(K)$ instead of $\mathbf{F}(K/k)$. If $K \to L$ is a morphism in $\mathfrak{C}_k$, for every element $a \in \mathbf{F}(K)$ we will denote by a_L the image of a under the map $\mathbf{F}(K) \longrightarrow \mathbf{F}(L)$. We shall say that a morphism $\mathbf{F} \longrightarrow \mathbf{G}$ between functors in $\mathfrak{F}_k$ is a SURJECTION if, for any field extension K/k, the corresponding map $\mathbf{F}(K) \longrightarrow \mathbf{G}(K)$ is a surjection of sets. If $\mathbf{F} : \mathfrak{C}_k \longrightarrow \mathbf{Sets}$ is an object of $\mathfrak{F}_k$ and if K/k is a field extension we will sometimes denote by $\mathbf{F}_K$ the functor $\mathbf{F}$ viewed as a functor over the category $\mathfrak{C}_K$. By a scheme over k, we mean a k-scheme of finite type.

Examples 1.1.

(1) The forgetful functor, denoted by $\mathbf{O}$, which assigns to each field extension K/k the underlying set of K and to each morphism its underlying map of sets, is an object of $\mathfrak{F}_k$.

(2) The stupid functor, denoted by $*$, sending a field K to a one-point set is an object of $\mathfrak{F}_k$.

(3) Let X be a scheme over k. It defines a "point functor", still denoted by X, in this way :

$$K \mapsto X(K) = \mathrm{Hom}(\mathrm{Spec}(K), X).$$

The set $X(K)$ is simply the set of all K-rational points of X.

(4) For any integer $n \geq 1$, we put $\mathbf{Q}_n(K)$ for the set of isomorphism classes of non-degenerate quadratic forms of dimension n over K. It is clear that $\mathbf{Q}_n$ defines an object of $\mathfrak{F}_k$.

(5) A K-algebra is called primitive if it is isomorphic to a quotient of $K[X]$. Every such algebra is thus of the form $K[X]/<f>$ for a single polynomial $f = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$. We denote by $\mathbf{Alg}_n(K)$ the set of isomorphism classes of n-dimensional primitive algebras. This also defines a functor $\mathbf{Alg}_n$ from $\mathfrak{C}_k$ to the category of sets.

(6) Let K be a field. We recall that an étale algebra over K is a finite dimensional commutative K-algebra A which satisfies the equality $\sharp\ \mathrm{Hom}_K(A, \overline{K}) = \dim_K A$, where $\overline{K}$ denotes an algebraic closure of K. This is equivalent to saying that $A \otimes_K \overline{K}$ is reduced or that A is a product of separable extensions of K. Moreover if K is infinite, A is étale over K if and only if $A \simeq K[X]/< f >$ where f has no multiple roots in $\overline{K}$. If A is étale over K and $K \to L$ is a field homomorphism then $A \otimes_K L$ is étale over L.
For any field extension K/k and any integer $n \geq 1$, let $\mathbf{\acute{E}t}_n(K)$ denote the set of isomorphism classes of n-dimensional étale algebras over K. It also defines an object of the category $\mathfrak{F}_k$. When the base field k is infinite $\mathbf{\acute{E}t}_n$ is a subfunctor of $\mathbf{Alg}_n$ and these functors are closely related for the essential dimension.

(7) Let G be a finite abstract group of order n and let K be a field. By a Galois G-algebra over K (or Galois K-algebra with group G) we mean an étale K-algebra L of dimension n such that G acts on L as a group of K-automorphisms and such that $L^G = K$. We denote by G-$\mathbf{Alg}(K)$ the set of G-isomorphism classes of Galois G-algebras over K. The assignment $K \mapsto G$-$\mathbf{Alg}(K)$ from $\mathfrak{C}_k$ to the category of sets defines an object of $\mathfrak{F}_k$.

(8) For every integer $d, n \geq 2$, define $\mathbf{F}_{d,n}(K)$ to be the set of all (non-trivial) homogenous forms over K of degree d in n variables modulo the $\mathbf{GL}_n(K)$-action and modulo the relation $f \sim \lambda f$ for $\lambda \in K^\times$. Once again $\mathbf{F}_{d,n}$ is an object of $\mathfrak{F}_k$.

(9) Let S be a pointed set with at least two elements and $d \geq 1$ an integer. We shall define the functor $\mathbf{F}_S^d$ in the following way :

$$\mathbf{F}_S^d(K) = \begin{cases} S & \text{if } \operatorname{trdeg}(K:k) \geq d \\ \\ * & \text{otherwise} \end{cases}$$

and, for an extension K'/K, the obvious constant map of pointed sets $\mathbf{F}_S^d(K) \longrightarrow \mathbf{F}_S^d(K')$.

(10) Let L/k be an arbitrary field extension. Then, the (covariant) representable functor h_L given by $h_L(K) = \operatorname{Hom}(L, K)$ defines also an object of $\mathfrak{F}_k$.

One natural question is to ask how many parameters are needed to describe a given structure. For example, any n-dimensional quadratic form in characteristic not 2, is determined by n parameters since it can be reduced to a diagonal form.
A quadratic algebra will certainly be described by one parameter since it can always be written as $k[X]/< X^2 + a >$ when $\frac{1}{2}$ exists. The natural notion of functor shall replace the word "structure" and the following crucial definition, which is due to A. Merkurjev, shall make precise the concept of "how many parameters" are needed to describe it.

Definition 1.2. *Let* $\mathbf{F}$ *be an object of* $\mathfrak{F}_k$, K/k *a field extension and* $a \in \mathbf{F}(K)$. *For* $n \in \mathbb{N}$, *we say that the* essential dimension of a is $\leq n$ *(and we write* $\operatorname{ed}(a) \leq n$*), if there exists a subextension* E/k *of* K/k *such that:*

i) $\operatorname{trdeg}(E:k) \leq n$,

ii) the element a *is in the image of the map* $\mathbf{F}(E) \longrightarrow \mathbf{F}(K)$.

We say that $\operatorname{ed}(a) = n$ *if* $\operatorname{ed}(a) \leq n$ *and* $\operatorname{ed}(a) \not\leq n-1$. *The* essential dimension of $\mathbf{F}$ *is the supremum of* $\operatorname{ed}(a)$ *for all* $a \in \mathbf{F}(K)$ *and for all* K/k. *The essential dimension of* $\mathbf{F}$ *will be denoted by* $\operatorname{ed}_k(\mathbf{F})$.

Examples 1.3.

(1) It is clear from the very definition that $\operatorname{ed}(*) = 0$ and $\operatorname{ed}(\mathbf{O}) = 1$. More generally, we may say that a functor $\mathbf{F}$ is flasque if, for any field extension K'/K, the map $\mathbf{F}(K) \longrightarrow \mathbf{F}(K')$ is surjective. Clearly every flasque functor $\mathbf{F}$ satisfies $\operatorname{ed}(\mathbf{F}) = 0$ and every constant functor is flasque.

(2) We shall do some very easy computations on polynomials of degree 2,3 and 4 in order to compute the essential dimension of $\mathbf{Alg}_2, \mathbf{Alg}_3$ and $\mathbf{Alg}_4$. We start with simple considerations on the functor $\mathbf{Alg}_n$ for arbitrary n. Let $A = K[X]/ < f >$ and $B = K[Y]/ < g >$ two n-dimensional primitive algebras. We denote by x and y the classes of X and Y respectively. A homomorphism $\varphi : A \longrightarrow B$ is determined by the image of x, say

$$\varphi(x) = c_{n-1}y^{n-1} + c_{n-2}y^{n-2} + \cdots + c_1 y + c_0,$$

satisfying $f(\varphi(x)) = 0$. Saying that φ is an isomorphism is nothing but saying that $\varphi(x)$ generates B. In this case we say that $\varphi(x)$ is a nondegenerate TSCHIRNHAUS TRANSFORMATION of f. Clearly a polynomial $f = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ is defined over $k(a_0, \ldots, a_{n-1})$ and thus computing the essential dimension of (the isomorphism class of) $K[X]/ < f >$ is the same as reducing the number of coefficients appearing in f by means of nondegenerate Tschirnhaus transformations. (This is the starting point of the paper [3]). It clearly suffices to do this on the "generic element" $X^n + t_{n-1}X^{n-1} + \cdots + t_1 X + t_0$ (where the t_i's are algebraically independent over k) since every other polynomial is a specialization of this one.

Now, when the characteristic of the ground field k does not divide n, the substitution $Y = X - \frac{t_{n-1}}{n}$ drops the coefficient t_{n-1} and hence

$$\mathrm{ed}(\mathbf{Alg}_n) \leq n - 1.$$

For the polynomial $X^2 + aX + b$ this says that one can reduce it to the form $X^2 + c$. Now the algebra $k(t)[X]/ < X^2 + t >$ is clearly not defined over an algebraic extension of k and hence

$$\mathrm{ed}(\mathbf{Alg}_2) = 1.$$

Now $X^3 + aX^2 + bX + c$ can be reduced to $X^3 + b'X + c'$ and, setting $Y = \frac{c'}{b'}X$, one makes the second and the third coefficient equal.
Thus one can reduce it to the form $X^3 + dX + d$. As before the algebra $k(t)[X]/ < X^3 + tX + t >$ is not defined over an algebraic extension of k and so

$$\mathrm{ed}(\mathbf{Alg}_3) = 1.$$

Similarly the generic polynomial of degree 4 can be reduced to the form $X^4 + sX^2 + tX + t$ and hence $\mathrm{ed}(\mathbf{Alg}_4) \leq 2$. We will see that it cannot be reduced, thus $\mathrm{ed}(\mathbf{Alg}_4) = 2$.

Remark 1.4. The notion of essential dimension depends on the ground field k. However, when the field k is fixed, there is no confusion by writing $\mathrm{ed}(\mathbf{F})$. When the context is not clear, or when we want to insist on some hypotheses made on the field, we shall write $\mathrm{ed}_k(\mathbf{F})$. In general, if k'/k is a field extension, every object $\mathbf{F}$ of $\mathfrak{F}_k$, can be viewed (by restriction) as an object of $\mathfrak{F}_{k'}$. The following proposition shows the behaviour of essential dimension under field extension.

Proposition 1.5. *Let k'/k a field extension and* $\mathbf{F}$ *an object of* $\mathfrak{F}_k$. *Then*

$$\mathrm{ed}_{k'}(\mathbf{F}) \leq \mathrm{ed}_k(\mathbf{F}).$$

Proof. If $\mathrm{ed}_k(\mathbf{F}) = \infty$, the result is obvious. Let $\mathrm{ed}_k(\mathbf{F}) = n$. Take K/k' a field extension and $a \in \mathbf{F}(K)$. There is a subextension $k \subseteq E \subseteq K$ with $\mathrm{trdeg}(E : k) \leq n$ such that a is in the image of the map $\mathbf{F}(E) \longrightarrow \mathbf{F}(K)$. The composite extension $E' = Ek'$ then satisfies $\mathrm{trdeg}(E' : k') \leq n$ and clearly a is in the image of the map $\mathbf{F}(E') \longrightarrow \mathbf{F}(K)$. Thus $\mathrm{ed}(a) \leq n$ and $\mathrm{ed}_{k'}(\mathbf{F}) \leq n$.

Remarks 1.6.

(1) The above proposition says that, for a fixed functor $\mathbf{F} \in \mathfrak{F}_k$, the map

$$\mathrm{ed}_{\_}(\mathbf{F}) : \mathfrak{C}_k \longrightarrow \mathbb{N} \cup \{\infty\}$$

is a contravariant functor where $\mathbb{N} \cup \{\infty\}$ is considered as a category by saying that there is a morphism $n \to m$ exactly when $n \leq m$. This implies that, if $\mathbf{F}$ is a functor defined over the category of *all* fields, to give an upper bound of $\mathrm{ed}_k(\mathbf{F})$ it is sufficient to give an upper bound over each prime field $\mathbb{F}_p$ when $\mathrm{char}(k) > 0$, and to give an upper bound over $\mathbb{Q}$ when $\mathrm{char}(k) = 0$.

(2) In general one does not have $\mathrm{ed}_k(\mathbf{F}) = \mathrm{ed}_{k'}(\mathbf{F})$ for any field extension k'/k. Example (9) above shows that the essential dimension can decrease considerably: one sees immediately that $\mathrm{ed}_{k'}(\mathbf{F}_S^d) = 0$ if $\mathrm{trdeg}(k' : k) \geq d$. This is due to the fact that the functor becomes constant over k' and hence its essential dimension is zero. On the other hand it is clear that $\mathrm{ed}_k(\mathbf{F}_S^d) = d$.

(3) Let L/k be an extension and H_L the corresponding representable functor of Example (10). Then one has $\mathrm{ed}_{k'}(\mathrm{H}_L) = \mathrm{trdeg}(L : k')$ if $k \subseteq k' \subseteq L$ and $\mathrm{ed}_{k'}(\mathrm{H}_L) = 0$ otherwise.

We shall see later on (Corollary 2.7 in Chapter II) an example of a functor for which the inequality of Proposition 1.5 is strict even if the extension k'/k is algebraic.

The behaviour of essential dimension with respect to subfunctors is not very clear. For example take for $\mathbf{G}$ the constant functor $\mathbf{G}(K) = S$ where S is a set with at least two elements. Then $\mathbf{F}_S^d$ is a subfunctor of $\mathbf{G}$ and the dimension of the former is d (which is arbitrarily large) whereas the dimension of $\mathbf{G}$ is zero. However there is a large class of subfunctors for which the essential dimension has a nice behaviour.

Definition 1.7. *Let* $\mathbf{G}$ *be an object of* $\mathfrak{F}_k$*. A subfunctor* $\mathbf{F} \subseteq \mathbf{G}$ *is called* saturated *if for any field extension* L/K *over* k *and any element* $a \in \mathbf{G}(K)$ *such that* $a_L \in \mathbf{F}(L)$ *there is an algebraic subextension* K'/K *such that* $a_{K'} \in \mathbf{F}(K')$.

Proposition 1.8. *Let* $\mathbf{F} \subseteq \mathbf{G}$ *be a saturated subfunctor. Then*

$$\mathrm{ed}(\mathbf{F}) \leq \mathrm{ed}(\mathbf{G}).$$

Proof. Let K/k be a field extension and $a \in \mathbf{F}(K)$. Assume that $\mathrm{ed}(\mathbf{G}) = n$. Then there is a subextension L/k and an element $b \in \mathbf{G}(L)$ such that $\mathrm{trdeg}(L : k) \leq n$ and $a = b_K$. Since $\mathbf{F}$ is saturated, there is an algebraic subextension E/L in K/L such that $b_E \in \mathbf{F}(E)$. Thus $a \in \mathrm{im}(\mathbf{F}(E) \longrightarrow \mathbf{F}(K))$ and since $\mathrm{trdeg}(E : k) \leq n$ this shows that $\mathrm{ed}(\mathbf{F}) \leq n$.

We continue our investigation with some very simple lemmas concerning the functorial properties of $\mathrm{ed} : \mathfrak{F}_k \longrightarrow \mathbb{N} \cup \{\infty\}$.

Lemma 1.9. *Let* $f : \mathbf{F} \twoheadrightarrow \mathbf{G}$ *be a surjection in* $\mathfrak{F}_k$*. Then*

$$\mathrm{ed}(\mathbf{G}) \leq \mathrm{ed}(\mathbf{F}).$$

Proof. Let K/k be an extension and $b \in \mathbf{G}(K)$. By assumption, there is an element $a \in \mathbf{F}(K)$ such that $f_K(a) = b$. Suppose that $\mathrm{ed}(\mathbf{F}) = n$. Take a subextension $k \subseteq E \subseteq K$ such that $\mathrm{trdeg}(E : k) \leq n$ and such that $a \in \mathrm{im}\big(\mathbf{F}(E) \longrightarrow \mathbf{F}(K)\big)$. The lemma now follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{F}(K) \arrow[r, two heads, "f_K"] & \mathbf{G}(K) \\
\mathbf{F}(E) \arrow[u] \arrow[r, two heads, "f_E"] & \mathbf{G}(E) \arrow[u]
\end{tikzcd}$$

Thus essential dimension is functorial (in a contravariant way) over the category of functors in $\mathfrak{F}_k$ with *surjections* as morphisms. Nevertheless we will not restrict ourselves to that category, since this would not be very natural. For instance, we will always consider products and coproducts in the category of functors with *all* morphisms. The next lemma shows that the essential dimension preserves coproducts.

Lemma 1.10. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be two objects of* $\mathfrak{F}_k$. *Then*

$$\operatorname{ed}(\mathbf{F} \coprod \mathbf{G}) = \max\{\operatorname{ed}(\mathbf{F}), \operatorname{ed}(\mathbf{G})\}.$$

Proof. Let K/k be an extension and $a \in \mathbf{F}(K) \coprod \mathbf{G}(K)$. Clearly $\operatorname{ed}(a) \leq \operatorname{ed}(\mathbf{F})$ or $\operatorname{ed}(a) \leq \operatorname{ed}(\mathbf{G})$ and hence $\operatorname{ed}(\mathbf{F} \coprod \mathbf{G}) \leq \max\{\operatorname{ed}(\mathbf{F}), \operatorname{ed}(\mathbf{G})\}$. The opposite inequality is clear since $\mathbf{F}$ and $\mathbf{G}$ are both saturated subfunctors of $\mathbf{F} \coprod \mathbf{G}$.

Lemma 1.11. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be two objects of* $\mathfrak{F}_k$. *Then*

$$\operatorname{ed}(\mathbf{F} \times \mathbf{G}) \leq \operatorname{ed}(\mathbf{F}) + \operatorname{ed}(\mathbf{G}).$$

Proof. Take K/k a field extension and $(a, a') \in \mathbf{F}(K) \times \mathbf{G}(K)$. Take two extensions $k \subseteq E, E' \subseteq K$ with $\operatorname{trdeg}(E : k) \leq \operatorname{ed}(\mathbf{F}), \operatorname{trdeg}(E' : k) \leq \operatorname{ed}(\mathbf{G})$ and such that a (respectively a') belongs to the image of $\mathbf{F}(E) \longrightarrow \mathbf{F}(K)$ (respectively $\mathbf{G}(E') \longrightarrow \mathbf{G}(K)$). So there exist $b \in \mathbf{F}(E)$ and $b' \in \mathbf{G}(E')$ such that $b_K = a$ and $b'_K = a'$. If we consider $L = EE'$ and denote by c (respectively c') the image of b in $\mathbf{F}(L)$ (respectively the image of b' in $\mathbf{G}(L)$) it is easily seen that (c, c') maps to (a, a'). Hence

$$\operatorname{ed}(a, a') \leq \operatorname{trdeg}(L : k) \leq \operatorname{trdeg}(E : k) + \operatorname{trdeg}(E' : k) \leq \operatorname{ed}(\mathbf{F}) + \operatorname{ed}(\mathbf{G}).$$

Thus $\operatorname{ed}(\mathbf{F} \times \mathbf{G}) \leq \operatorname{ed}(\mathbf{F}) + \operatorname{ed}(\mathbf{G})$.

A slight generalization of the previous inequality can be performed for functors which are in some kind of “fibration position”.

First recall that an action of a *set* Y over a set X is nothing but a map $Y \times X \to X$. If $y \in Y$ and $x \in X$ we shall write $y \cdot x$ for the image of (y, x) under this map. We say that a functor $\mathbf{F} : \mathfrak{C}_k \longrightarrow \mathbf{Sets}$ acts over a functor $\mathbf{G} : \mathfrak{C}_k \longrightarrow \mathbf{Sets}$ if, for every extension K/k, the set $\mathbf{F}(K)$ acts over $\mathbf{G}(K)$ and if the obvious compatibility condition holds: for each morphism $K \to L$ and for all elements $y \in \mathbf{F}(K)$ and $x \in \mathbf{G}(K)$, one has $(y \cdot x)_L = y_L \cdot x_L$. We shall say that the action of the functor $\mathbf{F}$ over the functor $\mathbf{G}$ is transitive if for every K/k the action of the set $\mathbf{F}(K)$ is transitive over $\mathbf{G}(K)$ (that is there is only one orbit). Recall also that, if $\pi : \mathbf{G} \longrightarrow \mathbf{H}$ is a morphism of functors in $\mathfrak{F}_k$ and K/k is an extension, each element $a \in \mathbf{H}(K)$ gives rise to a functor $\pi^{-1}(a)$, defined over the category $\mathfrak{C}_K$, by setting $\pi_L^{-1}(a) = \{x \in \mathbf{G}(L) \mid \pi_L(x) = a_L\}$ for every extension L/K.

Definition 1.12. *Let* $\pi : \mathbf{G} \twoheadrightarrow \mathbf{H}$ *be a surjection in* $\mathfrak{F}_k$. *We say that a functor* $\mathbf{F}$ *is in* fibration position *for* π *if* $\mathbf{F}$ *acts transitively on each fiber of* π. *More precisely, for every extension* K/k *and every* $a \in \mathbf{H}(K)$, *we require that the functor* $\mathbf{F}$ *(viewed over the category* $\mathfrak{C}_K$*) acts transitively on* $\pi^{-1}(a)$. *When* $\mathbf{F}$ *is in fibration position for* π *we simply write* $\mathbf{F} \rightsquigarrow \mathbf{G} \xrightarrow{\pi} \mathbf{H}$ *and call this a* fibration of functors.

In the following proposition we insist on the fact that all the functors involved *do not necessarly* take values in the category of groups.

Proposition 1.13. *Let* $\mathbf{F} \rightsquigarrow \mathbf{G} \xrightarrow{\pi} \mathbf{H}$ *be a fibration of functors. Then*

$$\mathrm{ed}(\mathbf{G}) \leq \mathrm{ed}(\mathbf{F}) + \mathrm{ed}(\mathbf{H}).$$

Proof. Let K/k a field extension and $a \in \mathbf{G}(K)$. By definition there is a field extension E with $k \subseteq E \subseteq K$, satisfying $\mathrm{trdeg}(E:k) \leq \mathrm{ed}(\mathbf{H})$, and an element $b' \in \mathbf{H}(E)$ such that $b'_K = \pi_K(a)$. Since π_E is surjective there exists $a' \in \mathbf{G}(E)$ such that $\pi_E(a') = b'$. Now clearly $\pi_K(a'_K) = \pi_K(a)$ and thus a'_K and a are in the same fiber. By assumption there exists an element $c \in \mathbf{F}(K)$ such that $a'_K \cdot c = a$. Now there exists an extension E' with $k \subseteq E' \subseteq K$ and $\mathrm{trdeg}(E':k) \leq \mathrm{ed}(\mathbf{F})$ such that c is in the image of the map $\mathbf{F}(E') \longrightarrow \mathbf{F}(K)$. We take $c' \in \mathbf{F}(E')$ such that $c'_K = c$. Considering now the composite extension $E'' = EE'$ and setting $d = a'_{E''} \cdot c'_{E''} \in \mathbf{G}(E'')$ we have, since the action is functorial,

$$d_K = (a'_{E''} \cdot c'_{E''})_K = a'_K \cdot c'_K = a'_K \cdot c = a,$$

and thus

$$\mathrm{ed}(a) \leq \mathrm{trdeg}(E'':k) \leq \mathrm{trdeg}(E:k) + \mathrm{trdeg}(E':k) \leq \mathrm{ed}(\mathbf{H}) + \mathrm{ed}(\mathbf{F}).$$

Since this is true for an arbitrary element a the desired inequality follows.

Remark 1.14. The inequality $\mathrm{ed}(\mathbf{F} \times \mathbf{G}) \leq \mathrm{ed}(\mathbf{F}) + \mathrm{ed}(\mathbf{G})$ is a consequence of this proposition. Indeed for $a \in \mathbf{G}(K)$ the fiber of the projection is $\mathbf{F}(K) \times \{a\}$ and the *set* $\mathbf{F}(K)$ acts transitively by simply setting $x \cdot (y, a) = (x, a)$.

Corollary 1.15. *Let* $1 \longrightarrow \mathbf{F} \longrightarrow \mathbf{G} \longrightarrow \mathbf{H} \longrightarrow 1$ *be a short exact sequence of group-valued functors. Then*

$$\mathrm{ed}(\mathbf{G}) \leq \mathrm{ed}(\mathbf{F}) + \mathrm{ed}(\mathbf{H}).$$

Proof. This is clear since $\mathbf{H}(K) \cong \mathbf{G}(K)/\mathbf{F}(K)$ and the set $\mathbf{F}(K)$ acts transitively on equivalence classes by group multiplication.

Remarks 1.16.

a) One can have $\mathrm{ed}(\mathbf{G}) < \mathrm{ed}(\mathbf{F}) + \mathrm{ed}(\mathbf{H})$ as is shown by the following example: For every field extension K let $\mathbf{F}(K) = K^{\times 2}$ be the subgroup of $\mathbf{G}(K) = K^\times$ consisting of all the squares and $\mathbf{H}(K) = K^\times / K^{\times 2}$ the corresponding quotient (as groups). It is not difficult to see that $\mathrm{ed}(\mathbf{F}) = \mathrm{ed}(\mathbf{G}) = \mathrm{ed}(\mathbf{H}) = 1$, and thus $1 = \mathrm{ed}(\mathbf{G}) < \mathrm{ed}(\mathbf{F}) + \mathrm{ed}(\mathbf{H}) = 2$ (but note that $\mathbf{G} \ncong \mathbf{F} \times \mathbf{H}$ as functors).

b) For a product of functors, since $\mathbf{F} \times \mathbf{G}$ maps onto both $\mathbf{F}$ and $\mathbf{G}$, we have

$$\max\{\mathrm{ed}(\mathbf{F}), \mathrm{ed}(\mathbf{G})\} \leq \mathrm{ed}(\mathbf{F} \times \mathbf{G}) \leq \mathrm{ed}(\mathbf{F}) + \mathrm{ed}(\mathbf{G}).$$

However, even the behaviour of products with respect to essential dimension is not clear. Consider for instance the following two examples :

• Consider the functor $\mathbf{F}_S^d$ of example (9) above. Clearly $\mathbf{F}_S^d \times \mathbf{F}_S^d = \mathbf{F}_{S\times S}^d$ and hence

$$\mathrm{ed}(\mathbf{F}_S^d \times \mathbf{F}_S^d) = \mathrm{ed}(\mathbf{F}_{S\times S}^d) = d = \mathrm{ed}(\mathbf{F}_S^d).$$

Thus it is possible to have $\mathrm{ed}(\mathbf{F} \times \cdots \times \mathbf{F}) = \mathrm{ed}(\mathbf{F})$.

• In contrast with the previous example consider $\mathbf{O}$ the forgetful functor. Then

$$\mathrm{ed}(\underbrace{\mathbf{O} \times \cdots \times \mathbf{O}}_{n \text{ times}}) = n$$

and hence $\mathrm{ed}\big(\prod_{n\in\mathbb{N}} \mathbf{O}\big) = \infty$.

The geometric class of functors introduced in example (3) has an easy essential-dimensional behaviour. This is treated in the following

Proposition 1.17. *Let X be a scheme over k. Then*

$$\mathrm{ed}(X) = \dim(X).$$

Proof. Let K/k and $a \in X(K) = \mathrm{Hom}(\mathrm{Spec}(K), X)$. If x denotes the corresponding point, we have an inclusion $k(x) \hookrightarrow K$, where $k(x)$ is the residue field at x. But

$$\dim(X) = \sup_{x\in X} \mathrm{trdeg}\,\big(k(x) : k\big),$$

hence $\dim(X) = \mathrm{ed}(X)$.

Definition 1.18. *Let $\mathbf{F}$ be an object of $\mathfrak{F}_k$. A* classifying scheme of $\mathbf{F}$ *is a k-scheme X such that there is a surjection $X \twoheadrightarrow \mathbf{F}$.*

Corollary 1.19. *If X is a classifying scheme of $\mathbf{F}$ then*

$$\mathrm{ed}(\mathbf{F}) \leq \dim(X).$$

Proof. This is clear from the definition and the previous considerations.

Examples 1.20.

• Consider $\mathbb{G}_m$ the multiplicative group scheme over k. If $\mathrm{char}(k) \neq 2$, then every quadratic form is diagonalizable, thus there is a surjective morphism of functors $\mathbb{G}_m^n \twoheadrightarrow \mathbf{Q}_n$ given by

$$\begin{aligned} \mathbb{G}_m^n(K) &\twoheadrightarrow \mathbf{Q}_n(K) \\ (a_1, \ldots, a_n) &\longmapsto \langle a_1, \ldots, a_n \rangle. \end{aligned}$$

Hence $\mathbb{G}_m^n$ is a classifying scheme of $\mathbf{Q}_n$. This shows that $\mathrm{ed}_k(\mathbf{Q}_n) \leq n$ if $\mathrm{char}(k) \neq 2$.

• For example (6) above, when k is infinite, there is also a classifying scheme X. Take $A = k[t_1, \ldots, t_n, \frac{1}{d(f)}]$ where $f = x^n + t_1 x^{n-1} + \cdots + t_n$ and $d(f)$ is the discriminant of f. It now suffices to take $X = \mathrm{Spec}(A)$. Hence $\mathrm{ed}_k(\mathbf{\acute{E}t}_n) \leq n$.

• In example (8) we easily see that every homogenous form of degree d with n variables can be written with at most $m = \binom{d+n-1}{n-1}$ coefficients. So one has a very rough classifying scheme $\mathbb{P}^{m-1}$ and thus

$$\mathrm{ed}(\mathbf{F}_{d,n}) \leq m - 1.$$

Moreover there is a fibration of functors

$$X_n \rightsquigarrow \mathbb{P}^{m-1} \twoheadrightarrow \mathbf{F}_{d,n}$$

where X_n is $\mathbf{PGL}_n$ viewed as a scheme over k. Thus, by Proposition 1.13, we have $\mathrm{ed}(\mathbb{P}^{m-1}) \leq \mathrm{ed}(X_n) + \mathrm{ed}(\mathbf{F}_{d,n})$. Since $\mathrm{ed}(\mathbb{P}^{m-1}) = m - 1$ and $\mathrm{ed}(X_n) = n^2 - 1$ it follows that

$$\mathrm{ed}(\mathbf{F}_{d,n}) \geq m - n^2.$$

In the case $n = 2$ one can easily show that $\mathrm{ed}(\mathbf{F}_{d,2}) \leq d - 2$ and the above inequality tells us that $\mathrm{ed}(\mathbf{F}_{d,2}) \geq d - 3$. Hence

$$d - 3 \leq \mathrm{ed}(\mathbf{F}_{d,2}) \leq d - 2.$$

For a discussion of the essential dimension of cubics in few variables, see a forthcoming paper of the authors.

• In example (8) one could have preferred considering homogenous forms only up to $\mathbf{GL}_n$ and not up to a scalar. Denote by $\mathbf{G}_{d,n}$ this new functor. There is a simple relationship between $\mathrm{ed}(\mathbf{F}_{d,n})$ and $\mathrm{ed}(\mathbf{G}_{d,n})$. Indeed there is an obvious surjection of functors

$$\mathbf{G}_{d,n} \twoheadrightarrow \mathbf{F}_{d,n}$$

sending a class modulo $\mathbf{GL}_n$ to its class in $\mathbf{F}_{d,n}$. But the fiber of a form $[f] \in \mathbf{F}_{d,n}(K)$ is clearly the subset $\{[\lambda f] \in \mathbf{G}_{d,n}(K) \,|\, \lambda \in K^\times\}$ and thus $K^\times$ acts transitively on each fiber. We hence obtain a fibration of functors

$$X \rightsquigarrow \mathbf{G}_{d,n} \twoheadrightarrow \mathbf{F}_{d,n}$$

where X is the scheme $\mathbb{A}^1 \setminus \{0\}$ viewed as a functor. This gives the inequality

$$\mathrm{ed}(\mathbf{G}_{d,n}) \leq \mathrm{ed}(\mathbf{F}_{d,n}) + \mathrm{ed}(X) = \mathrm{ed}(\mathbf{F}_{d,n}) + 1.$$

Remark 1.21. In this section all the basic concepts are introduced by Merkurjev in [12] with complete proofs. We have completed these results with Lemma 1.10, Definition 1.12, Corollary 1.15 and some trivial results. The discussion on $\mathbf{F}_{d,n}$ is also new.

2. Galois cohomology

We introduce an important class of functors using Galois cohomology. These functors will be the center of our considerations. Their essential dimension was first introduced by Reichstein, over an algebraically closed field, in terms of compressions. See [14] for details. The standard reference for Galois cohomology is Serre's book [19].

Let G be a k-group scheme (always of finite type). Take K/k a field extension and K_s a separable closure. The group $\Gamma_K = \mathrm{Gal}(K_s/K)$ acts on $G(K_s)$ compatibly with the G-action. The Galois cohomology set $H^1\big(\Gamma_K, G(K_s)\big) =: H^1(K,G)$ is then well defined, i.e. does not depend on the choice of the separable closure. Moreover $H^1(\_,G)$ is a functor in the first variable and thus is an object of $\mathfrak{F}_k$ (see [19] page 83). This allows us to set the following definition.

Definition 2.1. *Let G be a k-group scheme. The* essential dimension of *G is defined as*

$$\mathrm{ed}_k(G) = \mathrm{ed}_k(H^1(\_,G)).$$

A big portion of this paper is dedicated to the study of the essential dimension of certain group schemes. A certain number of techniques are developed in order to estimate it. In the sequel all group schemes are assumed for simplicity to be affine. We will mostly restrict ourselves to algebraic groups over k, that is smooth affine group schemes over k whose Hopf algebra is finitely generated.

We briefly recall the following interpretation of Galois cohomology (see [19] pages 128-129) which shows that many functors $\mathbf{F} : \mathfrak{C}_k \longrightarrow \mathbf{Sets}$ can be viewed as Galois cohomology functors.

Proposition 2.2. *Let (V_0, x_0) be an algebraic structure over k (in the sense of* [19]*). For any field extension K/k let $G(K) = \mathrm{Aut}_K(V_0 \otimes_k K)$ be the group of K-automorphisms which preserve the structure. Then the set $H^1(k,G)$ classifies the k-isomorphism classes of algebraic structures over k which become isomorphic to (V_0, x_0) over a separable closure.*

First examples. It is well known that $H^1(K, \mathbf{GL}_n) = 1$ for every field K. For $n = 1$ this is the so-called *Hilbert 90 Theorem*. Thus $\mathrm{ed}_k(\mathbf{GL}_n) = 0$ for every field k. Moreover the short exact sequence

$$1 \longrightarrow \mathbf{SL}_n \longrightarrow \mathbf{GL}_n \longrightarrow \mathbb{G}_m \longrightarrow 1$$

induces an exact sequence in cohomology showing that $H^1(K, \mathbf{SL}_n) = 1$ for every field K. Thus one also has $\mathrm{ed}_k(\mathbf{SL}_n) = 0$ for every field k. It is also known that $H^1(K, \mathbb{G}_a)$ is trivial for every field K. It follows that $\mathrm{ed}_k(\mathbb{G}_a) = 0$.

Example of $H^1(k, \mathcal{S}_n)$. We consider the symmetric group $G = \mathcal{S}_n$ as a constant group scheme over k.

Take $V_0 = k \times \cdots \times k = k^n$ with its product k-algebra structure. It is easily computed that $\mathcal{S}_n = \mathrm{Aut}_{K\text{-alg}}(V_0 \otimes_k K)$. Thus, by the preceding proposition, we have that $H^1(k, \mathcal{S}_n)$ is the set of isomorphism classes of k-algebras A such that there exists a separable extension L/k with $A \otimes_k L \cong L^n$. It is then easily checked that $H^1(\_, \mathcal{S}_n) \cong \mathbf{\acute{E}t}_n$ as functors and thus

$$\mathrm{ed}_k(\mathcal{S}_n) = \mathrm{ed}_k(\mathbf{\acute{E}t}_n).$$

Galois algebras. Let G any arbitrary finite constant group scheme over k. For any field extension K/k there is a bijection from G-$\mathbf{Alg}(K)$ to $H^1(K, G)$ given as follows: let L be a Galois G-algebra over K. The set E_L of K-algebra homomorphisms $L \to K_s$ is finite with $\dim_K L$ elements. One shows easily that E_L is a principal homogenous space under Γ_K and G. Sending $[L]$ to $[E_L]$ yields a well defined map from G-$\mathbf{Alg}(K)$ to $H^1(K, G)$ which one can show to be a bijection (see [11] for details). Thus G-$\mathbf{Alg} \cong H^1(\_, G)$.

Examples 2.3.

• The group μ_n.

Let k be a field and consider $\mu_n = \mathrm{Spec}\big(k[X]/ < X^n - 1 > \big)$ the k-group scheme of the n-th roots of the unity.

– Suppose that n is prime to the characteristic of k. Then it is well known that for any field extension L/k one has a functorial isomorphism $H^1(L, \mu_n) \cong L^\times / L^{\times n}$. It thus follows that $\mathrm{ed}_k(\mu_n) = 1$.

– If $n = \mathrm{char}(k)$, then μ_n has trivial cohomology and thus $\mathrm{ed}_k(\mu_n) = 0$.

• The group $\mathbb{Z}/p$.

Let k be a field, p a prime number and denote by $\mathbb{Z}/p$ the constant k-group scheme represented by $\mathrm{Spec}(k^{\mathbb{Z}/p})$.

– If $\mathrm{char}(k) \neq p$ and k contains all the p-th roots of unity we can identify the group scheme $\mathbb{Z}/p$ with μ_p by choosing a primitive root of unity. In this case one finds $\mathrm{ed}_k(\mathbb{Z}/p) = 1$. When the field does not contain all the p-th roots of unity, the computation of $\mathrm{ed}_k(\mathbb{Z}/p)$ is much harder as we shall see later.

– When $\mathrm{char}(k) = p$ the situation is easier. The long exact sequence in cohomology induced by the short exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{G}_a \longrightarrow \mathbb{G}_a \longrightarrow 0$$

gives a functorial isomorphism $H^1(L, \mathbb{Z}/p) \cong L/\wp(L)$ where $\wp(x) = x^p - x$ for $x \in L$. It now clearly follows that $\mathrm{ed}_k(\mathbb{Z}/p) = 1$.

Remark 2.4. When $\mathrm{char}(k) = p$, the group $\mathbb{Z}/p^n$ fits into a short exact sequence of k-group schemes analogous to the previous one, but using Witt vectors:

$$0 \longrightarrow \mathbb{Z}/p^n \longrightarrow W_n \longrightarrow W_n \longrightarrow 0$$

where $W_n(k)$ is the additive group of Witt vectors of length n (see [20]). Applying again cohomology and using the fact that $H^1(k, W_n) = 0$, one finds that W_n is a classifying scheme for $\mathbb{Z}/p^n$ and hence

$$\mathrm{ed}_k(\mathbb{Z}/p^n) \leq n.$$

Another proof of the inequality $\mathrm{ed}_k(\mathbb{Z}/p^n) \leq n$ is performed by looking at the exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^{n-1} \longrightarrow 0.$$

It induces a long exact sequence in Galois cohomology but, when the base field k is of characteristic p one has $H^2(K, \mathbb{Z}/p) = 0$ for every extension K/k (see [19] page 86), and thus it reduces to a short exact sequence of group-valued functors

$$0 \longrightarrow H^1(\_, \mathbb{Z}/p) \longrightarrow H^1(\_, \mathbb{Z}/p^n) \longrightarrow H^1(\_, \mathbb{Z}/p^{n-1}) \longrightarrow 0.$$

Then, by Corollary 1.15, one has

$$\mathrm{ed}_k(\mathbb{Z}/p^n) \leq \mathrm{ed}_k(\mathbb{Z}/p^{n-1}) + \mathrm{ed}_k(\mathbb{Z}/p)$$

and, since $\mathrm{ed}_k(\mathbb{Z}/p) = 1$, we are done by induction.

• The circle.

We are interested in the group scheme $S^1 = \mathrm{Spec}\left(k[X,Y]/< X^2 + Y^2 - 1>\right)$ with its usual group structure. We first notice that when -1 is a square and $\mathrm{char}(k) \neq 2$, the rings $k[X,Y]/ < X^2 + Y^2 - 1 >$ and $k[t, t^{-1}]$ are isomorphic. In that case it follows that the algebraic groups S^1 and $\mathbb{G}_m$ are isomorphic and hence $\mathrm{ed}_k(S^1) = 0$. When -1 is not a square we will see that the essential dimension increases.

Actually we will solve the problem for a wider class of algebraic groups.

Let k be a field and L an étale algebra over k. One defines the group scheme $\mathbb{G}^1_{m,L}$ by the exact sequence

$$1 \longrightarrow \mathbb{G}^1_{m,L} \longrightarrow \mathrm{R}_{L/k}(\mathbb{G}_{m,L}) \xrightarrow{N_{L/k}} \mathbb{G}_m \longrightarrow 1,$$

where $\mathrm{R}_{L/k}$ denotes the Weil restriction (see [11] p.329 where it is called corestriction).

In the sequel, we will prove the following result:

Theorem 2.5. *Let L/k be an étale algebra of dimension $n \geq 1$. Then*

$$\mathrm{ed}_k(\mathbb{G}^1_{m,L}) = \begin{cases} 0 & \text{if } L \text{ is isomorphic to a product of field} \\ & \text{extensions of relatively prime degrees} \\ 1 & \text{otherwise.} \end{cases}$$

The above sequence induces, for any extension K/k, the exact sequence in cohomology

$$(L \otimes K)^\times \xrightarrow{N_K} K^\times \longrightarrow H^1(K, \mathbb{G}^1_{m,L}) \longrightarrow 1$$

where N_K is a short notation for $N_{L\otimes K/K}$. This gives an isomorphism

$$H^1(K, \mathbb{G}^1_{m,L}) \simeq K^\times / N_{L\otimes K/K}(L \otimes K)^\times.$$

In particular one has $\mathrm{ed}_k(\mathbb{G}^1_{m,L}) \leq 1$ for every field k.

Since the case $n = 1$ is trivial, we may assume until the end of this section that $n \geq 2$.

We start with the following lemma:

Lemma 2.6. *Let k be a field, let L be a finite dimensional étale k-algebra of dimension $n \geq 2$, and let t be a transcendental element over k. Then t belongs to the norm group of $L \otimes k(t)/k$ if and only if L is isomorphic to a product of some finite separable field extensions of k those degrees are relatively prime.*

Proof. Assume that there exists $\alpha \in L \otimes k(t)$ such that $N_{L\otimes k(t)/k(t)}(\alpha) = t$. In the sequel, we will write $L(t)$ instead of $L \otimes k(t)$ in order to simplify notation. Write $\alpha = \dfrac{1}{Q(t)} \cdot \displaystyle\sum_{i=0}^{m} \lambda_i t^i$, for some $\lambda_i \in L$, with $\lambda_m \neq 0$ and some nonzero polynomial $Q(t) \in k[t]$ of degree $d \geq 0$. Assume first that L is a field. Then $L(t)/k(t)$ is again a separable field extension, and we have

$$Q(t)^n t = N_{L(t)/k(t)}(Q(t) \cdot \alpha) = \prod_\sigma \left(\sum_{i=0}^{m} \sigma(\lambda_i) \otimes t^i \right),$$

where σ describes $\mathrm{Hom}_k(L, k_s)$. Since L is a field and $\lambda_m \neq 0$, the leading coefficient of the right hand side term is equal to $N_{L/k}(\lambda_m) t^{mn}$. Since $Q(t)^n t$ is a polynomial of degree $nd + 1$ and $n \geq 2$, we get a contradiction.
Hence $L \simeq L_1 \times \cdots \times L_r$ for $r \geq 2$, where L_i/k is a finite separable field extension of degree n_i. We then have

$$t = N_{L_1(t)/k(t)}(\alpha_1) \cdots N_{L_r(t)/k(t)}(\alpha_r)$$

for some $\alpha_i \in L_i(t)^\times$. As above write $\alpha_i = \dfrac{1}{Q_i(t)} \cdot \displaystyle\sum_{j=0}^{m_i} \lambda_j^{(i)} \otimes t^j$, where $\lambda_{m_i}^{(i)} \neq 0$.

Since L_i is a field, the computation above shows that the leading coefficient of $Q_1(t)^{n_1} \cdots Q_r(t)^{n_r} t$ is

$$N_{L_1/k}(\lambda^{(1)}_{m_1}) t^{m_1 n_1} \cdots N_{L_r/k}(\lambda^{(r)}_{m_r}) t^{m_r n_r},$$

which has degree $m_1 n_1 + \cdots + m_r n_r$. By assumption, this degree is equal to $1 + n_1 d_1 + \cdots + n_r d_r$. It follows immediately that the n_i's are relatively prime. The converse is clear.

We now prove Theorem 2.5. Assume first that $\mathrm{ed}(\mathbb{G}^1_{m,L}) = 0$. Then the class of t in $H^1(k(t), \mathbb{G}^1_{m,L})$ is defined over k. That is there exist an element $a \in k$ such that $t = a N_{k(t)}(\alpha)$ for some $\alpha \in L \otimes k(t)$. Then $u = \frac{t}{a}$ is a transcendental element over k which belongs to the norm group of $L \otimes k(u)$. Applying the previous lemma shows that L is isomorphic to a product of some finite separable field extensions of k those degrees are relatively prime. Conversely, if L is isomorphic to a product of some finite separable field extensions of k those degrees are relatively prime, then one can easily see that N_K is surjective for any field extension K/k, so $\mathrm{ed}(\mathbb{G}^1_{m,L}) = 0$.

Corollary 2.7. *Let k be a field. Then*

$$\mathrm{ed}_k(S^1) = \begin{cases} 1 & \textit{if } \mathrm{char}(k) \neq 2 \textit{ and } -1 \notin k^{\times 2} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $\mathrm{char}(k) \neq 2$, apply the previous theorem with $L = k[X]/(X^2 - 1)$. If $\mathrm{char}(k) = 2$, it is easy to see that for any field extension K/k, we have $S^1(K_s) = \{(x, x+1) \mid x \in K_s\}$. In particular $S^1(K_s) \simeq K_s$ as Galois modules and $H^1(\_, S^1) = 0$, showing that $\mathrm{ed}_k(S^1) = 0$.

Remark 2.8. In this section new results are Remark 2.4, Theorem 2.5 and Corollary 2.7.

3. Cohomological invariants

One way of giving lower bounds of essential dimension of functors is to use cohomological invariants. This idea can be found in [14]. The advantage of Merkurjev's functorial point of view is that the definitions are natural and that one could in theory apply these methods to a broader class of invariants.

Definition 3.1. *Let* $\mathbf{F}$ *be an object of* $\mathfrak{F}_k$ *and* $n \geq 1$ *an integer. We say that* $\mathbf{F}$ *is* n-simple *if there exists a field extension* $\tilde{k}/k$ *such that for any extension* $K/\tilde{k}$ *with* $\mathrm{trdeg}(K : \tilde{k}) < n$ *the set* $\mathbf{F}(K)$ *consists of one element.*

Example 3.2. Let M be a discrete torsion Γ_k-module and $n \geq 1$ an integer. Then it is known that $H^n(K, M) = 0$ if K contains an algebraically closed field and is of transcendence degree $< n$ over this field (see [19], Proposition 11, page 93). Taking for $\tilde{k}$ an algebraic closure of k one sees that $H^n(\_, M)$ is n-simple.

Definition 3.3. *A morphism of functors $f : \mathbf{F} \to \mathbf{G}$ is called* non-constant *if for any field extension K/k there exists an extension L/K and elements $a \in \mathbf{F}(K)$, $a' \in \mathbf{F}(L)$ such that $f_L(a_L) \neq f_L(a')$.*

Proposition 3.4. *Let $f : \mathbf{F} \to \mathbf{G}$ be a non-constant morphism and suppose that $\mathbf{G}$ is n-simple. Then $\mathrm{ed}_k(\mathbf{F}) \geq n$.*

Proof. Let $\tilde{k}$ be the field in the definition of n-simplicity of $\mathbf{G}$. Suppose that $\mathrm{ed}_k(\mathbf{F}) < n$. Since $\mathrm{ed}_{\tilde{k}}(\mathbf{F}) \leq \mathrm{ed}_k(\mathbf{F})$ one has $\mathrm{ed}_{\tilde{k}}(\mathbf{F}) < n$ too. Since f is non-constant there exists an extension $L/\tilde{k}$ and elements $a \in \mathbf{F}(\tilde{k})$, $a' \in \mathbf{F}(L)$ such that $f_L(a_L) \neq f_L(a')$. Since $\mathrm{ed}_{\tilde{k}}(\mathbf{F}) < n$ there exists a subextension $\tilde{k} \subseteq E \subseteq L$ of transcendence degree $< n$ over $\tilde{k}$ such that $a' \in \mathrm{im}(\mathbf{F}(E) \longrightarrow \mathbf{F}(L))$ that is $a' = a''_L$ for some $a'' \in \mathbf{F}(E)$.
Since the diagram

$$\begin{tikzcd}
\mathbf{F}(L) \arrow[r, "f_L"] & \mathbf{G}(L) \\
\mathbf{F}(E) \arrow[r, "f_E"] \arrow[u] & \mathbf{G}(E) \arrow[u] \\
\mathbf{F}(\tilde{k}) \arrow[r, "f_{\tilde{k}}"] \arrow[u] & \mathbf{G}(\tilde{k}) \arrow[u]
\end{tikzcd}$$

is commutative, and since $f_L(a_L) \neq f_L(a')$ it follows that $f_E(a_E) \neq f_E(a'')$. This contradicts the fact that $\mathbf{G}(E)$ consists of one element.

Definition 3.5. *Let k be a field and $\mathbf{F}$ be a covariant functor from $\mathfrak{C}_k$ to the category of pointed sets. A* cohomological invariant of degree n of $\mathbf{F}$ *is a morphism of pointed functors $\varphi : \mathbf{F} \longrightarrow H^n(\_, M)$, where M is a discrete torsion Γ_k-module. (Here $H^n(\_, M)$ is pointed by 0, the class of the trivial cocycle.) We say that it is* non-trivial *if for any extension K/k there exists $L \supseteq K$ and $a \in \mathbf{F}(L)$ such that $\varphi_L(a) \neq 0$ in $H^n(L, M)$.*

Corollary 3.6. *Let k be an arbitrary field and $\mathbf{F}$ be a functor from $\mathfrak{C}_k$ to the category of pointed sets. If $\mathbf{F}$ has a non-trivial cohomological invariant φ of degree n, then $\mathrm{ed}_k(\mathbf{F}) \geq n$.*

Proof. Clearly any non-trivial cohomological invariant is a non-constant morphism.

We will apply the above corollary to a special class of algebraic groups: finite constant abelian groups. Recall that such a group G can always be written as $G \cong \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_n$ where $d_1 \mid d_2 \mid \cdots \mid d_n$. The number n is called the rank of G and is denoted by $\operatorname{rank}(G)$.

Proposition 3.7. *Let G be a finite abelian group and k a field such that* $\operatorname{char}(k) \nmid \exp(G)$. *Then* $\operatorname{ed}_k(G) \geq \operatorname{rank}(G)$.

Proof. For the proof one can suppose that k is algebraically closed. We will define a cohomological invariant φ of degree n for $H^1(\_, G)$. There is an isomorphism

$$H^1(K,G) \cong H^1(K, \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_n) \cong H^1(K, \mathbb{Z}/d_1) \times \cdots \times H^1(K, \mathbb{Z}/d_n),$$
$$c \longmapsto (c_1, \ldots, c_n)$$

which, composed with the cup product

$$H^1(K, \mathbb{Z}/d_1) \times \cdots \times H^1(K, \mathbb{Z}/d_n) \to H^n(K, \mathbb{Z}/d_1 \otimes \cdots \otimes \mathbb{Z}/d_n)$$
$$(c_1, \ldots, c_n) \longmapsto c_1 \cup \cdots \cup c_n$$

defines a cohomological invariant

$$\varphi : H^1(\_, G) \longrightarrow H^n(\_, \mathbb{Z}/d_1)$$

since $\mathbb{Z}/d_1 \otimes \cdots \otimes \mathbb{Z}/d_n \cong \mathbb{Z}/d_1$. It suffices to show that it is non-trivial. We have to show that, for a field extension K/k, there exists $L \supseteq K$ and $a \in H^1(L, G)$ such that $\varphi_L(a) \neq 0$. We take $L = K(t_1, \ldots, t_n)$ and set (t_i)= class of t_i in $L^\times / L^{\times d_i} \cong H^1(L, \mathbb{Z}/d_i)$ (this isomorphism holds since k is algebraically closed). Then, the image of

$$a = \big((t_1), \ldots, (t_n)\big) \in H^1(L, \mathbb{Z}/d_1) \times \cdots \times H^1(L, \mathbb{Z}/d_n) \cong H^1(L, G)$$

is the element $\varphi(a) = (t_1) \cup \cdots \cup (t_n) \in H^n(L, \mathbb{Z}/d_1)$. We show that this element is $\neq 0$ by induction on n :

– For $n = 1$, $(t_1) \in K(t_1)^\times / K(t_1)^{\times d_1}$ is clearly non-zero.

– Suppose that $n > 1$:

We use a more general fact (see [1]). If K is a field equipped with a discrete valuation $\upsilon : K^\times \longrightarrow \mathbb{Z}$, then there is the so-called residue homomorphism

$$\partial_\upsilon : H^n(K, \mathbb{Z}/d) \longrightarrow H^{n-1}(\kappa(\upsilon), \mathbb{Z}/d)$$

where $\kappa(\upsilon)$ denotes the residue field of υ. This homomorphism has the following property :

If $\upsilon(a_1) = \cdots = \upsilon(a_{n-1}) = 0$ and $\upsilon(a_n) = 1$ (i.e. $a_i \in \mathcal{O}_\upsilon^\times$ for $i < n$) then

$$\partial_\upsilon\big((a_1) \cup \cdots \cup (a_{n-1}) \cup (a_n)\big) = (\overline{a}_1) \cup \cdots \cup (\overline{a}_{n-1}) \in H^{n-1}(\kappa(\upsilon), \mathbb{Z}/d)$$

where $\overline{a}_i$ is the class of a_i in $\mathcal{O}_\upsilon / \mathfrak{m}_\upsilon = \kappa(\upsilon)$.

In our case, we take for v the t_n-adic valuation on L. We thus have

$$\partial_v\big((t_1)\cup\cdots\cup(t_n)\big)=(t_1)\cup\cdots\cup(t_{n-1})\in H^{n-1}\big(K(t_1,\ldots,t_{n-1}),\mathbb{Z}/d_1\big).$$

By induction hypothesis this element is non-zero, hence $(t_1)\cup\cdots\cup(t_n)\neq 0$ and $\mathrm{ed}(G)=n$.

Remark 3.8. This shows that $\mathrm{ed}_k(G)\geq\mathrm{rank}_p(G)$ for any field k with $\mathrm{char}(k)\neq p$. Here $\mathrm{rank}_p(G)$ denotes the rank of the largest p-elementary subgroup of G.

If $\mathrm{char}(k)=p$ this result is no longer true. Indeed, consider the group $\mathbb{Z}/p\times\cdots\times\mathbb{Z}/p$ (n copies). If one takes for k a field containing $\mathbb{F}_{p^n}$ there is a short exact sequence

$$0\to\mathbb{Z}/p\times\cdots\times\mathbb{Z}/p\to\mathbb{G}_a\to\mathbb{G}_a\to 0$$

where the map $\mathbb{G}_a\to\mathbb{G}_a$ is given by $x\mapsto x^p-x$. This gives in cohomology an exact sequence

$$\mathbb{G}_a(K)\to H^1(K,\mathbb{Z}/p\times\cdots\times\mathbb{Z}/p)\to\underbrace{H^1(K,\mathbb{G}_a)}_{=0}\to\cdots$$

Thus $\mathbb{G}_a$ is a classifying scheme for $\mathbb{Z}/p\times\cdots\times\mathbb{Z}/p$, when the field k contains $\mathbb{F}_{p^n}$, and it follows that $\mathrm{ed}_k(\mathbb{Z}/p\times\cdots\times\mathbb{Z}/p)=1$.

COROLLARY 3.9. *Let n be an integer and k a field with $\mathrm{char}(k)\nmid n$. Then*

$$\mathrm{ed}_k(\underbrace{\mu_n\times\cdots\times\mu_n}_{r\ times})=r.$$

PROOF. Since $H^1(K,\mu_n\times\cdots\times\mu_n)=K^\times/K^{\times n}\times\cdots\times K^\times/K^{\times n}$, one has a surjection of functors

$$\mathbb{G}_m\times\cdots\times\mathbb{G}_m\twoheadrightarrow H^1(\_,\mu_n\times\cdots\times\mu_n)$$

and thus $\mathrm{ed}_k(\mu_n\times\cdots\times\mu_n)\leq r$. For the opposite inequality it suffices to remark that over an algebraic closure the group $\mu_n\times\cdots\times\mu_n$ is isomorphic to the constant group $\mathbb{Z}/n\times\cdots\times\mathbb{Z}/n$ and apply Proposition 3.7.

Applying the same cohomological-invariant techniques to quadratic forms one can prove the following result which can be found in [14].

THEOREM 3.10. *Assume that* $\mathrm{char}(k)\neq 2$. *Then* $\mathrm{ed}_k(\mathbf{Q}_n)=n$.

Proof. We have already shown that $\mathrm{ed}(\mathbf{Q}_n) \leq n$. We prove that $\mathrm{ed}(\mathbf{Q}_n) = n$ using a non-trivial cohomological invariant: the Delzant's Stiefel-Whitney class (see [5]) denoted by ω_n.

For any field extension K/k take $L = K(t_1, \ldots, t_n)$ and let $q = \langle t_1, \ldots, t_n \rangle$. One has $\omega_n(q) = (t_1) \cup \cdots \cup (t_n) \in H^n(L, \mathbb{Z}/2)$ which is non-zero, as it was checked before. Hence ω_n is a non-trivial cohomological invariant of degree n. It follows that $\mathrm{ed}(\mathbf{Q}_n) = n$.

One of the most interesting features of the use of cohomological invariants is the following application to the symmetric group. This was originally found in [3].

Corollary 3.11. *If* $\mathrm{char}(k) \neq 2$ *one has* $\mathrm{ed}(\mathcal{S}_n) \geq \left[\frac{n}{2}\right]$.

Proof. We have already seen that $H^1(K, \mathcal{S}_n) = \mathbf{\acute{E}t}_n(K)$. By Proposition 1.5, one can assume that k is algebraically closed. Consider now the functorial morphism

$$\begin{aligned} \mathbf{\acute{E}t}_n(K) &\longrightarrow \mathbf{Q}_n(K) \\ A &\longmapsto \left(\mathcal{T}_{A/K} : x \mapsto \mathrm{Tr}_{A/K}(x^2)\right) \end{aligned}$$

and $\omega_m : \mathbf{Q}_n(K) \longrightarrow H^m(K, \mathbb{Z}/2)$ with $m = \left[\frac{n}{2}\right]$. We show that the composite

$$\mathbf{\acute{E}t}_n(K) \longrightarrow H^m(K, \mathbb{Z}/2)$$

is a non-trivial cohomological invariant. For any field extension K/k take $L = K(t_1, \ldots, t_m)$ and let

$$A \cong \begin{cases} L(\sqrt{t_1}) \times \cdots \times L(\sqrt{t_m}) & \text{if } n = 2m \\ L(\sqrt{t_1}) \times \cdots \times L(\sqrt{t_m}) \times L & \text{if } n = 2m+1. \end{cases}$$

Clearly the matrix of the trace form expressed in the basis $\{1, \sqrt{t}\}$ is $\begin{pmatrix} 2 & 0 \\ 0 & 2t_i \end{pmatrix}$. Hence

$$\begin{aligned} \mathcal{T}_{A/L} &\simeq \begin{cases} \langle 2, 2t_1, \ldots, 2, 2t_m \rangle & \text{if } n = 2m \\ \langle 2, 2t_1, \ldots, 2, 2t_m, 1 \rangle & \text{if } n = 2m+1 \end{cases} \\ &\simeq \langle t_1, \ldots, t_m \rangle \perp \langle 1, \ldots, 1 \rangle, \end{aligned}$$

since k is algebraically closed. Thus

$$\omega_m(\mathcal{T}_{A/L}) = \omega_m(\langle t_1, \ldots, t_m, 1, \ldots, 1 \rangle) = (t_1) \cup \cdots \cup (t_m) \neq 0.$$

4. Free actions and torsors

We recall here some facts about actions of group schemes and torsors in order to estimate $\mathrm{ed}(G)$. The main reference is the book of Demazure-Gabriel [6].

Let G be a group scheme over a scheme S and let X be an S-scheme. We say that G ACTS ON X if there is a morphism of S-schemes

$$\begin{aligned} G \times_S X &\longrightarrow X \\ (g, x) &\longmapsto x \cdot g \end{aligned}$$

which satisfy the categorical conditions of a usual group (right) action. It follows in particular that for any morphism $T \to S$ there is an action of the group $G(T)$ on the set $X(T)$.

Recall that a group G acts freely on a set X if the stabilizer of any point of X is trivial. One can mimic this and say that a group scheme G acts FREELY on a scheme X if for any S-scheme $T \longrightarrow S$ the group $G(T)$ acts freely on the set $X(T)$. One can also define the stabilizer of a point of X in the following way: Let $x \in X$ be any point. The SCHEME-THEORETIC STABILIZER OF x is the pull-back of the diagram

$$\begin{tikzcd} & G \times_S \{x\} \arrow[d] \\ \mathrm{Spec}(k(x)) \arrow[r, "x"] & X \end{tikzcd}$$

where the vertical map is the composite $G \times_S \{x\} \longrightarrow G \times_S X \longrightarrow X$. We denote it by G_x. It is a group scheme over $\mathrm{Spec}(k(x))$ and is a closed group subscheme of $G \times_S \{x\}$.

Once the vocabulary is established one has the following lemma.

LEMMA 4.1. *Let X and G be as above, everything being of finite type over $S = \mathrm{Spec}(k)$. Then the following are equivalent*
(i) G acts freely,
(ii) $G_x = \{1\}$ for all points $x \in X$.

PROOF. See [6], III, §2 Corollary 2.3.

One can also check these conditions on $\bar{k}$-points, where $\bar{k}$ is an algebraic closure of k.

Recall first that, for an algebraic group G over k, the Lie algebra can be defined as the kernel of the map $G(k[\tau]) \to G(k)$ where $k[\tau]$ is the algebra $k[t]/t^2$ and the map $k[\tau] \to k$ is given by $\tau \mapsto 0$. Let x be a point of a scheme X and denoted by $K = k(x)$ its residue field. The point x is then viewed as an element of $X(K) = \mathrm{Hom}(\mathrm{Spec}(K), X)$ and thus also as an element of $X(K[\tau])$ which we will denote by $x_{K[\tau]}$.

Lemma 4.2. *Let G be a group scheme of finite type over k acting on a k-scheme X of finite type.*

(i) *Suppose* $\mathrm{char}(k) = 0$*. Then G acts freely on X if and only if the group $G(\bar{k})$ acts freely on $X(\bar{k})$.*

(i') *Suppose* $\mathrm{char}(k) > 0$*. Then G acts freely on X if and only if the group $G(\bar{k})$ acts freely on $X(\bar{k})$, and for any closed point $x \in X$ the Lie algebra* $\mathrm{Lie}(G_x)$ *is trivial.*

Proof. See [6], III, §2 Corollary 2.5 and Corollary 2.8. The Lie algebra $\mathrm{Lie}(G_x)$ is called the Lie stabilizer of x.

Remark 4.3. The second part of condition (i') can checked easily using the following description of $\mathrm{Lie}(G_x)$ (see [6], III, §2, proof of Prop. 2.6.): let K be the residue field of x, and let $K[\tau]$ be the K-algebra $K[X]/(X^2)$. Then we have

$$\mathrm{Lie}(G_x) = \{g \in \mathrm{Lie}(G) \otimes K[\tau] \mid g \cdot x_{K[\tau]} = x_{K[\tau]}\}.$$

Remark 4.4. Let G act on X as above. For every scheme T consider the quotient map of sets $\pi : X(T) \longrightarrow Y(T) := X(T)/G(T)$. Sending a pair $(g, x) \in G(T) \times X(T)$ to $(x, x \cdot g)$ gives a mapping

$$G(T) \times X(T) \longrightarrow X(T) \times_{Y(T)} X(T).$$

If G acts freely this map is easily seen to be an isomorphism. It also says that the fibers of π are principal homogenous spaces under $G(T)$ (at least when they are non-empty). The notion of G-torsor generalizes this remark in the category of schemes and is the suitable definition for defining "parametrized" principal homogenous spaces.

Definition 4.5. *Let G be a group scheme over Y which is flat and locally of finite type over Y. We say that a morphism of schemes $X \to Y$ is a* (flat) G-torsor over Y *if G acts on X, the morphism $X \to Y$ is flat and locally of finite type, and the map $\varphi : G \times_Y X \to X \times_Y X$ defined by*

$$G \times_Y X \to X \times_Y X$$

$$(g, x) \mapsto (x, x \cdot g)$$

is an isomorphism.

This condition is equivalent to the existence of a covering $(U_i \to Y)$ for the flat topology on Y such that $X \times_Y U_i$ is isomorphic to $G \times_Y U_i$ for each i (see [13], Chapter III, Proposition 4.1). This means that X is "locally" isomorphic to G for the flat topology on Y. When the group G is smooth over Y it follows by faithfully flat descent that X is also smooth.

A morphism between two G-torsors $f : X \to Y$ and $f' : X' \to Y$ defined over the same base is simply a G-equivariant morphism $\varphi : X \to X'$ such that $f' \circ \varphi = f$. Again by faithfully flat descent it follows that any morphism between G-torsors is an isomorphism.

Remark 4.6. Notice that if $X \to Y$ is a G-torsor, then G acts freely on X. Indeed, take $x \in X$, then the fiber of the point $(x, x) \in X \times_Y X$ under the map $\varphi : G \times_Y X \to X \times_Y X$ is isomorphic to G_x. Since φ is an isomorphism it follows that G_x is trivial for every x.

We then consider the *contravariant* functor

$$G\text{-}\mathbf{Tors} : \text{Schemes} \longrightarrow \mathbf{Sets},$$

defined by

$$G\text{-}\mathbf{Tors}(Y) = \text{ isomorphism classes of } G\text{-torsors over } Y.$$

For every morphism $f : Y' \longrightarrow Y$ the corresponding map G-$\mathbf{Tors}(f)$ is defined as follows: if $X \to Y$ a G-torsor over Y, then the image of this torsor under G-$\mathbf{Tors}(f)$ is the pull-back of the diagram

$$\begin{tikzcd} & X \arrow[d] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

which is easily checked to be a G-torsor over Y'.

When Y is a point, say $Y = \mathrm{Spec}(K)$, and G is smooth over K then any G-torsor $X \to \mathrm{Spec}(K)$ gives rise to a principal homogeneous space over K. Indeed X is smooth and thus $X(K_s) \neq \emptyset$ is a principal homogenous space under $G(K_s)$, thus an element of $H^1(K, G)$. We may thus consider G-$\mathbf{Tors}$ as a generalization of the first Galois cohomology functor over the category of fields.

Now that the notion of torsor is well-defined we have to overcome the problem of quotients.

Let G act on a S-scheme X. A morphism $\pi : X \longrightarrow Y$ is called a CATEGORICAL QUOTIENT of X by G if π is (isomorphic to) the *push-out* of the diagram

$$\begin{array}{ccc} G \times_S X & \longrightarrow & X \\ {\scriptstyle \mathrm{pr}_2}\big\downarrow & & \\ X & & \end{array}$$

In general such a quotient does not exist in the category of schemes. When it exists the scheme Y is denoted by X/G. We will not give a detailed account on the existence of quotients. We will only need the existence of a *generic quotient*, that is a G-invariant dense open subscheme U of X for which the quotient $U \longrightarrow U/G$ exists. Moreover, we will need one non-trivial fact due to P. Gabriel (which can be found in [7]) which asserts the existence of a generic quotient which is also a G-torsor.

THEOREM 4.7. *Let G act freely on a S-scheme of finite type X such that the second projection $G \times_S X \to X$ is flat and of finite type. Then there exists a (non-empty) G-invariant dense open subscheme U of X satisfying the following properties:*

i) There exists a quotient map $\pi : U \longrightarrow U/G$ in the category of schemes.

ii) π is onto, open and U/G is of finite type over S.

iii) $\pi : U \longrightarrow U/G$ is a flat G-torsor.

PROOF. This follows from [7], Exposé V, Théorème 8.1, p.281 where the statement is much more general and deals with groupoids. In order to recover it we make a translation: in our context the groupoid is that of §2 Exemple a) p.255 which simply defines the equivalence relation on the scheme X under the G-action. The fact that our action is free implies that the morphism $G \times_S X \longrightarrow X \times_S X$ is quasi-finite, which is one of the hypotheses of Théorème 8.1.

We thank J.-P. Serre for pointing out to us this result and an alternative proof which can be found in a paper of Thomason ([22]).

DEFINITION 4.8. *Let G act on X. An open subscheme U which satisfies the conclusion of the above theorem will be called a* FRIENDLY *open subscheme of X.*

From now on take $S = \mathrm{Spec}(k)$ where k is a field and G an algebraic group over k, that is we require G to be *smooth* and of finite type over k, and all the morphisms between schemes will be of finite type. Unless otherwise specified, when we say that $X \longrightarrow Y$ is a G-torsor we mean that $X \longrightarrow Y$ is a G_Y-torsor where G_Y is the group scheme obtained from G by base change $Y \to \mathrm{Spec}(k)$. In this case this says that there is an isomorphism $G \times_k X \simeq X \times_Y X$.

Definition 4.9. *Let $\pi : X \to Y$ be a G-torsor. For any field extension K/k we define a map*

$$\partial : Y(K) \longrightarrow H^1(K, G)$$

as follows: for any $y \in Y(K)$, the fiber X_y of $\pi : X \to Y$ at y is a twisted form of G (that is locally isomorphic to G for the flat topology) and thus smooth over K. Hence X_y has a K_s-rational point x. We then set $\partial(y) =$ isomorphism class of $X_y(K_s)$.

We can paraphrase the definition in terms of cocycles: for all $\gamma \in \Gamma_K$ we have

$$\pi(\gamma \cdot x) = \gamma \cdot \pi(x) = \gamma \cdot y = y.$$

Hence $\gamma \cdot x$ belongs to $X_y(K_s)$. Since $X \to Y$ is a G-torsor, there exists a unique $g(\gamma) \in G(K_s)$ such that $\gamma \cdot x = x \cdot g(\gamma)$. The assignment $\gamma \longmapsto g(\gamma)$ is then a 1-cocycle and the map ∂ sends y to the class of that cocycle in $H^1(K, G)$.

Definition 4.10. *We say that G acts* generically freely *on X if there exists a non-empty G-stable open subscheme U of X on which G acts freely.*

The previous considerations show in particular that, if G acts generically freely on X, then there exists a friendly open subscheme $U \subset X$ on which G acts freely (take for U the intersection of a dense open subset on which G acts freely and a friendly open subscheme). Hence the statement of the following proposition is consistent.

Proposition 4.11. *Let G be an algebraic group over k acting linearly and generically free on an affine space $\mathbb{A}(V)$, where V is a finite dimensional k-vector space. Let U be a non-empty friendly open subscheme of $\mathbb{A}(V)$ on which G acts freely. Then U/G is a classifying scheme of $H^1(\_, G)$. In particular we have*

$$\mathrm{ed}(G) \leq \dim(V) - \dim(G).$$

Proof. It is sufficient to show that, for any field extension K/k, the map $\partial : U/G(K) \longrightarrow H^1(K, G)$ is surjective. Let $g \in Z^1(K, G)$. We twist the action of Γ_K over $V(K_s)$ by setting

$$\gamma * v = \gamma \cdot v \cdot g(\gamma)^{-1}$$

for all $\gamma \in \Gamma_K$ and $v \in V(K_s)$. Clearly this action is Γ_K-semilinear, that is $\gamma * (\lambda v) = \gamma(\lambda)(\gamma * v)$ for all $\lambda \in K_s$. Hence $V(K_s)^{(\Gamma_K, *)}$ is Zariski-dense in $V(K_s)$. Since U is open, there exists an invariant point $v_0 \in U(K_s)$ for the new action $*$. We thus have

$$v_0 = \gamma * v_0 = \gamma \cdot v_0 \cdot g(\gamma)^{-1}$$

and hence $v_0 \cdot g(\gamma) = \gamma \cdot v_0$. In particular, we have for any $\gamma \in \Gamma_K$

$$\gamma \cdot \pi(v_0) = \pi(\gamma \cdot v_0) = \pi(v_0 \cdot g(\gamma)) = \pi(v_0),$$

hence $\pi(v_0) \in U/G(K)$ and maps to g under ∂.

Remark 4.12. Any algebraic group G acts linearly and generically freely over some vector space. Indeed, since G is isomorphic to a closed subgroup of some $\mathbf{GL}_n$, one can assume that $G \subset \mathbf{GL}_n$. Let $V = \mathrm{M}_n(k)$. The group G then acts linearly on $\mathbb{A}(V)$ by (right) matrix multiplication. Now let $U = \mathbf{GL}_n$, viewed as an open subscheme of $\mathbb{A}(V)$. Clearly, the stabilizer of any matrix $M \in U(\bar{k})$ is trivial. Moreover, since the action of $\mathrm{Lie}(G)$ is obtained by restriction of the action of $G(k[\tau])$ (where $\tau^2 = 0$), the Lie stabilizer of any closed point of U is also trivial. Hence G acts freely on U. The previous proposition then shows that the essential dimension of G is finite.

Our next aim is to deal with finite group schemes. Recall that a group scheme over k is called ÉTALE if its Hopf algebra is a finitely generated separable algebra over k (see [6] p.234–238 for an account on étale group schemes).

Proposition 4.13. *Let G be an étale group scheme over k and let V be a finite dimensional k-vector space. Then*
i) G acts linearly and generically freely on $\mathbb{A}(V)$ if and only if G is isomorphic to a closed subgroup of $\mathbf{GL}(V)$.
ii) G acts linearly and generically freely on $\mathbb{P}(V)$ if and only if G is isomorphic to a closed subgroup of $\mathbf{PGL}(V)$.

Proof. We only prove the statement ii) since i) is similar. We have to find an open subscheme U of $\mathbb{P}(V)$ such that the group G acts freely on U. We first consider the action of $G(k_s)$ on $\mathbb{P}(V)(k_s)$. For each $g \in G(k_s)$ consider the linear subspace $S_g = \{x \in \mathbb{P}(V)(k_s) \mid g \cdot x = x\}$ and let $S = \bigcup_{g \in G(k_s)} S_g$. This is an algebraic subvariety of $\mathbb{P}(V)(k_s)$ which is invariant under the absolute Galois group Γ_k. By descent theory (see [23] pp.131-138) there exists a closed subscheme X of $\mathbb{P}(V)$ defined over k such that $X(k_s) = S$. Moreover, always by descent theory, the group scheme G acts on X since $G(k_s)$ acts on $X(k_s) = S$. The desired open subscheme is then $U = \mathbb{P}(V) \setminus X$. To prove this, by Lemma 4.1, we have to check that for all points $x \in U$ the stabilizer G_x is trivial. By construction we have that $G_x(k_s) = 1$ for all $x \in U$. But G is étale and hence G_x too. It then follows that $G_x = 1$.

We now study more carefully the case of finite *constant* group schemes. The following lemma is probably well-known, but we have not found any reference for it, so we give a proof for the convenience of the reader.

Lemma 4.14. *Let G be a constant group scheme, and let H be any algebraic group scheme defined over k. Then the map*

$$\mathrm{Hom}(G, H) \to \mathrm{Hom}(G(k), H(k))$$

sending $\Phi \in \mathrm{Hom}(G, H)$ to Φ_k is a bijection. Moreover, Φ is injective if and only if Φ_k is injective.

Proof. Given a morphism $\varphi : G(k) \longrightarrow H(k)$, we have to show that there exists a unique morphism of group schemes $\Phi : G \longrightarrow H$ such that $\Phi_k = \varphi$. We thus have to define, in a natural way, a group homomorphism $\Phi_R : G(R) \longrightarrow H(R)$ for every k-algebra R. Since $G(\prod R_i) = \prod G(R_i)$ and since every commutative ring is product of connected rings one may assume that R is connected. In this case, since G is constant, one has $G(R) = G = G(k)$ and one then defines Φ_R to be the composite $G(R) = G(k) \to H(k) \to H(R)$. This proves the first part of the statement.

Since Φ is a natural map and $G(\bar{k}) = G(k)$, it follows that $\Phi_{\bar{k}}$ is the composite of Φ_k and of the inclusion $H(k) \hookrightarrow H(\bar{k})$. Hence, if Φ_k is injective, then $\Phi_{\bar{k}}$ is also injective. Since the Lie algebra of a constant group scheme is trivial, Proposition 22.2 of [11] implies that Φ is injective.

Proposition 4.15. *Let V be a finite dimensional k-vector space, and let G be a finite constant group scheme over k. Then G acts linearly and generically freely on $\mathbb{A}(V)$ if and only if the abstract group G is isomorphic to a subgroup of $\mathbf{GL}(V)(k)$. In this case, we have*

$$\mathrm{ed}_k(G) \leq \dim(V).$$

Proof. If G is isomorphic to a subgroup of $\mathbf{GL}(V)(k)$, then there exists a group morphism $\varphi : G(k) \hookrightarrow \mathbf{GL}(V)(k)$. By Lemma 4.14 above there exists a unique injective morphism of group schemes $\Phi : G \longrightarrow \mathbf{GL}(V)$ extending φ. Proposition 4.13 then shows that G acts linearly and generically freely on $\mathbb{A}(V)$. The converse is clear. The inequality $\mathrm{ed}_k(G) \leq \dim(V)$ is then a direct application of the Proposition 4.11.

Proposition 4.15 helps in the computation of the essential dimension of finite abelian groups over sufficiently big fields.

Corollary 4.16. *Let G be a finite abelian group and k a field with $\mathrm{char}(k) \nmid \exp(G)$. If the field k contains all the $\exp(G)$-th roots of unity, then*

$$\mathrm{ed}_k(G) = \mathrm{rank}(G).$$

In particular, if G is cyclic then $\mathrm{ed}_k(G) = 1$.

Proof. By Proposition 3.7 we only have to prove that $\mathrm{ed}_k(G) \leq \mathrm{rank}(G)$. Let $n = \mathrm{rank}(G)$ and write $G \cong \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_n$ where $d_1 \,|\, d_2 \,|\cdots|\, d_n$. By hypothesis, we have $k \supset \mu_{d_m} \supset \cdots \supset \mu_{d_1}$. We then have the following injection

$$\begin{aligned} G \cong \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_n &\longrightarrow \mathbf{GL}_n(k) \\ ([m_1], \ldots, [m_n]) &\mapsto \begin{pmatrix} \zeta_1^{m_1} & & 0 \\ & \ddots & \\ 0 & & \zeta_n^{m_n} \end{pmatrix} \end{aligned}$$

where ζ_i denotes a primitive d_i-th root of unity. Now apply the above proposition.

We will see later on that the computation is much more complicated when no roots of unity are assumed to be in the base field.

An action of an algebraic group G on a scheme X is called faithful if G is isomorphic to a subgroup of $\mathrm{Aut}(X)$ via this action. Proposition 4.15 above then shows that for a finite constant group G, faithful actions on a vector space V correspond to generically free actions on V.

As a little application of faithful actions we give some bounds on the essential dimension of dihedral groups $D_n = \mathbb{Z}/n \rtimes \mathbb{Z}/2$. We will use the classical presentation $D_n =< \sigma, \tau \mid \sigma^n = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} >$.

Corollary 4.17. *Let k be a field of characteristic $p \geq 0$. Let n be a natural integer such that $p \nmid n$ and suppose that $\mu_n \subset k^\times$. Then $\mathrm{ed}_k(D_n) \leq 2$.*

Proof. Let ζ a n-th primitive root of unity and define an homomorphism $D_n \longrightarrow \mathbf{GL}_2(k)$ by sending σ to $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ and τ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One can easily show that this gives an injective group homomorphism, and then apply Proposition 4.15.

For the groups D_4 and D_6, one can even drop the assumptions on the field at least when $\mathrm{char}(k) \neq 2$. Actually,

$$\begin{aligned} D_4 &\longrightarrow \mathbf{GL}_2(k) \\ \sigma &\longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \tau &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} D_6 &\longrightarrow \mathbf{GL}_2(k) \\ \sigma &\longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \\ \tau &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

are both faithful representations. Hence $\mathrm{ed}_k(D_4) \leq 2$ and $\mathrm{ed}_k(D_6) \leq 2$ for any field k of characteristic $\neq 2$.

In the sequel we will not only deal with faithful linear representations but also with projective ones. The following lemma is an immediate consequence of Proposition 4.13 and Lemma 4.14. We state it here for further reference.

Lemma 4.18. *Let G be a finite constant group scheme over k. Then G acts generically freely on $\mathbb{P}(V)$ if and only if the abstract group G is isomorphic to a subgroup of $\mathbf{PGL}(V)(k)$.*

Remark 4.19. This section is directly inspired by the work of Merkurjev. In particular Propositions 4.11, 4.15 and Corollary 4.16 can be found in [12]. However, all the previous presentation takes care of many technical details which were not pointed out in Merkurjev's paper. Proofs are consequently a little bit longer and a great attention is given to working without any assumption on the characteristic of the ground field. Some trivial results about dihedral groups have been added. Proposition 4.13 has been proved for a future computation on cubics (see the forthcoming paper of the authors: "Essential dimension of cubics").

5. Versal pairs and Rost's definition

In this section we define another notion of essential dimension and compare it with the one introduced at the beginning. The ideas described below are based on the paper [16] where Rost computes $\mathrm{ed}(\mathbf{PGL}_4)$. We therefore call it Rost's essential dimension.

Let k be a field and $\mathfrak{A}_k$ be the category of *all* (associative and unital) commutative k-algebras with homomorphism of k-algebras (sending 1 to 1) as morphisms. Every functor $\mathbf{F} : \mathfrak{A}_k \longrightarrow \mathbf{Sets}$ by restriction defines a functor $\mathfrak{C}_k \longrightarrow \mathbf{Sets}$ hence an object of $\mathfrak{F}_k$. We shall define the notion of essential dimension for a special class of functors $\mathbf{F} : \mathfrak{A}_k \longrightarrow \mathbf{Sets}$.

Let K/k be an object of $\mathfrak{C}_k$. For a local k-subalgebra $\mathcal{O}$ of K, with maximal ideal $\mathfrak{m}$, we will write $\kappa(\mathcal{O}) = \mathcal{O}/\mathfrak{m}$ for its residue field and $\pi : \mathcal{O} \longrightarrow \kappa(\mathcal{O})$ for the quotient map.

Definition 5.1. *Let K and L be two extensions of k. A* PSEUDO k-PLACE *$f : K \rightsquigarrow L$ is a pair $(\mathcal{O}_f, \alpha_f)$ where $\mathcal{O}_f$ is a local k-subalgebra of K and $\alpha_f : \kappa(\mathcal{O}_f) \to L$ is a morphism in $\mathfrak{C}_k$.*
Let $\mathbf{F} : \mathfrak{A}_k \longrightarrow \mathbf{Sets}$ be a functor and take $f : K \rightsquigarrow L$ a pseudo k-place. We say that an element $a \in \mathbf{F}(K)$ is UNRAMIFIED *in f if a belongs to the image of the map $\mathbf{F}(\mathcal{O}_f) \longrightarrow \mathbf{F}(K)$. In this case we define the* SET OF SPECIALIZATIONS OF *a to be*

$$f^*(a) = \big\{\mathbf{F}(\alpha_f \circ \pi)(c) \mid c \in \mathbf{F}(\mathcal{O}_f) \text{ with } c_K = a\big\}.$$

We say that a pair (a, K) with $a \in \mathbf{F}(K)$ is a VERSAL PAIR FOR $\mathbf{F}$ *(over k) if for every extension L/k and every element $b \in \mathbf{F}(L)$ there exists a pseudo k-place $f : K \rightsquigarrow L$ such that a is unramified in f and such that $b \in f^*(a)$.*

Here is a picture of the situation:

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[d, "\pi"'] & K \\ \kappa(\mathcal{O}) \arrow[r, "\alpha_f"] & L \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathbf{F}(\mathcal{O}) \arrow[r] \arrow[d] & \mathbf{F}(K) \ni a \\ \mathbf{F}(\kappa(\mathcal{O})) \arrow[r] & \mathbf{F}(L) \ni b \end{tikzcd}$$

Example 5.2. Let X be an irreducible k-scheme, $k(X)$ its function field and denote by $\eta : \mathrm{Spec}(k(X)) \longrightarrow X$ the unique morphism whose image is the generic point of X. Then $(\eta, k(X))$ is a versal pair for X.
Indeed, take $x : \mathrm{Spec}(L) \to X$ an element in $X(L)$. Then the local ring $\mathcal{O}_{X,x}$ at the point x is naturally a subring of $k(X)$ and there is a canonical morphism from the residue field $k(x)$ to L giving a pseudo k-place $k(X) \rightsquigarrow L$ with the desired property.

Definition 5.3. *Let* $\mathbf{F} : \mathfrak{A}_k \longrightarrow \mathbf{Sets}$ *be a functor which has a versal pair. We define its* (Rost's) essential dimension *(denoted by* $\mathrm{ed}'(\mathbf{F})$*) to be the minimum of the transcendence degree of the field of definition for versal pairs. More precisely* $\mathrm{ed}'(\mathbf{F}) = \min \mathrm{trdeg}(K : k)$ *for all* K/k *such that there exists an element* $a \in \mathbf{F}(K)$ *making* (a, K) *into a versal pair for* $\mathbf{F}$.

Remark 5.4. In the paper of Rost ([16]) the notion is a little bit different. What is called k-place in his context is a pseudo k-place where $\mathcal{O}$ is required to be a valuation ring. Every k-place is then trivially a pseudo k-place. However the converse is not true in general. Indeed for a local ring $\mathcal{O}$ in a field K one can always find a valuation whose local ring $\mathcal{O}_v$ dominates it but there is no control on the residue field.

Definition 5.5. *Let* $\mathbf{F} : \mathfrak{A}_k \longrightarrow \mathbf{Sets}$ *be a functor which has a versal pair. We say that a versal pair* (a, K) *is* nice *if for any* $L \subset K$ *and* $a' \in \mathbf{F}(L)$ *such that* $a = a'_K$*, the pair* (a', L) *is versal. We say that* $\mathbf{F}$ *is* nice *if it has a nice versal pair.*

Proposition 5.6. *Let* $\mathbf{F} : \mathfrak{A}_k \longrightarrow \mathbf{Sets}$ *be a functor which has a versal pair. Then we have*

$$\mathrm{ed}_k(\mathbf{F}) \leq \mathrm{ed}'_k(\mathbf{F})$$

where on the left $\mathbf{F}$ *is viewed as a functor on* $\mathfrak{C}_k$*. Moreover, if* $\mathbf{F}$ *is nice, then*

$$\mathrm{ed}'_k(\mathbf{F}) = \mathrm{ed}_k(\mathbf{F}) = \mathrm{ed}(a),$$

where (a, K) *is any nice versal pair.*

PROOF. Let L/k be any field extension, and let $b \in \mathbf{F}(L)$. Let (a, K) be a versal pair such that $\mathrm{trdeg}(K : k) = \mathrm{ed}'_k(\mathbf{F})$. Since (a, K) is versal, then b comes from an element of $\mathbf{F}(\kappa(\mathcal{O}))$ for some local ring $\mathcal{O}$. Then

$$\mathrm{ed}(b) \leq \mathrm{trdeg}(\kappa(\mathcal{O}) : k) \leq \mathrm{trdeg}(K : k).$$

This proves the first assertion.

Let now (a, K) be a nice versal pair (notice that $\mathrm{trdeg}(K : k)$ is not necessarily minimal). Take a subextension $k \subset L \subset K$ with an element $a' \in \mathbf{F}(L)$ such that $a = a'_L$ and $\mathrm{trdeg}(L : k) = \mathrm{ed}(a)$. By assumption, (a', L) is versal, so $\mathrm{ed}'_k(\mathbf{F}) \leq \mathrm{trdeg}(L : k) = \mathrm{ed}(a) \leq \mathrm{ed}_k(\mathbf{F})$. This concludes the proof.

Remark 5.7. All the present section is new but is inspired by the work of Rost which can be found in [16].

6. GENERIC TORSORS AND COMPRESSIONS

Now that we have seen the notion of versal pairs we want to apply it to $H^1(\_, G)$ when viewed as a functor over $\mathfrak{A}_k$. That is we consider the functor G-**Tors** over the category of affine k-schemes. This section deals with compressions of torsors and is closely related to Reichstein's original discussion. Compare with [14] where everything is done over an algebraically closed field. For the definition of generic torsors we follow [9].

Let G be an algebraic group over k. If G acts linearly and generically freely on a vector space V, there exists an open subscheme $U \subseteq \mathbb{A}(V)$ such that $\pi : U \longrightarrow U/G = Y$ is a G-torsor. We have defined a map (see Definition 4.9)

$$\partial : Y(K) \longrightarrow H^1(K, G)$$

and proved that ∂ is surjective (see Proposition 4.11). Actually, we have shown a little more: for every torsor $P \in H^1(K, G)$, there exists a non-empty subset S of Y such that the isomorphism class of $\pi^{-1}(y)$ is equal to P for every $y \in S(K)$. Such an S is a Zariski-dense subset of Y if K is infinite.

This leads naturally to the following definition:

DEFINITION 6.1. *Let $f : X \to Y$ be a G-torsor with Y irreducible. We say that it is* CLASSIFYING FOR G *if, for any field extension k'/k with k' infinite and for any principal homogenous space P' of G over k'/k, the set of points $y \in Y(k')$ such that P' is isomorphic to the fiber $f^{-1}(y)$ is dense in Y. In particular we have a surjection of functors $Y \twoheadrightarrow H^1(\_, G)$ showing that Y is a classifying scheme of G.*

Remark 6.2. Proposition 4.11 and Remark 4.12 show that a classifying G-torsor always exist for any algebraic group G. Moreover one can always find a reduced classifying torsor for G. Indeed take $X \to Y$ a classifying torsor for G and let $\varphi : Y_{\text{red}} \to Y$ the reduced scheme of Y with its canonical map. Then pulling back $X \to Y$ along φ gives a torsor which is isomorphic to $X_{\text{red}} \to Y_{\text{red}}$ and which is also classifying.

Definition 6.3. *We call* generic torsor over G *the generic fiber of a classifying G-torsor $X \to Y$, i.e. the pullback of*

$$\begin{tikzcd} & X \arrow[d] \\ \mathrm{Spec}(k(Y)) \arrow[r] & Y \end{tikzcd}$$

where $\mathrm{Spec}(k(Y)) \to Y$ *is the generic point. If* $P \longrightarrow \mathrm{Spec}(k(Y))$ *is such a generic torsor it can be viewed as an element of* $H^1(k(Y), G)$*.*

*More precisely one can restate the definition in the following way. Let G be an algebraic group over k, K a field extension of k and $P \longrightarrow \mathrm{Spec}(K)$ a G-torsor. We say that P is k-*versal *or k-*generic *if*

i) there exists an irreducible scheme Y (whose generic point is denoted by η) with function field $k(Y) \simeq K$ (such a scheme is called a model of K) and a G-torsor $f : X \longrightarrow Y$ whose generic fiber $f^{-1}(\eta) \longrightarrow \mathrm{Spec}(K)$ is isomorphic to $P \longrightarrow \mathrm{Spec}(K)$. In other words

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & Y \end{tikzcd}$$

is a pull-back.

ii) For every extension k'/k with k' infinite, for every non-empty open set U of Y and for every G-torsor $P' \longrightarrow \mathrm{Spec}(k')$, there exists a k'-rational point $x \in U$ such that $f^{-1}(x) \simeq P'$.

Remark 6.4. If $f : X \to Y$ is a classifying G-torsor, then, for any non-empty open subset U of Y, the map $f : f^{-1}(U) \to U$ is also a classifying torsor. This says that generic torsors over G correspond bijectively to birational classes of classifying torsors for G.

Lemma 6.5. *Let $P \to \mathrm{Spec}(k(Y))$ be a generic torsor. Then $(P, k(Y))$ is a versal pair for G-**Tors**.*

Proof. Take $T \to \mathrm{Spec}(L)$ any torsor defined over L/k. Since $X \to Y$ is a classifying torsor there exist a L-rational point $y : \mathrm{Spec}(L) \to Y$ such that $T \to \mathrm{Spec}(L)$ fits into a pull-back

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(L) \arrow[r] & Y
\end{tikzcd}$$

Take $\mathcal{O}_{Y,y}$ the local ring at the point y and let $\varphi : \mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$ be the canonical morphism. Consider $P' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the torsor obtained by pulling-back $X \to Y$ along φ. The local ring $\mathcal{O}_{Y,y}$ is naturally a sub-k-algebra of $k(Y)$ and we have a diagram

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & X \arrow[dd] \\
 & P' \arrow[ur] \arrow[dd] & \\
\mathrm{Spec}(k(Y)) \arrow[rr] \arrow[dr] & & Y \\
 & \mathrm{Spec}(\mathcal{O}_{Y,y}) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

showing that $P \to \mathrm{Spec}(k(Y))$ comes from a torsor over $\mathrm{Spec}(\mathcal{O}_{Y,y})$. Moreover the morphism $y : \mathrm{Spec}(L) \to Y$ factorizes through $\mathrm{Spec}(k(y))$ and, if we denote by $P'' \to \mathrm{Spec}(k(y))$ the torsor obtained by pulling-back $P' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ along the morphism $\mathrm{Spec}(k(y)) \to \mathrm{Spec}(\mathcal{O}_{Y,y})$, one has the following diagram

$$\begin{tikzcd}
 & T \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & X \arrow[dl] \arrow[dd] \\
P'' \arrow[rr] \arrow[dd] & & P' \arrow[dd] & \\
 & \mathrm{Spec}(L) \arrow[rr, "y"] \arrow[dl] & & Y \arrow[dl, "\varphi"] \\
\mathrm{Spec}(k(y)) \arrow[rr] & & \mathrm{Spec}(\mathcal{O}_{Y,y}) &
\end{tikzcd}$$

This shows that $T \to \mathrm{Spec}(L)$ comes from $P'' \to \mathrm{Spec}(k(y))$. Thus the local ring $\mathcal{O}_{Y,y}$ toghether with the morphism $k(y) \to L$ form the desired pseudo k-place showing that $(P, k(Y))$ is a versal pair.

Remark 6.6. In the proof of the preceeding lemma the density hypothesis in the definition of a classifying torsor is not used. This hypothesis will be used when talking about compressions.

Remark 6.7. Notice that when Y is smooth over k, the local ring $\mathcal{O}_{Y,y}$ of any point of Y is dominated by a valuation ring whose residue field is equal to $k(y)$. It follows in this case that any pseudo k-place defines a k-place in the sense of Rost (see [16]). Since we do not have a precise reference for this result we have decided to deal only with pseudo k-places.

Actually we will see that a generic torsor give rise to a nice versal pair for the functor G-**Tors**. We first need a definition

Definition 6.8. *Let $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ be two G-torsors. We say that f' is a* compression *of f if there is a diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed, "h"'] & Y' \end{tikzcd}$$

where g is a G-equivariant rational dominant morphism and h is a rational morphism too. The essential dimension *of a G-torsor f is the smallest dimension of Y' in a compression f' of f. We still denote this by* $\mathrm{ed}(f)$.

Remark 6.9. Take as above a compression of $f : X \longrightarrow Y$ and let $U \subseteq Y$ the open subscheme on which h is defined. Taking the pull-back of $X' \longrightarrow Y'$ along h one obtains a G-torsor $f'' : P \longrightarrow U$ which fits into a diagram

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "f"'] & P \arrow[r] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed] & U \arrow[r] & Y' \end{tikzcd}$$

and f'' is a compression too.

The following simple result will be helpful in the sequel.

Lemma 6.10. *Let $g : X \dashrightarrow X'$ be a rational dominant G-equivariant morphism between generically free schemes. Then there exist X_0 (resp. X_0') a friendly open subscheme of X (resp. of X') such that g induces a compression of torsors*

$$\begin{tikzcd} X_0 \arrow[r, dashed, "g"] \arrow[d] & X_0' \arrow[d] \\ X_0/G \arrow[r, dashed, "h"'] & X_0'/G \end{tikzcd}$$

Proof. Take U some friendly open subscheme of X. Since g is dominant one can find U', open subscheme of X', which lies in the image of g. Intersecting U' with some friendly open set of X' gives a friendly open set X'_0 in the image of U. Then $X_0 = g^{-1}(X'_0)$ is the desired open set.

Lemma 6.11. *Let $f : X \longrightarrow Y$ be a G-torsor with Y irreducible and reduced. Let $T \longrightarrow \mathrm{Spec}(k(Y))$ be its generic fiber. Then* $\mathrm{ed}(f) = \mathrm{ed}(T)$.

Proof. Let f and T be as above. Let $f' : X' \longrightarrow Y'$ be a compression of f and $T' \longrightarrow \mathrm{Spec}(k(Y'))$ its generic fiber. By Remark 6.9 above, and since the generic fiber of f is isomorphic to the generic fiber of f'', one can suppose that the compression is a pull-back. The cube

$$\begin{tikzcd}
& T \arrow[rr] \arrow[dl] \arrow[dd] & & X \arrow[dl] \arrow[dd] \\
T' \arrow[rr] \arrow[dd] & & X' \arrow[dd] & \\
& \mathrm{Spec}(k(Y)) \arrow[rr] \arrow[dl] & & Y \arrow[dl] \\
\mathrm{Spec}(k(Y')) \arrow[rr] & & Y' &
\end{tikzcd}$$

then shows that T' maps to T under $H^1(k(Y'), G) \longrightarrow H^1(k(Y), G)$. This shows that $\mathrm{ed}(T) \leq \mathrm{ed}(f)$.

Conversely suppose there is a subextension $k \subseteq K' \subseteq K := k(Y)$ together with a principal homogenous space T' over K' such that T' maps to T under $H^1(K', G) \longrightarrow H^1(k(Y), G)$. We have to find a G-torsor $f' : X' \longrightarrow Y'$ such that T' is isomorphic to its generic fiber and a compression from f to f'.

First remark that one can suppose everything to be affine. Indeed the generic point of Y lies in some open affine subset U and T is also the generic fiber of the G-torsor $f^{-1}(U) \longrightarrow U$.

Now rewrite the problem in terms of rings: say $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $T = \mathrm{Spec}(P)$, $T' = \mathrm{Spec}(P')$ and let $k[G]$ denote the algebra of G. We know that K is the field of fractions of A (since Y is reduced), that $P \simeq B \otimes_A K$ and $P \simeq P' \otimes_{K'} K$. We have to find a subring A' of K' whose field of fractions is K', a G-torsor B'/A' such that $P' \simeq B' \otimes_{A'} K'$ and a rational compression from B'/A' to B/A.

Since K is of finite type over k we can write it as $K = k(\alpha)$ where (α) is a short notation for $(\alpha_1, \ldots, \alpha_n)$. Similarly, since P is of finite type over K we write it $P = K[\beta]$ for some $\beta_1, \ldots, \beta_m$. In the same way we write $K' = k(\alpha')$ and $P' = K'[\beta']$.

We will take for A' a localisation of the ring $k[\alpha']$ for which the isomorphism $P' \otimes_{K'} P' \simeq P' \otimes_k k[G]$ is defined. More precisely, since both $P' \otimes_{K'} P'$ and $P' \otimes_k k[G]$ are finitely generated algebras over K' one can find a polynomial f in the α'_i such that $B' \otimes_{A'} B' \simeq B' \otimes_k k[G]$ where $A' = k[\alpha']_f$ and $B' = A'[\beta']$

(since there is only a finite number of polynomials to invert in order to define the isomorphism).

Now obviously $P' \simeq B' \otimes_{A'} K'$ and we just have to find a rational morphism from A' to A and this will induce a rational compression from B'/A' to B/A. This is easily done since the image of A' under the map $A' \subset K' \subset K$ lies in a subring of the form $k[\alpha]_g$ for some polynomial g in the α_i (again one has only to invert the polynomials that appear in the image of the α'_i which are only finite in number). Now $A = k[\alpha]_h$ for some polynomial h and we have a natural map $A' \to k[\alpha]_g \to (k[\alpha]_g)_h = A_g$. In the same way one finds a rational map $B' \to B_p$ compatible with the previous one.

It follows that $\mathrm{ed}(f) \leq \mathrm{ed}(T)$ and the proof is complete.

Remark 6.12. The hypothesis "reduced" on Y can be dropped easily arguing with $A/\mathrm{Nil}(A)$ rather then A. Since Remark 6.2 tells that one can always find a reduced classifying torsor this will not be proved.

Lemma 6.13. *Let $f' : X' \longrightarrow Y'$ be a compression of a classifying torsor $f : X \longrightarrow Y$. Then f' is also classifying.*

Proof. Let

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed, "h"'] & Y' \end{tikzcd}$$

be such a compression. Let k'/k be a field extension with k' infinite and let $P' \in H^1(k', G)$. Since f is classifying one can find a k'-rational point $y \in Y(k')$ which lies in U, the open set on which h is defined, such that $f^{-1}(y) \simeq P'$. Then the fiber of f' at $h(y)$ clearly gives a torsor isomorphic to P'.

Corollary 6.14. *Let $T \to \mathrm{Spec}(K)$ be a generic G-torsor, $K' \subset K$ and $T' \to \mathrm{Spec}(K')$ such that $T'_K = T$. Then T' is also a generic torsor.*

Proof. Take a classifying G-torsor $X \longrightarrow Y$ which is a model for T. Then, by the proof of Lemma 6.11, defining T over a smaller field means compressing the torsor $X \longrightarrow Y$. Since the compression of a classifying torsor is again classifying it follows that T comes from a generic torsor.

Corollary 6.15. *The functor G-**Tors** is nice.*

Proof. We have to show G-**Tors** has a nice versal pair. But a generic torsor defines a versal pair and niceness is ensured by the previous corollary.

Corollary 6.16. *Let G be an algebraic group over k and let $T \in H^1(K,G)$ be a generic torsor. Then* $\mathrm{ed}'_k(G) = \mathrm{ed}_k(G) = \mathrm{ed}(T)$.

Proof. As pointed out above, any generic torsor gives rise to a nice versal pair and we can apply Proposition 5.6.

Proposition 6.17. *Let G be an algebraic group acting linearly and generically freely on $\mathbb{A}(V)$ where V is some vector space. Suppose that the G-action induced on $\mathbb{P}(V)$ is again generically free. Then*

$$\mathrm{ed}(G) \leq \dim(V) - \dim(G) - 1.$$

Proof. The map $\mathbb{A}(V) \setminus \{0\} \to \mathbb{P}(V)$ gives a rational G-equivariant map from $\mathbb{A}(V) \to \mathbb{P}(V)$ which gives a compression of the corresponding torsors in view of Lemma 6.10 above.

Corollary 6.18. *Let G be a finite constant group scheme over k. Suppose that, for an integer $n \geq 2$, there is an injective map $\rho : G \hookrightarrow \mathbf{GL}_n(k)$ such that $\pi \circ \rho$ stays injective where $\pi : \mathbf{GL}_n(k) \to \mathbf{PGL}_n(k)$ is the canonical projection. Then* $\mathrm{ed}(G) \leq n-1$.

Proof. Indeed G acts generically freely on $\mathbb{A}^n$ by Proposition 4.15 and by Lemma 4.18 on $\mathbb{P}^{n-1}$ too. We can thus apply the above result.

Using compressions we are able to explain the behaviour of the essential dimension of G with respect to a closed subgroup.

Theorem 6.19. *Let G be an algebraic group and H a closed algebraic subgroup of G. Then*

$$\mathrm{ed}(H) + \dim(H) \leq \mathrm{ed}(G) + \dim(G).$$

In particular, if G is finite, we have

$$\mathrm{ed}(H) \leq \mathrm{ed}(G).$$

Proof. Let $\mathbb{A}(V)$ be an affine space on which G acts generically freely. Take U open in $\mathbb{A}(V)$ such that U/G and U/H both exist and are torsors. Now take

$$\begin{tikzcd} U \arrow[r, dashed, "g"] \arrow[d] & X \arrow[d] \\ U/G \arrow[r, dashed, "h"'] & Y \end{tikzcd}$$

a G-compression such that $\dim(Y) = \mathrm{ed}(G)$. Since the stabilizer in H of a point x is a subgroup of G_x it follows that H acts generically freely on U and on X too. Now g is also H-equivariant and by the Lemma 6.10 above g gives rise to an H-compression of $U \longrightarrow U/H$. It then follows that

$$\begin{aligned} \mathrm{ed}(H) &\leq \dim(X) - \dim(H) \\ &= \dim(Y) + \dim(G) - \dim(H) \\ &= \mathrm{ed}(G) + \dim(G) - \dim(H). \end{aligned}$$

This provides another proof of the following

Corollary 6.20. *If* $\mathrm{char}(k) \neq 2$ *one has* $\mathrm{ed}(\mathcal{S}_n) \geq [\frac{n}{2}]$.

Proof. We have $H = \underbrace{\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2}_{[\frac{n}{2}] \text{ times}} \subset \mathcal{S}_n$. But we have seen (Corollary 4.16) that the essential dimension of a finite 2-torsion elementary abelian group is equal to its rank if $\mathrm{char}(k) \neq 2$. One concludes using the preceeding theorem.

Proposition 6.21. *Let G be an algebraic group over k and denote by G^0 its connected component. If* $\mathrm{ed}_k(G) = 1$ *then G/G^0 is isomorphic to a finite subgroup of* $\mathbf{PGL}_2$.

Proof. The fact that the group G/G^0 is finite is well-known. Assume now that $\mathrm{ed}_k(G) = 1$. Let $\mathbb{A}(V)$ be an affine space on which G acts generically freely. Let $U \subseteq \mathbb{A}(V)$ be a friendly open subscheme and let $X \longrightarrow Y$ a G-torsor together with a compression of the generic torsor $U \longrightarrow U/G$

$$\begin{tikzcd} U \arrow[r, dashed] \arrow[d] & X \arrow[d] \\ U/G \arrow[r, dashed] & Y \end{tikzcd} .$$

Now G acts freely on X (by Remark 4.6) and hence G^0 too. Then the quotient X/G^0 exists and G/G^0 acts freely on it. It follows that there is a monomorphism of group schemes $G/G^0 \to \mathrm{Aut}(X/G^0)$. Now $\mathbb{A}(V)$ is rational and thus X/G^0 is unirational. But

$$\dim(X/G^0) = \dim(X) - \dim(G^0) = \dim(X) - \dim(G) = \dim(Y) = 1$$

and then by a theorem of Lüroth X/G^0 is birationnaly equivalent to $\mathbb{P}^1$. It follows that $\mathrm{Aut}(X/G^0) \cong \mathbf{PGL}_2$. Thus G/G^0 is isomorphic to a subgroup of $\mathbf{PGL}_2$.

Remark 6.22. The above discussion is longer than Merkurjev's one. Many details are given and proofs are completed. However the philosophy introduced here is due to Merkurjev which was himself inspired by Reichstein's work. The discussion about the niceness of G-**Tors** is new. Proposition 6.21 is a new result which was pointed out to us by J.-P. Serre.

7. Some finite groups

In this section we will compute the essential dimension of some constant group schemes. We first deal with some generalities and an application to the symmetric group (which can originally be found in [3]). Groups of the form $\mathbb{Z}/n$ and dihedral groups are then studied more carefully.

In what follows G will denote a finite constant group scheme over k.

We first recall that if G is such a group, then any linear generically free action on a vector space V is actually a faithful representation (see Proposition 4.15). Since G is finite and acts faithfully on the field of functions $k(V)$, this gives rise to a Galois extension $k(V)/k(V)^G$. This is indeed a generic torsor for G by our previous considerations. Now any subfield $E \subseteq k(V)$ on which G acts faithfully gives rise in the same way to a Galois extension E/E^G. From this remark we have the following proposition which is the definition of essential dimension in [3]

Proposition 7.1. *Let G be a finite constant group scheme over k acting faithfully on a k-vector space V. Then the essential dimension of G is the minimum of the* $\mathrm{trdeg}(E:k)$ *for all the fields $E \subseteq k(V)$ on which G acts faithfully.*

Application to $\mathcal{S}_n$.

In this example we suppose that $\mathrm{char}(k) \neq 2$.

With this assumption on the ground field, $\mathcal{S}_n$ acts faithfully on the hyperplane $H = \{\ x \in \mathbb{A}^n_k \mid x_1 + \cdots + x_n = 0\ \}$ and thus on $k(x_1, \ldots, x_{n-1})$. But on $k(x_1, \ldots, x_{n-1})$ we have a multiplicative action, i.e. a $\mathbb{G}_m$-action, given by $\lambda \cdot x_i = \lambda x_i$ for all $\lambda \in \mathbb{G}_m(k)$ and all $i = 1, \ldots, n-1$. This action commutes with the action of $\mathcal{S}_n$. We easily see that

$$k(x_1, \ldots, x_{n-1})^{\mathbb{G}_m} = k\big(x_1/x_{n-1}, \ldots, x_{n-2}/x_{n-1}\big).$$

Now, if $n \geq 3$, the group $\mathcal{S}_n$ acts faithfully on the latter field. The transcendence degree of $k\big(x_1/x_{n-1}, \ldots, x_{n-2}/x_{n-1}\big)$ being equal to $n-2$, one concludes that $\mathrm{ed}(\mathcal{S}_n) \leq n-2$ for $n \geq 3$.

In particular we find $\mathrm{ed}(\mathcal{S}_3) = 1$ and $\mathrm{ed}(\mathcal{S}_4) = 2$.

If now we suppose $n \geq 5$, we show that $\mathrm{ed}(\mathcal{S}_n) \leq n-3$.

The group $\mathbf{PGL}_2(k)$ acts on $k(x_1, \ldots, x_n)$ in the following way :

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot x_i = \frac{ax_i + b}{cx_i + d} \qquad \forall\ i = 1, \ldots, n.$$

If now i, j, k, ℓ are distinct, the cross-sections $[x_i, x_j, x_k, x_\ell] = \frac{(x_i - x_k)(x_j - x_\ell)}{(x_j - x_k)(x_i - x_\ell)}$ are $\mathbf{PGL}_2$-invariant. Hence we have

$$k([x_i, x_j, x_k, x_\ell]) \subset k(x_1, \ldots, x_n)^{\mathbf{PGL}_2(k)}$$

where $k([x_i, x_j, x_k, x_\ell])$ is a short notation for the field generated by the biratios $[x_i, x_j, x_k, x_\ell]$ for i, j, k, l all distinct. But $k([x_i, x_j, x_k, x_\ell])$ is generated by the biratios $[x_1, x_2, x_3, x_i]$ with $i = 4, \ldots, n$.
Hence $k([x_i, x_j, x_k, x_\ell]) \cong k(y_1, \ldots, y_{n-3})$. But, if $n \geq 5$, every $\sigma \in \mathcal{S}_n \setminus \{1\}$ moves at least one of the $[x_i, x_j, x_k, x_\ell]$'s. Consequently, since the above action commutes with the $\mathcal{S}_n$-action, $\mathcal{S}_n$ acts faithfully on $k(y_1, \ldots, y_{n-3})$.

This shows that $\mathrm{ed}(\mathcal{S}_n) \leq n - 3$ for all $n \geq 5$.

In particular we have $\mathrm{ed}(\mathcal{S}_5) = 2$ and $\mathrm{ed}(\mathcal{S}_6) = 3$.

The question is still open concerning $\mathcal{S}_7$. Do we have $\mathrm{ed}(\mathcal{S}_7) = 3$ or 4 ?

The following lemma is an immediate consequence of Proposition 6.21. We restate it here in the case of finite groups and reprove it using an algebraic argument. Compare with [3] Theorem 6.2.

Lemma 7.2 (Useful Lemma). *Let G be a finite constant group. If* $\mathrm{ed}_k(G) = 1$, *then G is isomorphic to a subgroup of* $\mathbf{PGL}_2(k)$.

Proof. Let G act faithfully on a vector space V and let $k(V)/k(V)^G$ be the corresponding Galois extension. Saying that $\mathrm{ed}_k(G) = 1$ means that there is a subextension K/k where $\mathrm{trdeg}(K : k) = 1$ with G acting faithfully on K. Since K is a subextension of $k(V)$, which is rational, and since $\mathrm{trdeg}(K : k) = 1$, by Lüroth's theorem K is also rational. Thus $K \cong k(t)$. Since G acts faithfully on $k(t)$ this means that G is a subgroup of $\mathrm{Aut}(k(t)) \cong \mathbf{PGL}_2(k)$.

We continue this section studying more carefully the groups $\mathbb{Z}/n$ and D_n.

We recall first of all that, if the field k contains the n-th roots of unity, one has $\mathrm{ed}_k(\mathbb{Z}/n) = 1$ and that the inequality $\mathrm{ed}_k(\mathbb{Z}/n) \geq 1$ holds for any field. Upper bounds are usually given by actions or representations and these will essentially depend on the ground field. Furthermore lower bounds are generally difficult to find. We begin with some easy considerations in order to understand the problem.

Consider $\mathbb{Z}/n$ as a constant $\mathbb{R}$-group scheme. Then one has a faithful representation

$$\mathbb{Z}/n \longrightarrow \mathrm{SL}_2(\mathbb{R})$$

given by sending the generator of $\mathbb{Z}/n$ to the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

representing the rotation of angle $\theta = 2\pi/n$. Hence $\mathrm{ed}_{\mathbb{R}}(\mathbb{Z}/n) \leq 2$ for every n. Clearly this holds for an arbitrary field k containing $\mathbb{R}$. The question becomes particularly interesting when the field is $\mathbb{Q}$. For a better results on the essential dimension of cyclic and dihedral groups over $\mathbb{Q}$ see the work of A. Ledet in [10] where for example the equality $\mathrm{ed}_{\mathbb{Q}}(\mathbb{Z}/7) = 2$ is proven.

But linear representations do not always give the best possible upper bounds. Recall that if G is a finite subgroup of $\mathbf{GL}_n(k)$ for some n and if its image in $\mathbf{PGL}_n(k)$ is still G then $\mathrm{ed}_k(G) \leq n-1$ (see Corollary 6.18).

As we shall see, in the study of cyclic groups there is a gap between groups of odd and even order.

Lemma 7.3 (Simple Lemma). *Let n be an integer, k a field such that* $\mathrm{char}(k) \nmid n$ *and $\zeta \in \overline{k}$ a primitive n-th root of the unity. Suppose that $\zeta + \zeta^{-1} \in k$. Let $S = \begin{pmatrix} \zeta + \zeta^{-1} & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the order of S in $\mathbf{GL}_2(k)$ equals n and the subgroup generated by S and T is isomorphic to the dihedral group D_n. Moreover, if n is odd, the same holds in $\mathbf{PGL}_2(k)$ for the classes of S and T.*

Proof. Let $P = \begin{pmatrix} 1 & \zeta^{-1} \\ 1 & \zeta \end{pmatrix}$. Then $S = P^{-1} \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} P$ showing that S has order n. Moreover easily $TST^{-1} = S^{-1}$.
Now assume that n is odd. We only have to check that $S^i \neq \lambda I$ for all $\lambda \in k$ and all $i = 1, \ldots, n-1$. Suppose that $S^i = \lambda I$ for some $\lambda \in k$ and some $i = 1, \ldots, n-1$. This would mean that $\zeta^i = \lambda$ and $\zeta^{-i} = \lambda$. Thus $\zeta^{2i} = 1$. This means that $n \mid 2i$ which is impossible.

This lemma gives us already the exact value of $\mathrm{ed}_k(\mathbb{Z}/n)$, with n odd, when the field contains $\zeta + \zeta^{-1}$.

Proposition 7.4. *Let n be an odd integer, k a field such that* $\mathrm{char}(k) \nmid n$ *and ζ a primitive n-th root of the unity. If $\zeta + \zeta^{-1} \in k$ then*

$$\mathrm{ed}_k(\mathbb{Z}/n) = 1.$$

Proof. We only have to prove that $\mathrm{ed}_k(\mathbb{Z}/n) \leq 1$. But the lemma above shows that $\mathbb{Z}/n$ injects into $\mathbf{GL}_2(k)$ and that this map stays injective when passing to $\mathbf{PGL}_2(k)$. Thus $\mathrm{ed}_k(G) \leq 2 - 1 = 1$ by Corollary 6.18.

This gives the essential dimension of $\mathbb{Z}/3$:

Corollary 7.5. *For any field k one has* $\mathrm{ed}_k(\mathbb{Z}/3) = 1$.

Proof. Clearly every field contains $\zeta + \zeta^{-1} = -1$ and hence, if the characteristic of k is $\neq 3$, one can apply the above argument. In characteristic 3 we already know the result (see Examples 2.3).

The tough problem is to deal with groups of the form $\mathbb{Z}/2n$ where n is even. The following theorem gives an answer for $n = 2$. We postpone its proof until the end of the present section.

Theorem 7.6. *Let k be a field of characteristic $\neq 2$. Then*

$$\mathrm{ed}_k(\mathbb{Z}/4) = \begin{cases} 1 & \textit{if } -1 \textit{ is a square in } k \\ \\ 2 & \text{otherwise.} \end{cases}$$

The result was already known by Serre in [21] (see Exercice 1.2) even though the notion of essential dimension was not defined. More recently in [17] Rost computed the essential dimension of a twisted form of $\mathbb{Z}/4$ generalizing the present result.

The above Simple Lemma has a converse statement when n is prime.

Lemma 7.7. *Let $p > 2$ a prime, k a field of characteristic $\neq p$ and $\zeta \in \overline{k}$ a primitive p-th root of unity. If $\mathbf{PGL}_2(k)$ has an element of order p then $\zeta + \zeta^{-1} \in k$.*

Proof. Let $M \in \mathbf{GL}_2(k)$ of order p in $\mathbf{PGL}_2(k)$. There is a $\lambda \in k^\times$ such that $M^p = \lambda I$, thus the minimal polynomial m_M divides $X^p - \lambda$. Hence $X^p - \lambda$ is not irreducible (otherwise $p = \deg(m_M) \leq 2$) and therefore $\lambda = \mu^p$ for some $\mu \in k^\times$. Thus we can suppose that $\lambda = 1$. In that case, the eigenvalues of M are of the form ζ^i. Let ζ^i and ζ^j be the two eigenvalues of M. We have $\det(M) = \zeta^{i+j} \in k^\times$. Suppose that $i+j \not\equiv 0 \mod p$, then $< \zeta^{i+j} >= \mu_p \subset k^\times$ and hence $\zeta + \zeta^{-1} \in k^\times$. Suppose that $i + j \equiv 0 \mod p$, then $j \equiv -i$. If $i \equiv 0$ then $M = I$ which is impossible, hence $i \not\equiv 0$ and the eigenvalues are disctinct. Thus

$$M = P^{-1} \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^{-i} \end{pmatrix} P \in \mathbf{GL}_2(k)$$

for some invertible matrix P. But since $i \not\equiv 0$, there exists j such that $ij \equiv 1 \mod p$. Then $M^j = P^{-1} \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} P$ belongs to $\mathbf{GL}_2(k)$ and it follows that $\zeta + \zeta^{-1} = \mathrm{Tr}(M^j) \in k$.

Corollary 7.8. *Let p be a prime, k a field such that $\mathrm{char}(k) \neq p$ and suppose that $\zeta + \zeta^{-1} \notin k$. Then*

$$\mathrm{ed}_k(\mathbb{Z}/p) \geq 2.$$

Proof. Suppose that $\mathrm{ed}(\mathbb{Z}/p) = 1$, then by the Useful Lemma we would have an injection $\mathbb{Z}/p \longrightarrow \mathbf{PGL}_2(k)$ which is impossible by the above lemma.

We now have the exact value of the essential dimension of $\mathbb{Z}/5$.

Corollary 7.9. *Let k a field such that* $\mathrm{char}(k) \neq 5$ *and ζ a primitive 5-th root of unity. Then*

$$\mathrm{ed}_k(\mathbb{Z}/5) = \begin{cases} 1 & \text{if } \zeta + \zeta^{-1} \in k \\ \\ 2 & \text{otherwise.} \end{cases}$$

Proof. If $\zeta + \zeta^{-1} \in k$ apply Proposition 7.4. If $\zeta + \zeta^{-1} \notin k$ then by the above corollary we have $\mathrm{ed}_k(\mathbb{Z}/5) \geq 2$. It then suffices to show that $\mathrm{ed}_k(\mathcal{S}_5) \leq 2$ since $\mathbb{Z}/5$ is a subgroup of $\mathcal{S}_5$ and thus $\mathrm{ed}_k(\mathbb{Z}/5) \leq \mathrm{ed}_k(\mathcal{S}_5) = 2$. If $\mathrm{char}(k) \neq 2$ this has been proven at the beginning of this section.
Assume now that $\mathrm{char}(k) = 2$. It suffices to show that the generic torsor for $\mathcal{S}_5$ is defined over a field of transcendence degree at most 2. By [2] Proposition 4.4, the generic polynomial defining the generic torsor can be reduced to the form $X^5 + aX^2 + bX + c$. If $b = 0$ we are done. If $b \neq 0$ replacing X by $\frac{c}{b}X$ gives the conclusion.

Another application of the Useful Lemma concerns $\mathbb{Z}/p^2$ in characteristic p. Recall that we already know that $\mathrm{ed}_k(\mathbb{Z}/p^2) \leq 2$ in that case as it was shown in Section 2.

Proposition 7.10. *If* $\mathrm{char}(k) = p$ *then* $\mathrm{ed}_k(\mathbb{Z}/p^2) = 2$.

Proof. By the Useful Lemma we know that if $\mathrm{ed}_k(G) = 1$ then G is isomorphic to a subgroup of $\mathbf{PGL}_2(k)$. Thus it suffices to show that, if $\mathrm{char}(k) = p$, there are no elements of order p^2 in $\mathbf{PGL}_2(k)$. We leave it as an easy exercise to the reader.

One can handle in a similar way the computation of some essential dimensions for the dihedral groups D_n.

Corollary 7.11. *Let n be odd, k a field such that* $\mathrm{char}(k) \nmid n$ *and ζ a primitive n-th root of the unity. If $\zeta + \zeta^{-1} \in k$ then* $\mathrm{ed}_k(D_n) = 1$.

Proof. It readily follows from Simple Lemma above and Corollary 6.18.

Corollary 7.12. *Let n be an integer. Then*

$$\mathrm{ed}_{\mathbb{R}}(D_n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. By the Simple Lemma, there is a real 2-dimensional faithful representation of D_n for every n. Hence $\mathrm{ed}_{\mathbb{R}}(D_n) \leq 2$. Moreover, when n is even D_n contains $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ as a subgroup, according to whether n is congruent to 0 or 2 modulo 4. Thus the statement is a consequence of Theorem 7.6 and Proposition 3.7.

One very interesting result for finite groups can be found in [10] and concerns the essential dimension of $G \times \mathbb{Z}/2$. We give here this result without proof.

Theorem 7.13 (Jensen, Ledet, Yui). *Let k be a field of characteristic 0 containg the primitive pth roots of unity, for a prime p, and let G be a finite group. Assume that k does not contain the primitive rth root of unity for any prime $r \neq p$ dividing $|Z(G)|$. Then*

$$\mathrm{ed}_k(G \times \mathbb{Z}/2) = \mathrm{ed}_k(G) + 1.$$

This result gives for example $\mathrm{ed}_{\mathbb{Q}}(G \times \mathbb{Z}/2) = \mathrm{ed}_{\mathbb{Q}}(G) + 1$ for any finite group G. The same holds for $\mathbb{R}$.

Corollary 7.14. *Let n be an odd integer. Then*

$$\mathrm{ed}_{\mathbb{Q}}(\mathbb{Z}/2n) = \mathrm{ed}_{\mathbb{Q}}(\mathbb{Z}/n) + 1.$$

The same holds for $\mathbb{R}$.

Using this result and Theorem 7.6 the computation over the real numbers for cyclic groups is complete:

Corollary 7.15. *Let $n \neq 2$ be an integer. Then*

$$\mathrm{ed}_{\mathbb{R}}(\mathbb{Z}/n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

Proof. We already know that $\mathrm{ed}_{\mathbb{R}}(\mathbb{Z}/n) \leq 2$. If n is odd, Proposition 7.4 tells that $\mathrm{ed}_{\mathbb{R}}(\mathbb{Z}/n) = 1$. If n is even, two cases arise: either $n = 2m$ with m odd and one applies the above corollary, or $n = 4m$ and in this case $\mathbb{Z}/n$ contains $\mathbb{Z}/4$ as a subgroup. Then Theorem 7.6 shows that $\mathrm{ed}_{\mathbb{R}}(\mathbb{Z}/n) \geq 2$.

As promised, we finish the section with a proof of Theorem 7.6 which gives the essential dimension of $\mathbb{Z}/4$.

Notice first that when -1 is a square in k (and $\mathrm{char}(k) \neq 2$) then Corollary 4.16 tells that $\mathrm{ed}_k(\mathbb{Z}/4) = 1$.

Notice also that one always has $\mathrm{ed}_k(\mathbb{Z}/4) \leq 2$. Indeed let k be a field of characteristic $\neq 2$ and let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since A is of order 4, this gives a faithful representation $\mathbb{Z}/4 \longrightarrow \mathbf{GL}_2$ and one concludes that $\mathrm{ed}_k(\mathbb{Z}/4) \leq 2$ using Proposition 4.15.

It thus suffices to prove that $\mathrm{ed}_k(\mathbb{Z}/4) \geq 2$ when $-1 \notin k^{\times 2}$. Our proof is based on the following parametrization of cyclic extensions of degree 4 (see [8]).

Proposition 7.16. *Let K be a field of characteristic $\neq 2$. Let $D \in K^\times \setminus K^{\times 2}$. Then $K(\sqrt{D})/K$ is contained in a cyclic field extension of degree 4 if and only if D is a sum of two squares in K. Let $D = a^2 + b^2, a, b \in K$. Then $K(\sqrt{q(D + a\sqrt{D})})$, $q \in K^\times$ is a parametrization of all cyclic extensions of degree 4 with discriminant D. The trace form of $K(\sqrt{q(D + a\sqrt{D})})$ over K is $\langle 1, D, q, q\rangle$.*

This result tells us that the trace form essentially "depends on two parameters". Let L_0 be the Galois algebra $K(\sqrt{q(D + a\sqrt{D})})$ described in the above proposition. Let $K = k(s,t)$ the function field in two variables and set $D = s^2+1, q = t$ (here $a = s, b = 1$ in the notation of the proposition). Now the algebra L_0 can be viewed as an element of $H^1(k(s,t), \mathbb{Z}/4)$. To prove $\mathrm{ed}(L_0) = 2$ it is sufficient to show that the trace form $q = \langle 1, s^2 + 1, t, t\rangle$ is not defined over a subfield $K \subset k(s,t)$ of transcendence degree 1. We will show that this is the case when k is a field in which -1 is not a square using an idea of Rost.

We begin by making some easy observations on the first residue map of quadratic forms. For convenience we recall briefly its definition following [18].

Let (F, υ) be a field of characteristic different from 2 equipped with a discrete valuation, and let π denotes a prime element (i.e. an element such that $\upsilon(\pi) = 1$). We denote by $\mathcal{O}_\upsilon$ the valuation ring of υ and by $\kappa(\upsilon)$ the residue field.

Any quadratic form q defined over F can be diagonalized as

$$q \simeq \langle a_1, \ldots, a_m, \pi a_{m+1}, \ldots, \pi a_n\rangle,$$

with $a_i \in \mathcal{O}_\upsilon^\times$. Then the map $\partial_\upsilon : W(F) \longrightarrow W(\kappa(\upsilon))$ defined by

$$\partial_\upsilon(q) := \langle \bar{a}_1, \cdots, \bar{a}_m\rangle$$

is a well-defined group homomorphism which is independent of the choice of π, called the first residue map.

Now let $K \subset F$, and let $\omega = \upsilon_{|K}$. If ω is trivial over K, then $K \subset \kappa(\upsilon)$ and it follows from the definition that for any $q \in W(K)$ we have $\partial_\upsilon(q_F) = q_{\kappa(\upsilon)}$.

If ω is non-trivial over K, then any prime element π' of (K, ω) can be written as $\pi' = u\pi^e$ for some $u \in \mathcal{O}_\upsilon^\times$ and some non-negative integer e. The integer e is well-defined and called the ramification index of (K, ω) in (F, υ). If $e = 1$, we say that the extension $(F, \upsilon)/(K, \omega)$ is unramified. Moreover in this case, we have an inclusion $\kappa(\omega) \subset \kappa(\upsilon)$.

If e is odd, then for any $q \in W(K)$, one easily checks that in $W(\kappa(\upsilon))$ the equality $\partial_\upsilon(q_F) = \partial_\upsilon(q)_{\kappa(\upsilon)}$ holds.

Let now k a field in which -1 is not a square. We consider υ the t-adic valuation on the field $F = k(s,t)$ and υ' the (s^2+1)-adic valuation on $\kappa(\upsilon) \cong k(s)$ (note that since -1 is not a square we can consider this valuation).

Suppose now that q is defined over a subfield $K \subset k(s,t)$ with $\mathrm{trdeg}(K:k) = 1$, and write $q = q'_F$ for some quadratic form q' defined over K. Notice that, since $\mathrm{trdeg}(K:k) = 1$, then $\mathrm{trdeg}(F:K) = 1$, and it follows that F/K is a purely transcendental extension.

If the valuation $\omega = \upsilon_{|K}$ is trivial we have

$$\partial_\upsilon(q) = \partial_\upsilon(q'_F) = q'_{\kappa(\upsilon)}.$$

Since $\kappa(\upsilon) = k(s) \subset F$, by scalar extension we obtain the following equality in $W(F)$

$$\partial_\upsilon(q)_F = q.$$

It follows that $\langle 1, 1+s^2\rangle = \langle 1, 1+s^2, t, t\rangle$, showing that $\langle t, t\rangle$ is hyperbolic over F. Then, comparing discriminants, one finds that -1 is a square in $F = k(s,t)$, hence in k, which is a contradiction. Thus the valuation ω is non-trivial over K.

Notice now that $\kappa(\omega)$ is a finite extension of k, since any discrete k-valuation over a field extension of transcendence degree 1 over k is associated to some irreducible polynomial with coefficients in k. Since $\kappa(\omega) \subset k(s)$, this implies that $\kappa(\omega) = k$. It follows, by [4], Prop. 2, p. 327, that ω and υ has same value group, that is $(F,\upsilon)/(K,\omega)$ is unramified. In particular, we have

$$\partial_\upsilon(q) = \partial_\upsilon(q'_F) = \partial_\omega(q')_{\kappa(\upsilon)}.$$

Since $\partial_\omega(q') \in W(\kappa(\omega)) = W(k)$, we then get $\partial_{\upsilon'}(\partial_\upsilon(q)) = \partial_\omega(q')$, so we finally obtain the equality

$$\partial_{\upsilon'}(\partial_\upsilon(q))_{\kappa(\upsilon)} = \partial_\upsilon(q),$$

that is $\langle 1\rangle = \langle 1, 1+s^2\rangle$ in $W(k(s))$, which is a contradiction.

This shows that $\mathrm{ed}(\langle 1, s^2+1, t, t\rangle) = 2$ when -1 is not a square. It follows that $\mathrm{ed}(L_0) = 2$ and consequently $\mathrm{ed}_k(\mathbb{Z}/4) \geq 2$ in that case. This completes the proof of Theorem 7.6.

Remark 7.17. Most of the results of the present section were known to Buhler and Reichstein over an algebraically closed field of characteristic 0. Emphasis is given here to the computation of the essential dimension over arbitrary fields.

8. Homotopy invariance

In this section we shall prove the so-called *homotopy invariance* (that is $\mathrm{ed}_k(G) = \mathrm{ed}_{k(t)}(G)$) for algebraic groups defined over infinite fields. We first begin with some considerations on places of the form $k(t) \rightsquigarrow k$. Unadorned $\otimes$ will always mean $\otimes_k$.

Let k be any field, $a(t) \in k(t)$ and $\tau \in k$. We say that $a(t)$ is UNRAMIFIED at τ if $a(t) \in k[t]_{\mathfrak{m}_\tau}$ where $\mathfrak{m}_\tau$ denotes the maximal ideal $< t-\tau >$ of $k[t]$. When $a(t)$ is unramified at τ one can evaluate or specialize it at τ by simply replacing t by τ. Actually every $\tau \in k$ defines a pseudo k-place $k(t) \rightsquigarrow k$ denoted by $(\mathcal{O}_\tau, \alpha_\tau)$ where the local ring $\mathcal{O}_\tau$ is $k[t]_{\mathfrak{m}_\tau}$ and the morphism α_τ is the isomorphism $k[t]/\mathfrak{m}_\tau \simeq k$. Saying that $a(t)$ is unramified at τ is then the same than saying that $a(t)$ (viewed as an element of $\mathbf{F}(k(t))$ where $\mathbf{F}$ is the forgetful functor) is unramified in the place $(\mathcal{O}_\tau, \alpha_\tau)$ and $a(\tau)$ the specialization of $a(t)$ at τ is nothing but the image of $a(t)$ under the map

$$s_\tau : \mathcal{O}_\tau = k[t]_{\mathfrak{m}_\tau} \to k[t]_{\mathfrak{m}_\tau}/\mathfrak{m}_\tau \simeq k[t]/\mathfrak{m}_\tau \simeq k.$$

These considerations extend naturally to vector spaces as follows:

Definition 8.1. *Let A be a k-vector space (not necessarily finite dimensional). Let t be an indeterminate over k, and let $\tau \in k$. We say that an element $a(t) \in A \otimes k(t)$ is* UNRAMIFIED *at τ if $a \in A \otimes \mathcal{O}_\tau$. Let $s_\tau : \mathcal{O}_\tau \longrightarrow k$ be the above morphism.The* SPECIALIZATION *of $a(t)$, denoted by $a(\tau)$, is the image of $a(t)$ under the map $\mathrm{Id}_A \otimes s_\tau : A \otimes \mathcal{O}_\tau \to A \otimes k \simeq A$.*

Let $B \subset A$ be a k-subspace. Recall that the maps $B \otimes k(t) \to A \otimes k(t)$, $B \otimes \mathcal{O}_\tau \longrightarrow B \otimes k(t)$ etc are injective.

We need the following result:

Lemma 8.2. *Let $b(t) \in B \otimes k(t)$. Assume that $b(t)$, viewed as an element of $A \otimes k(t)$ is unramified at τ. Then $b(t)$, viewed as an element of $B \otimes k(t)$, is unramified at τ, and the two corresponding specializations coincide. In particular $b(\tau)$ is in B.*

Proof. This follows from the formula $(A \otimes \mathcal{O}_\tau) \cap (B \otimes k(t)) = B \otimes \mathcal{O}_\tau$.

We continue with some considerations on torsors. Let $X \longrightarrow Y$ be a G-torsor over k and let E/k be any field extension. Pulling back everything along

$\mathrm{Spec}(E) \to \mathrm{Spec}(k)$ one obtains $X_E \longrightarrow Y_E$ a G-torsor over E:

$$\begin{tikzcd}
X_E \arrow[r] \arrow[d] & X \arrow[d] \\
Y_E \arrow[r] \arrow[d] & Y \arrow[d] \\
\mathrm{Spec}(E) \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

Now, for any field extension L/E and any G-torsor $T \to \mathrm{Spec}(L)$ there is a one-to-one correspondence between the set of L-rational points of Y having T as a fiber and the set of L-rational points of Y_E having T as a fiber. Indeed if $y : \mathrm{Spec}(L) \to Y$ is such a point, we have a diagram

$$\begin{tikzcd}
T \arrow[d] \arrow[dr, dashed] \arrow[drr, bend left] & & \\
\mathrm{Spec}(L) \arrow[dr, dashed] \arrow[drr, bend left, "y"] \arrow[ddr, bend right] & X_E \arrow[r] \arrow[d] & X \arrow[d] \\
& Y_E \arrow[r] \arrow[d] & Y \arrow[d] \\
& \mathrm{Spec}(E) \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

by the universal property of the pull-backs involved.

From now on we will deal with $E = k(t)$ and we shall write $X(t) \longrightarrow Y(t)$ instead of $X_{k(t)} \longrightarrow Y_{k(t)}$.

Lemma 8.3. *Let $X \to Y$ be a classifying torsor over an infinite field k. Then the torsor $X(t) \to Y(t)$ is a classifying torsor over $k(t)$.*

Proof. First notice that one can suppose Y to be affine. Let now $L/k(t)$ be a field extension and $T \to \mathrm{Spec}(L)$ be any G-torsor. Let $Z \subset Y$ be the dense subset of Y such that for every $y : \mathrm{Spec}(L) \to Z$ the fiber of $X \to Y$ at y is T. Denote by $Z(t)$ the corresponding subset of $Y(t)$. We have to show that $Z(t)$ is dense. Write $Y = \mathrm{Spec}(A)$ for some k-algebra A. We have that $Y(t) = \mathrm{Spec}(A \otimes k(t))$ and the bijection between the sets Z and $Z(t)$ says that every point $\mathfrak{p}(t) \in Z(t)$ is of the form $\mathfrak{p} \otimes k(t)$ for exactely one $\mathfrak{p} \in Z$. Saying that $Z \subset Y$ is dense means that for every non-zero element f of A there exists $\mathfrak{p} \in Z$ such that $f \notin \mathfrak{p}$. Take $f(t) \in A \otimes k(t)$ a non-zero element and suppose that $Z(t)$ is not dense, that is $f(t) \in \mathfrak{p}(t)$ for all $\mathfrak{p}(t) \in Z(t)$. Since k is infinite one can find $\tau \in k$ such that $f(t)$ is unramified at τ and $f(\tau) \neq 0$. Now Lemma 8.2 tells that $f(\tau) \in \mathfrak{p}$ for all $\mathfrak{p} \in Z$ contradicting the fact that Z is dense in Y.

Theorem 8.4 (Homotopy invariance).
Let G be an algebraic group over an infinite field k. Then

$$\mathrm{ed}_k(G) = \mathrm{ed}_{k(t)}(G).$$

Proof. We only have to prove $\mathrm{ed}_k(G) \leq \mathrm{ed}_{k(t)}(G)$. Let $X \longrightarrow Y$ a classifying G-torsor over k with Y minimal for the dimension (that is $\dim(Y) = \mathrm{ed}_k(G)$). Pulling back everything along $\mathrm{Spec}(k(t))$ one obtains $X(t) \longrightarrow Y(t)$ which is again a classifying torsor in view of the preceeding lemma.

Suppose now that $\mathrm{ed}_{k(t)}(G) < \mathrm{ed}_k(G)$. This means that the torsor $X(t) \longrightarrow Y(t)$ can be further compressed over $k(t)$. That means that there exists a G-torsor $X' \longrightarrow Y'$ with $\dim\ Y' < \dim\ Y(t) = \dim\ Y$ fitting into a pull-back

$$\begin{tikzcd} X(t) \arrow[r] \arrow[d] & X' \arrow[d] \\ Y(t) \arrow[r] & Y' \end{tikzcd}$$

But now, one can find $\varphi \in k[t]$ such that the above pull-back is defined over $\mathrm{Spec}\big(k[t, \frac{1}{\varphi}]\big)$. Now take $\xi : \mathrm{Spec}(k) \to \mathrm{Spec}\big(k[t, \frac{1}{\varphi}]\big)$ a k-rational point. Such a point exists since k is infinite. Now Y'_ξ, the fiber of Y' over ξ, is closed in Y' and thus satisfies $\dim\ Y'_\xi \leq \dim\ Y'$. Pulling back the above square along ξ one has

$$\begin{tikzcd} X(t)_\xi \arrow[r] \arrow[d] & X'_\xi \arrow[d] \\ Y(t)_\xi \arrow[r] & Y'_\xi \end{tikzcd}$$

But $X(t)_\xi \simeq X$, so the torsor $X \longrightarrow Y$ can be compressed into a torsor $X'_\xi \longrightarrow Y'_\xi$ with $\dim\ Y'_\xi \leq \dim\ Y' < \dim\ Y$ contradicting the minimality of Y.

For the moment we do not know if homotopy invariance holds for finite fields.

Remark 8.5. To our knowledge the homotopy invariance is a new result.

References

[1] J. Arason, *Cohomologische Invarianten quadratischer Formen.* J. of Algebra 36 (1975), 446–491

[2] A.–M. Bergé, J. Martinet, *Formes quadratiques et extensions en caractéristique 2.* Ann. Inst. Fourier (2), 35 (1985) 57–77

[3] J. Buhler, Z. Reichstein, *On the essential dimension of a finite group.* Comp. Math. 106 (1997), 159–179

[4] P.M. Cohn, *Algebra, vol. 2.* John Wiley & Sons Ed., London (1977)

[5] A. Delzant, *Définition des classes de Stiefel-Whitney d'un module quadratique sur un corps de caractéristique différente de* 2. C.R Acad. Sci. Paris 255 (1962), 1366–1368

[6] M. Demazure, P. Gabriel, *Groupes algébriques, vol. 1.* Ed. Masson & Cie (1970)

[7] M. Demazure, A. Grothendieck, *Schémas en groupes vol. 1.* SGA 3, Springer-Verlag (1970)

[8] C. Drees, M. Epkenhans, M. Krüskemper, *Computation of the trace form of Galois extensions.* J. of Algebra 192 (1997), 209–234

[9] R. Garibaldi, A. Merkurjev, J.-P. Serre, *Cohomological invariants in Galois cohomology.* AMS University Lecture Series 28 (2003)

[10] C Jensen, A Ledet, N Yui, *Generic Polynomials* MSRI Publ. 45, Cambridge University Press (2002)

[11] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, *The book of involutions.* AMS Coll. Pub. 44 (1998)

[12] A. Merkurjev, *Essential dimension.* Private notes (1999), Lecture notes (2000)

[13] J.S. Milne, *Étale Cohomology.* Princeton Math. Series 33, Princeton University Press, Princeton, N.J. (1980)

[14] Z. Reichstein, *On the notion of essential dimension for algebraic groups.* Transformation Groups 5, no. 3 (2000), 265–304

[15] M. Rosenlicht, *Some basic theorems on algebraic groups.* American J. of Math. 78 (1963), 401–443

[16] M. Rost, *Computation of some essential dimensions.* Preprint (2000). Available on http://www.math.ohio-state.edu/˜rost/ed.html

[17] M. Rost, *Essential dimension of twisted* C_4. Preprint (2000). Available on http://www.math.ohio-state.edu/˜rost/ed.html

[18] W. Scharlau, *Quadratic and hermitian forms.* Grund. Math. Wiss. 270, Springer-Verlag, Berlin-Heidelberg (1985)

[19] J.-P. Serre, *Cohomologie Galoisienne.* Cinquième éd. Lecture Notes 5, Springer-Verlag (1997)

[20] J.-P. Serre, *Corps locaux.* Hermann, Paris (1962)

[21] J.-P. Serre, *Topics in Galois Theory.* Research notes in Math. 1, Jones and Bartlett Pib., Boston, MA (1992)

[22] R. W. Thomason, *Comparison of equivariant algebraic and topological K-theory.* Duke Math. J. 53, No 3, (1986), 795–825

[23] W. C. Waterhouse. *Introduction to Affine Group Schemes.* GTM 66, Springer-Verlag (1979)

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$J_1(p)$ Has Connected Fibers

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Abstract. We study resolution of tame cyclic quotient singularities on arithmetic surfaces, and use it to prove that for any subgroup $H \subseteq (\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ the map $X_H(p) = X_1(p)/H \to X_0(p)$ induces an injection $\Phi(J_H(p)) \to \Phi(J_0(p))$ on mod p component groups, with image equal to that of H in $\Phi(J_0(p))$ when the latter is viewed as a quotient of the cyclic group $(\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$. In particular, $\Phi(J_H(p))$ is always Eisenstein in the sense of Mazur and Ribet, and $\Phi(J_1(p))$ is trivial: that is, $J_1(p)$ has connected fibers. We also compute tables of arithmetic invariants of optimal quotients of $J_1(p)$.

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Contents

1 Introduction

Let p be a prime and let $X_1(p)_{/\mathbf{Q}}$ be the projective smooth algebraic curve over $\mathbf{Q}$ that classifies elliptic curves equipped with a point of exact order p. Let $J_1(p)_{/\mathbf{Q}}$ be its Jacobian. One of the goals of this paper is to prove:

Theorem 1.1.1. *For every prime p, the Néron model of $J_1(p)_{/\mathbf{Q}}$ over $\mathbf{Z}_{(p)}$ has closed fiber with trivial geometric component group.*

This theorem is obvious when $X_1(p)$ has genus 0 (*i.e.*, for $p \leq 7$), and for $p = 11$ it is equivalent to the well-known fact that the elliptic curve $X_1(11)$ has j-invariant with a simple pole at 11 (the j-invariant is $-2^{12}/11$). The strategy of the proof in the general case is to show that $X_1(p)_{/\mathbf{Q}}$ has a regular proper model $\mathcal{X}_1(p)_{/\mathbf{Z}_{(p)}}$ whose closed fiber is geometrically integral. Once we have such a model, by using the well-known dictionary relating the Néron model of a generic-fiber Jacobian with the relative Picard scheme of a regular proper model (see [9, Ch. 9], esp. [9, 9.5/4, 9.6/1], and the references therein), it follows that the Néron model of $J_1(p)$ over $\mathbf{Z}_{(p)}$ has (geometrically) connected closed fiber, as desired. The main work is therefore to prove the following theorem:

Theorem 1.1.2. *Let p be a prime. There is a regular proper model $\mathcal{X}_1(p)$ of $X_1(p)_{/\mathbf{Q}}$ over $\mathbf{Z}_{(p)}$ with geometrically integral closed fiber.*

What we really prove is that if $X_1(p)^{\mathrm{reg}}$ denotes the minimal regular resolution of the normal (typically non-regular) coarse moduli scheme $X_1(p)_{/\mathbf{Z}_{(p)}}$, then a minimal regular contraction $\mathcal{X}_1(p)$ of $X_1(p)^{\mathrm{reg}}$ has geometrically integral closed fiber; after all the contractions of -1-curves are done, the component that remains corresponds to the component of $X_1(p)_{/\mathbf{F}_p}$ classifying étale order-p subgroups. When $p > 7$, so the generic fiber has positive genus, such a minimal regular contraction is the unique minimal regular proper model of $X_1(p)_{/\mathbf{Q}}$.

Theorem 1.1.2 provides natural examples of a finite map π between curves of arbitrarily large genus such that π does not extend to a morphism of the minimal regular proper models. Indeed, consider the natural map

$$\pi : X_1(p)_{/\mathbf{Q}} \to X_0(p)_{/\mathbf{Q}}.$$

When $p = 11$ or $p > 13$, the target has minimal regular proper model over $\mathbf{Z}_{(p)}$ with reducible geometric closed fiber [45, Appendix], while the source has minimal regular proper model with (geometrically) integral closed fiber, by Theorem 1.1.2. If the map extended, it would be proper and dominant (as source and target have unique generic points), and hence surjective. On the level of closed fibers, there cannot be a surjection from an irreducible scheme onto a reducible scheme. By the valuative criterion for properness, π is defined in codimension 1 on minimal regular proper models, so there are finitely many points of $\mathcal{X}_1(p)$ in codimension 2 where π cannot be defined.

Note that the fiber of $J_1(p)$ at infinity need not be connected. More specifically, a modular-symbols computation shows that the component group of $J_1(p)(\mathbf{R})$ has order 2 for $p = 17$ and $p = 41$. In contrast, A. Agashe has observed that [47, §1.3] implies that $J_0(p)(\mathbf{R})$ is always connected.

Rather than prove Theorem 1.1.2 directly, we work out the minimal regular model for $X_H(p)$ over $\mathbf{Z}_{(p)}$ for any subgroup $H \subseteq (\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ and use this to study the mod p component group of the Jacobian $J_H(p)$; note that $J_H(p)$ usually does not have semistable reduction. Our basic method is to use a variant on the classical Jung–Hirzebruch method for complex surfaces, adapted to the case of a proper curve over an arbitrary discrete valuation ring. We refer the reader to Theorem 2.4.1 for the main result in this direction; this is the main new theoretical contribution of the paper. This technique will be applied to prove:

THEOREM 1.1.3. *For any prime p and any subgroup H of $(\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$, the natural surjective map $J_H(p) \to J_0(p)$ of Albanese functoriality induces an injection on geometric component groups of mod-p fibers, with the component group $\Phi(\mathcal{J}_H(p)_{/\overline{\mathbf{F}}_p})$ being cyclic of order $|H|/\gcd(|H|, 6)$. In particular, the finite étale component-group scheme $\Phi(\mathcal{J}_H(p)_{/\mathbf{F}_p})$ is constant over $\mathbf{F}_p$.*

If we view the constant cyclic component group $\Phi(\mathcal{J}_0(p)_{/\mathbf{F}_p})$ as a quotient of the cyclic $(\mathbf{Z}/p)^\times/\{\pm 1\}$, then the image of the subgroup $\Phi(\mathcal{J}_H(p)_{/\mathbf{F}_p})$ in this quotient is the image of $H \subseteq (\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ in this quotient.

Remark 1.1.4. The non-canonical nature of presenting one finite cyclic group as a quotient of another is harmless when following images of subgroups under maps, so the final part of Theorem 1.1.3 is well-posed.

The constancy in Theorem 1.1.3 follows from the injectivity claim and the fact that $\Phi(\mathcal{J}_0(p)_{/\mathbf{F}_p})$ is constant. Such constancy was proved by Mazur-Rapoport [45, Appendix], where it is also shown that this component group for $J_0(p)$ is cyclic of the order indicated in Theorem 1.1.3 for $H = (\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$.

Since the Albanese map is compatible with the natural map $\mathbf{T}_H(p) \to \mathbf{T}_0(p)$ on Hecke rings and Mazur proved [45, §11] that $\Phi(\mathcal{J}_0(p)_{/\overline{\mathbf{F}}_p})$ is Eisenstein as a $\mathbf{T}_0(p)$-module, we obtain:

COROLLARY 1.1.5. *The Hecke module $\Phi(\mathcal{J}_H(p)_{/\overline{\mathbf{F}}_p})$ is Eisenstein as a $\mathbf{T}_H(p)$-module* (i.e., *T_ℓ acts as $1 + \ell$ for all $\ell \neq p$ and $\langle d\rangle$ acts trivially for all $d \in (\mathbf{Z}/p\mathbf{Z})^\times$*).

In view of Eisenstein results for component groups due to Edixhoven [18] and Ribet [54], [55] (where Ribet gives examples of non-Eisenstein component groups), it would be of interest to explore the range of validity of Corollary 1.1.5 when auxiliary prime-to-p level structure of $\Gamma_0(N)$-type is allowed. A modification of the methods we use should be able to settle this more general problem. In fact, a natural approach would be to aim to essentially reduce to the Eisenstein results in [54] by establishing a variant of the above injectivity result on component groups when additional $\Gamma_0(N)$ level structure is allowed away from p. This would require a new idea in order to avoid the crutch of cyclicity (the case of $\Gamma_1(N)$ seems much easier to treat using our methods because the relevant groups tend to be cyclic, though we have not worked out the details for $N > 1$), and preliminary calculations of divisibility among orders of component groups are consistent with such injectivity.

In order to prove Theorem 1.1.3, we actually first prove a surjectivity result:

THEOREM 1.1.6. *The map of Picard functoriality $J_0(p) \to J_H(p)$ induces a surjection on mod p component groups, with the mod p component group for $J_H(p)$ having order $|H|/\gcd(|H|, 6)$.*

In particular, each connected component of $\mathcal{J}_H(p)_{/\mathbf{F}_p}$ *contains a multiple of the image of* $(0)-(\infty) \in \mathcal{J}_0(p)(\mathbf{Z}_{(p)})$ *in* $\mathcal{J}_H(p)(\mathbf{F}_p)$.

Let us explain how to deduce Theorem 1.1.3 from Theorem 1.1.6. Recall [28, Exposé IX] that for a discrete valuation ring R with fraction field K and an abelian variety A over K over R, Grothendieck's biextension pairing sets up a bilinear pairing between the component groups of the closed fibers of the Néron models of A and its dual A'. Moreover, under this pairing the component-group map induced by a morphism $f : A \to B$ (to another abelian variety) has as an adjoint the component-group map induced by the dual morphism $f' : B' \to A'$. Since Albanese and Picard functoriality maps on Jacobians are dual to each other, the surjectivity of the Picard map therefore implies the injectivity of the Albanese map provided that the biextension pairings in question are perfect pairings (and then the description of the image of the resulting Albanese injection in terms of H as in Theorem 1.1.3 follows immediately from the order calculation in Theorem 1.1.6).

In general the biextension pairing for an abelian variety and its dual need not be perfect [8], but once it is known to be perfect for the $J_H(p)$'s then surjectivity of the Picard map in Theorem 1.1.6 implies the injectivity of the Albanese map as required in Theorem 1.1.3. To establish the desired perfectness, one can use either that the biextension pairing is always perfect in case of generic characteristic 0 with a perfect residue field [6, Thm. 8.3.3], or that surjectivity of the Picard map ensures that $J_H(p)$ has mod p component group of order prime to p, and the biextension pairing is always perfect on primary components prime to the residue characteristic [7, §3, Thm. 7].

It is probable that the results concerning the component groups $\Phi(\mathcal{J}_H(p)_{/\overline{\mathbf{F}}_p})$ and the maps between them that are proved in this article via models of $X_H(p)$ over $\mathbf{Z}_{(p)}$ can also be proved using [20, 5.4, Rem. 1], and the well-known stable model of $X_1(p)$ over $\mathbf{Z}_{(p)}[\zeta_p]$ that one can find for example in [30]. (This observation was prompted by questions of Robert Coleman.) However, such an approach does not give information on regular models of $X_H(p)$ over $\mathbf{Z}_{(p)}$. Hence we prefer the method of this paper.

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1.2 Outline

Section 1.3 contains a few background notational remarks. In Section 2 we develop the basic Jung–Hirzebruch resolution technique in the context of tame cyclic quotient surface singularities. This includes mod-p singularities on many (coarse) modular curves when $p > 3$ and the p-power level structure is only on p-torsion. In Section 3, we recall some general results on moduli problems for elliptic curves and coarse moduli schemes for such problems. In Section 4, we use the results of Sections 2 and 3 to locate all the non-regular points on the coarse moduli scheme $X_H(p)_{/\mathbf{Z}_{(p)}}$ (*e.g.*, when H is trivial this is the set of $\mathbf{F}_p$-rational points $(E, 0)$ with $j = 0, 1728$). In Section 5, we use the Jung–Hirzebruch formulas to compute the minimal regular resolution $X_H(p)^{\mathrm{reg}}$ of $X_H(p)_{/\mathbf{Z}_{(p)}}$, and we use use a series of intersection number computations to obtain a regular proper model for $X_H(p)_{/\mathbf{Q}}$; from this, the desired results on component groups follow. We conclude in Section 6 with some computer computations concerning the arithmetic of $J_1(p)$ for small p, where (among other things) we propose a formula for the order of the torsion subgroup of $J_1(p)(\mathbf{Q})$.

To avoid using Weierstrass equations in proofs, we have sometimes argued more abstractly than is strictly necessary, but this has the merit of enabling us to treat cusps by essentially the same methods as the other points. We would prefer to avoid mentioning j-invariants, but it is more succinct to say "cases with $j = 0$" than it is to say "cases such that $\mathrm{Aut}(E_{/k})$ has order 6."

Because we generally use methods of abstract deformation theory, the same approach should apply to Drinfeld modular curves, as well as to cases with auxiliary level structure away from p (including mod p component groups of suitable Shimura curves associated to indefinite quaternion algebras over $\mathbf{Q}$, with p not dividing the discriminant). However, since a few additional technicalities arise, we leave these examples to be treated at a future time.

1.3 Notation and terminology

Throughout this paper, p denotes an arbitrary prime unless otherwise indicated. Although the cases $p \leq 3$ are not very interesting from the point of view of our main results, keeping these cases in mind has often led us to more conceptual proofs. We write $\Phi_p(T) = (T^p - 1)/(T - 1) \in \mathbf{Z}[T]$ to denote the pth cyclotomic polynomial (so $\Phi_p(T + 1)$ is p-Eisenstein).

We write $V^\vee$ to denote the dual of a vector space V, and we write $\mathcal{F}^\vee$ to denote the dual of a locally free sheaf $\mathcal{F}$.

If X and S' are schemes over a scheme S then $X_{/S'}$ and $X_{S'}$ denote $X \times_S S'$. If S is an integral scheme with function field K and X is a K-scheme, by a *model* of X (over S) we mean a flat S-scheme with generic fiber X.

By an *S-curve* over a scheme S we mean a flat separated finitely presented map $X \to S$ with fibers of pure dimension 1 (the fibral dimension condition need only be checked on generic fibers, thanks to [27, IV$_3$, 13.2.3] and a reduction to the noetherian case). Of course, when a map of schemes $X \to S$ is proper flat and finitely presented with geometrically connected generic fibers, then the other fibers are automatically geometrically connected (via reduction to the noetherian case and a Stein factorization argument). For purely technical reasons, we do *not* require S-curves to be proper or to have geometrically connected fibers. The main reason for this is that we want to use étale localization arguments on X without having to violate running hypotheses. The use of Corollary 2.2.4 in the proof of Theorem 2.4.1 illustrates this point.

2 Resolution of singularities

Our eventual aim is to determine the component groups of Jacobians of intermediate curves between $X_1(p)$ and $X_0(p)$. Such curves are exactly the quotient curves $X_H(p) = X_1(p)/H$ for subgroups $H \subseteq (\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$, where we identify the group $\mathrm{Aut}_{\mathbf{Q}}(X_1(p)/X_0(p)) = \mathrm{Aut}_{\overline{\mathbf{Q}}}(X_1(p)/X_0(p))$ with $(\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ via the diamond operators (in terms of moduli, $n \in (\mathbf{Z}/p\mathbf{Z})^\times$ sends a pair (E, P) to the pair $(E, n \cdot P)$). The quotient $X_H(p)_{/\mathbf{Z}_{(p)}}$ is an arithmetic surface with tame cyclic quotient singularities (at least when $p > 3$).

After some background review in Section 2.1 and some discussion of generalities in Section 2.2, in Section 2.3 we will describe a class of curves that give rise to (what we call) *tame cyclic quotient singularities*. Rather than work with global quotient situations X/H, it is more convenient to require such quotient descriptions only on the level of complete local rings. For example, this is what one encounters when computing complete local rings on coarse modular curves: the complete local ring is a subring of invariants of the universal deformation ring under the action of a finite group, but this group-action might not be induced by an action on the global modular curve. In Section 2.4 we establish the Jung–Hirzebruch continued-fraction algorithm that minimally resolves tame cyclic quotient singularities on curves over an arbitrary discrete valuation ring. The proof requires the Artin approximation theorem, and for this reason we need to define the concept of a *curve* as in Section 1.3 without requiring properness or geometric connectivity of fibers.

We should briefly indicate here why we need to use Artin approximation to compute minimal resolutions. Although the end result of our resolution process is intrinsic and of étale local nature on the curve, the mechanism by which the proof gets there depends on coordinatization and is not intrinsic (*e.g.*, we do not blow-up at points, but rather along certain codimension-1 subschemes). The only way we can relate the general case to a coordinate-dependent calculation in a special case is to use Artin approximation to find a common étale neighborhood over the general case and a special case (coupled with the étale local nature of the intrinsic minimal resolution that we are seeking to describe).

These resolution results are applied in subsequent sections to compute a regular proper model of $X_H(p)_{/\mathbf{Q}}$ over $\mathbf{Z}_{(p)}$ in such a way that we can compute both the mod-p geometric component group of the Jacobian $J_H(p)$ and the map induced by $J_0(p) \to J_H(p)$ on mod-p geometric component-groups. In this way, we will prove Theorem 1.1.6 (as well as Theorem 1.1.2 in the case of trivial H).

2.1 Background review

Some basic references for intersection theory and resolution of singularities for connected proper flat regular curves over Dedekind schemes are [29, Exposé X], [13], and [41, Ch. 9].

If S is a connected Dedekind scheme with function field K and X is a normal S-curve, when S is excellent we can construct a resolution of singularities as follows: blow-up the finitely many non-regular points of X (all in codimension 2), normalize, and then repeat until the process stops. That this process always stops is due to a general theorem of Lipman [40]. For more general (*i.e.*, possibly non-excellent) S, and $X_{/S}$ with *smooth* generic fiber, the same algorithm works (including the fact that the non-regular locus consists of only finitely many closed points in closed fibers). Indeed, when $X_{/K}$ is smooth then the non-smooth locus of $X \to S$ is supported on finitely many closed fibers, so we may assume $S = \mathrm{Spec}(R)$ is local. We can then use Lemma 2.1.1 below to bring results down from $X_{/\widehat{R}}$ since $\widehat{R}$ is excellent.

See Theorem 2.2.2 for the existence and uniqueness of a canonical minimal regular resolution $X^{\mathrm{reg}} \to X$ for any connected Dedekind S when $X_{/K}$ smooth. A general result of Lichtenbaum [39] and Shafarevich [61] ensures that when $X_{/S}$ is also proper (with smooth generic fiber if S isn't excellent), by beginning with X^{reg} (or any regular proper model of $X_{/K}$) we can successively blow down -1-curves (see Definition 2.2.1) in closed fibers over S until there are no more such -1-curves, at which point we have reached a relatively minimal model among the regular proper models of $X_{/K}$. Moreover, when $X_{/K}$ is in addition geometrically integral with positive arithmetic genus (*i.e.*, $\mathrm{H}^1(X_{/K}, \mathcal{O}) \neq 0$), this is the unique relatively minimal regular proper model, up to unique isomorphism.

In various calculations below with proper curves, it will be convenient to work over a base that is complete with algebraically closed residue field. Since passage from $\mathbf{Z}_{(p)}$ to $W(\overline{\mathbf{F}}_p)$ involves base change to a strict henselization followed by base change to a completion, in order to not lose touch with the situation over $\mathbf{Z}_{(p)}$ it is useful to keep in mind that formation of the minimal regular proper model (when the generic fiber is smooth with positive genus) is compatible with base change to a completion, henselization, and strict henselization on the base. We will not really require these results, but we do need to use the key fact in their proof: certain base changes do not destroy regularity or normality (and so in particular commute with formation of normalizations). This is given by:

LEMMA 2.1.1. *Let R be a discrete valuation ring with fraction field K and let X be a locally finite type flat R-scheme that has* regular *generic fiber. Let $R \to R'$ be an extension of discrete valuation rings for which $\mathfrak{m}_R R' = \mathfrak{m}_{R'}$ and the residue field extension $k \to k'$ is separable. Assume either that the fraction* *field extension $K \to K'$ is separable or that $X_{/K}$ is smooth* (*so either way, $X_{/K'}$ is automatically regular*).

For any $x' \in X' = X \times_R R'$ lying over $x \in X$, the local ring $\mathcal{O}_{X',x'}$ is regular (*resp. normal*) *if and only if the local ring $\mathcal{O}_{X,x}$ is regular* (*resp. normal*).

Proof. Since $\mathfrak{m}_R R' = \mathfrak{m}_{R'}$, the map $\pi : X' \to X$ induces $\pi_k : X_{/k} \times_k k' \to X_{/k}$ upon reduction modulo $\mathfrak{m}_R$. The separability of k' over k implies that π_k is a regular morphism. Thus, if x and x' lie in the closed fibers then $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat with regular fiber ring $\mathcal{O}_{X',x'}/\mathfrak{m}_x$. Consequently, X is regular at x if and only if X' is regular at x' [44, 23.7]. Meanwhile, if x and x' lie in the generic fibers then they are both regular points since the generic fibers are regular. This settles the regular case.

For the normal case, when X' is normal then the normality of X follows from the faithful flatness of π [44, Cor. to 23.9]. Conversely, when X is normal then to deduce normality of X' we use Serre's "$R_1 + S_2$" criterion. The regularity of X' in codimensions ≤ 1 is clear at points on the regular generic fiber. The only other points of codimension ≤ 1 on X' are the generic points of the closed fiber, and these lie over the (codimension 1) generic points of the closed fiber of X. Such points on X are regular since X is now being assumed to be normal, so the desired regularity on X' follows from the preceding argument. This takes care of the R_1 condition. It remains to check that points $x' \in X'$ in codimensions ≥ 2 contain a regular sequence of length 2 in their local rings. This is clear if x' lies on the regular generic fiber, and otherwise x' is a point of codimension ≥ 1 on the closed fiber. Thus, $x = \pi(x')$ is either a generic point of $X_{/k}$ or is a point of codimension ≥ 1 on $X_{/k}$. In the latter case the normal local ring $\mathcal{O}_{X,x}$ has dimension at least 2 and hence contains a regular sequence of length 2; this gives a regular sequence in the faithfully flat extension ring $\mathcal{O}_{X',x'}$. If instead x is a generic point of $X_{/k}$ then $\mathcal{O}_{X,x}$ is a regular ring. It follows that $\mathcal{O}_{X',x'}$ is regular, so we again get the desired regular sequence (since $\dim \mathcal{O}_{X',x'} \geq 2$). ∎

We wish to record an elementary result in intersection theory that we will use several times later on. First, some notation needs to be clarified: if X is a connected regular proper curve over a discrete valuation ring R with residue field k, and D and D' are two *distinct* irreducible and reduced divisors in the closed fiber, then

$$D.D' := \dim_k \mathrm{H}^0(D \cap D', \mathcal{O}) = \sum_{d \in D \cap D'} \dim_k \mathcal{O}_{D \cap D', d}.$$

This is generally larger than the length of the artin ring $\mathrm{H}^0(D \cap D', \mathcal{O})$, and is called the *$k$-length* of $D \cap D'$. If $F = \mathrm{H}^0(D, \mathcal{O}_D)$, then $D \cap D'$ is also an F-scheme, and so it makes sense to define

$$D ._F D' = \dim_F \mathrm{H}^0(D \cap D', \mathcal{O}) = D.D'/[F : k].$$

We call this the *F-length* of $D \cap D'$. We can likewise define $D._{F'}D'$ for the field $F' = \mathrm{H}^0(D', \mathcal{O})$. If $D' = D$, we define the relative self-intersection $D._F D$ to be $(D.D)/[F : k]$ where $D.D$ is the usual self-intersection number on the k-fiber.

THEOREM 2.1.2. *Let X be a connected regular proper curve over a discrete valuation ring, and let $P \in X$ be a closed point in the closed fiber. Let C_1, C_2 be two (possibly equal) effective divisors supported in the closed fiber of X, with each C_j passing through P, and let C'_j be the strict transform of C_j under the blow-up $\pi : X' = \mathrm{Bl}_P(X) \to X$. We write $E \simeq \mathbf{P}^1_{k(P)}$ to denote the exceptional divsor.*

We have $\pi^{-1}(C_j) = C'_j + m_j E$ where $m_j = \mathrm{mult}_P(C_j)$ is the multiplicity of the curve C_j at P. Also, $m_j = (C'_j)._{k(P)}E$ and

$$C_1.C_2 = C'_1.C'_2 + m_1 m_2 [k(P) : k].$$

Proof. Recall that for a regular local ring R of dimension 2 and any non-zero non-unit $g \in R$, the 1-dimensional local ring R/g has multiplicity (*i.e.*, leading coefficient of its Hilbert-Samuel polynomial) equal to the unique integer $\mu \geq 1$ such that $g \in \mathfrak{m}_R^{\mu}$, $g \notin \mathfrak{m}_R^{\mu+1}$.

We have $\pi^{-1}(C_j) = C'_j + m_j E$ for some positive integer m_j that we must prove is equal to the multiplicity $\mu_j = \mathrm{mult}_P(C_j)$ of C_j at P. We have $E._{k(P)}E = -1$, so $E.E = -[k(P) : k]$, and we also have $\pi^{-1}(C_j).E = 0$, so $m_j = (C'_j.E)/[k(P) : k] = (C'_j)._{k(P)}E$. The strict transform C'_j is the blow-up of C_j at P, equipped with its natural (closed immersion) map into X'. The number m_j is the $k(P)$-length of the scheme-theoretic intersection $C'_j \cap E$; this is the fiber of $\mathrm{Bl}_P(C_j) \to C_j$ over P. Intuitively, this latter fiber is the scheme of tangent directions to C_j at P, but more precisely it is $\mathrm{Proj}(S_j)$, where

$$S_j = \bigoplus_{n \geq 0} \mathfrak{m}_j^n / \mathfrak{m}_j^{n+1},$$

and $\mathfrak{m}_j$ is the maximal ideal of $\mathcal{O}_{C_j,P} = \mathcal{O}_{X,P}/(f_j)$, with f_j a local equation for C_j at P. We have $\mathfrak{m}_j = \mathfrak{m}/(f_j)$ with $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{X,P}$. Since $f_j \in \mathfrak{m}^{\mu_j}$ and $f_j \notin \mathfrak{m}^{\mu_j+1}$,

$$S_j \simeq \mathrm{Sym}_{k(P)}(\mathfrak{m}/\mathfrak{m}^2)/\overline{f}_j = k(P)[u,v]/(\overline{f}_j)$$

with $\overline{f}_j$ denoting the nonzero image of f_j in degree μ_j. We conclude that $\mathrm{Proj}(S_j)$ has $k(P)$-length μ_j, so $m_j = \mu_j$. Thus, we may compute

$$\begin{aligned} C_1.C_2 = \pi^{-1}(C_1).\pi^{-1}(C_2) &= C'_1.C'_2 + 2m_1 m_2 [k(P) : k] + m_1 m_2 E.E \\ &= C'_1.C'_2 + m_1 m_2 [k(P) : k]. \end{aligned}$$

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2.2 Minimal resolutions

It is no doubt well-known to experts that the classical technique of resolution for cyclic quotient singularities on complex surfaces [25, §2.6] can be adapted to the case of tame cyclic quotient singularities on curves over a complete equicharacteristic discrete valuation ring. We want the case of an arbitrary discrete valuation ring, and this seems to be less widely known (it is not addressed in the literature, and was not known to an expert in log-geometry with whom we consulted). Since there seems to be no adequate reference for this more general result, we will give the proof after some preliminary work (*e.g.*, we have to define what we mean by a *tame cyclic quotient singularity*, and we must show that this definition is applicable in many situations. Our first step is to establish the existence and uniqueness of a minimal regular resolution in the case of relative curves over a Dedekind base (the case of interest to us); this will eventually serve to make sense of the *canonical resolution* at a point.

Since we avoid properness assumptions, to avoid any confusion we should explicitly recall a definition.

Definition 2.2.1. Let $X \to S$ be a regular S-curve, with S a connected Dedekind scheme. We say that an integral divisor $D \hookrightarrow X$ in a closed fiber X_s is a -1-*curve* if D is proper over $k(s)$, $\mathrm{H}^1(D, \mathcal{O}_D) = 0$, and $\deg_k \mathcal{O}_D(D) = -1$, where $k = \mathrm{H}^0(D, \mathcal{O}_D)$ is a finite extension of $k(s)$.

By Castelnuovo's theorem, a -1-curve $D \hookrightarrow X$ as in Definition 2.2.1 is k-isomorphic to a projective line over k, where $k = \mathrm{H}^0(D, \mathcal{O}_D)$.

The existence and uniqueness of minimal regular resolutions is given by:

Theorem 2.2.2. *Let $X \to S$ be a normal S-curve over a connected Dedekind scheme S. Assume either that S is excellent or that $X_{/S}$ has smooth generic fiber.*

There exists a birational proper morphism $\pi : X^{\mathrm{reg}} \to X$ such that X^{reg} is a regular S-curve and there are no -1-curves in the fibers of π. Such an X-scheme is unique up to unique isomorphism, and every birational proper morphism $X' \to X$ with a regular S-curve X' admits a unique factorization through π. Formation of X^{reg} is compatible with base change to $\operatorname{Spec} \mathcal{O}_{S,s}$ *and* $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ *for closed points $s \in S$. For local S, there is also compatibility with ind-étale base change $S' \to S$ with local S' whose closed point is residually trivial over that of S.*

We remind that reader that, for technical reasons in the proof of Theorem 2.4.1, we avoid requiring curves to be proper and we do not assume the generic fiber to be geometrically connected. The reader is referred to [41, 9/3.32] for an alternative discussion in the proper case.

Proof. We first assume S to be excellent, and then we shall use Lemma 2.1.1 and some descent considerations to reduce the general case to the excellent case by passage to completions.

As a preliminary step, we wish to reduce to the proper case (to make the proof of uniqueness easier). By Nagata's compactification theorem [43] and the finiteness of normalization for excellent schemes, we can find a schematically dense open immersion $X \hookrightarrow \overline{X}$ with $\overline{X}_{/S}$ normal, proper, and flat over S (hence a normal S-curve). By resolving singularities along $\overline{X} - X$, we may assume the non-regular locus on $\overline{X}$ coincides with that on X. Thus, the existence and uniqueness result for X will follow from that for $\overline{X}$. The assertion on regular resolutions (uniquely) factorizing through π goes the same way. Hence, we now assume (for excellent S) that $X_{/S}$ is proper. We can also assume X to be connected.

By Lemma 2.1.1 and resolution for excellent surfaces, there exists a birational *proper* morphism $X' \to X$ with X' a regular proper S-curve. If there is a -1-curve in the fiber of X' over some (necessarily closed) point of X, then by Castelnuovo we can blow down the -1-curve and $X' \to X$ will factor through the blow-down. This blow-down process cannot continue forever, so we get the existence of $\pi : X^{\rm reg} \to X$ with no -1-curves in its fibers.

Recall the Factorization Theorem for birational *proper* morphisms between regular connected S-curves: such maps factor as a composite of blow-ups at closed points in closed fibers. Using the Factorization Theorem, to prove uniqueness of π and the (unique) factorization through π for any regular resolution of X we just have to show that if $X'' \to X' \to X$ is a tower of birational proper morphisms with regular S-curves X' and X'' such that X' has no -1-curves in its fibers over X, then any -1-curve C in a fiber of $X'' \to X$ is necessarily contracted by $X'' \to X'$. Also, via Stein factorization we can assume that the proper normal connected S-curves X, X', and X'' with common generic fiber over S have geometrically connected fibers over S. We may assume that S is local. Since the map $q : X'' \to X'$ is a composite of blow-ups, we may assume that C meets the exceptional fiber E of the first blow-down $q_1 : X'' \to X''_1$ of a factorization of q. If $C = E$ we are done, so we may assume $C \neq E$. In this case we will show that X is regular, so again uniqueness holds (by the Factorization Theorem mentioned above).

The image $q_1(C)$ is an irreducible divisor on X''_1 with strict transform C, so by Theorem 2.1.2 we conclude that $q_1(C)$ has non-negative self-intersection number, so this self-intersection must be zero. Since $X''_1 \to S$ is its own Stein factorization, and hence has geometrically connected closed fiber, $q_1(C)$ must be the entire closed fiber of X''_1. Thus, X''_1 has irreducible closed fiber, and so the (surjective) proper birational map $X''_1 \to X$ is quasi-finite and hence finite. Since X and X''_1 are normal and connected (hence integral), it follows that $X''_1 \to X$ must be an isomorphism. Thus, X is regular, as desired.

With $X^{\rm reg}$ unique up to (obviously) unique isomorphism, for the base change compatibility we note that the various base changes $S' \to S$ being considered (to completions on S, or to local S' ind-étale surjective over local S and residually trivial at closed points), the base change $X^{\rm reg}_{/S'}$ is regular and proper birational over the normal curve $X_{/S'}$ (see Lemma 2.1.1). Thus, we just have to check that the fibers of $X^{\rm reg}_{/S'} \to X_{/S'}$ do not contain -1-curves. The closed-fiber situation is identical to that before base change, due to the residually trivial condition at closed points, so we are done.

Now suppose we do not assume S to be excellent, but instead assume $X_{/S}$ has smooth generic fiber. In this case all but finitely many fibers of $X_{/S}$ are smooth. Thus, we may reduce to the local case $S = \mathrm{Spec}(R)$ with a discrete valuation ring R. Consider $X_{/\widehat{R}}$, a normal $\widehat{R}$-curve by Lemma 2.1.1. Since $\widehat{R}$ is excellent, there is a minimal regular resolution

$$\pi : (X_{/\widehat{R}})^{\rm reg} \to X_{/\widehat{R}}.$$

By [40, Remark C, p. 155], the map π is a blow-up along a 0-dimensional closed subscheme $\widehat{Z}$ physically supported in the non-regular locus of $X_{/\widehat{R}}$. This $\widehat{Z}$ is therefore physically supported in the closed fiber of $X_{/\widehat{R}}$, yet $\widehat{Z}$ is artinian and hence lies in some infinitesimal closed fiber of $X_{/\widehat{R}}$. Since $X \times_R \widehat{R} \to X$ induces isomorphisms on the level of nth infinitesimal closed-fibers for all n, there is a unique 0-dimensional closed subscheme Z in X with $Z_{/\widehat{R}} = \widehat{Z}$ inside of $X_{/\widehat{R}}$.

Since the blow-up $\mathrm{Bl}_Z(X)$ satisfies

$$\mathrm{Bl}_Z(X)_{/\widehat{R}} \simeq \mathrm{Bl}_{\widehat{Z}}(X_{/\widehat{R}}) = (X_{/\widehat{R}})^{\rm reg},$$

by Lemma 2.1.1 we see that $\mathrm{Bl}_Z(X)$ is a regular S-curve. There are no -1-curves in its fibers over X since $\mathrm{Spec}\,\widehat{R} \to \mathrm{Spec}\,R$ is an isomorphism over $\mathrm{Spec}\,R/\mathfrak{m}$. This establishes the existence of $\pi : X^{\rm reg} \to X$, as well as its compatibility with base change to completions on S. To establish uniqueness of π, or more generally its universal factorization property, we must prove that certain birational maps from regular S-curves to $X^{\rm reg}$ are morphisms. This is handled by a standard graph argument that can be checked after faithfully flat base change to $\widehat{R}$ (such base change preserves regularity, by Lemma 2.1.1). Thus, the uniqueness results over the excellent base $\widehat{R}$ carry over to our original R. The same technique of base change to $\widehat{R}$ shows compatibility with ind-étale base change that is residually trivial over closed points.

□

One mild enhancement of the preceding theorem rests on a pointwise definition:

Definition 2.2.3. Let $X_{/S}$ be as in Theorem 2.2.2, and let $\Sigma \subseteq X$ be a finite set of closed points in closed fibers over S. Let U be an open in X containing Σ such that U does not contain the finitely many non-regular points of X outside of Σ. We define the *minimal regular resolution along* Σ to be the morphism $\pi_\Sigma : X_\Sigma \to X$ obtained by gluing $X - \Sigma$ with the part of $X^{\rm reg}$ lying over U (note: the choice of U does not matter, and X_Σ is not regular if there are non-regular points of X outside of Σ).

It is clear that the minimal regular resolution along Σ is compatible with local residually-trivial ind-étale base change on a local S, as well as with base change to a (non-generic) complete local ring on S. It is also uniquely characterized among normal S-curves C equipped with a proper birational morphism $\varphi : C \to X$ via the following conditions:

- π_Σ is an isomorphism over $X - \Sigma$,
- X_Σ is regular at points over Σ,
- X_Σ has no -1-curves in its fibers over Σ.

This yields the crucial consequence that (under some mild restrictions on residue field extensions) formation of X_Σ is étale-local on X. This fact is ultimately the reason we did not require properness or geometrically connected fibers in our definition of *S-curve*:

COROLLARY 2.2.4. *Let $X_{/S}$ be a normal S-curve over a connected Dedekind scheme S, and let $\Sigma \subseteq X$ be a finite set of closed points in closed fibers over S. Let $X' \to X$ be étale (so X' is an S-curve), and let Σ' denote the preimage of Σ. Assume that S is excellent or $X_{/S}$ has smooth generic fiber.*

If $X_\Sigma \to X$ denotes the minimal regular resolution along Σ, and $X' \to X$ is residually trivial over Σ, then the base change $X_\Sigma \times_X X' \to X'$ is the minimal regular resolution along Σ'.

Remark 2.2.5. The residual triviality condition over Σ is satisfied when S is local with separably closed residue field, as then all points of Σ have separably closed residue field (and so the étale $X' \to X$ must induce trivial residue field extensions over such points).

Proof. Since $X_\Sigma \times_X X'$ is étale over X_Σ, we conclude that $X_\Sigma \times_X X'$ is an S-curve that is regular along the locus over $\Sigma' \subseteq X'$, and its projection to X' is proper, birational, and an isomorphism over $X' - \Sigma'$. It remains to check that

$$X_\Sigma \times_X X' \to X' \tag{2.2.1}$$

has no -1-curves in the proper fibers over Σ'. Since $X' \to X$ is residually trivial over Σ (by hypothesis), so this is clear.

□

2.3 NIL-SEMISTABLE CURVES

In order to compute minimal regular resolutions of the sort that arise on $X_H(p)$'s, it is convenient to study the following concept before we discuss resolution of singularities. Let S be a connected Dedekind scheme and let X be an S-curve.

Definition 2.3.1. For a closed point $s \in S$, a closed point $x \in X_s$ is *nil-semistable* if the reduced fiber-curve X_s^{red} is semistable over $k(s)$ at x and all of the analytic branch multiplicities through x are not divisible by $\mathrm{char}(k(s))$. If X_s^{red} is semistable for all closed points $s \in S$ and all irreducible components of X_s have multiplicity not divisible by $\mathrm{char}(k(s))$, X is a *nil-semistable curve* over S.

Considerations with excellence of the fiber X_s show that the number of analytic branches in Definition 2.3.1 may be computed on the formal completion at a point over x in $X_{s/k'}$ for any separably closed extension k' of $k(s)$. We will use the phrase "analytic branch" to refer to such (formal) branches through a point over x in such a geometric fiber over s.

As is well-known from [34], many fine moduli schemes for elliptic curves are nil-semistable.

Fix a closed point $s \in S$. From the theory of semistable curves over fields [24, III, §2], it follows that when $x \in X_s^{\mathrm{red}}$ is a semistable non-smooth point then the finite extension $k(x)/k(s)$ is separable. We have the following analogue of the classification of semistable curve singularities:

Lemma 2.3.2. *Let $x \in X_s$ be a closed point and let $\pi_s \in \mathcal{O}_{S,s}$ be a uniformizer.*

If x is a nil-semistable point at which X is regular, then the underlying reduced scheme of the geometric closed fiber over s has either one or two analytic branches at a geometric point over x, with these branches smooth at x. When moreover $k(x)/k(s)$ is separable and there is exactly one analytic branch at $x \in X_s$, with multiplicity m_1 in $\mathcal{O}^{\mathrm{sh}}_{X_s,x}$, then

$$\widehat{\mathcal{O}^{\mathrm{sh}}_{X,x}} \simeq \widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}[\![t_1, t_2]\!]/(t_1^{m_1} - \pi_s). \tag{2.3.1}$$

If there are two analytic branches (so $k(x)/k(s)$ is automatically separable), say with multiplicities m_1 and m_2 in $\mathcal{O}^{\mathrm{sh}}_{X_s,x}$, then

$$\widehat{\mathcal{O}^{\mathrm{sh}}_{X,x}} \simeq \widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}[\![t_1, t_2]\!]/(t_1^{m_1} t_2^{m_2} - \pi_s). \tag{2.3.2}$$

Conversely, if $\widehat{\mathcal{O}^{\mathrm{sh}}_{X,x}}$ admits one of these two explicit descriptions with the exponents not divisible by $\mathrm{char}(k(s))$, then x is a nil-semistable regular point on X with $k(x)/k(s)$ separable.

In view of this lemma, we call the exponents in the formal isomorphisms (2.3.1) and (2.3.2) the *analytic geometric multiplicities* of X_s at x (this requires $k(x)/k(s)$ to be separable). We emphasize that these exponents can be computed after base change to any separably closed extension of $k(s)$ when x is nil-semistable with $k(x)/k(s)$ separable.

Proof. First assume $x \in X_s^{\mathrm{red}}$ is a non-smooth semistable point and X is regular at x. Since $k(x)$ is therefore finite separable over $k(s)$, we can make a base change to the completion of a strict henselization of $\mathcal{O}_{S,s}$ to reduce to the case $S = \mathrm{Spec}(W)$ with a complete discrete valuation ring W having separably closed residue field k such that x a k-rational point. Since $\widehat{\mathcal{O}}_{X,x}$ is a 2-dimensional complete regular local W-algebra with residue field k, it is a quotient of $W[\![t_1, t_2]\!]$ and hence has the form $W[\![t_1, t_2]\!]/(f)$ where f is a regular parameter. The semistability condition and non-smoothness of $X^{\mathrm{red}}_{/k}$ at x imply

$$k[\![t_1, t_2]\!]/\mathrm{rad}(\overline{f}) = (k[\![t_1, t_2]\!]/(\overline{f}))_{\mathrm{red}} \simeq \widehat{\mathcal{O}}_{X^{\mathrm{red}}_{/k},x} \simeq k[\![u_1, u_2]\!]/(u_1 u_2)$$

where $\overline{f} = f \bmod \mathfrak{m}_W$, so $\overline{f}$ has exactly two distinct irreducible factors and these have distinct (non-zero) tangent directions in $X^{\mathrm{red}}_{/k}$ through x. We can choose t_1 and t_2 to lift these tangent directions, so upon replacing f with a unit multiple we may assume $\overline{f} = t_1^{m_1} t_2^{m_2} \bmod \mathfrak{m}_W$ for some $m_1, m_2 \geq 1$ not divisible by $p = \mathrm{char}(k) \geq 0$. Let π be a uniformizer of W, so $f = t_1^{m_1} t_2^{m_2} - \pi g$ for some g, and g must be a unit since f is a regular parameter. Since some m_j is not divisible by p, and hence the unit g admits an m_jth root, by unit-rescaling of the corresponding t_j we get to the case $g = 1$.

In the case when X_s^{red} is smooth at x and $k(x)/k(s)$ is separable, we may again reduce to the case in which $S = \mathrm{Spec}\, W$ with complete discrete valuation ring W having separably closed residue field k and $k(x) = k$. In this case, there is just one analytic branch and we see by a variant of the preceding argument that the completion of $\mathcal{O}^{\mathrm{sh}}_{X,x}$ has the desired form.

The converse part of the lemma is clear. □

In Definition 2.3.6, we shall give a local definition of the class of curve-singularities that we wish to resolve, but we will first work through some global considerations that motivate the relevance of the local Definition 2.3.6.

Assume X is *regular*, and let H be a finite group and assume we are given an action of H on $X_{/S}$ that is free on the scheme of generic points (*i.e.*, no non-identity element of H acts trivially on a connected component of X). A good example to keep in mind is the (affine) fine moduli scheme over $S = \mathrm{Spec}(\mathbf{Z}_{(p)})$ of $\Gamma_1(p)$-structures on elliptic curves equipped with auxiliary full level ℓ-structure for an odd prime $\ell \neq p$, and $H = \mathrm{GL}_2(\mathbf{F}_\ell)$ acting in the usual manner (see Section 3 for a review of these basic level structures).

We wish to work with a quotient S-curve $X' = X/H$, so we now also assume that X is quasi-projective Zariski-locally on S. Clearly $X \to X'$ is a finite H-equivariant map with the expected universal property; in the above modular-curve example, this quotient X' is the coarse moduli scheme $Y_1(p)$ over $\mathbf{Z}_{(p)}$. We also now assume that S is excellent or $X_{/K}$ is smooth, so that there are only finitely many non-regular points (all in codimension 2) and various results centering on resolution of singularities may be applied.

The S-curve X' has regular generic fiber (and even smooth generic fiber when $X_{/S}$ has smooth generic fiber), and X' is regular away from finitely many closed points in the closed fibers. Our aim is to understand the *minimal regular resolution* ${X'}^{\mathrm{reg}}$ of X', or rather to describe the geometry of the fibers of $X'^{\text{reg}} \to X'$ over non-regular points x' satisfying a mild hypothesis on the structure of $X \to X'$ over x'.

We want to compute the minimal regular resolution for $X' = X/H$ at non-regular points x' that satisfy several conditions. Let $s \in S$ be the image of x', and let $p \geq 0$ denote the common characteristic of $k(x')$ and $k(s)$. Pick $x \in X$ over x'.

- We assume that X is nil-semistable at x (by the above hypotheses, X is also regular at x).
- We assume that the inertia group $H_{x|x'}$ in H at x (*i.e.*, the stablizer in H of a geometric point over x) has order not divisible by p (so this group acts semi-simply on the tangent space at a geometric point over x).
- When there are two analytic branches through x, we assume $H_{x|x'}$ does not interchange them.

These conditions are independent of the choice of x over x' and can be checked at a geometric point over x, and when they hold then the number of analytic branches through x coincides with the number of analytic branches through x' (again, we are really speaking about analytic branches on a geometric fiber over s).

Since p does not divide $|H_{x|x'}|$, it follows that $k(x')$ is the subring of invariants under the action of $H_{x|x'}$ on $k(x)$, so a classical theorem of Artin ensures that $k(x)/k(x')$ is separable (and even Galois). Thus, $k(x)/k(s)$ is separable if and only if $k(x')/k(s)$ is separable, and such separability holds when the point $x \in X_s^{\text{red}}$ is semistable but not smooth. Happily for us, this separability condition over $k(s)$ is always satisfied (we are grateful to Lorenzini for pointing this out):

Lemma 2.3.3. *With notation and hypotheses as above, particularly with $x' \in X' = X/H$ a non-regular point, the extension $k(x')/k(s)$ is separable.*

Proof. Recall that, by hypothesis, $x \in X_s^{\text{red}}$ is either a smooth point or an ordinary double point. If x is a non-smooth point on the curve X_s^{red}, then the desired separability follows from the theory of ordinary double point singularities. Thus, we may (and do) assume that x is a smooth point on X_s^{red}.

We may also assume S is local and strictly henselian, so $k(s)$ is separably closed and hence $k(x)$ and $k(x')$ are separably closed. Thus, $k(x) = k(x')$ and $H_{x|x'}$ is the physical stabilizer of the point $x \in X$. We need to show that the common residue field $k(x) = k(x')$ is separable over $k(s)$. If we let $X'' = X/H_{x|x'}$, then the image x'' of x in X'' has complete local ring isomorphic to that of $x' \in X'$, so we may replace X' with X'' to reduce to the case when H has order not divisible by p and x is in the fixed-point locus of H. By [20, Prop. 3.4], the fixed-point locus of H in X admits a closed-subscheme structure in X that is smooth over S. On the closed fiber this smooth scheme is finite and hence étale over $k(s)$, so its residue fields are separable over $k(s)$. ∎

The following refinement of Lemma 2.3.2 is adapted to the $H_{x|x'}$-action, and simultaneously handles the cases of one and two (geometric) analytic branches through x'.

Lemma 2.3.4. *With hypotheses as above, there is an* $\widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}$*-isomorphism*

$$\widehat{\mathcal{O}^{\mathrm{sh}}_{X,x}} \simeq \widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}[\![t_1,t_2]\!]/(t_1^{m_1}t_2^{m_2}-\pi_s)$$

(with $m_1>0$, $m_2\geq 0$) such that the $H_{x|x'}$-action looks like $h(t_j)=\chi_j(h)t_j$ for characters $\chi_1,\chi_2 : H_{x|x'}\to \widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}^{\times}$ that are the Teichmüller lifts of characters giving a decomposition of the semisimple $H_{x|x'}$-action on the 2-dimensional cotangent space at a geometric point over x. Moreover, $\chi_1^{m_1}\chi_2^{m_2}=1$.

The characters χ_j also describe the action of $H_{x|x'}$ on the tangent space at (a geometric point over) x. There are two closed-fiber analytic branches through x when m_1 and m_2 are positive, and then the branch with formal parameter t_2 has multiplicity m_1 since

$$(k[\![t_1,t_2]\!]/(t_1^{m_1}t_2^{m_2}))[1/t_2]=k(\!(t_2)\!)[t_1]/(t_1^{m_1})$$

has length m_1. Likewise, when $m_2>0$ it is the branch with formal parameter t_1 that has multiplicity m_2.

Proof. We may assume $S=\operatorname{Spec} W$ with W a complete discrete valuation ring having separably closed residue field k and uniformizer π, so x is k-rational. Let $R=\widehat{\mathcal{O}^{\mathrm{sh}}_{X,x}}=\widehat{\mathcal{O}}_{X,x}$. We have seen in Lemma 2.3.2 that there is an isomorphism of the desired type as W-algebras, but we need to find better such t_j's to linearize the $H_{x|x'}$-action.

We first handle the easier case $m_2=0$. In this case there is only one minimal prime (t_1) over (π), so $h(t_1)=u_ht_1$ for a unique unit $u_h\in R^\times$. Since $t_1^{m_1}=\pi$ is $H_{x|x'}$ invariant, we see that $u_h\in\mu_{m_1}(R)$ is a Teichmüller lift from k (since $p\nmid m_1$). Thus, $h(t_1)=\chi_1(h)t_1$ for a character $\chi_1 : H_{x|x'}\to R^\times$ that is a lift of a character for $H_{x|x'}$ on $\mathrm{Cot}_x(X)$. Since $H_{x|x'}$ acts semisimply on the 2-dimensional cotangent space $\mathrm{Cot}_x(X)$ and there is a stable line spanned by $t_1 \bmod \mathfrak{m}_x^2$, we can choose t_2 to lift an $H_{x|x'}$-stable line complementary to the one spanned by $t_1 \bmod \mathfrak{m}_x^2$. If χ_2 denotes the Teichmüller lift of the character for $H_{x|x'}$ on this complementary line, then

$$h(t_2)=\chi_2(h)(t_2+\delta_h)$$

with $\delta_h\in\mathfrak{m}_x^i$ for some $i\geq 2$. It is straightfoward to compute that

$$h\mapsto \delta_h \bmod \mathfrak{m}_x^{i+1}$$

is a 1-cocycle with values in the twisted $H_{x|x'}$-module $\chi_2^{-1}\otimes(\mathfrak{m}_x^i/\mathfrak{m}_x^{i+1})$. Changing this 1-cocycle by a 1-coboundary corresponds to adding an element of $\mathfrak{m}_x^i/\mathfrak{m}_x^{i+1}$ to $t_2 \bmod \mathfrak{m}_x^{i+1}$. Since

$$\mathrm{H}^1(H_{x|x'},\chi_2^{-1}\otimes(\mathfrak{m}_x^i/\mathfrak{m}_x^{i+1}))=0,$$

we can successively increase $i \geq 2$ and pass to the limit to find a choice of t_2 such that $H_{x|x'}$ acts on t_2 through the character χ_2. That is, $h(t_1) = \chi_1(h)t_1$ and $h(t_2) = \chi_2(h)t_2$ for all $h \in H_{x|x'}$. This settles the case $m_2 = 0$.

Now we turn to the more interesting case when also $m_2 > 0$, so there are two analytic branches through x. By hypothesis, the $H_{x|x'}$-action preserves the two minimal primes (t_1) and (t_2) over (π) in R. We must have $h(t_1) = u_h t_1$, $h(t_2) = v_h t_2$ for unique units $u_h, v_h \in R^\times$. Since $t_1^{m_1} t_2^{m_2} = \pi$, by applying h we get $u_h^{m_1} v_h^{m_2} = 1$.

Consider what happens if we replace t_2 with a unit multiple $t_2' = vt_2$, and then replace t_1 with the unit multiple $t_1' = v^{-m_2/m_1} t_1$ so as to ensure ${t'}_1^{m_1} {t'}_2^{m_2} = \pi$. Note that an m_1th root v^{-m_2/m_1} of the unit v^{-m_2} makes sense since k is separably closed and $p \nmid m_1$. The resulting map $W[\![t_1', t_2']\!]/({t'}_1^{m_1} {t'}_2^{m_2} - \pi) \to R$ is visibly surjective, and hence is an isomorphism for dimension reasons. Switching to these new coordinates on R has the effect of changing the 1-cocycle $\{v_h\}$ by a 1-coboundary, and *every* 1-cocycle cohomologous to $\{v_h\}$ is reached by making such a unit multiple change on t_2.

By separately treating residue characteristic 0 and positive residue characteristic, an inverse limit argument shows that $\mathrm{H}^1(H_{x|x'}, U)$ vanishes, where $U = \ker(R^\times \twoheadrightarrow k^\times)$. Thus, the natural map $\mathrm{H}^1(H_{x|x'}, R^\times) \to \mathrm{H}^1(H_{x|x'}, k^\times)$ is injective. The $H_{x|x'}$-action on $k^\times$ is trivial since $H_{x|x'}$ acts trivially on W, so

$$\mathrm{H}^1(H_{x|x'}, k^\times) = \mathrm{Hom}(H_{x|x'}, k^\times) = \mathrm{Hom}(H_{x|x'}, k^\times_{\mathrm{tors}}),$$

with all elements in the torsion subgroup $k^\times_{\mathrm{tors}}$ of order not divisible by p and hence uniquely multiplicatively lifting into R. Thus,

$$\mathrm{H}^1(H_{x|x'}, R^\times) \to \mathrm{H}^1(H_{x|x'}, k^\times)$$

is bijective, and so replacing t_1 and t_2 with suitable unit multiples allows us to assume $h(t_2) = \chi_2(h)t_2$, with $\chi_2 : H_{x|x'} \to W^\times_{\mathrm{tors}}$ some homomorphism of order not divisible by p (since $H_{x|x'}$ acts trivially on $k^\times$ and $p \nmid |H_{x|x'}|$).

Since

$$1 = u_h^{m_1} v_h^{m_2} = u_h^{m_1} \chi_2(h)^{m_2}$$

and $p \nmid m_1$, we see that u_h is a root of unity of order not divisible by p. Viewing $k^\times_{\mathrm{tors}} \subseteq R^\times$ via the Teichmüller lifting, we conclude that $u_h \in k^\times_{\mathrm{tors}} \subseteq R^\times$. Thus, we can write $h(t_1) = \chi_1(h)t_1$ for a homomorphism $\chi_1 : H_{x|x'} \to W^\times_{\mathrm{tors}}$ also necessarily of order not divisible by p. The preceding calculation also shows that $\chi_1^{m_1} \chi_2^{m_2} = 1$ since $u_h^{m_1} v_h^{m_2} = 1$.

□

Although Lemma 2.3.4 provides good (geometric) coordinate systems for describing the inertia action, one additional way to simplify matters is to reduce to the case in which the tangent-space characters χ_1 and χ_2 are powers of each other. We wish to explain how this special situation is essentially the general case (in the presence of our running assumption that H acts freely on the scheme of generic points of X).

First, observe that $H_{x|x'}$ acts faithfully on the tangent space $T_x(X)$ at x. Indeed, if an element in $H_{x|x'}$ acts trivially on the tangent space $T_x(X)$, then by Lemma 2.3.4 it acts trivially on the completion of $\mathcal{O}^{\mathrm{sh}}_{X,x}$ and hence acts trivially on the corresponding connected component of the normal X. By hypothesis, H acts freely on the scheme of generic points of X, so we conclude that the product homomorphism

$$\chi_1 \times \chi_2 : H_{x|x'} \hookrightarrow k(x)^{\times}_{\mathrm{sep}} \times k(x)^{\times}_{\mathrm{sep}}, \tag{2.3.3}$$

is *injective* (where $k(x)_{\mathrm{sep}}$ is the separable closure of $k(x)$ used when constructing $\mathcal{O}^{\mathrm{sh}}_{X,x}$). In particular, $H_{x|x'}$ is a product of two cyclic groups (one of which might be trivial).

LEMMA 2.3.5. *Let $\kappa_j = |\ker(\chi_j)|$. The characters $\chi_1^{\kappa_2}$ and $\chi_2^{\kappa_1}$ factor through a common quotient of $H_{x|x'}$ as faithful characters. When $H_{x|x'}$ is cyclic, this quotient is $H_{x|x'}$.*

In addition, $\kappa_2|m_1$ and $\kappa_1|m_2$.

The cyclicity condition on $H_{x|x'}$ will hold in our application to modular curves, as then even H is cyclic.

Proof. The injectivity of (2.3.3) implies that χ_1 is faithful on $\ker(\chi_2)$ and χ_2 is faithful on $\ker(\chi_1)$. Since $\chi_1^{m_1}\chi_2^{m_2} = 1$, we get $\kappa_2|m_1$ and $\kappa_1|m_2$ (even if $m_2 = 0$).

For the proof that the indicated powers of the χ_j's factor as faithful characters of a common quotient of $H_{x|x'}$, it is enough to focus attention on ℓ-primary parts for a prime ℓ dividing $|H_{x|x'}|$ (so $\ell \neq p$). More specifically, if G is an finite ℓ-group that is either cyclic or a product of two cyclic groups, and $\psi_0, \psi_1 : G \to \mathbf{Z}/\ell^n\mathbf{Z}$ are homomorphisms such that $\psi_0 \times \psi_1$ is injective (*i.e.*, $\ker(\psi_0)\cap\ker(\psi_1) = \{1\}$), then we claim that the $\psi_j^{\kappa_{1-j}}$'s factor as faithful characters on a common quotient of G, where $\kappa_j = |\ker(\psi_j)|$. If one of the ψ_j's is faithful (or equivalently, if the ℓ-group G is cyclic), this is clear. This settles the case in which G is cyclic, so we may assume G is a product of two non-trivial cyclic ℓ-groups and that both ψ_j's have non-trivial kernel. Since the ℓ-torsion subgroups $\ker(\psi_j)[\ell]$ must be non-trivial with trivial intersection, these must be distinct lines spanning $G[\ell]$. Passing to group $G/G[\ell]$ and the characters ψ_j^{ℓ} therefore permits us to induct on $|G|$.

□

By the lemma, we conclude that the characters $\chi_1' = \chi_1^{\kappa_2}$ and $\chi_2' = \chi_1^{\kappa_1}$ both factor faithfully through a common (cyclic) quotient $H'_{x|x'}$ of $H_{x|x'}$. Define $t_1' = t_1^{\kappa_2}$ and $t_2' = t_2^{\kappa_1}$. Since formation of $H_{x|x'}$-invariants commutes with passage to quotients on $\widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}$-modules, Lemma 2.3.4 shows that in order to compute the $H_{x|x'}$-invariants of $\widehat{\mathcal{O}}^{\mathrm{sh}}_{X',x'}$ it suffices to compute invariants on the level of $\widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}[\![t_1, t_2]\!]$ and then pass to a quotient. The subalgebra of invariants in

$\widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}[\![t_1, t_2]\!]$ under the subgroup generated by $\ker(\chi_1)$ and $\ker(\chi_2)$ is $\widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}[\![t'_1, t'_2]\!]$, and $H_{x|x'}$ acts on this subalgebra through the quotient $H'_{x|x'}$ via the characters χ'_1 and χ'_2. Letting $m'_1 = m_1/\kappa_2$ and $m'_2 = m_2/\kappa_1$ (so $m'_2 = 0$ in the case of one analytic branch), we obtain the description

$$\widehat{\mathcal{O}^{\mathrm{sh}}_{X',x'}} = (\widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}[\![t'_1, t'_2]\!]/({t'_1}^{m'_1}{t'_2}^{m'_2} - \pi_s))^{H'_{x|x'}} \tag{2.3.4}$$

Obviously $\chi'_2 = {\chi'_1}^{r_{x|x'}}$ for a unique $r_{x|x'} \in (\mathbf{Z}/|H'_{x|x'}|\mathbf{Z})^\times$, as the characters χ'_j are both faithful on $H'_{x|x'}$.

Since $|H'_{x|x'}|$ and $r_{x|x'} \in (\mathbf{Z}/|H'_{x|x'}|\mathbf{Z})^\times$ are intrinsic to $x' \in X' = X/H$ and do not depend on x (or on a choice of $k(x)_{\mathrm{sep}}$), we may denote these two integers $n_{x'}$ and $r_{x'}$ respectively. We have $m'_1 + m'_2 r'_{x'} \equiv 0 \bmod n_{x'}$ since $1 = {\chi'_1}^{m'_1}{\chi'_2}^{m'_2} = {\chi'_1}^{m'_1 + m'_2 r_{x'}}$ with χ'_1 faithful. Theorem 2.3.9 below shows that $n_{x'} > 1$, since x' is the non-regular.

If S were a smooth curve over $\mathbf{C}$, then the setup in (2.3.4) would be the classical cyclic surface quotient-singularity situation whose minimal regular resolution is most readily computed via toric varieties. That case motivates what to expect for minimal regular resolutions with more general S in §2.4, but rather than delve into a relative theory of toric varieties we can just use the classical case as a guide.

To define the class of singularities we shall resolve, let $X'_{/S}$ now be a *normal* (not necessarily connected) curve over a connected Dedekind scheme S. Assume moreover that either S is excellent or that $X'_{/S}$ has smooth generic fiber, so there are only finitely many non-regular points (all closed in closed fibers). Consider a closed point $s \in S$ with residue characteristic $p \geq 0$, and pick a closed point $x' \in X'_s$ such that X'_s has one or two (geometric) analytic branches at x'.

Definition 2.3.6. We say that a closed point x' in a closed fiber X'_s is a *tame cyclic quotient singularity* if there exists a positive integer $n > 1$ not divisible by $p = \mathrm{char}(k(s))$, a unit $r \in (\mathbf{Z}/n\mathbf{Z})^\times$, and integers $m'_1 > 0$ and $m'_2 \geq 0$ satisfying $m'_1 \equiv -rm'_2 \bmod n$ such that $\widehat{\mathcal{O}^{\mathrm{sh}}_{X',x'}}$ is isomorphic to the subalgebra of $\mu_n(k(s)_{\mathrm{sep}})$-invariants in $\widehat{\mathcal{O}^{\mathrm{sh}}_{S,s}}[\![t'_1, t'_2]\!]/({t'_1}^{m'_1}{t'_2}^{m'_2} - \pi_s)$ under the action $t'_1 \mapsto \zeta t'_1$, $t'_2 \mapsto \zeta^r t'_2$.

Remark 2.3.7. Note that when $X'_{/S}$ has a tame cyclic quotient singularity at $x' \in X'_s$, then $k(x')/k(s)$ is separable and x' is non-regular (by Theorem 2.3.9 below). Also, it is easy to check that the exponents m'_1 and m'_2 are necessarily the analytic branch multiplicities at x'. Note that the data of n and r is merely part of a presentation of $\widehat{\mathcal{O}}_{X',x'}$ as a ring of invariants, so it is not clear *a priori* that n and r are intrinsic to $x' \in X'$. The fact that n and r are uniquely determined by x' follows from Theorem 2.4.1 below, where we show that n and r arise from the structure of the minimal regular resolution of X' at x'.

Using notation as in the preceding global considerations, there is a very simple criterion for a nil-semistable $x' \in X/H$ to be a non-regular point: there should not be a line in $T_x(X)$ on which the inertia group $H_{x|x'}$ acts trivially. To prove this, we recall Serre's pseudo-reflection theorem [57, Thm. 1′]. This requires a definition:

Definition 2.3.8. Let V be a finite-dimensional vector space over a field k. An element σ of $\mathrm{Aut}_k(V)$ is called a *pseudo-reflection* if $\mathrm{rank}(1-\sigma) \leq 1$.

Theorem 2.3.9 (Serre). *Let A be a noetherian regular local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let G be a finite subgroup of* $\mathrm{Aut}(A)$*, and let A^G denote the local ring of G-invariants of A. Suppose that:*

1. *The characteristic of k does not divide the order of G,*
2. *G acts trivially on k, and*
3. *A is a finitely generated A^G-module.*

Then A^G is regular if and only if the image of G in $\mathrm{Aut}_k(\mathfrak{m}/\mathfrak{m}^2)$ *is generated by pseudo-reflections.*

In fact, the "only if" implication is true without hypotheses on the order of G, provided A^G has residue field k (*which is automatic when k is algebraically closed*)*.*

Remark 2.3.10. By Theorem 3.7(i) of [44] with $B = A$ and $A = A^G$, hypothesis 3 of Serre's theorem forces A^G to be noetherian. Serre's theorem ensures that x' as in Definition 2.3.6 is necessarily non-regular.

Proof. Since this result is not included in Serre's Collected Works, we note that a proof of the "if and only if" assertion can be found in [68, Cor. 2.13, Prop. 2.15]. The proof of the "only if" implication in [68] works without any conditions on the order of G as long as one knows that A^G has the same residue field as A. Such equality is automatic when k is algebraically closed. Indeed, the case of characteristic 0 is clear, and for positive characteristic we note that k is a priori finite over the residue field of A^G, so if equality were to fail then the residue field of A^G would be of positive characteristic with algebraic closure a finite extension of degree > 1, an impossibility by Artin-Schreier.

To see why everything still works without restriction on the order of G when we assume A^G is regular, note first that regularity of A^G ensures that $A^G \to A$ must be finite free, so even without a Reynolds operator we still have $(A \otimes_{A^G} A)^G = A$, where G acts on the left tensor factor. Hence, the proof of [68, Lemma 2.5] still works. Meanwhile, equality of residue fields for A^G and A makes the proof of [68, Prop. 2.6] still work, and then one easily checks that the proofs of [68, Thm. 2.8, Prop. 2.15(i)⇒(ii)] go through unchanged.

□

The point of the preceding study is that in a *global* quotient situation $X' = X/H$ as considered above, one always has a tame cyclic quotient singularity at the image x' of a nil-semistable point $x \in X_s$ when x' is not regular (by Lemma 2.3.3, both $k(x)$ and $k(x')$ are automatically separable over $k(s)$ when such non-regularity holds). Thus, when computing complete local rings at geometric closed points on a coarse modular curve (in residue characteristic > 3), we will naturally encounter a situation such as in Definition 2.3.6. The ability to explicitly (minimally) resolve tame cyclic quotient singularities in general will therefore have immediate applications to modular curves.

2.4 Jung–Hirzebruch resolution

As we noted in Remark 2.3.7, it is natural to ask whether the numerical data of n and $r \in (\mathbf{Z}/n\mathbf{Z})^\times$ in Definition 2.3.6 are intrinsic to $x' \in X'$. We shall see in the next theorem that this data is intrinsic, as it can be read off from the minimal regular resolution over x'.

Theorem 2.4.1. *Let $X'_{/S}$ be a normal curve over a local Dedekind base S with closed point s. Assume either that S is excellent or that $X'_{/S}$ has smooth generic fiber. Assume X' has a tame cyclic quotient singularity at a closed point $x' \in X'_s$ with parameters n and r (in the sense of Definition 2.3.6), where we represent $r \in (\mathbf{Z}/n\mathbf{Z})^\times$ by the unique integer r satisfying $1 \le r < n$ and $\gcd(r, n) = 1$. Finally, assume either that $k(s)$ is separably closed or that all connected components of the regular compactification $\overline{X}'_K$ of the regular generic-fiber curve X'_K have positive arithmetic genus.*

Consider the Jung–Hirzebruch continued fraction expansion

$$\frac{n}{r} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_\lambda}}} \tag{2.4.1}$$

with integers $b_j \ge 2$ for all j.

The minimal regular resolution of X' along x' has fiber over $k(x')_{\mathrm{sep}}$ whose underlying reduced scheme looks like the chain of E_j's as shown in Figure 1, where:

- *all intersections are transverse, with $E_j \simeq \mathbf{P}^1_{k(x')_{\mathrm{sep}}}$;*
- *$E_j.E_j = -b_j < -1$ for all j;*
- *E_1 is transverse to the strict transform $\widetilde{X}'_1$ of the global algebraic irreducible component X'_1 through x' with multiplicity m'_2 (along which t'_1 is a cotangent direction), and similarly for E_λ and the component $\widetilde{X}'_2$ with multiplicity m'_1 in the case of two analytic branches.*

Remark 2.4.2. The case $X'_2 = X'_1$ can happen, and there is no $\widetilde{X}'_1$ in case of one analytic branch (*i.e.*, in case $m'_2 = 0$).

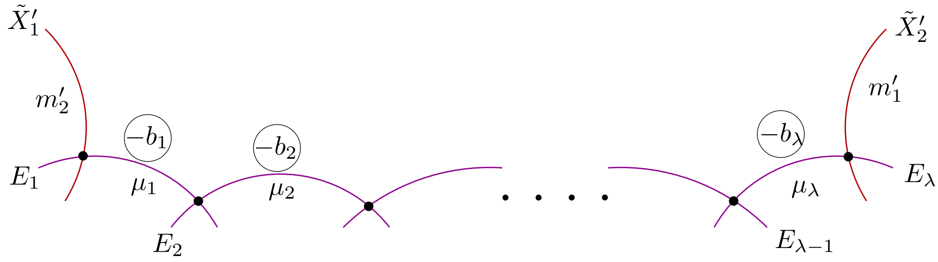

Figure 1: Minimal regular resolution of x'

We will also need to know the multiplicities μ_j of the components E_j in Figure 1, but this will be easier to give after we have proved Theorem 2.4.1; see Corollary 2.4.3.

The labelling of the E_j's indicates the order in which they arise in the resolution process, with each "new" E_j linking the preceding ones to the rest of the closed fiber in the case of one initial analytic branch. Keeping this picture in mind, we see that it is always the strict transform $\widetilde{X}'_2$ of the initial component with formal parameter t'_2 that occurs at the end of the chain, and this is the component whose multiplicity is m'_1.

Proof. We may assume S is local, and if S is not already excellent then (by hypothesis) X'_K is smooth and all connected components of its regular compactification have positive arithmetic genus. We claim that this positivity assumption is preserved by extension of the fraction field K. That is, if $\overline{C}$ is a connected regular proper curve over a field k with $\mathrm{H}^1(\overline{C}, \mathcal{O}_{\overline{C}}) \neq 0$ and C is a dense open in $\overline{C}$ that is k-smooth, then for any extension k'/k we claim that all connected components C'_i of the regular k'-curve $C' = C_{/k'}$ have compactification $\overline{C}'_i$ with $\mathrm{H}^1(\overline{C}'_i, \mathcal{O}_{\overline{C}'_i}) \neq 0$. Since the field $\mathrm{H}^0(\overline{C}, \mathcal{O}_{\overline{C}})$ is clearly finite separable over k, by using Stein factorization for $\overline{C}$ we may assume $\overline{C}$ is geometrically connected over k. Thus, $\overline{C}' = \overline{C}_{/k'}$ is a connected proper k'-curve with $\mathrm{H}^1(\overline{C}', \mathcal{O}_{\overline{C}'}) \neq 0$ and there is a dense open C' that is k'-smooth, and we want to show that the normalization of $\overline{C}'_{\mathrm{red}}$ has positive arithmetic genus. Since $\overline{C}'$ is generically reduced, the map from $\mathcal{O}_{\overline{C}'}$ to the normalization sheaf of $\mathcal{O}_{\overline{C}'_{\mathrm{red}}}$ has kernel and cokernel supported in dimension 0, and so the map on H^1's is an isomorphism. Thus, the normalization of $\overline{C}'_{\mathrm{red}}$ indeed has positive arithmetic genus.

We conclude that Lemma 2.1.1 and the base-change compatibility of Definition 2.2.3 (via Theorem 2.2.2) permit us to base-change to $\widehat{\mathcal{O}}_{S,s}$ without losing any hypotheses. Thus, we may assume $S = \operatorname{Spec} W$ with W a complete (hence excellent) discrete valuation ring. This brings us to the excellent case with all connected components of the regular compactification of X'_K having positive arithmetic genus when the residue field is not separably closed. If in addition

$k(s)$ is not separably closed, then we claim that base-change to $\operatorname{Spec} W^{\mathrm{sh}}$ preserves all hypotheses, and so we can always get to the case of a separably closed residue field (in particular, we get to the case with $k(x')$ separably closed); see [24, p. 17] for a proof that strict henselization preserves excellence. We need to show that base change to W^{sh} commutes with the formation of the minimal regular resolution. This is a refinement on Theorem 2.2.2 because such base change is generally not residually trivial.

From the proof of Theorem 2.2.2 in the excellent case, we see that if $X' \hookrightarrow \overline{X}'$ is a Nagata compactification then the minimal resolution $X \to X'$ of X' is the part of the minimal regular resolution of $\overline{X}'$ that lies over X'. Hence, the base-change problem for $W \to W^{\mathrm{sh}}$ is reduced to the proper case. We may assume that X' is connected, so $\widetilde{W} = \mathrm{H}^0(X', \mathcal{O}_{X'})$ is a complete discrete valuation ring finite over W. Hence, $\widetilde{W}^{\mathrm{sh}} \simeq \widetilde{W} \otimes_W W^{\mathrm{sh}}$, so we may reduce to the case when $X' \to \operatorname{Spec} W$ is its own Stein factorization. In this proper case, the positivity condition on the arithmetic genus of the generic fiber allows us to use [41, 9/3.28] (which rests on a dualizing-sheaf criterion for minimality) to conclude that formation of the minimal regular resolution of X' is compatible with étale localization on W. A standard direct limit argument that chases the property of having a -1-curve in a fiber over X' thereby shows that the formation of the minimal regular resolution is compatible with ind-étale base change (such as $W \to W^{\mathrm{sh}}$). Thus, we may finally assume that W is excellent and has a separably closed residue field, and so we no longer need to impose a positivity condition on arithmetic genera of the connected components of the generic-fiber regular compactification.

The intrinsic numerical data for the *unique* minimal resolution (that is, the self-intersection numbers and multiplicities of components in the exceptional divisor for this resolution) may be computed in an étale neighborhood of x', by Corollary 2.2.4 and Remark 2.2.5, and the Artin approximation theorem is the ideal tool for finding a convenient étale neighborhood in which to do such a calculation. We will use the Artin approximation theorem to construct a special case that admits an étale neighborhood that is also an étale neighborhood of our given x', and so it will be enough to carry out the resolution in the special case. The absence of a good theory of minimal regular resolutions for complete 2-dimensional local noetherian rings prevents us from carrying out a proof entirely on $\widehat{\mathcal{O}}_{X',x'}$, and so forces us to use the Artin approximation theorem. It is perhaps worth noting at the outset that the reason we have to use Artin approximation is that the resolution process to be used in the special case will not be intrinsic (we blow up certain codimension-1 subschemes that depend on coordinates).

Here is the special case that we wish to analyze. Let $n > 1$ be a positive integer that is a unit in W, and choose $1 \leq r < n$ with $\gcd(r, n) = 1$. Pick integers $m_1 \geq 1$ and $m_2 \geq 0$ satisfying $m_1 \equiv -rm_2 \bmod n$. For technical reasons, we do not require either of the m_j's to be units in W. To motivate things, let us temporarily assume that the residue field k of W contains a full set of nth roots of unity. Let $\mu_n(k)$ act on the *regular domain* $A = W[t_1, t_2]/(t_1^{m_1} t_2^{m_2} - \pi)$ via

$$[\zeta](t_1) = \zeta t_1, \;\; [\zeta](t_2) = \zeta^r t_2. \tag{2.4.2}$$

Since the $\mu_n(k)$-action in (2.4.2) is clearly free away from $t_1 = t_2 = \pi = 0$, the quotient

$$Z = (\mathrm{Spec}(A))/\mu_n(k) = \mathrm{Spec}(B)$$

(with $B = A^{\mu_n(k)}$) is normal and also is regular away from the image point $z \in Z$ of $t_1 = t_2 = \pi = 0$.

To connect up the special situation (Z, z) and the tame cyclic quotient singularity $x' \in X'_{/S}$, note that Lemma 2.3.4 shows that our situation is formally isomorphic to the algebraic $Z = \mathrm{Spec}(B)$ for a suitable such B and $n \in W^\times$. By the Artin approximation theorem, there is a common (residually trivial) connected étale neighborhood (U, u) of (Z, z) and (X', x'). That is, there is a pointed connected affine W-scheme $U = \mathrm{Spec}(A)$ that is a residually-trivial étale neighborhood of x' and of z. In particular, U is a connected normal W-curve. We can assume that u is the only point of U over z, and also the only point of U over x'. Keep in mind (*e.g.*, if $\gcd(m_1, m_2) > 1$) that the field K might not be separably closed in the function fields of U or Z, so the generic fibers of U and $Z = \mathrm{Spec}(B)$ over W might not be geometrically connected and U is certainly not proper over W in general.

The étale-local nature of the minimal regular resolution, as provided by Corollary 2.2.4 and Remark 2.2.5, implies that the minimal regular resolutions of (X', x') and (Z, z) have pullbacks to (U, u) that coincide with the minimal regular resolution of U along $\{u\}$. The fibers over u, x', z are all the same due to residual-triviality, so the geometry of the resolution fiber at x' is the same as that over z. Hence, we shall compute the minimal regular resolution $Z' \to Z$ at z, and will see that the fiber of Z' over z is as in Figure 1.

Let us now study (Z, z). Since n is a unit in W, the normal domain $B = A^{\mu_n(k)}$ is a quotient of $W[t_1, t_2]^{\mu_n(k)}$ via the natural map. Since the action of $\mu_n(k)$ as in (2.4.2) sends each monomial $t_1^{e_1} t_2^{e_2}$ to a constant multiple of itself, the ring of invariants $W[t_1, t_2]^{\mu_n(k)}$ is spanned over W by the invariant monomials. Clearly $t_1^{e_1} t_2^{e_2}$ is $\mu_n(k)$-invariant if and only if $e_1 + re_2 = nf$ for some integer f (so $e_2 \le (n/r)f$), in which case $t_1^{e_1} t_2^{e_2} = u^f v^{e_2}$, where $u = t_1^n$ and $v = t_2/t_1^r$ are $\mu_n(k)$-invariant elements in the fraction field of $W[t_1, t_2]$. Note that even though v does not lie in $W[t_1, t_2]$, for any pair of integers i, j satisfying $0 \le j \le (n/r)i$ we have $u^i v^j \in W[t_1, t_2]$ and

$$W[t_1, t_2]^{\mu_n(k)} = \bigoplus_{0 \le j \le (n/r)i} W u^i v^j.$$

We have $t_1^{m_1} t_2^{m_2} = u^\mu v^{m_2}$ with $m_1 + rm_2 = n\mu$ (so $m_2 \le (n/r)\mu$). Thus,

$$B = \frac{\bigoplus_{0 \le j \le (n/r)i} W u^i v^j}{(u^\mu v^{m_2} - \pi)}. \tag{2.4.3}$$

Observe that (2.4.3) makes sense as a definition of finite-type W-algebra, without requiring n to be a unit and without requiring that k contain any non-trivial roots of unity. It is clear that (2.4.3) is W-flat, as it has a W-module basis given by monomials $u^i v^j$ with $0 \leq j \leq (n/r)i$ and either $i < \mu$ or $j < m_2$. It is less evident if (2.4.3) is normal for any n, but we do not need this fact. We will inductively compute certain blow-ups on (2.4.3) *without restriction on* n or on the residue field, and the process will end at a resolution of singularities for $\operatorname{Spec} B$.

Before we get to the blowing-up, we shall show that $\operatorname{Spec} B$ is a W-curve and we will infer some properties of its closed fiber. Note that the map $K(u,v) \to K(t_1,t_2)$ defined by $u \mapsto t_1^n$, $v \mapsto t_2/t_1^r$ induces a W-algebra injection

$$\bigoplus_{0 \leq j \leq (n/r)i} W u^i v^j \to W[t_1, t_2] \tag{2.4.4}$$

that is *finite* because $t_1^n = u$ and $t_2^n = u^r v^n$. Thus, the left side of (2.4.4) is a 3-dimensional noetherian domain and passing to the quotient by $u^\mu v^{m_2} - \pi = t_1^{m_1} t_2^{m_2} - \pi$ yields a finite surjection

$$\operatorname{Spec}(W[t_1,t_2]/(t_1^{m_1} t_2^{m_2} - \pi)) \to \operatorname{Spec}(B). \tag{2.4.5}$$

Passing to the generic fiber and recalling that B is W-flat, we infer that $\operatorname{Spec}(B)$ is a W-curve with irreducible generic fiber, so $\operatorname{Spec}(B)$ is 2-dimensional and connected. We also have a finite surjection modulo π,

$$\operatorname{Spec}(k[t_1,t_2]/(t_1^{m_1} t_2^{m_2})) \to \operatorname{Spec}(B/\pi), \tag{2.4.6}$$

so the closed fiber of $\operatorname{Spec}(B)$ consists of at most two irreducible components (or just one when $m_2 = 0$), to be called the images of the t_1-axis and t_2-axis (where we omit mention of the t_1-axis when $m_2 = 0$). Since the t_2-axis is the preimage of the zero-scheme of $u = t_1^n$ under (2.4.6), we conclude that when $m_2 > 0$ the closed fiber $\operatorname{Spec}(B/\pi)$ does have two distinct irreducible components.

Inspired by the case of toric varieties, we will now compute the blow-up Z' of the W-flat $Z = \operatorname{Spec}(B)$ along the ideal (u, uv). Since

$$\operatorname{Spec}(W[t_1,t_2]/(t_1^{m_1} t_2^{m_2} - \pi, t_1^n, t_1^{n-r} t_2)) \to \operatorname{Spec}(B/(u,uv))$$

is a finite surjection and the source is supported in the t_2-axis of the closed fiber over $\operatorname{Spec}(W)$, it follows that $\operatorname{Spec}(B/(u,uv))$ is supported in the image of the t_2-axis of the closed fiber of $\operatorname{Spec}(B)$ over $\operatorname{Spec}(W)$. In particular, blowing up Z along (u, uv) does not affect the generic fiber of Z over W. Since Z is W-flat, it follows that the proper blow-up map $Z' \to Z$ is surjective.

There are two charts covering Z', $D_+(u)$ and $D_+(uv)$, where we adjoin the ratios $uv/u = v$ and $u/uv = 1/v$ respectively. Thus,

$$D_+(u) = \operatorname{Spec}(B[v]) = \operatorname{Spec}(W[u,v]/(u^\mu v^{m_2} - \pi))$$

is visibly regular and connected, and $D_+(uv) = \mathrm{Spec}(B[1/v])$ with

$$B[1/v] = \frac{\bigoplus_{j \le (n/r)i,\, 0 \le i} W u^i v^j}{(u^\mu v^{m_2} - \pi)}.$$

We need to rewrite this latter expression in terms of a more useful set of variables. We begin by writing (as one does when computing the Jung–Hirzebruch continued fraction for n/r)

$$n = b_1 r - r'$$

with $b_1 \ge 2$ and either $r = 1$ with $r' = 0$ or else $r' > 0$ with $\gcd(r, r') = 1$ (since $\gcd(n, r) = 1$). We will first treat the case $r' = 0$ (proving that $B[1/v]$ is also regular) and then we will treat the case $r' > 0$. Note that there is no reason to expect that p cannot divide r or r', even if $p \nmid n$, and it is for this reason that we had to recast the definition of B in a form that avoids the assumption that n is a unit in W. For similar reasons, we must avoid assuming m_1 or m_2 is a unit in W.

Assume $r' = 0$, so $r = 1$, $b_1 = n$, and $b_1\mu - m_2 = m_1$. Let $i' = b_1 i - j$ and $j' = i$, so i' and j' vary precisely over non-negative integers and $u^i v^j = (1/v)^{i'} (uv^{b_1})^{j'}$. Thus, letting $u' = 1/v$ and $v' = uv^{b_1}$ yields

$$B[1/v] = W[u', v']/({u'}^{b_1\mu - m_2} {v'}^\mu - \pi) = W[u', v']/({u'}^{m_1} {v'}^\mu - \pi),$$

which is regular. In the closed fiber of $Z' = \mathrm{Bl}_{(u,uv)}(Z)$ over $\mathrm{Spec}(W)$, let D_1 denote the v'-axis in $D_+(uv) = \mathrm{Spec}\, B[1/v]$ and when $m_2 > 0$ let D_2 denote the u-axis in $D_+(u)$. The multiplicities of D_1 and D_2 in Z'_k are respectively $m_1 = b_1\mu - m_2$ and m_2 (with multiplicity $m_2 = 0$ being a device for recording that there is no D_2). The exceptional divisor E is a projective line over k (with multiplicity μ and gluing data $u' = 1/v$) and hence the uniformizer π has divisor on $Z' = \mathrm{Bl}_{(u,uv)}(Z)$ given by

$$\mathrm{div}_{Z'}(\pi) = (b_1\mu - m_2)D_1 + \mu E + m_2 D_2 = m_1 D_1 + \mu E + m_2 D_2$$

(when $m_2 = 0$, the final term really is omitted).

It is readily checked that the D_j's each meet E transversally at a single k-rational point (suppressing D_2 when $m_2 = 0$). The intersection product $\mathrm{div}_{Z'}(\pi).E$ makes sense since E is proper over k, even though Z is not proper over W, and it must vanish because $\mathrm{div}_{Z'}(\pi)$ is principal, so by additivity of intersection products in the first variable (restricted to effective Cartier divisors for a fixed proper second variable such as E) we have

$$0 = \mathrm{div}_{Z'}(\pi).E = b_1\mu - m_2 + \mu(E.E) + m_2.$$

Thus, $E.E = -b_1$.

Now assume $r' > 0$. Since $n = b_1 r - r'$, the condition $0 \le j \le (n/r)i$ can be rewritten as $0 \le i \le (r/r')(b_1 i - j)$. Letting $j' = i$ and $i' = b_1 i - j$, we have $u^i v^j = u'^{i'} v'^{j'}$ with $u' = 1/v$ and $v' = uv^{b_1}$. In particular, $u^\mu v^{m_2} = u'^{b_1\mu - m_2} v'^\mu$. Thus,

$$B[1/v] = \frac{\bigoplus_{0 \le j' \le (r/r')i'} W u'^{i'} v'^{j'}}{(u'^{b_1\mu - m_2} v'^\mu - \pi)}. \tag{2.4.7}$$

Note the similarity between (2.4.3) and (2.4.7) up to modification of parameters: replace (n, r, m_1, m_2, μ) with $(r, r', m_1, \mu, b_1\mu - m_2)$. The blow-up along $(u', u'v')$ therefore has closed fiber over $\mathrm{Spec}(W)$ with the following irreducible components: the v'-axis D_1 in $D_+(uv)$ with multiplicity $b_1\mu - m_2$, the u-axis D_2 in $D_+(u)$ with multiplicity m_2 (so this only shows up when $m_2 > 0$), and the exceptional divisor E that is a projective line (via gluing $u' = 1/v$) having multiplicity μ and meeting D_1 (as well as D_2 when $m_2 > 0$) transversally at a single k-rational point. We will focus our attention on $D_+(uv)$ (as we have already seen that the other chart $D_+(u)$ is regular), and in particular we are interested in the "origin" in the closed fiber of $D_+(uv)$ over $\mathrm{Spec}(W)$ where the projective line E meets D_1; near this origin, $D_+(uv)$ is an affine open that is given by the spectrum of (2.4.7).

If r were also a unit in W then $D_+(uv)$ would be the spectrum of the ring of $\mu_r(k)$-invariants in $W[t'_1, t'_2]/(t_1'^{m_1} t_2'^{\mu} - \pi)$ with the action $[\zeta](t'_1) = \zeta t'_1$ and $[\zeta](t'_1) = \zeta^{r'} t'_2$ (this identification uses the identity $m_1 + r'\mu = r(b_1\mu - m_2)$), and without any restriction on r we at least see that (2.4.7) is an instance of the general (2.4.3) and that there is a natural finite surjection

$$\mathrm{Spec}(k[t'_1, t'_2]/(t_1'^{m_1} t_2'^{\mu})) \to D_+(uv)_k.$$

On $D_+(uv)_k$, the component E of multiplicity μ is the image of the t'_1-axis and the component D_1 with multiplicity m_1 is the image of the t'_2-axis. As a motivation for what follows, note also that if $r \in W^\times$ then since $r > 1$ we see that the "origin" in $D_+(uv)_k$ is necessarily a non-regular point in the total space over $\mathrm{Spec}(W)$ (by Serre's Theorem 2.3.9).

We conclude (without requiring any of our integer parameters to be units in W) that if we make the change of parameters

$$(n, r, m_1, m_2, \mu) \rightsquigarrow (r, r', m_1, \mu, b_1\mu - m_2) \tag{2.4.8}$$

then $D_+(uv)$ is like the original situation (2.4.3) with a revised set of initial parameters. In particular, n is replaced by the strictly smaller $r > 1$, so the process will eventually end. Moreover, since $\mu > 0$ we see that the case $m_2 = 0$ is now "promoted" to the case $m_2 > 0$. When we make the blow-up at the origin in $D_+(uv)_k$, the strict transform E_1 of E plays the same role that D_2 played above, so E_1 is entirely in the regular locus and the new exceptional divisor E' has multiplicity $b_1\mu - m_2$ (this parameter plays the role for the second blow-up that μ played for the first blow-up, as one sees by inspecting our change of parameters in (2.4.8)).

As the process continues, nothing more will change around E_1, so inductively we conclude from the descriptions of the regular charts that the process ends at a regular connected W-curve with closed-fiber Weil divisor

$$\cdots + (b_1\mu - m_2)E' + \mu E_1 + m_2 D_2 + \dots \tag{2.4.9}$$

(where we have abused notation by writing E' to denote the strict transform of E' in the final resolution, and this strict transform clearly has generic multiplicity $b_1\mu - m_2$). The omitted terms in (2.4.9) do not meet E_1, so we may form the intersection against E_1 to solve

$$0 = (b_1\mu - m_2) + \mu(E_1.E_1) + m_2$$

just as in the case $r' = 0$ (*i.e.*, $r = 1$), so $E_1.E_1 = -b_1$. Since

$$\frac{n}{r} = b_1 - \frac{1}{r/r'},$$

by induction on the length of the continued fraction we reach a regular resolution in the expected manner, with $E_j.E_j = -b_j$ for all j and the final resolution having fiber over $z \in Z$ looking exactly like in Figure 1. Note also that each new blow-up separates all of the previous exceptional lines from the (strict transform of the initial) component through z with multiplicity m_1. Since $-b_j \leq -2 < -1$ for all j, we conclude that at no stage of the blow-up process before the end did we have a regular scheme (otherwise there would be a -1-curve in a fiber over the original base Z). Thus, we have computed the minimal regular resolution at z.

□

We now compute the multiplicity μ_j in the closed fiber of X'^{reg} for each fibral component E_j over $x' \in X'$ in Figure 1. In order to compute the μ_j's, we introduce some notation. Let $n/r > 1$ be a reduced-form fraction with positive integers n and r, so we can write

$$n/r = [b_1, b_2, \dots, b_\lambda]_{\mathrm{JH}} := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_\lambda}}}$$

as a Jung–Hirzebruch continued fraction, where $b_j \geq 2$ for all j. Define $P_j = P_j(b_1, \dots, b_\lambda)$ and $Q_j = Q_j(b_1, \dots, b_\lambda)$ by

$$P_{-1} = 0,\ \ Q_{-1} = -1,\ \ P_0 = 1,\ \ Q_0 = 0,$$

$$P_j = b_j P_{j-1} - P_{j-2},\ \ Q_j = b_j Q_{j-1} - Q_{j-2}$$

for all $j \geq 1$. Clearly P_j and Q_j are universal polynomials in $b_1, \dots, b_j$, and by induction $P_j Q_{j-1} - Q_j P_{j-1} = -1$ and $Q_j > Q_{j-1}$ for all $j \geq 0$, so in particular $Q_j > 0$ for all $j > 0$. Thus,

$$[b_1, \dots, b_\lambda]_{\mathrm{JH}} = \frac{P_\lambda(b_1, \dots, b_\lambda)}{Q_\lambda(b_1, \dots, b_\lambda)}$$

makes sense and P_λ/Q_λ is in reduced form. Thus, $P_\lambda = n$ and $Q_\lambda = r$ since the Q_j's are necessarily positive.

Corollary 2.4.3. *With hypotheses and notation as in Theorem 2.4.1, let μ_j denote the multiplicity of E_j in the fiber of ${X'}^{\mathrm{reg}}$ over $k(x')_{\mathrm{sep}}$. The condition $r = 1$ happens if and only if $\lambda = 1$, in which case $\mu_1 = (m'_1 + m'_2)/n$.*

If $r > 1$ (so $\lambda > 1$), then the μ_j's are the unique solution to the equation

$$\begin{pmatrix} b_1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & b_2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & b_3 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & b_{\lambda-1} & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & b_\lambda \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \vdots \\ \mu_\lambda \end{pmatrix} = \begin{pmatrix} m'_2 \\ 0 \\ \vdots \\ 0 \\ m'_1 \end{pmatrix}. \tag{2.4.10}$$

Keeping the condition $r > 1$, define $P'_j = P_j(b_{\lambda-j+1}, \dots, b_\lambda)$, so $P'_\lambda = n$ and $P'_{\lambda-1} = Q_\lambda(b_1, \dots, b_\lambda) = r$. If we let $\widetilde{m}_2 = P'_{\lambda-1}m'_2 + m'_1 = rm'_2 + m'_1$, then the μ_j's are also the unique solution to

$$\begin{pmatrix} P'_\lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ -P'_{\lambda-2} & P'_{\lambda-1} & 0 & \dots & 0 & 0 & 0 \\ 0 & -P'_{\lambda-3} & P'_{\lambda-2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -P'_1 & P'_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & P'_1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \vdots \\ \mu_\lambda \end{pmatrix} = \begin{pmatrix} \widetilde{m}_2 \\ m'_1 \\ \vdots \\ m'_1 \\ m'_1 \end{pmatrix}. \tag{2.4.11}$$

In particular, $\mu_1 = (rm'_2 + m'_1)/n$.

Note that in the applications with $X' = X/H$ as at the beginning of §2.3, the condition $\chi'_1 \neq \chi'_2$ (*i.e.*, $H'_{x|x'}$ does not act through scalars) is equivalent to the condition $r > 1$ in Corollary 2.4.3.

Proof. The value of μ_1 when $r = 1$ was established in the proof of Theorem 2.4.1, so now assume $r > 1$. On ${X'}^{\mathrm{reg}}$ (or rather, its base change to $\mathcal{O}^{\mathrm{sh}}_{S,s}$) we have

$$\mathrm{div}(\pi_s) = m'_1\widetilde{X}'_2 + \sum_{j=1}^{\lambda} \mu_j E_j + m'_2\widetilde{X}'_1 + \dots \tag{2.4.12}$$

where

- the $\widetilde{X}'_1$-term does not appear if there is only one analytic branch through x' (recall we also set $m'_2 = 0$ in this case),
- the $\widetilde{X}'_j$-terms are a single term when there are two analytic branches but only one global irreducible (geometric) component (in which case $m'_1 = m'_2$),

- the omitted terms "…" on the right side of (2.4.12) are not in the fiber over x' (and in particular do not intersect the E_j's).

Thus, the equations $E_j.\mathrm{div}(\pi_s) = 0$ and the intersection calculations in the proof of Theorem 2.4.1 (as summarized by Figure 1, including transversalities) immediately yield (2.4.10). By solving this system of equations by working up from the bottom row, an easy induction argument yields the reformulation (2.4.11).

□

To prove Theorems 1.1.2 and 1.1.6, the preceding general considerations will provide the necessary intersection-theoretic information on a minimal resolution. To apply Theorem 2.4.1 and Corollary 2.4.3 to the study of singularities at points x' on modular curves, we need to find the value of the parameter $r_{x'}$ in each case. This will be determined by studying universal deformation rings for moduli problems of elliptic curves.

3 The Coarse moduli scheme $X_1(p)$

Let p be a prime number. In this section we review the construction of the coarse moduli scheme $X_1(p)$ attached to $\Gamma_1(p)$ in terms of an auxiliary finite étale level structure which exhibits $X_1(p)$ as the compactification of a quotient of a fine moduli scheme. It is the fine moduli schemes whose completed local rings are well understood through deformation theory (as in [34]), and this will provide the starting point for our subsequent calculations of regular models and component groups.

3.1 Some general nonsense

As in [34, Ch. 4], for a scheme T we let (Ell/T) be the category whose objects are elliptic curves over T-schemes and whose morphisms are cartesian diagrams. The moduli problem $[\Gamma_1(p)]$ is the contravariant functor $(\mathrm{Ell}) \to (\mathrm{Sets})$ that to an elliptic curve $E_{/S}$ attaches the set of $P \in E(S)$ such that the relative effective Cartier divisor

$$[0] + [P] + [2P] + \cdots + [(p-1)P],$$

viewed as a closed subscheme of E, is a closed subgroup scheme. For any moduli problem $\mathcal{P}$ on (Ell/T) and any object $E_{/S}$ over a T-scheme, we define the functor $\mathcal{P}_{E/S}(S') = \mathcal{P}(E_{/S'})$ to classify "$\mathcal{P}$-structures" on base changes of $E_{/S}$. If $\mathcal{P}_{E/S}$ is representable (with some property $\mathbf{P}$ relative to S) for every $E_{/S}$, we say that $\mathcal{P}$ is *relatively representable* (with property $\mathbf{P}$). For example, $[\Gamma_1(p)]$ is relatively representable and finite locally free of degree p^2-1 on (Ell) for every prime p.

For $p \geq 5$, the moduli problem $[\Gamma_1(p)]_{/\mathbf{Z}[1/p]}$ is representable by a smooth affine curve over $\mathbf{Z}[1/p]$ [34, Cor. 2.7.3, Thm. 3.7.1, and Cor. 4.7.1]. For any elliptic curve $E_{/S}$ over an $\mathbf{F}_p$-scheme S, the point $P = 0$ is fixed by the automorphism -1 of $E_{/S}$, and is in $[\Gamma_1(p)](E/S)$ because $[0]+[P]+\cdots+[(p-1)P]$ is the kernel of the relative Frobenius morphism $F\colon E \to E^{(p)}$. Thus, $[\Gamma_1(p)]_{/\mathbf{Z}_{(p)}}$ is not rigid, so it is not representable.

As there is no fine moduli scheme associated to $[\Gamma_1(p)]_{/\mathbf{Z}_{(p)}}$ for any prime p, we let $X_1(p)$ be the compactified coarse moduli scheme $\overline{M}([\Gamma_1(p)]_{/\mathbf{Z}_{(p)}})$, as constructed in [34, Ch. 8]. This is a proper normal $\mathbf{Z}_{(p)}$-model of a smooth and geometrically connected curve $X_1(p)_{/\mathbf{Q}}$, but $X_1(p)$ is usually not regular. Nevertheless, the complete local rings on $X_1(p)$ are computable in terms of abstract deformation theory. Since $(\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ acts on isomorphism classes of $\Gamma_1(p)$-structures via

$$(E, P) \mapsto (E, a \cdot P) \simeq (E, -a \cdot P),$$

we get a natural action of this group on $X_1(p)$ which is readily checked to be a faithful action (*i.e.*, non-identity elements act non-trivially). Thus, for any subgroup $H \subseteq (\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ we get the modular curve $X_H(p) = X_1(p)/H$ which is a normal proper connected $\mathbf{Z}_{(p)}$-curve with smooth generic fiber $X_H(p)_{/\mathbf{Q}}$. When $p > 3$, the curve $X_H(p)$ has tame cyclic quotient singularities at its non-regular points.

In order to compute a minimal regular model for these normal curves, we need more information than is provided by abstract deformation theory: we need to keep track of *global* irreducible components on the geometric fiber mod p, whereas deformation theory will only tell us about the analytic branches through a point. Fortunately, in the case of modular curves $X_H(p)$, distinct analytic branches through a closed-fiber geometric point always arise from distinct global (geometric) irreducible components through the point. In order to review this fact, as well as to explain the connection between complete local rings on $X_H(p)$ and rings of invariants in universal deformation rings, we need to recall how $X_1(p)$ can be constructed from fine moduli schemes. Let us briefly review the construction process.

Pick a representable moduli problem $\mathcal{P}$ that is finite, étale, and Galois over $(\mathrm{Ell}/\mathbf{Z}_{(p)})$ with Galois group $G_{\mathcal{P}}$, and for which $M(\mathcal{P})$ is affine. For example (cf. [34, §4.5–4.6]) if $\ell \neq p$ is a prime with $\ell \geq 3$, we can take $\mathcal{P}$ to be the moduli problem $[\Gamma(\ell)]_{/\mathbf{Z}_{(p)}}$ that attaches to $E_{/S}$ the set of isomorphisms of S-group schemes

$$\phi\colon (\mathbf{Z}/\ell\mathbf{Z})^2_S \simeq E[\ell];$$

the Galois group $G_{\mathcal{P}}$ is $\mathrm{GL}_2(\mathbf{F}_\ell)$. Let $Y_1(p; \mathcal{P})$ be the fine moduli scheme $M([\Gamma_1(p)]_{/\mathbf{Z}_{(p)}}, \mathcal{P})$ that classifies pairs consisting of a $\Gamma_1(p)$-structure and a $\mathcal{P}$-structure on elliptic curves over variable $\mathbf{Z}_{(p)}$-schemes. The scheme $Y_1(p; \mathcal{P})$ is a flat affine $\mathbf{Z}_{(p)}$-curve. Let $Y_1(p)$ be the quotient of $Y_1(p; \mathcal{P})$ by the $G_{\mathcal{P}}$-action.

We introduce the global $\mathcal{P}$ rather than just use formal deformation theory throughout because on characteristic-p fibers we need to retain a connection between closed fiber irreducible components of global modular curves and closed fiber "analytic" irreducible components of formal deformation rings. The precise connection between global $\mathcal{P}$'s and infinitesimal deformation theory is given by the well-known:

Theorem 3.1.1. *Let k be an algebraically closed field of characteristic p and let $W = W(k)$ be its ring of Witt vectors. Let $z \in Y_1(p)_{/k}$ be a rational point. Let* $\mathrm{Aut}(z)$ *denote the finite group of automorphisms of the* (*non-canonically unique*) *$\Gamma_1(p)$-structure over k underlying z. Choose a $\mathcal{P}$-structure on the elliptic curve underlying z, with $\mathcal{P}$ as above, and let $z' \in Y_1(p;\mathcal{P})(k)$ be the corresponding point over z.*

The ring $\widehat{\mathcal{O}}_{Y_1(p;\mathcal{P})_W,z'}$ is naturally identified with the formal deformation ring of z. Under the resulting natural action of $\mathrm{Aut}(z)$ *on $\widehat{\mathcal{O}}_{Y_1(p;\mathcal{P})_W,z'}$, the subring of* $\mathrm{Aut}(z)$*-invariants is $\widehat{\mathcal{O}}_{Y_1(p)_W,z}$.*

For any subgroup $H \subseteq (\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$ equipped with its natural action on $Y_1(p)$, the stabilizer $H_{z'|z}$ of z' in H acts faithfully on the universal deformation ring $\widehat{\mathcal{O}}_{Y_1(p;\mathcal{P})_W,z'}$ of z in the natural way, with subring of invariants $\widehat{\mathcal{O}}_{Y_H(p)_W,z}$.

Proof. Since $\mathcal{P}$ is étale and $Y_1(p;\mathcal{P})_W$ is a fine moduli scheme, the interpretation of $\widehat{\mathcal{O}}_{Y_1(p;\mathcal{P})_W,z'}$ as a universal deformation ring is immediate. Since $Y_1(p)_W$ is the quotient of $Y_1(p;\mathcal{P})_W$ by the action of $G_\mathcal{P}$, it follows that $\widehat{\mathcal{O}}_{Y_1(p)_W,z}$ is identified with the subring of invariants in $\widehat{\mathcal{O}}_{Y_1(p;\mathcal{P})_W,z'}$ for the action of the stabilizer of z' for the $G_\mathcal{P}$-action on $Y_1(p;\mathcal{P})_W$. We need to compute this stabilizer subgroup.

If $z' = (E_z, P_z, \iota)$ with supplementary $\mathcal{P}$-structure ι, then $g \in G_\mathcal{P}$ fixes z' if and only if (E_z, P_z, ι) is isomorphic to $(E_z, P_z, g(\iota))$. This says exactly that there exists an automorphism α_g of (E_z, P_z) carrying ι to $g(\iota)$, and such α_g is clearly unique if it exists. Moreover, any two $\mathcal{P}$-structures on E_z are related by the action of a unique $g \in G_\mathcal{P}$ because of the definition of $G_\mathcal{P}$ as the Galois group of $\mathcal{P}$ (and the fact that z is a geometric point). Thus, the stabilizer of z in $G_\mathcal{P}$ is naturally identified with $\mathrm{Aut}(E_z, P_z) = \mathrm{Aut}(z)$ (*compatibly* with actions on the universal deformation ring of z). The assertion concerning the H-action is clear. $\square$

Since $Y_1(p;\mathcal{P})$ is a regular $\mathbf{Z}_{(p)}$-curve [34, Thm. 5.5.1], it follows that its quotient $Y_1(p)$ is a normal $\mathbf{Z}_{(p)}$-curve. Moreover, by [34, Prop. 8.2.2] the natural map $j : Y_1(p) \to \mathbf{A}^1_{\mathbf{Z}_{(p)}}$ is finite, and hence it is also *flat* [44, 23.1]. In [34], $X_1(p)$ is *defined* to be the normalization of $Y_1(p)$ over the compactified j-line $\mathbf{P}^1_{\mathbf{Z}_{(p)}}$. Both $X_1(p)$ and $Y_1(p)$ are independent of the auxiliary choice of $\mathcal{P}$. The complex analytic theory shows that $X_1(p)$ has geometrically connected fibers over $\mathbf{Z}_{(p)}$, so the same is true for $Y_1(p)$ since the complete local rings at the cusps are analytically irreducible mod p (by the discussion in §4.2, especially the self-contained Lemma 4.2.4 and Lemma 4.2.5).

3.2 Formal parameters

To do deformation theory computations, we need to recall some canonical formal parameters in deformation rings. Fix an algebraically closed field k of characteristic p and let $W = W(k)$ denote its ring of Witt vectors. Let $z \in Y_1(p)_{/k}$ be a k-rational point corresponding to an elliptic curve $E_{z/k}$ with $\Gamma_1(p)$-structure P_z.

For later purposes, it is useful to give a conceptual description of the 1-dimensional "reduced" cotangent space $\mathfrak{m}/(p, \mathfrak{m}^2)$ of $\mathcal{R}_z^0$, or equivalently the cotangent space to the equicharacteristic formal deformation functor of E_z:

Theorem 3.2.1. *The cotangent space to the equicharacteristic formal deformation functor of an elliptic curve E over a field k is canonically isomorphic to* $\mathrm{Cot}_0(E)^{\otimes 2}$.

Proof. This is just the dual of the Kodaira-Spencer isomorphism. More specifically, the cotangent space is isomorphic to $\mathrm{H}^1(E, (\Omega^1_{E/k})^\vee)^\vee$, and Serre duality identifies this latter space with

$$\mathrm{H}^0(E, (\Omega^1_{E/k})^{\otimes 2}) \xleftarrow{\simeq} \mathrm{H}^0(E, \Omega^1_{E/k})^{\otimes 2} = \mathrm{Cot}_0(E)^{\otimes 2},$$

the first map being an isomorphism since $\Omega^1_{E/k}$ is (non-canonically) trivial. $\square$

Let

$$\mathbf{E}_z \to \mathrm{Spec}(\mathcal{R}_z^0)$$

denote an algebraization of the universal deformation of E_z, so non-canonically $\mathcal{R}_z^0 \simeq W[\![t]\!]$ and (by Theorem 3.1.1) there is a unique local W-algebra map $\mathcal{R}_z^0 \to \mathcal{R}_z$ to the universal deformation ring $\mathcal{R}_z$ of (E_z, P_z) such that there is a (necessarily unique) isomorphism of deformations between the base change of $\mathbf{E}_z$ over $\mathcal{R}_z$ and the universal elliptic curve underlying the algebraized universal $\Gamma_1(p)$-structure deformation at z.

Now make the additional hypothesis $P_z = 0$, so upon choosing a formal coordinate $\underline{x}$ for the formal group of $\mathbf{E}_z$ it makes sense to consider the coordinate

$$x = \underline{x}(\mathbf{P}_z) \in \mathcal{R}_z$$

of the "point" $\mathbf{P}_z$ in the universal $\Gamma_1(p)$-structure over $\mathcal{R}_z$. We thereby get a natural local W-algebra map

$$W[\![x, t]\!] \to \mathcal{R}_z. \tag{3.2.1}$$

Theorem 3.2.2. *The natural map* (3.2.1) *is a surjection with kernel generated by an element f_z that is part of a regular system of parameters of the regular local ring $W[\![x, t]\!]$. Moreover, x and t span the* 2*-dimensional cotangent space of the target ring.*

Proof. The surjectivity and cotangent-space claims amount to the assertion that an artinian deformation whose $\Gamma_1(p)$-structure vanishes and whose t-parameter vanishes necessarily has $p = 0$ in the base ring (so we then have a constant deformation). The vanishing of p in the base ring is [34, 5.3.2.2]. Since the deformation ring $\mathcal{R}_z$ is a 2-dimensional regular local ring, the kernel of the surjection (3.2.1) is a height-1 prime that must therefore be principal with a generator that is part of a regular system of parameters.

□

3.3 Closed-fiber description

For considerations in Section 5, we will need some more refined information, particularly a description of $f_z \bmod p$ in Theorem 3.2.2. To this end, we first need to recall some specialized moduli problems in characteristic p.

Definition 3.3.1. If $E_{/S}$ is an elliptic curve over an $\mathbf{F}_p$-scheme S, and $G \hookrightarrow E$ is a finite locally free closed subgroup scheme of order p, we shall say that G is a $(1,0)$-*subgroup* if G is the kernel of the relative Frobenius map $F_{E/S} : E \to E^{(p)}$ and G is a $(0,1)$-*subgroup* if the order p group scheme $E[p]/G \hookrightarrow E/G$ is the kernel of the relative Frobenius for the quotient elliptic curve E/G over S.

Remark 3.3.2. This is a special case of the more general concept of (a,b)-*cyclic subgroup* which is developed in [34, §13.4] for describing the mod p fibers of modular curves. On an ordinary elliptic curve over a field of characteristic p, an (a,b)-cyclic subgroup has connected-étale sequence with connected part of order p^a and étale part of order p^b.

Let $\mathcal{P}$ be a representable moduli problem over $(\mathrm{Ell}/\mathbf{Z}_{(p)})$ that is finite, étale, and Galois with $M(\mathcal{P})$ affine (as in §3.1). For $(a,b) = (1,0), (0,1)$, it makes sense to consider the subfunctor

$$[[\Gamma_1(p)]\text{-}(a,b)\text{-cyclic}, \mathcal{P}] \tag{3.3.1}$$

of points of $[\Gamma_1(p)_{/\mathbf{F}_p}, \mathcal{P}]$ whose $\Gamma_1(p)$-structure generates an (a,b)-cyclic subgroup. By [34, 13.5.3, 13.5.4], these subfunctors (3.3.1) are represented by closed subschemes of $Y_1(p;\mathcal{P})_{/\mathbf{F}_p}$ that intersect at exactly the supersingular points and have ordinary loci that give a covering of $Y_1(p;\mathcal{P})^{\mathrm{ord}}_{/\mathbf{F}_p}$ by open subschemes. Explicitly, we have an $\mathbf{F}_p$-scheme isomorphism

$$M([\Gamma_1(p)]\text{-}(0,1)\text{-cyclic}, \mathcal{P}) \simeq M([\mathrm{Ig}(p)], \mathcal{P}) \tag{3.3.2}$$

with a smooth (possibly disconnected) Igusa curve, where $[\mathrm{Ig}(p)]$ is the moduli problem that classifies $\mathbf{Z}/p\mathbf{Z}$-generators of the kernel of the relative Verschiebung $V_{E/S} : E^{(p)} \to E$, and the line bundle ω of relative 1-forms on the universal elliptic curve over $M(\mathcal{P})_{/\mathbf{F}_p}$ provides the description

$$M([\Gamma_1(p)]\text{-}(1,0)\text{-cyclic}, \mathcal{P}) \simeq \mathrm{Spec}((\mathrm{Sym}_{M(\mathcal{P})_{/\mathbf{F}_p}}\omega)/\omega^{\otimes(p-1)}) \tag{3.3.3}$$

as the cover obtained by locally requiring a formal coordinate of the level-p structure to have $(p-1)$th power equal to zero. The scheme (3.3.3) has generic multiplicity $p-1$ and has smooth underlying reduced curve $M(\mathcal{P})_{/\mathbf{F}_p}$.

We conclude that $Y_1(p;\mathcal{P})$ is $\mathbf{Z}_{(p)}$-smooth at points in

$$M([\Gamma_1(p)]\text{-}(0,1)\text{-cyclic}, \mathcal{P})^{\mathrm{ord}},$$

and near points in $M([\Gamma_1(p)]\text{-}(1,0)\text{-cyclic}, \mathcal{P})$ we can use a local trivialization of ω to find a nilpotent function X with a moduli-theoretic interpretation as the formal coordinate of the point in the $\Gamma_1(p)$-structure (with X^{p-1} arising as $\Phi_p(X+1) \bmod p$ along the ordinary locus). Thus, we get the "ordinary" part of:

Theorem 3.3.3. *Let k be an algebraically closed field of characteristic p, and $z \in Y_1(p)_{/k}$ a rational point corresponding to a $(1,0)$-subgroup of an elliptic curve E over k. Choose $z' \in Y_1(p;\mathcal{P})_{/k}$ over z. Let f_z be a generator of the kernel of the surjection $W[\![x,t]\!] \twoheadrightarrow \widehat{\mathcal{O}}_{Y_1(p;\mathcal{P}),z'}$ in* (3.2.1).

We can choose f_z so that

$$f_z \bmod p = \begin{cases} x^{p-1} & \text{if } E \text{ is ordinary,} \\ x^{p-1}t' & \text{if } E \text{ is supersingular,} \end{cases}$$

with p, x, t' a regular system of parameters in the supersingular case. In particular, $Y_1(p;\mathcal{P})^{\mathrm{red}}_{/k}$ has smooth irreducible components, ordinary double point singularities at supersingular points, and no other non-smooth points.

The significance of Theorem 3.3.3 for our purposes is that it ensures the regular $\mathbf{Z}_{(p)}$-curve $Y_1(p;\mathcal{P})_{\mathbf{Z}_{(p)}}$ is nil-semistable in the sense of Definition 2.3.1. In particular, for $p > 3$ and any subgroup $H \subseteq (\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\}$, the modular curve $X_H(p)$ has tame cyclic quotient singularities away from the cusps.

Proof. The geometric irreducible components of $Y_1(p,\mathcal{P})^{\mathrm{red}}_{/k}$ are smooth curves (3.3.2) and (3.3.3) that intersect at exactly the supersingular points, and (3.3.3) settles the description of $f_z \bmod p$ in the ordinary case. It remains to verify the description of $f_z \bmod p$ at supersingular points z, for once this is checked then the two minimal primes (x) and (t') in the deformation ring at z must correspond to the k-fiber irreducible components of the smooth curves (3.3.2) and $(3.3.3)_{\mathrm{red}}$ through z', and these two primes visibly generate the maximal ideal at z' in the k-fiber so (3.3.2) and $(3.3.3)_{\mathrm{red}}$ intersect transversally at z' as desired.

Consider the supersingular case. The proof of [34, 13.5.4] ensures that we can choose f_z so that

$$f_z \bmod p = g_{(1,0)}g_{(0,1)}, \tag{3.3.4}$$

with $k[\![x,t]\!]/g_{(0,1)}$ the complete local ring at z' on the closed subscheme (3.3.2) and likewise for $k[\![x,t]\!]/g_{(1,0)}$ and (3.3.3). By (3.3.3), we can take $g_{(1,0)} = x^{p-1}$, so by (3.3.4) it suffices to check that the formally smooth ring $k[\![x,t]\!]/g_{(0,1)}$ does not have t as a formal parameter. In the proof of [34, 12.8.2], it is shown that there is a natural isomorphism between the moduli stack of Igusa structures and the moduli stack of $(p-1)$th roots of the Hasse invariant of elliptic curves over $\mathbf{F}_p$-schemes. Since the Hasse invariant commutes with base change and the Hasse invariant on the the universal deformation of a supersingular elliptic curve over $k[\![t]\!]$ has a simple zero [34, 12.4.4], by extracting a $(p-1)$th root we lose the property of t being a formal parameter if $p > 2$. We do not need the theorem for the supersingular case when $p = 2$, so we leave this case as an exercise for the interested reader.

□

4 Determination of non-regular points

Since the quotient $X_H(p)$ of the normal proper $\mathbf{Z}_{(p)}$-curve $X_1(p;\mathcal{P})$ is normal, there is a finite set of non-regular points in codimension-2 on $X_H(p)$ that we have to resolve to get a regular model. We will prove that the non-regular points on the nil-semistable $X_H(p)$ are certain *non-cuspidal* $\mathbf{F}_p$-rational points with j-invariants 0 and 1728, and that these singularities are tame cyclic quotient singularities when $p > 3$, so Jung–Hirzebruch resolution in Theorem 2.4.1 will tell us everything we need to know about the minimal regular resolution of $X_H(p)$.

4.1 Analysis away from cusps

The only possible non-regular points on $X_H(p)$ are closed points in the closed fiber. We will first consider those points that lie in $Y_H(p)$, and then we will study the situation at the cusps. The reason for treating these cases separately is that the deformation theory of generalized elliptic curves is a little more subtle than that of elliptic curves. One can also treat the situation at the cusps by using Tate curves instead of formal deformation theory; this is the approach used in [34].

In order to determine the non-regular points on $Y_H(p)$, by Lemma 2.1.1 we only need to consider geometric points. By Theorem 3.1.1, we need a criterion for detecting when a finite group acting on a regular local ring has regular subring of invariants. The criterion is provided by Serre's Theorem 2.3.9 and leads to:

THEOREM 4.1.1. *A geometric point $z = (E_z, P_z) \in Y_1(p)$ has non-regular image in $Y_H(p)$ if and only if it is a point in the closed fiber such that $|\mathrm{Aut}(E_z)| > 2$, $P_z = 0$, and $2|H| \nmid |\mathrm{Aut}(E_z)|$.*

In particular, when $p > 3$ there are at most two non-regular points on $Y_H(p)$ and such points are $\mathbf{F}_p$-rational, while for $p \le 3$ (so H is trivial) the unique ($\mathbf{F}_p$-rational) supersingular point is the unique non-regular point.

Proof. Let k be an algebraically closed field of characteristic p and define $W = W(k)$; we may assume that z is a k-rational point. By Lemma 2.1.1, we may consider the situation after base change by $\mathbf{Z}_{(p)} \to W$. A non-regular point z must be a closed point on the closed fiber. Let z' be a point over z in $Y_1(p;\mathcal{P})(k)$. Let (E_z, P_z) be the structure arising from z.

First suppose $p > 3$ and H is trivial. The group $\mathrm{Aut}_k(E_z)$ is cyclic of order prime to p, so the automorphism group $\mathrm{Aut}(z)$ of the $\Gamma_1(p)$-structure underlying z is also cyclic of order prime to p. By Theorems 3.1.1 and 2.3.9, the regularity of $\widehat{\mathcal{O}}_{Y_1(p)_W,z}$ is therefore equivalent to the existence of a stable line under the action of $\mathrm{Aut}(z)$ on the 2-dimensional cotangent space to the regular universal deformation ring $\mathcal{R}_z = \widehat{\mathcal{O}}_{Y_1(p;\mathcal{P})_W,z'}$ of the $\Gamma_1(p)$-structure z.

When the $\Gamma_1(p)$-structure z is étale (*i.e.*, $P_z \neq 0$), then the formal deformation theory for z is the same as for the underlying elliptic curve $E_z/\langle P_z\rangle$, whence the universal deformation ring is isomorphic to $W[\![t]\!]$. In such cases, p spans an $\mathrm{Aut}(z)$-invariant line in the cotangent space of the deformation ring. Even when H is not assumed to be trivial, this line is stable under the action of the stabilizer of z' the preimage of H in $(\mathbf{Z}/p\mathbf{Z})^\times$). Hence, we get regularity at z for any H when $p > 3$ and $P_z \neq 0$.

Still assuming $p > 3$, now drop the assumption of triviality on H but suppose that the $\Gamma_1(p)$-structure is not étale, so $z = (E_z, 0)$ and $\mathrm{Aut}(z) = \mathrm{Aut}_k(E_z)$. The preimage $H' \subseteq (\mathbf{Z}/p\mathbf{Z})^\times$ of H acts on the deformation ring $\mathcal{R}_z$ since $P_z = 0$. By Theorem 3.1.1 and Theorem 3.2.2, the cotangent space to $\mathcal{R}_z$ is canonically isomorphic to

$$\mathrm{Cot}_0(E_z) \oplus \mathrm{Cot}_0(E_z)^{\otimes 2}, \tag{4.1.1}$$

where this decomposition corresponds to the lines spanned by the images of x and t respectively. Conceptually, the first line in (4.1.1) arises from equicharacterisitc deformations of the point of order p on constant deformations of the elliptic curve E_z, and the second line arises from deformations of the elliptic curve without deforming the vanishing level structure P_z. These identifications are compatible with the natural actions of $\mathrm{Aut}(z) = \mathrm{Aut}(E_z)$.

Since $p > 3$, the action of $\mathrm{Aut}(E_z) = \mathrm{Aut}(z)$ on the line $\mathrm{Cot}_0(E_z)$ is given by a faithful (non-trivial) character $\overline{\chi}_{\mathrm{id}}$, and the other line in (4.1.1) is acted upon by $\mathrm{Aut}(E_z)$ via the character χ_{id}^2. The resulting representation of $\mathrm{Aut}(z)$ on $\mathrm{Cot}_0(E_z)^{\otimes 2}$ is trivial if and only if $\overline{\chi}_{\mathrm{id}}^2 = 1$, which is to say (by faithfulness) that $\mathrm{Aut}(E_z)$ has order 2 (*i.e.*, $j(E_z) \neq 0, 1728$). Since the H'-action is trivial on the line $\mathrm{Cot}_0(E_z)^{\otimes 2}$ (due to H' only acting on the level structure) and we are passing to invariants by the action of the group $H' \times \mathrm{Aut}(E_{z/k})$, by Serre's theorem we get regularity without restriction on H when $j(E_z) \neq 0, 1728$.

If $j(E_z) \in \{0, 1728\}$ then $|\mathrm{Aut}(E_z)| > 2$ and the cyclic H' acts on (4.1.1) through a representation $\psi \oplus 1$ with ψ a faithful character. The cyclic $\mathrm{Aut}(z)$ acts through a representation $\chi \oplus \chi^2$ with χ a faithful character, so $\chi^2 \neq 1$. The commutative group of actions on (4.1.1) generated by H' and $\mathrm{Aut}(z)$ is generated by pseudo-reflections if and only if the action of the cyclic $\mathrm{Aut}(z)$ on the first line is induced by the action of a subgroup of H'. That is, the order of χ must divide the order of ψ, or equivalently $|\operatorname{Aut}(z)|$ must divide $|H'| = 2|H|$. This yields exactly the desired conditions for non-regularity when $p > 3$.

Now suppose $p \leq 3$, so H is trivial. If $\operatorname{Aut}(E_{z/k}) = \{\pm 1\}$, so z is an ordinary point, then for $p = 3$ we can use the preceding argument to deduce regularity at z. Meanwhile, for $p = 2$ we see that $\mathcal{R}_z$ is formally smooth by Theorem 3.3.3, so the subring of invariants at z is formally smooth (by [34, p. 508]). It remains to check non-regularity at the unique (supersingular) point $z \in Y_1(p)_{/k}$ with $j = 0 = 1728$ in k.

By Serre's theorem, it suffices to check that the action of $\operatorname{Aut}(z) = \operatorname{Aut}(E_z)$ on (4.1.1) is not generated by pseudo-reflections, where E_z is the unique supersingular elliptic curve over k (up to isomorphism). The action of $\operatorname{Aut}(E_z)$ is through 1-dimensional characters, so the p-Sylow subgroup must act trivially. In both cases ($p = 2$ or 3) the group $\operatorname{Aut}(E_z)$ has order divisible by only two primes p and p', with the p'-Sylow of order > 2. This p'-Sylow must act through a faithful character on $\operatorname{Cot}_0(E_z)$ (use [20, Lemma 3.3] or [68, Lemma 2.16]), and hence this group also acts non-trivially on $\operatorname{Cot}_0(E_z)^{\otimes 2}$. It follows that this action is not generated by pseudo-reflections.

□

4.2 Regularity along the cusps

Now we check that $X_H(p)$ is regular along the cusps, so we can focus our attention on $Y_H(p)$ when computing the minimal regular resolution of $X_H(p)$. We will again use deformation theory, but now in the case of generalized elliptic curves. Throughout this section, p is an arbitrary prime.

Recall that a *generalized elliptic curve* over a scheme S is a proper flat map $\pi : E \to S$ of finite presentation equipped with a section $e : S \to E^{\mathrm{sm}}$ into the relative smooth locus and a map

$$+ : E^{\mathrm{sm}} \times_S E \to E$$

such that

- the geometric fibers of π are smooth genus 1 curves or Néron polygons;
- $+$ restricts to a commutative group scheme structure on E^{sm} with identity section e;
- $+$ is an action of E^{sm} on E such that on singular geometric fibers with at least two "sides", the translation action by each rational point in the smooth locus induces a rotation on the graph of irreducible components.

Since the much of the basic theory of Drinfeld structures was developed in [34, Ch. 1] for arbitrary smooth separated commutative group schemes of relative dimension 1, it can be applied (with minor changes in proofs) to the smooth locus of a generalized elliptic curve. In this way, one can merge the "affine" moduli-theoretic $\mathbf{Z}$-theory in [34] with the "proper" moduli-theoretic $\mathbf{Z}[1/N]$-theory in [15]. We refer the reader to [21] for further details on this synthesis.

The main deformation-theoretic fact we need is an analogue of Theorem 3.2.1:

Theorem 4.2.1. *An irreducible generalized elliptic curve C_1 over a perfect field k of characteristic $p > 0$ admits a universal deformation ring that is abstractly isomorphic to $W[\![t]\!]$, and the equicharacteristic cotangent space of this deformation ring is canonically isomorphic to* $\mathrm{Cot}_0(C_1^{\mathrm{sm}})^{\otimes 2}$.

Proof. The existence and abstract structure of the deformation ring are special cases of [15, III, 1.2]. To describe the cotangent space intrinsically, we wish to put ourselves in the context of deformation theory of proper flat curves. Infinitesimal deformations of C_1 admit a unique generalized elliptic curve structure once we fix the identity section [15, II, 2.7], and any two choices of identity section are uniquely related by a translation action. Thus, the deformation theory for C_1 as a generalized elliptic (*i.e.*, marked) curve coincides with its deformation theory as a flat (unmarked) curve. In particular, the tangent space to this deformation functor is canonically identified with $\mathrm{Ext}^1_{C_1}(\Omega^1_{C_1/k}, \mathcal{O}_{C_1})$ [56, §4.1.1].

Since the natural map $\Omega^1_{C_1/k} \to \omega_{C_1/k}$ to the invertible relative dualizing sheaf is injective with finite-length cokernel (supported at the singularity),

$$\mathrm{Ext}^1_{C_1}(\omega_{C_1/k}, \mathcal{O}_{C_1}) \simeq \mathrm{Ext}^1_{C_1}(\omega^{\otimes 2}_{C_1/k}, \omega_{C_1/k}) \simeq \mathrm{H}^0(C_1, \omega^{\otimes 2}_{C_1/k})^{\vee},$$

with the final isomorphism provided by Grothendieck duality. Thus, the cotangent space to the deformation functor is identified with $\mathrm{H}^0(C_1, \omega^{\otimes 2}_{C_1/k})$. Since $\omega_{C_1/k}$ is (non-canonically) trivial, just as for elliptic curves, we get a canonical isomorphism

$$\mathrm{H}^0(C_1, \omega^{\otimes 2}_{C_1/k}) \simeq \mathrm{H}^0(C_1, \omega_{C_1/k})^{\otimes 2} \simeq \mathrm{Cot}_0(C_1^{\mathrm{sm}})^{\otimes 2}$$

(the final isomorphism defined via pullback along the identity section).

□

Definition 4.2.2. A $\Gamma_1(N)$*-structure* on a generalized elliptic curve $E \to S$ is an "S-ample" Drinfeld $\mathbf{Z}/N\mathbf{Z}$-structure on E^{sm}; *i.e.*, a section $P \in E^{\mathrm{sm}}[N](S)$ such that the relative effective Cartier divisor

$$D = \sum_{j \in \mathbf{Z}/N\mathbf{Z}} [jP]$$

in E^{sm} is a subgroup scheme which meets all irreducible components of all geometric fibers.

If $E_{/S}$ admits a $\Gamma_1(N)$-structure, then the non-smooth geometric fibers must be d-gons for various $d|N$. In case $N = p$ is prime, this leaves p-gons and 1-gons as the only options. The importance of Definition 4.2.2 is the following analogue of Theorem 3.1.1:

Theorem 4.2.3. *Let k be an algebraically closed field of characteristic $p > 0$, and $W = W(k)$. The points of $X_1(p)_{/k} - Y_1(p)_{/k}$ correspond to isomorphism classes of $\Gamma_1(p)$-structures on degenerate generalized elliptic curves over k with 1 or p sides.*

For $z \in X_1(p)_{/k} - Y_1(p)_{/k}$, there exists a universal deformation ring $\mathcal{S}_z$ for the $\Gamma_1(p)$-structure z, and $\widehat{\mathcal{O}}_{X_1(p)_W,z}$ is the subring of $\mathrm{Aut}(z)$-invariants in $\mathcal{S}_z$.

Proof. In general, $\Gamma_1(p)$-structures on generalized elliptic curves form a proper flat Deligne-Mumford stack $\overline{M}_{\Gamma_1(p)}$ over $\mathbf{Z}_{(p)}$ of relative dimension 1, and this stack is smooth over $\mathbf{Q}$ and is normal (as one checks via abstract deformation theory). For our purposes, the important point is that if we choose an odd prime $\ell \neq p$ then we can define an evident $[\Gamma_1(p), \Gamma(\ell)]$-variant on Definition 4.2.2 (imposing an ampleness condition on the combined level structure), and the open locus of points with trivial geometric automorphism group is a scheme (as it is an algebraic space quasi-finite over the j-line). This locus fills up the entire stack $\overline{M}_{[\Gamma_1(p),\Gamma(\ell)]}$ over $\mathbf{Z}_{(p)}$, so this stack is a scheme.

The resulting normal $\mathbf{Z}_{(p)}$-flat proper scheme $\overline{M}_{[\Gamma_1(p),\Gamma(\ell)]}$ is finite over the j-line, whence it must *coincide* with the scheme $X_1(p;[\Gamma(\ell)])$ as constructed in [34] by the *ad hoc* method of normalization of the fine moduli scheme $Y_1(p;[\Gamma(\ell)])$ over the j-line. We therefore get a map

$$\overline{M}_{[\Gamma_1(p),\Gamma(\ell)]} = X_1(p;[\Gamma(\ell)]) \to X_1(p)$$

that *must* be the quotient by the natural $\mathrm{GL}_2(\mathbf{F}_\ell)$-action on the source. Since complete local rings at geometric points on a Deligne-Mumford stack coincide with universal formal deformation rings, we may conclude as in the proof of Theorem 3.1.1.

□

We are now in position to argue just as in the elliptic curve case: we shall work out the deformation rings in the various possible cases and for $p \neq 2$ we will use Serre's pseudo-reflection theorem to deduce regularity of $X_1(p)$ along the cusps on the closed fiber. A variant on the argument will also take care of $p = 2$.

As in the elliptic curve case, it will suffice to consider geometric points. Thus, there will be two types of $\Gamma_1(p)$-structures (E, P) to deform: E is either a p-gon or a 1-gon.

Lemma 4.2.4. *Let E_0 be a p-gon over an algebraically closed field k of characteristic p, and $P_0 \in E_0^{\mathrm{sm}}(k)$ a $\Gamma_1(p)$-structure. The deformation theory of (E_0, P_0) coincides with the deformation theory of the 1-gon generalized elliptic curve $E_0/\langle P_0\rangle$.*

Note that in the p-gon case, the point $P_0 \in E_0^{\mathrm{sm}}(k)$ generates the order-p constant component group of E_0^{sm}, so the group scheme $\langle P_0\rangle$ generated by P_0 is visibly étale and the quotient $E_0/\langle P_0\rangle$ makes sense (as a generalized elliptic curve) and is a 1-gon.

Proof. For any infinitesimal deformation (E, P) of (E_0, P_0), the subgroup scheme H generated by P is finite étale, and it makes sense to form the quotient E/H as a generalized elliptic curve deformation of the 1-gon E_0/H_0 (with $H_0 = \langle P_0 \rangle$). Since any finite étale cover of a generalized elliptic curve admits a unique compatible generalized elliptic curve structure once we fix a lift of the identity section and demand geometric connectedness of fibers over the base [15, II, 1.17], we see that the deformation theory of (E_0, H_0) (ignoring P) is equivalent to the deformation theory of the 1-gon E_0/H_0. The deformation theory of a 1-gon is formally smooth of relative dimension 1 [15, III, 1.2], and upon specifying (E, H) deforming (E_0, H_0) the étaleness of H ensures the existence and uniqueness of the choice of $\Gamma_1(p)$-structure P generating H such that P lifts P_0 on E_0. That is, the universal deformation ring for (E_0, P_0) coincides with that of E_0/H_0.

□

In the 1-gon case, there is only one (geometric) possibility up to isomorphism: the pair $(C_1, 0)$ where C_1 is the standard 1-gon (over an algebraically closed field k of characteristic p). For this, we have an analogue of (4.1.1):

Lemma 4.2.5. *The universal deformation ring of the $\Gamma_1(p)$-structure $(C_1, 0)$ is isomorphic to the regular local ring $W[\![t]\!][\![X]\!]/\Phi_p(X+1)$, with cotangent space canonically isomorphic to*

$$\mathrm{Cot}_0(C_1^{\mathrm{sm}}) \oplus \mathrm{Cot}_0(C_1^{\mathrm{sm}})^{\otimes 2}.$$

Proof. Since the p-torsion on C_1^{sm} is isomorphic to μ_p, upon fixing an isomorphism $C_1^{\mathrm{sm}}[p] \simeq \mu_p$ there is a unique compatible isomorphism $C^{\mathrm{sm}}[p] \simeq \mu_p$ for any infinitesimal deformation C of C_1. Thus, the deformation problem is that of endowing a $\mathbf{Z}/p\mathbf{Z}$-generator to the μ_p inside of deformations of C_1 (as a generalized elliptic curve). By Theorem 4.2.3, this is the scheme of generators of μ_p over the universal deformation ring $W[\![t]\!]$ of C_1.

The scheme of generators of μ_p over $\mathbf{Z}$ is $\mathbf{Z}[Y]/\Phi_p(Y)$, so we obtain $W[\![t]\!][Y]/\Phi_p(Y)$ as the desired (regular) deformation ring. Now just set $X = Y - 1$. The description of the cotangent space follows from Theorem 4.2.1.

□

Since C_1 has automorphism group (as a generalized elliptic curve) generated by the unique extension $[-1]$ of inversion from C_1^{sm} to all of C_1, we conclude that $\mathrm{Aut}(C_1, 0)$ is generated by $[-1]$. This puts us in position to carry over our earlier elliptic-curve arguments to prove:

Theorem 4.2.6. *The scheme $X_H(p)$ is regular along its cusps.*

Proof. As usual, we may work after making a base change by $W = W(k)$ for an algebraically closed field k of characteristic $p > 0$. Let $z \in X_1(p)_{/k}$ be a cusp whose image z_H in $X_H(p)_{/k}$ we wish to study. Let H' be the preimage of H in $(\mathbf{Z}/p\mathbf{Z})^\times$, and let H'_z be the maximal subgroup of H' that acts on the deformation space for z (*e.g.*, $H'_z = H'$ if the level structure P_z vanishes). By Theorem 4.2.3, the ring $\widehat{\mathcal{O}}_{X_H(p),z_H}$ is the subring of invariants under the action of $\mathrm{Aut}(z) \times H'_z$ on the formal deformation ring for z. By Theorem 4.2.1 and Lemma 4.2.4 (as well as [34, p. 508]), this deformation ring is regular (even formally smooth) in the p-gon case. In the 1-gon case, Lemma 4.2.5 ensures that the deformation ring is regular (and even formally smooth when $p = 2$). Thus, for $p \neq 2$ we may use Theorem 2.3.9 to reduce the problem for $p \neq 2$ to checking that the action of $\mathrm{Aut}(z) \times H'_z$ on the 2-dimensional cotangent space to the deformation functor has an invariant line.

In the p-gon case, the deformation ring is $W[\![t]\!]$ and the cotangent line spanned by p is invariant. In the 1-gon case, Lemma 4.2.5 provides a functorial description of the cotangent space to the deformation functor and from this it is clear that the involution $[-1]$ acts with an invariant line $\mathrm{Cot}_0(z)^{\otimes 2}$ when $p \neq 2$ and that H'_z also acts trivially on this line.

To take care of $p = 2$ (for which H is trivial), we just have to check that any non-trivial W-algebra involution ι of $W[\![T]\!]$ has regular subring of invariants. In fact, for $T' = T\iota(T)$ the subring of invariants is $W[\![T']\!]$ by [34, p. 508].

□

5 The Minimal resolution

We now are ready to compute the minimal regular resolution $X_H(p)^{\mathrm{reg}}$ of $X_H(p)$. Since $X_H(p)_{/\mathbf{Q}}$ is a projective line when $p \leq 3$, both Theorem 1.1.2 and Theorem 1.1.6 are trivial for $p \leq 3$. Thus, from now on we assume $p > 3$. We have found all of the non-regular points (Theorem 4.1.1): the $\mathbf{F}_p$-rational points of $(1,0)$-type such that $j \in \{0, 1728\}$, provided that $|H|$ is not divisible by 3 (resp. 2) when $j = 0$ (resp. $j = 1728$). Theorem 3.3.3 provides the necessary local description to carry out Jung–Hirzebruch resolution at these points. These are tame cyclic quotient singularities (since $p > 3$). Moreover, the closed fiber of $X_H(p)$ is a nil-semistable curve that consists of two irreducible components that are geometrically irreducible, as one sees by considering the (1,0)-cyclic and (0,1)-cyclic components.

5.1 General considerations

There are four cases, depending on $p \equiv \pm 1, \pm 5 \bmod 12$ as this determines the behavior of the j-invariants 0 and 1728 in characteristic p (*i.e.*, supersingular or ordinary). This dichotomy between ordinary and supersingular cases corresponds to Jung–Hirzebruch resolution with either one or two analytic branches.

Pick a point $z = (E, 0) \in X_1(p)(\mathbf{F}_p)$ with $j = 0$ or 1728 corresponding an elliptic curve E over $\overline{\mathbf{F}}_p$ with automorphism group of order > 2. Let $z_H \in X_H(p)(\mathbf{F}_p)$ be the image of z. By Theorem 4.1.1, we know that z_H is non-regular if and only if $|H|$ is odd for $j(E) = 1728$, and if and only if $|H|$ is not divisible by 3 for $j(E) = 0$.

There is a single irreducible component through z_H in the ordinary case (arising from either (3.3.2) or (3.3.3)), while there are two such (transverse) components in the supersingular case, and to compute the generic multiplicities of these components in $X_H(p)_{/\overline{\mathbf{F}}_p}$ we may work with completions because the irreducible components through z_H are analytically irreducible (even smooth) at z_H.

Let C' and C denote the irreducible components of $X_H(p)_{/\overline{\mathbf{F}}_p}$, with C' corresponding to étale level p-structures. Since the preimage of H in $(\mathbf{Z}/p\mathbf{Z})^\times$ (of order $2|H|$) acts generically freely (resp. trivially) on the preimage of C' (resp. of C) in a fine moduli scheme over $X_H(p)_{/\overline{\mathbf{F}}_p}$ obtained by adjoining some prime-to-p level structure, ramification theory considerations and Theorem 3.3.3 show that the components C' and C in $X_H(p)_{/\overline{\mathbf{F}}_p}$ have respective multiplicities of 1 and $(p-1)/2|H| = [(\mathbf{Z}/p\mathbf{Z})^\times/\{\pm 1\} : H]$. Moreover, by Theorem 3.3.3 we see that z_H lies on C when it is an ordinary point.

5.2 The case $p \equiv -1 \bmod 12$

We are now ready to resolve the singularities on $X_H(p)_{/W}$ with $W = W(\overline{\mathbf{F}}_p)$. We will first carry out the calculation in the case $p \equiv -1 \pmod{12}$, so 0 and 1728 are supersingular j-values. In this case $(p-1)/2$ is not divisible by 2 or 3, so $|H|$ is automatically not divisible by 2 or 3 (so we have two non-regular points).

Write $p = 12k-1$ with $k \geq 1$. By the Deuring Mass Formula [34, Cor. 12.4.6] the components C and C' meet in $(p-11)/12 = k-1$ geometric points away from the two supersingular points with $j = 0, 1728$. Consider one of the two non-regular supersingular points z_H. The complete local ring at z_H on $X_H(p)_W$ is the subring of invariants for the commuting actions of $\mathrm{Aut}(z)$ and the preimage $H' \subseteq (\mathbf{Z}/p\mathbf{Z})^\times$ of H on the universal deformation ring $\mathcal{R}_z$ of the $\Gamma_1(p)$-structure z. Note that the actions of H' and $\mathrm{Aut}(z)$ on $\mathcal{R}_z$ have a common involution. The action of H' on the tangent space fixes one line and acting through a faithful character on the other line (see the proof of Theorem 4.1.1), so by Serre's Theorem 2.3.9 the subring of H'-invariants in $\mathcal{R}_z$ is regular. By Lemma 2.3.5 and the subsequent discussion there, the subring of H'-invariant has the form $W[\![x', t']\!]/({x'}^{(p-1)/|H'|}t' - p)$ with $\mathrm{Aut}(z)/\{\pm 1\}$ acting on the tangent space via $\chi^{|H|} \oplus \chi$ for a faithful character χ of $\mathrm{Aut}(z)/\{\pm 1\}$. Let $h = |H|$, so $\rho := (p-1)/2h$ is the multiplicity of C in $X_H(p)_{/\overline{\mathbf{F}}_p}$.

When $j(z_H) = 1728$ the character χ is quadratic, so we apply Theorem 2.4.1 and Corollary 2.4.3 with $n = 2$, $r = 1$, $m'_1 = 1$, $m'_2 = \rho$. The resolution has a single exceptional fiber D' that is transverse to the strict transforms $\overline{C}$ and $\overline{C}'$, and D' has self-intersection -2 and multiplicity $(m'_1 + m'_2)/2 = (\rho+1)/2$. When $j(z_H) = 0$ the character χ is cubic, so we apply Theorem 2.4.1 with $n = 3$, $m'_1 = 1$, $m'_2 = \rho$, and $r = h \bmod 3$. That is, $r = 1$ when $h \equiv 1 \bmod 6$ and $r = 2$ when $h \equiv -1 \bmod 6$. In the case $r = 1$ we get a single exceptional fiber E' in the resolution, transverse to $\overline{C}$ and $\overline{C}'$ with self-intersection -3 and

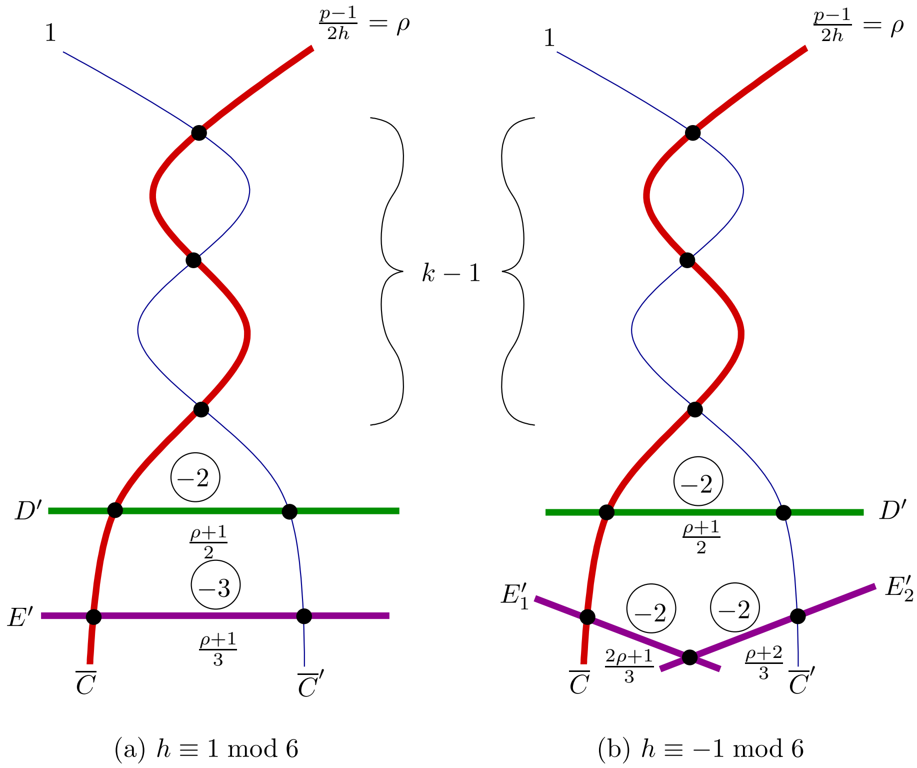

Figure 2: Minimal regular resolution $X_H(p)'$ of $X_H(p)$, $p = 12k - 1$, $k \geq 1$, $h = |H|$

multiplicity $(\rho + 1)/3$ (by Corollary 2.4.3). This is illustrated in Figure 2(a). In the case $r = 2$ we use the continued fraction $3/2 = 2 - 1/2$ to see that the resolution of z_H has exceptional fiber with two components E'_1 and E'_2, and these have self-intersection -2 and transverse intersections as shown in Figure 2(b) with respective multiplicities $(2\rho+1)/3$ and $(\rho+2)/3$ by Corollary 2.4.3. This completes the computation of the minimal regular resolution $X_H(p)'$ of $X_H(p)$ when $p \equiv -1 \bmod 12$.

To compute the intersection matrix for the closed fiber of $X_H(p)'$, we need to compute some more intersection numbers. For $h \equiv 1 \bmod 6$ we let μ and ν denote the multiplicities of D' and E' in $X_H(p)'$, and for $h \equiv -1 \bmod 6$ we define μ in the same way and let ν_j denote the multiplicity of E'_j in $X_H(p)'$. In other words,

$$\mu = (\rho+1)/2,\ \nu = (\rho+1)/3,\ \nu_1 = (2\rho+1)/3,\ \nu_2 = (\rho+2)/3.$$

Thus,

$$\overline{C}' + \rho\overline{C} + \mu D' + \nu E' \equiv 0, \tag{5.2.1}$$

so if we intersect (5.2.1) with $\overline{C}$ and use the identities

$$\rho = (6k-1)/h, \ \ \overline{C}'.\overline{C} = k - 1 = (h\rho - 5)/6,$$

we get

$$\overline{C}.\overline{C} = -1 - (h - \varepsilon)/6$$

where $\varepsilon = \pm 1 \equiv h \bmod 6$. In particular, $\overline{C}.\overline{C} < -1$ unless $h = 1$ (*i.e.*, unless H is trivial). We can also compute the self-intersection for $\overline{C}'$, but we do not need it.

When H is trivial, so $\overline{C}$ is a -1-curve, we can contract $\overline{C}$ and then by Theorem 2.1.2 and Figure 2 the self-intersection numbers for the components D' and E' drop to -1 and -2 respectively. Then we may contract D', so E' becomes a -1-curve, and finally we end with a single irreducible component (coming from $\overline{C}'$). This proves Theorem 1.1.2 when $p \equiv -1 \bmod 12$.

Returning to the case of general H, let us prove Theorem 1.1.6 for $p \equiv -1 \bmod 12$. Since $\overline{C}'$ has multiplicity 1 in the closed fiber of $X_H(p)'$, we can use the following special case of a result of Lorenzini [9, 9.6/4]:

Lemma 5.2.1 (Lorenzini). *Let X be a regular proper flat curve over a complete discrete valuation ring R with algebraically closed residue field and fraction field K. Assume that $X_{/K}$ is smooth and geometrically connected. Let $X_1, \dots, X_m$ be the irreducible components of the closed fiber $\overline{X}$ and assume that some component X_{i_0} occurs with multiplicity 1 in the closed fiber divisor.*

The component group of the Néron model of the Jacobian $\mathrm{Pic}^0_{X_K/K}$ has order equal to the absolute value of the $(m-1)\times(m-1)$ minor of the intersection matrix $(X_i.X_j)$ obtained by deleting the i_0th row and column.

The intersection submatrices formed by the ordered set $\{\overline{C}, D', E'\}$ for $h \equiv 1 \bmod 6$ and by $\{\overline{C}, D', E_1', E_2'\}$ for $h \equiv -1 \bmod 6$ are given in Figure 3. The absolute value of the determinant is h in each case, so by Lemma 5.2.1 the order of the component group $\Phi(\mathcal{J}_H(p)_{/\mathbf{F}_p})$ is $h = |H| = |H|/\gcd(|H|, 6)$.

To establish Theorem 1.1.6 for $p \equiv -1 \bmod 12$, it remains to show that the natural Picard map $J_0(p) \to J_H(p)$ induces a surjection on mod-p geometric component groups. We outline a method that works for general p but that we will (for now) carry out only for $p \equiv -1 \bmod 12$, as we have only computed the intersection matrix in this case.

The component group for $J_0(p)$ is generated by $(0)-(\infty)$, where (0) classifies the 1-gon with standard subgroup $\mu_p \hookrightarrow \mathbf{G}_m$ in the smooth locus, and (∞) classifies the p-gon with subgroup $\mathbf{Z}/p\mathbf{Z} \hookrightarrow (\mathbf{Z}/p\mathbf{Z}) \times \mathbf{G}_m$ in the smooth locus. The generic-fiber Picard map induced by the coarse moduli scheme map

$$X_H(p)_{/\mathbf{Z}_{(p)}} \to X_0(p)_{/\mathbf{Z}_{(p)}}$$

$$\begin{array}{c} \\ \overline{C} \\ D' \\ E' \end{array}\begin{array}{c} \begin{array}{ccc} \overline{C} & D' & E' \end{array} \\ \begin{pmatrix} -1-\frac{(h-1)}{6} & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix} \end{array}$$

(a) $h \equiv 1 \bmod 6$

$$\begin{array}{c} \\ \overline{C} \\ D' \\ E_1' \\ E_2' \end{array}\begin{array}{c} \begin{array}{cccc} \overline{C} & D' & E_1' & E_2' \end{array} \\ \begin{pmatrix} -1-\frac{(h+1)}{6} & 1 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \end{array}$$

(b) $h \equiv -1 \bmod 6$

Figure 3: Submatrices of intersection matrix for $X_H(p)'$, $p \equiv -1 \bmod 12$

pulls $(0) - (\infty)$ back to a divisor

$$P \; - \sum_{j=1}^{(p-1)/2|H|} P_i' \tag{5.2.2}$$

where the P_i''s are $\mathbf{Q}$-rational points whose (cuspidal) reduction lies in the component $\overline{C}'$ classifying étale level-structures and P is a point with residue field $(\mathbf{Q}(\zeta_p)^+)^H$ whose (cuspidal) reduction lies in the component $\overline{C}$ classifying multiplicative level-structures. This description is seen by using the moduli interpretation of cusps (*i.e.*, Néron polygons) and keeping track of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-actions, and it is valid for any prime p (*e.g.*, the $\Gamma_1(p)$-structures on the standard 1-gon consistute a principal homogenous space for the action of $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$, so they give a single closed point P on $X_H(p)_{/\mathbf{Q}}$ with residue field $(\mathbf{Q}(\zeta_p)^+)^H$).

To apply (5.2.2), we need to recall some general facts (see [9, 9.5/9, 9.6/1]) concerning the relationship between the closed fiber of a regular proper model X of a smooth geometrically connected curve X_η and the component group Φ of (the Néron model of) the Jacobian of X_η, with the base equal to the spectrum of a discrete valuation ring R with algebraically closed residue field. If $\{X_i\}_{i\in I}$ is the set of irreducible components in the closed fiber of X, then we can form a complex

$$\mathbf{Z}^I \xrightarrow{\alpha} \mathbf{Z}^I \xrightarrow{\beta} \mathbf{Z}$$

where $\mathbf{Z}^I$ is the free group on the X_i's, the map α is defined by the intersection matrix $(X_i.X_j)$, and β sends each standard basis vector to the multiplicity of the corresponding component in the closed fiber. The cokernel $\ker(\beta)/\mathrm{im}(\alpha)$ is naturally identified with the component group Φ via the map $\mathrm{Pic}(X) \to \mathbf{Z}^I$ that assigns to each invertible sheaf $\mathcal{L}$ its tuple of partial degrees $\deg_{X_i}(\mathcal{L})$.

By using [9, 9.1/5] to compute such line-bundle degrees, one finds that the Néron-model integral point associated to the pullback divisor in (5.2.2) has reduction whose image in $\Phi(\mathcal{J}_H(p)_{/\overline{\mathbf{F}}_p})$ is represented by

$$\frac{[\mathbf{Q}(P):\mathbf{Q}]}{\mathrm{mult}(\overline{C})}\cdot\overline{C}-\sum_{i=1}^{(p-1)/2|H|}\overline{C}'=\overline{C}-\frac{p-1}{2|H|}\cdot\overline{C}' \tag{5.2.3}$$

when this component group is computed by using the regular model $X_H(p)'$ that we have found for $p\equiv -1 \bmod 12$ (the same calculation will work for all other p's, as we shall see).

The important property emerging from this calculation is that one of the coefficients in (5.2.3) is ± 1, so an element in $\ker(\beta)$ that is a $\mathbf{Z}$-linear combination of $\overline{C}$ and $\overline{C}'$ *must* be a multiple of (5.2.3) and hence is in the image of $\Phi(\mathcal{J}_0(p))$ under the Picard map. Thus, to prove that the component group for $J_0(p)$ surjects onto the component group for $J_H(p)$, it suffices to check that any element in $\ker(\beta)$ can be modified modulo $\mathrm{im}(\alpha)$ to lie in the $\mathbf{Z}$-span of $\overline{C}$ and $\overline{C}'$.

Since the matrix for α is the intersection matrix, it suffices (and is even necessary) to check that the submatrix $M_{\overline{C},\overline{C}'}$ of the intersection matrix given by the rows labelled by the irreducible components other than $\overline{C}$ and $\overline{C}'$ is a *surjective* matrix over $\mathbf{Z}$. Indeed, such surjectivity ensures that we can always subtract a suitable element of $\mathrm{im}(\alpha)$ from any element of $\ker\beta$ to kill coefficients away from $\overline{C}$ and $\overline{C}'$ in a representative for an element in $\Phi\simeq\ker(\beta)/\mathrm{im}(\alpha)$. The surjectivity assertion over $\mathbf{Z}$ amounts to requiring that the matrix $M_{\overline{C},\overline{C}'}$ have top-degree minors with gcd equal to 1. It is enough to check that those minors that avoid the column coming from $\overline{C}'$ have gcd equal to 1. Thus, it is enough to check that in Figure 3 the matrix of rows beneath the top row has top-degree minors with gcd equal to 1. This is clear in both cases. In particular, this calculation (especially the analysis of (5.2.3)) yields the following result when $p\equiv -1 \bmod 12$:

Corollary 5.2.2. *Let $\rho=(p-1)/2|H|$. The degree-0 divisor $\overline{C}-\rho\overline{C}'$ represents a generator of the mod-p component group of $J_H(p)$.*

The other cases $p\equiv 1,\pm 5 \bmod 12$ will behave similarly, with Corollary 5.2.2 being true for all such p. The only differences in the arguments are that cases with $|H|$ divisible by 2 or 3 can arise and we will sometimes have to use the "one branch" version of Jung–Hirzebruch resolution to resolve non-regular ordinary points.

5.3 The case $p\equiv 1 \bmod 12$.

We have $p=12k+1$ with $k\geq 1$, so $(p-1)/2=6k$. In this case 0 and 1728 are both ordinary j-invariants, so the number of supersingular points is $(p-1)/12=k$ by the Deuring Mass Formula. The minimal regular resolution $X_H(p)'$ of $X_H(p)$ is illustrated in Figure 4, depending on the congruence class of $h=|H|$ modulo 6. When h is divisible by 6 there are no non-regular points, so $X_H(p)' = X_H(p)_{/W}$ is as in Figure 4(a). When h is even but not divisible by 3 there is only the non-regularity at $j = 0$ to be resolved, as shown in Figures 4(b),(c). The case of odd h is given in Figures 4(d)–(f), and these are all easy applications of Theorem 2.4.1 and Corollary 2.4.3. We illustrate by working out the case $h \equiv 5 \bmod 6$, for which there are two ordinary singularities to resolve.

Arguing much as in the case $p \equiv -1 \bmod 12$, but now with a "one branch" situation at ordinary points, the ring to be resolved is formally isomorphic to the ring of invariants in $W[\![x', t']\!]/(x'^{(p-1)/2|H|} - p)$ under an action of the cyclic $\mathrm{Aut}(z)/\{\pm 1\}$ with a tangent-space action of $\chi^{|H|} \oplus \chi$ for a faithful character χ. At a point with $j = 1728$ we have quadratic χ, $n = 2$, $r = 1$. Using the "one branch" version of Theorem 2.4.1 yields the exceptional divisor D' as illustrated in Figure 4(f), transverse to $\overline{C}$ with self-intersection -2 and multiplicity $\rho/2$. At a point with $j = 0$ we have a cubic χ, so $n = 3$. Since $h \equiv 2 \bmod 3$ when $h \equiv 5 \bmod 6$, we have $r = 2$. Since $3/2 = 2 - 1/2$, we get exceptional divisors E_1' and E_2' with transverse intersections as shown and self-intersections of -2. The "outer" component E_1' has multiplicity $\rho/3$ and the "inner" component E_2' has multiplicity $2\rho/3$. Once again we will suppress the calculation of $\overline{C}'.\overline{C}'$ since it is not needed.

We now proceed to analyze the component group for each value of $h \bmod 6$. Since $\overline{C}'$ has multiplicity 1 in the closed fiber, we can carry out the same strategy that was used for $p \equiv -1 \bmod 12$, resting on Lemma 5.2.1. When $h \equiv 0 \bmod 6$, there are only the components $\overline{C}$ and $\overline{C}'$ in the closed fiber of $X_H(p)' = X_H(p)$, with $\overline{C}.\overline{C} = -h/6$. Thus, the component group has the expected order $|H|/6$ and since there are no additional components we are done in this case.

If $h \equiv 1 \bmod 6$, one finds that the submatrix of the intersection matrix corresponding to the ordered set $\{\overline{C}, D', E'\}$ is

$$\begin{pmatrix} -(h+5)/6 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$$

with absolute determinant $h = |H|/\gcd(|H|, 6)$ as desired, and the bottom two rows have 2×2 minors with gcd equal to 1. Moreover, in the special case $h = 1$ we see that $\overline{C}$ is a -1-curve, and after contracting this we contract D' and E' in turn, leaving us with only the component $\overline{C}'$. This proves Theorem 1.1.2 for $p \equiv 1 \bmod 12$.

For $h \equiv 2 \bmod 6$, the submatrix indexed by $\{\overline{C}, E_1', E_2'\}$ is

$$\begin{pmatrix} -(h+4)/6 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

with absolute determinant $h/2 = |H|/\gcd(|H|, 6)$, and the bottom two rows have 2×2 minors with gcd equal to 1. The cases $h \equiv 3, 4 \bmod 6$ are even

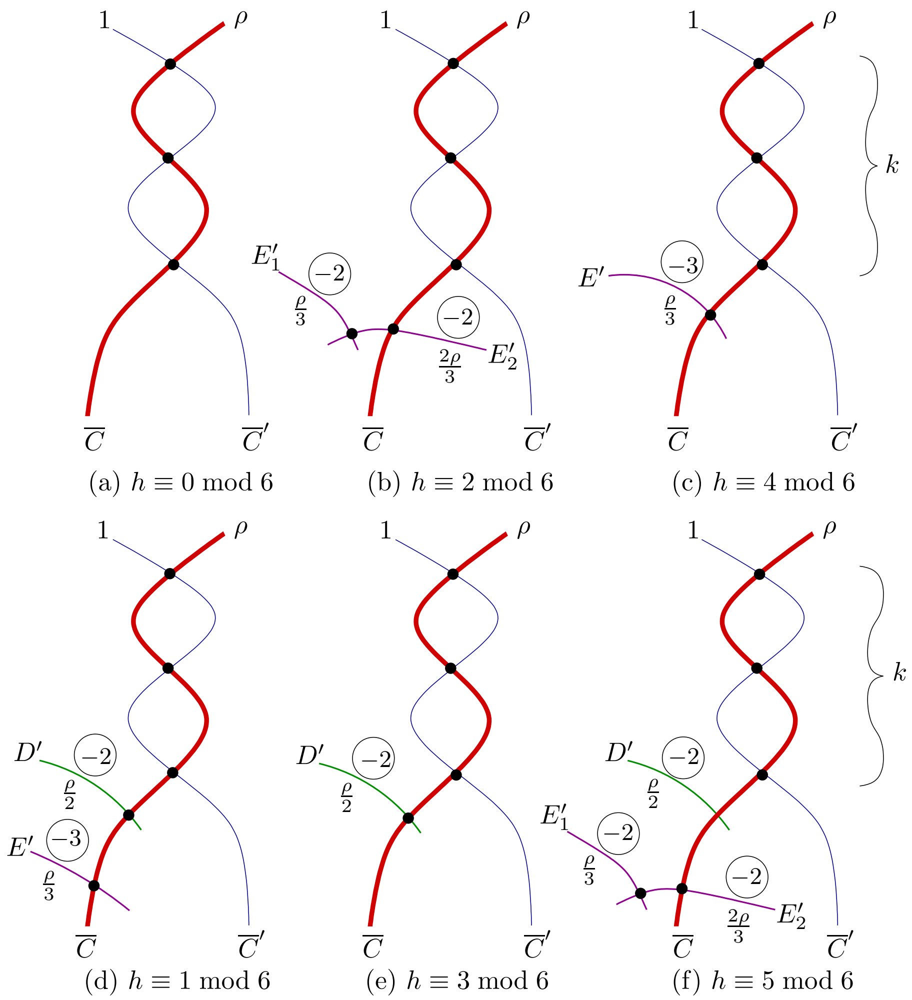

Figure 4: Minimal regular resolution $X_H(p)'$, $p = 12k+1$, $k \geq 1$, $h = |H|$, $\rho = (p-1)/2h$

easier, since there are just two components to deal with, $\{\overline{C}, D'\}$ and $\{\overline{C}, E'\}$ with corresponding matrices

$$\begin{pmatrix} -(h+3)/6 & 1 \\ 1 & -2 \end{pmatrix}, \quad \begin{pmatrix} -(h+2)/6 & 1 \\ 1 & -3 \end{pmatrix}$$

that yield the expected results.

For the final case $h \equiv -1 \bmod 6$, the submatrix indexed by the ordered set of components $\{\overline{C}, D', E_1', E_2'\}$ is

$$\begin{pmatrix} -(h+7)/6 & 1 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

with absolute determinant $h = |H|/\gcd(|H|, 6)$ and gcd 1 for the 3×3 minors along the bottom three rows. The case $p \equiv 1 \bmod 12$ is now settled.

5.4 The cases $p \equiv \pm 5 \bmod 12$

With $p = 12k + 5$ for $k \geq 0$, we have $(p-1)/2 = 6k + 2$, so $h = |H|$ is not divisible by 3. Thus, the supersingular $j = 0$ is always non-regular and the ordinary $j = 1728$ is non-regular for even h.

Using Theorem 2.4.1 and Corollary 2.4.3, we obtain a minimal regular resolution depending on the possibilities for $h \bmod 6$ not divisible by 3, as given in Figure 5.

From Figure 5 one easily carries out the computations of the absolute determinant and the gcd of minors from the intersection matrix, just as we have done in earlier cases, and in all cases one gets $|H|/\gcd(|H|, 6)$ for the absolute determinant and the gcd of the relevant minors is 1. Also, the case $h = 1$ has $\overline{C}$ as a -1-curve, and successive contractions end at an integral closed fiber, so we have established Theorems 1.1.2 and 1.1.6 for the case $p \equiv 5 \bmod 12$.

When $p = 12k - 5$ with $k \geq 1$, so $(p-1)/2 = 6k - 3$ is odd, we have that $h = |H|$ is odd. Thus, $j = 1728$ does give rise to a non-regular point, but the behavior at $j = 0$ depends on $h \bmod 6$. The usual applications of Jung–Hirzebruch resolution go through, and the minimal resolution has closed-fiber diagram as in Figure 6, depending on odd $h \bmod 6$, and both Theorem 1.1.2 and Theorem 1.1.6 drop out just as in the preceding cases.

6 The Arithmetic of $J_1(p)$

Our theoretical results concerning component groups inspired us to carry out some arithmetic computations in $J_1(p)$, and this section summarizes this work.

In Section 6.1 we recall the Birch and Swinnerton-Dyer conjecture, as this motivates many of our computations, and then we describe some of the theory behind the computations that went into computing the tables of Section 6.6.

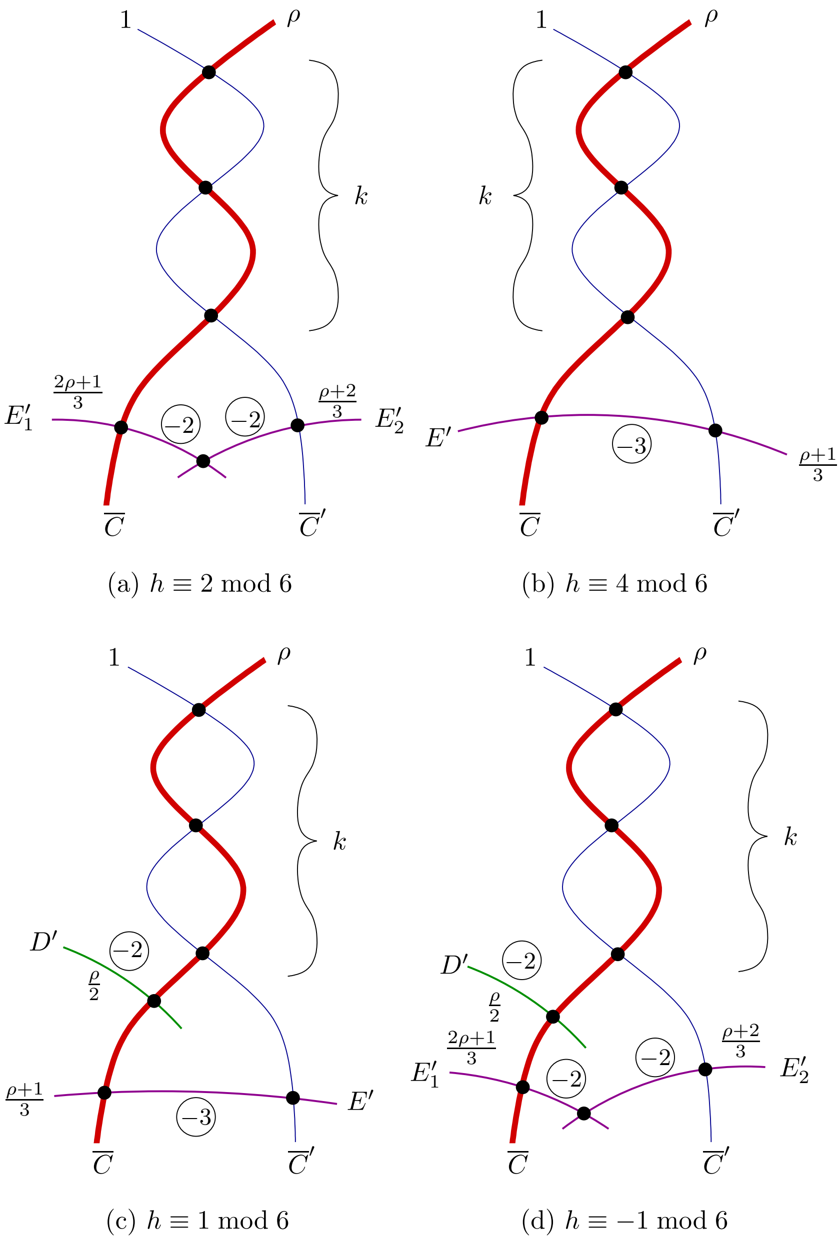

Figure 5: Minimal regular resolution $X_H(p)'$, $p = 12k+5$, $k \geq 0$, $h = |H|$, $\rho = (p-1)/2h$

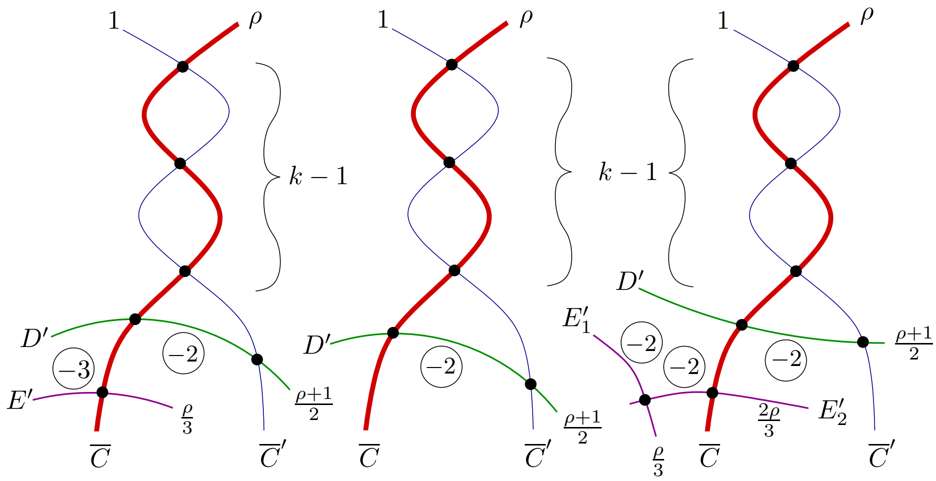

Figure 6: Minimal regular resolution $X_H(p)'$, $p = 12k - 5$, $k \geq 1$, $h = |H|$, $\rho = (p-1)/2h$

In Section 6.2 we find all p such that $J_1(p)$ has rank 0. We next discuss tables of certain arithmetic invariants of $J_1(p)$ and we give a conjectural formula for $|J_1(p)(\mathbf{Q})_{\mathrm{tor}}|$, along with some evidence. In Section 6.3 we investigate Jacobians of intermediate curves $J_H(p)$ associated to subgroups of $(\mathbf{Z}/p\mathbf{Z})^\times$, and in Section 6.4 we consider optimal quotients A_f of $J_1(p)$ attached to newforms. In Section 6.4.1 we describe the lowest-level modular abelian variety that (assuming the Birch and Swinnerton-Dyer conjecture) should have infinite Mordell-Weil group but to which the general theorems of Kato, Kolyvagin, *et al.*, do not apply.

6.1 Computational methodology

We used the third author's modular symbols package for our computations; this package is part of [10] V2.10-6. See Section 6.5 for a description of how to use Magma to compute the tables. For the general theory of computing with modular symbols, see [14] and [63].

Remark 6.1.1. Many of the results of this section assume that a Magma program running on a computer executed correctly. Magma is complicated software that runs on physical hardware that is subject to errors from both programming mistakes and physical processes, such as cosmic radiation. We thus make the running *assumption* for the rest of this section that the computations below were performed correctly. To decrease the chance of hardware errors such as the famous Pentium bug (see [17]), we computed the tables in Section 6.6 on three separate computers with different CPU architectures (an

AMD Athlon 2000MP, a Sun Fire V480 which was donated to the third author by Sun Microsystems, and an Intel Pentium 4-M laptop).

Let A be a modular abelian variety over $\mathbf{Q}$, *i.e.*, a quotient of $J_1(N)$ for some N. We will make frequent reference to the following special case of the general conjectures of Birch and Swinnerton-Dyer:

Conjecture 6.1.2 (BSD Conjecture). *Let $Ш(A)$ be the Shafarevich-Tate group of A, let $c_p = |\Phi_{A,p}(\mathbf{F}_p)|$ be the Tamagawa number at p for A, and let Ω_A be the volume of $A(\mathbf{R})$ with respect to a generator of the invertible sheaf of top-degree relative differentials on the Néron model $A_{/\mathbf{Z}}$ of A over $\mathbf{Z}$. Let $A^\vee$ denote the abelian variety dual of A. The group $Ш(A)$ is finite and*

$$\frac{L(A,1)}{\Omega_A} = \frac{|Ш(A)| \cdot \prod_{p|N} c_p}{|A(\mathbf{Q})| \cdot |A^\vee(\mathbf{Q})|},$$

where we interpret the right side as 0 in case $A(\mathbf{Q})$ is infinite.

Remark 6.1.3. The hypothesis that A is modular implies that $L(A,s)$ has an analytic continuation to the whole complex plane and a functional equation of a standard type. In particular, $L(A,1)$ makes sense. Also, when $L(A,1) \neq 0$, [32, Cor. 14.3] implies that $Ш(A)$ is finite.

Let $\{f_1, \ldots, f_n\}$ be a set of newforms in $S_2(\Gamma_1(N))$ that is $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-stable. Let I be the Hecke-algebra annihilator of the subspace generated by $f_1, \ldots, f_n$. *For the rest of Section* 6.1*, we assume that $A = A_I = J_1(N)/IJ_1(N)$ for such an I.* Note that A is an *optimal quotient* in the sense that $IJ_1(N) = \ker(J_1(N) \to A)$ is an abelian subvariety of $J_1(N)$.

6.1.1 Bounding the torsion subgroup

To obtain a multiple of the order of the torsion subgroup $A(\mathbf{Q})_{\mathrm{tor}}$, we proceed as follows. For any prime $\ell \nmid N$, the algorithm of [3, §3.5] computes the characteristic polynomial $f \in \mathbf{Z}[X]$ of Frob_ℓ acting on any p-adic Tate module of A with $p \neq \ell$. To compute $|A(\mathbf{F}_\ell)|$, we observe that

$$|A(\mathbf{F}_\ell)| = \deg(\mathrm{Frob}_\ell - 1) = \det(\mathrm{Frob}_\ell - 1),$$

and this is the value of the characteristic polynomial of Frob_ℓ at 1. For any prime $\ell \nmid 2N$, the reduction map $A(\mathbf{Q})_{\mathrm{tor}} \to A(\mathbf{F}_\ell)$ is injective, so $|A(\mathbf{Q})_{\mathrm{tor}}|$ divides

$$T = \gcd\{|A(\mathbf{F}_\ell)| : \ \ell < 60 \text{ and } \ell \nmid 2N\}.$$

(If N is divisible by all primes up to 60, let $T = 0$. In all of the examples in this paper, N is prime and so $T \neq 0$.) The injectivity of reduction mod ℓ on the finite group $A(\mathbf{Q})_{\mathrm{tor}}$ for any prime $\ell \neq 2$ is well known and follows from the determination of the torsion in a formal group (see, *e.g.*, the appendix to [33] and [59, §IV.6–9]).

The cardinality $|A(\mathbf{F}_\ell)|$ does not change if A is replaced by a $\mathbf{Q}$-isogenous abelian variety B, so we do not expect in general that $|A(\mathbf{Q})_{\mathrm{tor}}| = T$. (For much more on relationships between $|A(\mathbf{Q})_{\mathrm{tor}}|$ and T, see [33, p. 499].) When we refer to an upper bound on torsion, T is the (multiplicative) upper bound that we have in mind.

The number 60 has no special significance; we had to make some choice to do computations, and in practice the sequence of partial gcd's rapidly stabilizes. For example, if $A = J_1(37)$, then the sequence of partial gcd's is:

$$15249085236272475, 802583433488025, 160516686697605, \ldots$$

where the term 160516686697605 repeats for all $\ell < 1000$.

6.1.2 The Manin index

Let p be a prime, let $\Omega_{A/\mathbf{Z}}$ denote the sheaf of relative 1-forms on the Néron model of A over $\mathbf{Z}$, and let I be the annihilator of A in the Hecke algebra $\mathbf{T} \subset \mathrm{End}(J_1(N))$. For a subring $R \subset \mathbf{C}$, let $S_2(\Gamma_1(N), R)$ be the R-module of cusp forms whose Fourier expansion at ∞ lies in $R[\![q]\!]$. The natural surjective Hecke-equivariant morphism $J_1(N) \to J_1(N)/IJ_1(N) = A$ induces (by pullback) a Hecke-equivariant injection $\Psi_A : \mathrm{H}^0(A_{/\mathbf{Z}}, \Omega_{A/\mathbf{Z}}) \hookrightarrow S_2(\Gamma_1(N), \mathbf{Q})$ whose image lies in $S_2(\Gamma_1(N), \mathbf{Q})[I]$. (Here we identify $S_2(\Gamma_1(N), \mathbf{Q})$ with $\mathrm{H}^0(X_1(N), \Omega_{X_1(N)/\mathbf{Q}}) = \mathrm{H}^0(J_1(N), \Omega_{J_1(N)/\mathbf{Q}})$ in the usual manner.)

Definition 6.1.4 (Manin index). The *Manin index of A* is

$$c = [S_2(\Gamma_1(N), \mathbf{Z})[I] : \Psi_A(\mathrm{H}^0(A_{/\mathbf{Z}}, \Omega_{A/\mathbf{Z}}))] \in \mathbf{Q}.$$

Remark 6.1.5. We name c after Manin, since he first studied c, but only in the context of elliptic curves. When $X_0(N) \to A$ is an optimal elliptic-curve quotient attached to a newform f, the usual Manin constant of A is the rational number c such that $\pi^*(\omega_A) = \pm c \cdot f \mathrm{d}q/q$, where ω_A is a basis for the differentials on the Néron model of A. The usual Manin constant equals the Manin index, since $S_2(\Gamma_1(N), \mathbf{Z})[I]$ is generated as a $\mathbf{Z}$-module by f.

A priori, the index in Definition 6.1.4 is only a generalized lattice index in the sense of [12, Ch. 1, §3], which we interpret as follows. In [12], for any Dedekind domain R, the *lattice index* is defined for any two finite free R-modules V and W of the same rank ρ that are embedded in a ρ-dimensional $\mathrm{Frac}(R)$-vector space U. The lattice index is the fractional R-ideal generated by the determinant of any automorphism of U that sends V isomorphically onto W. In Definition 6.1.4, we take $R = \mathbf{Z}$, $U = S_2(\Gamma_1(N), \mathbf{Q})[I]$, $V = S_2(\Gamma_1(N), \mathbf{Z})[I]$, and $W = \Psi_A(\mathrm{H}^0(A_{/\mathbf{Z}}, \Omega_{A/\mathbf{Z}}))$. Thus, c is the absolute value of the determinant of any linear transformation of $S_2(\Gamma_1(N), \mathbf{Q})[I]$ that sends $S_2(\Gamma_1(N), \mathbf{Z})[I]$ onto $\Psi_A(\mathrm{H}^0(A_{/\mathbf{Z}}, \Omega_{A/\mathbf{Z}}))$. In fact, it is not necessary to consider lattice indexes, as the following lemma shows (note we will use lattices indices later in the statement of Proposition 6.1.10).

Lemma 6.1.6. *The Manin index c of A is an integer.*

Proof. Let $X_\mu(N)$ be the coarse moduli scheme over $\mathbf{Z}$ that classifies isomorphism classes of pairs $(E/S, \alpha)$, with $\alpha : \mu_N \hookrightarrow E^{\mathrm{sm}}$ a closed subgroup in the smooth locus of a generalized elliptic curve E with irreducible geometric fibers E_s. This is a smooth $\mathbf{Z}$-curve that is not proper, and it is readily constructed by combining the work of Katz-Mazur and Deligne-Rapoport (see §9.3 and §12.3 of [16]). There is a canonical $\mathbf{Z}$-point $\infty \in X_\mu(N)(\mathbf{Z})$ defined by the standard 1-gon equipped with the canonical embedding of μ_N into the smooth locus $\mathbf{G}_m$, and the theory of the Tate curve provides a canonical isomorphism between $\mathrm{Spf}(\mathbf{Z}[\![q]\!])$ and the formal completion of $X_\mu(N)$ along ∞.

There is an isomorphism between the smooth proper curves $X_1(N)$ and $X_\mu(N)$ over $\mathbf{Z}[1/N]$ because the open modular curves $Y_1(N)$ and $Y_\mu(N)$ coarsely represent moduli problems that may be identified over the category of $\mathbf{Z}[1/N]$-schemes via the map

$$(E, P) \mapsto (E/\langle P\rangle, E[N]/\langle P\rangle),$$

where $E[N]/\langle P\rangle$ is identified with μ_N via the Weil pairing on $E[N]$. For our purposes, the key point (which follows readily from Tate's theory) is that under the moduli-theoretic identification of the analytification of the $\mathbf{C}$-fiber of $X_\mu(N)$ with the analytic modular curve $X_1(N)$ via the trivialization of $\mu_N(\mathbf{C})$ by means of $\zeta_N = e^{\pm 2\pi\sqrt{-1}/N}$, the formal parameter q at the $\mathbf{C}$-point ∞ computes the standard analytic q-expansion for weight-2 cusp forms on $\Gamma_1(N)$. The reason we consider $X_\mu(N)$ rather than $X_1(N)$ is simply because we want a smooth $\mathbf{Z}$-model in which the analytic cusp ∞ descends to a $\mathbf{Z}$-point.

Let $\phi : J_1(N) \to A$ be the Albanese quotient map over $\mathbf{Q}$, and pass to Néron models over $\mathbf{Z}$ (without changing the notation). Since $X_\mu(N)$ is $\mathbf{Z}$-smooth, there is a morphism $X_\mu(N) \to J_1(N)$ over $\mathbf{Z}$ that extends the usual morphism sending ∞ to 0. We have a map $\Psi : \mathrm{H}^0(A, \Omega) \to \mathbf{Z}[\![q]\!]\mathrm{d}q/q$ of $\mathbf{Z}$-modules defined by composition

$$\mathrm{H}^0(A, \Omega) \to \mathrm{H}^0(J_1(N), \Omega) \to \mathrm{H}^0(X_\mu(N), \Omega) \xrightarrow{q-\exp} \mathbf{Z}[\![q]\!]\frac{\mathrm{d}q}{q}.$$

The map Ψ is injective, since it is injective after base extension to $\mathbf{Q}$ and each group above is torsion free. The image of Ψ in $\mathbf{Z}[\![q]\!]\mathrm{d}q/q$ is a finite free $\mathbf{Z}$-module, contained in the image of $S = S_2(\Gamma_1(N), \mathbf{Z})$, the sub-$\mathbf{Z}$-module of $S_2(\Gamma_1(N), \mathbf{C})$ of those elements whose analytic q-expansion at ∞ has coefficients in $\mathbf{Z}$. Since Ψ respects the action of Hecke operators, the image of Ψ is contained in $S[I]$, so the lattice index c is an integer. $\square$

We make the following conjecture:

Conjecture 6.1.7. *If $A = A_f$ is a quotient of $J_1(N)$ attached to a single Galois-conjugacy class of newforms, then $c = 1$.*

Manin made this conjecture for one-dimensional optimal quotients of $J_0(N)$. Mazur bounded c in some cases in [46], Stevens considered c for one-dimensional quotients of $J_1(N)$ in [65], González and Lario considered c for $\mathbf{Q}$-curves in [26], Agashe and Stein considered c for quotients of $J_0(N)$ of dimension bigger than 1 in [4], and Edixhoven proved integrality results in [19, Prop. 2] and [22, §2].

Remark 6.1.8. We only make Conjecture 6.1.7 when A is attached to a *single* Galois-conjugacy class of newforms, since the more general conjecture is false. Adam Joyce [31] has recently used failure of multiplicity one for $J_0(p)$ to produce examples of optimal quotients A of $J_1(p)$, for $p = 431$, 503, and 2089, whose Manin indices are divisible by 2. Here, A is isogenous to a product of two elliptic curves, so A is not attached to a single Galois-orbit of newforms.

Remark 6.1.9. The question of whether or not c is an isogeny-invariant is not meaningful in the context of this paper because we only define the Manin index for optimal quotients.

6.1.3 Computing L-ratios

There is a formula for $L(A_f,1)/\Omega_{A_f}$ in [3, §4.2] when A_f is an optimal quotient of $J_0(N)$ attached to a single Galois conjugacy class of newforms. In this section we describe that formula; it applies to our quotient A of $J_1(N)$.

Recall our running hypothesis that $A = A_I$ is an optimal (new) quotient of $J_1(N)$ attached to a Galois conjugacy class of newforms $\{f_1,\ldots,f_n\}$. Let

$$\Psi : \mathrm{H}_1(X_1(N),\mathbf{Q}) \to \mathrm{Hom}(S_2(\Gamma_1(N))[I],\mathbf{C})$$

be the linear map that sends a rational homology class γ to the functional $\int_\gamma$ on the subspace $S_2(\Gamma_1(N))[I]$ in the space of holomorphic 1-forms on $X_1(N)$.

Let $\mathbf{T} \subset \mathrm{End}(\mathrm{H}_1(X_1(N),\mathbf{Q}))$ be the ring generated by all Hecke operators. Since the $\mathbf{T}$-module $H = \mathrm{Hom}(S_2(\Gamma_1(N))[I],\mathbf{C})$ has a natural $\mathbf{R}$-structure (and even a natural $\mathbf{Q}$-structure), it admits a natural $\mathbf{T}$-linear and $\mathbf{C}$-semilinear action by complex conjugation. If M is a $\mathbf{T}$-submodule of H, let M^+ denote the $\mathbf{T}$-submodule of M fixed by complex conjugation.

Let c be the Manin index of A as in Section 6.1.2, let c_∞ be the number of connected components of $A(\mathbf{R})$, let Ω_A be the volume of $A(\mathbf{R})$ as in Conjecture 6.1.2, and let $\{0,\infty\} \in \mathrm{H}_1(X_1(N),\mathbf{Q})$ be the rational homology class whose integration functional is integration from 0 to $i\infty$ along the i-axis (for the precise definition of $\{0,\infty\}$ and a proof that it lies in the rational homology see [38, Ch. IV §1–2]).

Proposition 6.1.10. *Let $A = A_I$ be an optimal quotient of $J_1(N)$ attached to a Galois-stable collection of newforms. With notation as above, we have*

$$c_\infty \cdot c \cdot \frac{L(A,1)}{\Omega_A} = [\Psi(\mathrm{H}_1(X_1(N),\mathbf{Z}))^+ : \Psi(\mathbf{T}\{0,\infty\})], \tag{6.1.1}$$

where the index is a lattice index as discussed in Section 6.1.7 (*in particular,* $L(A,1) = 0$ *if and only if* $\Psi(\mathbf{T}\{0,\infty\})$ *has smaller rank than* $\mathrm{H}_1(X_1(N),\mathbf{Z})^+$).

Proof. It is straightforward to adapt the argument of [3, §4.2] with $J_0(N)$ replaced by $J_1(N)$ (or even $J_H(N)$), but one must be careful when replacing A_f with A. The key observation is that if $f_1, \dots, f_n$ is the unique basis of normalized newforms corresponding to A, then $L(A,s) = L(f_1,s)\cdots L(f_n,s)$. □

Remark 6.1.11. This equality (6.1.1) need not hold if oldforms are involved, even in the $\Gamma_0(N)$ case. For example, if $A = J_0(22)$, then $L(A,s) = L(J_0(11),s)^2$, but two copies of the newform corresponding to $J_0(11)$ do not form a basis for $S_2(\Gamma_0(22))$.

We finish this section with some brief remarks on how to compute the rational number $c \cdot L(A,1)/\Omega_A$ using (6.1.1) and a computer. Using modular symbols, one can explicitly compute with $\mathrm{H}_1(X_1(N), \mathbf{Z})$. Though the above lattice index involves two lattices in a complex vector space, the index is unchanged if we replace Ψ with any linear map to a $\mathbf{Q}$-vector space such that the kernel is unchanged (see [3, §4.2]). Such a map may be computed via standard linear algebra by finding a basis for $\mathrm{Hom}(\mathrm{H}_1(X_1(N),\mathbf{Q}),\mathbf{Q})[I]$.

To compute c_∞, use the following well-known proposition; we include a proof for lack of an adequate published reference.

Proposition 6.1.12. *For an abelian variety A over $\mathbf{R}$,*

$$c_\infty = 2^{\dim_{\mathbf{F}_2} A[2](\mathbf{R}) - d},$$

where $d = \dim A$ and $c_\infty := |A(\mathbf{R})/A^0(\mathbf{R})|$.

Proof. Let $\Lambda = \mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})$, so the exponential uniformization of $A(\mathbf{C})$ provides a short exact sequence

$$0 \to \Lambda \to \mathrm{Lie}(A(\mathbf{C})) \to A(\mathbf{C}) \to 0.$$

There is an evident action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on all terms via the action on $A(\mathbf{C})$, and this short exact sequence is Galois-equivariant because A is defined over $\mathbf{R}$. Let Λ^+ be the subgroup of Galois-invariants in Λ, so we get an exact cohomology sequence

$$0 \to \Lambda^+ \to \mathrm{Lie}(A(\mathbf{R})) \to A(\mathbf{R}) \to \mathrm{H}^1(\mathrm{Gal}(\mathbf{C}/\mathbf{R}), \Lambda) \to 0$$

because higher group cohomology for a finite group vanishes on a $\mathbf{Q}$-vector space (such as the Lie algebra of $A(\mathbf{C})$). The map $\mathrm{Lie}(A(\mathbf{R})) \to A(\mathbf{R})$ is the exponential map for $A(\mathbf{R})$, and so its image is $A(\mathbf{R})^0$. Thus, Λ^+ has $\mathbf{Z}$-rank equal to $\dim A$ and

$$A(\mathbf{R})/A(\mathbf{R})^0 \simeq \mathrm{H}^1(\mathrm{Gal}(\mathbf{C}/\mathbf{R}), \Lambda).$$

To compute the size of this H^1, consider the short exact sequence

$$0 \to \Lambda \xrightarrow{2} \Lambda \to \Lambda/2\Lambda \to 0$$

of Galois-modules. Since $\Lambda/n\Lambda \simeq A[n](\mathbf{C})$ as Galois-modules for any $n \neq 0$, the long-exact cohomology sequence gives an isomorphism

$$A[2](\mathbf{R})/(\Lambda^+/2\Lambda^+) \simeq \mathrm{H}^1(\mathrm{Gal}(\mathbf{C}/\mathbf{R}), \Lambda).$$

□

Remark 6.1.13. Since the canonical isomorphism

$$A[n](\mathbf{C}) \simeq \mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})/n\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})$$

is $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant, we can identify $A[2](\mathbf{R})$ with the kernel of $\overline{\tau} - 1$ where $\overline{\tau}$ is the mod-2 reduction of the involution on $\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})$ induced by the action τ of complex conjugation on $A(\mathbf{C})$. In the special case when A is a quotient of some $J_1(N)$, and we choose a connected component of $\mathbf{C} - \mathbf{R}$ to uniformize $Y_1(N)$ in the usual manner, then via the $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant isomorphism $\mathrm{H}_1(J_1(N)(\mathbf{C}), \mathbf{Z}) \simeq \mathrm{H}_1(X_1(N)(\mathbf{C}), \mathbf{Z})$ we see that $\mathrm{H}_1(A(\mathbf{C}), \mathbf{Z})$ may be computed by modular symbols and that the action of τ on the modular symbol is $\{\alpha, \beta\} \mapsto \{-\alpha, -\beta\}$. This makes $A[2](\mathbf{R})$, and hence c_∞, readily computable via modular symbols.

6.2 ARITHMETIC OF $J_1(p)$

6.2.1 THE TABLES

For $p \leq 71$, the first part of Table 1 (on page 399) lists the dimension of $J_1(p)$ and the rational number $L = c \cdot L(J_1(p), 1)/\Omega_{J_1(p)}$. Table 1 also gives an upper bound T (in the sense of divisibility) on $|J_1(p)(\mathbf{Q})_{\mathrm{tor}}|$ for $p \leq 71$, as discussed in §6.1.1.

When $L \neq 0$, Conjecture 6.1.2 and the assumption that $c = 1$ imply that the numerator of L divides $c_p \cdot |\text{Ш}(A)|$, that in turn divides T^2L. For every $p \neq 29$ with $p \leq 71$, we found that $T^2L = 1$. For $p = 29$, we have $T^2L = 2^{12}$; it would be interesting if the isogeny-invariant T overestimates the order of $J_1(29)(\mathbf{Q})_{\mathrm{tor}}$ or if $\text{Ш}(J_1(29))$ is nontrivial.

6.2.2 DETERMINATION OF POSITIVE RANK

PROPOSITION 6.2.1. *The primes p such that $J_1(p)$ has positive rank are the same as the primes for which $J_0(p)$ has positive rank:*

$$p = 37, 43, 53, 61, 67, \text{ and all } p \geq 73.$$

Proof. Proposition 2.8 of [45, §III.2.2, p. 147] says: "Suppose $g^+ > 0$ (which is the case for all $N > 73$, as well as $N = 37, 43, 53, 61, 67$). Then the Mordell-Weil group of J_+ is a torsion-free group of infinite order (*i.e.* of positive rank)." Here, N is a prime, g^+ is the genus of the Atkin-Lehner quotient $X_0(N)^+$ of $X_0(N)$, and J_+ is isogenous to the Jacobian of $X_0(N)^+$. This is essentially correct, except for the minor oversight that $g^+ > 0$ also when $N = 73$ (this is stated correctly on page 34 of [45]).

By Mazur's proposition $J_0(p)$ has positive algebraic rank for all $p \geq 73$ and for $p = 37, 43, 53, 61, 67$. The sign in the functional equation for $L(J_+, s)$ is -1, so

$$L(J, 1) = L(J_+, 1)L(J_-, 1) = 0 \cdot L(J_-, 1) = 0$$

for all p such that $g^+ > 0$. Using (6.1.1) we see that $L(J, 1) \neq 0$ for all p such that $g^+ = 0$, which by Kato (see [32, Cor. 14.3]) or Kolyvagin–Logachev (see [36]) implies that J has rank 0 whenever $g^+ = 0$. Thus $L(J_0(p), 1) = 0$ if and only if $J_0(p)$ has positive rank.

Work of Kato (see [32, Cor. 14.3]) implies that if $J_1(p)$ has analytic rank 0, then $J_1(p)$ has algebraic rank 0. It thus suffices to check that $L(J_1(p), 1) \neq 0$ for the primes p such that $J_0(p)$ has rank 0. We verify this by computing $c \cdot L(J_1(p), 1)/\Omega_{J_1(p)}$ using (6.1.1), as illustrated in Table 1.

□

If we instead consider composite level, it is not true that $J_0(N)$ has positive analytic rank if and only if $J_1(N)$ has positive analytic rank. For example, using (6.1.1) we find that $J_0(63)$ has analytic rank 0, but $J_1(63)$ has positive analytic rank. Closer inspection using Magma (see the program below) shows that there is a two-dimensional new quotient A_f with positive analytic rank, where $f = q + (\omega - 1)q^2 + (-\omega - 2)q^3 + \cdots$, and $\omega^3 = 1$. It would be interesting to prove that that the algebraic rank of A_f is positive.

```
> M := ModularSymbols(63,2);
> S := CuspidalSubspace(M);
> LRatio(S,1);        // So J_0(63) has rank 0
1/384

> G<a,b> := DirichletGroup(63,CyclotomicField(6));
> e := a^5*b;
> M := ModularSymbols([e],2,+1);
> S := CuspidalSubspace(M);
> LRatio(S,1);     // This step takes some time.
0
> D := NewformDecomposition(S);
> LRatio(D[1],1);
0
> qEigenform(D[1],5);
q + (-2*zeta_6 + 1)*q^2 + (-2*zeta_6 + 1)*q^3 - q^4 + O(q^5)
```

6.2.3 Conjectural order of $J_1(\mathbf{Q})_{\mathrm{tor}}$

For any Dirichlet character ε modulo N, define Bernoulli numbers $B_{2,\varepsilon}$ by

$$\sum_{a=1}^{N} \frac{\varepsilon(a)te^{at}}{e^{Nt}-1} = \sum_{k=0}^{\infty} \frac{B_{k,\varepsilon}}{k!}t^k.$$

We make the following conjecture.

CONJECTURE 6.2.2. *Let $p \geq 5$ be prime. The rational torsion subgroup $J_1(p)(\mathbf{Q})_{\mathrm{tor}}$ is generated by the differences of $\mathbf{Q}$-rational cusps on $X_1(p)$. Equivalently (see below), for any prime $p \geq 5$,*

$$|J_1(p)(\mathbf{Q})_{\mathrm{tor}}| = \frac{p}{2^{p-3}} \cdot \prod_{\varepsilon \neq 1} B_{2,\varepsilon} \tag{6.2.1}$$

where the product is over the nontrivial even Dirichlet characters ε of conductor dividing p.

Due to how we defined $X_1(p)$, its $\mathbf{Q}$-rational cusps are exactly its cusps lying over the cusp $\infty \in X_0(p)(\mathbf{Q})$ (corresponding to the standard 1-gon equipped with the subgroup μ_p in its smooth locus $\mathbf{G}_m$) via the second standard degeneracy map

$$(E, P) \mapsto (E/\langle P\rangle, E[p]/\langle P\rangle).$$

In [49] Ogg showed that $|J_1(13)(\mathbf{Q})| = 19$, verifying Conjecture 6.2.2 for $p = 13$. The results of [37] are also relevant to Conjecture 6.2.2, and suggest that the rational torsion of $J_1(p)$ is cuspidal. Let $C(p)$ be the conjectural order of $J_1(p)(\mathbf{Q})_{\mathrm{tor}}$ on the right side of (6.2.1). In [37, p. 153], Kubert and Lang prove that $C(p)$ is equal to the order of the group generated by the differences of $\mathbf{Q}$-rational cusps on $X_1(p)$ (in their language, these are viewed as the cusps that lie over $0 \in X_0(p)(\mathbf{Q})$ via the first standard degeneracy map

$$(E, P) \mapsto (E, \langle P\rangle)),$$

and so $C(p)$ is *a priori* an integer that moreover divides $|J_1(p)(\mathbf{Q})_{\mathrm{tor}}|$.

Table 1 provides evidence for Conjecture 6.2.2. Let $T(p)$ be the upper bound on $J_1(p)(\mathbf{Q})_{\mathrm{tor}}$ (see Table 1). For all $p \leq 157$, we have $C(p) = T(p)$ except for $p = 29$, 97, 101, 109, and 113, where $T(p)/C(p)$ is 2^6, 17, 2^4, 3^7, and $2^{12} \cdot 3^2$, respectively. Thus Conjecture 6.2.2 is true for $p \leq 157$, except possibly in these five cases, where the deviation is consistent with the possibility that $T(p)$ is a nontrivial multiple of the true order of the torsion subgroup (recall that $T(p)$ is an isogeny-invariant, and so it is not surprising that it may be too large).

6.3 Arithmetic of $J_H(p)$

For each divisor d of $p-1$, let $H = H_d$ denote the unique subgroup of $(\mathbf{Z}/p\mathbf{Z})^\times$ of order $(p-1)/d$. The group of characters whose kernel contains H_d is exactly the group of characters of order dividing d. Since the linear fractional transformation associated to $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ acts trivially on the upper half plane, we lose nothing (for the computations that we will do in this section) if we assume that $-1 \in H$, and so $|H|$ is even.

For any subgroup H of $(\mathbf{Z}/p\mathbf{Z})^\times$ as above, let J_H be the Jacobian of $X_H(p)$, as in Section 1. For each $p \leq 71$, Table 2 lists the dimension of $J_H = J_H(p)$, the rational number $L = c \cdot L(J_H, 1)/\Omega_{J_H}$, an upper bound T on $|J_H(\mathbf{Q})_{\mathrm{tor}}|$, the conjectural multiple T^2L of $|\text{Ш}(J_H)| \cdot c_p$, and $c_p = |\Phi(J_H)|$. We compute $|\Phi(J_H)(\mathbf{F}_p)| = |\Phi(J_H)(\overline{\mathbf{F}}_p)|$ using Theorem 1.1.3. Note that Table 2 omits the data for $d = (p-1)/2$, since $J_H = J_1(p)$ for such d, so the corresponding data is therefore already contained in Table 1.

When $L \neq 0$, we have $T^2L = |\Phi(J_H)|$ in all but one case. The exceptional case is $p = 29$ and $d = 7$, where $T^2L = 2^6$, but $|\Phi(J_H)| = 1$; probably T overestimates the torsion in this case. In the following proposition we use this observation to deduce that $|\text{Ш}(J_H)| = c = 1$ in some cases.

Proposition 6.3.1. *Suppose that $p \leq 71$ is a prime and $d \mid (p-1)$ with $(p-1)/d$ even. Let J_H be the Jacobian of $X_H(p)$, where H is the subgroup of $(\mathbf{Z}/p\mathbf{Z})^\times$ of order $(p-1)/d$. Assume that Conjecture* 6.1.2 *is true, and if $p = 29$ then assume that $d \neq 7, 14$. If $L(J_H, 1) \neq 0$, then $|\text{Ш}(J_H)| = 1$ and $c = 1$.*

It is not interesting to remove the condition $p \leq 71$ in the statement of the proposition, since when $p > 71$ the quantity $L(J_H, 1)$ automatically vanishes (see Proposition 6.2.1). It is probably not always the case that $|\text{Ш}(J_H)| = 1$; for example, Conjecture 6.1.2 and the main result of [1] imply that 7^2 divides $|\text{Ш}(J_0(1091))|$.

Proof. We deduce the proposition from Tables 1–3 as follows. Using Conjecture 6.1.2 we have

$$c \cdot |\text{Ш}(J_H)| = c \cdot \frac{L(J_H, 1)}{\Omega_{J_H} \cdot |\Phi(J_H)|} \cdot |J_H(\mathbf{Q})_{\mathrm{tor}}|^2. \tag{6.3.1}$$

Let T denote the torsion bound on $J_H(\mathbf{Q})_{\mathrm{tor}}$ as in Section 6.1.1 and let $L = c \cdot L(J_H, 1)/\Omega_{J_H}$, so the right side of (6.3.1) divides $T^2L/|\Phi(J_H)|$. An inspection of the tables shows that $T^2L/|\Phi(J_H)| = 1$ for J_H satisfying the hypothesis of the proposition (in the excluded cases $p = 29$ and $d = 7, 14$, the quotient equals 2^6 and 2^{12}, respectively). Since $c \in \mathbf{Z}$, we conclude that $c = |\text{Ш}(J_H)| = 1$.

□

Remark 6.3.2. Theorem 1.1.3 is an essential ingredient in the proof of Proposition 6.3.1 because we used Theorem 1.1.3 to compute the Tamagawa factor c_p.

6.4 Arithmetic of newform quotients

Tables 4–5 at the end of this paper contain arithmetic information about each newform abelian variety quotient A_f of $J_1(p)$ with $p \leq 71$.

The first column gives a label determining a Galois-conjugacy class of newforms $\{f, \ldots\}$, where **A** corresponds to the first class, **B** to the second, *etc.*, and the classes are ordered first by dimension and then in lexicographical order by the sequence of nonegative integers $|\operatorname{tr}(a_2(f))|, |\operatorname{tr}(a_3(f))|, |\operatorname{tr}(a_5(f))|, \ldots$. (WARNING: This ordering does not agree with the one used by Cremona in [14]; for example, our **37A** is Cremona's **37B**.) The next two columns list the dimension of A_f and the order of the Nebentypus character of f, respectively. The fourth column lists the rational number $L = L(A_f, 1)/\Omega_{A_f}$, and the fifth lists the product $T^2 L$, where T is an upper bound (as in Section 6.1.1) on the order of $A_f(\mathbf{Q})_{\text{tor}}$. The sixth column, labeled "modular kernel", lists invariants of the group of $\overline{\mathbf{Q}}$-points of the kernel of the polarization $A_f^\vee \hookrightarrow J_1(p) \to A_f$; this kernel is computed by using an algorithm based on Proposition 6.4.1 below. The elementary divisors of the kernel are denoted with notation such as $[2^2 14^2]$ to denote

$$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/14\mathbf{Z} \times \mathbf{Z}/14\mathbf{Z}.$$

Proposition 6.4.1. *Suppose $A = A_I$ is an optimal quotient of $J = J_1(N)$ attached to the annihilator I of a Galois-stable collection of newforms. The group of $\overline{\mathbf{Q}}$-points of the kernel of the natural map $A^\vee \hookrightarrow J \to A$ is isomorphic to the cokernel of the natural map*

$$\operatorname{Hom}(\mathrm{H}_1(X_1(N), \mathbf{Z}), \mathbf{Z})[I] \to \operatorname{Hom}(\mathrm{H}_1(X_1(N), \mathbf{Z})[I], \mathbf{Z}).$$

Proof. The proof is the same as [35, Prop. 1]. □

It is possible to compute the modular kernel by using the formula in this proposition, together with modular symbols and standard algorithms for computing with finitely generated abelian groups.

We do not give T in Tables 4–5, since in all but six cases $T^2 L \neq 0$, hence $T^2 L$ and L determine T. The remaining six cases are **37B**, **43A**, **53A**, **61A**, **61B**, and **67C**, and in all these cases $T = 1$.

Remark 6.4.2. If $A = A_f$ is an optimal quotient of $J_1(p)$ attached to a newform, then the tables do not include the toric, additive, and abelian ranks of the closed fiber of the Néron model of A over $\mathbf{F}_p$, since they are easy to determine from other data about A as follows. If $\varepsilon(f) = 1$, then the toric rank is $\dim(A)$, since A is isogenous to an abelian subvariety of $J_0(p)$ and so A has purely toric reduction over $\mathbf{Z}_p$. Now suppose that $\varepsilon(f)$ is nontrivial, so A is isogenous to an abelian subvariety of the abelian variety $J_1(p)/J_0(p)$ that has potentially good reduction at p. Hence the toric rank of A is zero, and inertia $I_p \subset G_p = \operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts with finite image on the $\mathbf{Q}_\ell$-adic Tate module V_ℓ of A for any $\ell \neq p$. Hence V_ℓ splits as a nontrivial direct sum of simple representations of I_p. Let V' be a factor of V_ℓ corresponding to a simple summand K

of $\mathbf{T} \otimes \mathbf{Q}_\ell$, where $\mathbf{T}$ is the Hecke algebra. Since the Artin conductor of the 2-dimensional K-representation V'_ℓ is p, the $\overline{\mathbf{Q}}_\ell[I_p]$-module $\overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}_\ell} V'$ is the direct sum of the trivial representation and the character $\varepsilon(f) : (\mathbf{Z}/p\mathbf{Z})^\times \to \overline{\mathbf{Q}}_\ell^\times$ viewed as a character of G_p via the identification $\mathrm{Gal}(\mathbf{Q}_p(\zeta_p)/\mathbf{Q}_p) = (\mathbf{Z}/p\mathbf{Z})^\times$. This implies that the abelian rank as well as the additive rank are both equal to half of the dimension of A.

6.4.1 The Simplest example not covered by general theory

The prime $p = 61$ is the only prime $p \leq 71$ such that the maximal quotient of $J_1(p)$ with positive analytic rank is not a quotient of $J_0(p)$. Let ε be a Dirichlet character of conductor 61 and order 6. Consider the abelian variety A_f attached to the newform

$$f = q + (e^{2\pi i/3} - 1)q^2 - 2q^3 + \cdots$$

that lies in the 6-dimensional $\mathbf{C}$-vector space $S_2(\Gamma_1(61), \varepsilon)$. Using Proposition 6.1.10, we see that $L(f, 1) = 0$.

It would be interesting to show that A_f has positive algebraic rank, since A_f is not covered by the general theorems of Kolyvagin, Logachev, and Kato concerning Conjecture 6.1.2. This example is the simplest example in the following sense: every elliptic curve over $\mathbf{Q}$ is a quotient of some $J_0(N)$, and an inspection of Tables 4–5 for any integer $N < 61$ shows that the maximal quotient of $J_1(N)$ with positive analytic rank is also a quotient of $J_0(N)$.

The following observation puts this question in the context of $\mathbf{Q}$-curves, and may be of some use in a direct computation to show that A_f has positive algebraic rank. Since $\overline{f} = f \otimes \varepsilon^{-1}$, Shimura's theory (see [62, Prop. 8]) supplies an isogeny $\varphi : A_f \to A_f$ defined over the degree-6 abelian extension of $\mathbf{Q}$ cut out by $\ker(\varepsilon)$. Using φ, one sees that A_f is isogenous to a product of two elliptic curves. According to Enrique Gonzalez-Jimenez (personal communication) and Jordi Quer, if $t^6 + t^5 - 25t^4 + 8t^3 + 123t^2 - 126t + 27 = 0$, so t generates the degree 6 subfield of $\mathbf{Q}(\zeta_{61})$ corresponding to ε, then one of the elliptic-curve factors of A_f has equation $y^2 = x^3 + c_4 x + c_6$, where

$$\begin{aligned} c_4 &= \frac{1}{3}(-321 + 738t - 305t^2 - 196t^3 + 47t^4 + 13t^5), \\ c_6 &= \frac{1}{3}(-4647 + 6300t + 996t^2 - 1783t^3 - 432t^4 - 14t^5). \end{aligned}$$

6.4.2 Can Optimal Quotients Have Nontrivial Component Group?

Let p be a prime. Component groups of optimal quotients of $J_0(p)$ are well-understood in the sense of the following theorem of Emerton [23]:

Theorem 6.4.3 (Emerton). *If $A_1, \ldots, A_n$ are the distinct optimal quotients of $J_0(p)$ attached the Galois-orbits of newforms, then the product of the orders of the component groups of the A_i's equals the order of the component* *group of* $J_0(p)$*,* i.e.*, the numerator of* $(p-1)/12$*. Moreover, the natural maps* $\Phi(J_0(p)) \to \Phi(A_i)$ *are surjective.*

Shuzo Takehashi asked a related question about $J_1(p)$:

QUESTION 6.4.4 (TAKEHASHI). Suppose $A = A_f$ is an optimal quotient of $J_1(p)$ attached to a newform. What can be said about the component group of A? In particular, is the component group of A necessarily trivial?

Since $J_1(p)$ has trivial component group (see Theorem 1.1.1), the triviality of the component group of A is equivalent to the surjectivity of the natural map from $\Phi(J_1(p))$ to $\Phi(A_f)$.

The data in Tables 4–5 sheds little light on Question 6.4.4. The following are the A_f's that have nonzero $L = c \cdot L(A_f, 1)/\Omega$ with numerator divisible by an odd prime: **37D**, **37F**, **43C**, **43F**, **53D**, **61E**, **61F**, **61G**, **61J**, **67D**, **67E**, and **67G**. For each of these, Conjecture 6.1.2 implies that $c \cdot Ш(A_f) \cdot c_p$ is divisible by an odd prime. However, it seems difficult to deduce which factors in the product are not equal to 1. We remark that for each A_f listed above such that the numerator of L is exactly divisible by p, there is a rank-1 elliptic curve E over $\mathbf{Q}$ such that $E[p] \subset A$, so methods as in [2] may shed light on this problem.

6.5 USING MAGMA TO COMPUTE THE TABLES

In this section, we describe how to use MAGMA V2.10-6 (or later) to compute the entries in Tables 1–5 at the end of this paper.

6.5.1 COMPUTING TABLE 1: ARITHMETIC OF $J_1(p)$

Let p be a prime. The following MAGMA code illustrates how to compute the two rows in Table 1 corresponding to p $(= 19)$. Note that the space of cuspidal modular symbols has dimension $2 \dim J_1(p)$.

```
> p := 19;
> M := ModularSymbols(Gamma1(p));
> S := CuspidalSubspace(M);
> S;
Modular symbols space of level 19, weight 2, and dimension
14 over Rational Field (multi-character)
> LRatio(S,1);
1/19210689
> Factorization(19210689);
[ <3, 4>, <487, 2> ]
> TorsionBound(S,60);
4383
```

Remark 6.5.1. It takes less time and memory to compute $c \cdot L(J_1(p), 1)/\Omega$ in $\mathbf{Q}^\times/2^{\mathbf{Z}}$, and this is done by replacing `M:=ModularSymbols(Gamma1(p))` with

`M:=ModularSymbols(Gamma1(p),2,+1)`. A similar remark applies to all computations of L-ratios in the sections below.

6.5.2 Computing Tables 2–3: Arithmetic of $J_H(p)$

Let p be a prime, d a divisor of $p-1$ such that $(p-1)/d$ is even, and H the subgroup of $(\mathbf{Z}/N\mathbf{Z})^\times$ of order $(p-1)/d$. We use Theorem 1.1.3 and commands similar to the ones in Section 6.5.1 to fill in the entries in Tables 2–3. The following code illustrates computation of the second row of Table 2 for $p = 19$.

```
> p := 19;
> [d : d in Divisors(p-1) | IsEven((p-1) div d)];
[ 1, 3, 9 ]
> d := 3;
> M := ModularSymbolsH(p,(p-1) div d, 2, 0);
> S := CuspidalSubspace(M);
> S;
Modular symbols space of level 19, weight 2, and dimension 2
over Rational Field (multi-character)
> L := LRatio(S,1); L;
1/9
> T := TorsionBound(S,60); T;
3
> T^2*L;
1
> Phi := d / GCD(d,6);  Phi;
1
```

It takes about ten minutes to compute all entries in Table 2–3 using an Athlon 2000MP-based computer.

6.5.3 Computing Tables 4–5

Let p be a prime number. To compute the modular symbols factors corresponding to the newform optimal quotients A_f of $J_1(p)$, we use the `NewformDecomposition` command. To compute the modular kernel, we use the command `ModularKernel`. The following code illustrates computation of the second row of Table 4 corresponding to $p = 19$.

```
> p := 19;
> M := ModularSymbols(Gamma1(19));
> S := CuspidalSubspace(M);
> D := NewformDecomposition(S);
> D;
[
    Modular symbols space for Gamma_0(19) of weight 2 and
    dimension 2 over Rational Field,
```

```
    Modular symbols space of level 19, weight 2, and
    dimension 12 over Rational Field (multi-character)
]
> A := D[2];
> Dimension(A) div 2;
6
> Order(DirichletCharacter(A));
9
> L := LRatio(A,1); L;
1/2134521
> T := TorsionBound(A,60);
> T^2*L;
1
> Invariants(ModularKernel(A));
[ 3, 3 ]
```

It takes about 2.5 hours to compute all entries in Tables 4–5, except that the entries corresponding to $p = 71$, using an Athlon 2000MP-based computer. The $p = 71$ entry takes about 3 hours.

6.6 Arithmetic tables

The notation in Tables 1–5 below is explained in Section 6.

Table 1: Arithmetic of $J_1(p)$

$J_1(p)$	dim	$c \cdot L(J_1(p),1)/\Omega$
11	1	$1/5^2$
13	2	$1/19^2$
17	5	$1/2^6\cdot 73^2$
19	7	$1/3^4\cdot 487^2$
23	12	$1/11^2\cdot 37181^2$
29	22	$1/2^{12}\cdot 3^2\cdot 7^2\cdot 43^2\cdot 17837^2$
31	26	$1/2^4\cdot 5^4\cdot 7^2\cdot 11^2\cdot 2302381^2$
37	40	0
41	51	$1/2^8\cdot 5^2\cdot 13^2\cdot 31^4\cdot 431^2\cdot 250183721^2$
43	57	0
47	70	$1/23^2\cdot 139^2\cdot 82397087^2\cdot 12451196833^2$
53	92	0
59	117	$1/29^2\cdot 59^2\cdot 9988553613691393812358794271^2$
61	126	0
67	155	0
71	176	$1/5^2\cdot 7^2\cdot 31^2\cdot 113^2\cdot 211^2\cdot 281^2\cdot 701^4\cdot 12713^2\cdot$ $130708499192256557290 61^2$

$J_1(p)$	Torsion Bound
11	5
13	19
17	$2^3\cdot 73$
19	$3^2\cdot 487$
23	$11\cdot 37181$
29	$2^{12}\cdot 3\cdot 7\cdot 43\cdot 17837$
31	$2^2\cdot 5^2\cdot 7\cdot 11\cdot 2302381$
37	$3^2\cdot 5\cdot 7\cdot 19\cdot 37\cdot 73\cdot 577\cdot 17209$
41	$2^4\cdot 5\cdot 13\cdot 31^2\cdot 431\cdot 250183721$
43	$2^2\cdot 7\cdot 19\cdot 29\cdot 463\cdot 1051\cdot 416532733$
47	$23\cdot 139\cdot 82397087\cdot 12451196833$
53	$7\cdot 13\cdot 85411\cdot 96331\cdot 379549\cdot 641949283$
59	$29\cdot 59\cdot 9988553613691393812358794271$
61	$5\cdot 7^2\cdot 11^2\cdot 19\cdot 31\cdot 2081\cdot 2801\cdot 40231\cdot 411241\cdot 514216621$
67	$11\cdot 67\cdot 193\cdot 661^2\cdot 2861\cdot 8009\cdot 11287\cdot 9383200455691459$
71	$5\cdot 7\cdot 31\cdot 113\cdot 211\cdot 281\cdot 701^2\cdot 12713\cdot 1307084991922565 5729061$

Table 2: Arithmetic of $J_H(p)$

p	d	dim	$L = c \cdot L(J_H,1)/\Omega$	T = Torsion Bound	T^2L	$\lvert\Phi(J_H)\rvert$
11	1	1	$1/5$	5	5	5
13	1	0	1	1	1	1
	2	0	1	1	1	1
	3	0	1	1	1	1
17	1	1	$1/2^2$	2^2	2^2	2^2
	2	1	$1/2^3$	2^2	2	2
	4	1	$1/2^4$	2^2	1	1
19	1	1	$1/3$	3	3	3
	3	1	$1/3^2$	3	1	1
23	1	2	$1/11$	11	11	11
29	1	2	$1/7$	7	7	7
	2	4	$1/3^2 \cdot 7$	$3 \cdot 7$	7	7
	7	8	$1/2^6 \cdot 7^2 \cdot 43^2$	$2^6 \cdot 7 \cdot 43$	2^6	1
31	1	2	$1/5$	5	5	5
	3	6	$1/2^4 \cdot 5 \cdot 7^2$	$2^2 \cdot 5 \cdot 7$	5	5
	5	6	$1/5^4 \cdot 11^2$	$5^2 \cdot 11$	1	1
37	1	2	0	3	0	3
	2	4	0	$3 \cdot 5$	0	3
	3	4	0	$3 \cdot 7$	0	1
	6	10	0	$3 \cdot 5 \cdot 7 \cdot 37$	0	1
	9	16	0	$3^2 \cdot 7 \cdot 19 \cdot 577$	0	1
41	1	3	$1/2 \cdot 5$	$2 \cdot 5$	$2 \cdot 5$	$2 \cdot 5$
	2	5	$1/2^6 \cdot 5$	$2^3 \cdot 5$	5	5
	4	11	$1/2^8 \cdot 5 \cdot 13^2$	$2^4 \cdot 5 \cdot 13$	5	5
	5	11	$1/2 \cdot 5^2 \cdot 431^2$	$2 \cdot 5 \cdot 431$	2	2
	10	21	$1/2^6 \cdot 5^2 \cdot 31^4 \cdot 431^2$	$2^3 \cdot 5 \cdot 31^2 \cdot 431$	1	1

Table 3: Arithmetic of $J_H(p)$ (continued)

p	d	dim	$L = c \cdot L(J_H, 1)/\Omega$	T = Torsion Bound	T^2L	$\lvert\Phi(J_H)\rvert$
43	1	3	0	7	0	7
	3	9	0	$2^2 \cdot 7 \cdot 19$	0	7
	7	15	0	$7 \cdot 29 \cdot 463$	0	1
47	1	4	$1/23$	23	23	23
53	1	4	0	13	0	13
	2	8	0	$7 \cdot 13$	0	13
	13	40	0	$13 \cdot 96331 \cdot 379549$	0	1
59	1	5	$1/29$	29	29	29
61	1	4	0	5	0	5
	2	8	0	$5 \cdot 11$	0	5
	3	12	0	$5 \cdot 7 \cdot 19$	0	5
	5	16	0	$5 \cdot 2801$	0	1
	6	26	0	$5 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$	0	5
	10	36	0	$5 \cdot 11^2 \cdot 2081 \cdot 2801$	0	1
	15	56	0	$5 \cdot 7 \cdot 19 \cdot 2801 \cdot$ 514216621	0	1
67	1	5	0	11	0	11
	3	15	0	$11 \cdot 193$	0	11
	11	45	0	$11 \cdot 661 \cdot 2861 \cdot 8009$	0	1
71	1	6	$1/5 \cdot 7$	$5 \cdot 7$	$5 \cdot 7$	$5 \cdot 7$
	5	26	$1/5^2 \cdot 7 \cdot 31^2 \cdot 211^2$	$5 \cdot 7 \cdot 31 \cdot 211$	7	7
	7	36	$1/5 \cdot 7^2 \cdot 113^2 \cdot 12713^2$	$5 \cdot 7 \cdot 113 \cdot 12713$	5	5

Table 4: Arithmetic of Optimal Quotients A_f of $J_1(p)$

A_f	dim	ord(ε)	$L = c \cdot L(A_f, 1)/\Omega$	T^2L	modular kernel
11A	1	1	$1/5^2$	1	[]
13A	2	6	$1/19^2$	1	[]
17A	1	1	$1/2^4$	1	$[2^2]$
17B	4	8	$1/2^2 \cdot 73^2$	1	$[2^2]$
19A	1	1	$1/3^2$	1	$[3^2]$
19B	6	9	$1/3^2 \cdot 487^2$	1	$[3^2]$
23A	2	1	$1/11^2$	1	$[11^2]$
23B	10	11	$1/37181^2$	1	$[11^2]$
29A	2	2	$1/3^2$	1	$[14^4]$
29B	2	1	$1/7^2$	1	$[2^2 14^2]$
29C	6	7	$1/2^6 \cdot 43^2$	2^6	$[2^{10} 14^2]$
29D	12	14	$1/2^6 \cdot 17837^2$	2^6	$[2^8 14^4]$
31A	2	1	$1/5^2$	1	$[3^2 15^2]$
31B	4	5	$1/5^2 \cdot 11^2$	1	$[3^6 15^2]$
31C	4	3	$1/2^4 \cdot 7^2$	1	$[5^4 15^4]$
31D	16	15	$1/2302381^2$	1	$[15^8]$
37A	1	1	$1/3^2$	1	$[12^2]$
37B	1	1	0	0	$[36^2]$
37C	2	2	$2/5^2$	2	$[18^4]$
37D	2	3	$3/7^2$	3	$[6^2 18^2]$
37E	4	6	$1/37^2$	1	$[3^4 18^4]$
37F	6	9	$3/577^2$	3	$[2^6 6^2 102^4]$
37G	6	9	$1/3^2 \cdot 19^2$	1	$[2^8 34^2 102^2]$
37H	18	18	$1/73^2 \cdot 17209^2$	1	$[2^{12} 6^{12}]$
41A	2	2	$1/2^4$	1	$[20^4]$
41B	3	1	$1/2^2 \cdot 5^2$	1	$[2^2 20^4]$
41C	6	4	$1/2^2 \cdot 13^2$	1	$[5^2 10^{10}]$
41D	8	10	$1/31^4$	1	$[4^{12} 20^4]$
41E	8	5	$1/431^2$	1	$[4^{12} 20^4]$
41F	24	20	$1/250183721^2$	1	$[2^{20} 10^{12}]$
43A	1	1	0	0	$[42^2]$
43B	2	1	$2/7^2$	2	$[3^2 42^2]$
43C	2	3	$3/2^4$	3	$[35^2 105^2]$
43D	4	3	$1/19^2$	1	$[7^4 105^4]$
43E	6	7	$1/29^2$	1	$[3^8 39^2 273^2]$
43F	6	7	$7/463^2$	7	$[3^8 39^2 273^2]$
43G	36	21	$1/1051^2 \cdot 416532733^2$	1	$[3^{12} 21^{12}]$

Table 5: Arithmetic of Optimal Quotients A_f of $J_1(p)$ (continued)

A_f	dim	ord(ε)	$L = c \cdot L(A_f, 1)/\Omega$	T^2L	modular kernel
47A	4	1	$1/23^2$	1	$[23^6]$
47B	66	23	$1/139^2 \cdot 82397087^2 \cdot$ 12451196833^2	1	$[23^6]$
53A	1	1	0	0	$[52^2]$
53B	3	1	$2/13^2$	2	$[2^2 26^2 52^2]$
53C	4	2	$2/7^2$	2	$[26^8]$
53D	36	13	$13/96331^2 \cdot 379549^2$	13	$[2^{66} 26^6]$
53E	48	26	$1/85411^2 \cdot 641949283^2$	1	$[2^{64} 26^8]$
59A	5	1	$1/29^2$	1	$[29^8]$
59B	112	29	$1/59^2 \cdot$ $99885536136913938123587942 71^2$	1	$[29^8]$
61A	1	1	0	0	$[60^2]$
61B	2	6	0	0	$[55^4]$
61C	3	1	$2/5^2$	2	$[6^2 30^2 60^2]$
61D	4	2	$2/11^2$	2	$[30^8]$
61E	8	3	$3/7^2 \cdot 19^2$	3	$[10^8 30^8]$
61F	8	6	$11^2/7^2 \cdot 31^2$	11^2	$[10^8 30^4 330^4]$
61G	12	5	$5/2801^2$	5	$[6^{18} 30^6]$
61H	16	10	$1/11^2 \cdot 2081^2$	1	$[3^8 6^{16} 30^8]$
61I	32	15	$1/514216621^2$	1	$[2^{40} 6^8 30^{16}]$
61J	40	30	$5^2/40231^2 \cdot 411241^2$	5^2	$[2^{32} 6^{12} 30^{20}]$
67A	1	1	1	1	$[165^2]$
67B	2	1	$2^2/11^2$	2^2	$[6^2 330^2]$
67C	2	1	0	0	$[66^4]$
67D	10	11	$11/2861^2$	11	$[3^{16} 7521^2 82731^2]$
67E	10	3	$3^2/193^2$	3^2	$[11^{10} 33^{10}]$
67F	10	11	$1/661^2$	1	$[3^{16} 4623^2 50853^2]$
67G	20	11	$11/8009^2$	11	$[3^{36} 240999^4]$
67H	100	33	$1/67^2 \cdot 661^2 \cdot 11287^2 \cdot$ $93832004556914 59^2$	1	$[3^{60} 33^{20}]$
71A	3	1	$1/7^2$	1	$[5^2 35^2 315^2]$
71B	3	1	$1/5^2$	1	$[7^2 35^2 315^2]$
71C	20	5	$1/31^2 \cdot 211^2$	1	$[7^{30} 35^{10}]$
71D	30	7	$1/113^2 \cdot 12713^2$	1	$[5^{50} 35^{10}]$
71E	120	35	$1/281^2 \cdot 701^4 \cdot$ $1307084991922565572906 1^2$	1	$[5^{20} 35^{40}]$

REFERENCES

[1] A. Agashé, *On invisible elements of the Tate-Shafarevich group*, C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), no. 5, 369–374.

[2] A. Agashe and W. Stein, *Visibility of Shafarevich-Tate groups of abelian varieties*, J. Number Theory 97 (2002), no. 1, 171–185.

[3] ______, *Visible evidence for the Birch and Swinnerton-Dyer conjecture for modular abelian varieties of analytic rank* 0, to appear in Math. Comp.

[4] ______, *The Manin constant, congruence primes, and the modular degree*, in progress.

[5] M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. Math. IHES, 36 (1969), 23–58.

[6] L. Bégueri, *Dualité sur un corps local à corps résiduel algébriquement clos*, Mém. Soc. Math. France (1980/81), no. 4.

[7] A. Bertapelle, *On perfectness of Grothendieck's pairing for the ℓ-parts of component groups*, J. Reine Angew. Math., 538 (2001), 223–236.

[8] A. Bertapelle and S. Bosch, *Weil restriction and Grothendieck's duality conjecture*, Journal of algebraic geometry, 9 (2000), 155–164.

[9] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Springer-Verlag, Berlin, 1990.

[10] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24 (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993).

[11] J. W. S. Cassels and E. V. Flynn, *Prolegomena to a middlebrow arithmetic of curves of genus* 2, Cambridge University Press, Cambridge, 1996.

[12] J. W. S. Cassels, A. Fröhlich, *Algebraic Number Theory*, Academic Press, London, 1967.

[13] T. Chinburg "Minimal models of curves over Dedekind rings" in *Arithmetic Geometry* (Cornell/Silverman ed.), Springer-Verlag, Berlin, 1986.

[14] J. E. Cremona, *Algorithms for modular elliptic curves*, second ed., Cambridge University Press, Cambridge, 1997.

[15] P. Deligne and M. Rapoport, "Les schémas de modules de courbes elliptiques" in *Modular Functions of One Variables II*, Lecture Notes in Mathematics 349, Springer-Verlag, Berlin, 1973.

[16] F. Diamond and J. Im, *Modular forms and modular curves*, Seminar on Fermat's Last Theorem, Providence, RI, 1995, pp. 39–133.

[17] A. Edelman, *The mathematics of the Pentium division bug*, SIAM Rev. 39 (1997), no. 1, 54–67.

[18] S. J. Edixhoven, *L'action de l'algèbre de Hecke sur les groupes de composantes des jacobiennes des courbes modulaires est "Eisenstein"*, Astérisque (1991), no. 196–197, 7–8, 159–170 (1992), Courbes modulaires et courbes de Shimura (Orsay, 1987/1988).

[19] ______, *On the Manin constants of modular elliptic curves*, Arithmetic algebraic geometry (Texel, 1989), Birkhäuser Boston, Boston, MA, 1991, pp. 25–39.

[20] ______, *Néron models and tame ramification*, Compositio Math., 81 (1992), 291–306.

[21] ______, *Modular parameterizations at primes of bad reduction*, in preparation.

[22] ______, *Comparison of integral structures on spaces of modular forms of weight two, and computation of spaces of forms mod* 2 *of weight one*, preprint.

[23] M. Emerton, *Optimal Quotients of Modular Jacobians*, to appear in Math. Ann.

[24] E. Freitag and R. Kiehl, *Étale cohomology and the Weil conjectures*, Springer-Verlag, Berlin, 1988.

[25] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies 131, Princeton University Press, 1993.

[26] J. González and J.-C. Lario, $\mathbf{Q}$*-curves and their Manin ideals*, Amer. J. Math. 123 (2001), no. 3, 475–503.

[27] A. Grothendieck, *Éléments de géométrie algébrique.* IV_4. Étude locale des schémas et des morphismes de schémas, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28.

[28] ______, *Groupes de monodromie en géométrie algébrique. I*, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Lecture Notes in Mathematics, Vol. 288.

[29] ______, *Groupes de monodromie en géométrie algébrique. II*, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz, Lecture Notes in Mathematics, Vol. 340.

[30] B. H. Gross, *A tameness criterion for Galois representations associated to modular forms* (*mod* p), Duke Math. J. 61 (1990), no. 2, 445–517.

[31] A. Joyce, *The Manin Constant of an Optimal Quotient of* $J_0(431)$, preprint, 2003.

[32] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, 244 page preprint.

[33] N. M. Katz, *Galois properties of torsion points on abelian varieties*, Invent. Math. 62 (1981), no. 3, 481–502.

[34] N. M. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*, Princeton University Press, Princeton, N.J., 1985.

[35] D. R. Kohel and W. A. Stein, *Component Groups of Quotients of* $J_0(N)$, Proceedings of the 4th International Symposium (ANTS-IV), Leiden, Netherlands, July 2–7, 2000 (Berlin), Springer, 2000.

[36] V. A. Kolyvagin and D. Y. Logachev, *Finiteness of the Shafarevich-Tate group and the group of rational points for some modular abelian varieties*, Algebra i Analiz 1 (1989), no. 5, 171–196.

[37] D. S. Kubert and S. Lang, *Modular units*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 244, Springer-Verlag, New York, 1981.

[38] S. Lang, *Introduction to modular forms*, Springer-Verlag, Berlin, 1995, With appendixes by D. Zagier and W. Feit, Corrected reprint of the 1976 original.

[39] S. Lichtenbaum, *Curves over discrete valuation rings*, American Journal of Mathematics, 90 (1968), 380–405.

[40] J. Lipman, *Desingularization of two-dimensional schemes*, Annals of Mathematics, 107 (1978), 151–207.

[41] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford University Press, Oxford, 2002.

[42] Q. Liu and D. Lorenzini, *Models of curves and finite covers*, Compositio Math., 118 (1999), 61–102.

[43] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Mathematica, 80 (1993), 95–111.

[44] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid.

[45] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 33–186.

[46] ———, *Rational isogenies of prime degree* (*with an appendix by D. Goldfeld*), Invent. Math. 44 (1978), no. 2, 129–162.

[47] L. Merel, *L'accouplement de Weil entre le sous-groupe de Shimura et le sous-groupe cuspidal de $J_0(p)$*, J. Reine Angew. Math. 477 (1996), 71–115.

[48] A. P. Ogg, *Hyperelliptic modular curves*, Bull. Soc. Math. France 102 (1974), 449–462.

[49] ———, *Rational points on certain elliptic modular curves*, Analytic number theory (Proc. Sympos. Pure Math., Vol XXIV, St. Louis Univ., St. Louis, Mo., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 221–231.

[50] A. P. Ogg, *Rational points of finite order on elliptic curves*, Invent. Math. 12 (1971), 105–111.

[51] B. Poonen and M. Stoll, *The Cassels–Tate pairing on polarized abelian varieties*, Ann. of Math. (2) 150 (1999), no. 3, 1109–1149.

[52] D. Popescu, *General Néron desingularization and approximation*, Nagoya Math Journal, 104 (1986), pp. 85–115.

[53] Michel Raynaud, *Schémas en groupes de type $(p, \ldots, p)$*, Bull. Soc. Math. France 102 (1974), 241–280.

[54] K. Ribet, "On the component groups and the Shimura subgroup of $J_0(N)$", exposé 6, Sém. Th. Nombres, Université Bordeaux, 1987–88.

[55] ———, "Irreducible Galois representations arising from component groups of Jacobians" in *Elliptic curves, modular forms, and Fermat's Last Theorem*, International Press, 1995.

[56] M. Schlessinger, *Infinitesimal deformations of singularities*, 1964 Harvard Ph.D. thesis, unpublished.

[57] J-P. Serre, *Groupes finis d'automorphismes d'anneaux locaux réguliers*, Colloque d'Algèbre (Paris, 1967), Exp. 8 (1968), 11.

[58] ———, *Cohomologie des groupes discrets*, Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970), Princeton Univ. Press, Princeton, N.J., 1971, pp. 77–169. Ann. of Math. Studies, No. 70.

[59] ———, *Lie algebras and Lie groups*, second ed., Lecture Notes in Mathematics, vol. 1500, Springer-Verlag, Berlin, 1992, 1964 lectures given at Harvard University.

[60] ———, *Œuvres. Collected papers. IV*, Springer-Verlag, Berlin, 2000, 1985–1998.

[61] I. Shafarevich, *Lectures on minimal models and birational transformations of two-dimensional schemes*, Tata Institute: Bombay, 1966.

[62] G. Shimura, *On the factors of the jacobian variety of a modular function field*, J. Math. Soc. Japan 25 (1973), no. 3, 523–544.

[63] ______, *An introduction to computing modular forms using modular symbols*, to appear in an MSRI proceedings.

[64] ______, *Shafarevich-Tate groups of nonsquare order*, to appear in proceedings of MCAV 2002, Progress of Mathematics.

[65] G. Stevens, *Stickelberger elements and modular parametrizations of elliptic curves*, Invent. Math. 98 (1989), no. 1, 75–106.

[66] R. Swan, *Néron-Popescu desingularization*, Lectures in Algebra and Geometry 2, International Press, Cambridge (1998), pp. 135–192.

[67] J. Tate, *Duality theorems in Galois cohomology over number fields*, Proc. Internat. Congr. Mathematicians (Stockholm, 1962), Inst. Mittag-Leffler, Djursholm, 1963, pp. 288–295.

[68] J. Watanabe, *Some remarks on Cohen-Macaulay rings with many zero divisors and an application*, J. Algebra 39 (1976), no. 1, 1–14.

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Enriched Functors and Stable Homotopy Theory

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Abstract. In this paper we employ enriched category theory to construct a convenient model for several stable homotopy categories. This is achieved in a three-step process by introducing the pointwise, homotopy functor and stable model category structures for enriched functors. The general setup is shown to describe equivariant stable homotopy theory, and we recover Lydakis' model category of simplicial functors as a special case. Other examples – including motivic homotopy theory – will be treated in subsequent papers.

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1 Introduction

An appropriate setting to study stable phenomena in algebraic topology is the stable homotopy theory of spectra described in [1]. More recently, much research has been focused on rebuilding the foundation of stable homotopy theory. There are now several equivalent model categories to study structured ring spectra and their modules. These frameworks are important in many aspects and make powerful tools from algebra applicable to "brave new rings". For the purpose of this paper, the relevant constructions are those of symmetric spectra [9] and simplicial functors [11].

In [8], Hovey considers the notion of spectra for general model categories. This level of generality allows one to use techniques from stable homotopy theory in traditionally unrelated subjects. Of particular interest, where further applications are expected, is algebraic geometry and Voevodsky's motivic stable homotopy category [16]. We are interested in an approach to this subject where all coherence problems which arise when one tries to make a smash product are encoded in the underlying category. This is different from the popular means of attack through symmetric spectra, where the controlling categories are much more restricted. Our point of view is analogous to Lydakis' work [11] on simplicial functors as a model for ordinary spectra. But the theory we develop here is complicated by the fact that we do not assume properties which are particular to simplicial sets. Let us state a tentative version of the main theorem in this paper. Due to their technical nature, we defer on listing all the required assumptions. The basic input is a monoidal model category $\mathcal{V}$ and a finitely presentable cofibrant object T in $\mathcal{V}$, the 1-sphere. See Sections 6 and 7 for precise statements.

Theorem 1. *There is a monoidal model category* $(\mathcal{F}, \wedge, \mathbb{I})$ *which satisfies the monoid axiom, and a right Quillen equivalence from* $\mathcal{F}$ *to the stable model category of* T*-spectra.*

A T-spectrum is a sequence $(E_0, E_1, \ldots)$ of objects in $\mathcal{V}$ together with structure maps $T \otimes E_n \longrightarrow E_{n+1}$. An object in $\mathcal{F}$ is a functor X from a category of finitely presentable objects in $\mathcal{V}$ to $\mathcal{V}$, which is "continuous" or *enriched* in the sense that for finitely presentable objects v and w there is a natural map

$$v \otimes X(w) \longrightarrow X(v \otimes w).$$

Using this map, it follows that any enriched functor yields a T-spectrum by evaluating at spheres $T^{\otimes n}$. We show that the induced functor from $\mathcal{F}$ to the category of T-spectra is a right Quillen equivalence. The monoidal structure is a special case of a result due to Day [4]. By construction, the sphere spectrum or unit $\mathbb{I}$ is the inclusion of the subcategory of finitely presentable objects. For $\mathcal{V}$ the category of simplicial sets, Lydakis [11] has shown that $\mathcal{F}$ models the classical stable homotopy category. Our theorem extends this result to a wide range of model categories. In the sequel [5] we construct a model for Voevodsky's motivic stable homotopy category. Motivic cohomology has a natural description as an algebra in this model. The monoid axiom implies that also categories of algebras and modules in $\mathcal{F}$ have model structures [15].

As a guide to this paper, it seems appropriate to summarize the content of each section. In Section 2 we recall categorical precursors and Day's smash product for enriched functors. This material is included to make the paper reasonably self-contained and to set notation. Next we record a general isomorphism between enriched functor categories build from spheres and symmetric spectra. Moreover, under this isomorphism the corresponding smash products are shown to agree. Section 3 recalls some frequently used notions in homotopical algebra. An expert could skip most of this part. We introduce a class of model categories dubbed *weakly finitely generated* and show that weak equivalences and fibrant objects are closed under filtered colimits. Such a model structure is cofibrantly generated, with additional finiteness conditions on the generating cofibrations and acyclic cofibrations which are satisfied in many cases of interest. This introductory part ends with a discussion of fibrant replacement functors. Quillen's small object argument is the usual device for replacing objects in a model category by fibrant objects. Some modifications are necessary in the enriched setting. If the monoid axiom holds and the model category is weakly finitely generated, we construct enriched fibrant replacements. Our constructions are primarily of a technical interest and might be omitted on a first cursory reading. However, we should remark that much of the following relies on this input.

In the remaining sections we study homotopical algebra for enriched functor categories. First we construct the *pointwise model structure* where the fibrations and weak equivalences are defined pointwise. This gives an example of a weakly finitely generated model structure provided some weak assumptions are satisfied. In many cases of interest, we prove the monoid axiom and that smashing with a cofibrant object preserves weak equivalences. The latter result requires further assumptions on $\mathcal{F}$, and is similar to the algebraic fact that tensoring with a projective module preserves short exact sequences. These two results are important for the model structures we construct later on.

One drawback with the pointwise model structure is that it has far too many homotopy types. For example, a weak equivalence $v \xrightarrow{\sim} w$ does not necessarily induce a pointwise weak equivalence $\mathcal{V}(w,-) \longrightarrow \mathcal{V}(v,-)$ on the level of representable functors. However, for all fibrant objects u in $\mathcal{V}$ the map $\mathcal{V}(w,u) \longrightarrow \mathcal{V}(v,u)$ is a weak equivalence. We therefore enlarge the class of pointwise weak equivalences by looking at fibrant objects as input only. The result is the *homotopy functor model structure* which has the same cofibrations as the pointwise model structure. The fibrant functors are precisely the pointwise fibrant functors which preserve weak equivalences, thus any enriched functor is weakly equivalent in the homotopy functor model structure to a homotopy functor. It seems to be of considerable interest to discuss a motivic version of Goodwillie's calculus of functors. Let us remark that the homotopy functor model structure is a first step in this direction.

In Section 6, the *stable model structure* is constructed by means of a general stabilization process. Theorem 6.26 lists conditions for the stable model structure to exist. The stable fibrations are characterized by pointwise homotopy pullback squares, and stable acyclic fibrations are precisely the pointwise acyclic fibrations. To prove these results we compare with spectra [8]. The stabilization we use does not coincide with the usual stabilization for spectra, and it requires some cruel details to compare them. These can be found in Appendix A. We note that the monoid axiom holds under an additional assumption on the source of the functor category. For a particular choice of the source category, which is explained in Section 7, the evaluation functor is the right adjoint in a Quillen equivalence. It follows that the highly structured category of en-

riched functors describes the same homotopy theory as spectra in many cases of interest. In Section 8, we give a short summary of the important algebraic consequences of the previous sections.
In the last section we discuss equivariant homotopy theory for finite groups and we prove the following theorem. The general machinery gives deeper structure than stated, but we refer the reader to Section 9 for more details.

Theorem 2. *Let G be a finite group. Then there is a monoidal model category $(G\mathcal{F}, \wedge, \mathbb{S}^G)$ satisfying the monoid axiom and a right Quillen equivalence from $G\mathcal{F}$ to the category of G-spectra.*

The general framework may seem abstract, but we obtain a common footing for applications. A project in progress suggests that the approach anticipated in the present paper is relevant for the theory of motives. We hope the reader finds results herein which he or she can prove to have further applications.

2 Enriched Categories

This section contains an introduction to enriched categories, Day's work on enriched functor categories [4], and simplicial homotopies in categories enriched over simplicial sets. In the last part we show that spectra and symmetric spectra are isomorphic to enriched functor categories build from spheres.

2.1 Introduction

Entry points to the literature on enriched category theory include [2] and [12]. A *monoidal* category consists of a category $\mathcal{V}$, and

- a functor $\otimes\colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ and natural associativity isomorphisms

  $$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

  subject to the coherence law [2, 6.1],

- an object e of $\mathcal{V}$ called the *unit*, and natural unit isomorphisms

  $$l_A\colon e \otimes A \longrightarrow A \quad \text{and} \quad r_A\colon A \otimes e \longrightarrow A$$

  such that [2, 6.2] holds.

The functor $\otimes$ is the *tensor* or *monoidal* product of $\mathcal{V}$. A monoidal category is *symmetric monoidal* if there is a natural isomorphism $\sigma_{A,B}\colon A \otimes B \longrightarrow B \otimes A$ subject to the coherence laws [2, 6.3, 6.4, 6.5]. A symmetric monoidal category $(\mathcal{V}, \otimes, e)$ is *closed* if there exists a right adjoint $\mathrm{Hom}_{\mathcal{V}}(A, -)\colon \mathcal{V} \longrightarrow \mathcal{V}$ to the endofunctor $- \otimes A$ for every object A of $\mathcal{V}$. The categories of sets **Set** and pointed sets $\mathbf{Set}_*$ are both closed symmetric monoidal categories. Let Δ denote the simplicial category. Its objects are the finite ordered sets $[n] =$

$\{0 < 1 < \cdots < n\}$ for $n \geq 0$, and morphisms are order-preserving maps. Consider $\mathbf{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$, the category of simplicial sets. Its monoidal product is the categorical product, formed degree-wise in $\mathbf{Set}$. If $K, L \in \mathbf{sSet}$, the simplicial set of maps $\mathrm{Hom}_{\mathbf{sSet}}(K, L)$ has n-simplices the set of maps from $K \times \Delta^n$ to L. Here Δ^n is the simplicial set represented by $[n]$. The unit of the product is the terminal object Δ^0.

Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal category. Then a *$\mathcal{V}$-category* $\mathcal{C}$, or a category *enriched over* $\mathcal{V}$, consists of a class $\mathrm{Ob}\,\mathcal{C}$ of objects and

- for any pair (a, b) of objects in $\mathcal{C}$, an object $\mathcal{V}_{\mathcal{C}}(a, b)$ of $\mathcal{V}$ called the *$\mathcal{V}$-object of maps in* $\mathcal{C}$,
- a composition $\mathcal{V}_{\mathcal{C}}(b, c) \otimes \mathcal{V}_{\mathcal{C}}(a, b) \longrightarrow \mathcal{V}_{\mathcal{C}}(a, c)$, an identity or unit map $e \longrightarrow \mathcal{V}_{\mathcal{C}}(a, a)$ subject to the associativity and unit coherence laws listed in [2, 6.9 and 6.10].

Categories in the usual sense are the $\mathbf{Set}$-categories. If $\mathcal{C}$ is a category, let $\mathbf{Set}_{\mathcal{C}}(a, b)$ denote the set of maps in $\mathcal{C}$ from a to b. A closed symmetric monoidal category $\mathcal{V}$ is a $\mathcal{V}$-category due to its internal Hom objects [2, 6.2.6]. Let $\mathcal{V}(A, B)$ denote the $\mathcal{V}$-object $\mathrm{Hom}_{\mathcal{V}}(A, B)$ of maps in $\mathcal{V}$. Any $\mathcal{V}$-category $\mathcal{C}$ defines a $\mathbf{Set}$-category $\mathcal{UC}$. Its class of objects is $\mathrm{Ob}\,\mathcal{C}$, the morphism sets are $\mathbf{Set}_{\mathcal{UC}}(a, b) = \mathbf{Set}_{\mathcal{V}}(e, \mathcal{V}_{\mathcal{C}}(a, b))$. For example, the $\mathbf{Set}$-category obtained from a $\mathbf{sSet}$-category $\mathcal{C}$ has morphism sets $\mathbf{Set}_{\mathbf{sSet}}(\Delta^0, \mathbf{sSet}_{\mathcal{C}}(a, b)) = \mathbf{sSet}_{\mathcal{C}}(a, b)_0$ the zero-simplices of the simplicial sets of maps.

A *$\mathcal{V}$-functor* F from $\mathcal{C}$ to $\mathcal{D}$ is an assignment from $\mathrm{Ob}\,\mathcal{C}$ to $\mathrm{Ob}\,\mathcal{D}$ together with morphisms $\hom^F_{a,b}\colon \mathcal{V}_{\mathcal{C}}(a, b) \longrightarrow \mathcal{V}_{\mathcal{D}}(F(a), F(b))$ in $\mathcal{V}$ which preserve composition and identities. A small $\mathbf{sSet}$-category defines a simplicial object in the category $\mathbf{Cat}$ of small categories. With this description, a $\mathbf{sSet}$-functor is a natural transformation of functors from Δ^{op} to $\mathbf{Cat}$ [11, 3.2].

If F and G are $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{D}$, a *$\mathcal{V}$-natural transformation* $t\colon F \longrightarrow G$ consists of the following data: There is a morphism $t(a)\colon F(a) \longrightarrow G(a)$ in $\mathcal{UD}$ for every $a \in \mathrm{Ob}\,\mathcal{C}$, and all the diagrams of the following form commute.

$$\begin{tikzcd}
\mathcal{V}_{\mathcal{C}}(a,b) \arrow[r, "\hom^F_{a,b}"] \arrow[d, "\hom^G_{a,b}"'] & \mathcal{V}_{\mathcal{D}}(F(a),F(b)) \arrow[d, "\mathcal{V}_{\mathcal{D}}(F(a),t(b))"] \\
\mathcal{V}_{\mathcal{D}}(G(a),G(b)) \arrow[r, "\mathcal{V}_{\mathcal{D}}(t(a),G(b))"] & \mathcal{V}_{\mathcal{D}}(F(a),G(b))
\end{tikzcd}$$

The $\mathcal{V}$-natural isomorphisms and $\mathcal{V}$-adjoint pairs of $\mathcal{V}$-functors are defined as for $\mathcal{V} = \mathbf{Set}$. The adjoint pair of endofunctors $(- \otimes A, \mathcal{V}(A, -))$ on $\mathcal{V}$ explicate a $\mathcal{V}$-adjoint pair. Denote the unit of the adjunction by $\eta_A\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow \mathcal{V}(A, - \otimes A)$, and the counit by $\epsilon_A\colon \mathcal{V}(A, -) \otimes A \longrightarrow \mathrm{Id}_{\mathcal{V}}$. Details concerning $\hom^{-\otimes A}$ and $\hom^{\mathcal{V}(A,-)}$ can be found in Appendix A.

Note that any $\mathcal{V}$-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ gives a functor $\mathcal{U}F\colon \mathcal{UC} \longrightarrow \mathcal{UD}$ with the same effect on objects as F, and similarly for $\mathcal{V}$-natural transformations. That is, one can consider $\mathcal{U}$ as a 2-functor from the 2-category of small $\mathcal{V}$-categories,

$\mathcal{V}$-functors and $\mathcal{V}$-natural transformations to the 2-category of small categories, functors and natural transformations. A $\mathcal{V}$-category $\mathcal{C}$ is called small provided $\mathcal{UC}$ is small. In fact, $\mathcal{U}$ is the base change along the lax symmetric monoidal functor $\mathbf{Set}_{\mathcal{V}}(e,-)\colon \mathcal{V} \longrightarrow \mathbf{Set}$, see [2, 6.4]. If no confusion can arise, we will omit $\mathcal{U}$ from the notation.

The *monoidal product* $\mathcal{C} \otimes \mathcal{D}$ of two $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ is the $\mathcal{V}$-category where $\mathrm{Ob}\,(\mathcal{C}\otimes\mathcal{D}) := \mathrm{Ob}\,\mathcal{C}\times\mathrm{Ob}\,\mathcal{D}$ and $\mathcal{V}_{\mathcal{C}\otimes\mathcal{D}}((a,x),(b,y)) := \mathcal{V}_{\mathcal{C}}(a,b)\otimes\mathcal{V}_{\mathcal{D}}(x,y)$. Note that the monoidal product in $\mathcal{V}$ induces a $\mathcal{V}$-functor mon: $\mathcal{V}\otimes\mathcal{V} \longrightarrow \mathcal{V}$.

A $\mathcal{V}$-category $\mathcal{C}$ is a *right* $\mathcal{V}$*-module* if there is a $\mathcal{V}$-functor act: $\mathcal{C}\otimes\mathcal{V} \longrightarrow \mathcal{C}$, denoted $(c,A) \longmapsto c\oslash A$ and a $\mathcal{V}$-natural unit isomorphism $r_c\colon \mathrm{act}(c,e) \longrightarrow c$ subject to the following conditions.

- There are natural coherent associativity isomorphisms

$$\mathrm{act}(c, A\otimes B) \longrightarrow \mathrm{act}(\mathrm{act}(c,A),B).$$

- The isomorphisms $\mathrm{act}(c, e\otimes A) \rightrightarrows \mathrm{act}(c,A)$ coincide.

A right $\mathcal{V}$-module $(\mathcal{C},\mathrm{act},r)$ is *closed* if there is a $\mathcal{V}$-functor

$$\mathrm{coact}\colon \mathcal{V}^{\mathrm{op}}\otimes\mathcal{C} \longrightarrow \mathcal{C}$$

such that for all $A\in\mathrm{Ob}\,\mathcal{V}$ and $c\in\mathrm{Ob}\mathcal{C}$, the $\mathcal{V}$-functor $\mathrm{act}(-,A)\colon\mathcal{C}\longrightarrow\mathcal{C}$ is left $\mathcal{V}$-adjoint to $\mathrm{coact}(A,-)$ and $\mathrm{act}(c,-)\colon\mathcal{V}\longrightarrow\mathcal{C}$ is left $\mathcal{V}$-adjoint to $\mathcal{V}_{\mathcal{C}}(c,-)$.

A *monoidal* $\mathcal{V}$-category consists of a $\mathcal{V}$-category $\mathcal{C}$ equipped with a $\mathcal{V}$-functor $\diamond\colon\mathcal{C}\otimes\mathcal{C}\longrightarrow\mathcal{C}$, a unit $u\in\mathrm{Ob}\,\mathcal{C}$, a $\mathcal{V}$-natural associativity isomorphism and two $\mathcal{V}$-natural unit isomorphisms satisfying the conditions mentioned for $\mathcal{V}=\mathbf{Set}$. Symmetric monoidal and closed symmetric monoidal $\mathcal{V}$-categories are defined similarly.

2.2 Categories of enriched functors

If $\mathcal{C}$ is a small $\mathcal{V}$-category, $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ and their $\mathcal{V}$-natural transformations form the category $[\mathcal{C},\mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. If $\mathcal{V}$ is complete, then $[\mathcal{C},\mathcal{V}]$ is also a $\mathcal{V}$-category. Denote this $\mathcal{V}$-category by $\mathcal{F}(\mathcal{C})$, or $\mathcal{F}$ if no confusion can arise. The morphism $\mathcal{V}$-object $\mathcal{V}_{\mathcal{F}}(X,Y)$ is the end

$$\int_{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(X(c),Y(c)).$$

See [2, 6.3.1] for details. Note that $\mathcal{UF}$ is $[\mathcal{C},\mathcal{V}]$. One can compare $\mathcal{F}$ with $\mathcal{C}$ and $\mathcal{V}$ as follows: Given $c\in\mathrm{Ob}\,\mathcal{C}$, $X\longmapsto X(c)$ defines the $\mathcal{V}$-functor $\mathrm{Ev}_c\colon\mathcal{F}\longrightarrow\mathcal{V}$ called "evaluation at c". The assignment $c\longmapsto\mathcal{V}_{\mathcal{C}}(c,-)$ from $\mathcal{C}$ to $\mathcal{F}$ is again a $\mathcal{V}$-functor $\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{F}$, called the $\mathcal{V}$*-Yoneda embedding* [2, 6.3.6]. $\mathcal{V}_{\mathcal{C}}(c,-)$ is a representable functor, represented by c.

Lemma 2.1 (Enriched Yoneda lemma). *Let $\mathcal{V}$ be a complete closed symmetric monoidal category and $\mathcal{C}$ a small $\mathcal{V}$-category. For every $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ and every $c \in \operatorname{Ob}\mathcal{C}$, there is a $\mathcal{V}$-natural isomorphism $X(c) \cong \mathcal{V}_{\mathcal{F}}(\mathcal{V}_{\mathcal{C}}(c,-), X)$.*

The isomorphism in 2.1 is called the *Yoneda* isomorphism [2, 6.3.5]. It follows from 2.1 that every $\mathcal{V}$-functor can be expressed as a colimit of representable functors [2, 6.6.13, 6.6.17]:

Lemma 2.2. *If $\mathcal{V}$ is a bicomplete closed symmetric monoidal category and $\mathcal{C}$ is a small $\mathcal{V}$-category, then $[\mathcal{C}, \mathcal{V}]$ is bicomplete. (Co)limits are formed pointwise.*

Corollary 2.3. *Assume $\mathcal{V}$ is bicomplete, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Then any $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ is $\mathcal{V}$-naturally isomorphic to the coend*

$$\int^{\operatorname{Ob}\mathcal{C}} \mathcal{V}_{\mathcal{C}}(c,-) \otimes X(c).$$

See [2, 6.6.18] for a proof of 2.3.

Proposition 2.4. *Let $\mathcal{V}$ be a closed symmetric monoidal category, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Then $\mathcal{F}$ is a closed $\mathcal{V}$-module.*

Proof. There is an obvious "pointwise" closed $\mathcal{V}$-module structure. The $\mathcal{V}$-functor $\mathcal{F} \otimes \mathcal{V} \longrightarrow \mathcal{F}$ defined by $(X, A) \longmapsto (- \otimes A) \circ X$ gives the action of $\mathcal{V}$ on $\mathcal{F}$. Next, the assignment $(A, X) \longmapsto \mathcal{V}(A, -) \circ X$ defines the coaction. There are $\mathcal{V}$-natural isomorphisms $\mathcal{V}_{\mathcal{F}}((-\otimes A) \circ X, Y) \cong \mathcal{V}_{\mathcal{F}}(X, \mathcal{V}(A,-) \circ Y) \cong \mathcal{V}(A, \mathcal{V}_{\mathcal{F}}(X, Y))$ induced from the natural closed $\mathcal{V}$-module structure on $\mathcal{V}$. From this, a routine check finishes the proof. $\square$

Recall the notion of *left Kan extensions*:

Proposition 2.5. *Fix a bicomplete closed symmetric monoidal category $\mathcal{V}$, and a $\mathcal{V}$-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ of small $\mathcal{V}$-categories. For any $\mathcal{V}$-functor $X\colon \mathcal{C} \longrightarrow \mathcal{V}$, there exists a $\mathcal{V}$-functor $F_*(X)\colon \mathcal{D} \longrightarrow \mathcal{V}$ and a $\mathcal{V}$-natural isomorphism*

$$\mathcal{V}_{\mathcal{F}(\mathcal{D})}(F_* X, Y) \cong \mathcal{V}_{\mathcal{F}(\mathcal{C})}(X, Y \circ F).$$

In other words, there exists a $\mathcal{V}$-adjoint pair of $\mathcal{V}$-functors

$$F_*\colon \mathcal{F}(\mathcal{C}) \rightleftarrows \mathcal{F}(\mathcal{D})\colon F^*$$

where F^* denotes pre-composition with F.

See [2, 6.7.7] for a proof of 2.5. The $\mathcal{V}$-functor $F_* X$ is the left Kan extension of X along F. An explicit expression is given by the coend

$$F_* X = \int^{\operatorname{Ob}\mathcal{C}} \mathcal{V}_{\mathcal{D}}(F(c), -) \otimes X(c).$$

2.3 SMASH PRODUCT OF ENRICHED FUNCTORS

Let $(\mathcal{C}, \diamond, u)$ be a small symmetric monoidal $\mathcal{V}$-category where $\mathcal{V}$ is bicomplete. In [4], B. Day constructed a closed symmetric monoidal product $\wedge$ on the category $[\mathcal{C}, \mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$. For $X, Y \in \mathrm{Ob}\,[\mathcal{C}, \mathcal{V}]$, there is the $\mathcal{V}$-functor

$$X\overline{\wedge}Y : \mathcal{C} \otimes \mathcal{C} \xrightarrow{X \otimes Y} \mathcal{V} \otimes \mathcal{V} \xrightarrow{\mathrm{mon}} \mathcal{V}.$$

The smash product $X \wedge Y \in \mathrm{Ob}\,[\mathcal{C}, \mathcal{V}]$ is the left Kan extension

$$\diamond_*(X\overline{\wedge}Y) = \int^{\mathrm{Ob}\,(\mathcal{C}\otimes\mathcal{C})} \mathcal{V}_{\mathcal{C}}(c \diamond d, -) \otimes (X(c) \otimes Y(d)) : \mathcal{C} \longrightarrow \mathcal{V}.$$

The next result is a special case of [4, 3.3], cf. [4, 3.6, 4.1].

THEOREM 2.6 (DAY). *Let $(\mathcal{V}, \otimes, e)$ be a bicomplete closed symmetric monoidal category and $(\mathcal{C}, \diamond, u)$ a small symmetric monoidal $\mathcal{V}$-category. Then the category $([\mathcal{C}, \mathcal{V}], \wedge, \mathcal{V}_{\mathcal{C}}(u, -))$ is closed symmetric monoidal.*

The $\mathcal{V}$-category $\mathcal{F}$ of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ is also a closed symmetric monoidal $\mathcal{V}$-category. The internal Hom functor, right adjoint to $- \wedge X$, is given by

$$\mathcal{F}(X, Y)(c) = \mathcal{V}_{\mathcal{F}}(X, Y(c \diamond -)) = \int_{d \in \mathrm{Ob}\,\mathcal{C}} \mathcal{V}(X(d), Y(c \diamond d)).$$

Concerning smash products of representable functors, one has the following result.

LEMMA 2.7. *The smash product of representable functors is again representable. There is a natural isomorphism $\mathcal{V}_{\mathcal{C}}(c, -) \wedge \mathcal{V}_{\mathcal{C}}(d, -) \cong \mathcal{V}_{\mathcal{C}}(c \diamond d, -)$.*

In the following, a sub-$\mathcal{V}$-category means a sub-$\mathcal{V}$-category of $\mathcal{V}$. We use 2.7 to define assembly maps if $\mathcal{C}$ is a full sub-$\mathcal{V}$-category containing the unit and closed under the monoidal product. In this case, the inclusion $\mathbb{I} : \mathcal{C} \hookrightarrow \mathcal{V}$ can be chosen as the unit of $[\mathcal{C}, \mathcal{V}]$. The *composition* $X \circ Y$ of two $\mathcal{V}$-functors $X, Y : \mathcal{C} \longrightarrow \mathcal{V}$ is given by $\mathbb{I}_* X \circ Y$. Up to coherent natural isomorphisms, the composition is associative with unit $\mathbb{I}$.

COROLLARY 2.8. *Given functors X and Y in $[\mathcal{C}, \mathcal{V}]$, there exists a natural assembly map $X \wedge Y \longrightarrow X \circ Y = \mathbb{I}_* X \circ Y$ which is an isomorphism if Y is representable.*

Proof. One can define the assembly map objectwise via the composition

$$\mathbb{I}_* X(c) \otimes Y(d) \xrightarrow{\mathrm{sw}^{\mathbb{I}_* X}_{Y(d)}(c)} \mathbb{I}_* X(Y(d) \otimes c) \xrightarrow{\mathbb{I}_* X(\mathrm{sw}^{Y}_{c}(d))} \mathbb{I}_* X(Y(c \otimes d))$$

where $\mathrm{sw}^{Z}_{c} : Z \otimes c \longrightarrow Z(c \otimes -)$ is 'the' natural map described in Appendix A. Here is another description via representable functors. Suppose $X = \mathcal{V}(c, -)$ and $Y = \mathcal{V}(d, -)$, for $c, d \in \mathrm{Ob}\,\mathcal{C}$. By 2.7, $X \wedge Y$ is naturally isomorphic to

$\mathcal{V}(c \otimes d, -)$, i.e. to $\mathcal{V}(c, -) \circ \mathcal{V}(d, -)$. If X is arbitrary, it follows from 2.3 that $X \wedge \mathcal{V}(d, -)$ is naturally isomorphic to $\mathbb{I}_* X \circ \mathcal{V}(d, -)$. If also Y is arbitrary, apply 2.3 and consider the natural map

$$\int^{\operatorname{Ob}\mathcal{C}} (\mathbb{I}_* X \circ \mathcal{V}(c, -)) \otimes Y(c) \longrightarrow \mathbb{I}_* X \circ \int^{\operatorname{Ob}\mathcal{C}} \mathcal{V}(c, -) \otimes Y(c).$$

□

2.4 Categories enriched over simplicial sets

A functor $F\colon \mathcal{V} \longrightarrow \mathcal{W}$ of monoidal categories $(\mathcal{V}, \otimes, e)$ and $(\mathcal{W}, \otimes', e')$ is *lax monoidal* if there is a natural transformation $t_{A,B}\colon F(A) \otimes' F(B) \longrightarrow F(A \otimes B)$ and a morphism $e' \longrightarrow F(e)$, such that the diagrams [2, 6.27, 6.28] commute. The word "lax" is replaced by "*strict*" if $t_{A,B}$ is a natural isomorphism and $e' \longrightarrow F(e)$ is an isomorphism. F is *lax symmetric monoidal* if $t_{B,A} \circ \sigma_{FA,FB} = F(\sigma_{A,B}) \circ t_{A,B}$. In this case, every $\mathcal{W}$-category is a $\mathcal{V}$-category by [2, 6.4.3]. We used this fact for the forgetful 2-functor $\mathcal{U}$ induced by $\mathbf{Set}_{\mathcal{V}}(e, -)$. The assembly map makes $\operatorname{Id}_{[\mathcal{C},\mathcal{V}]}$ into a lax monoidal functor from the monoidal category $([\mathcal{C}, \mathcal{V}], \circ, \mathbb{I})$ to the closed symmetric monoidal category $([\mathcal{C}, \mathcal{V}], \wedge, \mathbb{I})$. Suppose that $F\colon \mathbf{sSet} \longrightarrow \mathcal{V}$ is a lax symmetric monoidal functor. One can then lift the notion of *simplicial homotopy equivalence* from $\mathbf{sSet}$-categories to $\mathcal{V}$-categories.

Definition 2.9. Let $\mathcal{C}$ be a $\mathbf{sSet}$-category and $f, f'\colon \Delta^0 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(c, d)$ maps in $\mathcal{C}$. Then $H\colon \Delta^1 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(c, d)$ is a *simplicial homotopy* from f to f' if the following diagram commutes, where i_0 and i_1 are the canonical inclusions.

$$\begin{tikzcd}
\Delta^0 \arrow[r, "i_0"] \arrow[dr, "f"'] & \Delta^1 \arrow[d, "H"] & \Delta^0 \arrow[l, "i_1"'] \arrow[dl, "f'"] \\
& \mathbf{sSet}_{\mathcal{C}}(c, d) &
\end{tikzcd}$$

The map f is called a *simplicial homotopy equivalence* if there exists a map $g\colon \Delta^0 \longrightarrow \mathbf{sSet}_{\mathcal{C}}(d, c)$, and simplicial homotopies from $g \circ f$ to id_c and from $f \circ g$ to id_d. Let the symbol $\simeq$ denote simplicial homotopy equivalences.

Simplicial homotopy equivalence is in general not an equivalence relation. If $\mathcal{C}$ is a closed $\mathbf{sSet}$-module with action $(c, K) \longmapsto c \oslash K$ and coaction $(c, K) \longmapsto c^K$, a simplicial homotopy may also be described by maps $c \oslash \Delta^1 \longrightarrow d$ or $c \longrightarrow d^{\Delta^1}$. If $\mathcal{C}$ has pushouts, the *simplicial mapping cylinder* factors any map f as follows: Let C_f denote the pushout of the diagram

$$c \oslash \Delta^1 \xleftarrow{c \oslash i_1} c \oslash \Delta^0 \xleftarrow{\cong} c \xrightarrow{f} d.$$

The maps $c \oslash \Delta^1 \xrightarrow{c \oslash s} c \oslash \Delta^0 \xrightarrow{\cong} c \xrightarrow{f} d$ and $\mathrm{id}_d : d \longrightarrow d$ induce the simplicial homotopy equivalence $p_f\colon C_f \longrightarrow d$. Its homotopy inverse is the canonical map. Denote the composition $c \longrightarrow c \oslash \Delta^0 \xrightarrow{c \oslash i_0} c \oslash \Delta^1 \longrightarrow C_f$ by $i_f: c \longrightarrow C_f$. Note the factorization $p_f \circ i_f = f$. The relevance of this will become clear in the context of simplicial model categories. In this case, i_f is a cofibration provided c is cofibrant. It is easy to prove the next result.

Lemma 2.10. *A* **sSet***-functor preserves simplicial homotopies, and therefore simplicial homotopy equivalences.*

Corollary 2.11. *Assume F:* **sSet** $\longrightarrow \mathcal{V}$ *is a lax monoidal functor. Then any $\mathcal{V}$-functor preserves simplicial homotopy equivalences.*

2.5 Spectra as enriched functors

Let $(\mathcal{V}, \otimes, e)$ denote a bicomplete closed symmetric monoidal category with initial object $\emptyset$. For $T \in \operatorname{Ob}\mathcal{V}$, one can consider T-spectra in $\mathcal{V}$, see [8, 1.1]. A *T-spectrum* E is a sequence $E_0, E_1, \cdots$ of objects in $\mathcal{V}$, together with structure maps $e_n: E_n \otimes T \longrightarrow E_{n+1}$ for all n. If E and F are T-spectra, a *map of T-spectra* $g: E \longrightarrow F$ is a collection of maps $g_n: E_n \longrightarrow F_n$ such that

$$\begin{tikzcd} E_n \otimes T \arrow[r, "e_n"] \arrow[d, "g_n \otimes T"'] & E_{n+1} \arrow[d, "g_{n+1}"] \\ F_n \otimes T \arrow[r, "f_n"] & F_{n+1} \end{tikzcd}$$

commutes for all n. Thus T-spectra in $\mathcal{V}$ form a category $\operatorname{Sp}(\mathcal{V}, T)$, see [8, 1.3]. We claim $\operatorname{Sp}(\mathcal{V}, T)$ can be viewed as an enriched functor category, cf. [11, 4.3]. Its domain category is the $\mathcal{V}$-category $T\mathrm{Sph}$. The objects in $T\mathrm{Sph}$ are the objects T^n for $n \geq 0$, where $T^0 = e$ and $T^n := T \otimes T^{n-1}$ for $n > 0$. The $\mathcal{V}$-objects of morphisms are $\mathcal{V}_{T\mathrm{Sph}}(T^m, T^n) := T^{n-m}$ for $n \geq m \geq 0$ and $\mathcal{V}_{T\mathrm{Sph}}(T^m, T^n) := \emptyset$ for $n < m$. Note that there are canonical unit maps $\mathrm{id}_{T^0}: T^0 \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^n, T^n)$ for all $n \geq 0$. It remains to describe the composition. For $k, l, m \geq 0$, the map

$$\mathcal{V}_{T\mathrm{Sph}}(T^{l+m}, T^{k+l+m}) \otimes \mathcal{V}_{T\mathrm{Sph}}(T^m, T^{l+m}) \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^m, T^{k+l+m})$$

is the associativity isomorphism $\alpha_{k,l}: T^k \otimes T^l \longrightarrow T^{k+l}$. In all other cases, the composition is uniquely determined. It follows that $T\mathrm{Sph}$ is a $\mathcal{V}$-category, using the associativity and unit coherence laws in $\mathcal{V}$.
To elaborate on this definition, let us describe a $\mathcal{V}$-functor $\pi: T\mathrm{Sph} \longrightarrow \mathcal{V}$. Define π to be the identity on objects. Concerning morphisms, it suffices to give $\hom^{\pi}_{T^m, T^{k+m}}: \mathcal{V}_{T\mathrm{Sph}}(T^m, T^{k+m}) \longrightarrow \mathcal{V}(T^m, T^{k+m})$ for $k, m \geq 0$, which we choose as

$$i_{k,m}: T^k \xrightarrow{\eta_{T^m} T^k} \mathcal{V}(T^m, T^k \otimes T^m) \xrightarrow{\mathcal{V}(T^m, \alpha_{k,m})} \mathcal{V}(T^m, T^{k+m}).$$

Associativity coherence and a calculation with adjoints imply that the composition $T^k \otimes T^l \xrightarrow{\alpha_{k,l}} T^{k+l} \xrightarrow{i_{k+l,m}} \mathcal{V}(T^m, T^{k+l+m})$ is the same as the composition

$T^k \otimes T^l \xrightarrow{i_{k,l+m}\otimes i_{l,m}} \mathcal{V}(T^{l+m}, T^{k+l+m}) \otimes \mathcal{V}(T^m, T^{l+m}) \xrightarrow{\text{comp}} \mathcal{V}(T^m, T^{k+l+m})$. Hence π preserves composition, and it clearly preserves identites. In our cases of interest, the maps $\eta_{T^m}T^k$ are monomorphisms so that $T\mathrm{Sph}$ can be regarded as a sub-$\mathcal{V}$-category.

Proposition 2.12. *The categories* $\mathrm{Sp}(\mathcal{V}, T)$ *and* $[T\mathrm{Sph}, \mathcal{V}]$ *are isomorphic.*

Proof. Let $X: T\mathrm{Sph} \longrightarrow \mathcal{V}$ be a $\mathcal{V}$-functor. Define $\Psi(X)$ to be the spectrum with $\Psi(X)_n := X(T^n)$ and structure maps $\Psi(X)_n \otimes T \longrightarrow \Psi(X)_{n+1}$ adjoint to $T = \mathcal{V}_{T\mathrm{Sph}}(T^n, T^{n+1}) \longrightarrow \mathcal{V}(X(T^n), X(T^{n+1}))$. If $f: X \longrightarrow Y$ is a $\mathcal{V}$-natural transformation, let $\Psi(f)_n := f(T^n): X(T^n) \longrightarrow Y(T^n)$. The diagram

$$\begin{array}{ccc} X(T^n) \otimes T & \longrightarrow & X(T^{n+1}) \\ {\scriptstyle f(T^n)\otimes T}\downarrow & & \downarrow{\scriptstyle f(T^{n+1})} \\ Y(T^n) \otimes T & \longrightarrow & Y(T^{n+1}) \end{array}$$

commutes by $\mathcal{V}$-naturality, and Ψ respects identities and composition.
Define $\Phi: T\mathrm{Sph} \longrightarrow \mathcal{V}$ by $\Phi(E)(T^n) := E_n$. If $n = m + k$, $m \geq 0$ and $k \geq 1$, $\hom^{\Phi(E)}_{T^m,T^n}: T^k = \mathcal{V}_{T\mathrm{Sph}}(T^m, T^n) \longrightarrow \mathcal{V}(E_m, E_n)$ is the adjoint of the composition $E_m \otimes T^k \xrightarrow{\alpha^{-1}_{E_m,T,T^{k-1}}} (E_m \otimes T) \otimes T^{k-1} \xrightarrow{e_m \otimes T^{k-1}} E_{m+1} \otimes T^{k-1} \longrightarrow \cdots \longrightarrow E_n$. The maps $\hom^{\Phi(E)}_{T^n,T^n}$ are determined by the property that $\Phi(E)$ has to preserve identities. To prove that $\Phi(E)$ is a $\mathcal{V}$-functor, it remains to note that

$$\begin{array}{ccc} T^k \otimes T^l & \xrightarrow{\qquad\qquad \alpha_{k,l} \qquad\qquad} & T^{k+l} \\ {\scriptstyle \hom^{\Phi(E)}_{T^{l+m},T^{k+l+m}}}\downarrow{\scriptstyle \otimes \hom^{\Phi(E)}_{T^m,T^{l+m}}} & & \downarrow{\scriptstyle \hom^{\Phi(E)}_{T^m,T^{k+l+m}}} \\ \mathcal{V}(E_{l+m}, E_{k+l+m}) \otimes \mathcal{V}(E_m, E_{l+m}) & \xrightarrow{\text{comp}} & \mathcal{V}(E_m, E_{k+l+m}) \end{array}$$

commutes for $m \geq 0$ and $k, l \geq 1$. This uses $\epsilon_{E_m}(-\otimes E_m) \circ \eta_{E_m} \otimes E_m = \mathrm{id}_{-\otimes E_m}$, associativity coherence, and associativity of composition in $\mathcal{V}$. If g is a map, let $\Phi(g)(T^n)$ be g_n. Then $\Phi(g)$ is $\mathcal{V}$-natural, and functoriality of Φ follows.
Note that $\Phi(\Psi(X)) = X$ on objects, i.e. for all powers of T. The structure maps of X and $\Phi(\Psi(X))$ coincide, since the adjointness isomorphism that defines the structure maps of $\Phi(\Psi(X))$ is inverse to the adjointness isomorphism that defines the structure maps of $\Psi(X)$. The equality $\Phi(\Psi(f)) = f$ is obvious, hence $\Phi \circ \Psi$ is the identity functor. Likewise, one finds $\Psi \circ \Phi = \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)}$. □

Remark 2.13. In the following, we will identify $\mathrm{Sp}(\mathcal{V}, T)$ with $[T\mathrm{Sph}, \mathcal{V}]$ via 2.12. Then $\mathrm{Sp}(\mathcal{V}, T)$ is a closed $\mathcal{V}$-module by 2.4. A consequence of A.1 is that the functor "suspension with T" obtained from the action of $\mathcal{V}$ on $\mathrm{Sp}(\mathcal{V}, T)$ can also be defined as the prolongation [8, 1.5] of the functor $-\otimes T: \mathcal{V} \longrightarrow \mathcal{V}$ using the natural transformation $t: (-\otimes T) \circ (-\otimes T) \longrightarrow (-\otimes T) \circ (-\otimes T)$, which twists the factors. In detail, the latter is defined by the coherence isomorphism

$(A\otimes T)\otimes T \xrightarrow{\alpha_{A,T,T}} A\otimes(T\otimes T) \xrightarrow{A\otimes\sigma_{T,T}} A\otimes(T\otimes T) \xrightarrow{\alpha^{-1}_{A,T,T}} (A\otimes T)\otimes T$. Another functor "suspension with T" – which we denote by Σ_T – is obtained as the prolongation of $-\otimes T$ using the identity natural transformation $\mathrm{id}_{(-\otimes T)\circ(-\otimes T)}$. If $X\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ is a $\mathcal{V}$-functor, then the nth structure map in the associated spectrum of $X\otimes T$ is the left hand side composition in the following diagram.

$$\begin{tikzcd}
& (X_n\otimes T)\otimes T \arrow[dl, "\sigma_{X_n\otimes T,T}"'] \arrow[dr, "\sigma_{X_n,T\otimes T}"] & \\
T\otimes(X_n\otimes T) \arrow[rr, "\alpha^{-1}_{T,X_n,T}"] \arrow[d, "\hom^X_{n,n+1}\otimes(X_n\otimes T)"'] & & (T\otimes X_n)\otimes T \arrow[d, "(\hom^X_{n,n+1}\otimes X_n)\otimes T"] \\
\mathcal{V}(X_n,X_{n+1})\otimes(X_n\otimes T) \arrow[rr, "\alpha^{-1}_{\mathcal{V}(X_n,X_{n+1}),X_n,T}"] \arrow[d, "\hom^{-\otimes T}_{X_n,X_{n+1}}"'] & & (\mathcal{V}(X_n,X_{n+1})\otimes X_n)\otimes T \arrow[d, "(\epsilon_{X_n}X_{n+1})\otimes T"] \\
\mathcal{V}(X_n\otimes T,X_{n+1}\otimes T)\otimes(X_n\otimes T) \arrow[rr, "\epsilon_{X_n\otimes T}(X_{n+1}\otimes T)"] & & X_{n+1}\otimes T
\end{tikzcd}$$

The right hand side composition is the structure map of the spectrum $\Sigma_T X$. The lower square commutes by A.1, the middle square commutes by naturality, but the triangle does not commute in general. This will cause some complications in our comparison of stable model categories, cp. Section 6.

The monoidal product $\otimes$ defines a $\mathcal{V}$-functor $\mathrm{mon}\colon \mathcal{V}\otimes\mathcal{V} \longrightarrow \mathcal{V}$ where

$$\hom^{\mathrm{mon}}_{(A_1,A_2)(B_1,B_2)}\colon \mathcal{V}(A_1,B_1)\otimes\mathcal{V}(A_2,B_2) \longrightarrow \mathcal{V}(A_1\otimes A_2, B_1\otimes B_2)$$

is the adjoint of the composition

$$\begin{tikzcd}
\mathcal{V}(A_1,B_1)\otimes\mathcal{V}(A_2,B_2)\otimes A_1\otimes A_2 \arrow[d, "\mathcal{V}(A_1,B_1)\otimes"', "\sigma_{\mathcal{V}(A_2,B_2),A_1}\otimes A_2"] \\
\mathcal{V}(A_1,B_1)\otimes A_1\otimes\mathcal{V}(A_2,B_2)\otimes A_2 \arrow[d, "\epsilon_{A_1}B_1\otimes"', "\epsilon_{A_2}B_2"] \\
B_1\otimes B_2.
\end{tikzcd}$$

Now suppose that the symmetric monoidal product in $\mathcal{V}$ induces a $\mathcal{V}$-functor $\mathrm{mon}\colon T\mathrm{Sph}\otimes T\mathrm{Sph} \longrightarrow T\mathrm{Sph}$. On objects we have that $\mathrm{mon}(T^k,T^l) = T^k\otimes T^l = T^{k+l}$, while for $\mathcal{V}$-objects of morphisms there is a map f from $\mathcal{V}_{T\mathrm{Sph}}(T^m,T^k\otimes T^m)\otimes\mathcal{V}_{T\mathrm{Sph}}(T^n,T^l\otimes T^n)$ to $\mathcal{V}_{T\mathrm{Sph}}(T^m\otimes T^n,T^k\otimes T^m\otimes T^l\otimes T^n)$ rendering the following diagram commutative.

$$\begin{tikzcd}
T^k\otimes T^l \arrow[r, "f"] \arrow[d, "\eta_{T^m}T^k\otimes\eta_{T^n}T^l"'] & T^k\otimes T^l \arrow[d, "\eta_{T^m\otimes T^n}(T^k\otimes T^l)"] \\
\mathcal{V}(T^m,T^k\otimes T^m)\otimes\mathcal{V}(T^n,T^l\otimes T^n) \arrow[r, "\hom^{\mathrm{mon}}"] & \mathcal{V}(T^m\otimes T^n,T^k\otimes T^m\otimes T^l\otimes T^n)
\end{tikzcd}$$

Reverting to adjoints, a straightforward calculation shows that the maps

$$T^{k+l+m+n} = T^k \otimes T^l \otimes T^m \otimes T^n \underset{T^k \otimes \sigma_{T^l,T^m} \otimes T^n}{\overset{f \otimes T^m \otimes T^n}{\rightrightarrows}} T^k \otimes T^m \otimes T^l \otimes T^n = T^{k+l+n+m}$$

must coincide. This is only possible if $\sigma_{T,T} = \mathrm{id}_{T^2}$ and $f = \mathrm{id}$. In other words, the product $\otimes$ does not necessarily restrict to a monoidal product of $T\mathrm{Sph}$ via $\pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$. In the next section, we show – following [11, 5.15] – how to remedy this by enlarging $T\mathrm{Sph}$.

2.6 Symmetric spectra as enriched functors

For ease of notation, we will leave out associativity and unit isomorphisms throughout this section. If $n \geq 1$, let $\overline{n}$ be short for $\{1, \cdots, n\}$ and let $\overline{0}$ denote the empty set. Let Inj be the category with objects the sets $\overline{n}$ for all $n \geq 0$, and injective maps as morphisms. If $m \leq n$, define

$$\mathrm{Inj}(m,n) := \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\overline{m},\overline{n})} T^0$$

where T^0 is the unit of $\mathcal{V}$. Note that $\mathrm{Inj}(n,n)$ is a group object in $\mathcal{V}$, the symmetric group on n letters. By regarding $\mathrm{Inj}(n,n)$ as a $\mathcal{V}$-category with a single object, a left $\mathrm{Inj}(n,n)$-action on $A \in \mathrm{Ob}\,\mathcal{V}$ is a $\mathcal{V}$-functor $\mathrm{Inj}(n,n) \longrightarrow \mathcal{V}$ with value A. One gets a left $\mathrm{Inj}(n,n)$-action on $T^n = T_n \otimes \cdots \otimes T_2 \otimes T_1$ using iterations of the commutativity isomorphism $\sigma_{T,T}$.

A *symmetric T-spectrum X in $\mathcal{V}$* as defined in [8, 7.2] consists of a sequence $X_0, X_1, \cdots, X_n, \cdots$, where X_n is an object of $\mathcal{V}$ with a left $\mathrm{Inj}(n,n)$-action, and with structure maps $X_n \otimes T \longrightarrow X_{n+1}$ such that the following composition $X_n \otimes T^m \longrightarrow X_{n+1} \otimes T^{m-1} \longrightarrow \cdots \longrightarrow X_{n+m}$ is $\mathrm{Inj}(n,n) \otimes \mathrm{Inj}(m,m)$-equivariant. A map of symmetric T-spectra consists of maps $X_n \longrightarrow Y_n$ that are compatible with the $\mathrm{Inj}(n,n)$-action and the structure maps. We will show that the category $\mathrm{Sp}^\Sigma(\mathcal{V},T)$ of symmetric T-spectra in $\mathcal{V}$ is isomorphic to a category of $\mathcal{V}$-functors with codomain $\mathcal{V}$ and domain $T\mathrm{Sph}^\Sigma$.

The objects in $T\mathrm{Sph}^\Sigma$ are the objects in $T\mathrm{Sph}$, but the morphism objects are different. If $n = k+m$ with $m \geq 0, k \geq 0$, define the $\mathcal{V}$-object $\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^m, T^n)$ to be $\mathrm{Inj}(m,n) \otimes T^k$. If $n < m$, define $\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^m, T^n)$ to be the initial object. The unit map $T^0 \longrightarrow \mathrm{Inj}(n,n) = \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n,T^n)$ is the canonical map to the summand corresponding to $\mathrm{id}_{\overline{n}}$. Next, to describe the composition, identify $\mathrm{Inj}(m, k+m) \otimes T^k$ indexed by $\beta\colon \overline{m} \longrightarrow \overline{k+m}$ with $T_{i_1^\beta} \otimes \cdots \otimes T_{i_k^\beta} = T^k$. Here $\{i_1^\beta, i_2^\beta, \cdots, i_k^\beta\}$ is the reordering of $\overline{k+m} \setminus \beta(\overline{m})$ which satisfies that $i_1^\beta > i_2^\beta > \cdots > i_k^\beta$. If $n = k+l+m$ with $k,l,m \geq 0$, we define the map

$$\mathrm{Inj}(l+m,n) \otimes T^k \otimes \mathrm{Inj}(m,l+m) \otimes T^l \longrightarrow \mathrm{Inj}(m,n) \otimes T^{k+l}$$

in two steps. First, we identify the source of the map with the coproduct $\coprod_{\mathbf{Set}_{\mathrm{Inj}}(\overline{l+m},\overline{n}) \times \mathbf{Set}_{\mathrm{Inj}}(\overline{m},\overline{l+m})} T^{k+l}$. For the second step, consider the unique isomorphism $T^{k+l} \longrightarrow T^{k+l}$ induced by the permutation that reorders the set $\{i_1^\gamma, i_2^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \gamma(i_2^\beta), \cdots, \gamma(i_l^\beta)\}$, i.e. the set $\overline{k+l+m} \setminus (\gamma \circ \beta)(\overline{m})$, as $\{i_1^{\gamma\circ\beta}, i_2^{\gamma\circ\beta}, \cdots, i_{k+l}^{\gamma\circ\beta}\}$. This isomorphism maps the summand $T_{i_1^\gamma} \otimes T_{i_2^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes T_{i_1^\beta} \otimes T_{i_2^\beta} \otimes \cdots \otimes T_{i_l^\beta} = T^{k+l}$ indexed by (γ, β) to the summand $T_{i_1^{\gamma\circ\beta}} \otimes T_{i_2^{\gamma\circ\beta}} \otimes \cdots \otimes T_{i_{k+l}^{\gamma\circ\beta}} = T^{k+l}$ indexed by $\gamma \circ \beta$.

Lemma 2.14. *$T\mathrm{Sph}^\Sigma$ is a symmetric monoidal $\mathcal{V}$-category.*

Proof. The composition in $T\mathrm{Sph}^\Sigma$ is clearly unital. Associativity follows, since the permutation reordering the set $\{i_1^\gamma, i_2^\gamma, \ldots, i_k^\gamma, \gamma(i_1^\beta), \gamma(i_2^\beta), \ldots, \gamma(i_l^\beta)\}$ is unique.

On objects, $(T^l, T^k) \longmapsto T^{k+l}$ defines the monoidal product in $T\mathrm{Sph}^\Sigma$. For morphisms, consider injections $\beta\colon \overline{n} \longrightarrow \overline{l+n}$ and $\gamma\colon \overline{m} \longrightarrow \overline{k+m}$. Define $\beta \triangleright \gamma\colon \overline{m+n} \longrightarrow \overline{k+l+m+n}$ by concatenation. That is, $\beta \triangleright \gamma(i)$ is defined as $\gamma(i)$ for $i \leq m$ and as $\beta(i-m)+k+m$ for $i > m$. Note that the ordered set $i_1^\beta + k + m > \cdots > i_l^\beta + k + m > i_1^\gamma > \cdots > i_k^\gamma$ coincides with $i_1^{\beta\triangleright\gamma} > \cdots > i_{k+l}^{\beta\triangleright\gamma}$. On $\mathcal{V}$-objects of morphisms, the map

$$\mathrm{Inj}(n, l+n) \otimes T^l \otimes \mathrm{Inj}(m, k+m) \otimes T^k \longrightarrow \mathrm{Inj}(m+n, k+l+m+n) \otimes T^{k+l}$$

sends the summand $T^l \otimes T^k = T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}$ indexed by $(\overline{n} \xrightarrow{\beta} \overline{l+n}, \overline{m} \xrightarrow{\gamma} \overline{k+m})$ via the identity onto the summand T^{k+l} indexed by $\beta \triangleright \gamma$. To see this, one can rewrite the indices according to the above equality of ordered sets, that is, changing $T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}$ into $T_{i_1^\beta+k+m} \otimes \cdots \otimes T_{i_l^\beta+k+m} \otimes T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma}$.

Let us check that this defines a $\mathcal{V}$-functor $\mathrm{mon}^\Sigma\colon T\mathrm{Sph}^\Sigma \otimes T\mathrm{Sph}^\Sigma \longrightarrow T\mathrm{Sph}^\Sigma$. The reason is essentially that concatenation $\triangleright\colon \mathrm{Inj} \times \mathrm{Inj} \longrightarrow \mathrm{Inj}$ is a functor. Since $\mathrm{id}_{\overline{n}} \triangleright \mathrm{id}_{\overline{m}} = \mathrm{id}_{\overline{m+n}}$, if follows that mon^Σ preserves identities. To check compatibility with composition, fix four injective maps $\alpha\colon \overline{m} \longrightarrow \overline{l+m}$, $\beta\colon \overline{l+m} \longrightarrow \overline{k+l+m =: n}$, $\gamma\colon \overline{r} \longrightarrow \overline{q+r}$ and $\delta\colon \overline{q+r} \longrightarrow \overline{p+q+r}$. The equality $(\delta \circ \gamma) \triangleright (\beta \circ \alpha) = (\delta \triangleright \beta) \circ (\gamma \triangleright \alpha)$ implies that it suffices to consider only one summand, say $M = \{i_1^\delta, \cdots, i_p^\delta, i_1^\beta, \cdots, i_k^\beta, i_1^\gamma, \cdots, i_q^\gamma, i_1^\alpha, \cdots, i_l^\alpha\}$. First, by rewriting M we find $\{i_1^\delta, \cdots, i_p^\delta, \delta(i_1^\gamma), \cdots, \delta(i_q^\gamma), i_1^\beta, \cdots, i_k^\beta, \beta(i_1^\alpha), \cdots, \beta(i_l^\alpha)\}$, and next by reordering we get $\{i_1^{\delta\circ\gamma}, \cdots, i_{p+q}^{\delta\circ\gamma}, i_1^{\beta\circ\alpha}, \cdots, i_{k+l}^{\beta\circ\alpha}\}$. The monoidal product rewrites the latter as $M' = \{i_1^{\delta\circ\gamma} + n, \cdots, i_{p+q}^{\delta\circ\gamma} + n, i_1^{\beta\circ\alpha}, \cdots, i_{k+l}^{\beta\circ\alpha}\}$ and there is a bijection $M \xrightarrow{\cong} M'$. On the other hand, the monoidal product rewrites M as $\{i_1^\delta + n, \cdots, i_p^\delta + n, i_1^\beta, \cdots, i_k^\beta, i_1^\gamma + l + m, \cdots, i_q^\gamma + l + m, i_1^\alpha, \cdots, i_l^\alpha\}$, and composition rewrites this set as $\{i_1^\delta + n, \cdots, i_p^\delta + n, i_1^\beta, \cdots, i_k^\beta, \delta \triangleright \beta(i_1^\gamma + l + m), \cdots, \delta \triangleright \beta(i_q^\gamma + l + m), \delta \triangleright \beta(i_1^\alpha), \cdots, \delta \triangleright \beta(i_l^\alpha).\}$ By definition of $\triangleright$, this set coincides with

$$\{i_1^\delta + n, \cdots, i_p^\delta + n, i_1^\beta, \cdots, i_k^\beta, \delta(i_1^\gamma) + n, \cdots, \delta(i_q^\gamma) + n, \beta(i_1^\alpha), \cdots, \beta(i_l^\alpha)\}$$

which is then reordered as $M' = \{i_1^{(\delta \triangleright \beta)\circ(\gamma \triangleright \alpha)}, \cdots, i_{k+l+p+q}^{(\delta \triangleright \beta)\circ(\gamma \triangleright \alpha)}\}$. The corresponding bijection $M \xrightarrow{\cong} M'$ is thus the same as above. Hence mon^Σ is a $\mathcal{V}$-functor.
By definition, $\mathrm{mon}^\Sigma(T^0, T^m) = T^m = \mathrm{mon}^\Sigma(T^m, T^0)$. So T^0 is a strict unit in $T\mathrm{Sph}^\Sigma$. Similarly, strict associativity holds. The commutativity isomorphism $T^0 \longrightarrow \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^{m+n}, T^{m+n})$ is the canonical map on the summand indexed by the permutation $\overline{m+n} \longrightarrow \overline{n+m}$ which interchanges m and n. Next, the coherence conditions [2, 6.3,6.4,6.5] follow by a straightforward calculation with permutations of $\overline{l+m+n}$. This ends the proof. $\square$

To explain the composition in $T\mathrm{Sph}^\Sigma$ more throughly, we will define $\mathcal{V}$-functors $\nu\colon T\mathrm{Sph} \longrightarrow T\mathrm{Sph}^\Sigma$ and $\sigma\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ such that $\sigma \circ \nu = \pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$. Both ν and σ are the respectively identities on objects. The map

$$\mathrm{hom}^\nu_{T^m, T^{l+m}}\colon T^l \longrightarrow \mathrm{Inj}(m, l+m) \otimes T^l \cong \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\overline{m}, \overline{l+m})} T^l$$

hits the summand indexed by the inclusion $\overline{m} \hookrightarrow \overline{l+m}$. It is then clear that ν preserves identities. For the composition, consider

$$\begin{tikzcd}
T^k \otimes T^l \arrow[rr, "\mathrm{id}"] \arrow[d, "\mathrm{hom}^\nu_{T^{l+m}, T^{k+l+m}} \otimes \mathrm{hom}^\nu_{T^m, T^{l+m}}"'] & & T^{k+l} \arrow[d, "\mathrm{hom}^\nu_{T^m, T^{k+l+m}}"] \\
\mathrm{Inj}(l+m, k+l+m) \otimes T^k \otimes \mathrm{Inj}(m, l+m) \otimes T^l \arrow[rr, "\mathrm{comp}"] & & \mathrm{Inj}(m, k+l+m) \otimes T^{k+l}
\end{tikzcd}$$

and observe that the left vertical map hits the summand

$$(T_{k+l+m} \otimes T_{k+l+m-1} \otimes \cdots \otimes T_{l+m+1}) \otimes (T_{l+m} \otimes T_{l+m-1} \otimes \cdots \otimes T_{m+1})$$

indexed by the inclusions $(\overline{l+m} \hookrightarrow \overline{k+l+m}, \overline{m} \hookrightarrow \overline{l+m})$. Composition in $T\mathrm{Sph}^\Sigma$ maps this summand by the identity to the summand $T_{k+l+m} \otimes \cdots \otimes T_{m+1}$ indexed by the inclusion $\overline{m} \hookrightarrow \overline{l+m} \hookrightarrow \overline{k+l+m}$, since the indices are ordered in the prescribed way. So the diagram commutes and ν is a $\mathcal{V}$-functor. This also explains the ordering $i_1^\beta > i_2^\beta > \cdots > i_k^\beta$ of the set $\overline{k+m} \setminus \beta(\overline{k})$.
The adjoint $\mathrm{Inj}(m, l+m) \otimes T^l \otimes T^m \cong \coprod_{\mathbf{Set}_{\mathrm{Inj}}(\overline{m}, \overline{l+m})} T^l \otimes T^m \longrightarrow T^{l+m}$ of $\mathrm{hom}^\sigma_{T^m, T^{l+m}}\colon \mathrm{Inj}(m, l+m) \otimes T^l \longrightarrow \mathcal{V}(T^m, T^{l+m})$ is defined as follows: Consider the summand $T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_m \otimes \cdots \otimes T_1$ indexed by β. First rewrite the indices as $T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes T_{\beta(m)} \otimes \cdots \otimes T_{\beta(1)}$, and then map to $T_{l+m} \otimes \cdots \otimes T_2 \otimes T_1$ by the unique permutation which reorders the set $\{i_1^\beta, \cdots, i_l^\beta, \beta(m), \cdots, \beta(1)\}$. To conclude that σ is a $\mathcal{V}$-functor, one has to check that reordering the set $\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \cdots, \gamma(i_l^\beta), \gamma(\beta(m)), \cdots, \gamma(\beta(1))\}$ is the same as first reordering $\{i_1^\gamma, \cdots, i_l^\gamma, i_1^\beta, \cdots, i_k^\beta, \beta(m), \cdots, \beta(1)\}$ as $\{i_1^\gamma, \cdots, i_l^\gamma, l+m, \cdots, 2, 1\}$, and then reordering $\{i_1^\gamma, \cdots, i_l^\gamma, \gamma(l+m), \cdots, \gamma(2), \gamma(1)\}$. However,

the monoidal product mon^Σ on $T\mathrm{Sph}^\Sigma$ is such that the diagram

$$\begin{tikzcd} T\mathrm{Sph}^\Sigma \otimes T\mathrm{Sph}^\Sigma \arrow[r, "\sigma\otimes\sigma"] \arrow[d, "\mathrm{mon}^\Sigma"'] & \mathcal{V}\otimes\mathcal{V} \arrow[d, "\mathrm{mon}"] \\ T\mathrm{Sph}^\Sigma \arrow[r, "\sigma"] & \mathcal{V} \end{tikzcd}$$

commutes, so σ is in fact a strict symmetric monoidal $\mathcal{V}$-functor.

Proposition 2.15. *The category $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ is isomorphic to the category of symmetric T-spectra in $\mathcal{V}$, and ν induces the forgetful functor from symmetric T-spectra to T-spectra. The smash product on $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ corresponds to the smash product on $\mathrm{Sp}^\Sigma(\mathcal{V}, T)$.*

Proof. This is similar to 2.12, and some details will be left out in the proof. The functor $\Phi: [T\mathrm{Sph}^\Sigma, \mathcal{V}] \longrightarrow \mathrm{Sp}^\Sigma(\mathcal{V}, T)$ maps $X: T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ to the sequence $XT^0, XT^1, \cdots$. It is clear that XT^n has a left $\mathrm{Inj}(n,n)$-action. The adjoint of the structure map $XT^n \otimes T \longrightarrow XT^{n+1}$ is given by $\mathrm{hom}^X_{T^n,T^{n+1}} \circ \mathrm{hom}^\nu_{T^n,T^{n+1}}$. More generally, the composition of structure maps $XT^n \otimes T^k \longrightarrow XT^{n+k}$ is the adjoint of $T^k \xrightarrow{\mathrm{hom}^\nu_{T^n,T^{k+n}}} \mathrm{Inj}(n, k+n) \otimes T^k \xrightarrow{\mathrm{hom}^X_{T^n,T^{k+n}}} \mathcal{V}(XT^n, XT^{k+n})$. This proves the required equivariance. The definition of Φ on $\mathcal{V}$-natural transformations is clear, and also functoriality.

The inverse $\Psi: \mathrm{Sp}^\Sigma(\mathcal{V}, T) \longrightarrow [T\mathrm{Sph}^\Sigma, \mathcal{V}]$ is harder to define. If $X_0, X_1, \cdots$ is a symmetric T-spectrum with structure maps $x^1_n: X_n \otimes T \longrightarrow X_{n+1}$, define $\Psi(X): T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ on objects by $T^n \longmapsto X_n$. Since X_n has a left $\mathrm{Inj}(n,n)$-action, there is the map $\mathrm{hom}^{\Psi(X)}_{T^n,T^n}: \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n, T^n) = \mathrm{Inj}(n,n) \longrightarrow \mathcal{V}(X_n, X_n)$. Choose an injection $\beta: \overline{n} \longrightarrow \overline{k+n}$, and define

$$\mathrm{hom}^{\Psi(X)}_{T^n,T^{k+n}}: \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n, T^{k+n}) = \mathrm{Inj}(n, k+n) \longrightarrow \mathcal{V}(X_n, X_{k+n})$$

on the summand $T_{i^\beta_1} \otimes \cdots \otimes T_{i^\beta_k}$ as the adjoint map of the following composition $T_{i^\beta_1} \otimes \cdots \otimes T_{i^\beta_k} \otimes X_n \xrightarrow{\sigma_{T^k, X_n}} X_n \otimes T_{i^\beta_1} \otimes \cdots \otimes T_{i^\beta_k} \xrightarrow{x^k_n} X_{k+n} \longrightarrow X_{k+n}$ where x^k_n is defined by the structure maps. The right hand map is the isomorphism associated to the permutation of the set $\overline{k+n}$ that changes $\{k+n, \cdots, 2, 1\}$ to $\{i^\beta_1, \cdots, i^\beta_k, \beta(n), \cdots, \beta(1)\}$ and reorders this as $\{k+n, \cdots, 2, 1\}$. For the latter we use the left $\mathrm{Inj}(k+n, k+n)$-action on X_{k+n}.

The $\mathrm{Inj}(n,n)$-action on X_n is unital, so $\Psi(X)$ preserves identities. To prove that $\Psi(X): T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ is a $\mathcal{V}$-functor, pick injective maps $\beta: \overline{m} \longrightarrow \overline{l+m}$,

$\gamma\colon \overline{l+m} \longrightarrow \overline{k+l+m}$ and consider the following diagram.

$$\begin{tikzcd}
T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes T_{i_1^\beta} \otimes \cdots \otimes T_{i_l^\beta} \otimes X_m \arrow[rr, "T^k \otimes (x_m^l \circ \sigma_{T^l, X_m})"] \arrow[d, "f \otimes X_m"'] \arrow[dr, "x_m^{k+l} \circ \sigma_{T^{k+l}, X_m}"] & & T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes X_{l+m} \arrow[dl, "x_{l+m}^k \circ \sigma_{T^k, X_{l+m}}"'] \arrow[d, "T^k \otimes b"] \\
T_{i_1^{\gamma\circ\beta}} \otimes \cdots \otimes T_{i_{k+l}^{\gamma\circ\beta}} \otimes X_m \arrow[d, "x_m^{k+l} \circ \sigma_{T^{k+l}, X_m}"'] & X_{k+l+m} \arrow[d, "f'"] & T_{i_1^\gamma} \otimes \cdots \otimes T_{i_k^\gamma} \otimes X_{l+m} \arrow[d, "x_{l+m}^k \circ \sigma_{T^k, X_{l+m}}"] \\
X_{k+l+m} \arrow[dr, "cb"] & X_{k+l+m} \arrow[d, "cb"] & X_{k+l+m} \arrow[dl, "c"'] \\
& X_{k+l+m} &
\end{tikzcd}$$

Here $b\colon X_{l+m} \longrightarrow X_{l+m}$ is the isomorphism obtained from reordering the set $\{i_1^\beta, \cdots, i_k^\beta, \beta(m), \cdots, \beta(1)\}$ as $\{l+m, \cdots, 2, 1\}$, and similarly for c and cb. The isomorphism $f\colon T^{k+l} \longrightarrow T^{k+l}$ is induced by the permutation

$$\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \cdots, \gamma(i_l^\beta)\} \xrightarrow{\cong} \{i_1^{\gamma\circ\beta}, \cdots, i_{k+l}^{\gamma\circ\beta}\}.$$

Similarly, $f'\colon X_{k+l+m} \longrightarrow X_{k+l+m}$ is induced by the permutation

$$\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \cdots, \gamma(i_l^\beta), m, \cdots, 1\} \xrightarrow{\cong} \{i_1^{\gamma\circ\beta}, \cdots, i_{k+l}^{\gamma\circ\beta}, m, \cdots, 1\}.$$

Since x_m^{k+l} is $\mathrm{Inj}(k+l, k+l) \otimes \mathrm{Inj}(m,m)$-equivariant, the left parallelogram commutes. The upper triangle commutes by commutativity coherence. Finally, the right parallelogram commutes, because x_{l+m}^k is $\mathrm{Inj}(k,k) \otimes \mathrm{Inj}(l+m, l+m)$-equivariant, the $\mathrm{Inj}(k+l+m, k+l+m)$-action on X_{k+l+m} is associative and the permutation obtained from reordering $\{i_1^\gamma, \cdots, i_k^\gamma, i_1^\beta, \cdots, i_l^\beta, \beta(m), \cdots, \beta(1)\}$ as $\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(l+m), \cdots, \gamma(1)\}$ and then as $\{k+l+m, \cdots, 2, 1\}$ equals the permutation obtained from $\{i_1^\gamma, \cdots, i_k^\gamma, \gamma(i_1^\beta), \cdots, \gamma(i_l^\beta), \gamma(\beta(m)), \cdots, \gamma(\beta(1))\}$. The definition of Ψ on maps is clear and functoriality follows easily. The proof of 2.12 shows that $\Phi \circ \Psi = \mathrm{Id}_{\mathrm{Sp}^\Sigma(\mathcal{V},T)}$, but an extra argument is required to prove the equality $\Psi \circ \Phi = \mathrm{Id}_{[T\mathrm{Sph}^\Sigma, \mathcal{V}]}$. The only point which is not obvious is whether the maps $\hom^X_{T^m, T^{k+m}}\colon \mathrm{Inj}(m, k+m) \otimes T^k \longrightarrow \mathcal{V}(XT^m, XT^{m+k})$ and $\hom^{\Psi(\Phi(X))}_{T^m, T^{k+m}}\colon \mathrm{Inj}(m, k+m) \otimes T^k \longrightarrow \mathcal{V}(XT^m, XT^{m+k})$ coincide. To prove this, we fix an injection $\beta\colon \overline{m} \longrightarrow \overline{k+m}$ and let $\iota\colon \overline{m} \hookrightarrow \overline{k+m}$ denote the inclusion. The permutation $\gamma\colon \overline{k+m} \longrightarrow \overline{k+m}$ obtained from rewriting the set $\{k+m, \cdots, 1+m, m, \cdots, 1\}$ as $\{i_1^\beta, \cdots, i_k^\beta, \beta(m), \cdots, \beta(1)\}$ and reordering this set as $\{k+m, \cdots, 2, 1\}$ has the property that $\gamma \circ \iota = \beta$. Since X is a $\mathcal{V}$-functor, the map $\hom^X_{T^m, T^{k+m}}$ is determined by its restriction to the summand indexed by $\overline{m} \hookrightarrow \overline{k+m}$. This shows that $\hom^X_{T^m, T^{k+m}} = \hom^{\Psi(\Phi(X))}_{T^m, T^{k+m}}$.

The claim concerning $[\nu, \mathcal{V}]\colon [T\mathrm{Sph}^\Sigma, \mathcal{V}] \longrightarrow [T\mathrm{Sph}, \mathcal{V}]$ is clear by the above. It remains to prove compatibility of the smash products. The smash product $\wedge'$

of symmetric T-spectra satisfies $F_mT^0 \wedge' F_nT^0 \cong F_{m+n}T^0$ by the remark below [8, 7.3]. On the other hand, the monoidal product in $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ is determined by representable functors, and from 2.7 there is the natural isomorphism

$$\mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^m, -) \wedge \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^n, -) \cong \mathcal{V}_{T\mathrm{Sph}^\Sigma}(T^{m+n}, -).$$

This completes the proof. □

Corollary 2.16. *Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category. Assume $\mathcal{C}$ is closed under the monoidal product, and contains the unit and T. Then $\sigma\colon T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{V}$ factors over $\mathcal{C} \hookrightarrow \mathcal{V}$, and the induced $\mathcal{V}$-functor $T\mathrm{Sph}^\Sigma \longrightarrow \mathcal{C}$ induces a lax symmetric monoidal functor $[\mathcal{C}, \mathcal{V}] \longrightarrow \mathrm{Sp}^\Sigma(\mathcal{V}, T)$ which has a strict symmetric monoidal left adjoint.*

Proof. The factorization of σ is clear. Under the isomorphism in 2.15, the closed symmetric monoidal product on $[T\mathrm{Sph}^\Sigma, \mathcal{V}]$ coincides with the closed symmetric monoidal product on symmetric T-spectra. For formal reasons, the $\mathcal{V}$-functor induced by the first factor of σ is lax symmetric monoidal. By checking on representable functors, it follows that its left $\mathcal{V}$-adjoint obtained by an enriched Kan extension 2.5 is strict symmetric monoidal. □

3 Model Categories

The term *model category* is to be understood in the sense of [7, 1.1.4]. We denote weak equivalences by $\xrightarrow{\sim}$, fibrations by $\twoheadrightarrow$ and cofibrations by $\rightarrowtail$.

3.1 Types of model categories

The model structures we will consider on enriched functor categories require a *cofibrantly generated* model category $(\mathcal{C}, I, J)$ as input. For a definition of this type of model categories and related terminology, consider [7, 2.1]. Maps in $I = \{i\colon si \rightarrowtail ti\}_{i\in I}$ are called *generating cofibrations*, and maps in $J = \{j\colon sj \overset{\sim}{\rightarrowtail} tj\}_{j\in J}$ are called *generating acyclic cofibrations*. The (co)domains of I and J may have additional properties.

Definition 3.1. An object $A \in \mathrm{Ob}\,\mathcal{C}$ is *finitely presentable* if the set-valued Hom-functor $\mathbf{Set}_\mathcal{C}(A, -)$ commutes with all filtered colimits. If $\mathcal{C}$ is a $\mathcal{V}$-category, $A \in \mathrm{Ob}\,\mathcal{C}$ is *$\mathcal{V}$-finitely presentable* if the $\mathcal{V}$-valued Hom-functor $\mathcal{V}_\mathcal{C}(A, -)$ commutes with all filtered colimits.

A set is finitely presentable in the category of sets if and only if it is a finite set. If the unit e in $\mathcal{V}$ is finitely presentable, any $\mathcal{V}$-finitely presentable object of $\mathcal{V}$ is finitely presentable. See [8, 4.1] for 3.2 and 3.3.

Definition 3.2. A cofibrantly generated model category $\mathcal{C}$ is *finitely generated* if I and J can be chosen such that the (co)domains of the maps in I and J are κ-small for a finite cardinal κ.

Finitely generated model categories are not necessarily closed under Bousfield localization, cf. [8, §4]. The following definition was suggested by Voevodsky.

Definition 3.3. The cofibrantly generated model category $\mathcal{C}$ is *almost finitely generated* if I and J can be chosen such that the (co)domains of the maps in I are κ-small for a finite cardinal κ, and there exists a subset J' of J for which

- the domains and the codomains of the maps in J' are κ-small for a finite cardinal κ,
- a map $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that B is fibrant is a fibration if and only if it is contained in J'-inj.

The left Bousfield localization with respect to a set with **sSet**-small domains and codomains preserves the structure of almost finitely generated cellular left proper simplicial model categories [6, Chapters 3,4]. For almost finitely generated model categories, the classes of weak equivalences and fibrant objects are closed under sequential colimits. We require these classes to be closed under filtered colimits, which holds for model categories of the following type.

Definition 3.4. A cofibrantly generated model category $\mathcal{V}$ is *weakly finitely generated* if I and J can be chosen such that the following conditions hold.

- The domains and the codomains of the maps in I are finitely presentable.
- The domains of the maps in J are small.
- There exists a subset J' of J of maps with finitely presentable domains and codomains, such that a map $f\colon A \longrightarrow B$ in $\mathcal{V}$ with fibrant codomain B is a fibration if and only if it is contained in J'-inj.

The choices of the sets I, J and J' will often be left implicit in the following. A weakly finitely generated model category is almost finitely generated. Examples include simplicial sets, simplicial sets with an action of a finite group, cp. 9.5, and the category of pointed simplicial presheaves on the smooth Nisnevich site. The latter will be discussed in [5].

Lemma 3.5. *Assume $\mathcal{V}$ is a weakly finitely generated model category. Then the following classes are closed under filtered colimits: weak equivalences, acyclic fibrations, fibrations with fibrant codomain, and fibrant objects.*

Proof. Since $\mathcal{V}$ is cofibrantly generated, [6, 11.6.1] shows that $\mathrm{Fun}(\mathcal{I},\mathcal{V})$ supports a cofibrantly generated model structure for any small category $\mathcal{I}$. Fibrations and weak equivalences are defined pointwise. Any weak equivalence in $\mathrm{Fun}(\mathcal{I},\mathcal{V})$ factors as an acyclic cofibration g composed with an acyclic fibration $p\colon T \twoheadrightarrow B$. Consider the induced factorization $\mathrm{colim} p \circ \mathrm{colim} g$. Note that $\mathrm{colim} g$ is an acyclic cofibration, since colim is a left Quillen functor. Therefore, the second claim will imply the first claim. If $\mathcal{I}$ is filtered, then colim$p$ is an acyclic fibration: Let

$$\begin{tikzcd} si \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathrm{colim}T \arrow[d, "\mathrm{colim}p"] \\ ti \arrow[r, "\beta"] & \mathrm{colim}B \end{tikzcd}$$

be a lifting problem, where $i \in I$. The existence of a lift in this diagram for all choices of α and β is equivalent to surjectivity of the canonical map

$$\phi\colon \mathbf{Set}_{\mathcal{V}}(ti, \mathrm{colim}T) \longrightarrow \mathbf{Set}_{\mathcal{V}}(si, \mathrm{colim}T) \times_{\mathbf{Set}_{\mathcal{V}}(si,\mathrm{colim}B)} \mathbf{Set}_{\mathcal{V}}(ti, \mathrm{colim}B).$$

Since si and ti are finitely presentable by assumption and filtered colimits commute with pullbacks in $\mathbf{Set}$, the canonical map ϕ is the filtered colimit of the canonical maps

$$\phi_d\colon \mathbf{Set}_{\mathcal{V}}(ti, T(d)) \longrightarrow \mathbf{Set}_{\mathcal{V}}(si, T(d)) \times_{\mathbf{Set}_{\mathcal{V}}(si,B(d))} \mathbf{Set}_{\mathcal{V}}(ti, B(d))$$

induced by composition with $p(d)$ and pre-composition with i. Note that ϕ_d is surjective since i is a cofibration and p is a pointwise acyclic fibration. It follows that ϕ is surjective and colim$p$ is an acyclic fibration. The proof of the third claim – including the last claim as a special case – is analogous. □

Definition 3.6. Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal category, and $\mathcal{C}$ a right $\mathcal{V}$-module with action $(C, A) \longmapsto C \oslash A$. Consider, for $f\colon C \longrightarrow D$ a map in $\mathcal{C}$ and $g\colon A \longrightarrow B$ a map in $\mathcal{V}$, the diagram:

$$\begin{tikzcd} C \oslash A \arrow[r, "C\oslash g"] \arrow[d, "f\oslash A"'] & C \oslash B \arrow[d, "f\oslash B"] \\ D \oslash A \arrow[r, "D\oslash f"] & D \oslash B \end{tikzcd}$$

If $\mathcal{C}$ has pushouts, denote the induced map from the pushout of the diagram to the terminal corner by $f\Box g\colon D \oslash A \cup_{C\oslash A} C \oslash B \longrightarrow D \oslash B$. This is the *pushout product map* of f and g.

Note that 3.6 applies to $(\mathcal{V}, \otimes, e)$ considered as a right $\mathcal{V}$-module. Recall the pushout product axiom [15, 3.1].

Definition 3.7. Let $(\mathcal{V}, \otimes, e)$ be a closed symmetric monoidal category and a model category. It is a *monoidal model category* if the pushout product $f\Box g$ of two cofibrations f and g is a cofibration, which is acyclic if either f or g is acyclic.

Monoidal model categories with a cofibrant unit are symmetric monoidal model categories in the sense of [7, 4.2.6]. See [15, 3.3] for the following definition.

Definition 3.8. Let $\mathcal{V}$ be a monoidal model category. For a class $\mathcal{K}$ of maps in $\mathcal{V}$, define $\mathcal{K} \otimes \mathcal{V}$ as the class of maps $f \otimes A$, where f is a map in $\mathcal{K}$ and $A \in \operatorname{Ob} \mathcal{V}$. Let $\operatorname{aCof}(\mathcal{V})$ be the class of acyclic cofibrations in $\mathcal{V}$. The *monoid axiom* holds if every map in $\operatorname{aCof}(\mathcal{V}) \otimes \mathcal{V}$-cell is a weak equivalence.

In the proof of 4.2, we will use the monoid axiom to construct the pointwise model structure on enriched functor categories. Let us end this section with the definition of two other types of model categories.

Definition 3.9. Let $\mathcal{V}$ be a monoidal model category, $\mathcal{C}$ a closed $\mathcal{V}$-module and a model category. The action of $\mathcal{V}$ on $\mathcal{C}$ allows us to consider the pushout product of a map in $\mathcal{C}$ and a map in $\mathcal{V}$. Then $\mathcal{C}$ is a *$\mathcal{V}$-model category* if the pushout product of a cofibration f in $\mathcal{C}$ and a cofibration g in $\mathcal{V}$ is a cofibration in $\mathcal{C}$, which is acyclic if either f or g is acyclic.

Note that a simplicial model category is a **sSet**-model category.

Lemma 3.10. *A simplicial homotopy equivalence in a* **sSet***-model category is a weak equivalence.*

Proof. This follows from [6, 9.5.16]. Simplicial homotopy equivalences in 2.9 are also simplicial homotopy equivalences as defined in [6, 9.5.8]. ☐

Definition 3.11. Let $F\colon \mathcal{V} \longrightarrow \mathcal{W}$ be a strict symmetric monoidal functor of monoidal model categories. If F is a left Quillen functor, $\mathcal{W}$ is called a *monoidal $\mathcal{V}$-model category*.

A monoidal $\mathcal{V}$-model category is clearly a $\mathcal{V}$-model category.

3.2 Homotopy pullback squares

Homotopy pullback squares will be used to characterize fibrations. Definition 3.12 is equivalent to [7, 7.1.12].

Definition 3.12. Let $\mathcal{C}$ be a model category. A commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

in $\mathcal{C}$ is a *homotopy pullback square* if for any commutative diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\sim"'] & D \arrow[d, "\sim"'] & B \arrow[l, "f"'] \arrow[d, "\sim"'] \\ C' \arrow[r] & D' & B' \arrow[l, two heads] \end{tikzcd}$$

where C' and D' are fibrant and $B' \twoheadrightarrow D'$ is a fibration, the canonical map $A \longrightarrow C' \times_{D'} B'$ is a weak equivalence.

This definition seems to by asymmetric, but one of the squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "i"] \\ B \arrow[r, "h"] & D \end{tikzcd}$$

is a homotopy pullback square if and only if the other is a homotopy pullback square. We list some elementary properties.

Lemma 3.13. *All diagrams below are commutative diagrams in $\mathcal{C}$.*

1. *The diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "\sim"] & D \end{tikzcd}$$

is a homotopy pullback square if and only if f is a weak equivalence.

2. *Consider a natural transformation $f: A \longrightarrow B$ of diagrams*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_2 \arrow[r] & A_3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_0 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ B_2 \arrow[r] & B_3 \end{tikzcd}$$

which is a pointwise weak equivalence. That is, $f_i: A_i \longrightarrow B_i$ is a weak equivalence for all $i \in \{0, 1, 2, 3\}$. Then A is a homotopy pullback square if and only if B is a homotopy pullback square.

3. *Let*

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & A_2 \arrow[d] \\ B_0 \arrow[r] & B_1 \arrow[r] & B_2 \end{tikzcd}$$

be a diagram where (2) is a homotopy pullback square. Then the composed square (12) is a homotopy pullback square if and only (1) is a homotopy pullback square.

Lemma 3.14. *Assume $\mathcal{C}$ is right proper. Then*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

is a homotopy pullback square if and only if for some factorization of f as a weak equivalence $B \xrightarrow{\sim} E$ followed by a fibration $E \twoheadrightarrow D$, the induced map $A \longrightarrow C \times_D E$ is a weak equivalence.

Proof. Since $\mathcal{C}$ is right proper, the dual of the gluing lemma holds. The statement follows easily. □

3.3 Fibrant replacement functors

In every model category, any object may be replaced by a fibrant object in a functorial way up to an acyclic cofibration. It is often desirable to explicate fibrant replacement functors. Quillen's small object argument is the classical method 3.3.1. We place emphasis on enriched fibrant replacement functors 3.3.2. Another fibrant replacement functor is constructed in 3.3.3 as a certain filtered colimit.

3.3.1 Classical

Fix a cocomplete category $\mathcal{V}$ and a set $K = \{sk \longrightarrow tk\}_{k \in K}$ of maps in $\mathcal{V}$ with finitely presentable domains and finitely presentable codomains. The set K gives rise to a natural transformation of endofunctors on $\mathcal{V}$, namely

$$\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} sk \xrightarrow{\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} k} \coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} tk.$$

Consider also

$$\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} sk \xrightarrow{\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} f} \coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,-)} \mathrm{Id}_{\mathcal{V}} \xrightarrow{\text{codiagonal}} \mathrm{Id}_{\mathcal{V}},$$

and take the pushout of the two natural transformations. Let $\iota_1^K\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow F_1^K$ denote the canonically induced map. This construction can be iterated. Suppose there is a natural transformation $\iota_n^K\colon F_{n-1}^K \longrightarrow F_n^K$ of endofunctors of $\mathcal{V}$. Next, define F_{n+1}^K as the pushout of

$$\coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,F_n(-))} tk \longleftarrow \coprod_{k\in K}\coprod_{f\in \mathbf{Set}_{\mathcal{V}}(sk,F_n(-))} sk \longrightarrow F_n.$$

The colimit of the sequence $\mathrm{Id}_{\mathcal{V}} = F_0 \xrightarrow{\iota_1^K} F_1^K \xrightarrow{\iota_2^K} \cdots$ is denoted as $F^K\colon \mathcal{V} \longrightarrow \mathcal{V}$, and $\iota^K\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow F^K$ is the canonical natural transformation. The following statement is a special case of [7, 2.1.14].

Lemma 3.15. *For every object A in $\mathcal{V}$, the map $\iota^K(A)\colon A \longrightarrow F^K(A)$ is in K-cell, and the map $F^K(A) \longrightarrow *$ is in K-inj.*

Corollary 3.16. *Suppose that $\mathcal{V}$ is a weakly finitely generated model category. Then $\iota^{J'}\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow F^{J'}$ is a fibrant replacement functor, i.e. $F^{J'}(A)$ is fibrant and the natural map $\iota^{J'}(A)\colon A \longrightarrow F^{J'}(A)$ is an acyclic cofibration for all $A \in \mathrm{Ob}\,\mathcal{V}$.*

This fibrant replacement functor yields "big" objects. We will make this more precise after recalling some definitions from [6, Chapter 10].

Definition 3.17. A relative K-cell complex $f\colon A \longrightarrow B$ is called *finite* if f is a composition $A = B_0 \xrightarrow{f_1} B_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} B_n = B$ where each f_m is a cobase change of a map in K, i.e. f is obtained by attaching finitely many cells from K.

Definition 3.18. A relative K-cell complex $f\colon A \longrightarrow B$ is called *presented* if there is an explicit choice of the data [6, 10.6.3]. In detail, one chooses a limit ordinal λ, a λ-sequence $A = B_0 \xrightarrow{f_1} B_1 \longrightarrow \cdots \xrightarrow{f_{\beta+1}} B_{\beta+1} \longrightarrow \cdots$ whose sequential composition is f, and $f_{\beta+1}\colon B_\beta \longrightarrow B_{\beta+1}$ is gotten from the pushout of the following diagram for every $\beta < \lambda$:

$$\coprod_{m \in M_\beta} ti_m \xleftarrow{\coprod i_m} \coprod_{m \in M_\beta} si_m \longrightarrow B_\beta$$

We omit the choice from the notation. Let $f\colon A \longrightarrow B$ be a presented relative K-cell complex. A *subcomplex* of f is a presented K-cell complex $g\colon A \longrightarrow C$ relative to A such that the explicit choice relevant for g is a subset of the explicit choice for f, see [6, 10.6.7]. In particular, there exists a map $h\colon C \longrightarrow B$ in K-cell such that $h \circ g = f$. A subcomplex is called *finite* if it is a finite K-cell complex relative to A, using the explicit choice.

Any relative K-cell complex can be turned into a presented one. If we consider $\iota^K(A)$ as a presented relative K-cell complex, we will use the explicit choices appearing in its construction.

Lemma 3.19. *Let $f\colon A \longrightarrow B$ be a finite relative K-cell complex. Then f has the structure of a finite subcomplex of $\iota^K(K)\colon A \longrightarrow F^K(A)$.*

Proof. We will prove this by induction on the number of cells. By convention, a map obtained by attaching no cells is an identity map. Suppose the following is true. If $f\colon A \longrightarrow B$ is obtained by attaching n cells from K, with $n \geq 0$, then f has the structure of a finite subcomplex of the presented relative K-cell complex $A \xrightarrow{\iota_1^K(A)} \cdots \xrightarrow{\iota_n^K(A)} F_n^K(A)$. Assume $f\colon A \longrightarrow B$ is obtained by attaching $n+1$ cells from K. By definition, there is a factorization $A \xrightarrow{g} C \xrightarrow{h} B$ where g is obtained by attaching n cells from K and h is the cobase change of some $k \in K$ along a map $\alpha\colon sk \longrightarrow C$. From the induction hypothesis, α induces a map $sk \longrightarrow C \longrightarrow F_nA$, hence an element in $\mathbf{Set}_{\mathcal{V}}(sk, F_nA)$. It follows that there is a map $\beta\colon tk \longrightarrow F_{n+1}A$ rendering the diagram

$$\begin{tikzcd} sk \arrow[r, "\alpha"] \arrow[d, "k"', "\sim"] & C \arrow[r] & F_n^K(A) \arrow[d, "\iota_{n+1}^K(A)"] \\ tk \arrow[rr, "\beta"] & & F_{n+1}^K(A) \end{tikzcd}$$

commutative. Since h is the cobase change of k along α, there is a unique induced map $B \longrightarrow F_{n+1}^K(A)$ which gives f the structure of a subcomplex of the complex $A \xrightarrow{\iota_1^K(A)} \cdots \xrightarrow{\iota_{n+1}^K(A)} F_{n+1}^K(A)$. □

3.3.2 Enriched

Suppose that $\mathcal{V}$ is a cocomplete and closed symmetric monoidal category, and that K is a set of maps with finitely presentable domains and codomains. The fibrant replacement functor defined in 3.3.1 is a priori not a $\mathcal{V}$-functor, but one can remedy this as follows. Each k in K induces a $\mathcal{V}$-natural transformation $\mathcal{V}(sk,-)\otimes k\colon \mathcal{V}(sk,-)\otimes sk \longrightarrow \mathcal{V}(sk,-)\otimes tk$ of endo-$\mathcal{V}$-functors. On the other hand, the counit $\epsilon_{sk}\colon \mathcal{V}(sk,-)\otimes sk \longrightarrow \mathrm{Id}_{\mathcal{V}}$ is also a $\mathcal{V}$-natural transformation. By taking the coproduct over all $k \in K$, one gets the diagram of $\mathcal{V}$-functors

$$\coprod_{k\in K} \mathcal{V}(sk,-)\otimes tk \xleftarrow{\coprod \mathcal{V}(sk,-)\otimes k} \coprod_{k\in K} \mathcal{V}(sk,-)\otimes sk \longrightarrow \mathrm{Id}_{\mathcal{V}}.$$

Denote the pushout by R_1^K. It is clear that one can iterate this construction. Given a $\mathcal{V}$-natural transformation $\rho_n^K\colon R_{n-1}^K \longrightarrow R_n^K$ of endo-$\mathcal{V}$-functors of $\mathcal{V}$, let R_{n+1}^K be the pushout of

$$\coprod_{k\in K} \mathcal{V}(sk,R_n^K(-))\otimes tk \xleftarrow{\coprod_{k\in K} \mathcal{V}(sk,R_n^K(-))\otimes k} \coprod_{k\in K} \mathcal{V}(sk,R_n^K(-))\otimes sk \longrightarrow R_n^K.$$

The colimit of the diagram $\mathrm{Id}_{\mathcal{V}} = R_0 \xrightarrow{\rho_1^K} R_1^K \xrightarrow{\rho_2^K} \cdots$ is called $R^K\colon \mathcal{V} \longrightarrow \mathcal{V}$. Let $\rho^K\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow R^K$ be the canonical $\mathcal{V}$-natural transformation.

Lemma 3.20. *Given any object A in $\mathcal{V}$, the map $\rho^K(A)\colon A \longrightarrow R^K(A)$ is in $K\otimes\mathcal{V}$-cell, and the map $R^K(A) \longrightarrow *$ is in K-inj.*

Proof. The first statement is obvious. To prove that $R^K(A) \longrightarrow *$ is in K-inj, consider a lifting problem for $k \in K$:

$$\begin{tikzcd} sk \arrow[r, "f"] \arrow[d, "k"', "\sim"] & R^K(A) \arrow[d] \\ tk \arrow[r] & * \end{tikzcd}$$

Note that f factors as $sk \xrightarrow{g} R^K_n(A) \longrightarrow R^K(A)$ for some n, since sk is finitely presentable and R is a sequential colimit. The adjoint of g, tensored with sk, induces the map $h\colon sk \longrightarrow \mathcal{V}(sk, R^K_n(A)) \otimes sk$ such that the following diagram commutes.

$$\begin{tikzcd} & \mathcal{V}(sk, R^K_n(A)) \otimes sk \arrow[dr, "\epsilon_{sk}(R^K_n(A))"] & \\ sk \arrow[ur, "h"] \arrow[rr, "g"] & & R^K_n(A) \end{tikzcd}$$

Let h' denote the canonical map to the coproduct given by h. Then the diagram

$$\begin{tikzcd} sk \arrow[r, "h'"] \arrow[d, "k"', "\sim"] & \coprod_{k\in K} \mathcal{V}(sk, R^K_n(A)) \otimes sk \arrow[r] \arrow[d] & R^K_n(A) \arrow[d] \\ tk \arrow[r] & \coprod_{k\in K} \mathcal{V}(sk, R^K_n(A)) \otimes tk \arrow[r] & R^K_{n+1}(A) \end{tikzcd}$$

shows that $tk \longrightarrow \coprod_{k\in K} \mathcal{V}(sk, R^K_n(A)) \longrightarrow R^K_{n+1}(A) \longrightarrow R^K(A)$ solves the lifting problem above. This proves that $R^K(A) \longrightarrow *$ is in K-inj. $\square$

Corollary 3.21. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category satisfying the monoid axiom. Then $\rho^{J'}\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow R^{J'}$ is a fibrant replacement $\mathcal{V}$-functor, i.e. $R^{J'}(A)$ is fibrant and $\rho^{J'}(A)$ is a weak equivalence for all A.*

Proof. It remains to prove that the natural map $\rho^{J'}_A\colon A \longrightarrow R^{J'}(A)$ is a weak equivalence for every $A \in \mathrm{Ob}\,\mathcal{V}$. Note that $\rho^{J'}_A$ is contained in $\mathrm{aCof}(\mathcal{V}) \otimes \mathcal{V}$-cell. The monoid axiom for $\mathcal{V}$ implies that $\rho^{J'}_A$ is a weak equivalence. $\square$

Remark 3.22. The map $\rho^{J'}(A)$ is not a cofibration in general, even if A is cofibrant. But if all objects in $\mathcal{V}$ are cofibrant, then $\rho^{J'}(A)$ is a cofibration.

3.3.3 Filtered

Let $\mathcal{V}$ denote a cocomplete closed symmetric monoidal category, and K a set of maps with finitely presentable domains and codomains. We will define for each object A of $\mathcal{V}$ three categories $\mathrm{ac}^K(A)$, $\mathrm{ac}^K(A,R)$ and $\mathrm{ac}^K(A,F)$, and functors $U_R\colon \mathrm{ac}^K(A,R) \longrightarrow \mathrm{ac}^K(A)$, $U_F\colon \mathrm{ac}^K(A,F) \longrightarrow \mathrm{ac}^K(A)$.

Objects in $\mathrm{ac}^K(A)$ are finite K-cell complexes $\beta\colon A \longrightarrow B$ relative to A. The morphisms in $\mathrm{ac}^K(A)$ from $\beta\colon A \longrightarrow B$ to $\gamma\colon A \longrightarrow C$ are finite K-cell complexes $\tau\colon B \longrightarrow C$ for which $\tau\circ\beta = \gamma$. The identity id_A is the initial object of $\mathrm{ac}^K(A)$. Consider the functor $\Psi_A\colon \mathrm{ac}^K(A) \longrightarrow \mathcal{V}$ which sends $\beta\colon A \longrightarrow B$ to B and $\tau\colon B \longrightarrow C$ to τ. The colimit of Ψ_A will define the desired fibrant replacement of A, up to isomorphism.

Objects in $\mathrm{ac}^K(A,R)$ are pairs $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow R^K(A))$ such that β is a finite K-cell complex relative to A, and $\beta' \circ \beta = \rho^K(A)$ holds. A map from $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow R^K(A))$ to $(\gamma\colon A \overset{\sim}{\rightarrowtail} C, \gamma'\colon C \xrightarrow{\sim} R^K(A))$ is a finite K-cell complex $\tau\colon B \rightarrowtail C$ such that

$$\begin{tikzcd} & B \arrow[dd, "\tau"] \arrow[dr, "\beta'"] & \\ A \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & R^K(A) \\ & C \arrow[ur, "\gamma'"'] & \end{tikzcd}$$

commutes. The initial object of $\mathrm{ac}^K(A,R)$ is (id_A, ρ_A). Denote the forgetful functor $\mathrm{ac}^K(A,R) \longrightarrow \mathrm{ac}^K(A)$ which maps the pair (β,β') to β by U_R.
The category $\mathrm{ac}^K(A,F)$ has objects finite subcomplexes of the relative K-cell complex $\iota^K(A)\colon A \longrightarrow F^K(A)$. Such a subcomplex $\beta\colon A \longrightarrow B$ comes with a map $\beta'\colon B \longrightarrow F^K(A)$ in K-cell such that $\beta' \circ \beta = \iota^K(A)$, whence the objects in $\mathrm{ac}^K(A,F)$ are also denoted (β,β'). Maps are defined as for $\mathrm{ac}^K(A,R)$. Here $(\mathrm{id}_A, \iota(A))$ is the initial object. Let $U_F\colon \mathrm{ac}^K(A,F) \longrightarrow \mathrm{ac}^K(A)$ be the forgetful functor. Recall from [12, IX.3] the notion of a final functor.

Lemma 3.23. *If maps in K-cell are monomorphisms, then U_R and U_F are final functors, and $\mathrm{ac}^K(A,F)$ is a small filtered preorder for all $A \in \mathrm{Ob}\,\mathcal{V}$.*

Proof. To prove that U_R is final, let $\beta\colon A \longrightarrow B$ be an object of $\mathrm{ac}^K(A)$. Since $R^K(A) \longrightarrow *$ is in K-inj, there exists a lift in the following diagram.

$$\begin{tikzcd} A \arrow[r, "\rho^K(A)"] \arrow[d, "\beta"'] & R^K(A) \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

Hence the comma category $\beta \downarrow U_R$ is nonempty. Consider objects (γ,γ') and (δ,δ') in $\mathrm{ac}^K(A,R)$, and maps $\sigma\colon \beta \longrightarrow \gamma$, $\tau\colon \beta \longrightarrow \delta$ in $\mathrm{ac}^K(A)$. The pushout

$$\begin{tikzcd} B \arrow[r, "\tau"] \arrow[d, "\sigma"'] & D \arrow[d] \\ C \arrow[r] & C \cup_B D \end{tikzcd}$$

yields a finite K-cell complex $\alpha\colon A \longrightarrow C \cup_B D$. The maps γ' and δ' induce a map $\alpha'\colon C \cup_B D \longrightarrow R^K(A)$ turning (α,α') into an object of $\mathrm{ac}^K(A,R)$. The cobase changes of σ and τ are maps $(\gamma,\gamma') \longrightarrow (\alpha,\alpha')$ and $(\delta,\delta') \longrightarrow (\alpha,\alpha')$ in $\mathrm{ac}^K(A,R)$. This implies that $\beta \downarrow U_R$ is connected. Hence U_R is final. Next we consider U_F. Let $\beta\colon A \longrightarrow B$ be an object of $\mathrm{ac}^K(A)$. By 3.19, $\beta \downarrow U_F$ is nonempty. Since maps in K-cell are monomorphisms, the union of two finite subcomplexes is again a finite subcomplex [6, 12.2.1]. Connectness follows.
A category is a preorder if there is at most one map between any two objects. Suppose that there exist two different maps $\sigma\colon B \longrightarrow C$ and $\tau\colon B \longrightarrow C$

from $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow F^K(A))$ to $(\gamma\colon A \longrightarrow C, \gamma'\colon C \longrightarrow F^K(A))$. By definition, we have $\gamma' \circ \sigma = \beta' = \gamma' \circ \tau$. Since γ' is a monomorphism by assumption, it follows that $\sigma = \tau$. Note that $\mathrm{ac}^K(A,F)$ is nonempty, since it contains the initial object (id_A, ι_A). Hence to prove that $\mathrm{ac}^K(A,F)$ is filtered, it remains to observe that any two objects have a common upper bound, given by the union of subcomplexes, cf. [6, 12.2.1] ☐

The colimits of Ψ_A, $\Psi_A \circ U_R$ and $\Psi_A \circ U_F$ are isomorphic via the canonical maps $\mathrm{colim}\Psi_A \circ U_R \longrightarrow \mathrm{colim}\Psi_A \longleftarrow \mathrm{colim}\Psi_A \circ U_F$ by Theorem 1 of [12, IX.3]. To have a natural comparison with the $\mathcal{V}$-functor R^K, we let $\Phi^K(A)$ be the colimit of the functor $\Psi_A \circ U_R\colon \mathrm{ac}^K(A,R) \longrightarrow \mathcal{V}$. There is a canonical map $\varphi_A\colon A \longrightarrow \Phi^K A$ induced by the object $(\mathrm{id}_A, \rho^K(A))$ of $\mathrm{ac}^K(A,R)$. Likewise there is a canonical map $\omega_A\colon \Phi^K(A) \longrightarrow R^K(A)$ induced by the maps β'.

Proposition 3.24. *Suppose that relative K-cell complexes are monomorphisms. Then $\Phi^K(A) \longrightarrow *$ is in K-inj, and $\varphi_A\colon A \longrightarrow \Phi^K(A)$ is a filtered colimit of finite relative K-cell complexes. Moreover, the assignment $A \longmapsto \Phi^K(A)$ defines a functor Φ^K, and the maps $A \xrightarrow{\varphi_A} \Phi^K(A)$ and $\Phi^K(A) \xrightarrow{\omega_A} R^K(A)$ are natural transformations $\mathrm{Id}_{\mathcal{V}} \xrightarrow{\varphi} \Phi^K \xrightarrow{\omega} R^K$ such that $\omega \circ \varphi = \rho^K$.*

Proof. The object $\Phi^K(A)$ is isomorphic to $\tilde{A} := \mathrm{colim}\Psi_A \circ U_F$. Hence to prove the first claim, it suffices to consider a lifting problem

$$\begin{tikzcd} sk \arrow[r, "f"] \arrow[d, "k"', "\sim"] & \tilde{A} \arrow[d] \\ tk \arrow[r] & * \end{tikzcd}$$

The object sk is finitely presentable, and $\tilde{A}$ is the colimit of a filtered diagram by 3.23. Hence f factors as $sk \xrightarrow{g} a \longrightarrow \tilde{A}$ for some finite subcomplex $(\beta\colon A \longrightarrow B, \beta'\colon B \longrightarrow F^K(A))$ in $\mathrm{ac}^K(A,F)$. Take the pushout of

$$\begin{tikzcd} sk \arrow[r, "g"] \arrow[d, "k"'] & B \arrow[d, "\tau"] \\ tk \arrow[r, "h"] & C \end{tikzcd}$$

and define $\gamma := \tau \circ \beta\colon A \longrightarrow C$. Since τ is the cobase change of $k \in K$, 3.19 implies that $\gamma\colon A \longrightarrow C$ is a finite subcomplex of ι_A such that $\tau\colon \beta \longrightarrow \gamma$ is a map in $\mathrm{ac}^K(A,F)$. The canonical map $C \longrightarrow \tilde{A}$ belongs to γ, and the map $tk \xrightarrow{h} C \longrightarrow \tilde{A}$ solves the lifting problem. Hence $\tilde{A} \longrightarrow *$ is in K-inj.
To prove the second assertion, let $\mathrm{ac}^K(A,F) \longrightarrow \pi_0\mathrm{ac}^K(A,F)$ be the canonical projection onto the set of connected components of $\mathrm{ac}^K(A,F)$. This functor is final, and the constant diagram $c_A\colon \mathrm{ac}^K(A,F) \longrightarrow \mathcal{V}$ with value A factors it. Since $\mathrm{ac}^K(A,F)$ is connected, it follows that $\mathrm{colim} c_A \cong A$. There

is a natural transformation of diagrams $c_A \longrightarrow \Psi_A \circ U_F$ with value β at (β, β'). The induced map $\operatorname{colim} c_A \longrightarrow \Phi^K(A)$ coincides with the canonical map $\varphi_A \colon A \longrightarrow \Phi^K(A)$ up to the isomorphism above. The proof of the remaining claim is clear and will be left to the reader. □

Corollary 3.25. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category satisfying the monoid axiom. If relative J'-cell complexes are monomorphisms, then*

$$A \xrightarrow{\varphi_A} \Phi^{J'}(A) \xrightarrow{\omega_A} R^{J'}(A)$$

are weak equivalences with fibrant codomains.

If $\mathcal{V}$ is pointed by $*$, any endo-$\mathcal{V}$-functor on $\mathcal{V}$ maps the point to the point. On the other hand, ι_* is not an isomorphism and there may be non-trivial finite subcomplexes of ι_*. So Φ^K is not a $\mathcal{V}$-functor in general. To define the stable model structure we need Φ to be "enriched with respect to spheres". This requires a natural map $T \otimes \Phi(A) \longrightarrow \Phi(T \otimes A)$ for some finitely presentable object T, such that

$$\begin{tikzcd} T \otimes \Phi^K(A) \arrow[r] \arrow[d, "T\otimes\omega_A"'] & \Phi^K(T \otimes A) \arrow[d, "\omega_{T\otimes A}"] \\ T \otimes R^K(A) \arrow[r, "\mathrm{sw}^R_T(A)"] & R^K(T \otimes A) \end{tikzcd}$$

is commutative. Here the lower horizontal map is the adjoint of

$$T \xrightarrow{\eta_A(T)} \mathcal{V}(A, T \otimes A) \xrightarrow{\hom^{R^K}_{A,T\otimes A}} \mathcal{V}(R^K(A), R^K(T \otimes A)).$$

More details can be found in Appendix A. Suppose $T \otimes -$ maps finite K-cell complexes relative to A to finite K-cell complexes relative to $T \otimes A$, for all $A \in \operatorname{Ob}\mathcal{V}$. Then tensoring with T defines a functor $\Theta_A \colon \mathrm{ac}^K(A, R) \longrightarrow \mathrm{ac}^K(T \otimes A, R)$ by sending $(\beta \colon A \longrightarrow B, \beta' \colon B \longrightarrow R^K(A))$ to

$$(T \otimes A \xrightarrow{T\otimes\beta} T \otimes B, T \otimes B \xrightarrow{T\otimes\beta'} T \otimes R^K(A) \xrightarrow{r(T,A)} R^K(T \otimes A)).$$

This functor induces a map

$$\begin{aligned} T \otimes \Phi^K(A) &= T \otimes \operatorname{colim}_{\mathrm{ac}^K(A,R)} \Psi_A \circ U_R \\ &\cong \operatorname{colim}_{\mathrm{ac}^K(A,R)} (T \otimes -) \circ \Psi_A \circ U_R \\ &= \operatorname{colim}_{\mathrm{ac}^K(A,R)} \Psi_{T\otimes A} \circ U_R \circ \Theta_A \\ &\longrightarrow \operatorname{colim}_{\mathrm{ac}(T\otimes A,R)} \Psi_{T\otimes A} \circ U_R \\ &= \Phi^K(T \otimes A) \end{aligned}$$

via the identity natural transformation $(T \otimes -) \circ \Psi_A \circ U_R \longrightarrow \Psi_{T\otimes A} \circ U_R \circ \Theta_A$.

Lemma 3.26. *Let $\mathcal{V}$ be a monoidal model category satisfying the monoid axiom. Suppose that $\mathcal{V}$ is weakly finitely generated, and that relative J'-cell complexes are monomorphisms. If A is an object of $\mathcal{V}$, the map $\theta_A\colon T \otimes \Phi^{J'}(A) \longrightarrow \Phi^{J'}(T \otimes A)$ is a weak equivalence making the diagram*

$$\begin{tikzcd} T \otimes \Phi^{J'}(A) \arrow[r, "\theta_A"] \arrow[d, "T\otimes\omega_A"'] & \Phi^{J'}(T \otimes A) \arrow[d, "\omega_{T\otimes A}"] \\ T \otimes R^{J'}(A) \arrow[r, "r(T,A)"] & R^{J'}(T \otimes A) \end{tikzcd}$$

commutative. Furthermore, θ_A is natural in A.

Proof. Commutativity and naturality follow by the construction of θ_A. Note that $\rho_{T\otimes A}$ and $T\otimes\rho_A^{J'} \in \mathrm{aCof}(\mathcal{V})\otimes\mathcal{V}$-cell are weak equivalences. It follows that $r(T,A)$ is a weak equivalence since $r(T,A) \circ T \otimes \rho_A^{J'} = \rho_{T\otimes A}^{J'}$. The map $\varphi_{T\otimes A}$ is a weak equivalence, and $T \otimes \varphi_A$ is a filtered colimit of weak equivalences. Hence the vertical maps in the diagram are weak equivalences. □

Finally, we relate $\mathcal{V}$-functors and Φ^K in the case where $\mathbb{I}\colon \mathcal{C} \hookrightarrow \mathcal{V}$ is a full sub-$\mathcal{V}$-category and all objects in $\mathcal{C}$ are $\mathcal{V}$-finitely presentable.

Lemma 3.27. *Suppose that relative K-cell complexes are monomorphisms. Let $X\colon \mathcal{C} \longrightarrow \mathcal{V}$ be a $\mathcal{V}$-functor and let $A \in \mathrm{Ob}\,\mathcal{C}$. There is an isomorphism*

$$\mathbb{I}_*X(\Phi^K(A)) \cong \mathrm{colim}(X \circ \Psi_A \circ U_R\colon \mathrm{ac}^K(A,R) \longrightarrow \mathcal{V})$$

which is natural in X.

Proof. To prove this, use the canonical expression of a $\mathcal{V}$-functor as a coend of representables 2.3, 3.23 and $\mathcal{V}$-finiteness of the objects in $\mathcal{C}$. Then

$$\mathbb{I}_*X(\Phi^K(A)) = \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c, \Phi^K(A)) \otimes X(c) \cong \mathrm{colim} \int^{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(c, \Psi_A \circ U_R) \otimes X(c),$$

and the claim follows. □

4 The pointwise model structure

Let $\mathcal{V}$ be a weakly finitely generated monoidal model category. If $\mathcal{C}$ is a small $\mathcal{V}$-category and the monoid axiom holds in $\mathcal{V}$, we introduce the pointwise model structure on $[\mathcal{C}, \mathcal{V}]$. Of particular interest are the cases where $\mathcal{C}$ is a full sub-$\mathcal{V}$-category, and $\mathcal{C}$ satisfies the following properties.

f0 Every object of $\mathcal{V}$ is a filtered colimit of objects in $\mathcal{C}$.

f1 Every object of $\mathcal{C}$ is $\mathcal{V}$-finitely presentable.

f2 The unit e is in $\mathcal{C}$, and $\mathcal{C}$ is closed under the monoidal product in $\mathcal{V}$.

4.1 The general case

Our pointwise notions of weak equivalences, fibrations and cofibrations are as follows.

Definition 4.1. A morphism f in $[\mathcal{C},\mathcal{V}]$ is a

- *pointwise weak equivalence* if $f(c)$ is a weak equivalence in $\mathcal{V}$ for all $c \in \mathrm{Ob}\,\mathcal{C}$,
- *pointwise fibration* if $f(c)$ is a fibration in $\mathcal{V}$ for all $c \in \mathrm{Ob}\,\mathcal{C}$,
- *cofibration* if f has the left lifting property with respect to all pointwise acyclic fibrations.

Theorem 4.2. *Let $\mathcal{V}$ be a weakly finitely generated monoidal model category, and let $\mathcal{C}$ be a small $\mathcal{V}$-category. Suppose the monoid axiom holds in $\mathcal{V}$. Then $[\mathcal{C},\mathcal{V}]$, with the classes of maps in 4.1, is a weakly finitely generated model category.*

Proof. We will use [7, 2.1.19]. The category $[\mathcal{C},\mathcal{V}]$ is bicomplete by 2.2. The class of pointwise weak equivalences is closed under retracts and satisfies the "two out of three" or saturation axiom. Let I be the generating cofibrations in $\mathcal{V}$, and J the generating acyclic cofibrations in $\mathcal{V}$. Let $\mathcal{P}_I$ be the set of maps

$$\{\mathcal{V}_{\mathcal{C}}(c,-)\otimes si \xrightarrow{\mathcal{V}_{\mathcal{C}}(c,-)\otimes i} \mathcal{V}_{\mathcal{C}}(c,-)\otimes ti \mid i \in I, c \in \mathrm{Ob}\,\mathcal{C}\}.$$

Likewise, let $\mathcal{P}_J$ denote the set of maps

$$\{\mathcal{V}_{\mathcal{C}}(c,-)\otimes sj \xrightarrow{\mathcal{V}_{\mathcal{C}}(c,-)\otimes j} \mathcal{V}_{\mathcal{C}}(c,-)\otimes tj \mid j \in J, c \in \mathrm{Ob}\,\mathcal{C}\}.$$

Since $\mathcal{V}$ is cofibrantly generated, it follows from adjointness that $\mathcal{P}_J$-inj coincides with the class of pointwise fibrations, and $\mathcal{P}_I$-inj coincides with the class of pointwise acyclic fibrations. If A is finitely presentable (small) in $\mathcal{V}$, then $\mathcal{V}_{\mathcal{C}}(c,-)\otimes A$ is finitely presentable (small) in $[\mathcal{C},\mathcal{V}]$ for any $c \in \mathrm{Ob}\,\mathcal{C}$, since colimits are computed pointwise according to 2.2. Hence the smallness conditions listed in 3.4 are satisfied. It remains to show that maps in $\mathcal{P}_J$-cell are pointwise weak equivalences. Every map in $\mathcal{P}_J$-cell is pointwise a map in $J\otimes\mathcal{V}$-cell, and the latter class consists of weak equivalences by the monoid axiom. □

We refer to the model structure in 4.2 as the *pointwise* model structure. Note that the evaluation functor preserves fibrations and acyclic fibrations.

Lemma 4.3. *Suppose the pointwise model structure exists. Then the functor from $\mathcal{V}$ to $[\mathcal{C},\mathcal{V}]$ which maps A to $\mathcal{V}_{\mathcal{C}}(c,-)\otimes A$ is a left Quillen functor, with right adjoint Ev_c for all $c \in \mathrm{Ob}\,\mathcal{C}$.*

If the unit in $\mathcal{V}$ is cofibrant, then the representable functors are cofibrant in the pointwise model structure.

Theorem 4.4. *Consider $\mathcal{V}$ and $\mathcal{C}$ as in 4.2. Then the pointwise model structure gives $[\mathcal{C},\mathcal{V}]$ the structure of a $\mathcal{V}$-model category. Likewise, $[\mathcal{C},\mathcal{V}]$ is a monoidal $\mathcal{V}$-model category provided $\mathcal{C}$ is a symmetric monoidal $\mathcal{V}$-category, and the monoid axiom holds.*

Proof. Recall from 2.4 that $[\mathcal{C},\mathcal{V}]$ is a closed $\mathcal{V}$-module. By [7, 4.2.5] it suffices to check the following conditions.

- Let $\mathcal{V}_{\mathcal{C}}(c,-)\otimes i\colon \mathcal{V}_{\mathcal{C}}(c,-)\otimes si \rightarrowtail \mathcal{V}_{\mathcal{C}}(c,-)\otimes ti$ be a map in $\mathcal{P}_I$, and let $j\colon sj \rightarrowtail tj$ be a map in I. Then the pushout product $(\mathcal{V}_{\mathcal{C}}(c,-)\otimes i)\square j$ is a cofibration.

- If either i or j in the above sentence are generating acyclic cofibrations, then $(\mathcal{V}_{\mathcal{C}}(c,-)\otimes i)\square j$ is a pointwise acyclic cofibration.

Since $\mathcal{V}_{\mathcal{C}}(c,-)\otimes -$ is a left adjoint, the pushout product map in question is of the form $\mathcal{V}_{\mathcal{C}}(c,-)\otimes(i\square j)$. Hence the conditions hold, because $\mathcal{V}$ is a monoidal model category and 4.3 holds.
The monoidality statement is proven similarly using [7, 4.2.5]. Note from 2.7 and the compatibility of $\wedge$ and $\otimes$, that the pushout product map of $\mathcal{V}_{\mathcal{C}}(c,-)\otimes i$ and $\mathcal{V}_{\mathcal{C}}(d,-)\otimes j$ is isomorphic to $\mathcal{V}_{\mathcal{C}}(c\diamond d,-)\otimes(i\square j)$ where $\diamond$ denotes the monoidal product in $\mathcal{C}$. Let u be the unit of $\mathcal{C}$. Then $\mathcal{V}_{\mathcal{C}}(u,-)\otimes -$ is a strict symmetric monoidal functor and a left Quillen functor by 4.3.
It remains to prove the monoid axiom. Abbreviate $[\mathcal{C},\mathcal{V}]$ by $\mathcal{F}$. Since $\mathcal{F}$ is cofibrantly generated, it suffices to check that every map in the class $\mathcal{P}_J\wedge\mathcal{F}$-cell is a pointwise weak equivalence. Let $c\in\operatorname{Ob}\mathcal{C}$, $j\in J$ and $X\colon\mathcal{C}\longrightarrow\mathcal{V}$ a $\mathcal{V}$-functor. Then $(\mathcal{V}(c,-)\otimes j)\wedge X$ coincides up to isomorphism with the map $(\mathcal{V}(c,-)\wedge X)\otimes j$. In particular, $((\mathcal{V}(c,-)\wedge X)\otimes j)(d)=(\mathcal{V}(c,-)\wedge X)(d)\otimes j$ is contained in $J\otimes\mathcal{V}$ for every $d\in\operatorname{Ob}\mathcal{C}$, $X\in\operatorname{Ob}\mathcal{F}$. For a map f in $\mathcal{P}_J\otimes\mathcal{F}$-cell, $f(d)$ belongs to $J\otimes\mathcal{V}$-cell because colimits are formed pointwise. Since the monoid axiom holds in $\mathcal{V}$, f is a pointwise weak equivalence. $\square$

Remark 4.5. Via 2.12, the pointwise model structure on T-spectra corresponds to the pointwise model structure on $[T\mathrm{Sph},\mathcal{V}]$.

For a discussion of properness of the pointwise model structure, we introduce the following definition. Let $\mathrm{Cof}(\mathcal{V})$ denote the class of cofibrations in $\mathcal{V}$.

Definition 4.6. A monoidal model category $\mathcal{V}$ is *strongly left proper* if the cobase change of a weak equivalence along any map in $\mathrm{Cof}(\mathcal{V})\otimes\mathcal{V}$-cell is again a weak equivalence.

Strongly left proper monoidal model categories are left proper. If a model category has only cofibrant objects, it is left proper. If a monoidal model category has only cofibrant objects, it is strongly left proper. The relevance of 4.6 is explained by the following lemma.

Lemma 4.7. *Consider $\mathcal{V}$ and $\mathcal{C}$ as in 4.2. If f is a cofibration in $[\mathcal{C},\mathcal{V}]$, then $f(c)$ is a retract of a map in $\mathrm{Cof}(\mathcal{V})\otimes\mathcal{V}$-cell for every $c\in\operatorname{Ob}\mathcal{C}$.*

Proof. Any cofibration in $[\mathcal{C},\mathcal{V}]$ is a retract of a relative $\mathcal{P}_I$-cell complex, and $\mathcal{V}_\mathcal{C}(d,c)\otimes i$ is a map in $I\otimes\mathcal{V}$-cell for every $c\in\mathrm{Ob}\,\mathcal{C}$. The claim follows. □

Corollary 4.8. *The pointwise model structure on $[\mathcal{C},\mathcal{V}]$ is right proper if $\mathcal{V}$ is right proper, and left proper if $\mathcal{V}$ is strongly left proper.*

4.2 The subcategory case

The goal in this section is to develop conditions under which smashing with cofibrant $\mathcal{V}$-functors preserves pointwise weak equivalences. This fact will be used to prove the monoid axiom for the stable model structure. Recall that if $\mathcal{C}$ is a full sub-$\mathcal{V}$-category of $\mathcal{V}$, the left Kan extension along the inclusion functor $\mathbb{I}\colon\mathcal{C}\hookrightarrow\mathcal{V}$ is

$$\mathbb{I}_*X=\int^{\mathrm{Ob}\,\mathcal{C}}\mathcal{V}(c,-)\otimes X(c).$$

Lemma 4.9. *Assume $\mathcal{V}$ is a weakly finitely generated monoidal model category, and $\mathcal{C}$ is a small and full sub-$\mathcal{V}$-category satisfying* **f0** *and* **f1**. *If f is a pointwise weak equivalence in $[\mathcal{C},\mathcal{V}]$, then so is $\mathbb{I}_*f$.*

Proof. We have to check that, for any $A\in\mathrm{Ob}\,\mathcal{V}$, $\mathbb{I}_*f(A)$ is a pointwise weak equivalence. Write A as the colimit of $C\colon\mathcal{I}\longrightarrow\mathcal{C}$ for $\mathcal{I}$ filtering by **f0**. Since coends commute with colimits and **f1** holds, it follows that

$$\begin{aligned}
\mathbb{I}_*f(A) &= \int^{\mathrm{Ob}\,(\mathcal{C})}\mathcal{V}(B,A)\otimes f(B)\\
&\cong \int^{\mathrm{Ob}\,(\mathcal{C})}\mathcal{V}(B,\mathrm{colim}_{i\in\mathcal{I}}C)\otimes f(B)\\
&\cong \mathrm{colim}_{i\in\mathcal{I}}\int^{\mathrm{Ob}\,(\mathcal{C})}\mathcal{V}(B,C(i))\otimes f(B)\\
&\cong \mathrm{colim}_{i\in\mathcal{I}}\ f\circ C.
\end{aligned}$$

Note that, since f is a pointwise weak equivalence, $\mathbb{I}_*f(A)$ is a filtered colimit of weak equivalences and hence a weak equivalence by 3.5. □

Corollary 4.10. *Let $\mathcal{V}$ and $\mathcal{C}$ be as in 4.9. If $\mathcal{C}$ satisfies* **f2** *and f is a pointwise weak equivalence in $[\mathcal{C},\mathcal{V}]$, then $f\wedge\mathcal{V}(c,-)$ is a pointwise weak equivalence for all $c\in\mathrm{Ob}\,\mathcal{C}$.*

Proof. Axiom **f2** implies that the smash product of two enriched functors exists. By 2.8, $f\wedge\mathcal{V}(c,-)$ is isomorphic to $\mathbb{I}_*f\circ\mathcal{V}(c,-)$. Since $\mathbb{I}_*f$ is a pointwise weak equivalence by 4.9, $\mathbb{I}_*f(\mathcal{V}(c,d))$ is a weak equivalence for all $d\in\mathrm{Ob}\,\mathcal{C}$. □

Theorem 4.11. *Let $\mathcal{V}$ and $\mathcal{C}$ be as in 4.10. Assume $\mathcal{V}$ satisfies the monoid axiom and is strongly left proper, and tensoring with the domains and the codomains of the generating cofibrations in $\mathcal{V}$ preserves weak equivalences. Then smashing with a cofibrant object in $[\mathcal{C},\mathcal{V}]$ preserves pointwise weak equivalences.*

Proof. Let $f\colon X \longrightarrow Y$ be a pointwise weak equivalence, and let Z be a cofibrant $\mathcal{V}$-functor. Since Z is cofibrant, it is a retract of some $\mathcal{V}$-functor Z', such that $* \longrightarrow Z'$ is the sequential composition of a sequence

$$* = Z'_0 \xrightarrow{g_0} Z'_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} Z'_n \xrightarrow{g_n} \cdots,$$

where g_n is the cobase change of a coproduct of maps in $\mathcal{P}_I$. It suffices to consider sequences indexed by the natural numbers, since the domains of the maps in $\mathcal{P}_I$ are finitely presentable. The map $f \wedge Z$ is a retract of $f \wedge Z'$, hence it remains to prove that the latter is a pointwise weak equivalence. We will prove this by induction on n. Consider the following diagram.

$$\begin{tikzcd}
X \wedge \coprod_{m\in M} \mathcal{V}(c_m,-)\otimes ti_m \arrow[d, "f\wedge \coprod \mathcal{V}(c_m,-)\otimes ti_m"'] & X \wedge \coprod_{m\in M} \mathcal{V}(c_m,-)\otimes si_m \arrow[l, "X\wedge \coprod \mathcal{V}(c_m,-)\otimes i_m"'] \arrow[r] \arrow[d, "f\wedge \coprod \mathcal{V}(c_m,-)\otimes si_m"'] & X \wedge Z_n \arrow[d, "f\wedge Z_n"] \\
Y \wedge \coprod_{m\in M} \mathcal{V}(c_m,-)\otimes ti_m & Y \wedge \coprod_{m\in M} \mathcal{V}(c_m,-)\otimes si_m \arrow[l, "X\wedge \coprod \mathcal{V}(c_m,-)\otimes i_m"'] \arrow[r] & Y \wedge Z_n.
\end{tikzcd}$$

The map induced on the pushouts of the upper and lower row is $f \wedge Z_{n+1}$. Suppose $f \wedge Z'_n$ is a pointwise weak equivalence. By 4.10 and the hypothesis on I, it follows that both of the other vertical maps are pointwise weak equivalences. The horizontal maps on the left hand side are not necessarily cofibrations. However, evaluation at any object gives maps in $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell. Since $\mathcal{V}$ is strongly left proper, this implies that the map induced on the pushouts – which is computed pointwise – is a pointwise weak equivalence. □

Remark 4.12. Let $\mathcal{V}$ be a weakly finitely generated monoidal model category. Suppose that $-\otimes si$ and $-\otimes ti$ preserve weak equivalences for every $i \in I$, and $\mathcal{V}$ is strongly left proper. Then $-\otimes A$ preserves weak equivalences for any cofibrant object A in $\mathcal{V}$, cp. 4.11. We say a strongly left proper monoidal model category is *strongly monoidal* if $-\otimes A$ preserves weak equivalences for A either cofibrant or a domain or codomain of the generating cofibrations, if they exist. A monoidal model category in which every object is cofibrant satisfies this condition and also the monoid axiom.

5 The homotopy functor model structure

Suppose $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor of categories with chosen subclasses of weak equivalences. If F maps weak equivalences to weak equivalences, then F is called a *homotopy functor*. As a first step towards the stable model structure on enriched functors, we define a model structure in which every enriched functor is weakly equivalent to a homotopy functor.

Let $\mathcal{V}$ be a weakly finitely generated strongly monoidal **sSet**-model category. Additionally, assume the following for $\mathcal{V}$: the monoid axiom holds, the unit is cofibrant, Δ^1 is finitely presentable in $\mathcal{V}$, filtered colimits commute with pullbacks, and cofibrations are monomorphisms. The simplicial structure is used for the simplicial mapping cylinder construction. Let $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ be a small full sub-$\mathcal{V}$-category such that the following axioms hold.

f1 Every object of $\mathbf{f}\mathcal{V}$ is $\mathcal{V}$-finitely presentable.

f2 The unit e is in $\mathbf{f}\mathcal{V}$, and $\mathbf{f}\mathcal{V}$ is closed under the monoidal product.

f3 If $tj \xleftarrow{\sim} sj \longrightarrow v$ is a diagram in $\mathcal{V}$ where $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$ and $j \in J'$, then the pushout $tj \cup_{sj} v$ is in $\mathbf{f}\mathcal{V}$.

Consider also the following additional axioms.

f0 Every object of $\mathcal{V}$ is a filtered colimit of objects in $\mathbf{f}\mathcal{V}$.

f4 All objects in $\mathbf{f}\mathcal{V}$ are cofibrant.

f5 The simplicial mapping cylinder exists in $\mathbf{f}\mathcal{V}$.

Objects in $\mathbf{f}\mathcal{V}$ will usually be denoted by small letters, since $\mathbf{f}\mathcal{V}$ should be thought of as a category of small objects. Let $\mathcal{F}$ be short for $[\mathbf{f}\mathcal{V}, \mathcal{V}]$. Recall the left Kan extension $\mathbb{I}_*X$ of $X \in \operatorname{Ob}\mathcal{F}$ along $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$, cp. 2.5.

5.1 EQUIVALENCES OF HOMOTOPY FUNCTORS

Let $\Phi\colon \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$ denote the functor induced by the fibrant replacement functor $\Phi^{J'}$ from 3.3.3. In general it is not a $\mathcal{V}$-functor. Denote by $\hbar(X)\colon \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$ the composition $\mathbb{I}_*X \circ \Phi$, and by $\hbar\colon \mathcal{F} \longrightarrow \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ the induced functor. There is a natural transformation $X \longrightarrow \hbar(X)$ induced by the canonical maps $\varphi_v\colon v \longrightarrow \Phi(v)$ where v varies through the set of objects in $\mathbf{f}\mathcal{V}$.

LEMMA 5.1. *The functor $\hbar$ commutes with colimits and the action of $\mathcal{V}$. The natural transformations $X \longrightarrow \hbar(X)$ define a natural transformation from the forgetful functor $\mathcal{F} \longrightarrow \operatorname{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ to $\hbar$.*

DEFINITION 5.2. A map f is an *hf-equivalence* if $\hbar(f)(v)$ is a weak equivalence in $\mathcal{V}$ for all $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$.

LEMMA 5.3. *Any pointwise weak equivalences is an hf-equivalence. The class of hf-equivalences is saturated.*

Proof. The first statement follows as in 4.9, since $\hbar(f)(v) = \mathbb{I}_*(f)(\Phi(v))$ is a filtered colimit of weak equivalences provided f is a pointwise weak equivalence. The second statement follows from 5.1 and the analogous fact in $\mathcal{V}$. □

LEMMA 5.4. *Let f be a cofibration in $\mathcal{F}$. Then $\hbar(f)(v)$ is a retract of a map in $\operatorname{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell for every $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$.*

Proof. If $\mathcal{V}(v,-) \otimes i$ is a generating cofibration, then $\hbar(\mathcal{V}(v,w) \otimes i)$ coincides with $\mathcal{V}(v,\Phi(w)) \otimes i$, which is in $\operatorname{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell. The general case follows since $\hbar$ commutes with colimits. □

5.2 Fibrations of homotopy functors

If $\phi\colon v \longrightarrow w$ is an acyclic cofibration in $\mathbf{f}\mathcal{V}$, the simplicial mapping cylinder factors $\mathcal{V}(\phi,-)\colon \mathcal{V}(w,-) \longrightarrow \mathcal{V}(v,-)$ as a cofibration $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ followed by a simplicial homotopy equivalence. This uses that $\mathcal{V}(w,-)$ is a cofibrant functor (since the unit in $\mathcal{V}$ is cofibrant), and that $\mathcal{V}$ (and hence $\mathcal{F}$ by 4.4) is a **sSet**-model category. Take a generating cofibration $i\colon si \longrightarrow ti$ in $\mathcal{V}$ and form the pushout product

$$c_\phi \Box i\colon \mathcal{V}(w,-)\otimes ti \cup_{\mathcal{V}(w,-)\otimes si} C_\phi \otimes si \longrightarrow C_\phi \otimes ti.$$

Let $\mathcal{H}$ denote the set $\{c_\phi \Box i\}$, where ϕ runs through the set of acyclic cofibrations in $\mathbf{f}\mathcal{V}$ and i runs through the set I of generating cofibrations in $\mathcal{V}$.

Definition 5.5. A map is an *hf-fibration* if it is a pointwise fibration having the right lifting property with respect to $\mathcal{H}$.

Lemma 5.6. *Let $f\colon X \longrightarrow Y$ be a pointwise fibration. Then f is an hf-fibration if and only if the following diagram is a homotopy pullback square in $\mathcal{V}$ for every acyclic cofibration $\phi\colon v \overset{\sim}{\rightarrowtail} w$ in $\mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd} X(v) \arrow[r, "X(\phi)"] \arrow[d, "f(v)"'] & X(w) \arrow[d, "f(w)"] \\ Y(v) \arrow[r, "Y(\phi)"] & Y(w) \end{tikzcd}$$

Proof. Let $\mathcal{V}_{\mathcal{F}}(X,Y)$ denote the $\mathcal{V}$-object of maps in $\mathcal{F}$ from X to Y. For a map of $\mathcal{V}$-functors $f\colon X \longrightarrow Y$, the square in the statement of the lemma is naturally isomorphic, by the Yoneda lemma 2.1, to the square

$$\begin{tikzcd} \mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-),X) \arrow[r, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(\phi,-)\text{,}X)"] \arrow[d, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-)\text{,}f)"'] & \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),X) \arrow[d, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-)\text{,}f)"] \\ \mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-),Y) \arrow[r, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(\phi,-)\text{,}Y)"] & \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),Y). \end{tikzcd}$$

The factorization of $\mathcal{V}(\phi,-)$ as a cofibration $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ followed by a simplicial homotopy equivalence $C_\phi \longrightarrow \mathcal{V}(v,-)$ induces a factorization of the square above into two squares. Since $\mathcal{V}_{\mathcal{F}}(-,X)$ preserves simplicial homotopy equivalences by 2.11, which are pointwise weak equivalences by 3.10, the square above is a homotopy pullback square if and only if

$$\begin{tikzcd} \mathcal{V}_{\mathcal{F}}(C_\phi,X) \arrow[r, "\mathcal{V}_{\mathcal{F}}(c_\phi\text{,}X)"] \arrow[d, "\mathcal{V}_{\mathcal{F}}(C_\phi\text{,}f)"'] & \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),X) \arrow[d, "\mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-)\text{,}f)"] \\ \mathcal{V}_{\mathcal{F}}(C_\phi,Y) \arrow[r, "\mathcal{V}_{\mathcal{F}}(c_\phi\text{,}Y)"] & \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),Y) \end{tikzcd}$$

is a homotopy pullback square. If $f: X \longrightarrow Y$ is a pointwise fibration, the induced map $g: \mathcal{V}_{\mathcal{F}}(C_\phi, X) \longrightarrow \mathcal{V}_{\mathcal{F}}(C_\phi, Y) \times_{\mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-),Y)} \mathcal{V}_{\mathcal{F}}(\mathcal{V}(w,-), X)$ is a fibration in $\mathcal{V}$. Here we use **f2**, so $\mathcal{F}$ is a monoidal model category when equipped with the pointwise model structure, see 4.4. The square in 5.6 is therefore a homotopy pullback square if and only if g has the right lifting property with respect to the generating cofibrations in $\mathcal{V}$. By adjointness, this holds if and only if f has the right lifting property with respect to $\mathcal{H}$. $\square$

Lemma 5.7. *Let $f: X \longrightarrow Y$ be an hf-fibration. Then*

$$\begin{tikzcd} X(v) \arrow[r] \arrow[d, "f(v)"'] & \hbar(X)(v) \arrow[d, "\hbar(f)(v)"] \\ Y(v) \arrow[r] & \hbar(Y)(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for every object v of $\mathbf{f}\mathcal{V}$.

Proof. Let $f: X \twoheadrightarrow Y$ be a pointwise fibration of pointwise fibrant functors. From 3.27, $\hbar(f)(v)$ is a filtered colimit of fibrations of fibrant objects in $\mathcal{V}$, and therefore a fibration of fibrant objects by 3.5. This uses properties **f1** and **f3**. The square in the lemma is therefore a homotopy pullback square if and only if $X(v) \longrightarrow Y(v) \times_{\mathrm{colim} Y(a)} \mathrm{colim} X(a)$ is a weak equivalence in $\mathcal{V}$. Up to isomorphism, the colimit is taken over the category of finite subcomplexes $(\alpha: v \overset{\sim}{\rightarrowtail} a, \alpha': a \overset{\sim}{\rightarrowtail} Fv)$ in $\mathrm{Ob}\,\mathrm{ac}^{J'}(v, F)$. Note that the colimit is filtered by 3.23. Filtered colimits commute with pullbacks in $\mathcal{V}$ by assumption, so the map in question is a filtered colimit of maps $X(v) \longrightarrow Y(v) \times_{Y(a)} X(a)$ for acyclic cofibrations $\alpha: v \overset{\sim}{\rightarrowtail} a$ in $\mathbf{f}\mathcal{V}$. If f is an hf-fibration, then by 5.6 the map in question is a filtered colimit of weak equivalences in $\mathcal{V}$, and hence a weak equivalence.

If $f: X \twoheadrightarrow Y$ is any pointwise fibration, use the factorizations in the pointwise model structure to construct a commutative square

$$\begin{tikzcd} X \arrow[r, tail, "\sim", "g"'] \arrow[d, two heads, "f"'] & X' \arrow[d, two heads, "f'"] \\ Y \arrow[r, tail, "\sim", "h"'] & Y' \end{tikzcd}$$

for f' a pointwise fibration of pointwise fibrant functors, and g and f pointwise acyclic cofibrations. Note that f is an hf-fibration if and only f' is, see 3.13 and 5.6. The maps g and h are pointwise weak equivalences, hence $\hbar(g)(v)$ and $\hbar(h)(v)$ are weak equivalences in $\mathcal{V}$ for every v. The square in question is therefore a homotopy pullback square by the previous case. $\square$

Corollary 5.8. *A map is an hf-fibration and an hf-equivalence if and only if it is a pointwise acyclic fibration.*

Proof. If f is an hf-fibration, it is a pointwise fibration by definition. If f is also an hf-equivalence, it is a pointwise weak equivalence by 3.13 and 5.7. A pointwise acyclic fibration is an hf-fibration by 3.13 and 5.6. $\square$

5.3 The homotopy functor theorem

Before we prove the existence of the homotopy functor model structure, let us first consider the maps in $\mathcal{H}$-cell.

Lemma 5.9. *The maps in $\mathcal{H}$-cell are hf-equivalences.*

Proof. Let $\phi\colon v \overset{\sim}{\rightarrowtail} w$ be an acyclic cofibration in $\mathbf{f}\mathcal{V}$. Then the induced map $\mathcal{V}(\phi,-)\colon \mathcal{V}(w,-) \longrightarrow \mathcal{V}(v,-)$ is an hf-equivalence, because $\hbar(\mathcal{V}(\phi,-))(v)$ is naturally isomorphic to $\mathcal{V}(\phi,\Phi(v))$, ϕ is an acyclic cofibration and $\Phi(v)$ is fibrant. Pointwise weak equivalences are hf-equivalences by 5.3, so the map $c_\phi\colon \mathcal{V}(w,-) \rightarrowtail C_\phi$ is an hf-equivalence.
Let $i\colon si \rightarrowtail ti$ be a generating cofibration in $\mathcal{V}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{V}(w,-)\otimes si \arrow[r, "c_\phi\otimes si"] \arrow[d, "\mathcal{V}(w,-)\otimes i"'] & C_\phi \otimes si \arrow[d, "C_\phi\otimes i"] \\
\mathcal{V}(w,-)\otimes ti \arrow[r, "c_\phi\otimes ti"] & C_\phi \otimes ti
\end{tikzcd}$$

and the pushout product map $c_\phi \square i$. The functor $\hbar$ commutes with pushouts and the action of $\mathcal{V}$ by 5.1, so $\hbar(c_\phi\square i)(u)$ is the pushout product map obtained from $\hbar(c_\phi)(u)$ and i. Since $\mathcal{V}$ is strongly monoidal, it follows that $\hbar(c_\phi)(u)\square i$ is a weak equivalence. Hence the maps in $\mathcal{H}$ are hf-equivalences. The general case of a map in $\mathcal{H}$-cell follows using similar arguments and 5.4. $\square$

Theorem 5.10. *Let $\mathcal{V}$ be a weakly finitely generated monoidal* **sSet***-model category, and let* $\mathbf{f}\mathcal{V}$ *be a full sub-$\mathcal{V}$-category satisfying* **f1**, **f2** *and* **f3**. *Suppose the monoid axiom holds in $\mathcal{V}$, pullbacks commute with filtered colimits in $\mathcal{V}$, and Δ^1 is finitely presentable in $\mathcal{V}$. Suppose also that $\mathcal{V}$ is strongly monoidal, and that cofibrations in $\mathcal{V}$ are monomorphisms. Then $\mathcal{F}$ is a weakly finitely generated model category, with hf-equivalences as weak equivalences, hf-fibrations as fibrations, and cofibrations as cofibrations.*

Proof. Again we use [7, 2.1.19]. The set of generating cofibrations is $\mathcal{P}_I$, and the set of generating acyclic cofibrations is the union $\mathcal{P}_J\cup\mathcal{H}$. It is clear that the class of hf-equivalences is saturated and closed under retracts. It is also clear that the domains of the maps in $\mathcal{H}$ are finitely presentable, because finitely presentable objects are closed under pushouts and tensoring with finitely presentable objects. Here we use that Δ^1 is finitely presentable in $\mathcal{V}$. The other properties which have to be checked are either obvious or follow from 5.8, 5.9 and the corresponding fact for the pointwise model structure 4.2. $\square$

The model structure in 5.10 is called the *homotopy functor* model structure. To emphasize the model structure, we use the notation $\mathcal{F}_{\mathrm{hf}}$. An hf-equivalence is denoted by $\xrightarrow{\sim \mathrm{hf}}$ and an hf-fibration by $\overset{\mathrm{hf}}{\twoheadrightarrow}$. Likewise, we use the notations $\mathcal{F}_{\mathrm{pt}}$, $\xrightarrow{\sim \mathrm{pt}}$ and $\overset{\mathrm{pt}}{\twoheadrightarrow}$ for the pointwise model structure.

Lemma 5.11. *The identity induces a left Quillen functor* $\mathrm{Id}_{\mathcal{F}}\colon \mathcal{F}_{\mathrm{pt}} \longrightarrow \mathcal{F}_{\mathrm{hf}}$.

Lemma 5.12. *Let $\mathcal{V}$ and $\mathbf{f}\mathcal{V}$ be as in 5.10, and assume $\mathbf{f}\mathcal{V}$ satisfies* **f4**. *Then $\mathcal{F}_{\mathrm{hf}}$ is a monoidal $\mathcal{F}_{\mathrm{pt}}$-model category.*

Proof. Condition **f2** is used to construct the smash product on $\mathcal{F}$, and 4.4 holds. To prove that the homotopy functor model structure is monoidal, it suffices to show that the pushout product map of a map $c_\phi \square i$ in $\mathcal{H}$ (where $\phi\colon v \overset{\sim}{\rightarrowtail} w$) and a generating cofibration $\mathcal{V}(u,-)\otimes j$ is an hf-equivalence. It is straightforward to check that this pushout product map coincides with the pushout product map $c_{\phi\otimes u}\square f$ where f is the pushout product map in $\mathcal{V}$ of i and j. Since $\mathcal{V}$ is a monoidal model category and $\mathbf{f}\mathcal{V}$ satisfies **f2** and **f4**, $\phi\otimes u$ is an acyclic cofibration in $\mathbf{f}\mathcal{V}$. Hence the map in question is an hf-equivalence. $\square$

Lemma 5.13. *The homotopy functor model structure is left proper. If $\mathcal{V}$ is right proper, then the homotopy functor model structure is right proper.*

Proof. For left properness, let $i\colon Y \longrightarrow Z\cup_X Y$ be the cobase change of an hf-equivalence $g\colon X \xrightarrow{\sim \mathrm{hf}} Z$ along a cofibration $f\colon X \rightarrowtail Y$. Factor g as a cofibration $h\colon X \rightarrowtail T$, followed by a pointwise acyclic fibration $p\colon T \overset{\sim}{\twoheadrightarrow} Z$. Then g is an hf-equivalence, hence an acyclic cofibration in the homotopy functor model structure. These maps are closed under cobase changes, so i factors as an acyclic cofibration, followed by the cobase change of p. The latter is a pointwise weak equivalence, since the cobase change of f along g is a cofibration and the pointwise model structure is left proper by 4.8 provided $\mathcal{V}$ is strongly left proper.

A slightly stronger property than right properness holds. Consider the maps $f\colon X \overset{\mathrm{pt}}{\twoheadrightarrow} Z$ and $g\colon Y \xrightarrow{\sim \mathrm{hf}} Z$. We claim the base change i of g along f is an hf-equivalence. Let us shorten the notation by setting $R = R^{J'}$. To prove that i is an hf-equivalence, factor $Rf\colon RX \longrightarrow RZ$ as $h\colon RX \overset{\sim \mathrm{pt}}{\rightarrowtail} T$ followed by $p\colon T \overset{\mathrm{pt}}{\twoheadrightarrow} RZ$. Then $\hbar(p)(v)$ is a fibration of fibrant objects for any v by 3.5. Moreover, $\hbar(h)(v)$ and $\hbar(\rho_{Z(v)})\colon \hbar(Z)(v) \longrightarrow \hbar(RZ)(v)$ are weak equivalences for any v. Hence the base change of the weak equivalence $\hbar(\rho_{Z(v)})\circ\hbar(g)(v)$ along $\hbar(b)(v)$ is a weak equivalence, using that $\mathcal{V}$ is right proper. Note that the base change map factors as $\hbar(h)(v)$ composed with the base change of $\hbar(g)(v)$ along $\hbar(f)(v)$, i.e. $\hbar(i)(v)$ since pullbacks commute with filtered colimits. It follows that i is an hf-equivalence. $\square$

5.4 Homotopy functors

We end this section with a discussion of homotopy functors. If **f4** and **f5** hold, then a fibrant $\mathcal{V}$-functor X in $\mathcal{F}_{\mathrm{hf}}$ is a homotopy functor for the following reasons: By 5.6, X maps acyclic cofibrations to weak equivalences. Using **f4** and **f5**, every weak equivalence in $\mathbf{f}\mathcal{V}$ can be factored as an acyclic cofibration in $\mathbf{f}\mathcal{V}$, followed by a simplicial homotopy equivalence in $\mathbf{f}\mathcal{V}$. It follows from 2.11 that X preserves arbitrary weak equivalences in $\mathbf{f}\mathcal{V}$. Conversely, any $\mathcal{V}$-functor which is pointwise fibrant and a homotopy functor is fibrant in $\mathcal{F}_{\mathrm{hf}}$. Therefore we regard the expressions "pointwise fibrant homotopy functor" and "fibrant in $\mathcal{F}_{\mathrm{hf}}$" as synonymous. Next we define a fibrant replacement functor in $\mathcal{F}_{\mathrm{hf}}$ which allows to replace our definition of hf-equivalence by a better one.

Definition 5.14. For $X \in \mathrm{Ob}\,\mathcal{F}$, define X^h as the composition $\mathbb{I}_* X \circ R$, where $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ is the inclusion and $R := R^{J'}$ is the fibrant replacement $\mathcal{V}$-functor constructed in 3.3.2.

For pointed simplicial sets, Lydakis [11, 8.6] uses the singular complex applied to the geometric realization as an enriched fibrant replacement functor. This functor preserves fibrations, weak equivalences and finite limits.

Lemma 5.15. *The map $X \longrightarrow X^h$ is a $\mathcal{V}$-natural transformation of $\mathcal{V}$-functors, and extends to a natural transformation $\mathrm{Id}_{\mathcal{F}} \longrightarrow (-)^h$. The functor $(-)^h$ commutes with colimits and the action of $\mathcal{V}$.*

Proof. The first two statements follow from 3.25 and properties of enriched Kan extension. For the last statement, use that coends commute with colimits and the action of $\mathcal{V}$, which are pointwise constructions. □

Lemma 5.16. *Assume* **f4** *and let $X \in \mathrm{Ob}\,\mathcal{F}$ be cofibrant. For every object v in $\mathbf{f}\mathcal{V}$, the weak equivalence $\omega_v\colon \Phi(v) \xrightarrow{\sim} R(v)$ induces a weak equivalence $\hbar(X)(v) \longrightarrow X^h(v)$.*

Proof. Since $\mathcal{V}$ is monoidal, **f4** implies that $\mathcal{V}(w, \Phi v) \longrightarrow \mathcal{V}(w, Rv)$ induced by the canonical weak equivalence $\omega_v\colon \Phi v \xrightarrow{\sim} R(v)$ is a weak equivalence for every $w \in \mathbf{f}\mathcal{V}$. Now express X as a retract of a $\mathcal{P}_I$-cell complex. The lemma follows then by induction, because $\mathcal{V}$ is strongly monoidal. □

Corollary 5.17. *Assume* **f4** *holds and X is cofibrant. Then $R \circ X^h$ is a pointwise fibrant homotopy functor and $X \longrightarrow R \circ X^h$ is an hf-equivalence. A map f of cofibrant functors is an hf-equivalence if and only if f^h is a pointwise weak equivalence.*

Proof. The second claim follows directly from 5.16, while the first claim requires just a slight variation of the proof of 5.16. □

Lemma 5.18. *Suppose $\mathcal{C}$ satisfies* **f0** *and* **f4**. *For any $X \in \mathrm{Ob}\,\mathcal{V}$, the canonical map $\hbar(X)(v) \longrightarrow X^h(v)$ induced by the weak equivalence $\omega_v\colon \Phi(v) \xrightarrow{\sim} R(v)$ is then a weak equivalence for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Note that 4.9 holds. Hence, by a cofibrant replacement in $\mathcal{F}_{\mathrm{pt}}$, it suffices to consider cofibrant functors. This case follows from 5.16. □

Corollary 5.19. *Assume* **f0** *and* **f4** *hold. A map f is then an hf-equivalence if and only if f^h is a pointwise weak equivalence. Furthermore, for any $X \in \operatorname{Ob}\mathcal{F}$, the functor $R \circ X^h$ is a pointwise fibrant homotopy functor and the natural map $X \longrightarrow R \circ X^h$ is an hf-equivalence.*

Proof. As in 5.17, using 4.9. □

Remark 5.20. If $\rho\colon \mathrm{Id}_{\mathcal{V}} \longrightarrow R$ has the property that its restriction to $\mathbf{f}\mathcal{V}$ takes values in cofibrant objects, then X^h is always a homotopy functor. The reason is that in a **sSet**-model category, any weak equivalence of fibrant and cofibrant objects is a simplicial homotopy equivalence, and $\mathcal{V}$-functors preserve them.

6 The stable model structure

We will construct the stable model structure on the category $\mathcal{F} = [\mathbf{f}\mathcal{V}, \mathcal{V}]$ with respect to some cofibrant object T of $\mathbf{f}\mathcal{V}$. For this, assume $\mathcal{V}$ and $\mathbf{f}\mathcal{V}$ are as in 5.10. In addition, $\mathcal{V}$ has to be right proper and cellular. We also require that $\mathbf{f}\mathcal{V}$ satisfies **f4**, in order to have a well-behaved fibrant replacement in $\mathcal{F}_{\mathrm{hf}}$. Finally, we assume the adjoint pair $(- \otimes T, \mathcal{V}(T, -))$ is a Quillen equivalence on the stable model structure on spectra described in 6.16. Since T is contained in $\mathbf{f}\mathcal{V}$ and **f2** holds, the canonical functor $\pi\colon T\mathrm{Sph} \longrightarrow \mathcal{V}$ factors over the inclusion as $i\colon T\mathrm{Sph} \longrightarrow \mathbf{f}\mathcal{V}$. Let (i_*, ev) denote the corresponding adjoint pair of functors.

6.1 Stable equivalences

We start by describing the stabilization process. For every object v in $\mathbf{f}\mathcal{V}$, the composition of the counit $\epsilon_T\mathcal{V}(v,-)\colon \mathcal{V}(T, \mathcal{V}(v,-)) \otimes T \longrightarrow \mathcal{V}(v,-)$ and the natural isomorphism $\mathcal{V}(T, \mathcal{V}(v,-)) \cong \mathcal{V}(T \otimes v, -)$ define a morphism $\tau_v\colon \mathcal{V}(T \otimes v, -) \otimes T \longrightarrow \mathcal{V}(v,-)$ which is natural in v. If X is a $\mathcal{V}$-functor, then the induced map $\mathcal{V}_{\mathcal{F}}(\tau_v, X)\colon \mathcal{V}_{\mathcal{F}}(\mathcal{V}(v,-), X) \longrightarrow \mathcal{V}_{\mathcal{F}}(\mathcal{V}(T \otimes v, -) \otimes T, X)$ is natural in v and X. Using the enriched Yoneda lemma 2.1, one obtains a map $t_X(v)\colon X(v) \longrightarrow \mathcal{V}(T, X(T \otimes v))$. Let $\mathrm{Sh}\colon \mathcal{F} \longrightarrow \mathcal{F}$ denote the 'shift' functor obtained by pre-composing with the $\mathcal{V}$-functor $T \otimes -\colon \mathbf{f}\mathcal{V} \longrightarrow \mathbf{f}\mathcal{V}$. Define $\mathbb{T}\colon \mathcal{F} \longrightarrow \mathcal{F}$ to be the composition $\mathcal{V}(T,-) \circ \mathrm{Sh}$, so that $\mathbb{T}(X)(v) = \mathcal{V}(T, X(T \otimes v))$. The collection of the maps $t_X(v)$ is a $\mathcal{V}$-natural transformation $t_X\colon X \longrightarrow \mathbb{T}(X)$. Let $\mathbb{T}^\infty(X)$ denote the colimit of the sequence $X \xrightarrow{t(X)} \mathbb{T}(X) \xrightarrow{\mathbb{T}(t(X))} \mathbb{T}(\mathbb{T}(X)) \longrightarrow \cdots$. The canonical map $t^\infty_X\colon X \longrightarrow \mathbb{T}^\infty(X)$ yields a natural transformation $t^\infty\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow \mathbb{T}^\infty$.
The definition of stable weak equivalences uses the fibrant replacement functor $\Phi^{J'}$ considered in 3.3.3. Let Φ be short notation for $\Phi^{J'}$, and similarly for the other fibrant replacement functors R and F. Recall that $\hbar(X)$ is not necessarily

a $\mathcal{V}$-functor for every $X \in \mathrm{Ob}\,\mathcal{F}$. But it can be stabilized, since there are natural weak equivalences $\theta_v\colon T \otimes \Phi(v) \longrightarrow \Phi(T \otimes v)$ according to 3.26. Let $\mathbb{T}'\colon \mathrm{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V}) \longrightarrow \mathrm{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ be the functor that maps X to the composition $\mathcal{V}(T, -) \circ X \circ (T \otimes -)\colon \mathbf{f}\mathcal{V} \longrightarrow \mathcal{V}$. Define the map $t'_{\hbar(X)}\colon \hbar(X) \longrightarrow \mathbb{T}'(\hbar(X))$ pointwise as the adjoint of

$$\mathbb{I}_* X(\Phi(v)) \otimes T \longrightarrow \mathbb{I}_* X(T \otimes \Phi(v)) \longrightarrow \mathbb{I}_* X(\Phi(T \otimes v)).$$

The map on the left hand side in this composition is adjoint to the composition $T \xrightarrow{\eta_{\Phi(v)}(T)} \mathcal{V}(\Phi(v), T \otimes \Phi(v)) \xrightarrow{\hom^{\mathbb{I}_* X}_{\Phi(v), T\otimes\Phi(v)}} \mathcal{V}(\mathbb{I}_* X(\Phi(v)), \mathbb{I}_* X(T \otimes \Phi(v)))$, and the map on the right hand side is $\mathbb{I}_* X(\theta_v)$

Lemma 6.1. *There is a natural transformation $t'_\hbar\colon \hbar \longrightarrow \mathbb{T}' \circ \hbar$. The natural transformation $u\colon U \longrightarrow \hbar$, where $U\colon \mathcal{F} \longrightarrow \mathrm{Fun}(\mathbf{f}\mathcal{V}, \mathcal{V})$ is the forgetful functor, makes the following diagram commutative.*

$$\begin{tikzcd} U \circ \mathrm{Id}_{\mathcal{F}} \arrow[r, "U\circ t"] \arrow[d, "u"'] & U \circ T = \mathbb{T}' \circ U \arrow[d, "\mathbb{T}'\circ u"] \\ \hbar \arrow[r, "t'_\hbar"] & \mathbb{T}' \circ \hbar. \end{tikzcd}$$

Proof. The claim follows since $t_X(v)\colon X(v) \longrightarrow \mathbb{T}(X)(v)$ can be defined as the adjoint (under tensoring with T) of the adjoint (under tensoring with $X(v)$) of $\hom^X_{v, T\otimes v} \circ \eta_v(T)\colon T \longrightarrow \mathcal{V}(v, T \otimes v) \longrightarrow \mathcal{V}(X(v), X(T \otimes v))$, cp. A.8. $\square$

Denote the colimit of $\hbar(X) \xrightarrow{t'_{\hbar(X)}} \mathbb{T}'(\hbar(X)) \xrightarrow{\mathbb{T}'(t'_{\hbar(X)})} \mathbb{T}'(\mathbb{T}'(\hbar(X))) \longrightarrow \cdots$ by $\mathbb{T}'^{\infty}(\hbar(X))$, and let $t'^{\infty}_{\hbar(X)}$ be the canonical map $\hbar(X) \longrightarrow \mathbb{T}'^{\infty}(\hbar(X))$.

Definition 6.2. A map f in $\mathcal{F}$ is a *stable equivalence* if $\mathbb{T}'^{\infty}(\hbar(Rf))(v)$ is a weak equivalence in $\mathcal{V}$ for every object v of $\mathbf{f}\mathcal{V}$.

Lemma 6.3. *Every hf-equivalence is a stable equivalence. The class of stable equivalences is saturated.*

There are canonical maps $X(v) \longrightarrow RX(v) \longrightarrow \hbar(RX)(v)$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$. Consider the induced map $\mathbb{T}^{\infty}(X) \longrightarrow \mathbb{T}'^{\infty}(\hbar(RX))$. The latter is sometimes a pointwise weak equivalence.

Lemma 6.4. *Assume $X \in \mathrm{Ob}\,\mathcal{F}$ is a pointwise fibrant homotopy functor. Then $\mathbb{T}^{\infty}(X)(v) \longrightarrow \mathbb{T}'^{\infty}(\hbar(RX))(v)$ is a weak equivalence in $\mathcal{V}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Note that $X(v) \longrightarrow \hbar(RX)(v)$ is a weak equivalence of fibrant objects. $\mathcal{V}(T, -)$ preserves weak equivalences of fibrant objects since T is cofibrant and $\mathcal{V}$ is a monoidal model category. The map is question is hence a sequential colimit of weak equivalences, so 3.5 concludes the proof. $\square$

Corollary 6.5. *A map f between pointwise fibrant homotopy functors is a stable equivalence if and only if $\mathbb{T}^{\infty}(f)$ is a pointwise weak equivalence.*

6.2 Stable fibrations

Let $\tau_v\colon \mathcal{V}(T \otimes v, -) \otimes T \longrightarrow \mathcal{V}(v, -)$ be the canonical map of $\mathcal{V}$-functors described in section 6.1. The simplicial mapping cylinder factors τ_v as a cofibration $d_v\colon \mathcal{V}(T \otimes v, -) \otimes T \rightarrowtail D_v$ followed by a simplicial homotopy equivalence. Take a generating cofibration $i\colon si \rightarrowtail ti \in I$ in $\mathcal{V}$ and form the pushout product $d_v \square i$. The set $\mathcal{D}$ of generating acyclic cofibrations for the class of stable equivalences is $\{d_v \square i\}$, where $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$ and $i \in I$.

Definition 6.6. A map is called a *stable fibration* if it is an hf-fibration having the right lifting property with respect to the set $\mathcal{D}$.

Lemma 6.7. *An hf-fibration $f\colon X \longrightarrow Y$ is a stable fibration if and only if*

$$\begin{tikzcd} X(v) \arrow[r, "t_X(v)"] \arrow[d, "f(v)"'] & \mathbb{T}(X)(v) \arrow[d, "\mathbb{T}(f)(v)"] \\ Y(v) \arrow[r, "t_Y(v)"] & \mathbb{T}(Y)(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for every object v of $\mathbf{f}\mathcal{V}$.

Proof. The proof is formally the same as for 5.6. $\square$

The rest of this section is devoted to prove that a stable fibration which is also a stable equivalence is a pointwise weak equivalence.

Lemma 6.8. *Assume $\mathcal{V}$ is right proper, and that filtered colimits commute with pullbacks in $\mathcal{V}$. Let $f\colon X \longrightarrow Y$ be a stable fibration. Then*

$$\begin{tikzcd} \hbar(RX)(v) \arrow[r, "t'_{\hbar(RX)}(v)"] \arrow[d, "\hbar(Rf)(v)"'] & \mathbb{T}'(\hbar(RX))(v) \arrow[d, "\mathbb{T}'(\hbar(Rf))(v)"] \\ \hbar(RY)(v) \arrow[r, "t'_{\hbar(RY)}(v)"] & \mathbb{T}'(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$.

The proof of 6.8 uses 6.10, 6.11 and 6.12. We start with a general fact about model categories.

Lemma 6.9. *Let $G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between right proper model categories which preserves pullbacks, fibrations and acyclic fibrations. Suppose*

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, "f"'] & B \arrow[d] \\ C \arrow[r, "\sim"] & D \end{tikzcd}$$

is a commutative diagram in $\mathcal{C}$, such that the horizontal maps are weak equivalences with fibrant targets, and such that $f\colon A \twoheadrightarrow C$ is a fibration. Then the image of this square under G is a homotopy pullback square.

Proof. By Ken Brown's lemma, we can assume that the horizontal maps are in fact acyclic cofibrations. Factor the composition $A \longrightarrow D$ as an acyclic cofibration $i: A \overset{\sim}{\rightarrowtail} E$ followed by a fibration $p: E \twoheadrightarrow D$. Then the map $B \longrightarrow D$ factors as $B \xrightarrow{\sim} E \overset{p}{\twoheadrightarrow} D$, by choosing a lift in the diagram

$$\begin{tikzcd} A \arrow[r, tail, "\sim", "i"'] \arrow[d, "\sim"'] & E \arrow[d, two heads, "p"] \\ B \arrow[r] & D. \end{tikzcd}$$

Define P to be the pullback of $C \xrightarrow{\sim} D \overset{p}{\twoheadleftarrow} E$ and call the map induced by f and i $h: A \longrightarrow P$. Since $C \xrightarrow{\sim} D$ and i are weak equivalences and $\mathcal{C}$ is right proper, h is a weak equivalence. Using the assumptions on G and right properness of $\mathcal{D}$ (3.14), we have to prove that $G(h)$ is a weak equivalence.

Let Q be the pullback of $A \overset{f}{\twoheadrightarrow} C \overset{p'}{\twoheadleftarrow} P$, where p' is the base change of p. The maps id_A and h induce a map $A \longrightarrow Q$ which can be factored as $A \underset{\sim}{\overset{s}{\rightarrowtail}} W \overset{q}{\twoheadrightarrow} Q$. After all, we have a factorization of h as $A \underset{\sim}{\overset{s}{\rightarrowtail}} W \overset{f' \circ q}{\twoheadrightarrow} P$, where f' is the base change of the fibration f. The map h is a weak equivalence, thus $f' \circ q$ is an acyclic fibration. In particular, $G(f' \circ q)$ is an acyclic fibration. The map $p'' \circ q: W \longrightarrow A$ is a fibration (where p'' is the base change of the fibration p') and has $s: A \overset{\sim}{\rightarrowtail} W$ as a section. Hence $p'' \circ q$ is an acyclic fibration, and so is $G(p'' \circ q)$. Now $G(s)$ is a section of $G(p'' \circ q)$, implying that $G(s)$, and therefore $G(h = (f' \circ q) \circ s)$, is a weak equivalence. □

Corollary 6.10. *Let $f: X \longrightarrow Y$ be a pointwise fibration, and assume $\mathcal{V}$ is right proper. Then the following is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd} \mathcal{V}(T, X(v)) \arrow[r, "\mathcal{V}(T,\rho_{X(v)})"] \arrow[d, "\mathcal{V}(T,f(v))"'] & \mathcal{V}(T, RX(v)) \arrow[d, "\mathcal{V}(T,Rf(v))"] \\ \mathcal{V}(T, Y(v)) \arrow[r, "\mathcal{V}(T,\rho_{Y(v)})"] & \mathcal{V}(T, RY(v)) \end{tikzcd}$$

Proof. Follows from 6.9, since $\mathcal{V}(T, -): \mathcal{V} \longrightarrow \mathcal{V}$ is a right Quillen functor. □

Corollary 6.11. *Suppose $\mathcal{V}$ is right proper and $f: X \longrightarrow Y$ is a stable fibration. Then the following is a homotopy pullback square for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

$$\begin{tikzcd} RX(v) \arrow[r, "t_{RX(v)}"] \arrow[d, "Rf(v)"'] & \mathbb{T}(RX)(v) \arrow[d, "\mathbb{T}(Rf)(v)"] \\ RY(v) \arrow[r, "t_{RY(v)}"] & \mathbb{T}(RY)(v) \end{tikzcd}$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
X(v) \arrow[rr, "t_X(v)"] \arrow[dr, "\rho_{X(v)}"'] \arrow[dd, "f(v)"'] & & \mathbb{T}(X)(v) \arrow[dr, "\mathbb{T}(\rho_{X(v)})"] \arrow[dd, "\mathbb{T}(f)(v)"' near start] & \\
& RX(v) \arrow[rr, "t_{RX}(v)" near start] \arrow[dd, "Rf(v)"' near start] & & \mathbb{T}(RX)(v) \arrow[dd, "\mathbb{T}(Rf)(v)"] \\
Y(v) \arrow[rr, "t_Y(v)" near end] \arrow[dr, "\rho_{Y(v)}"'] & & \mathbb{T}(Y)(v) \arrow[dr, "\mathbb{T}(\rho_{Y(v)})"] & \\
& RY(v) \arrow[rr, "t_{RY}(v)"] & & \mathbb{T}(RY)(v)
\end{tikzcd}$$

The right hand square of the cube is a homotopy pullback square by 6.10. Likewise for the square in the back using the assumption on f. In the left hand square the horizontal maps are weak equivalences. Hence the square in question is a homotopy pullback square by 3.13. $\square$

Lemma 6.12. *Assume $f\colon X \longrightarrow Y$ is a stable fibration of pointwise fibrant functors, and filtered colimits commute with pullbacks in $\mathcal{V}$. Then*

$$\begin{tikzcd}
\hbar(X)(v) \arrow[r, "t'_{\hbar(X)}(v)"] \arrow[d, "\hbar(f)(v)"'] & \mathbb{T}'(\hbar(X))(v) \arrow[d, "\mathbb{T}'(\hbar(f))(v)"] \\
\hbar(Y)(v) \arrow[r, "t'_{\hbar(Y)}(v)"] & \mathbb{T}'(\hbar(Y))(v)
\end{tikzcd}$$

is a homotopy pullback square for all $v \in \operatorname{Ob} \mathbf{f}\mathcal{V}$.

Proof. Up to isomorphism, the square above decomposes into two squares:

$$\begin{tikzcd}
\operatorname{colim}_{\mathrm{ac}(v,R)} X(a) \arrow[r, "\operatorname{colim} t_X(a)"] \arrow[d, "\operatorname{colim} f(a)"'] & \operatorname{colim}_{\mathrm{ac}(v,R)} \mathbb{T}(X)(a) \arrow[r, "c_X"] \arrow[d, "\operatorname{colim} \mathbb{T}(f)(a)"'] & \operatorname{colim}_{\mathrm{ac}(T\otimes v,R)} \mathcal{V}(T, X(b)) \arrow[d, "\operatorname{colim}"', "\mathcal{V}(T,f(b))"] \\
\operatorname{colim}_{\mathrm{ac}(v,R)} Y(a) \arrow[r, "\operatorname{colim} t_Y(a)"] & \operatorname{colim}_{\mathrm{ac}(v,R)} \mathbb{T}(Y)(a) \arrow[r, "c_Y"] & \operatorname{colim}_{\mathrm{ac}(T\otimes v,R)} \mathcal{V}(T, Y(b))
\end{tikzcd}$$

Recall from 3.3.3 that the objects in $\mathrm{ac}(T\otimes v, R)$ are $T\otimes v \overset{\sim}{\rightarrowtail} b \longrightarrow R(T\otimes v)$. The maps c_X and c_Y are obtained from $\Theta_v\colon \mathrm{ac}(v,R) \longrightarrow \mathrm{ac}(T\otimes v, R)$ which maps $v \overset{\sim}{\rightarrowtail} a \longrightarrow R(v)$ to $T\otimes v \overset{\sim}{\rightarrowtail} T\otimes a \longrightarrow T\otimes R(v) \longrightarrow R(T\otimes v)$, and the natural transformation $\mathcal{V}(T, X(T\otimes \Phi_v)) \longrightarrow \mathcal{V}(T, X(\Phi_{T\otimes v} \circ \Theta_v))$ which consists of identity maps. From 3.23, one can replace the indexing categories $\mathrm{ac}(v,R)$ and $\mathrm{ac}(T\otimes v, R)$ by filtered ones, namely $\mathrm{ac}(v,F)$ and $\mathrm{ac}(T\otimes v, F)$. Then all the vertical maps are fibrations of fibrant objects, because f is a pointwise fibration of pointwise fibrant functors and 3.5 holds. It follows that the left hand square is a homotopy pullback square if and only if the canonical map g from $\mathrm{colim}_{\mathrm{ac}(v,F)} X(a)$ to the pullback is a weak equivalence. Filtered colimits commute with pullbacks, so g is the filtered colimit of the canonical maps induced by the squares

$$\begin{tikzcd} X(a) \arrow[r, "t_X(a)"] \arrow[d, "f(a)"'] & \mathbb{T}(X)(a) \arrow[d, "\mathbb{T}(f)(a)"] \\ Y(a) \arrow[r, "t_Y(a)"] & \mathbb{T}(Y)(a) \end{tikzcd}$$

for finite sub-complexes $v \xrightarrow{\sim} a$ of $v \xrightarrow{\iota_v} Fv$. These squares are all homotopy pullback squares: f is a stable fibration, and the vertical maps are fibrations of fibrant objects. It follows that g is a filtered colimit of weak equivalences, so the left hand square is a homotopy pullback square.
That the right hand square is a homotopy pullback depends on whether f is an hf-fibration. We claim that

$$\begin{tikzcd} X(T\otimes a) \arrow[r] \arrow[d, "f(T\otimes a)"'] & \mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} X(b) \arrow[d, "\mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} f(b)"] \\ Y(T\otimes a) \arrow[r] & \mathrm{colim}_{\mathrm{ac}(T\otimes v,R)} Y(b) \end{tikzcd}$$

is a homotopy pullback square for all $v \xrightarrow[\alpha]{\sim} a \longrightarrow R(v)$ in $\mathrm{ac}(v,R)$. Denote the full subcategory of $\mathrm{ac}(T\otimes v,R)$ consisting of $T\otimes v \xrightarrow[\beta]{\sim} b \longrightarrow R(T\otimes v)$, where β factors as $T\otimes v \xrightarrow{T\otimes\alpha} T\otimes a \longrightarrow b$, by $\mathrm{ac}(T\otimes v,R)_a$. This category is a final subcategory of $\mathrm{ac}(T\otimes v,R)$. Hence we may assume the colimit in the square above is indexed by $\mathrm{ac}(T\otimes v,R)_a$. As in the proof of 5.7, it follows that the square above is a homotopy pullback square for all $v \xrightarrow[\alpha]{\sim} a \longrightarrow R(v)$ in $\mathrm{ac}(v,R)$. Here we use that f is an hf-fibration. The right hand square in the main diagram is then a homotopy pullback square, using the by now standard argument for filtered colimits of homotopy pullback squares. □

A proof of 6.8 follows:

Proof. By 6.11, we may assume f is a stable fibration of pointwise fibrant functors. The result is therefore a consequence of 6.12. □

Lemma 6.13. *Suppose that $\mathcal{V}$ is right proper and filtered colimits commute with pullbacks in $\mathcal{V}$. Let $f\colon X \longrightarrow Y$ be a stable fibration. Then*

$$\begin{tikzcd} X(v) \arrow[r] \arrow[d, "f(v)"'] & \mathbb{T}'^{\infty}(\hbar(RX))(v) \arrow[d, "\mathbb{T}'^{\infty}(\hbar(Rf))(v)"] \\ Y(v) \arrow[r] & \mathbb{T}'^{\infty}(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.

Proof. Let $f\colon X \longrightarrow Y$ be a stable fibration. Factor Rf as $RX \xrightarrow{\sim \mathrm{pt}} Z$ followed by $g\colon Z \xrightarrow{\mathrm{pt}} RY$. Here g is a stable fibration of pointwise fibrant functors by 5.6 and 6.11. The claim in 6.13 is equivalent to the statement that

$$\begin{tikzcd} Z(v) \arrow[r] \arrow[d, "g(v)"'] & \mathbb{T}'^{\infty}(\hbar(Z))(v) \arrow[d, "\mathbb{T}'^{\infty}(\hbar(g))(v)"] \\ RY(v) \arrow[r] & \mathbb{T}'^{\infty}(\hbar(RY))(v) \end{tikzcd}$$

is a homotopy pullback square. The vertical maps are fibrations of fibrant objects, and there is the decomposition

$$\begin{tikzcd} Z(v) \arrow[r] \arrow[d, "g(v)"'] & \hbar(Z)(v) \arrow[r] \arrow[d, "\hbar(g)(v)"] & \mathbb{T}'^{\infty}(\hbar(Z))(v) \arrow[d, "\mathbb{T}'^{\infty}(\hbar(g))(v)"] \\ RY(v) \arrow[r] & \hbar(RY)(v) \arrow[r] & \mathbb{T}'^{\infty}(\hbar(RY))(v) \end{tikzcd}$$

The left hand square is a homotopy pullback square by 5.7. The functor $\mathcal{V}(T,-)$ preserves homotopy pullback squares provided the vertical maps are fibrations of fibrant objects. It follows, using 6.8, that the right hand square is a homotopy pullback square. □

Corollary 6.14. *A map is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

Proof. Let $f\colon X \longrightarrow Y$ be a stable fibration and a stable equivalence. Stable fibrations are in particular pointwise fibrations, so it remains to prove that f is a pointwise weak equivalence. By 6.13, the following diagram is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.

$$\begin{tikzcd} X(v) \arrow[r] \arrow[d, "f(v)"'] & \mathbb{T}'^{\infty}(\hbar(RX))(v) \arrow[d, "\mathbb{T}'^{\infty}(\hbar(Rf))(v)"] \\ Y(v) \arrow[r] & \mathbb{T}'^{\infty}(\hbar(RY))(v) \end{tikzcd}$$

Since f is a stable equivalence, the right hand vertical map is a weak equivalence. It follows that f is a pointwise weak equivalence. Consider the other implication.

A pointwise acyclic fibration $f\colon X \longrightarrow Y$ is a stable equivalence according to 6.3. So 5.8 implies that f is an hf-fibration. By 6.7, it remains to prove that

$$\begin{tikzcd} X(v) \arrow[r, "t(X)(v)"] \arrow[d, "f(v)"'] & \mathbb{T}(X)(v) \arrow[d, "\mathbb{T}(f)(v)"] \\ Y(v) \arrow[r, "t(Y)(v)"] & \mathbb{T}(Y)(v) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{V}$ for all $v \in \mathbf{f}\mathcal{V}$. The maps $f(v)$ and $f(T \otimes v)$ are acyclic fibrations and $\mathcal{V}(T,-)$ preserves acyclic fibrations. This implies that $\mathbb{T}(f)(v)$ is an acyclic fibration. □

6.3 Comparison with T-spectra

To proceed with the stable model structure we will compare the stabilizations of enriched functors 6.1 and spectra [8, §4]. Recall the "suspension with T" functors $-\otimes T$ and Σ_T on $\mathrm{Sp}(\mathcal{V},T)$ from 2.13. Let $\mathrm{Ev}_n := \mathrm{Ev}_{T^n}$ denote the functor evaluating a spectrum on T^n. If $n \geq 0$, there is the commutative diagram:

$$\begin{tikzcd}
\mathrm{Sp}(\mathcal{V},T) \arrow[r, "-\otimes T"] \arrow[d, "\Sigma_T"'] & \mathrm{Sp}(\mathcal{V},T) \arrow[d, "\mathrm{Ev}_n"] \\
\mathrm{Sp}(\mathcal{V},T) \arrow[r, "\mathrm{Ev}_n"] & \mathcal{V}
\end{tikzcd}$$

Hence, for any spectrum E, $E\otimes T$ and $\Sigma_T E$ differ only in their structure maps. This statement carries over to the adjoints $\mathcal{V}(T,-)\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$ and $\Omega_T\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$. The composition $T\mathrm{Sph} \hookrightarrow \mathcal{V} \xrightarrow{T\otimes -} \mathcal{V}$ does not factor over the inclusion $T\mathrm{Sph} \hookrightarrow \mathcal{V}$. Hence the shift functor $\mathrm{Sh}\colon \mathcal{F} \longrightarrow \mathcal{F}$ does not have a compatible analog in the category of spectra. But there is the shift $\mathrm{sh}\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$ where $(\mathrm{sh}(E))_n := E_{n+1}$. The nth structure map of $\mathrm{sh}(E)$ is e_{n+1}, and the following diagram commutes for all $n \geq 0$.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathrm{Sh}"] \arrow[d, "\mathrm{ev}"'] & \mathcal{F} \arrow[r, "\mathrm{ev}"] & \mathrm{Sp}(\mathcal{V},T) \arrow[d, "\mathrm{Ev}_n"] \\
\mathrm{Sp}(\mathcal{V},T) \arrow[r, "\mathrm{sh}"] & \mathrm{Sp}(\mathcal{V},T) \arrow[r, "\mathrm{Ev}_n"] & \mathcal{V}
\end{tikzcd}$$

The stabilization for spectra uses that the structure maps of a spectrum E define a natural map $s(E)\colon E \longrightarrow \mathrm{sh}(\Omega_T E)$. Let us abbreviate the composition $\mathrm{sh}\circ\Omega_T$ by $S\colon \mathrm{Sp}(\mathcal{V},T) \longrightarrow \mathrm{Sp}(\mathcal{V},T)$. Then $s(E)_n\colon E_n \longrightarrow S(E)_n = \Omega_T E_{n+1}$ is the adjoint of e_n. The stabilization $S^\infty(E)$ of a spectrum E is the colimit of the diagram $E \xrightarrow{s(E)} S(E) \xrightarrow{S(s(E))} S(S(E)) \longrightarrow \cdots$. Let $s^\infty\colon E \longrightarrow S^\infty(E)$ be the canonical map. In [8, §4], the notation $\iota\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow \Theta$ is used instead of $s\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow S$, and $j\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow \Theta^\infty$ instead of $s^\infty\colon \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)} \longrightarrow S^\infty$.

Definition 6.15. A map $f\colon E \longrightarrow F$ of spectra is a *stable equivalence* if $S^\infty(R\circ f)$ is a pointwise weak equivalence, and a *stable fibration* if f is a pointwise fibration and

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "f"'] & R\circ E \arrow[r, "s^\infty(R\circ E)"] \arrow[d, "R\circ f"'] & S^\infty(R\circ E) \arrow[d, "S^\infty(R\circ f)"] \\
F \arrow[r] & R\circ F \arrow[r, "s^\infty(R\circ F)"] & S^\infty(R\circ F)
\end{tikzcd}$$

is a homotopy pullback square in the pointwise model structure.

We summarize the results [8, 4.12, 4.14, 6.5].

Theorem 6.16 (Hovey). *Let $\mathcal{V}$ be an almost finitely generated, pointed, proper, and cellular monoidal model category. Let T be some cofibrant and $\mathcal{V}$-finitely presentable object of $\mathcal{V}$. Assume sequential colimits commute with pullbacks in $\mathcal{V}$. Then $\mathrm{Sp}(\mathcal{V},T)$ is an almost finitely generated proper $\mathcal{V}$-model category with stable equivalences as weak equivalences, stable fibrations as fibrations and cofibrations as cofibrations.*

The condition that $\mathcal{V}$ be cellular might be weakened according to the remark after [8, 4.12]. An important input in the proof of 6.16 is the following lemma.

Lemma 6.17. *If E is pointwise fibrant, then $s^\infty(E)$ is a stable equivalence with a stably fibrant codomain.*

The natural maps $\mathrm{ev}(t^\infty(X))$ and $s^\infty(\mathrm{ev}(X))$ do not coincide. Compatibility of the two stabilization processes is therefore an issue, see 6.18 and 6.19 below.

Lemma 6.18. *If v is an object of $\mathbf{f}\mathcal{V}$, let $v^*\colon \mathcal{F} \longrightarrow \mathcal{F}$ be the functor where $X \longmapsto X \circ (v \otimes -)$ and let $\gamma_{v,n}$ be short for the coherence isomorphism*

$$T \otimes (v \otimes T^n) \xrightarrow{\alpha_{T,V,T^n}} (T \otimes v) \otimes T^n \xrightarrow{\sigma_{T,v\otimes T^n}} (v \otimes T) \otimes T^n \xrightarrow{\alpha^{-1}_{v,T,T^n}} v \otimes T^{n+1}.$$

Then the next diagram is commutative and natural in X.

$$\begin{tikzcd} X(v\otimes T^n) \arrow[r, "t(X)(v\otimes T^n)"] \arrow[dr, "s(\mathrm{ev}(v^*(X)))_n"'] & \mathcal{V}(T, X(T\otimes(v\otimes T^n))) = \mathbb{T}(X)(v\otimes T^n) \arrow[d, "\mathcal{V}(T,X(\gamma_{v,n}))"] \\ & \mathcal{V}(T, X(v\otimes T^{n+1})) \end{tikzcd}$$

Proof. By definition, $t(X)(v\otimes T^n)$ is the adjoint of

$$T \xrightarrow{\eta_{v\otimes T^n}(T)} \mathcal{V}(v\otimes T^n, T\otimes(v\otimes T^n)) \xrightarrow{\hom^X} \mathcal{V}(X(v\otimes T^n), X(T\otimes(v\otimes T^n))).$$

Likewise, the map $s(\mathrm{ev}(v^*(X)))_n$ is the adjoint of

$$T \xrightarrow{\eta_{T^n}(T)} \mathcal{V}(T^n, T^{n+1}) \xrightarrow{\hom^{v^*(X)}_{T^n,T^{n+1}}} \mathcal{V}(X(v\otimes T^n), X(v\otimes T^{n+1})).$$

Note that $\hom^{v^*(X)}_{T^n,T^{n+1}} = \hom^{X\circ(v\otimes -)}_{T^n,T^{n+1}} = \hom^X_{v\otimes T^n, v\otimes T^{n+1}} \circ \hom^{v\otimes -}_{T^n,T^{n+1}}$. The claim follows from A.2. □

Corollary 6.19. *For all $X \in \mathrm{Ob}\,\mathcal{F}$ and $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$, there exists an isomorphism $\gamma\colon \mathbb{T}^\infty(X)(v\otimes T^n) \longrightarrow S^\infty(\mathrm{ev}(v^*(X)))_n$ and the diagram below is commutative and natural in X.*

$$\begin{tikzcd} X(v\otimes T^n) \arrow[r, "t^\infty(X)(v\otimes T^n)"] \arrow[dr, "s^\infty(\mathrm{ev}(v^*(X)))_n"'] & \mathbb{T}^\infty(X)(v\otimes T^n) \arrow[d, "\gamma"] \\ & S^\infty(\mathrm{ev}(v^*(X)))_n \end{tikzcd}$$

Proof. The maps γ and $s^\infty(\mathrm{ev}(v^*(X)))_n$ are the vertical sequential compositions in the following commutative diagram.

$$\begin{tikzcd}
X(v\otimes T^n) \arrow[r, "\mathrm{id}_{X(v\otimes T^n)}"] \arrow[d, "t(X)(v\otimes T^n)"'] & X(v\otimes T^n) \arrow[d, "s(\mathrm{ev}(v^*(X)))_n"] \\
\mathbb{T}(X)(v\otimes T^n) \arrow[r, "\mathcal{V}(T,X(\gamma^1))"] \arrow[d, "\mathbb{T}(t(X))(v\otimes T^n)"'] & \mathcal{V}(T,X(v\otimes T^{n+1})) \arrow[d, "\mathcal{V}(T,s(\mathrm{ev}(v^*(X)))_{n+1})"] \\
\mathbb{T}(\mathbb{T}(X))(v\otimes T^n) \arrow[r, "\mathcal{V}(T,\mathcal{V}(T,X(\gamma^2)))"] \arrow[d] & \mathcal{V}(T,\mathcal{V}(T,X(v\otimes T^{n+2}))) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Here γ^1 is the map $\gamma_{v,n}$ in 6.18. Note that the second square above is $\mathcal{V}(T,-)$ applied to the diagram

$$\begin{tikzcd}
X(T\otimes(v\otimes T^n)) \arrow[r, "t(X)(t\otimes(v\otimes T^n))"] \arrow[d, "\mathcal{V}(T,X(\gamma_1))"'] & \mathbb{T}(X)(T\otimes(v\otimes T^n)) \arrow[d, "\mathcal{V}(T,X(\gamma_2))"] \\
X(v\otimes T^{n+1}) \arrow[r, "s(\mathrm{ev}(v^*(X)))_{n+1}"] & \mathcal{V}(T,X(v\otimes T^{n+1})).
\end{tikzcd}$$

Let γ^2 be defined as $T\otimes(T\otimes(v\otimes T^n)) \xrightarrow{T\otimes\gamma_{v,n}} T\otimes(v\otimes T^{n+1}) \xrightarrow{\gamma_{v,n+1}} v\otimes T^{n+1}$. Then the lower square commutes because of naturality and 6.18. Likewise one constructs γ^n inductively, and puts γ to be $\mathrm{colim}_n\gamma^n$. The result follows. $\square$

If $v\in\mathrm{Ob}\,\mathbf{f}\mathcal{V}$, the composition $\mathbb{I}_*X\circ\Phi\circ(v\otimes -)=\hbar(X)\circ(v\otimes -)$ determines a T-spectrum $\overline{v^*X}$ for every $\mathcal{V}$-functor X. The nth term of $\overline{v^*X}$ is $\mathbb{I}_*X(\Phi(v\otimes T^n))$ and the structure map $\overline{v^*X}_n\otimes T \longrightarrow \overline{v^*X}_{n+1}$ is the composition

$$\mathbb{I}_*X(\Phi(v\otimes T^n))\otimes T \xrightarrow{\mathrm{sw}_T^{\mathbb{I}_*X}} \mathbb{I}_*X(\Phi(v\otimes T^n)\otimes T) \xrightarrow{\mathbb{I}_*X(\theta_{v\otimes T^n})} \mathbb{I}_*X(\Phi(v\otimes T^{n+1}))$$

up to an associativity isomorphism. This construction is functorial and commutes with colimits and the closed $\mathcal{V}$-module structures.

Lemma 6.20. *A map $f\colon X\longrightarrow Y$ in $\mathcal{F}$ is a stable equivalence if and only if $\overline{v^*f}\colon\overline{v^*X}\longrightarrow\overline{v^*Y}$ is a stable equivalence of T-spectra for all $v\in\mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Let $f\colon X\longrightarrow Y$ be a stable equivalence, and pick $v\in\mathrm{Ob}\,\mathbf{f}\mathcal{V}$. The map $\mathbb{T}'^\infty(\hbar(R\circ f))(v\otimes T^n)$ is a weak equivalence in $\mathcal{V}$ by definition. The isomorphism mentioned in 6.19 implies that this map is isomorphic to $S^\infty(v^*(R\circ f)')_n$. Thus $\overline{v^*(R\circ f)}$ is a stable equivalence of T-spectra. Since this is a pointwise fibrant replacement of $\overline{v^*f}$ in $\mathrm{Sp}(\mathcal{V},T)$, it follows that $\overline{v^*f}$ is a stable equivalence. The converse holds by running the argument backwards. $\square$

If additional conditions are satisfied, the characterization of stable equivalences can be improved in that $\hbar$ becomes redundant. Note that the last characterization uses the axiom **f4**.

Corollary 6.21. *A map $f\colon X \longrightarrow Y$ of pointwise fibrant homotopy functors is a stable equivalence if and only if* $\mathrm{ev}(v^*(f))$ *is a stable equivalence of spectra for every object v of $\mathbf{f}\mathcal{V}$.*

Proof. By 6.4, f is a stable equivalence if and only if $\mathbb{T}^\infty(f)(w)$ is a weak equivalence in $\mathcal{V}$ for every w. The proof proceeds as in 6.20. ☐

Corollary 6.22. *A map f of cofibrant functors is a stable equivalence if and only if* $\mathrm{ev}(v^*(f^h))$ *is a stable equivalence of spectra for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$.*

Proof. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\sim\mathrm{hf}"] \arrow[d, "f"'] & RX^h \arrow[d, "Rf^h"] \\ Y \arrow[r, "\sim\mathrm{hf}"] & RY^h. \end{tikzcd}$$

¿From 5.17 – which uses **f4** – and 6.3 we have that f is a stable equivalence if and only if Rf^h is a stable equivalence. Corollary 6.21 shows that f is a stable equivalence if and only if $\mathrm{ev}(v^*(Rf^h))$ is a stable equivalence in $\mathrm{Sp}(\mathcal{V}, T)$ for all $v \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$. Since $\mathrm{ev}(v^*(Rf^h)) = R\,\mathrm{ev}(v^*(f^h))$ and pointwise weak equivalences of spectra are stable equivalences, it follows that f is a stable equivalence if and only if $\mathrm{ev}(v^*(f^h))$ is a stable equivalence of spectra. ☐

6.4 The generating stable equivalences

Recall the stable model structure on $\mathrm{Sp}(\mathcal{V}, T)$ from 6.16. In this structure, (Σ_T, Ω_T) is a Quillen equivalence by [8, 3.9]. In general, it is not clear whether $(-\otimes T, \mathcal{V}(T, -))$ – which is more natural to consider when viewing spectra as $\mathcal{V}$-functors – is a Quillen equivalence. As explained in [8, 10.3], this holds if T is *symmetric*, which roughly means that the cyclic permutation on $T \otimes T \otimes T$ is homotopic to the identity. We formulate a working hypothesis.
Hypothesis: The adjoint functor pair $(-\otimes T, \mathcal{V}(T, -))$ is a Quillen equivalence for the model category $\mathrm{Sp}(\mathcal{V}, T)$ described in 6.16.

Lemma 6.23. *The maps in $\mathcal{D}$ are stable equivalences.*

Proof. A map $d_v \square i$ in $\mathcal{D}$ is a cofibration of cofibrant functors, and by 6.22 a stable equivalence if and only if $\mathrm{ev}(w^*(d_v \square i)^h) = \mathrm{ev}(w^*(d_v)^h)\square i$ is a stable equivalence of spectra for all $w \in \mathrm{Ob}\,\mathbf{f}\mathcal{V}$. We claim the latter holds if $f := \mathrm{ev}(w^*(d_v)^h)$ is a stable equivalence.
Factor f as a cofibration g followed by a pointwise acyclic fibration p. The stable model structure on spectra is a $\mathcal{V}$-model structure by 6.16, hence $f\square i$ factors as a cobase change of the stable acyclic cofibration $g\square i$, followed by the

map $p\Box i$. Since $\mathcal{V}$ is assumed to be strongly monoidal, $p\Box i$ is a pointwise weak equivalence.
It remains to prove that d_v is a stable equivalence. This condition is equivalent to τ_v being a stable equivalence. The latter factors by definition as

$$\mathcal{V}(T\otimes v,-)\otimes T \xrightarrow{\cong} \mathcal{V}(T,\mathcal{V}(v,-))\otimes T \xrightarrow{\epsilon_T V(v,-)} \mathcal{V}(v,-)$$

where ϵ_T is the counit $(-\otimes T)\circ\mathcal{V}(T,-) \longrightarrow \mathrm{Id}_{\mathcal{F}}$. We are reduced to prove that $\epsilon_T V(v,-)$ is a stable equivalence. By 6.22, this map is a stable equivalence if and only if $\mathrm{ev}(w^*(\epsilon_T\mathcal{V}(v,-))^h)$ is a stable equivalence of spectra for every w. Since ev commutes with the action and coaction of $\mathcal{V}$, $\mathrm{ev}(w^*(\epsilon_T\mathcal{V}(v,-))^h)$ coincides with $\epsilon_T\mathrm{ev}(w^*(\mathcal{V}(v,-))^h)$.
Let $q\colon Q \longrightarrow \mathrm{Id}_{\mathrm{Sp}(\mathcal{V},T)}$ be a cofibrant replacement functor in the category of spectra, so $q(E)_n$ is an acyclic fibration in $\mathcal{V}$ for every spectrum E and $n\geq 0$. Consider the following diagram, where the notation is simplified.

$$\begin{tikzcd}
(Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))))\otimes T \arrow[r, "q\otimes T"] \arrow[d, "Q\epsilon_T"'] & \mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))))\otimes T \arrow[d, "\epsilon_T"] \\
Q\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))) \arrow[r, "q"] & \mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))
\end{tikzcd}$$

The composition

$$s^\infty\circ q\circ Q\epsilon_T\colon (Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))))\otimes T \longrightarrow S^\infty(\mathrm{ev}(\mathcal{V}(v,R(w\otimes -))))$$

is a stable equivalence of spectra by the hypothesis. The target of $s^\infty\circ q\circ Q\epsilon_T$ is stably fibrant, and its domain is cofibrant. Recall that $\mathcal{V}(T,-)$ commutes with filtered colimits. Up to an isomorphism, the stable weak equivalence

$$s^\infty\circ q\colon Q\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))) \longrightarrow S^\infty(\mathcal{V}(T,\mathrm{ev}(\mathcal{V}(v,R(w\otimes -)))))$$

is an adjoint of $s^\infty\circ q\circ Q\epsilon_T$. Thus $q\circ Q\epsilon_T$ is a stable equivalence. The map $q\otimes T$ is a pointwise weak equivalence since $\mathcal{V}$ is strongly monoidal. Hence ϵ_T is a stable equivalence of spectra. This ends the proof. □

To consider maps in $\mathcal{D}$-cell, we need to record a property of the stable model structure of spectra.

Lemma 6.24. *Let $f\colon E \longrightarrow F$ be a stable equivalence of spectra such that f_n is a retract of a map in $\mathrm{Cof}(\mathcal{V})\otimes\mathcal{V}$-cell for every $n\geq 0$. Then any cobase change of f is a stable equivalence.*

Proof. Let $g\colon E \longrightarrow G$ be a map of spectra. Factor g as $i\colon E \rightarrowtail T$ followed by $p\colon T \overset{\sim\mathrm{pt}}{\twoheadrightarrow} G$, and consider the diagram:

$$\begin{tikzcd}
E \arrow[r, tail, "i"] \arrow[d, "f"'] & T \arrow[r, two heads, "p", "\sim\mathrm{pt}"'] \arrow[d, "f'"] & G \arrow[d, "f''"] \\
F \arrow[r, tail] & F\cup_E T \arrow[r, "p'"] & F\cup_E G
\end{tikzcd}$$

The stable model structure on spectra is left proper by 6.16, hence f' is a stable equivalence. Pushouts are formed pointwise, so f'_n is a retract of a map belonging to $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$-cell for every $n \geq 0$. By the assumption that $\mathcal{V}$ is strongly left proper, the cobase change p' of the pointwise weak equivalence p along f' is again a pointwise weak equivalence. Hence f'' is a stable equivalence. □

Lemma 6.25. *The maps in $\mathcal{D}$-cell are stable equivalences.*

Proof. Let $f: X \longrightarrow Y$ be a map in $\mathcal{D}$-cell. First suppose that X is cofibrant. Then Y is automatically cofibrant. By 6.22, it suffices to prove that $\mathrm{ev}(v^* f^h)$ is a stable equivalence of spectra for every v. The functors ev, v^* and $(-)^h$ preserve colimits, hence $\mathrm{ev}(v^* f^h)$ is in $\mathrm{ev}(v^*(\mathcal{D}^h))$-cell. Every map in $\mathrm{ev}(v^*(\mathcal{D}^h))$ is of the form considered in 6.24, so cobase changes of these are stable equivalences of spectra. The stable model structure on spectra is almost finitely generated, which implies that stable equivalences of spectra are closed under sequential compositions. This proves the lemma for maps in $\mathcal{D}$-cell with cofibrant domain.

For f arbitrary, we will construct a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\sim\mathrm{pt}"'] & Y' \arrow[d, "\sim\mathrm{pt}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where X' is cofibrant and f' is a map in $\mathcal{P}_J \cup \mathcal{D}$-cell. It allows to finish the proof using the special case treated above. Without loss of generality, f is the sequential composition of $X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$, where f_n is the cobase change of a coproduct of maps in $\mathcal{D}$. We construct f' as a sequential composition. Consider a cofibrant replacement $g_0: X'_0 = X' \overset{\sim\mathrm{pt}}{\twoheadrightarrow} X$. Assume f_0 is the cobase change of $z_0: sZ_0 \longrightarrow tZ_0$, and let $a_0: sZ_0 \longrightarrow X$ be the attaching map. The functor sZ_0 is cofibrant, so a_0 lifts to a map $a'_0: sZ_0 \longrightarrow X'_0$. Taking pushouts in the commutative diagram

$$\begin{tikzcd} tZ_0 \arrow[d, "\mathrm{id}"'] & sZ_0 \arrow[l, "z_0"'] \arrow[r, "a'_0"] \arrow[d, "\mathrm{Id}"'] & X'_0 \arrow[d, "g_0"'] \\ tZ_0 & sZ_0 \arrow[l, "z_0"'] \arrow[r, "a_0"] & X_0 \end{tikzcd}$$

gives a pointwise weak equivalence $tZ_0 \cup_{sZ_0} X'_0 \xrightarrow{\sim\mathrm{pt}} X_1$. It factors as a map in $\mathcal{P}_J$-cell followed by say $X'_1 \overset{\sim\mathrm{pt}}{\twoheadrightarrow} X_1$. By iterating this construction one finds pointwise acyclic fibrations $g_n: X'_n \overset{\sim\mathrm{pt}}{\twoheadrightarrow} X_n$ for all $n \geq 0$. Taking the colimit gives $f': X' \longrightarrow Y := \mathrm{colim}_n X'_n$. □

6.5 The main theorem

Before stating our main theorem, we summarize the list of assumptions. First, $\mathcal{V}$ is a weakly finitely generated monoidal **sSet**-model category for which the monoid axiom holds. Moreover, $\mathcal{V}$ is strongly monoidal as defined in 4.12, right proper and cellular. Assume that filtered colimits commute with pullbacks in $\mathcal{V}$, that Δ^1 is finitely presentable in $\mathcal{V}$, and that cofibrations are monomorphisms. We require $\mathbf{f}\mathcal{V}$ to satisfy

f1 Every object of $\mathbf{f}\mathcal{V}$ is $\mathcal{V}$-finitely presentable.

f2 The unit e is in $\mathbf{f}\mathcal{V}$, and $\mathbf{f}\mathcal{V}$ is closed under the monoidal product.

f3 If $tj \xleftarrow{\sim} sj \longrightarrow v$ is a diagram in $\mathcal{V}$ where $v \in \operatorname{Ob}\mathbf{f}\mathcal{V}$ and $j \in J'$, then the pushout $tj \cup_{sj} v$ is in $\mathbf{f}\mathcal{V}$.

f4 All objects in $\mathbf{f}\mathcal{V}$ are cofibrant.

In what follows, T is a cofibrant object of $\mathbf{f}\mathcal{V}$ with the property that $(-\otimes T, \mathcal{V}(T,-))$ is a Quillen equivalence in the stable model structure on $\operatorname{Sp}(\mathcal{V}, T)$.

Theorem 6.26. *Under the assumptions above, the classes of stable equivalences, stable fibrations and cofibrations give $\mathcal{F} = [\mathbf{f}\mathcal{V}, \mathcal{V}]$ the structure of a weakly finitely generated model category.*

Proof. The proof is analogous to the proof of 5.10, using 6.14 and 6.25. Nevertheless we give some details. The set of additional generating acyclic cofibrations is the set $\mathcal{D}$, and the domains and codomains of the maps in $\mathcal{D}$ are finitely presentable. Lemma 6.25 then shows that relative cell complexes built from the generating acyclic cofibrations are stable equivalences. By 6.14 and 4.2, the stable acyclic fibrations are detected by the generating cofibrations. With 6.3 we get that all criteria of [7, 2.1.19] are satisfied. □

We refer to the model structure in 6.26 as the *stable* model structure. If $\mathcal{F}$ is equipped with the stable model structure, we indicate this by the subscript "st".

Lemma 6.27. *The model category $\mathcal{F}_{\mathrm{st}}$ is a monoidal $\mathcal{F}_{\mathrm{hf}}$-model category.*

Proof. Axiom **f4** implies that the homotopy functor structure is monoidal. Let $d_v \square i$ be a map in $\mathcal{D}$, and let $\mathcal{V}(w,-)\otimes j$ be a map in $\mathcal{P}_I$. Then the pushout product map in $\mathcal{F}$ is isomorphic to $d_{v\otimes w}\square(i\square j)$, i.e. a stable equivalence. □

Lemma 6.28. *The stable model structure is proper.*

Proof. Left properness follows since the pointwise model structure is left proper. To prove right properness, it remains by 5.13 to check that the base change of a stable equivalence of pointwise fibrant homotopy functors along an hf-fibration of pointwise fibrant homotopy functors is a stable equivalence. This follows from 6.5 since $\mathbb{T}^\infty$ preserves pullbacks and pointwise fibrations of pointwise fibrant functors, and $\mathcal{F}_{\mathrm{pt}}$ is right proper. □

In the stable structure we have the following important result, analogous to and in fact an easy a consequence of Theorem 4.11.

Theorem 6.29. *If* $\mathbf{f}\mathcal{V}$ *satisfies* **f0**, *then smashing with a cofibrant* $\mathcal{V}$*-functor in* $\mathcal{F}$ *preserves stable equivalences.*

Proof. Factor a stable equivalence f as a stable acyclic cofibration followed by a stable acyclic fibration, i.e. a pointwise acyclic fibration. By 4.11, we may assume f is a stable acyclic cofibration. The claim follows since $\mathcal{F}_{\mathrm{st}}$ is monoidal. □

Lemma 6.30. *Suppose* $\mathbf{f}\mathcal{V}$ *satisfies* **f0**. *Then the monoid axiom holds in* $\mathcal{F}_{\mathrm{st}}$.

Proof. The domains of the generating acyclic cofibrations $\mathcal{D}' := \mathcal{P}_J \cup \mathcal{H} \cup \mathcal{D}$ for the stable model structure on $\mathcal{F}$ are cofibrant. Because **f0** holds, 4.11 implies that every map in $\mathcal{D}' \wedge \mathcal{F}$ is a stable equivalence. The case of a map in $\mathcal{D}' \wedge \mathcal{F}$-cell follows similarly as in the proof of 6.25. □

7 A Quillen equivalence

In this section, we will discuss two natural choices for the domain category $\mathbf{f}\mathcal{V}$. One of the choices gives a Quillen equivalence between the stable model structure $\mathrm{Sp}(\mathcal{V}, T)_{\mathrm{st}}$ on spectra and $\mathcal{F}_{\mathrm{st}}$.

7.1 The choices

Let $\mathbf{f}\mathcal{V}_{\max}$ be the category of all cofibrant $\mathcal{V}$-finitely presentable objects, and $\mathbf{f}\mathcal{V}_{\min}$ the full subcategory of $\mathbf{f}\mathcal{V}_{\max}$ given by the objects v for which there exists an acyclic cofibration $T^n \overset{\sim}{\rightarrowtail} v$ for some $n \geq 0$. In the applications, the category of $\mathcal{V}$-finitely presentable objects is equivalent to a small category, hence its subcategories are valid domain categories. Axioms **f1**, **f2** and **f4** hold in both cases. If the domains and codomains of the maps in J' are $\mathcal{V}$-finite, then **f3** holds. The minimal choice satisfies a property which does not hold for the maximal choice in general.

Lemma 7.1. *A map* f *of pointwise fibrant homotopy functors in* $[\mathbf{f}\mathcal{V}_{\min}, \mathcal{V}]$ *is a stable equivalence if and only if* $\mathrm{ev}(f)$ *is a stable equivalence of spectra.*

Proof. This follows by definition of $\mathbf{f}\mathcal{V}_{\min}$. □

The evaluation functor is a right Quillen functor for both choices.

Lemma 7.2. *Evaluation* $\mathrm{ev}\colon \mathcal{F}_{\mathrm{st}} \longrightarrow \mathrm{Sp}(\mathcal{V}, T)_{\mathrm{st}}$ *is a right Quillen functor.*

Proof. Pointwise fibrations and pointwise acyclic fibrations are preserved by ev. The characterizations of stable fibrations using homotopy pullback squares can be compared using 6.19, which implies that ev preserves stable fibrations. □

To deduce that ev is the right adjoint of a Quillen equivalence for the minimal choice, we prove a property of the stable model structure of spectra which is independent of the choice of $\mathbf{f}\mathcal{V}$.

7.2 The unit of the adjunction

The following lemma is a crucial observation which depends on the hypotheses on T and the stabilization functor of spectra 6.17. Recall that i_* is the left adjoint of ev.

Lemma 7.3. *The canonical map*

$$\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow (\mathcal{V}(T^n,-))^h = \mathrm{ev}((i_*\mathcal{V}_{T\mathrm{Sph}}(T^n,-))^h)$$

is a stable equivalence of T-spectra for all $n \geq 0$.

Proof. By 6.17, this follows if $\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow S^\infty((\mathcal{V}(T^n,-))^h)$ is a stable equivalence. The canonical map $\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes T^n \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^0,-)$ consists of isomorphisms in degree n and on, so it is a stable equivalence. Note that the map $\mathcal{V}_{T\mathrm{Sph}}(T^0,-) \longrightarrow \mathcal{V}(T^0,-)$ is the identity, and that $\mathcal{V}(T^0,-) \longrightarrow (\mathcal{V}(T^0,-))^h$ is even a pointwise weak equivalence. 6.17 shows that $\mathcal{V}(T^0,-)^h \longrightarrow S^\infty((\mathcal{V}(T^0,-))^h)$ is a stable equivalence whose codomain is a stably fibrant T-spectrum. This uses that $(\mathcal{V}(T^0,-))^h$ is pointwise fibrant: T^0 is cofibrant and the input is fibrant. Hence the composition

$$\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes T^n \longrightarrow \mathcal{V}_{T\mathrm{Sph}}(T^0,-) \longrightarrow \mathcal{V}(T^0,-)^h \longrightarrow S^\infty(\mathcal{V}(T^0,-)^h)$$

is a stable equivalence from the n-fold T-suspension of a cofibrant T-spectrum to a stably fibrant T-spectrum. The functor $-\otimes T$ is assumed to be a Quillen equivalence. Thus its adjoint $\mathcal{V}_{T\mathrm{Sph}}(T^n,-) \longrightarrow \mathcal{V}(T^n, S^\infty((\mathcal{V}(T^0,-))^h))$ is a stable equivalence. Since T is $\mathcal{V}$-finite, the latter T-spectrum is isomorphic to $S^\infty((\mathcal{V}(T^n,-))^h)$. It is straightforward to check that these observations imply the claim. $\square$

Corollary 7.4. *The canonical map $c_E\colon E \longrightarrow \mathrm{ev}((i_*E)^h)$ is a stable equivalence for every cofibrant T-spectrum E.*

Proof. For any T-spectrum E and $A \in \mathrm{Ob}\,\mathcal{V}$, the map $c_{E\otimes A}$ is isomorphic to $c_E \otimes A$. Tensoring with the domains and codomains of the generating cofibrations preserves stable equivalences of spectra since the analogues statement holds for $\mathcal{V}$. The cofibrant T-spectra are precisely the retracts of $\mathrm{Sph}(I)$-cell complexes, where $\mathrm{Sph}(I)$ denotes the set of generating cofibrations $\{\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes i\}_{n\geq 0, i\in I}$ from [8, 1.8]. So it suffices to consider $\mathrm{Sph}(I)$-cell complexes. Recall that ev, i_* and $(-)^h$ preserve colimits, and stable equivalences of T-spectra are closed under sequential compositions. This allows to use transfinite induction. The induction step follows from the diagram:

$$\begin{tikzcd}
\mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes ti \arrow[d, "\sim"'] & \mathcal{V}_{T\mathrm{Sph}}(T^n,-)\otimes si \arrow[l] \arrow[r] \arrow[d, "\sim"'] & E \arrow[d, "\sim"'] \\
\mathcal{V}(T^n,-)^h\otimes ti & \mathcal{V}(T^n,-)^h\otimes si \arrow[l, "f"'] \arrow[r] & \mathrm{ev}((i_*E)^h)
\end{tikzcd}$$

The right hand vertical map is a stable equivalence by the induction hypothesis, and likewise for the other vertical maps by 7.3 and the argument given above. Note that f is not necessarily a cofibration of spectra, but it is pointwise in $\mathrm{Cof}(\mathcal{V}) \otimes \mathcal{V}$. Finally, strong left properness of $\mathcal{V}$ implies that the map induced on the pushouts of the rows in the diagram is a stable equivalence. □

Corollary 7.5. *The functor* $\mathrm{ev}\colon [\mathbf{f}\mathcal{V}_{\min}, \mathcal{V}] \longrightarrow \mathrm{Sp}(\mathcal{V}, T)$ *is the right adjoint in a Quillen equivalence.*

Proof. Use 7.1 and 7.4. □

Let $\mathcal{V}$ be the usual model category $\mathbf{sSet}_* = \mathcal{S}$ of pointed simplicial sets, and let T be the circle $S^1 = \Delta^1/\partial\Delta^1$. The n-sphere S^n is the n-fold smash product of S^1. Then $\mathbf{f}\mathcal{S}_{\max}$ is the full subcategory given by the finitely presentable pointed simplicial sets, and $\mathbf{f}\mathcal{S}_{\min}$ is the full subcategory of pointed simplicial sets K for which there exists an acyclic cofibration $S^n \overset{\sim}{\rightarrowtail} K$ for some $n \geq 0$. By [11] implies that the canonical functor $[\mathbf{f}\mathcal{S}_{\max}, \mathcal{S}] \longrightarrow [\mathbf{f}\mathcal{S}_{\min}, \mathcal{S}]$ is the right adjoint in a Quillen equivalence of stable model categories. This uses that all pointed simplicial sets are generated by spheres.

In general, one needs to distinguish between using $\mathbf{f}\mathcal{V}_{\min}$ and $\mathbf{f}\mathcal{V}_{\max}$. For example, if $\mathcal{V} = \mathcal{S}$ and T is the coproduct $S^0 \vee S^0$, we claim the corresponding stable model categories are different. In this case, $\mathbf{f}\mathcal{S}_{\max}$ is as above, while $\mathbf{f}\mathcal{S}_{\min}$ is the full subcategory of finitely presentable pointed simplicial sets which are weakly equivalent to a discrete pointed simplicial set with $1+2^n$ points for some $n \geq 0$. To show that the resulting stable model structures are not Quillen equivalent via the restriction functor, we will describe a map $f\colon X \longrightarrow Y$ of $S^0 \vee S^0$-stably fibrant functors in $\mathbf{SF} := [\mathbf{f}\mathcal{S}_{\max}, \mathcal{S}]$ that is not a weak equivalence, although $f(K)$ is a weak equivalence for every discrete pointed simplicial set. Let Y be the constant functor with value $*$. Let X be the stably fibrant replacement of X', which maps K to the connected component of K containing the basepoint. Since $X'(K \wedge \Delta^n_+) = X'(K) \wedge \Delta^n_+$, X' is enriched over $\mathcal{S}$. Clearly, X' is a homotopy functor, so using an enriched fibrant replacement functor $R\colon \mathcal{S} \longrightarrow \mathcal{S}$, the $S^0 \vee S^0$-stably fibrant replacement X of X' maps K to

$$X(K) = \mathrm{colim}_n \mathcal{S}((S^0 \vee S^0)^n, RX'((S^0 \vee S^0)^n \wedge K)).$$

If K is discrete, $X(K) = *$, hence the map $X \longrightarrow *$ is a weak equivalence in $[\mathbf{f}\mathcal{S}_{\min}, \mathcal{S}]$. However, $X(S^1)$ is weakly equivalent to a countable product of a countable coproduct of S^1 with itself, and hence not contractible.

8 Algebraic structure

This section recalls the important algebraic structures which $\mathcal{F}_{\mathrm{st}}$ supports if the monoid axiom holds. Recall that $\mathcal{F}_{\mathrm{st}}$ satisfies the monoid axiom if $\mathbf{f}\mathcal{V}$ satisfies **f0**, cp. 6.30. Fix a $\mathcal{V}$ which satisfies the conditions listed in the beginning of Section 5, and a small full sub-$\mathcal{V}$-category $\mathbb{I}\colon \mathbf{f}\mathcal{V} \hookrightarrow \mathcal{V}$ which satisfies the axioms **f1**–**f4**.

8.1 Recollections

In a symmetric monoidal category like $(\mathcal{F}, \wedge, \mathbb{I})$ there are notions of algebras and modules over the algebras. Recall that an $\mathbb{I}$-algebra is a monoid in $(\mathcal{F}, \wedge, \mathbb{I})$ or just an $\mathcal{F}$-category with only one object.

If A is an $\mathbb{I}$-algebra, a (left) A-module M is an object in $\mathcal{F}$ with an appropriate action of A. It can alternatively be described as an $\mathcal{F}$-functor from A to $\mathcal{F}$. The category of A-modules mod_A is then an $\mathcal{F}$-category. There is a smash product $\wedge_A \colon \mathrm{mod}_{A^{\mathrm{op}}} \wedge \mathrm{mod}_A \longrightarrow \mathcal{F}$. If $M \in \mathrm{mod}_{A^{\mathrm{op}}}$ and $N \in \mathrm{mod}_A$, then $M \wedge_A N$ is the coequalizer of $M \wedge A \wedge N \rightrightarrows M \wedge N$. Likewise, for A-modules M and N the function object $\mathrm{mod}_A(M, N)$ in $\mathcal{F}$ is the equalizer of $\mathcal{F}(M, N) \rightrightarrows \mathcal{F}(A \wedge M, N)$.

If k is a *commutative* $\mathbb{I}$-algebra, recall that $\mathrm{mod}_k \cong \mathrm{mod}_{k^{\mathrm{op}}}$ and mod_k is a closed symmetric monoidal category under $\wedge_k$ with internal morphism object $\mathrm{mod}_k(M, N)$. A k-algebra is a monoid in $(\mathrm{mod}_k, \wedge_k, k)$ or a mod_k-category with one object. With this notation, notice that $\mathrm{mod}_{\mathbb{I}}$ is $\mathcal{F}$.

8.2 The model structures

Definition 8.1. If A is an $\mathbb{I}$-algebra and k is a commutative $\mathbb{I}$-algebra, then a map in mod_A or alg_k is called a *weak equivalence* (resp. *fibration*) if it is so when considered in $\mathcal{F}_{\mathrm{st}}$. Cofibrations are defined by the left lifting property.

Remark 8.2. Note that we chose the stable model structure as our basis. This is fixed in the following (at least on the top level), so the missing prefix “stable” from fibrations and weak equivalences should not be a source of confusion.

The next result is due to Schwede and Shipley [15, 4.1].

Theorem 8.3. *Suppose that $\mathcal{F}_{\mathrm{st}}$ satisfies the monoid axiom. With the structures described above, the following is true.*

- *Let $A \in \mathcal{F}$ be an $\mathbb{I}$-algebra. Then the category mod_A of (left) A-modules is a cofibrantly generated model category.*
- *Let $k \in \mathcal{F}$ be a commutative $\mathbb{I}$-algebra. Then the category of k-modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.*
- *Let $k \in \mathcal{F}$ be a commutative $\mathbb{I}$-algebra. Then the category alg_k of k-algebras is a cofibrantly generated model category.*

Note that we did not state the hypothesis that all objects in $\mathcal{F}$ are small. Since $\mathcal{V}$ is weakly finitely generated, $\mathcal{F}_{\mathrm{st}}$ is so too. The smallness of the domains and codomains of the generating cofibrations and generating acyclic cofibrations in $\mathcal{F}_{\mathrm{st}}$ carries over to the relevant smallness conditions needed to prove the theorem. See also [15, 2.4].

Lemma 8.4. *Suppose that* $\mathbf{f}\mathcal{V}$ *satisfies* $\mathbf{f0}$. *Let* A *be an* $\mathbb{I}$*-algebra. Then for any cofibrant* A*-module* N*, the functor* $- \wedge_A N$ *takes weak equivalences in* $\mathrm{mod}_{A^{\mathrm{op}}}$ *to weak equivalences in* $\mathcal{F}_{\mathrm{st}}$.

Proof. Given a weak equivalence in mod_A, factor it as an acyclic cofibration followed by an acyclic fibration. The only trouble is with the acyclic fibration, but this is a pointwise acyclic fibration, and the argument can be phrased in the analogous theory for A-modules built on the pointwise structure $\mathcal{F}_{\mathrm{pt}}$. In this case, the generating cofibrations in mod_A are of the form $A \wedge S \xrightarrow{A\wedge i} A \wedge T$ where $S \overset{i}{\rightarrowtail} T$ is a generating cofibration in $\mathcal{F}_{\mathrm{pt}}$. The argument of 4.11 goes through verbatim: smashing commutes with colimits and $A \wedge_A S \cong S$. $\square$

Lemma 8.4 and [15, 4.3] imply:

Corollary 8.5. *Suppose that* $\mathbf{f}\mathcal{V}$ *satisfies the axiom* $\mathbf{f0}$. *Let* $f\colon A \xrightarrow{\sim} B$ *be a weak equivalence of* $\mathbb{I}$*-algebras. Then extension and restriction of scalars define the Quillen equivalence*

$$\mathrm{mod}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{mod}_B.$$

If A *and* B *are commutative, there is the Quillen equivalence*

$$\mathrm{alg}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{alg}_B.$$

9 Equivariant Stable Homotopy Theory

Let $\mathcal{S}$ be the category of pointed simplicial sets, or *spaces* for short. The finitely presentable spaces are the ones with only finitely many non-degenerate simplices, thus we may call these *finite*. A simplicial functor in the sense of [11, 4.5] is an $\mathcal{S}$-functor from the category of finite spaces to the category of all spaces. In [11], Lydakis showed how simplicial functors give rise to a monoidal model category which is Quillen equivalent to the model category of spectra. Thus simplicial functors model the stable homotopy category. The purpose of this section is to use the machinery developed in the main part of the paper to give a functor model for the equivariant stable homotopy category. For technical reasons we will only consider finite groups. Fix a finite group G with multiplication $\mu\colon G \times G \longrightarrow G$.

9.1 Equivariant spaces

The category $G\mathcal{S}$ of G*-spaces* consists of pointed simplicial sets with a basepoint preserving left G-action. Note that G_+ is a $\mathcal{S}$-category with only one object and composition $G_+ \wedge G_+ \longrightarrow G_+$ induced by μ. One can identify $G\mathcal{S}$ with the category $[G_+, \mathcal{S}]$ of $\mathcal{S}$-functors $K : G_+ \longrightarrow \mathcal{S}$. We will often write (uK, a_K) for K to stress the underlying space $uK \in \mathrm{Ob}\,\mathcal{S}$, i.e. the value of K at the single object, and the left G-action $a_K\colon G_+ \wedge uK \longrightarrow uK$. Note that a_K is adjoint to $\hom^K\colon G_+ \longrightarrow \mathcal{S}(uK, uK)$, where $g \longmapsto g\colon uK \longrightarrow uK$. According to 2.4, $G\mathcal{S}$ is a closed $\mathcal{S}$-module. The functor $u\colon G\mathcal{S} \longrightarrow \mathcal{S}$ has a left $\mathcal{S}$-adjoint $G_+ \wedge -\colon \mathcal{S} \longrightarrow G\mathcal{S}$ by 2.5. Consider the G-space $G_+ \wedge K\colon G_+ \longrightarrow \mathcal{S}$. Its underlying space is $G_+ \wedge K$, with left G-action

$$G_+ \wedge (G_+ \wedge K) \xrightarrow{\cong} (G \times G)_+ \wedge K \xrightarrow{\mu_+ \wedge K} G_+ \wedge K.$$

Similarly, the right $\mathcal{S}$-adjoint of u is given by $K \longmapsto \times_{g\in G} K_g$, the G-fold product of K where $h \in G$ sends K_g to K_{hg} via the identity. Let $(-)^{\wedge G}\colon \mathcal{S} \longrightarrow G\mathcal{S}$ be the functor whose value on K is the G-fold smash product $K^{\wedge G} = \wedge_{g\in G} K_g$ of K, where G acts by permuting the factors as above. Another functor we consider is $\mathrm{ct}\colon \mathcal{S} \longrightarrow G\mathcal{S}$. The G-space $\mathrm{ct}K$ is constant, i.e. the underlying space is K and $\hom^{\mathrm{ct}K}\colon G_+ \longrightarrow \mathcal{S}(K, K)$ sends g to either the identity map or the trivial map.
Let $\Delta G\colon G \longrightarrow G \times G$ be the diagonal map. The *smash product* $K \wedge L$ of two G-spaces $K, L : G_+ \longrightarrow \mathcal{S}$ is given by the composition

$$G_+ \xrightarrow{\Delta G_+} (G \times G)_+ \cong G_+ \wedge G_+ \xrightarrow{K\wedge L} \mathcal{S} \wedge \mathcal{S} \xrightarrow{\wedge} \mathcal{S}.$$

The right hand smash product uses [2, 6.2.9]. In other words, the smash product of G-spaces is defined on the underlying spaces and G acts diagonally. For this reason we denote the smash product of G-spaces by $\wedge$. If G is commutative, another closed symmetric monoidal product of G-spaces exists by 2.6.

Proposition 9.1. *The category $(G\mathcal{S}, \wedge, \mathrm{ct}S^0)$ is closed symmetric monoidal. The functors u,* ct *and $(-)^{\wedge G}$ are strict symmetric monoidal, and $G_+ \wedge -$ is lax symmetric monoidal.*

Lemma 9.2. *Let $K\colon G_+ \longrightarrow \mathcal{S}$ be a G-space. The following are equivalent.*

1. *K is $G\mathcal{S}$-finitely presentable.*
2. *K is finitely presentable.*
3. *uK is finite.*

Proof. Let $\mathrm{Fix}(G, -)$ be the $\mathcal{S}$-functor that maps a G-space (uK, a_K) to the subspace $\mathrm{Fix}(G, K) = \{x \in uK \mid a_K\overline{(g, x)} = x \text{ for all } g \in G\}$ fixed under the action of G. Equivalently, $\mathrm{Fix}(G, K)$ is $\lim(K\colon G_+ \longrightarrow \mathcal{S})$. Note that $\mathrm{Fix}(G, -)$ commutes with filtered colimits since G_+ is a finite index category. $\mathrm{Fix}(G, -)$ is the right $\mathcal{S}$-adjoint of $\mathrm{ct}\colon \mathcal{S} \longrightarrow G\mathcal{S}$. In particular, the G-space $\mathrm{ct}S^0$ is finitely presentable. This proves the implication $1 \Rightarrow 2$. Likewise, the right adjoint of u commutes with filtered colimits, thus $2 \Rightarrow 3$.
It remains to prove $3 \Rightarrow 1$. Let $D\colon \mathcal{I} \longrightarrow G\mathcal{S}$ be a functor where $\mathcal{I}$ is filtered, and consider the canonical map $f_K\colon \mathrm{colim}_{\mathcal{I}} G\mathcal{S}(K, D) \longrightarrow G\mathcal{S}(K, \mathrm{colim}_{\mathcal{I}} D)$. Since colimits in $G\mathcal{S}$ are formed on underlying spaces, $u(f_K)$ is the canonical map $\mathrm{colim}_{\mathcal{I}} \mathcal{S}(uK, u \circ D) \longrightarrow \mathcal{S}(uK, \mathrm{colim}_{\mathcal{I}} u \circ D)$. If uK is finite, $u(f_K)$ is an isomorphism, which implies that f_K is an isomorphism since the G-action on the domain coincides with the G-action on the codomain. We are done. $\square$

The full subcategory of finitely presentable G-spaces is equivalent to a small category, which can be chosen to be closed under the smash product of 9.1. Often we will refer to *finite* G-spaces instead of finitely presentable ones.

Lemma 9.3. *Any G-space is the filtered colimit of its finite sub-G-spaces.*

9.2 Unstable equivariant homotopy theory

Theorems 4.2 and 4.4 give $G\mathcal{S}$ the *coarse* model structure, with weak equivalences and fibrations defined on underlying spaces. A cofibration is an injective map $f\colon K \longrightarrow L$ where G acts freely on the complement of $f(K)$ in L. Hence, the cofibrant G-spaces are the G-spaces with a free G-action away from the basepoint. In the following, we will consider another model structure on $G\mathcal{S}$. If H is a subgroup of G, let G/H_+ be the pointed G-space with action $g \cdot g'H := (gg')H$. Consider the $\mathcal{S}$-functor $G/H_+ \wedge\colon \mathcal{S}_+ \longrightarrow G\mathcal{S}_+$, $K \longmapsto G/H_+ \wedge K$, with trivial action on K. Its right $\mathcal{S}$-adjoint is $\mathrm{Fix}(H, -)\colon G\mathcal{S}_+ \longrightarrow \mathcal{S}_+$ which maps L to the space of fixed points under the action of H on L. Note that $\mathrm{Fix}(H, -)$ coincides with $\mathcal{S}_{G\mathcal{S}}(G/H_+, -)$. If H and H' are two subgroups of G, there is a natural isomorphism $\mathrm{Fix}(H', G/H_+ \wedge K) \cong \mathrm{Fix}(H', G/H_+) \wedge K$.

Definition 9.4. A map f in $G\mathcal{S}$ is a *G-weak equivalence* if $\mathrm{Fix}(H, f)$ is a weak equivalence in $\mathcal{S}$ for every subgroup H of G. Likewise for *G-fibrations*.

Theorem 9.5. *There is a proper monoidal model structure on $G\mathcal{S}$ with G-weak equivalences as weak equivalences and G-fibrations as fibrations. Cofibrations are the injective maps. One can choose generating acyclic cofibrations and generating cofibrations with finitely presentable domains and codomains.*

Proof. This result is well-known. A proof is included for completeness. To prove the existence of the model structure, we will apply [7, 2.1.19]. Let

$$I_G := \{G/H_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{n \geq 0,\, H \text{ subgroup of } G}$$

and

$$J_G := \{G/H_+ \wedge (\Lambda^n_i \hookrightarrow \Delta^n)_+\}_{n \geq 1,\, 0 \leq i \leq n,\, H \text{ subgroup of } G}.$$

It is clear by adjointness that a map is a G-fibration if and only if it is in J_G-inj, or a G-fibration and a G-weak equivalence if and only if it is in I_G-inj. From 9.2, the domains and codomains of the maps in I_G and J_G are finite. The natural isomorphism $\mathrm{Fix}(H', G/H_+ \wedge K) \cong \mathrm{Fix}(H', G/H_+) \wedge K$ for subgroups H and H' of G, shows that maps in J_G are G-weak equivalences. The existence of the model structure follows, if every map in J_G-cell is a G-weak equivalence. Since $\mathrm{Fix}(H', -)$ commutes with sequential colimits, it suffices to check that the cobase change of a map in J_G is a G-weak equivalence. Fix a subgroup H' and consider the pushout diagram:

$$\begin{tikzcd} A := (G/H \times \Lambda^n_i)_+ \arrow[r] \arrow[d, hook] & K \arrow[d, hook] \\ B := (G/H \times \Delta^n)_+ \arrow[r] & L \end{tikzcd}$$

The induced map $\mathrm{Fix}(H',B)\cup_{\mathrm{Fix}(H',A)}\mathrm{Fix}(H',K) \longrightarrow \mathrm{Fix}(H',L)$ is injective. Surjectivity follows since $\mathrm{Fix}(H',-)$ preserves injective maps. Thus any cobase change of a map in J_G is a G-weak equivalence, and the model structure exists. Any map in I_G-cell is clearly injective. Conversely, by considering fixed point spaces it follows that any injective map is contained in I_G-cell. The statement about the cofibrations follows, and also left properness. Right properness holds since $\mathrm{Fix}(H,-)$ commutes with pullbacks and $\mathcal{S}$ is right proper.
The pushout product map of injective maps is again injective, so consider the pushout product map

$$G/H_+ \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+)\Box G/H'_+ \wedge (\Lambda^m_i \hookrightarrow \Delta^m)_+ \cong (G/H \times G/H')_+ \wedge i,$$

where i is a weak equivalence of spaces. Since there is an isomorphism of spaces $\mathrm{Fix}(H'',(G/H \times G/H')_+ \wedge i) \cong \mathrm{Fix}(H'',(G/H \times G/H')_+) \wedge i$, the pushout product map of a generating cofibration and a generating acyclic cofibration is again acyclic. Hence the model structure is monoidal. The monoid axiom then holds, since all G-spaces are cofibrant. □

We will refer to the model structure in 9.5 as the *fine* model structure. The regular representation $S^{\wedge G}$ is the G-fold smash product of $S^1 = \Delta^1/\partial\Delta^1$ where G acts by permuting the factors. Its geometric realization is homeomorphic – as a G-space – to the one-point compactification of the real vector space $\mathbb{R}^G$ of maps $G \longrightarrow \mathbb{R}$. The G-space $S^{\wedge G}$ is finite, since G is a finite group.

9.3 Stable equivariant homotopy theory

Let $\mathbf{f}G\mathcal{S}$ denote the full sub-$G\mathcal{S}$-category given by the finite G-spaces. It is equivalent to a small $G\mathcal{S}$-category. Objects of the enriched functor category $G\mathcal{F} = [\mathbf{f}G\mathcal{S}, G\mathcal{S}]$ will be called *G-simplicial functors*. If G is the trivial group, then $G\mathcal{F}$ is Lydakis' category of simplicial functors [11, 4.4]. Let $\wedge$ denote the smash product of G-simplicial functors. The unit of $\wedge$ is the inclusion $\mathbb{S}^G = \mathbb{I}\colon \mathbf{f}G\mathcal{S} \hookrightarrow G\mathcal{S}$. All G-spaces are cofibrant in the fine model structure.

Definition 9.6. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is a

- *pointwise weak equivalence* if $f(K)$ is a G-weak equivalence for all finite G-spaces K,
- *pointwise fibration* if $f(K)$ is a G-fibration for all finite G-spaces K,
- *cofibration* if f has the left lifting property with respect to all pointwise acyclic fibrations.

Theorem 9.7. *The category $G\mathcal{F}$, equipped with the classes described in 9.6, is a monoidal proper model category satisfying the monoid axiom. Generating cofibrations and generating acyclic cofibrations can be chosen with finitely presentable domains and codomains. Finally, smashing with a cofibrant G-simplicial functor preserves pointwise weak equivalences.*

Proof. ¿From 4.2, $\mathcal{P}_{I_G} = \{G\mathcal{S}(K,-) \wedge i\}_{i \in I_G,\, K \text{ finite}}$ are the generating cofibrations, and $\mathcal{P}_{J_G} = \{G\mathcal{S}(K,-) \wedge j\}_{j \in J_G,\, K \text{ finite}}$ are the generating acyclic cofibrations. The model structure is monoidal and satisfies the monoid axiom by 4.4. Properness holds by 4.8. The functor $- \wedge X$ preserves pointwise weak equivalences for cofibrant X since 4.11 holds, cp. 9.3. □

Let us write $G\mathcal{F}_{\text{pt}}$ for the *pointwise* model structure. To define the homotopy functor model structure on $G\mathcal{F}$, let $\rho\colon \mathrm{Id}_{G\mathcal{S}} \longrightarrow R$ denote the enriched fibrant replacement functor from 3.3.2 applied to the set J_G. Denote by $\mathbb{I}_*X$ the enriched left Kan extension of $X\colon \mathbf{f}G\mathcal{S} \longrightarrow G\mathcal{S}$ along $\mathbb{I}\colon \mathbf{f}G\mathcal{S} \hookrightarrow G\mathcal{S}$, and by X^h the composition $\mathbb{I}_*X \circ R \circ \mathbb{I}$. Then X^h defines an endofunctor of $G\mathcal{F}$ and there is a natural transformation $\mathrm{Id}_{G\mathcal{F}} \longrightarrow (-)^h$.

Definition 9.8. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is an

- *hf-equivalence* if f^h is a pointwise weak equivalence.
- *hf-fibration* if f is a pointwise fibration and there is a homotopy pullback square in $G\mathcal{S}$

$$\begin{array}{ccc} XK & \longrightarrow & XL \\ {\scriptstyle f(K)}\downarrow & & \downarrow{\scriptstyle f(L)} \\ YK & \longrightarrow & YL \end{array}$$

  for every G-weak equivalence $K \xrightarrow{\sim} L$ of finitely presentable G-spaces.

Lemma 9.9. *A map of G-simplicial functors is a pointwise acyclic fibration if and only if it is an hf-fibration and an hf-equivalence.*

Proof. The definition of hf-equivalences in 5.2 uses the filtered fibrant replacement functor Φ^{J_G} from 3.3.3. All G-spaces are cofibrant, so the canonical map $\omega_K\colon \Phi^{J_G}K \longrightarrow RK$ is a G-weak equivalence of fibrant G-spaces, hence a simplicial homotopy equivalence. Recall that $\hbar(X) = \mathbb{I}_*X \circ \Phi^{J_G}$. If K is finite, the induced map $\mathbb{I}_*X(\omega_K)\colon \hbar(X)(K) \longrightarrow X^hK$ is a simplicial homotopy equivalence by 2.11, in particular a G-weak equivalence. It follows that f^h is a pointwise weak equivalence if and only if $\hbar(f)(K)$ is a G-weak equivalence for every finite G-space K. The arguments in 5.4 show that the hf-fibrations in 9.8 allow the same characterization as general hf-fibrations, cp. 5.6. Since every G-space is cofibrant, any G-weak equivalence of finite G-spaces can be factored as an acyclic cofibration of finite G-spaces and a simplicial homotopy equivalence. The lemma follows from 5.8. □

Theorem 9.10. *The category $G\mathcal{F}$, equipped with the classes of hf-equivalences, hf-fibrations and cofibrations, is a proper monoidal model category satisfying the monoid axiom. Smashing with a cofibrant G-simplicial functor preserves hf-equivalences. One can choose generating cofibrations and generating acyclic cofibrations with finitely presentable domains and codomains.*

Proof. The model structure exists according to 5.10, is monoidal by 5.12 and proper by 5.13. A factorization argument and 9.7 imply the claim concerning cofibrant G-simplicial functors. The monoid axiom follows easily.
The generating acyclic cofibrations are $\mathcal{H}_G \cup \mathcal{P}_{J_G}$. It remains to define $\mathcal{H}_G$. A G-weak equivalence of finite G-spaces $w\colon K \longrightarrow L$ induces an hf-equivalence $G\mathcal{S}(w,-)\colon G\mathcal{S}(L,-) \longrightarrow G\mathcal{S}(K,-)$. The simplicial mapping cylinder gives a cofibration $c_w : G\mathcal{S}(L,-) \rightarrowtail C_w$. Let $i \in I_G$. Then $\mathcal{H}_G$ is the set of pushout product maps $\{c_w \Box i\}$, cp. Subsection 5.2. It is also possible, without changing the model structure, to consider for w only acyclic cofibrations of finite G-spaces. □

Denote the model category in 9.10 by $G\mathcal{F}_{\mathrm{hf}}$. In this category, $X \longrightarrow R \circ X^h$ is a fibrant replacement of X. Recall the functor $\mathbf{S}\colon G\mathcal{F} \longrightarrow G\mathcal{F}$ mapping X to $G\mathcal{S}(S^{\wedge G}, X(S^{\wedge G} \wedge -))$ and the natural transformation $s\colon \mathrm{Id}_{G\mathcal{F}} \longrightarrow \mathbf{S}$ obtained pointwise as the adjoint of $\mathrm{sw}^X_{S^{\wedge G}}(K)\colon XK \wedge S^{\wedge G} \longrightarrow X(S^{\wedge G} \wedge K)$. This map is the adjoint of

$$S^{\wedge G} \xrightarrow{\eta_K S^{\wedge G}} G\mathcal{S}(K, S^{\wedge G} \wedge K) \xrightarrow{\mathrm{hom}^X_{K, S^{\wedge G} \wedge K}} G\mathcal{S}(XK, X(S^{\wedge G} \wedge K)).$$

Let $\mathbf{S}^\infty(X)$ denote the colimit of $X \xrightarrow{s(X)} \mathbf{S}(X) \xrightarrow{\mathbf{S}(s(X))} \mathbf{S}(\mathbf{S}(X)) \longrightarrow \cdots$, and write $s\colon \mathrm{Id}_{G\mathcal{F}} \longrightarrow \mathbf{S}^\infty$ for the canonically induced natural transformation.

Definition 9.11. A map $f\colon X \longrightarrow Y$ in $G\mathcal{F}$ is a

- *stable equivalence* if $\mathbf{S}^\infty(R \circ f^h)$ is a pointwise weak equivalence,
- *stable fibration* if f is an hf-fibration and

$$\begin{array}{ccc} X & \xrightarrow{s(X)} & \mathbf{S}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{S}(f)} \\ X & \xrightarrow{s(X)} & \mathbf{S}(X) \end{array}$$

  is a homotopy pullback square in $G\mathcal{F}_{\mathrm{pt}}$.

Lemma 9.12. *A map of G-simplicial functors is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

Proof. ¿From the proof of 9.9, one sees that the definition of stable equivalences in 9.11 agrees with 6.2. The result follows from 6.14. □

The definition of stable fibrations leads to a set of generating stable acyclic cofibrations as in section 6.4. Recall the definition of $S^{\wedge G}$-spectra in $G\mathcal{S}$.

Definition 9.13. A *G-spectrum* E consists of a sequence $E_0, E_1, \cdots$ of G-spaces, together with structure maps $E_n \wedge S^{\wedge G} \longrightarrow E_{n+1}$. A *map* $f\colon E \longrightarrow F$

of G-spectra is a sequence of maps $f_n\colon E_n \longrightarrow F_n$ making the diagram

$$\begin{tikzcd} E_n \wedge S^{\wedge G} \arrow[r] \arrow[d, "f_n \wedge S^{\wedge G}"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\ F_n \wedge S^{\wedge G} \arrow[r] & F_{n+1} \end{tikzcd}$$

commutative for every $n \geq 0$.

By 2.12, $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ is isomorphic to the enriched category $[S^{\wedge G}\mathrm{Sph}, G\mathcal{S}]$ of G-simplicial functors from the category of $S^{\wedge G}$-spheres $S^{\wedge G}\mathrm{Sph}$ to $G\mathcal{S}$. Thus 4.2 gives $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ a pointwise model structure. A fibrant replacement functor in this model structure is $E \longmapsto R \circ E$, where R is defined in 3.3.2.
Recall that the adjoints of the structure maps of a G-spectrum E can be viewed as a natural map $E \longrightarrow \Omega_G \mathrm{sh} E$, where the nth structure map of the G-spectrum $\Omega_G \mathrm{sh} E$ is $G\mathcal{S}(S^{\wedge G}, E_{n+1} \longrightarrow G\mathcal{S}(S^{\wedge G}, E_{n+2}))$, the $S^{\wedge G}$-loops of the $n+1$-th structure map of E. Denote this natural transformation by $\mathrm{st}\colon \mathrm{Id} \longrightarrow \mathrm{St} = \Omega_G \circ \mathrm{sh}$, and let $\mathrm{st}^\infty\colon \mathrm{Id} \longrightarrow \mathrm{St}^\infty$ be the colimit of $\mathrm{Id} \xrightarrow{\mathrm{st}} \mathrm{St} \xrightarrow{\mathrm{St(st)}} \mathrm{St}^2 \longrightarrow \cdots$.
The stable model structure on G-spectra, which has the same cofibrations as the pointwise model structure, is defined as follows.

Definition 9.14. A map $f\colon E \longrightarrow F$ of G-spectra is a *stable equivalence* if $\mathrm{St}^\infty(R \circ f)$ is a pointwise weak equivalence, and a *stable fibration* if f is a pointwise fibration such that

$$\begin{tikzcd} E \arrow[r, "\mathrm{st}"] \arrow[d, "f"'] & \mathrm{St}(E) \arrow[d, "\mathrm{St}(f)"] \\ F \arrow[r, "\mathrm{st}"] & \mathrm{St}(F) \end{tikzcd}$$

is a homotopy pullback square in the pointwise model structure.

¿From now on, we consider G-spectra with the stable model structure. Note that geometric realization induces a functor from $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ to the category of G-prespectra, cf. [3, 3.2]. It is plausible that this functor induces an equivalence of homotopy categories. Hence the homotopy category of $\mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ is the G-equivariant stable homotopy category. Let us illustrate this by introducing spectra which are indexed on more general representations.
A *G-representation* is a finite-dimensional euclidean vector space on which G acts via linear isometries. A G-representation V is *irreducible* if zero and V are the only sub-G-representations. Let $\mathrm{Irr}_G = \{W_1, \ldots, W_r\}$ be a complete set of pairwise non-isomorphic irreducible G-representations. Every G-representation is isomorphic to a direct sum of representations in Irr_G. That is, given a G-representation V, there exist unique natural numbers $(n_1, \ldots, n_r)$ such that V is isomorphic to $W_1^{\oplus n_1} \oplus \cdots \oplus W_r^{\oplus n_r}$.

If V is a G-representation, let S^V_{top} denote the one-point compactification of V, with ∞ as the (G-fixed) basepoint. This is a finite G-CW-complex. One has $S^{V\oplus W}_{\mathrm{top}} \cong S^V_{\mathrm{top}} \wedge S^W_{\mathrm{top}}$. Furthermore, one can choose a finite G-space S^V such that the geometric realization $|S^V|$ is homeomorphic to S^V_{top}. Let Rep be the $G\mathcal{S}$-category with objects smash products

$$S^{n_1,\dots,n_r} := S^{W_1} \wedge \cdots \wedge S^{W_1} \wedge S^{W_2} \wedge \cdots \wedge S^{W_r}$$

and morphisms $G\mathcal{S}_{\mathrm{Rep}}(S^{n_1,\dots,n_r}, S^{n_1+k_1,\dots,n_r+k_r}) := S^{k_1,\dots,k_r}$. Hence Rep contains essentially all G-representations. The G-sphere $S^{\wedge G}$ does not reside in this category, but it contains a G-space $\tilde{S}^{\wedge G}$ with homeomorphic realization. Consequently, the stable model categories $[S^{\wedge G}, G\mathcal{S}]$ and $[\tilde{S}^{\wedge G}, G\mathcal{S}]$ are Quillen equivalent, and there are inclusions $\tilde{S}^{\wedge G} \overset{j}{\hookrightarrow} \mathrm{Rep} \hookrightarrow \mathbf{f}G\mathcal{S}$ inducing functors

$$G\mathcal{F} = [\mathbf{f}G\mathcal{S}, G\mathcal{S}] \longrightarrow [\mathrm{Rep}, G\mathcal{S}] \xrightarrow{j^*} [\tilde{S}^{\wedge G}, G\mathcal{S}].$$

Note that j is a full inclusion.

It is straightforward to define the stabilization $\mathrm{St}^\infty_{\mathrm{Rep}}X$ of a pointwise fibrant $G\mathcal{S}$-functor $X\colon \mathrm{Rep} \longrightarrow G\mathcal{S}$. Since every G-representation is a direct summand of a direct sum of copies of the regular representation $\mathbb{R}^G$, $j^*(\mathrm{St}^\infty_{\mathrm{Rep}}X)$ is equivalent to $\mathrm{St}^\infty j^*X$. This shows that the stable model structure on $[\mathrm{Rep}, G\mathcal{S}]$ is Quillen equivalent to the stable model structure on G-spectra. In particular, smashing with S^V is a Quillen equivalence of G-spectra for every G-representation V.

Let us turn to the last ingredient needed in the proof of 9.16.

Proposition 9.15. *The functor* $- \wedge S^{\wedge G}\colon \mathrm{Sp}(G\mathcal{S}, S^{\wedge G}) \longrightarrow \mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ *is a Quillen equivalence.*

Proof. We will show that $S^{\wedge G}$ is G-weakly equivalent to a symmetric G-space. The result follows then by [8, 10.3]. A G-space K is symmetric if there exists a map H such that the following diagram commutes, where $\mathrm{cyc}\colon K^{\wedge 3} \longrightarrow K^{\wedge 3}$ is the cyclic permutation map.

$$\begin{tikzcd}
K^{\wedge 3} \wedge \mathrm{ct}S^0 \arrow[r, "K^{\wedge 3}\wedge \mathrm{ct}i_0"] \arrow[dr, "id_{K^{\wedge 3}\wedge \mathrm{ct}S^0}"'] & K^{\wedge 3} \wedge \mathrm{ct}\Delta^1_+ \arrow[d, "H"] & K^{\wedge 3} \wedge \mathrm{ct}S^0 \arrow[l, "K^{\wedge 3}\wedge \mathrm{ct}i_1"'] \arrow[dl, "\mathrm{cyc}\wedge \mathrm{ct}S^0"] \\
 & K^{\wedge 3} \wedge \mathrm{ct}S^0 &
\end{tikzcd}$$

Consider first the trivial group. The cyclic permutation map on the geometric realization $|S^1|^{\wedge 3}$ is homotopic to the identity. Thus the singular complex $K := \mathrm{sing}|S^1|$ is a symmetric space and there is a weak equivalence $S^1 \longrightarrow K$. We claim the induced map $S^{\wedge G} \longrightarrow K^{\wedge G}$ of G-fold smash products is a G-weak equivalence. To see this, choose a subgroup H of G and write the underlying set of G as the union of the cosets gH. Fix a space L, and an element $x = x_{g_1} \wedge \cdots \wedge x_{g_n}$ of $L^{\wedge G}$ where n is the order of G and $x_{g_k} \in L$ for

every k. Note that x is invariant under the action of all $h \in H$ if and only if $x_{g_k} = x_{g_j}$ for all g_k, g_j in the same coset of H. Thus $\mathrm{Fix}(H, L^{\wedge G})$ is – up to a natural isomorphism – the G/H-fold smash product of L. In particular, we get an expression of the "diagonal" $d_L\colon \mathrm{ct}L \longrightarrow L^{\wedge G}$ as the adjoint of the isomorphism $L \longrightarrow \mathrm{Fix}(G, L^{\wedge G})$.
We will use essentially three maps to define a homotopy from the identity map to the cyclic permutation map on $K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G}$: (1) the natural isomorphism, cp. 9.1, of G-spaces $f_{K,L}\colon K^{\wedge G} \wedge L^{\wedge G} \longrightarrow (K \wedge L)^{\wedge G}$ which rearranges the factors, (2) the diagonal $d = d_{\Delta^1_+}\colon \mathrm{ct}\Delta^1_+ \longrightarrow (\Delta^1_+)^{\wedge G}$, and (3) the homotopy $F : K \wedge K \wedge K \wedge \Delta^1_+ \longrightarrow K \wedge K \wedge K$ from the cyclic permutation map to the identity map. Consider now the composition

$$\begin{array}{c}
K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G} \wedge \mathrm{ct}\Delta^1_+ \\
\Big\downarrow {\scriptstyle \mathrm{id}\wedge d} \\
K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G} \wedge (\Delta^1_+)^{\wedge G} \\
\Big\downarrow {\scriptstyle \cong} \\
(K \wedge K \wedge K)^{\wedge G} \\
\Big\downarrow {\scriptstyle F^{\wedge G}} \\
(K \wedge K \wedge K \wedge \Delta^1_+)^{\wedge G} \\
\Big\downarrow {\scriptstyle \cong} \\
K^{\wedge G} \wedge K^{\wedge G} \wedge K^{\wedge G}.
\end{array}$$

This is the homotopy which shows that $K^{\wedge G}$ is a symmetric G-space. □

Theorem 9.16. *The category $G\mathcal{F}$ and the classes of stable equivalences, stable fibrations and cofibrations, is a proper monoidal model category satisfying the monoid axiom. One can choose generating cofibrations and generating acyclic cofibrations with finite domains and codomains. Smashing with a cofibrant G-simplicial functor preserves stable equivalences.*

Proof. The results above allow us to apply 6.26, 6.27, 6.28, 6.29 and 6.30. □

Let $G\mathcal{F}_{\mathrm{st}}$ refer to the *stable* model category in 9.16. We end this section by comparing $G\mathcal{F}_{\mathrm{st}}$ with the stable model category of G-spectra. By Section 2.5, $S^{\wedge G}\mathrm{Sph}$ is a sub-$G\mathcal{S}$-category of $\mathbf{f}G\mathcal{S}$. Let i_* be the enriched left Kan extension along the corresponding inclusion i. It is left adjoint to pre-composition with i, which we denote by ev. The next result follows from 7.2.

Lemma 9.17. $\mathrm{ev}\colon G\mathcal{F}_{\mathrm{st}} \longrightarrow \mathrm{Sp}(G\mathcal{S}, S^{\wedge G})$ *is a right Quillen functor.*

Lemma 7.4 implies that the unit of the adjunction has the following property.

Lemma 9.18. *The canonical map* $E \longrightarrow \mathrm{ev}(i_*E)^h$ *is a stable equivalence of* G*-spectra for every cofibrant* G*-spectrum* E.

If G is the trivial group, Lydakis proved that a map of homotopy functors is a stable equivalence if and only its evaluation is a stable equivalence of spectra. The proof uses the Blakers-Massey theorem. We will extend this result to any finite group using Spanier-Whitehead duality, cp. [14, 17.6].

Proposition 9.19. *Let* K *and* L *be finitely presentable* G*-spaces. The canonical map*

$$\mathrm{ev}(G\mathcal{S}(K,R(-))\wedge L) \longrightarrow \mathrm{ev}(G\mathcal{S}(K,R(L\wedge -)))$$

of G*-spectra is a stable equivalence.*

Proof. Let $E\star F$ denote the (closed) symmetric monoidal product in the equivariant stable homotopy category $\mathrm{SH}(G)$, with unit $\mathbb{S}$, and let $\mathrm{Hom}(E,-)$ denote the right adjoint of $-\star E$. A G-spectrum D is *dualizable* if the canonical map $\mathrm{Hom}(D,\mathbb{S})\star D \longrightarrow \mathrm{Hom}(D,D)$ is an isomorphism in $\mathrm{SH}(G)$. It follows that the canonical map $\mathrm{Hom}(D,E)\star F \longrightarrow \mathrm{Hom}(D,E\star F)$ is an isomorphism for all $E,F\in \mathrm{Ob}\,\mathrm{SH}(G)$ if D is dualizable [10, II, Section 1],
Suspension G-spectra of finite G-spaces are dualizable in $\mathrm{SH}(G)$ [3, 2.C], [10, II, 2.7]. In particular, given finite G-spaces K and L, the canonical map

$$\mathrm{Hom}(\mathrm{ev}(-\wedge K),\mathrm{ev}R(-))\star\mathrm{ev}(-\wedge L) \longrightarrow \mathrm{Hom}(\mathrm{ev}(-\wedge K),\mathrm{ev}R(-)\star\mathrm{ev}(-\wedge L))$$

is an isomorphism in $\mathrm{SH}(G)$. In this special situation, a map of G-spectra lifting this isomorphism can be given as

$$\mathrm{St}^\infty(R(G\mathcal{S}(K,\mathrm{ev}R(-))\wedge L \longrightarrow G\mathcal{S}(K,\mathrm{ev}R(L\wedge -)))).$$

In particular, this map is a stable equivalence. This finishes the proof, because the above is a stably fibrant replacement of the map in question. $\square$

Corollary 9.20. *Let* X *be a* G*-simplicial functor and* L *be a finite* G*-space. The canonical map* $\mathrm{ev}X^h\wedge L \longrightarrow \mathrm{ev}X^h(L\wedge -)$ *is a stable equivalence of* G*-spectra. In particular,* ev *reflects stable equivalences of homotopy functors.*

Proof. We sketch a proof, following the script for the trivial group [11, 11.7]. Consider the first statement. If X is a cofibrant G-simplicial functor, use 9.19 by attaching cells. If X is arbitrary, use a cofibrant replacement X^c. The second statement then follows from 6.21. $\square$

Corollary 9.21. *The stable model structure on* G*-simplicial functors is Quillen equivalent to the stable model structure on* G*-spectra via the right Quillen functor* ev.

The Quillen equivalence from 9.21 factors through a Quillen equivalence to the category of symmetric G-spectra. In fact, the work [8] of Hovey shows that G-spectra and symmetric G-spectra are Quillen equivalent via a zig-zag of Quillen equivalences, but not necessarily via the canonical forgetful functor. In this case, however, it is possible to conclude this by extending results of [9] to G-spaces. Here are the details.

Theorem 9.22. *The forgetful functor from symmetric G-spectra to G-spectra is the right adjoint of a Quillen equivalence.*

Proof. In both the categories of G-spectra and symmetric G-spectra, a map of fibrant objects is a weak equivalence if and only if it is a pointwise weak equivalence. Hence the forgetful functor U preserves and reflects weak equivalence of fibrant objects. Its left adjoint V preserves cofibrations and pointwise acyclic cofibrations. Further, if L resp. L^Σ is the set of maps of G-spectra resp. symmetric G-spectra that Hovey uses to localize the pointwise model structures, then V maps L to L^Σ (up to isomorphism). Hence V also preserves stable acyclic cofibrations by properties of Bousfield localization [6] and is thus a left Quillen functor.

It remains to prove that the canonical map

$$E \longrightarrow U(V(E)^f)$$

is a stable equivalence of G-spectra for E cofibrant. Here $(-)^f$ denotes a fibrant replacement in the stable model structure of symmetric G-spectra. In fact, since both U and V preserve colimits and homotopy pushouts, it suffices to prove this for E varying through the domains and codomains of the generating cofibrations. To do so, we use the functors $\mathrm{Fix}(H,-)$ on (symmetric) spectrum level. This means the following. If E is a G-spectrum, the sequence $(\mathrm{Fix}(H,E_0),\mathrm{Fix}(H,E_1),\dots)$ is a $Fix(H,S^{\wedge G}) = S^{|G/H|}$-spectrum of spaces. That is, it has structure maps

$$\mathrm{Fix}(H,S^{\wedge G})\wedge \mathrm{Fix}(H,E_n)\cong \mathrm{Fix}(H,S^{\wedge G}\wedge E_n)\longrightarrow \mathrm{Fix}(H,E_{n+1}).$$

The same construction works on the level of symmetric G-spectra, so we have a commutative diagram

$$\begin{tikzcd} \mathrm{Sp}^\Sigma(G\mathcal{S},S^{\wedge G}) \arrow[r, "\mathrm{Fix}(H,-)"] \arrow[d, "U"'] & \mathrm{Sp}^\Sigma(\mathcal{S},S^{|G/H|}) \arrow[d, "U_H"] \\ \mathrm{Sp}(G\mathcal{S},S^{\wedge G}) \arrow[r, "\mathrm{Fix}(H,-)"] & \mathrm{Sp}(\mathcal{S},S^{|G/H|}). \end{tikzcd}$$

The domains and codomains of the generating cofibrations of G-spectra are of the form $\mathrm{Fr}_n(K)$ (representable $\wedge K$) for K in a certain set of G-spaces, and similarly for the $S^{|G/H|}$-spectra of spaces. Since $\mathrm{Fix}(H,-)\colon G\mathcal{S}\longrightarrow \mathcal{S}$ commutes with the smash product (up to natural isomorphism), we get a natural isomorphism

$$V_H(\mathrm{Fr}_n(\mathrm{Fix}(H,K)))\cong V_H(\mathrm{Fix}(H,\mathrm{Fr}_n(K)))\cong \mathrm{Fix}(H,V(\mathrm{Fr}_n(K))).$$

This isomorphism is compatible with the units of the adjunctions (V, U) and (V_H, U_H), so that

$$\mathrm{Fix}(H, \mathrm{Fr}_n(K) \longrightarrow UV\mathrm{Fr}_n(K)) \cong \mathrm{Fr}_n\mathrm{Fix}(H, K) \longrightarrow U_H V_H(\mathrm{Fr}_n\mathrm{Fix}(H, K)).$$

The categories of (symmetric) $S^{|G/H|}$-spectra of spaces are just slight variations of the categories of (symmetric) spectra of spaces, which implies that (V_H, U_H) is a Quillen equivalence. To conclude the same for (V, U), it is sufficient, by the above, to prove the following two facts.

- A map f of G-spectra is a stable equivalence if so is $\mathrm{Fix}(H, f)$ for every subgroup H.
- If $j_E\colon E \overset{\sim}{\rightarrowtail} E^f$ is a stably fibrant replacement of the symmetric G-spectrum E, then $\mathrm{Fix}(H, j_E\colon E \overset{\sim}{\rightarrowtail} E^f)$ is a stably fibrant replacement of $\mathrm{Fix}(H, E)$.

Concerning the first fact: a map f of G-spectra is clearly a pointwise weak equivalence if and only if so is $\mathrm{Fix}(H, f)$ for every subgroup H. Since a stable equivalence of stably fibrant G-spectra is a pointwise weak equivalence, it suffices to prove that $\mathrm{Fix}(H, -)$ preserves stably fibrant replacements for G-spectra. For this purpose, we apply the small object argument to the following set. Let $J_{\mathrm{pt}} := \{\mathrm{Fr}_n(\wedge(G/H \times (\Lambda^m_i \hookrightarrow \Delta^m))_+\}_{n,H,m,i}$ be the set of generating pointwise acyclic cofibrations. Obtain $\tilde{J}_{\mathrm{st}}$ from $L = \{\mathrm{Fr}_{n+1}S^{\wedge G} \longrightarrow \mathrm{Fr}_n(S^0)\}_n$ by applying the simplicial mapping cylinder, and let J_{st} be the set

$$\tilde{J}_{\mathrm{st}} \Box \{(G/H \times (\partial\Delta^m \hookrightarrow \Delta^m))\}_{H,m}$$

of pushout product maps. Finally, $J = J_{\mathrm{pt}} \cup J_{\mathrm{st}}$ is the set we may use for a fibrant replacement[1]. Note first that $\mathrm{Fix}(H, \mathrm{Fr}_{n+1}S^{\wedge G} \longrightarrow \mathrm{Fr}_n(S^0)) \cong \mathrm{Fr}_{n+1}S^{|G/H|} \longrightarrow \mathrm{Fr}_n(S^0)$. Further, $\mathrm{Fix}(H, -)$ is compatible with the simplicial mapping cylinder construction, since it commutes with the smash product and with pushouts of diagrams containing a monomorphism. The latter fact was already used in the proof of 9.5. It follows that $\mathrm{Fix}(H, -)$ maps J to the corresponding set J_H in the category $\mathrm{Sp}(\mathcal{S}, S^{|G/H|})$. In particular, $\mathrm{Fix}(H, -)$ maps sequential compositions of cobase changes of maps in J to stable equivalences.

To conclude that $\mathrm{Fix}(H, -)$ preserves the fibrant replacement, we have to show that it preserves stably fibrant objects. One can see this by arguing that it is in fact a right Quillen functor, whose left adjoint l_H is determined by the requirement that $l_H\mathrm{Fr}_n(K) = \mathrm{Fr}_n(G/H_+ \wedge K)$ for any n and any space K. Using this description, one can see that l_H preserves generating (acyclic) cofibrations, hence is a left Quillen functor. It follows that $\mathrm{Fix}(H, -)$ preserves the fibrant replacement. This proof translates to the category of symmetric

[1] The set J is in fact a set of generating acyclic cofibrations.

G-spectra, which justifies the second fact in the list above. This finishes the proof. $\square$

Since the resulting Quillen equivalence (j_*, j^*) between G-simplicial functors and symmetric G-spectra has nice monoidal properties according to 2.16, the closed symmetric monoidal structure induced by the smash product of G-simplicial functors is the correct one. Given the above, comparisons of modules and algebras along the lines of [14] are possible. Note, however, that at present it is not clear how to compare commutative algebras [14, 0.9].

Corollary 9.23. *The model categories of symmetric ring G-spectra and of $\mathbb{I}$-algebras in $G\mathcal{F}$ are Quillen equivalent via the canonical adjoint pair (ι_*, ι^*). If R is a cofibrant $\mathbb{I}$-algebra, the model categories of R-modules and of $\iota^* R$-modules are Quillen equivalent. If Q is a cofibrant symmetric ring G-spectrum, the model categories of Q-modules and of j_*Q-modules are Quillen equivalent.*

Proof. First we observe that the model category of symmetric G-spectra satisfies the monoid axiom. Here are some details. A pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r] & Z \cup_X Y \end{tikzcd}$$

of symmetric G-spectra in which f is a pointwise cofibration is a homotopy pushout square. This, the fact that stable equivalences of symmetric G-spectra are closed under sequential colimits (see 3.5) and general arguments from [15] show that it suffices to prove the following. Let X be a symmetric G-spectrum and $j\colon sj \overset{\sim}{\rightarrowtail} tj$ a generating acyclic cofibration, then $X \wedge j$ is a stable equivalence and a pointwise cofibration. To see that the latter holds, note that the pushout product map of a pointwise cofibration and a cofibration is a pointwise cofibration, by comparing with the smash product of *symmetric sequences of G-spaces* as in [9, section 5.3]. Now one can use that the stable model structure on symmetric G-spectra is *stable* in the sense that suspension with S^1 is a Quillen equivalence. So $X \wedge j$ will be a stable equivalence if and only if $X \wedge (tj/sj)$ is stably equivalent to a point. Note that tj/sj is cofibrant. Hence it suffices to prove that the smash product of a cofibrant symmetric G-spectrum and a pointwise weak equivalence is a pointwise weak equivalence. By arguments which already appeared in this proof, one can reduce to the case of the domains and codomains of the generating cofibrations (which are cofibrant). These, however, are gotten directly from symmetric sequences of G-spaces. A comparison like [9, proof of 5.3.7] of the smash products of symmetric G-spectra and these sequences concludes the proof of the monoid axiom.

The hard work is done. Let $\mathbb{S}$ denote the unit in the category of symmetric G-spectra, and recall that the unit in $G\mathcal{F}$ is the inclusion $\mathbb{I}$. By 2.16, the canonical adjoint pair (j_*, j^*) induced by the inclusion $j\colon S^{\wedge G}\mathrm{Sph}^{\Sigma} \hookrightarrow \mathbf{f}G\mathcal{S}$

induces an adjoint pair

$$\iota_*\colon \mathrm{alg}_{\mathbb{S}} \rightleftarrows \mathrm{alg}_{\mathbb{I}}\colon \iota^*.$$

Since the forgetful functors with domain $\mathrm{alg}_{\mathbb{S}}$ resp. $\mathrm{alg}_{\mathbb{I}}$ detect weak equivalences and fibrations, (ι_*, ι^*) is a Quillen adjunction. Moreover, since j^* preserves and detects weak equivalences of fibrant G-simplicial functors, ι^* detects weak equivalences of fibrant $\mathbb{I}$-algebras. To conclude that (ι_*, ι^*) is a Quillen equivalence, it suffices to note that a cofibrant $\mathbb{S}$-algebra is in particular a cofibrant symmetric G-spectrum [15, 4.1].
The other two cases are similar, modulo an application of 8.4. □

It is desirable to compare the stable model category of G-simplicial functors also with the stable model category of orthogonal G-spectra [13]. As an intermediate step, we will use symmetric spectra of topological G-spaces. Let $\mathbf{T}$ denote the (closed symmetric monoidal) model category of pointed compactly generated topological spaces [7, 2.4.21], and let $G\mathbf{T}$ be the category of G-objects in $\mathbf{T}$. The latter is closed symmetric monoidal by an analog of 9.1, and it is a monoidal model category by transferring the model structure from 9.5 to the topological situation (see [13, II.1.8 and II.1.22]). The (strict symmetric monoidal) Quillen equivalence $|-|\colon \mathcal{S} \longrightarrow \mathbf{T}$ given by geometric realization extends to a (strict symmetric monoidal) Quillen equivalence $|-|\colon G\mathcal{S} \longrightarrow G\mathbf{T}$. In particular, we can regard $G\mathbf{T}$ as a $G\mathcal{S}$-model category. As one can check using results from [13, II.1], the model structure on $G\mathbf{T}$ is cellular, so the stable model category of symmetric spectra in $G\mathbf{T}$ with respect to $|S^{\wedge G}|$ exists by [8]. Further, we can apply [8, 9.3] to conclude that

$$|-|\colon \mathrm{Sp}^{\Sigma}(G\mathcal{S}, S^{\wedge G}) \longrightarrow \mathrm{Sp}^{\Sigma}(G\mathbf{T}, |S^{\wedge G}|)$$

is a Quillen equivalence. By inspection, this Quillen functor is strict symmetric monoidal, and its right adjoint is lax symmetric monoidal. Thus to obtain a variant of 9.23, it suffices to note that the stable model category $\mathrm{Sp}^{\Sigma}(G\mathbf{T}, |S^{\wedge G}|)$ satisfies the monoid axiom. A proof can be obtained by translating the proof of the monoid axiom for $\mathrm{Sp}^{\Sigma}(G\mathcal{S}, S^{\wedge G})$ to the topological situation. It remains to relate orthogonal G-spectra to symmetric G-spectra of topological spaces. For the definition of an orthogonal G-spectrum, which we take to be indexed on all G-representations, consider [13]. Any orthogonal G-spectrum X gives rise to a symmetric G-spectrum of topological spaces uX by neglect of structure, or rather – since both objects are simply enriched functors on certain domain $G\mathbf{T}$-categories – restriction. The restriction takes place both on objects (from all G-representations to direct sums of the regular representation $\mathbb{R}^G$) and morphisms (from orthogonal groups to symmetric groups). See [14, 4.4] (for the non-equivariant case) and [13, II.4]. Viewed as a restriction, u has a left adjoint v by enriched Kan extension.

Theorem 9.24. *The adjoint pair (v, u) is a Quillen equivalence. It induces Quillen equivalences between orthogonal ring G-spectra and symmetric ring G-spectra of topological spaces. If R is a cofibrant symmetric ring G-spectrum and*

P is a cofibrant orthogonal ring G-spectrum, (v,u) induces Quillen equivalences

$$\mathrm{mod}_R \rightleftarrows \mathrm{mod}_{vR} \qquad \mathrm{mod}_{uP} \rightleftarrows \mathrm{mod}_P.$$

Proof. The forgetful functor $U_{\mathbf{T}}\colon \mathrm{Sp}^\Sigma(G\mathbf{T},|S^{\wedge G}|) \longrightarrow \mathrm{Sp}(G\mathbf{T},|S^{\wedge G}|)$ has a left adjoint $V_{\mathbf{T}}$ fitting into a commutative diagram

$$\begin{tikzcd}
\mathrm{Sp}(G\mathcal{S},S^{\wedge G}) \arrow[r, "|-|"] \arrow[d, "V"'] & \mathrm{Sp}(G\mathbf{T},|S^{\wedge G}|) \arrow[d, "V_{\mathbf{T}}"] \\
\mathrm{Sp}^\Sigma(G\mathcal{S},S^{\wedge G}) \arrow[r, "|-|"] & Spe^\Sigma(G\mathbf{T},|S^{\wedge G}|)
\end{tikzcd}$$

in which the left vertical functor resp. the lower horizontal functor are Quillen equivalences by 9.22 resp. [8, 9.3]. The upper horizontal functor is a Quillen equivalence by [8, 5.7], hence the right vertical functor is a Quillen equivalence. Thus to conclude that (v,u) is a Quillen equivalence, it suffices to prove that the forgetful functor from orthogonal G-spectra to $\mathrm{Sp}(G\mathbf{T},|S^{\wedge G}|)$ is a Quillen equivalence.

By [13, III.4.16], the forgetful functor from orthogonal G-spectra to G-*prespectra* as defined in [13, II.1.2] is a Quillen equivalence. The category of G-prespectra so far is indexed on all G-representations. However, as observed in [13, II.2.2 and V.1.10], one can index both orthogonal G-spectra and G-prespectra on a collection of G-representations with is both closed under direct sum and cofinal in the collection of all G-representations without changing the homotopy theory. An acceptable candidate is the collection of direct sums of the regular representation. Hence the restriction from orthogonal G-spectra to $\mathrm{Sp}(G\mathbf{T},|S^{\wedge G}|)$ is a Quillen equivalence. This proves the first statement. The other statements then follow as in the proof of 9.23, since the monoid axiom holds for orthogonal G-spectra [13, III.7.4]. □

Hence for the purpose of studying the homotopy theory of algebras and modules, the category of G-simplicial functors is as good as the category of orthogonal G-spectra for a finite group G. Another comparison functor can be obtained as in [14, 19.11] by passing from $G\mathcal{F}$ to $G\mathbf{T}$-functors from an appropriate domain category (say, finite G-CW-complexes) to $G\mathbf{T}$ via geometric realization, and then restricting to orthogonal G-spectra. Up to geometric realization, this functor amounts to a neglect of structure.

A Calculations

This appendix looks into the proofs of the remaining claims in the main part of the paper. The structure map $\hom^{-\otimes T}_{A,B}\colon \mathcal{V}(A,B) \longrightarrow \mathcal{V}(A\otimes T, B\otimes T)$ of the $\mathcal{V}$-functor $-\otimes T\colon \mathcal{V} \longrightarrow \mathcal{V}$ is defined as the adjoint of the composition

$$\mathcal{V}(A,B)\otimes(A\otimes T) \xrightarrow[\cong]{\alpha^{-1}_{\mathcal{V}(A,B),A,T}} (\mathcal{V}(A,B)\otimes A)\otimes T \xrightarrow{(\epsilon_A B)\otimes T} T\otimes A$$

where $\alpha^{-1}_{\mathcal{V}(B,T),B,A}$ is the associativity isomorphism. The next lemma shows that there are in general two different suspension functors for T-spectra.

Lemma A.1. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(A,B)\otimes(A\otimes T) \arrow[r, "\hom^{-\otimes T}_{A,B}\otimes(A\otimes T)"] \arrow[d, "\alpha^{-1}_{\mathcal{V}(A,B),A,T}"'] & \mathcal{V}(A\otimes T,B\otimes T)\otimes(A\otimes T) \arrow[d, "\epsilon_{A\otimes T}(B\otimes T)"] \\
(\mathcal{V}(A,B)\otimes A)\otimes T \arrow[r, "(\epsilon_A B)\otimes T"] & B\otimes T
\end{tikzcd}$$

Proof. Use naturality and the triangular identity

$$\epsilon_{A\otimes T}(\mathcal{V}(A,B)\otimes(A\otimes T))\circ(\eta_{A\otimes T}\mathcal{V}(A,B))\otimes(A\otimes T)=\mathrm{id}_{\mathcal{V}(A,B)\otimes(A\otimes T)}.$$

□

A similar statement is used to show that the stabilization of enriched functors and the stabilization of spectra can be compared.

Lemma A.2. *The following diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "\eta_B A"] \arrow[d, "\eta_{B\otimes T}A"'] & \mathcal{V}(B,A\otimes B) \arrow[d, "\hom^{-\otimes T}_{B,A\otimes B}"] \\
\mathcal{V}(B\otimes T,A\otimes(B\otimes T)) \arrow[r, "\mathcal{V}(B\otimes T,\alpha^{-1}_{A,B,T})"] & \mathcal{V}(B\otimes T,(A\otimes B)\otimes T)
\end{tikzcd}$$

Proof. Similar to A.1, using $\epsilon_B(A\otimes B)\circ(\eta_B A)\otimes B=\mathrm{id}_{A\otimes B}$. □

Next we start the proof of: the two natural stabilization maps $X\longrightarrow T(X)$ described in 6.1 coincide. It is lengthy and perhaps not very illuminating. The map $\hom^{\mathcal{V}(T,-)}_{A,B}\colon\mathcal{V}(A,B)\longrightarrow\mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B))$ is given as the adjoint of comp: $\mathcal{V}(A,B)\otimes\mathcal{V}(T,A)\longrightarrow\mathcal{V}(T,B)$ which, up to an associativity isomorphism, is adjoint to $\mathcal{V}(A,B)\otimes(\mathcal{V}(T,A)\otimes T)\xrightarrow{\mathcal{V}(A,B)\otimes\epsilon_T A}\mathcal{V}(A,B)\otimes A\xrightarrow{\epsilon_A B}B$.

Lemma A.3. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(A,B) \arrow[rr, "\hom^{\mathcal{V}(T,-)}_{A,B}"] \arrow[dr, "\mathcal{V}(\epsilon_T A,B)"'] & & \mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B)) \\
& \mathcal{V}(\mathcal{V}(T,A)\otimes T,B) \arrow[ur, "\cong", "f"'] &
\end{tikzcd}$$

Proof. Here, f is $\mathcal{V}(\eta_T\mathcal{V}(T,A),\mathcal{V}(T,B))\circ\hom^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}$. The diagram

$$\begin{tikzcd}
\mathcal{V}(A,B) \arrow[r, "\hom^{\mathcal{V}(T,-)}_{A,B}"] \arrow[d, "\mathcal{V}(\epsilon_T A,B)"'] & \mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B)) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\epsilon_T A),\mathcal{V}(T,B))"] \\
\mathcal{V}(\mathcal{V}(T,A)\otimes T,B) \arrow[r, "\hom^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}"] & \mathcal{V}(\mathcal{V}(T,\mathcal{V}(T,A)\otimes T),\mathcal{V}(T,B))
\end{tikzcd}$$

commutes, because $\mathcal{V}(T,-)$ is a $\mathcal{V}$-functor. Therefore, $f\circ\mathcal{V}(\epsilon_T(A),B)$ coincides with $\mathcal{V}(\eta_T\mathcal{V}(T,A),\mathcal{V}(T,B))\circ\mathcal{V}(\mathcal{V}(T,\epsilon_T(A)),\mathcal{V}(T,B))\circ\hom^{\mathcal{V}(T,-)}_{\mathcal{V}(T,A)\otimes T,B}$ and the triangular identity $\mathcal{V}(T,\epsilon_T(A))\circ\eta_T\mathcal{V}(T,A)=\mathrm{id}_{\mathcal{V}(T,A)}$ completes the proof. □

Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category closed under $\otimes$. If $v\in\mathrm{Ob}\,\mathcal{C}$ and $X\colon\mathcal{C}\longrightarrow\mathcal{V}$ is a $\mathcal{V}$-functor, one can consider the adjoint $X(c)\otimes v\longrightarrow X(v\otimes c)$ of the map $v\xrightarrow{\eta_c v}\mathcal{V}(c,v\otimes c)\xrightarrow{\hom^X_{c,v\otimes c}}\mathcal{V}(X(c),X(v\otimes c))$. It defines a $\mathcal{V}$-natural transformation $\mathrm{sw}^X_v\colon X\otimes v\longrightarrow X\circ(v\otimes -)$ (X *swallows* v). One of the maps $X\longrightarrow\mathbb{T}(X)$ is defined using the map sw^X_T. Using a commutativity isomorphism, one can define a map $v\otimes X\longrightarrow X\circ(v\otimes -)$ which will also be denoted sw^X_v. An interesting case is $\mathcal{V}(T,-)\colon\mathcal{V}\longrightarrow\mathcal{V}$. Then $\mathrm{sw}^{\mathcal{V}(T,-)}_A(B)\colon A\otimes\mathcal{V}(T,B)\longrightarrow\mathcal{V}(T,A\otimes B)$ is the adjoint of $(A\otimes\mathcal{V}(T,B))\otimes T\xrightarrow{\alpha_{A,\mathcal{V}(T,B)}}A\otimes(\mathcal{V}(T,B)\otimes T)\xrightarrow{A\otimes\epsilon_T B}A\otimes B$.

Lemma A.4. *Let $A,B,T\in\mathrm{Ob}\,\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
A \arrow[r, "\eta_{\mathcal{V}(T,B)}A"] \arrow[d, "\eta_B A"'] & \mathcal{V}(\mathcal{V}(T,B),A\otimes\mathcal{V}(T,B)) \arrow[d, "\mathcal{V}(\mathcal{V}(T,B),\mathrm{sw}^{\mathcal{V}(T,-)}_A(B))"] \\
\mathcal{V}(B,A\otimes B) \arrow[r, "\hom^{\mathcal{V}(T,-)}_{B,A\otimes B}"] & \mathcal{V}(\mathcal{V}(T,B),V(T,A\otimes B))
\end{tikzcd}$$

Proof. Using the definition of $\hom^{\mathcal{V}(T,-)}$ and the description of $\mathrm{sw}^{\mathcal{V}(T,-)}_A$ from above, one gets a large diagram which commutes by naturality and the triangular identity $\epsilon_B(A\otimes B)\circ(\eta_B A)\otimes B=\mathrm{id}_{A\otimes B}$. □

Lemma A.5. *Let $A,B,T\in\mathrm{Ob}\,\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(A,B)\otimes\mathcal{V}(T,A) \arrow[r, "\mathrm{sw}^{\mathcal{V}(T,-)}_{\mathcal{V}(A,B)}(A)"] \arrow[d, "\hom^{\mathcal{V}(T,-)}_{A,B}\otimes\mathcal{V}(T,A)"'] & \mathcal{V}(T,\mathcal{V}(A,B)\otimes A) \arrow[d, "\mathcal{V}(T,\epsilon_A B)"] \\
\mathcal{V}(\mathcal{V}(T,A),\mathcal{V}(T,B))\otimes\mathcal{V}(T,A) \arrow[r, "\epsilon_{\mathcal{V}(T,A)}\mathcal{V}(T,B)"] & \mathcal{V}(T,B)
\end{tikzcd}$$

Proof. This is similar to the proof of A.4; the relevant triangular identity is $\epsilon_{\mathcal{V}(T,A)}(\mathcal{V}(A,B)\otimes\mathcal{V}(T,A))\circ(\eta_{\mathcal{V}(T,A)}\mathcal{V}(A,B))\otimes\mathcal{V}(T,A)=\mathrm{id}_{\mathcal{V}(A,B)\otimes\mathcal{V}(T,A)}$. □

Lemma A.6. *Let $A,B,T\in\mathrm{Ob}\,\mathcal{V}$. The following diagram commutes.*

$$\begin{tikzcd}
A\otimes\mathcal{V}(T,B) \arrow[r, "\eta_T A\otimes\mathcal{V}(T,B)"] \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_A(B)"'] & \mathcal{V}(T,A\otimes T)\otimes\mathcal{V}(T,B) \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,B)}(A\otimes T)"] \\
\mathcal{V}(T,A\otimes B) \arrow[d, "\mathcal{V}(T,\sigma_{A,B})"'] & \mathcal{V}(T,\mathcal{V}(T,B)\otimes(A\otimes T)) \arrow[d, "\mathcal{V}(T,c)"] \\
\mathcal{V}(T,B\otimes A) & \mathcal{V}(T,(\mathcal{V}(T,B)\otimes T)\otimes A) \arrow[l, "\mathcal{V}(T,(\epsilon_T B)\otimes A)"']
\end{tikzcd}$$

Here c is the composition of an associativity and a commutativity isomorphism.

Proof. Insert the description of $\mathrm{sw}^{\mathcal{V}(T,-)}$ and use naturality, associativity and commutativity coherence, and $\epsilon_T(A\otimes T)\circ(\eta_T A)\otimes T = \mathrm{id}_{A\otimes T}$. □

Lemma A.7. *Let $\mathcal{C}$ be a full sub-$\mathcal{V}$-category closed under $\otimes$, $X\colon\mathcal{C}\longrightarrow\mathcal{V}$ a $\mathcal{V}$-functor, and $f\colon\mathcal{V}(T,\mathcal{V}(v,w))\longrightarrow\mathcal{V}(T\otimes v,w)$ the adjointness isomorphism, with $T,v,w\in\operatorname{Ob}\mathcal{C}$. The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes(T\otimes Xv) \arrow[rr, "\alpha^{-1}"] \arrow[d, "f\otimes\mathrm{sw}^X_T(v)"'] & & (\mathcal{V}(T,\mathcal{V}(v,w))\otimes T)\otimes Xv \arrow[d, "(\epsilon_T\mathcal{V}(v,w))\otimes Xv"] \\
\mathcal{V}(T\otimes v,w)\otimes X(T\otimes v) \arrow[d, "\mathrm{hom}^X_{T\otimes v,w}\otimes X(T\otimes v)"'] & & \mathcal{V}(v,w)\otimes Xv \arrow[d, "\mathrm{hom}^X_{v,w}\otimes Xv"] \\
\mathcal{V}(X(T\otimes v),Xw)\otimes X(T\otimes v) \arrow[dr, "\epsilon_{X(T\otimes v)}Xw"'] & & \mathcal{V}(Xv,Xw)\otimes Xv \arrow[dl, "\epsilon_{Xv}Xw"] \\
& Xw &
\end{tikzcd}$$

Proof. The proof is divided into two steps. First we note that

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes(T\otimes Xv) \arrow[rr, "\alpha^{-1}"] \arrow[d, "\mathcal{V}(T,\mathrm{hom}^X_{v,w})\otimes(T\otimes Xv)"'] & & (\mathcal{V}(T,\mathcal{V}(v,w))\otimes T)\otimes Xv \arrow[d, "(\epsilon_T\mathcal{V}(v,w))\otimes Xv"] \\
\mathcal{V}(T,\mathcal{V}(Xv,Xw))\otimes(T\otimes Xv) \arrow[d, "g\otimes(T\otimes Xv)"'] & & \mathcal{V}(v,w)\otimes Xv \arrow[d, "\mathrm{hom}^X_{v,w}\otimes Xv"] \\
\mathcal{V}(T\otimes Xv,Xw)\otimes(T\otimes Xv) \arrow[dr, "\epsilon_{T\otimes Xv}Xw"'] & & \mathcal{V}(Xv,Xw)\otimes Xv \arrow[dl, "\epsilon_{Xv}Xw"] \\
& Xw &
\end{tikzcd}$$

commutes, where g is the adjointness isomorphism

$$\mathcal{V}(T\otimes Xv,\epsilon_{Xv}Xw)\circ\mathrm{hom}^{-\otimes Xv}_{T,\mathcal{V}(Xv,Xw)}.$$

Commutativity of the diagram follows from the definition of $\mathrm{hom}^{-\otimes Xv}$, naturality and the triangular identity $\epsilon_{T\otimes Xv}(-\otimes(T\otimes Xv))\circ(\eta_{T\otimes Xv})\otimes(T\otimes Xv)=\mathrm{id}$ applied to $\mathcal{V}(T,\mathcal{V}(Xv,Xw))\otimes(T\otimes Xv)$. In the second step, we prove that

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes T\otimes Xv \arrow[r, "\mathcal{V}(T,\mathrm{hom}^X_{v,w})\otimes T\otimes Xv"] \arrow[d, "f\otimes\mathrm{sw}^X_T"'] & \mathcal{V}(T,\mathcal{V}(Xv,Xw))\otimes T\otimes Xv \arrow[d, "g\otimes T\otimes Xv"] \\
\mathcal{V}(T\otimes v,w)\otimes X(T\otimes v) \arrow[d, "\mathrm{hom}^X_{T\otimes v,w}"', "\otimes X(T\otimes v)"] & \mathcal{V}(T\otimes Xv,Xw)\otimes T\otimes Xv \arrow[d, "\epsilon_{T\otimes Xv}Xw"] \\
\mathcal{V}(X(T\otimes v),Xw)\otimes X(T\otimes v) \arrow[r, "\epsilon_{X(T\otimes v)}Xw"] & Xw
\end{tikzcd}$$

commutes. The adjoint of $\epsilon_{X(T\otimes v)}Xw \circ (\mathrm{hom}^X_{T\otimes v,w} \otimes \mathrm{sw}^X_T)$ coincides with the composition $\mathrm{comp} \circ (\mathrm{hom}^X_{T\otimes v,w} \otimes \mathrm{hom}^X_{v,T\otimes v}) \circ \mathcal{V}(T\otimes v,w) \otimes \eta_v T$. Because X is a $\mathcal{V}$-functor, this map is the same as $\mathrm{hom}^X_{v,w} \circ \mathrm{comp} \circ \mathcal{V}(T\otimes v,w) \otimes \eta_v T$. Hence the diagram above commutes if and only if

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes T\otimes Xv \arrow[rr, "\mathcal{V}(T,\mathrm{hom}^X_{v,w})\otimes T\otimes Xv"] \arrow[d, "f\otimes\eta_v T\otimes Xv"'] & & \mathcal{V}(T,\mathcal{V}(Xv,Xw))\otimes T\otimes Xv \arrow[d, "g\otimes T\otimes Xv"] \\
\mathcal{V}(T\otimes v,w)\otimes\mathcal{V}(v,T\otimes v)\otimes Xv \arrow[d, "\mathrm{comp}\otimes Xv"'] & & \mathcal{V}(T\otimes Xv,Xw)\otimes T\otimes Xv \arrow[d, "\epsilon_{T\otimes Xv}Xw"] \\
\mathcal{V}(v,w)\otimes Xv \arrow[dr, "\mathrm{hom}^X_{v,w}\otimes Xv"'] & & Xw \\
 & \mathcal{V}(Xv,Xw)\otimes Xv \arrow[ur, "\epsilon_{Xv}Xw"'] &
\end{tikzcd}$$

commutes. The shortest composition is $\mathrm{hom}^X_{v,w} \otimes Xw \circ (\epsilon_T\mathcal{V}(v,w)) \otimes Xw$, as the triangular identity $\epsilon_{T\otimes Xv}(-\otimes(T\otimes Xv))\circ(\eta_{T\otimes Xv})\otimes(T\otimes Xv) = \mathrm{id}_{-\otimes(T\otimes Xv)}$ evaluated at $\mathcal{V}(T,\mathcal{V}(Xv,Xw))$ shows. Therefore it remains to prove that the map $\mathrm{comp} \circ (f \otimes \eta_v T)$ coincides with the map $\epsilon_T\mathcal{V}(v,w)$. It is equivalent to switch to the adjoints (under tensoring with v), and here naturality and the triangular identity $\epsilon_{T\otimes v}(-\otimes(T\otimes v))\circ(\eta_{T\otimes v})\otimes(T\otimes v) = \mathrm{id}_{-\otimes(T\otimes v)}$ evaluated at $\mathcal{V}(T,\mathcal{V}(v,w))$ give the desired identification. □

In the proof of 6.1, we used the next result.

Proposition A.8. *The two maps $X \longrightarrow \mathbb{T}(X)$ coincide.*

Proof. Most for notational convenience, we will often leave out associativity and commutativity constraints. The two maps in question are determined by $Xv \xrightarrow{\eta_T Xv} V(T, Xv\otimes T) \xrightarrow{V(T,\mathrm{sw}^X_T(v))} \mathcal{V}(T, X(T\otimes v))$ and (up to Yoneda isomorphism) $\tau_v(w)\colon \mathcal{V}(T\otimes v,w)\otimes T \xrightarrow{f\otimes T} \mathcal{V}(T,\mathcal{V}(v,w))\otimes T \xrightarrow{\epsilon_T\mathcal{V}(v,w)} \mathcal{V}(v,w)$. The Yoneda isomorphism

$$Xv \xrightarrow{\cong} \int_{\mathrm{Ob}\,\mathcal{C}} \mathcal{V}(\mathcal{V}(v,w),Xw)$$

is induced by the natural transformation $y^X_v(w)\colon Xv \longrightarrow \mathcal{V}(\mathcal{V}(v,w),Xw)$, that is, the composition

$$\begin{aligned}
Xv &\xrightarrow{\eta_{\mathcal{V}(v,w)}Xv} \mathcal{V}(\mathcal{V}(v,w), Xv\otimes\mathcal{V}(v,w)) \\
&\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),Xv\otimes\mathrm{hom}^X_{v,w})} \mathcal{V}(\mathcal{V}(v,w), Xv\otimes\mathcal{V}(Xv,Xw)) \\
&\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),\sigma_{Xv,\mathcal{V}(Xv,Xw)})} \mathcal{V}(\mathcal{V}(v,w), \mathcal{V}(Xv,Xw)\otimes Xv) \\
&\xrightarrow{\mathcal{V}(\mathcal{V}(v,w),\epsilon_{Xv}Xw)} \mathcal{V}(\mathcal{V}(v,w), Xw).
\end{aligned}$$

Hence it suffices to prove that, for every $w \in \mathrm{Ob}\,\mathcal{C}$, the composition

$$a(w) := y_{T\otimes v}^{\mathcal{V}(T,X)}(w) \circ \mathcal{V}(T, \mathrm{sw}_T^X(v)) \circ \eta_T Xv$$

coincides with the composition $b(w) := h \circ \mathcal{V}(\tau_v, Xw) \circ y_v^X(w)$. Here h denotes the adjointness isomorphism $\mathcal{V}(\mathcal{V}(T \otimes v, w) \otimes T, Xw) \longrightarrow \mathcal{V}(\mathcal{V}(T \otimes v, w), \mathcal{V}(T, Xw))$. The isomorphism $f\colon \mathcal{V}(T, \mathcal{V}(v,w)) \longrightarrow \mathcal{V}(T \otimes v, w)$ will be used in the proof. The diagram

$$\begin{tikzcd}
\mathcal{V}(\mathcal{V}(v,w), Xw) \arrow[r, "\mathrm{hom}^{\mathcal{V}(T,-)}"] \arrow[d, "\mathcal{V}(\tau_v, Xw)"'] & \mathcal{V}(\mathcal{V}(T, \mathcal{V}(v,w)), \mathcal{V}(T, Xw)) \\
\mathcal{V}(\mathcal{V}(T \otimes v, w)) \otimes T, Xw) \arrow[r, "\mathcal{V}(f \otimes T, Xw)"'] & \mathcal{V}(\mathcal{V}(T, \mathcal{V}(v,w)) \otimes T, Xw) \arrow[u, "\cong"']
\end{tikzcd}$$

commutes by A.3, where the vertical map on the right hand side is the adjointness isomorphism. Then by naturality and A.4, $b(w)$ coincides with the composition $\mathcal{V}(\mathcal{V}(T, \mathcal{V}(v,w)), \mathcal{V}(T, \epsilon_{Xv} Xw \circ \mathrm{hom}_{v,w}^X) \circ \mathrm{sw}_{Xv}^{\mathcal{V}(T,-)}) \circ \eta_{\mathcal{V}(T,\mathcal{V}(v,w))} Xv$. The diagram

$$\begin{tikzcd}
Xv \arrow[r, "\eta_{\mathcal{V}(T,\mathcal{V}(v,w))} Xv"] \arrow[d, "\eta_T Xv"'] & \mathcal{V}(\mathcal{V}(T, \mathcal{V}(v,w)), Xv \otimes \mathcal{V}(T, \mathcal{V}(v,w))) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)), (\eta_T Xv) \otimes \mathcal{V}(T,\mathcal{V}(v,w)))"] \\
\mathcal{V}(T, Xv \otimes T) \arrow[d, "\mathcal{V}(T, \mathrm{sw}_T^X(v))"'] & \mathcal{V}(\mathcal{V}(T, \mathcal{V}(v,w)), \mathcal{V}(T, Xv \otimes T) \otimes \mathcal{V}(T, \mathcal{V}(v,w))) \arrow[d, "\mathcal{V}(\mathcal{V}(T,\mathcal{V}(v,w)), \mathcal{V}(T, \mathrm{sw}_T^X(v)) \otimes \mathcal{V}(T,\mathcal{V}(v,w)))"] \\
\mathcal{V}(T, X(T \otimes v)) \arrow[r, "\eta_{\mathcal{V}(T,\mathcal{V}(v,w))}"] & \mathcal{V}(\mathcal{V}(T, \mathcal{V}(v,w)), \mathcal{V}(T, X(T \otimes v)) \otimes \mathcal{V}(T, \mathcal{V}(v,w)))
\end{tikzcd}$$

commutes by naturality. Hence the maps $a(w)$ and $b(w)$ coincide if

$$\begin{tikzcd}
\mathcal{V}(T, \mathcal{V}(v,w)) \otimes Xv \arrow[r, "\mathcal{V}(T,\mathcal{V}(v,w)) \otimes \eta_T Xv"] \arrow[d, "\mathrm{sw}_{Xv}^{\mathcal{V}(T,-)}"'] & \mathcal{V}(T, \mathcal{V}(v,w)) \otimes \mathcal{V}(T, Xv \otimes T) \arrow[d, "\mathcal{V}(T,\mathcal{V}(v,w)) \otimes \mathcal{V}(T, \mathrm{sw}_T^X(v))"] \\
\mathcal{V}(T, Xv \otimes \mathcal{V}(v,w)) \arrow[d, "\mathcal{V}(T, Xv \otimes \mathrm{hom}_{v,w}^X)"'] & \mathcal{V}(T, \mathcal{V}(v,w)) \otimes \mathcal{V}(T, X(T \otimes v)) \arrow[d, "f \otimes \mathcal{V}(T, X(T \otimes v))"] \\
\mathcal{V}(T, Xv \otimes \mathcal{V}(Xv, Xw)) \arrow[d, "\mathcal{V}(T, \sigma)"'] & \mathcal{V}(T \otimes v, w) \otimes \mathcal{V}(T, X(T \otimes v)) \arrow[d, "\mathrm{hom}^X \otimes \mathcal{V}(T, X(T \otimes v))"] \\
\mathcal{V}(T, \mathcal{V}(Xv, Xw) \otimes Xv) \arrow[d, "\mathcal{V}(T, \epsilon_{Xv} Xw)"'] & \mathcal{V}(X(T \otimes v), Xw) \otimes \mathcal{V}(T, X(T \otimes v)) \arrow[d, "\mathrm{hom}^{\mathcal{V}(T,-)} \otimes \mathcal{V}(T, X(T \otimes v))"] \\
\mathcal{V}(T, Xw) & \mathcal{V}(\mathcal{V}(T, X(T \otimes v)), V(T, Xw)) \otimes V(T, X(T \otimes v)) \arrow[l, "\epsilon_{\mathcal{V}(T, X(T \otimes v))}"']
\end{tikzcd}$$

commutes for all w. Now use A.5 (with $A = X(T \otimes v)$ and $B = Xw$), naturality of the map $\mathrm{sw}^{\mathcal{V}(T,-)}$ and the isomorphism f to replace the composition from the upper corner on the right hand side to the lower corner on the left hand side. The result is the diagram

$$\begin{tikzcd}
\mathcal{V}(T,\mathcal{V}(v,w))\otimes Xv \arrow[rr, "\mathcal{V}(T,\mathcal{V}(v,w))\otimes\eta_T Xv"] \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_{Xv}"'] & & \mathcal{V}(T,\mathcal{V}(v,w))\otimes\mathcal{V}(T,Xv\otimes T) \arrow[d, "\mathrm{sw}^{\mathcal{V}(T,-)}_{\mathcal{V}(T,\mathcal{V}(v,w))}"] \\
\mathcal{V}(T,Xv\otimes\mathcal{V}(v,w)) \arrow[d, "\mathcal{V}(T,Xv\otimes\,\mathrm{hom}^X_{v,w})"'] & & \mathcal{V}(T,\mathcal{V}(T,\mathcal{V}(v,w))\otimes X(T\otimes v)) \arrow[ll, "\mathcal{V}(T,Xv\otimes\epsilon_T\mathcal{V}(v,w))"'] \arrow[d, "\mathcal{V}(T,f\otimes X(T\otimes v))"] \\
\mathcal{V}(T,Xv\otimes\mathcal{V}(Xv,Xw)) \arrow[d, "\mathcal{V}(T,\sigma)"'] & & \mathcal{V}(T,\mathcal{V}(T\otimes v,w)\otimes X(T\otimes v)) \arrow[d, "\mathcal{V}(T,\mathrm{hom}^X\otimes X(T\otimes v))"] \\
\mathcal{V}(T,\mathcal{V}(Xv,Xw)\otimes Xv) \arrow[dr, "\mathcal{V}(T,\epsilon_{Xv}Xw)"] & & \mathcal{V}(T,\mathcal{V}(X(T\otimes v),Xw)\otimes X(T\otimes v)) \arrow[dl, "\mathcal{V}(T,\epsilon_{X(T\otimes v)}Xw)"'] \\
& \mathcal{V}(T,Xw). &
\end{tikzcd}$$

The upper part commutes by A.6 (with $A = Xv$ and $B = \mathcal{V}(v,w)$), the lower part is $\mathcal{V}(T,-)$ applied to a diagram which commutes by A.7. This completes the proof. □

References

[1] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

[2] Francis Borceux. *Handbook of categorical algebra. 2.* Cambridge University Press, Cambridge, 1994. Categories and structures.

[3] Gunnar Carlsson. A survey of equivariant stable homotopy theory. *Topology*, 31(1):1–27, 1992.

[4] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar, IV*, pages 1–38. Springer, Berlin, 1970.

[5] Bjørn Ian Dundas, Oliver Röndigs, and Paul Arne Østvær. Motivic functors. *Documenta Math.*, 8:489–525, 2003.

[6] Philip S. Hirschhorn. *Model categories and their localizations.* American Mathematical Society, Providence, RI, 2003.

[7] Mark Hovey. *Model categories.* American Mathematical Society, Providence, RI, 1999.

[8] Mark Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.

[9] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[10] L. Gaunce Lewis, Jr., J. Peter May, and Mark Steinberger. Equivariant stable homotopy theory. Volume 1213 of *Lecture Notes in Math.*. Springer, Berlin, 1986.

[11] Manos Lydakis. Simplicial functors and stable homotopy theory. Preprint 98-049, SFB 343, Bielefeld, June 1998.

[12] Saunders MacLane. *Categories for the working mathematician.* Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[13] Michael A. Mandell and J. Peter May Equivariant orthogonal spectra and S-modules *Mem. Amer. Math. Soc.*. 159(755):x+108, 2002.

[14] Michael A. Mandell, J. Peter May, Stefan Schwede, and Brooke E. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc. (3)*, 82(2):441–512, 2001.

[15] Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[16] Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, pages 579–604 (electronic), 1998.

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Motivic Functors

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Abstract. The notion of motivic functors refers to a motivic homotopy theoretic analog of continuous functors. In this paper we lay the foundations for a homotopical study of these functors. Of particular interest is a model structure suitable for studying motivic functors which preserve motivic weak equivalences and a model structure suitable for motivic stable homotopy theory. The latter model is Quillen equivalent to the category of motivic symmetric spectra.

There is a symmetric monoidal smash product of motivic functors, and all model structures constructed are compatible with the smash product in the sense that we can do homotopical algebra on the various categories of modules and algebras. In particular, motivic cohomology is naturally described as a commutative ring in the category of motivic functors.

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1 Introduction

One of the advantages of the modern formulations of algebraic topology is that invariants can be expressed, not merely as functors into groups, but actually as functors taking values in spaces. As such, the invariants are now themselves approachable by means of standard moves in algebraic topology; they can be composed or otherwise manipulated giving structure and control which cannot be obtained when looking at isolated algebraic invariants.

Although handling much more rigid objects, Voevodsky's motivic spaces [16] are modeled on topological spaces. The power of this approach lies in that many of the techniques and results from topology turn out to work in algebraic geometry. As in topology, many of the important constructions in the theory can be viewed as functors of motivic spaces. The functor $M\mathbb{Z}$ (called L in [16]) which defines motivic cohomology is an example: it accepts motivic spaces as input and gives a motivic space as output. Given the importance of such functors and the development of algebraic topology in the 1990s, it is ripe time for a thorough study of these functors.

In this paper we initiate such a program for functors in the category of motivic spaces. The functors we shall consider are the analogs of continuous functors: *motivic functors* ($M\mathbb{Z}$ is an example; precise definitions will appear below). This involves setting up a homological – or rather homotopical – algebra for motivic functors, taking special care of how this relates to multiplicative and other algebraic properties.

A large portion of our work deals with the technicalities involved in setting up a variety of model structures on the category **MF** of motivic functors, each localizing at different aspects of motivic functors.

One of the model structures we construct on **MF** is Quillen equivalent to the stable model category of motivic spectra as defined, for instance by Jardine [10] and by Hovey [8].

Just as in the topological case, this solution comes with algebraic structure in the form of a symmetric monoidal smash product $\wedge$. Furthermore, the algebra and homotopy cooperate so that a meaningful theory paralleling that of ring spectra and modules follows. A tentative formulation is

Theorem. *There exists a monoidal model category structure* $\mathbf{MF}_{\mathrm{sph}}$ *on* **MF** *satisfying the monoid axiom, and a lax symmetric monoidal Quillen equivalence between* $\mathbf{MF}_{\mathrm{sph}}$ *and the model category of motivic symmetric spectra.*

To be slightly more concrete, a motivic space in our context is just a pointed simplicial presheaf on the category of smooth schemes over a base scheme S. There is a preferred "sphere" given by the Thom space T of the trivial line bundle $\mathbb{A}^1_S$. A motivic spectrum is a sequence of motivic spaces $E_0, E_1, \cdots$ together with structure maps

$$T \wedge E_n \longrightarrow E_{n+1}.$$

We should perhaps comment on the continuous/motivic nature of our functors, since this aspect may be new to some readers. Let $\mathcal{M}$ be the category of motivic spaces and $\mathbf{f}\mathcal{M}$ the subcategory of finitely presentable motivic spaces. A motivic functor is a functor

$$X\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$$

which is "continuous" or "enriched" in the sense that it induces a map of internal hom objects. The enrichment implies that there is a natural map

$$A \wedge X(B) \longrightarrow X(A \wedge B).$$

As a consequence, any motivic functor X gives rise to a motivic spectrum $\mathrm{ev}(X)$ by "evaluating on spheres", that is

$$\mathrm{ev}(X)_n := X(T^{\wedge n})$$

with structure map

$$T \wedge \mathrm{ev}(X)_n = T \wedge X(T^{\wedge n}) \longrightarrow X(T \wedge T^{\wedge n}) = \mathrm{ev}(X)_{n+1}$$

given by the enrichment. The motivic functors $\mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ form the category $\mathbf{MF}$ mentioned in the main theorem, and the evaluation on spheres induces the Quillen equivalence. The inclusion $\mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M}$ is the unit in the monoidal structure and plays the rôle of the sphere spectrum.

The reader should keep in mind how simple our objects of study are: they are just functors of motivic spaces. All coherence problems one might conceive of in relation to multiplicative structure, and which are apparent if one works with e.g. motivic symmetric spectra, can safely be forgotten since they are taken care of by the coherence inherent to the category of motivic spaces. Furthermore, the smash product in our model is just like the usual tensor product in that, though it is slightly hard to picture $X \wedge Y$, it is very easy to say what the maps

$$X \wedge Y \longrightarrow Z$$

are: they are simply natural maps

$$X(A) \wedge Y(B) \longrightarrow Z(A \wedge B),$$

where the smash product is sectionwise the smash product of pointed simplicial sets; this is all we require to set up a simple motivic theory with multiplicative structure.

A *motivic ring* is a monoid in $\mathbf{MF}$. These are the direct analogs of ring spectra. The multiplicative structure of motivic cohomology comes from the fact that $M\mathbb{Z}$ is a commutative motivic ring. This means we can consider $M\mathbb{Z}$-modules and also $M\mathbb{Z}$-algebras. Our framework allows one to do homotopical algebra. For instance:

Theorem. *The category of $M\mathbb{Z}$-modules in $\mathbf{MF}_{\mathrm{sph}}$ acquires a monoidal model category structure and the monoid axiom holds.*

The "spherewise" structure $\mathbf{MF}_{\mathrm{sph}}$ is not the only interesting model structure there is on $\mathbf{MF}$. One aspect we shall have occasion to focus on is the fact that although most interesting motivic functors preserve weak equivalences (hence the name "homotopy functors"), categorical constructions can ruin this property. The standard way of getting around this problem is to consider only derived functors. While fully satisfying when considering one construction at the time, this soon clobbers up the global picture. A more elegant and functorially satisfying approach is to keep our category and its constructions as they are, but change our model structure. Following this idea we construct a model structure suitable for studying homotopy functors, and yet another model structure which is more suitable for setting up a theory of Goodwillie calculus for motivic spaces.

As with the stable model, these models respect the smash product and algebraic structure. The following statement gives an idea of what the homotopy functor model expresses

Theorem. *There exists a monoidal model category structure* $\mathbf{MF}_{\mathrm{hf}}$ *on* $\mathbf{MF}$ *satisfying the monoid axiom. In this structure every motivic functor is weakly equivalent to a homotopy functor, and a map of homotopy functors* $X \longrightarrow Y$ *is a weak equivalence if and only if for all finitely presentable motivic spaces* A *the evaluation* $X(A) \longrightarrow Y(A)$ *is a weak equivalence of motivic spaces.*

At this point it is interesting to compare with Lydakis' setup [11] for simplicial functors, and note how differently simplicial sets and motivic spaces behave. In the motivic case the theory fractures into many facets which coincide for simplicial sets. For instance, there is no reason why the notions of "stable" and "linear" (in Goodwillie and Waldhausen's sense) should coincide.

The paper is organized as follows. In section 2 we set up the model structures for unstable motivic homotopy theory suitable for our purposes.

In section 3 we present the four basic model structures on motivic functors. In the preprint version of this paper we allowed the source category of motivic functors to vary. This handy technical tool has been abandoned in this paper for the sake of concreteness. We thank the referee for this suggestion and other detailed comments.

All along the properties necessary for setting up a theory of rings and modules are taken care of, and the results are outlined in section 4.

Contents

2 Motivic Spaces

In this section we recall some facts about the category of *motivic spaces* and fix some notation. We briefly discuss the categorical properties, and then the homotopical properties.

For background in model category theory we refer to [7] while for enriched category theory we refer to [3] and [4].

Let S be a Noetherian scheme of finite Krull dimension. Denote by Sm/S the category of smooth S-schemes of finite type. Due to the finiteness condition, Sm/S is an essentially small category. Furthermore, it has pullbacks, a terminal object S and an initial object $\emptyset$, the empty scheme. If $U, V \in \mathrm{Ob}\,\mathrm{Sm}/S$, we denote the set of maps between U and V by $\mathbf{Set}_{\mathrm{Sm}/S}(U, V)$.

Let $\mathcal{S}$ be the closed symmetric monoidal category of pointed simplicial sets with internal hom objects $\mathcal{S}(-,-)$. Recall that the standard n-simplex Δ^n is the simplicial set represented by $[n] \in \Delta$.

Definition 2.1. A *motivic space* is a contravariant functor $A : \mathrm{Sm}/S \longrightarrow \mathcal{S}$. Let $\mathcal{M}_S$ (or just $\mathcal{M}$ if confusion is unlikely to result) denote the category of motivic spaces and natural transformations.

By reversal of priorities, $\mathcal{M}$ can alternatively be viewed as the category of pointed set-valued presheaves on $\mathrm{Sm}/S \times \Delta$. Denote by

$$\begin{aligned} \mathrm{Sm}/S &\longrightarrow \mathcal{M} \\ U &\longmapsto h_U \end{aligned}$$

the Yoneda functor $h_U(V) = \mathbf{Set}_{\mathrm{Sm}/S}(V, U)_+$ considered as a discrete pointed simplicial set (the plus denotes an added base point).

Recall the following facts about the functor category $\mathcal{M}$:

Proposition 2.2. *The category $\mathcal{M}$ is a locally finitely presentable bicomplete $\mathcal{S}$-category. The pointwise smash product gives $\mathcal{M}$ a closed symmetric monoidal structure.*

Since $\mathcal{M}$ is locally finitely presentable, it follows that finite limits commute with filtered colimits. To fix notation, we find it convenient to explicate some of this structure.

The pointwise smash $A \wedge B$ on $\mathcal{M}$ is given by

$$(A \wedge B)(U) = A(U) \wedge B(U).$$

The unit is the constant presheaf S^0. If $U \in \operatorname{Ob}\operatorname{Sm}/S$, then the evaluation functor

$$\mathrm{Ev}_U \colon \mathcal{M} \longrightarrow \mathcal{S}, \qquad \mathrm{Ev}_U(A) = A(U)$$

preserves limits and colimits. The left adjoint of Ev_U is the functor

$$\mathrm{Fr}_U \colon \mathcal{S} \longrightarrow \mathcal{M}, \qquad \mathrm{Fr}_U(K) = h_U \wedge K.$$

Note that, since $h_S(V) = S^0$, we will often write K instead of $\mathrm{Fr}_S K$. Checking the relevant conditions we easily get that the functors Fr_S and Ev_U are strict symmetric monoidal, while Fr_U is lax symmetric monoidal. The pair $(\mathrm{Fr}_U, \mathrm{Ev}_U)$ is an $\mathcal{S}$-adjoint pair.

Using Fr_S we get (co)actions ("(co)tensors") of $\mathcal{S}$ on $\mathcal{M}$: if $A \in \mathcal{M}$ and $K \in \mathcal{S}$ the functor $A \wedge K = A \wedge Fr_S K \in \mathcal{M}$ sends $U \in \operatorname{Ob}\operatorname{Sm}/S$ to $A(U) \wedge K \in \mathcal{S}$, and the functor A^K sends $U \in \operatorname{Ob}\operatorname{Sm}/S$ to $\mathcal{S}(K, A(U))$.

We let $\mathbf{Set}_{\mathcal{M}}(A, B)$ be the set of natural transformations from A to B in $\mathcal{M}$. The enrichment of $\mathcal{M}$ in $\mathcal{S}$ is defined by letting the pointed simplicial set of maps from A to B have n-simplices

$$\mathcal{S}_{\mathcal{M}}(A, B)_n := \mathbf{Set}_{\mathcal{M}}(A \wedge \Delta^n_+, B).$$

Its simplicial structure follows from functoriality of the assignment $[n] \longmapsto \Delta^n$. The internal hom object is in turn given by

$$\mathcal{M}(A, B)(U) = \mathcal{S}_{\mathcal{M}}(A \wedge h_U, B).$$

Definition 2.3. A motivic space A is *finitely presentable* if the set-valued hom functor $\mathbf{Set}_{\mathcal{M}}(A, -)$ commutes with filtered colimits. Similarly, A is $\mathcal{M}$-*finitely presentable* if the internal hom functor $\mathcal{M}(A, -)$ commutes with filtered colimits.

Recall that a pointed simplicial set is finitely presentable if and only if it is finite, that is, if it has only finitely many non-degenerate simplices. On the other hand, a pointed simplicial set K is finite if and only the $\mathcal{S}$-valued hom functor $\mathcal{S}(K, -)$ commutes with filtered colimits. The same holds for motivic spaces, as one can deduce from the following standard fact [3, 5.2.5].

Lemma 2.4. *Every motivic space is a filtered colimit of finite colimits of motivic spaces of the form $h_U \wedge \Delta^n_+$.*

Let K be a pointed simplicial set. Using Lemma 2.4, the natural isomorphism $\mathcal{M}(h_U \wedge K, A) \cong A(U \times_S -)^K$ and the fact that Ev_U commutes with colimits we get

Lemma 2.5. *Let K be a finite pointed simplicial set and $U \in \mathrm{Ob}\,\mathrm{Sm}/S$. Then $h_U \wedge K$ is $\mathcal{M}$-finitely presentable. The class of $\mathcal{M}$-finitely presentable motivic spaces is closed under retracts, finite colimits and the smash product. A motivic space is $\mathcal{M}$-finitely presentable if and only if it is finitely presentable.*

The finiteness condition imposed on objects of Sm/S implies that the full subcategory $\mathbf{f}\mathcal{M}$ of finitely presentable motivic spaces in $\mathcal{M}$ is equivalent to a small category [3, 5.3.8], cf. [3, 5.3.3] and the pointed version of [3, 5.2.2b]. Out of convenience, since $\mathbf{f}\mathcal{M}$ is the codomain of the functor category $\mathbf{MF}$ one could choose such an equivalence. This ends our discussion of categorical precursors.

2.1 Unstable homotopy theory

Summarizing this section we get a model structure $\mathcal{M}_{\mathrm{mo}}$ on $\mathcal{M}$ called the *motivic model structure* satisfying

1. $\mathcal{M}_{\mathrm{mo}}$ is weakly finitely generated.
2. $\mathcal{M}_{\mathrm{mo}}$ is proper.
3. The identity on $\mathcal{M}_{\mathrm{mo}}$ is a left Quillen equivalence to the Goerss-Jardine $\mathbb{A}^1$-model structure [10].
4. The smash product gives $\mathcal{M}_{\mathrm{mo}}$ a monoidal model structure.
5. The smash product preserves weak equivalences.
6. $\mathcal{M}_{\mathrm{mo}}$ satisfies the monoid axiom.

For the convenience of the reader we repeat briefly for $\mathcal{M}$ the definitions of the notions *weakly finitely generated, monoidal model structure* and the *monoid axiom*; for details, see for example [5, 3.4, 3.7, 3.8].

Weakly finitely generated means in particular that the cofibrations and acyclic cofibrations in $\mathcal{M}$ are generated by sets I and J, respectively [7, 2.1.7]. In addition, we require that I has finitely presented domains and codomains, the domains of J are small and that there exists a subset J' of J with finitely presented domains and codomains such that a map $A \longrightarrow B$ of motivic spaces with fibrant codomain is a fibration if and only if it has the right lifting property with respect to all objects of J'.

Let $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ be two maps in $\mathcal{M}$. The *pushout product* of f and g is the canonical map

$$f \,\Box\, g\colon A \wedge D \coprod_{A \wedge C} B \wedge C \longrightarrow C \wedge D.$$

That $\mathcal{M}$ is a *monoidal model category* means that the pushout product of two cofibrations in $\mathcal{M}$ is a cofibration, and an acyclic cofibration if either one of the two cofibrations is so. It implies that the smash product descends to the homotopy category of $\mathcal{M}$. If $\mathrm{aCof}(\mathcal{M})$ denotes the acyclic cofibrations of $\mathcal{M}$, then the *monoid axiom* means that all the maps in $\mathrm{aCof}(\mathcal{M}) \wedge \mathcal{M}$-cell are weak equivalences. Among other nice consequences mentioned below, the monoid axiom allows to lift model structures to categories of monoids and modules over a fixed monoid [14].

Definition 2.6. A map $A \longrightarrow B$ in $\mathcal{M}$ is a *schemewise weak equivalence* if, for all $U \in \mathrm{Ob}\,\mathrm{Sm}/S$, $A(U) \longrightarrow B(U)$ is a weak equivalence in $\mathcal{S}$. *Schemewise fibrations* and *schemewise cofibrations* are defined similarly. A *cofibration* is a map having the left lifting property with respect to all schemewise acyclic fibrations.

Note that the schemewise cofibrations are simply the monomorphisms. We get the following basic model structure.

Theorem 2.7. *The schemewise weak equivalences, schemewise fibrations and cofibrations equip $\mathcal{M}$ with the structure of a proper monoidal $\mathcal{S}$-model category. The sets*

$$\{h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{n \geq 0,\, U \in \mathrm{Ob}\,\mathrm{Sm}/S}$$

$$\{h_U \wedge (\Lambda^n_i \hookrightarrow \Delta^n)_+\}_{0 \leq i \leq n,\, U \in \mathrm{Ob}\,\mathrm{Sm}/S}$$

induced up from the corresponding maps in $\mathcal{S}$ are sets of generating cofibrations and acyclic cofibrations, respectively. The domains and codomains of the maps in these generating sets are finitely presentable. For any $U \in \mathrm{Ob}\,\mathcal{M}$, the pair $(\mathrm{Fr}_U, \mathrm{Ev}_U)$ is a Quillen pair.

Proof. The existence of the model structure follows from [7, 2.1.19], using the generating cofibrations and generating acyclic cofibrations described above. The properties which have to be checked are either straightforward or follow from 2.5 and properties of the standard model structure on simplicial sets. Properness follows from properness in $\mathcal{S}$, where we use that a cofibration is in particular a schemewise cofibration.

Clearly, Fr_U is a left Quillen functor for all $U \in \mathrm{Ob}\,\mathrm{Sm}/S$. Using the natural isomorphism

$$(h_U \wedge K) \wedge (h_V \wedge L) \cong h_{U \times_S V} \wedge (K \wedge L),$$

we see that for $f_j\colon K_j \longrightarrow L_j \in \mathcal{S}$ and $U_j \in \mathrm{Ob}\,\mathrm{Sm}/S$, $j = 1, 2$, we may identify the pushout product of $h_{U_1} \wedge f_1$ and $h_{U_2} \wedge f_2$ with the map

$$h_{U_1 \times_S U_2} \wedge \left((K_1 \wedge L_2) \coprod_{(K_1 \wedge K_2)} (L_1 \wedge K_2) \right) \longrightarrow h_{U_1 \times_S U_2} \wedge (L_1 \wedge L_2).$$

Hence the pushout product axiom in $\mathcal{S}$ implies the pushout product axiom for $\mathcal{M}$. It follows that $\mathcal{M}$ is a monoidal $\mathcal{S}$-model category via the functor Fr_S. $\square$

Notation 2.8. We let $\mathcal{M}_{\mathrm{sc}}$ denote the model structure of 2.7 on $\mathcal{M}$. Schemewise weak equivalences will be written $\xrightarrow{\sim \mathrm{sc}}$ and schemewise fibrations $\overset{\mathrm{sc}}{\twoheadrightarrow}$. Cofibrations are denoted by $\rightarrowtail$ (since not all schemewise cofibrations are cofibrations in $\mathcal{M}_{\mathrm{sc}}$). Choose a cofibrant replacement functor $(-)^c \longrightarrow \mathrm{Id}_{\mathcal{M}}$ in $\mathcal{M}_{\mathrm{sc}}$ so that for any motivic space A, there is a schemewise acyclic fibration $A^c \overset{\sim \mathrm{sc}}{\twoheadrightarrow} A$ with cofibrant domain. We note that every representable motivic space is cofibrant.

The following statements are easily verified.

Lemma 2.9. *Taking the smash product $-\wedge A$ or a cobase change along a schemewise cofibration preserves schemewise weak equivalences for all $A \in \mathrm{Ob}\,\mathcal{M}$. The monoid axiom holds in $\mathcal{M}_{\mathrm{sc}}$.*

It turns out that the properties in 2.7 and 2.9 hold in the model for motivic homotopy theory. The latter is obtained by considering Sm/S in its Nisnevich topology and by inverting the affine line $\mathbb{A}^1_S$. The following allows to incorporate Bousfield localization [6] in the motivic homotopy theory.

Recall that the Nisnevich topology is generated by *elementary distinguished squares* [12]. These are pullback squares of the form

$$Q \quad = \quad \begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \phi \\ U & \xrightarrow{\psi} & X \end{array}$$

where ϕ is étale, ψ is an open embedding and $\phi^{-1}(X - U) \longrightarrow (X - U)$ is an isomorphism of schemes (with the reduced structure).

Definition 2.10. A schemewise fibrant motivic space A is *motivically fibrant* if the following conditions hold.

- $A(\emptyset)$ is contractible.
- If Q is an elementary distinguished square, then $A(Q)$ is a homotopy pullback square of pointed simplicial sets.

- If $U \in \operatorname{Ob} \mathrm{Sm}/S$, the canonically induced map $A(U) \longrightarrow A(U \times_S \mathbb{A}^1_S)$ is a weak equivalence of pointed simplicial sets.

The first two conditions imply that A is a sheaf up to homotopy in the Nisnevich topology. The third condition implies that $\mathbb{A}^1_S \longrightarrow S$ is a weak equivalence in the following sense (where $(-)^c$ is the cofibrant replacement functor in $\mathcal{M}_{\mathrm{sc}}$ chosen in 2.8):

Definition 2.11. A map $f : A \longrightarrow B$ of motivic spaces is a *motivic weak equivalence* if, for every motivically fibrant Z, the map

$$\mathcal{S}_{\mathcal{M}}(f^c, Z) : \mathcal{S}_{\mathcal{M}}(B^c, Z) \longrightarrow \mathcal{S}_{\mathcal{M}}(A^c, Z)$$

is a weak equivalence of pointed simplicial sets.

In 2.17 we shall note that 2.11 agrees with the corresponding notion in [12].

Using either Smith's work on combinatorial model categories or by Blander's [1, 3.1], we have

Theorem 2.12. *The motivic weak equivalences and the cofibrations define a cofibrantly generated model structure on $\mathcal{M}$.*

Notation 2.13. We refer to the model structure in 2.12 as the *motivic model structure* and make use of the notation $\mathcal{M}_{\mathrm{mo}}$. Its weak equivalences will be denoted by $\xrightarrow{\sim}$ and its fibrations by $\twoheadrightarrow$. In accordance with 2.10, we refer to the fibrations as *motivic fibrations*, since a motivic space A is motivically fibrant if and only if $A \longrightarrow *$ is a motivic fibration.

Alas, this notation conflicts slightly with [10]. See 2.17.

Next we shall derive some additional properties of the motivic model structure, starting with a characterization of motivic fibrations with motivically fibrant codomain. As above, consider an elementary distinguished square:

$$Q \quad = \quad \begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \phi \\ U & \xrightarrow{\psi} & X \end{array}$$

Using the simplicial mapping cylinder we factor the induced map $h_P \longrightarrow h_Y$ as a cofibration $h_P \rightarrowtail C = (h_P \wedge \Delta^1_+) \coprod_{h_P} h_Y$ followed by a simplicial homotopy equivalence $C \xrightarrow{\simeq} h_Y$. Similarly we factor the canonical map $sq = h_U \coprod_{h_P} C \longrightarrow h_X$ as $sq \overset{q}{\rightarrowtail} tq \xrightarrow{\simeq} h_X$. Finally, we consider $h_{U \times_S \mathbb{A}^1_S} \longrightarrow h_U$ and the factorization $h_{U \times_S \mathbb{A}^1_S} \overset{u}{\rightarrowtail} C_u \xrightarrow{\simeq} h_U$.

Definition 2.14. Let $\mathcal{Q}$ denote the collection of all elementary distinguished squares in Sm/S. Since Sm/S is essentially small, we may consider a skeleton and form the set $\tilde{J}$ of maps

$$\{* \rightarrowtail h_\emptyset\} \cup \{q : sq \rightarrowtail tq\}_{Q \in \mathcal{Q}} \cup \{u : h_{U \times_S \mathbb{A}^1_S} \rightarrowtail C_u\}_{U \in \operatorname{Ob} \mathrm{Sm}/S}.$$

Let J' be the set of pushout product maps $f \square g$ where $f \in \tilde{J}$ and $g \in \{\partial\Delta^n_+ \hookrightarrow \Delta^n_+\}$.

Lemma 2.15. *A schemewise fibration with motivically fibrant codomain is a motivic fibration if and only if it has the right lifting property with respect to the set J' of 2.14.*

Proof. We note that the (simplicial) functor $\mathcal{S}_{\mathcal{M}}(B,-)$ preserves simplicial homotopy equivalences, which in particular are schemewise weak equivalences. From the definitions, it then follows that a schemewise fibrant motivic space A is motivically fibrant if and only if the canonical map $A \overset{\mathrm{sc}}{\twoheadrightarrow} *$ enjoys the right lifting property with respect to J'. The statement follows using properties of Bousfield localizations [6, 3.3.16]. $\square$

Corollary 2.16. *The model category $\mathcal{M}_{\mathrm{mo}}$ is weakly finitely generated. In particular, motivic weak equivalences and motivic fibrations with motivically fibrant codomains are closed under filtered colimits.*

In the symmetric spectrum approach due to Jardine [10] one employs a slightly different model structure on motivic spaces. The cofibrations in this model structure are the schemewise cofibrations, i.e. the monomorphisms, while the weak equivalences are defined by localizing the so-called Nisnevich local weak equivalences [9] with respect to a rational point $h_S \longrightarrow h_{\mathbb{A}^1_S}$. Let us denote this model structure by $\mathcal{M}_{\mathrm{GJ}}$. Corollary 2.16 shows an advantage of working with $\mathcal{M}_{\mathrm{mo}}$. On the other hand, in $\mathcal{M}_{\mathrm{GJ}}$ every motivic space is schemewise cofibrant. We compare these two model structures in

Theorem 2.17. *The weak equivalences in the model structures $\mathcal{M}_{\mathrm{mo}}$ and $\mathcal{M}_{\mathrm{GJ}}$ coincide. In particular, the identity $\mathrm{Id}_{\mathcal{M}} : \mathcal{M}_{\mathrm{mo}} \longrightarrow \mathcal{M}_{\mathrm{GJ}}$ is the left adjoint of a Quillen equivalence.*

Proof. The fibrations in the pointed version of the model structure in [9] are called *global fibrations*. A weak equivalence in this model structure is a local weak equivalence, and a cofibration is a schemewise cofibration. We say that a globally fibrant presheaf Z is i_0*-fibrant* if the map $\mathcal{M}(h_S, Z) \longrightarrow \mathcal{M}(h_{\mathbb{A}^1_S}, Z)$ induced by the zero-section $i_0 : S \longrightarrow \mathbb{A}^1_S$ is an acyclic global fibration. Since h_{i_0} is a monomorphism, this is equivalent to the pointed version of h_{i_0}-local simplicial presheaves in [9, §1.2].

A map $f\colon A \longrightarrow B$ is an i_0*-equivalence* if for all i_0-fibrant presheaf Z, the induced map of pointed simplicial sets $\mathcal{S}_{\mathcal{M}}(B,Z) \longrightarrow \mathcal{S}_{\mathcal{M}}(A,Z)$ is a weak equivalence. The i_0-equivalences are the weak equivalences in $\mathcal{M}_{GJ}$.

First we prove that any motivic weak equivalence is an i_0-equivalence. Suppose that $f\colon A \xrightarrow{\sim} B$ and Z is i_0-fibrant. Then Z is motivically fibrant, and thus $\mathcal{S}_{\mathcal{M}}(f^c, Z)$ is a weak equivalence. Since f^c is related to f via schemewise weak equivalences, it follows that f is an i_0-equivalence. This proves that motivic weak equivalences are i_0-equivalences.

Choose a motivically fibrant Z and suppose $f : A \longrightarrow B$ is an i_0-equivalence. According to [9] there exists a map $Z \xrightarrow{\sim \mathrm{sc}} Z'$ where Z' is globally fibrant. Since the domain and codomain of h_{i_0} are cofibrant, 2.7 implies that Z' is i_0-fibrant. Using the fact that $\mathcal{M}_{\mathrm{mo}}$ is an $\mathcal{S}$-model category, we get the following commutative diagram:

$$\begin{tikzcd} \mathcal{S}_{\mathcal{M}}(B^c, Z) \arrow[r, "\mathcal{S}_{\mathcal{M}}(f^c, Z)"] \arrow[d, "\sim"'] & \mathcal{S}_{\mathcal{M}}(A^c, Z) \arrow[d, "\sim"] \\ \mathcal{S}_{\mathcal{M}}(B^c, Z') \arrow[r, "\mathcal{S}_{\mathcal{M}}(f^c, Z')"] & \mathcal{S}_{\mathcal{M}}(A^c, Z') \end{tikzcd}$$

The map $\mathcal{S}_{\mathcal{M}}(f^c, Z')$ is a weak equivalence of spaces since f^c is an i_0-equivalence, i.e. f is a motivic weak equivalence. The Quillen equivalence follows. □

Lemma 2.18. *Smashing with a cofibrant motivic space preserves motivic weak equivalences.*

Proof. Suppose Z is motivically fibrant, that is, the canonical map $Z \twoheadrightarrow *$ is a schemewise fibration having the right lifting properties with respect to J'. If C is cofibrant, then $\mathcal{M}(C, Z)$ is schemewise fibrant according to 2.7. We claim $\mathcal{M}(C, Z)$ is motivically fibrant. For this, it suffices to prove for every generating cofibration

$$i := h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+,$$

the induced map $\mathcal{M}(i, Z)$ has the right lifting property with respect to J'. By adjointness, it suffices to prove that the pushout product of i and any map in J' is a composition of cobase changes of maps in J'. This holds by the following facts.

- $h_\emptyset \wedge h_U \cong h_\emptyset$
- Taking the product of an elementary distinguished square with any object $U \in \mathrm{Ob}\,\mathrm{Sm}/S$ yields an elementary distinguished square.
- $(h_{V \times_S \mathbb{A}^1} \longrightarrow h_V) \wedge h_U \cong h_{U \times_S V \times_S \mathbb{A}^1} \longrightarrow h_{U \times_S V}$
- The pushout product of $\partial\Delta^m \hookrightarrow \Delta^m$ and $\partial\Delta^n \hookrightarrow \Delta^n$ is an inclusion of simplicial sets, hence can be formed by attaching cells.

To conclude, it remains to note that for every motivically fibrant Z and every $f : A \xrightarrow{\sim} B$, the induced map $\mathcal{S}_{\mathcal{M}}((f \wedge C)^c, Z)$ is a weak equivalence. First, note that by the argument above, the map $\mathcal{S}_{\mathcal{M}}(f^c \wedge C, Z) \cong \mathcal{S}_{\mathcal{M}}(f^c, \mathcal{M}(C, Z))$

is a weak equivalence. This means that $f^c \wedge C$ is a motivic weak equivalence. But 2.9 and the commutative diagram

$$\begin{tikzcd}
(A\wedge C)^c \arrow[r, two heads, "\sim \mathrm{sc}"] \arrow[d, "(f\wedge C)^c"'] & A\wedge C \arrow[d, "f\wedge C"'] & A^c\wedge C \arrow[l, "\sim \mathrm{sc}"'] \arrow[d, "f^c\wedge C"'] \\
(B\wedge C)^c \arrow[r, two heads, "\sim \mathrm{sc}"] & B\wedge C & B^c\wedge C \arrow[l, "\sim \mathrm{sc}"']
\end{tikzcd}$$

show that $(f\wedge C)^c$ is a motivic weak equivalence if and only $f^c \wedge C$ is so. □

Corollary 2.19. *$\mathcal{M}_{\mathrm{mo}}$ is a monoidal $\mathcal{M}_{\mathrm{sc}}$-model category.*

Proof. We have to check that the pushout product of $h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+$ and a generating acyclic cofibration in $\mathcal{M}_{\mathrm{mo}}$ is a motivic weak equivalence for all $U \in \mathrm{Ob}\,\mathrm{Sm}/S$ and $n \geq 0$. Since h_U is cofibrant, the result follows from 2.18 and left properness of $\mathcal{M}_{\mathrm{mo}}$. □

We can now extend 2.18 to all motivic spaces.

Lemma 2.20. *Taking the smash product $-\wedge A$ or a cobase change along a schemewise cofibration preserves motivic weak equivalences for all $A \in \mathrm{Ob}\,\mathcal{M}$.*

Proof. For the first claim: we may replace A by A^c using 2.9 and hence conclude using 2.18. The second claim follows by factoring any motivic weak equivalence as a motivic acyclic cofibration followed by a schemewise acyclic fibration, and quoting 2.9 for the schemewise acyclic fibration. □

Lemma 2.21. *The monoid axiom holds in $\mathcal{M}_{\mathrm{mo}}$.*

Proof. Let f be an acyclic cofibration in $\mathcal{M}_{\mathrm{mo}}$ and let C be any motivic space. By 2.20, $f \wedge C$ is a schemewise cofibration and a motivic weak equivalence. It suffices to prove that the class of such maps is closed under cobase changes and sequential compositions. For this we use 2.20 and 2.16, respectively. □

Lemma 2.22. *The model category $\mathcal{M}_{\mathrm{mo}}$ is proper.*

Proof. Left properness of $\mathcal{M}_{\mathrm{mo}}$ is obvious since the cofibrations are not altered. To see that the model structure is right proper, one can either employ [1, 3.1], or mimic Jardine's proof of [10, A.5]. □

Remark 2.23. It is worth noticing that all of the results above hold more generally. One may replace Sm/S by any site with interval, see [12], in which the Grothendieck topology is generated by a bounded, complete and regular cd-structure [17]. An interesting example is the cdh-topology on the category Sch/S of schemes of finite type over S and representing interval the affine line.

2.2 Stable homotopy theory

The model category $\mathcal{M}_{\mathrm{mo}}$ has all the properties required to apply the results of [8, Section 4]. On the one hand, $\mathcal{M}_{\mathrm{mo}}$ is a cellular model category by [1], so Hirschhorn's localization methods work. On the other hand, one can also use Smith's combinatorial model categories for Bousfield localization. In any case, the category $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, A)$ of spectra of motivic spaces (with respect to some cofibrant finitely presentable motivic space A) has a stable model structure. For precise statements consult [8, 4.12 and 4.14].

We are interested in special motivic spaces A. The basic "sphere" in motivic homotopy theory is obtained in the same way as the circle in classical homotopy theory. It is defined as the Thom space $\mathbb{A}^1_S/\mathbb{A}^1_S - \{0\}$ of the trivial line bundle. Since $\mathbb{A}^1_S$ is contractible, the pushout $(\mathbb{A}^1_S - \{0\}, 1) \wedge S^1$ of the diagram

$$* \longleftarrow h_S \wedge S^1 \xrightarrow{h_{i_1} \wedge S^1} h_{\mathbb{A}^1_S - \{0\}} \wedge S^1$$

is weakly equivalent to $\mathbb{A}^1_S/\mathbb{A}^1_S - \{0\}$ [12, 3.2.2]. In the diagram, the map $i_1 : S \longrightarrow \mathbb{A}^1_S - \{0\}$ is induced by the closed point $1 \in \mathbb{A}^1_S(S)$. Note that although $h_{i_1} \wedge S^1$ is a schemewise cofibration (i.e. monomorphism), it need not be a cofibration in the motivic model structure $\mathcal{M}_{\mathrm{mo}}$.

Since the domain and codomain of $h_{i_1} \wedge S^1$ are cofibrant, we may factor this map using the simplicial mapping cylinder as a cofibration $h_S \wedge S^1 \rightarrowtail C$ and a simplicial homotopy equivalence. The quotient $T := C/h_S \wedge S^1$ is then cofibrant and a finitely presentable motivic space, schemewise weakly equivalent to the smash product $(\mathbb{A}^1_S - \{0\}, 1) \wedge S^1$. Up to motivic weak equivalence the choice of T is irrelevant. See [8, 5.7] and cp. 2.20. Now the identity $\mathrm{Id}_{\mathcal{M}}$ is a left Quillen equivalence from $\mathcal{M}_{\mathrm{mo}}$ to the pointed version of Jardine's model structure on $\mathcal{M}$ by 2.17. So that by [8, 5.7] the stable model structure on the category of motivic spectra $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ is Quillen equivalent to Jardine's model for the motivic stable homotopy category. Using Voevodsky's observation about cyclic permutations, we get

Lemma 2.24. *The functor* $- \wedge T \colon \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T) \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ *is a Quillen equivalence.*

Proof. The identity $\mathrm{id}_{\mathcal{M}}$ induces a commutative diagram of left Quillen functors

$$\begin{tikzcd}
\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T) \arrow[r] \arrow[d, "-\wedge T"'] & \mathrm{Sp}(\mathcal{M}_{\mathrm{GJ}}, T) \arrow[d, "-\wedge T"] \\
\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T) \arrow[r] & \mathrm{Sp}(\mathcal{M}_{\mathrm{GJ}}, T)
\end{tikzcd}$$

where the two horizontal arrows are Quillen equivalences. Here $\mathcal{M}_{GJ}$ denotes the pointed version of the Goerss-Jardine model structure on $\mathcal{M}$. In $\mathcal{M}_{GJ}$, the cofibrations are the schemewise cofibrations. Hence every presheaf is cofibrant. By [8, 10.3] it suffices to establish that T is weakly equivalent to a symmetric presheaf A, so that the next diagram commutates where $\mathrm{cyc}\colon A^{\wedge 3} \longrightarrow A^{\wedge 3}$ is the cyclic permutation map and H is a homotopy from the cyclic permutation to the identity; for details we refer to [8, 10.2].

$$\begin{tikzcd}
A^{\wedge 3} \wedge h_S \arrow[r, "A^{\wedge 3}\wedge h_{i_0}"] \arrow[dr, "\mathrm{id}_{A^{\wedge 3}\wedge h_S}"'] & A^{\wedge 3} \wedge h_{\mathbb{A}^1} \arrow[d, "H"'] & A^{\wedge 3} \wedge h_S \arrow[l, "A^{\wedge 3}\wedge h_{i_1}"'] \arrow[dl, "\mathrm{cyc}\wedge h_S"] \\
 & A^{\wedge 3} \wedge h_S &
\end{tikzcd}$$

The presheaf $\mathbb{A}^1_S/\mathbb{A}^1_S - \{0\}$ is weakly equivalent to T, and symmetric according to [10, 3.13]. Hence $-\wedge T$ on the right hand side is a Quillen equivalence, which implies the same statement for the functor $-\wedge T$ on the left hand side. $\square$

3 Motivic functors

In this section we shall introduce the category of motivic functors, describe its monoidal structure and display some of its useful homotopy properties. We do this in four steps. Each step involves giving a monoidal model structure to the category of motivic functors.

The first step is defining the pointwise model, which is of little practical value, but it serves as a building block for all the other models. The second step deals with the homotopy functor model. We advocate this as a tool for doing motivic homotopy theory on a functorial basis, mimicking the grand success in algebraic topology. The most interesting functors are homotopy invariant, but many natural constructions will take to functors which do not preserve weak equivalences. The homotopy functor model structure is a convenient way of handling these problems.

Thirdly we have the stable structure, which from our point of view is the natural generalization of stable homotopy theory from algebraic topology, but which unfortunately does not automatically agree with the other proposed models for stable motivic homotopy theory. Hence we are forced to park this theory in our technical garage for time being and introduce the fourth and final model structure: the spherewise model structure. Although technically not as nice as the stable model, the spherewise model is Quillen equivalent to the other models for motivic stable homotopy theory.

Many of the results in this section can be justified by inferring references to [5]. For the convenience of the reader we will indicate most proofs of these results.

3.1 The category of motivic functors

Recall the category of motivic spaces $\mathcal{M} = \mathcal{M}_S = [(\mathrm{Sm}/S)^{\mathrm{op}}, \mathcal{S}]$ discussed in the previous section. As a closed symmetric monoidal category, it is enriched over itself, hence an $\mathcal{M}$-category. Let $\mathbf{f}\mathcal{M}$ be the full sub-$\mathcal{M}$-category of finitely presentable motivic spaces.

Definition 3.1. A *motivic functor* is an $\mathcal{M}$-functor $X\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$. That is, X assigns to any finitely presentable motivic space A a motivic space XA together with maps of motivic spaces $\hom^X_{A,B}\colon \mathcal{M}(A,B) \longrightarrow \mathcal{M}(XA,XB)$ compatible with the enriched composition and identities. We let **MF** be the category of motivic functors and $\mathcal{M}$-natural transformations.

Since **MF** is a category of functors with bicomplete codomain, it is bicomplete and enriched over $\mathcal{M}$. If X and Y are motivic functors, let $\mathcal{M}_{\mathbf{MF}}(X,Y)$ be the motivic space of maps from X to Y. If A is a finitely presentable motivic space, then the motivic functor represented by A is given as

$$\mathcal{M}(A,-)\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}, \qquad \mathcal{M}(A,-)(B) = \mathcal{M}(A,B)$$

The enriched Yoneda lemma holds, and every motivic functor can be expressed in a canonical way as a colimit of representable functors.

Theorem 3.2 (Day). *The category of motivic functors is closed symmetric monoidal with unit the inclusion* $\mathbb{I}\colon \mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M}$.

This theorem is a special case of [4]; it is simple enough to sketch the basic idea. Denote the monoidal product of two motivic functors X and Y by $X \wedge Y$. Since every motivic functor is a colimit of representables, it suffices to fix the monoidal product on representable functors

$$\mathcal{M}(A,-) \wedge \mathcal{M}(B,-) := \mathcal{M}(A \wedge B,-).$$

The internal hom is defined by setting

$$\mathbf{MF}(X,Y)(A) = \mathcal{M}_{\mathbf{MF}}(X, Y(- \wedge A)).$$

Let us describe a special feature of the category of motivic functors, which makes the monoidal product more transparent. The point is just that motivic functors can be composed. Note that any motivic functor $X\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ can be extended – via enriched left Kan extension along the full inclusion $\mathbb{I}\colon \mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M}$ – to an $\mathcal{M}$-functor $\mathbb{I}_*X\mathcal{M} \longrightarrow \mathcal{M}$ satisfying $\mathbb{I}_*X \circ \mathbb{I} \cong X$. Since the category of motivic spaces is locally finitely presentable 2.2, this defines an equivalence between **MF** and the category of $\mathcal{M}$-functors $\mathcal{M} \longrightarrow \mathcal{M}$ that preserve filtered colimits. Given motivic functors X and Y, one defines their composition by setting

$$X \circ Y := \mathbb{I}_*X \circ Y.$$

Moreover, there is the natural *assembly map* $X \wedge Y \longrightarrow X \circ Y$ which is an isomorphism provided Y is representable [5, 2.8]. In fact, if both X and Y are representable, then the assembly map is the natural adjointness isomorphism

$$\mathcal{M}(A,-) \wedge \mathcal{M}(B,-) = \mathcal{M}(A \wedge B,-) \cong \mathcal{M}(A, \mathcal{M}(B,-)).$$

Remark 3.3. A *motivic ring* is a monoid in the category of motivic functors. Given the simple nature of the smash product in **MF** motivic rings can be described quite explicitly. Running through the definitions we see that a map $X \wedge X \longrightarrow X$ of motivic functors is the same as an $\mathcal{M}$-natural transformation of two variables $XA \wedge XB \longrightarrow X(A \wedge B)$, and so a motivic ring is a motivic functor X together with natural transformations $XA \wedge XB \longrightarrow X(A \wedge B)$ and $A \longrightarrow XA$ such that the relevant diagrams commute. Hence motivic rings are analogous to Bökstedt's *functors with smash product* [2].

Example 3.4. Let SmCor/S be the category of smooth correspondences over S. The special case $S = \mathrm{Spec}(k)$ is described in [18]. A *motivic space with transfers* is an additive functor, or an $\mathbf{Ab}$-functor, $F\colon (\mathrm{SmCor}/S)^{\mathrm{op}} \longrightarrow \mathbf{sAb}$ to the category of simplicial abelian groups. Let $\mathcal{M}^{\mathrm{tr}}$ be the category of motivic spaces with transfers. By forgetting the extra structure of having transfers and composing with the opposite of the graph functor $\Gamma\colon \mathrm{Sm}/S \longrightarrow \mathrm{SmCor}/S$ it results a forgetful functor $u\colon \mathcal{M}^{\mathrm{tr}} \longrightarrow \mathcal{M}$ with left adjoint $\mathbb{Z}_{\mathrm{tr}}\colon \mathcal{M} \longrightarrow \mathcal{M}^{\mathrm{tr}}$. The functor $\mathbb{Z}_{\mathrm{tr}}$ is determined by the property that $\mathbb{Z}_{\mathrm{tr}}(h_U \wedge \Delta^n_+) = \mathrm{Hom}_{\mathrm{SmCor}/S}(-, U) \otimes \mathbb{Z}(\Delta^n)$.

Let $M\mathbb{Z} \in \mathbf{MF}$ be the composite functor

$$\mathbf{f}\mathcal{M} \hookrightarrow \mathcal{M} \xrightarrow{\mathbb{Z}_{\mathrm{tr}}} \mathcal{M}^{\mathrm{tr}} \xrightarrow{u} \mathcal{M}.$$

We claim that $M\mathbb{Z}$ is a commutative monoid in **MF**. First, the unit $\mathbb{I} \longrightarrow M\mathbb{Z}$ is the unit of the adjunction between $\mathcal{M}$ and $\mathcal{M}^{\mathrm{tr}}$. To define a multiplication, we note using [4] and [15] that $\mathcal{M}^{\mathrm{tr}}$ is closed symmetric monoidal. Since the graph functor is strict symmetric monoidal and forgetting the addition is lax symmetric monoidal, general category theory implies $\mathbb{Z}_{\mathrm{tr}}$ is strict symmetric monoidal and u is lax symmetric monoidal. In particular, we get the natural multiplication map μ on $M\mathbb{Z}$, given by

$$u(\mathbb{Z}_{\mathrm{tr}}(A)) \wedge u(\mathbb{Z}_{\mathrm{tr}}(B)) \longrightarrow u(\mathbb{Z}_{\mathrm{tr}}(A) \otimes \mathbb{Z}_{\mathrm{tr}}(B)) \longrightarrow u(\mathbb{Z}_{\mathrm{tr}}(A \wedge B)).$$

To see that $M\mathbb{Z}$ is a motivic functor, consider the composition

$$\mathcal{M}(A,B) \wedge u\mathbb{Z}_{\mathrm{tr}}A \longrightarrow u\mathbb{Z}_{\mathrm{tr}}\mathcal{M}(A,B) \wedge u\mathbb{Z}_{\mathrm{tr}}A \longrightarrow u\mathbb{Z}_{\mathrm{tr}}(\mathcal{M}(A,B) \wedge A),$$

and note that $u\mathbb{Z}_{\mathrm{tr}}(\mathcal{M}(A,B) \wedge A)$ maps naturally to $u\mathbb{Z}_{\mathrm{tr}}B$. In 4.6 we show $M\mathbb{Z}$ represents Voevodsky's motivic Eilenberg-MacLane spectrum [16].

3.2 Evaluation on spheres

As explained in [5, Section 2.5], the category $\mathrm{Sp}(\mathcal{M}, T)$ of motivic spectra with respect to the T of 2.2 can be described as a category of $\mathcal{M}$-functors. Let $T\mathrm{Sph}$ be the sub-$\mathcal{M}$-category of $\mathcal{M}$ with objects the smash powers $T^0 = S^0, T, T^{\wedge 2} := T \wedge T, T^{\wedge 3} := T \wedge (T^{\wedge 2}), \cdots$ of T. If $k \geq 0$ the motivic space of morphisms in $T\mathrm{Sph}$ from $T^{\wedge n}$ to $T^{\wedge n+k}$ is $T^{\wedge k}$ considered by adjointness as a subobject of $\mathcal{M}(T^{\wedge n}, T^{\wedge n+k})$. If $k < 0$ the morphism space is trivial. Let $i\colon T\mathrm{Sph} \hookrightarrow \mathbf{f}\mathcal{M}$ be the inclusion. Hence every motivic functor X gives rise to a motivic spectrum $\mathrm{ev}(X) := X \circ i$.

Similarly, the category $\mathrm{Sp}^{\Sigma}(\mathcal{M}, T)$ of motivic symmetric spectra is isomorphic to the category of $\mathcal{M}$-functors (with values in $\mathcal{M}$) from a slightly larger sub-$\mathcal{M}$-category $j\colon T\mathrm{Sph}^{\Sigma} \hookrightarrow \mathbf{f}\mathcal{M}$, which is determined by the property that it is the smallest sub-$\mathcal{M}$-category containing $T\mathrm{Sph}$ and the symmetric group $\Sigma(n)_+ \subseteq \mathcal{M}(T^{\wedge n}, T^{\wedge n})$ for all n. Hence, if U denotes the forgetful functor, then the evaluation map $\mathrm{ev}\colon \mathbf{MF} \longrightarrow \mathrm{Sp}(\mathcal{M}, T)$ factors as

$$\mathbf{MF} \xrightarrow{\mathrm{ev}'} \mathrm{Sp}^{\Sigma}(\mathcal{M}, T) \xrightarrow{U} \mathrm{Sp}(\mathcal{M}, T).$$

Moreover ev' is lax symmetric monoidal and its left adjoint is strict symmetric monoidal. For further details we refer the reader to [5, Section 2.6].

3.3 The pointwise structure

We first define the pointwise model structure on $\mathbf{MF}$. As earlier commented, the pointwise structure is of no direct use for applications, but it is vital for the constructions of the useful structures to come.

Definition 3.5. A map $f\colon X \longrightarrow Y$ in $\mathbf{MF}$ is a

- *Pointwise weak equivalence* if for every object A in $\mathbf{f}\mathcal{M}$ the induced map $f(A)\colon X(A) \longrightarrow Y(A)$ is a weak equivalence in $\mathcal{M}_{\mathrm{mo}}$.
- *Pointwise fibration* if for every object A in $\mathbf{f}\mathcal{M}$ the induced map $f(A)\colon X(A) \longrightarrow Y(A)$ is a fibration in $\mathcal{M}_{\mathrm{mo}}$.
- *Cofibration* if f has the left lifting property with respect to all pointwise acyclic fibrations.

The category $\mathbf{MF}$, together with these classes of morphisms, is denoted $\mathbf{MF}_{\mathrm{pt}}$ and referred to as the *pointwise structure* on $\mathbf{MF}$.

Theorem 3.6. *The pointwise structure* $\mathbf{MF}_{\mathrm{pt}}$ *is a cofibrantly generated proper monoidal model category satisfying the monoid axiom.*

Proof. The model structure follows from [7, 2.1.19], where the monoid axiom for $\mathcal{M}_{\mathrm{mo}}$ is used to ensure that the generating acyclic cofibrations listed in 3.7, as well as sequential compositions of cobase changes of these, are pointwise weak equivalences. The form of the generating (acyclic) cofibrations, together with the behavior of $\wedge$ on representables, ensures that $\mathbf{MF}$ is a monoidal model category [7, 4.2.5]. Right properness follows at once from the fact that $\mathcal{M}_{\mathrm{mo}}$ is right proper 2.22. Left properness requires more than $\mathcal{M}_{\mathrm{mo}}$ being left proper, but follows from 2.20.

To prove the monoid axiom, let X be a motivic functor and consider the smash product

$$X \wedge \mathcal{M}(A,-) \wedge sj \xrightarrow{X \wedge \mathcal{M}(A,-) \wedge j} X \wedge \mathcal{M}(A,-) \wedge tj$$

with a generating acyclic cofibration, where j is a generating acyclic cofibration for $\mathcal{M}_{\mathrm{mo}}$. It is a pointwise weak equivalence by 2.20, and also pointwise a schemewise cofibration. In particular, any sequential composition of cobase changes of maps like these is a pointwise weak equivalence, which concludes the proof. □

Remark 3.7. If A varies over the set of isomorphism classes in $\mathbf{f}\mathcal{M}$ and $i : si \rightarrowtail ti$ varies over the generating (acyclic) cofibrations in $\mathcal{M}_{\mathrm{mo}}$, then the maps $\mathcal{M}(A,-) \wedge i : \mathcal{M}(A,-) \wedge si \longrightarrow \mathcal{M}(A,-) \wedge ti$ form a set of generating (acyclic) cofibrations for $\mathbf{MF}_{\mathrm{pt}}$. In particular, all representable motivic functors (for example the unit) are cofibrant.

The following theorem will help us to deduce the monoid axiom for some other model structures on motivic functors.

Theorem 3.8. *Smashing with a cofibrant object in $\mathbf{MF}_{\mathrm{pt}}$ preserves pointwise equivalences.*

Proof. If X is representable, say $X = \mathcal{M}(A,-)$ and $f \colon Y \longrightarrow Z$ is a pointwise weak equivalence, then the assembly map is an isomorphism

$$f \wedge \mathcal{M}(A,-) \cong f \circ \mathcal{M}(A,-) = \mathbb{I}_* f \circ \mathcal{M}(A,-).$$

Since $\mathbb{I}_* f$ commutes with filtered colimits and every motivic space is a filtered colimit of finitely presentable motivic spaces, 2.16 implies that $\mathbb{I}_* f(B)$ is a motivic weak equivalence for every motivic space B, e.g. for $B = \mathcal{M}(A, C)$.

For an arbitrary cofibrant motivic functor, the result follows from the previous case using induction on the attaching cells and the fact that cobase change along monomorphisms preserves motivic weak equivalences 2.20. □

3.4 The homotopy functor structure

The major caveat concerning the pointwise model structure is that a motivic weak equivalence $A \xrightarrow{\sim} B$ of finitely presentable motivic spaces does not necessarily induce a pointwise weak equivalence $\mathcal{M}(B,-) \longrightarrow \mathcal{M}(A,-)$ of representable motivic functors. To remedy this problem, we introduce a model structure in which every motivic functor is a homotopy functor up to weak equivalence. A homotopy functor is a functor preserving weak equivalences.

Recall that the pointwise structure is defined entirely in terms of the weakly finitely generated model structure $\mathcal{M}_{\mathrm{mo}}$. However, to define the homotopy functor structure it is also useful to consider the Quillen equivalent model structure $\mathcal{M}_{\mathrm{GJ}}$ in which all motivic spaces are cofibrant. The slogan is: “use

$\mathcal{M}_{\mathrm{GJ}}$ on the source and $\mathcal{M}_{\mathrm{mo}}$ on the target". This is the main difference from the general homotopy functor setup presented in [5].

Definition 3.9. Let M be the set of acyclic monomorphisms (i.e. maps that are both monomorphisms and motivic weak equivalences) of finitely presentable motivic spaces. For a motivic space A, let $\mathrm{ac}(A)$ be the following category. The objects of $\mathrm{ac}(A)$ are the maps $A \longrightarrow B \in \mathcal{M}$ that can be obtained by attaching finitely many cells from M. The set of morphisms from an object $\beta\colon A \longrightarrow B$ to another $\gamma\colon A \longrightarrow C$ is the set of maps $\tau\colon B \longrightarrow C$ that can be obtained by attaching finitely many cells from M such that $\tau\beta = \gamma$. Set

$$\Phi(A) := \operatorname*{colim}_{A \to B \in \mathrm{ac}(A)} B.$$

Note that the objects in $\mathrm{ac}(A)$ are acyclic cofibrations in $\mathcal{M}_{\mathrm{GJ}}$.

The techniques from [5, Section 3.3] ensure the following properties of this construction, see [5, 3.24]

Lemma 3.10. *For every motivic space A, the map $\Phi(A) \longrightarrow *$ has the right lifting property with respect to the maps in M. In particular, $\Phi(A)$ is fibrant in $\mathcal{M}_{\mathrm{mo}}$. Moreover, Φ is a functor and there exists a natural transformation $\varphi_A\colon A \longrightarrow \Phi(A)$ which is an acyclic monomorphism. If the motivic space A is finitely presentable, then $\Phi(A)$ is isomorphic to a filtered colimit of finitely presentable motivic spaces weakly equivalent to A.*

There are occasions where it is more convenient to employ M instead of the set J' introduced in 2.15. For example, every motivic weak equivalence of finitely presentable motivic spaces can be factored as a map in M, followed by a simplicial homotopy equivalence. Adjointness and 2.7 imply:

Lemma 3.11. *Suppose A is a motivic space such that $A \longrightarrow *$ has the right lifting property with respect to the maps in M. If $f\colon B \longrightarrow C$ is an acyclic monomorphism of finitely presentable motivic spaces, then the induced map $\mathcal{M}(C, A) \longrightarrow \mathcal{M}(B, A)$ is an acyclic fibration in $\mathcal{M}_{\mathrm{mo}}$.*

We define the (not necessarily motivic) functor $\hbar(X)\colon \mathbf{f}\mathcal{M} \longrightarrow \mathcal{M}$ by the composition

$$\hbar(X)(A) := \mathbb{I}_* X(\Phi(A)).$$

Note that $\varphi\colon \mathrm{Id}_{\mathcal{M}} \longrightarrow \Phi$ induces a natural transformations of functors $\mathrm{Id}_{\mathbf{MF}} \longrightarrow \hbar$.

Definition 3.12. A map $f\colon X \longrightarrow Y$ in $\mathbf{MF}$ is an

- hf-*weak equivalence* if the map $\hbar(X)(A)$ is a weak equivalence in $\mathcal{M}_{\mathrm{mo}}$ for all $A \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$.

- hf-*fibration* if f is a pointwise fibration and for all acyclic monomorphisms $\phi\colon A \xrightarrow{\sim} B \in \mathbf{f}\mathcal{M}$ the diagram

$$\begin{tikzcd} X(A) \arrow[r, "X(\phi)"] \arrow[d, "f(A)"'] & X(B) \arrow[d, "f(B)"] \\ Y(A) \arrow[r, "Y(\phi)"] & Y(B) \end{tikzcd}$$

  is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$.

In the following, the hf-weak equivalences and hf-fibrations together with the class of cofibrations, will be referred to as the *homotopy functor structure* $\mathbf{MF}_{\mathrm{hf}}$ on $\mathbf{MF}$.

Lemma 3.13. *A map in* $\mathbf{MF}$ *is both an* hf*-fibration and an* hf*-equivalence if and only if it is a pointwise acyclic fibration.*

Proof. One implication is clear.

If $f\colon X \longrightarrow Y$ is an hf-fibration and an hf-equivalence, choose $A \in \mathbf{f}\mathcal{M}$ and consider the induced diagram:

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d, "f(A)"'] & \mathbb{I}_*X(\Phi(A)) \arrow[d, "\mathbb{I}_*f(\Phi(A))"] \\ Y(A) \arrow[r] & \mathbb{I}_*Y(\Phi(A)) \end{tikzcd}$$

It remains to prove that $f(A)$ is a motivic weak equivalence. The right vertical map is a motivic weak equivalence by assumption, so it suffices to prove that the diagram is a homotopy pullback square. Since f is an hf-fibration and $\mathbb{I}_*Z$ commutes with filtered colimits for any motivic functor Z, 3.10 shows the square is a filtered colimit of homotopy pullback squares. By 2.16, homotopy pullback squares in $\mathcal{M}_{\mathrm{mo}}$ are closed under filtered colimits, which finishes the proof. □

Theorem 3.14. *The homotopy functor structure is a cofibrantly generated and proper monoidal model category.*

Proof. First we establish the weakly finitely generated model structure. This follows from [7, 2.1.19], where 3.13 and 3.11 are needed to check the relevant conditions. More precisely, 3.11 shows that the generating acyclic cofibrations listed in 3.17 below are hf-equivalences. By arguments which can be found in the proof of [5, 5.9], any sequential composition of cobase changes of the generating acyclic cofibrations is an hf-equivalence.

Concerning the monoidal part, the crucial observation is that if $f\colon A \longrightarrow B$ is an acyclic monomorphism in $\mathbf{f}\mathcal{M}$ and C is finitely presentable, then the map $f \wedge C\colon A \wedge C \longrightarrow B \wedge C$ is an acyclic monomorphism in $\mathbf{f}\mathcal{M}$. For details and also right properness, see [5, 5.12 and 5.13]. Left properness is clear. □

Theorem 3.15. *Smashing with a cofibrant motivic functor preserves* hf*-equivalences and* $\mathbf{MF}_{\mathrm{hf}}$ *satisfies the monoid axiom.*

Proof. We factor the hf-equivalence into an hf-acyclic cofibration followed by an hf-acyclic fibration. Now 3.13 shows that hf-acyclic fibrations are pointwise acyclic fibrations, and 3.8 shows smashing with a cofibrant object preserves pointwise weak equivalences. Hence we may assume the hf-equivalence is a cofibration. Since the model structure $\mathbf{MF}_{\mathrm{hf}}$ is monoidal, smashing with a cofibrant object preserves hf-acyclic cofibrations. This proves our first claim.

The monoid axiom is shown to hold as follows. Suppose that $X \xrightarrow{\sim \mathrm{hf}} Y$ is a generating hf-acyclic cofibration, and Z is an object of $\mathbf{MF}$ with cofibrant replacement $Z^c \xrightarrow{\sim \mathrm{pt}} Z$. Since X and Y are cofibrant, there is the diagram:

$$\begin{tikzcd} X \wedge Z^c \arrow[r, "\sim \mathrm{hf}"] \arrow[d, "\sim \mathrm{pt}"'] & Y \wedge Z^c \arrow[d, "\sim \mathrm{pt}"] \\ X \wedge Z \arrow[r] & Y \wedge Z \end{tikzcd}$$

This implies $X \wedge Z \xrightarrow{\sim \mathrm{hf}} Y \wedge Z$. The full monoid axiom follows as indicated in [5, 6.30]. □

Remark 3.16. Every motivic functor is an $\mathcal{S}$-functor since $\mathcal{M}_{\mathrm{mo}}$ is a monoidal $\mathcal{S}$-model category. As such, they preserve simplicial homotopy equivalences, see [5, 2.11]. Any motivic weak equivalence can be factored as the composition of an acyclic monomorphism and a simplicial homotopy equivalence. It follows that a pointwise fibration $f\colon X \xrightarrow{\mathrm{pt}} Y$ is an hf-fibration if and only if for every motivic weak equivalence $\phi\colon A \xrightarrow{\sim} B$ in $\mathbf{f}\mathcal{M}$ the following diagram is a homotopy pullback square in the motivic model structure:

$$\begin{tikzcd} X(A) \arrow[r, "X(\phi)"] \arrow[d, "f(A)"'] & X(B) \arrow[d, "f(B)"] \\ Y(A) \arrow[r, "Y(\phi)"] & Y(B) \end{tikzcd}$$

In particular, the fibrant functors in $\mathbf{MF}_{\mathrm{hf}}$ are the pointwise fibrant homotopy functors. On the other hand, we could have constructed the homotopy functor structure as a Bousfield localization with respect to the homotopy functors, avoiding $\hbar$ in 3.12. However, note that we have a characterization of arbitrary fibrations, as opposed to the situation for a general Bousfield localization.

Remark 3.17. The generating cofibrations for the pointwise and homotopy functor structures coincide. The generating acyclic cofibrations for $\mathbf{MF}_{\mathrm{hf}}$ may be chosen as follows. Consider an acyclic monomorphism $\phi\colon A \longrightarrow B \in \mathbf{f}\mathcal{M}$ and its associated factorization $\mathcal{M}(B,-) \xrightarrow{c_\phi} C_\phi \xrightarrow{\simeq} \mathcal{M}(A,-)$ obtained using the simplicial mapping cylinder. The hf-acyclic cofibrations are generated by the pointwise acyclic cofibrations of 3.7, together with the pushout product maps

$$c_\phi \square i\colon \mathcal{M}(B,-)\wedge ti \coprod_{\mathcal{M}(B,-)\wedge si} C_\phi \wedge si \longrightarrow C_\phi \wedge ti,$$

where ϕ varies over the (isomorphism classes of) acyclic monomorphisms in $\mathbf{f}\mathcal{M}$ and $i\colon si \rightarrowtail ti \in I$ varies over the generating cofibrations in $\mathcal{M}_{\mathrm{mo}}$. The domains and codomains of these pushout product maps are finitely presentable in $\mathbf{MF}$.

To end this section, we indicate why $\hbar(X)(A)$ has the correct homotopy type.

Lemma 3.18. *Let $X \xrightarrow{\sim \mathrm{hf}} X^{\mathrm{hf}}$ be a fibrant replacement in $\mathbf{MF}_{\mathrm{hf}}$. Then we have natural motivic weak equivalences*

$$\hbar(X)(A) \xrightarrow{\sim} \hbar(X^{\mathrm{hf}})(A) \xleftarrow{\sim} X^{\mathrm{hf}}(A).$$

Proof. The first map is a motivic weak equivalence by definition. The second map is a motivic weak equivalence because $\hbar(X^{\mathrm{hf}})(A) \cong \operatorname{colim}_{A \xrightarrow{\sim} B} X^{\mathrm{hf}}(B)$ and X^{hf} preserves motivic weak equivalences. □

3.5 The stable structure

We start with the hf-model structure and define the stable model structure more or less as for the general case in [5, Section 6]. The stable equivalences are the maps which become pointwise weak equivalences after a stabilization process, and the stably fibrant objects are morally the "Ω-spectra".

Let us repeat the stabilization process in the case of $\mathbf{MF}$ and the motivic space T of 2.2, weakly equivalent to $\mathbb{A}^1_S/(\mathbb{A}^1_S - \{0\})$. If X is a motivic functor and A is a finitely presentable motivic space, there is a map

$$t_X(A)\colon X(A) \longrightarrow \mathbb{T}(X)(A) := \mathcal{M}(T, X(T\wedge A))$$

natural in both X and A. It is adjoint to the map $X(A)\wedge T \longrightarrow X(T\wedge A)$ which in turn is adjoint to the composition

$$T \longrightarrow \mathcal{M}(A, T\wedge A) \xrightarrow{\hom^X_{A,T\wedge A}} \mathcal{M}(XA, X(T\wedge A)).$$

Let $\mathbb{T}^\infty(X)$ be the colimit of the sequence

$$X \xrightarrow{t_X} \mathbb{T}(X) \xrightarrow{\mathbb{T}(t_X)} \mathbb{T}(\mathbb{T}(X)) \longrightarrow \cdots,$$

and let $t^\infty_X : X \longrightarrow \mathbb{T}^\infty(X)$ be the canonically induced map.

We fix a fibrant replacement $X \xrightarrow{\sim \mathrm{hf}} X^{\mathrm{hf}}$ in $\mathbf{MF}_{\mathrm{hf}}$.

Definition 3.19. A morphism $f\colon X \longrightarrow Y$ in **MF** is a

- *Stable equivalence* if the induced map $\mathbb{T}^\infty(f^{\mathrm{hf}})\colon \mathbb{T}^\infty(X^{\mathrm{hf}}) \longrightarrow \mathbb{T}^\infty(Y^{\mathrm{hf}})$ is a pointwise weak equivalence.

- *Stable fibration* if f is an hf-fibration and the diagram

$$\begin{tikzcd} X(A) \arrow[r, "t_X(A)"] \arrow[d] & \mathbb{T}(X)(A) \arrow[d] \\ Y(A) \arrow[r, "t_Y(A)"] & \mathbb{T}(Y)(A) \end{tikzcd}$$

  is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$ for all $A \in \mathbf{f}\mathcal{M}$.

We denote by $\mathbf{MF}_{\mathrm{st}}$ the *stable structure* on **MF**, i.e. the category **MF** together with the classes of stable equivalences and stable fibrations.

Remark 3.20. The definition of stable equivalences in the general setting of [5, 6.2] involves the functor $\hbar(-)$ instead of $(-)^{\mathrm{hf}}$. By 3.18, this does not make any difference. In particular, the class of stable equivalences does not depend on the choice of $(-)^{\mathrm{hf}}$.

Lemma 3.21. *A map is a stable fibration and a stable equivalence if and only if it is a pointwise acyclic fibration.*

Proof. One implication is obvious.
If f is a stable fibration and a stable equivalence, then f^{hf} is also a stable equivalence. In general, f^{hf} will not be a pointwise fibration, but – as one can prove by comparing with $\hbar(f)$ – this is the only obstruction preventing f^{hf} from being a stable fibration. That is, the relevant squares appearing in the definition of an hf-fibration 3.12 and in the definition of a stable fibration 3.19 are homotopy pullback squares for f^{hf}. Details can be found in [5, Section 6.2]. Since homotopy pullback squares are closed under filtered colimits (like $\mathbb{T}^\infty$), the statement follows. $\square$

To prove that the stable structure is in fact a model structure, we will introduce generating stable acyclic cofibrations.

Definition 3.22. For a finitely presentable motivic space A, let τ_A be the composition

$$\mathcal{M}(T\wedge A,-)\wedge T \xrightarrow{\cong} \mathcal{M}(T,\mathcal{M}(A,-))\wedge T \xrightarrow{\epsilon_T\mathcal{M}(A,-)} \mathcal{M}(A,-),$$

where ϵ_T is the counit of the adjunction $(-\wedge T, \mathcal{M}(T,-))$ on **MF**. There exists a factorization $d_A\colon \mathcal{M}(T\wedge A,-)\wedge T \rightarrowtail D_A$ followed by a simplicial homotopy equivalence. Let $\mathcal{D}$ be the set of pushout product maps $d_A\Box i$, where $i\colon si \rightarrowtail ti$ is a generating cofibration in $\mathcal{M}_{\mathrm{mo}}$.

To deduce that the stable structure is a model structure, we need to know that the maps in $\mathcal{D}$-cell are stable equivalences. For this purpose, we compare with the stable model structure on $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ which exists by [8]. If X is a motivic functor and $A \in \mathbf{f}\mathcal{M}$, we can form the composition $X \circ (- \wedge A) \in \mathbf{MF}$.

Lemma 3.23. *Let $f\colon X \longrightarrow Y$ be a map of motivic functors. Then f is a stable equivalence if and only if* $\mathrm{ev}(f^{\mathrm{hf}} \circ (- \wedge B))$ *is a stable equivalence of motivic spectra for every $B \in \mathbf{f}\mathcal{M}$.*

Proof. Although the stabilizations in $\mathbf{MF}$ and $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ do not coincide under ev, they can be compared at each $B \in \mathbf{f}\mathcal{M}$ and shown to yield motivic weak equivalences

$$\mathbb{T}^\infty(f^{\mathrm{hf}})(B) \xrightarrow{\sim} (\Theta^\infty \mathrm{ev}(f^{\mathrm{hf}}(- \wedge B)))_0.$$

Here Θ^∞ is the stabilization defined in [8, 4.4]. Details are recorded in [5, Section 6.3]. This proves the claim. $\square$

Lemma 3.24. *The maps in $\mathcal{D}$-cell are stable equivalences.*

Proof. Our strategy is to note that 2.24 and 3.23 imply the maps in $\mathcal{D}$ are stable equivalences. To this end, it suffices to show – using 2-out-of-3 and 2.9 – that $\epsilon_T \mathcal{M}(A, -)$ is a stable equivalence for all $A \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$. Equivalently, according to 3.23, we may consider the map of motivic spectra $\mathrm{ev}((\epsilon_T \mathcal{M}(A, -))^{\mathrm{hf}} \circ (- \wedge B))$ for $B \in \mathrm{Ob}\,\mathbf{f}\mathcal{M}$. Write $X := \mathcal{M}(A, -)$. There is a zig-zag of pointwise weak equivalences connecting $(\epsilon_T X)^{\mathrm{hf}} \circ (- \wedge B)$ and $\epsilon_T (X^{\mathrm{hf}} \circ (- \wedge B))$. It can be constructed as follows. By naturality, the diagram

$$\begin{tikzcd}
 & \mathcal{M}(T, X) \wedge T \arrow[dl] \arrow[d, "\epsilon_T X"'] \arrow[dr, "\sim_{hf}"] & \\
\mathcal{M}(T, X^{\mathrm{hf}}) \wedge T \arrow[dr, "\epsilon_T(X^{hf})"'] & X \arrow[d, tail, "\sim \mathrm{hf}"'] & (\mathcal{M}(T, X) \wedge T)^{\mathrm{hf}} \arrow[dl, "(\epsilon_T X)^{\mathrm{hf}}"] \\
 & X^{\mathrm{hf}} &
\end{tikzcd}$$

commutes. Factor the map $\epsilon_T(X^{\mathrm{hf}})$ as a pointwise acyclic cofibration, followed by a pointwise fibration $Z \xrightarrow{\mathrm{pt}} X^{\mathrm{hf}}$. Then $Z \xrightarrow{\mathrm{pt}} X^{\mathrm{hf}}$ is in fact an hf-fibration. The reason is that X^{hf} is a pointwise fibrant homotopy functor, so $M(T, X^{\mathrm{hf}})$ is also a (pointwise fibrant) homotopy functor, since T is cofibrant. By 2.18, $\mathcal{M}(T, X^{\mathrm{hf}}) \wedge T$ is then a homotopy functor, hence the pointwise weak equivalence $\mathcal{M}(T, X^{\mathrm{hf}}) \wedge T \xrightarrow{\sim \mathrm{pt}} Z$ implies that Z is a homotopy functor. Any pointwise fibration of homotopy functors is an hf-fibration, thus $Z \xrightarrow{\mathrm{pt}} X^{\mathrm{hf}}$ is an hf-fibration. Hence there exists a lift $f\colon (\mathcal{M}(T, X) \wedge T)^{\mathrm{hf}} \longrightarrow Z$ in the diagram:

$$\begin{tikzcd}
\mathcal{M}(T,X)\wedge T \arrow[r] \arrow[d, tail, "\sim\mathrm{hf}"'] & \mathcal{M}(T,X^{\mathrm{hf}})\wedge T \arrow[r, tail, "\sim\mathrm{pt}"] & Z \arrow[d, "\mathrm{hf}"] \\
(\mathcal{M}(T,X)\wedge T)^{\mathrm{hf}} \arrow[rr, "(\epsilon_T X)^{\mathrm{hf}}"] \arrow[urr, dashed, "\mathfrak{f}"] & & X^{\mathrm{hf}}
\end{tikzcd}$$

We will prove that f is a pointwise weak equivalence. It suffices to prove that f is an hf-equivalence because both the domain and the codomain of f are homotopy functors. Hence by the 2-out-of-3 property it suffices to prove that $\mathcal{M}(T,X)\wedge T \longrightarrow \mathcal{M}(T,X^{\mathrm{hf}})\wedge T$ is an hf-equivalence. Since $-\wedge T$ preserves hf-equivalences, let us consider $\mathcal{M}(T,X) \longrightarrow \mathcal{M}(T,X^{\mathrm{hf}})$. We have to prove that for every finitely presentable motivic space C, $\hbar(\mathcal{M}(T,X) \longrightarrow \mathcal{M}(T,X^{\mathrm{hf}}))(C)$ is a motivic weak equivalence. Since T is finitely presentable and $\hbar$ can be described as a filtered colimit, the map in question is isomorphic to the map $\mathcal{M}(T,\hbar(X \xrightarrow{\sim\mathrm{hf}} X^{\mathrm{hf}})(C))$. The map $\hbar(X \xrightarrow{\sim\mathrm{hf}} X^{\mathrm{hf}})(C)$ is a motivic weak equivalence by definition, so it remains to observe that the domain and the codomain are both fibrant in $\mathcal{M}_{\mathrm{mo}}$. Now $X=\mathcal{M}(A,-)$ where A is finitely presentable, so the domain $\hbar(\mathcal{M}(A,-))(C)=\mathcal{M}(A,\Phi(C))$ is fibrant in $\mathcal{M}_{\mathrm{mo}}$. The codomain is isomorphic to a filtered colimit of fibrant objects, hence it is fibrant in $\mathcal{M}_{\mathrm{mo}}$.

We have constructed the diagram:

$$\begin{tikzcd}
\mathcal{M}(T,\mathcal{M}(A,-)^{\mathrm{hf}})\wedge T \arrow[r, "\sim\mathrm{pt}"] \arrow[dr, "\epsilon_T(\mathcal{M}(A,-)^{hf})"'] & Z \arrow[d] & (\mathcal{M}(T,\mathcal{M}(A,-))\wedge T)^{\mathrm{hf}} \arrow[l, "\sim\mathrm{pt}"'] \arrow[dl, "(\epsilon_T\mathcal{M}(A,-))^{\mathrm{hf}}"] \\
& \mathcal{M}(A,-)^{\mathrm{hf}} &
\end{tikzcd}$$

Pre-composing with $-\wedge B$ preserves pointwise weak equivalences so that we get the desired zig-zag of pointwise weak equivalences connecting the two maps $\epsilon_T(\mathcal{M}(A,-)^{\mathrm{hf}}\circ(-\wedge B))$ and $(\epsilon_T\mathcal{M}(A,-))^{\mathrm{hf}}\circ(-\wedge B)$. Since ev preserves pointwise weak equivalences, it suffices to check that

$$\mathrm{ev}(\epsilon_T(\mathcal{M}(A,-)^{\mathrm{hf}}\circ(-\wedge B))) = \epsilon_T(\mathrm{ev}(\mathcal{M}(A,-)^{\mathrm{hf}})\circ(-\wedge B))$$

is a stable equivalence. In what follows, let us abbreviate by E the pointwise fibrant motivic spectrum $\mathrm{ev}(\mathcal{M}(A,-)^{\mathrm{hf}}\circ(-\wedge B))$. Then $\iota_E\colon E \longrightarrow \Theta^\infty E$ is a stable equivalence whose codomain is a stably fibrant motivic spectrum [8, 4.12]. Moreover, since T is finitely presentable and cofibrant, the map $\mathcal{M}(T,\iota_E)\colon \mathcal{M}(T,E) \longrightarrow \mathcal{M}(T,\Theta^\infty E)$ is also a stable equivalence with stably fibrant codomain. Choose a cofibrant replacement $\mathcal{M}(T,\Theta^\infty E)^c \xrightarrow{\sim\mathrm{pt}} \mathcal{M}(T,E)$

and consider the induced commutative diagram:

$$\begin{tikzcd}
\mathcal{M}(T,E)^c \wedge T \arrow[r, "\sim"] \arrow[d, "\mathcal{M}(T,\iota_E)^c\wedge T"', "\sim"] & \mathcal{M}(T,E)\wedge T \arrow[r, "\epsilon_T E", "\sim"'] \arrow[d, "\mathcal{M}(T,\iota_E)\wedge T"', "\sim"] & E \arrow[d, "\iota_E"] \\
\mathcal{M}(T,\Theta^\infty E)^c \wedge T \arrow[r, "\sim"] & \mathcal{M}(T,\Theta^\infty E)\wedge T \arrow[r, "\epsilon_T \Theta^\infty E", "\sim"'] & \Theta^\infty E
\end{tikzcd}$$

Since $- \wedge T$ is a Quillen equivalence 2.24, the lower horizontal composition is a stable equivalence. Since $- \wedge T$ preserves pointwise weak equivalences 2.18, both horizontal maps on the left hand side are pointwise weak equivalences. The right vertical map is a stable equivalence by construction. By factoring a stable equivalence as a stable acyclic cofibration, followed by a pointwise acyclic fibration, one can see that $- \wedge T$ preserves all stable equivalences. Hence also the other two vertical maps are stable equivalences. It follows that the map in question is a stable equivalence. □

Theorem 3.25. *The stable structure* $\mathbf{MF}_{\mathrm{st}}$ *is a cofibrantly generated, proper and monoidal model category.*

Proof. The model structure follows easily from [7, 2.1.19], using 3.21 and 3.24. The smash product of $\mathcal{M}(T \wedge A, -) \wedge T \longrightarrow \mathcal{M}(A,-)$ and $\mathcal{M}(B,-)$ is isomorphic to the map $\mathcal{M}(T \wedge (A \wedge B), -) \wedge T \longrightarrow \mathcal{M}(A \wedge B, -)$. This implies that the pushout product map of a generating cofibration $\mathcal{M}(B,-) \wedge h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+$ and a generating stable acyclic cofibration is again a stable acyclic cofibration, which proves that the model structure is monoidal. Left properness is clear, for right properness we refer to [5, 6.28]. □

Remark 3.26. In the pointwise and stable model structures, the generating cofibrations coincide. The set of generating acyclic cofibrations for the stable structure is the union of the set of generating hf-acyclic cofibrations in 3.17, together with the set D described above. Note that all of the maps have cofibrant domains and codomains. Furthermore, the domains and codomains of the maps in D are finitely presentable.

Remark 3.27. In fact, by the proofs of [5, 5.13 and 6.28] stable equivalences are closed under base change along pointwise fibrations.

By a verbatim copy of the argument in the hf-structure 3.15, we get the monoid axiom for the stable structure.

Theorem 3.28. *Smashing with a cofibrant object in* $\mathbf{MF}_{\mathrm{st}}$ *preserves stable equivalences, and* $\mathbf{MF}_{\mathrm{st}}$ *satisfies the monoid axiom.*

Our goal now is to compare the stable model structure on motivic functors with the stable model structure on motivic spectra.

It is clear that $\mathrm{ev}\colon \mathbf{MF} \longrightarrow \mathrm{Sp}(\mathcal{M},T)$ preserves acyclic fibrations, and from Hovey's results [8, Section 4], ev preserves stable fibrations. Hence ev is a right Quillen functor, with left adjoint i_* defined by left Kan extension along the inclusion $i\colon T\mathrm{Sph} \hookrightarrow \mathbf{f}\mathcal{M}$. (In fact, ev preserves stable equivalences of motivic homotopy functors by 3.23.) We would like ev to be a Quillen equivalence, which according to [7, 1.3.16] is equivalent to the following two conditions.

- ev detects stable equivalences of stably fibrant motivic functors.
- If E is a cofibrant motivic spectrum and $(-)^{\mathrm{st}}$ denotes a stably fibrant replacement functor for motivic spectra, then the canonical map

$$E \longrightarrow \mathrm{ev}((i_*E)^{\mathrm{st}})$$

  is a stable equivalence.

Here is a proof of the second condition.

Lemma 3.29. *Let E be a cofibrant motivic spectrum. Then $E \longrightarrow \mathrm{ev}((i_*E)^{\mathrm{st}})$ is a stable equivalence of motivic spectra.*

Proof. Let us start by observing that, by 3.23, it is sufficient to show that the map $E \longrightarrow \mathrm{ev}((i_*E)^{\mathrm{hf}})$ is a stable equivalence. To describe $(-)^{\mathrm{hf}}$ in convenient terms, we will employ the enriched fibrant replacement functor $\mathrm{Id}_{\mathcal{M}_{\mathrm{mo}}} \longrightarrow R$ [5, 3.3.2]. Its construction uses an enriched small object argument. For our notations concerning spectra see [8].

First, consider the case $E = F_0T^0$. Then $i_*F_0T^0 \cong \mathcal{M}(T^0,-) \cong \mathbb{I}$, and we can choose $\mathbb{I}^{\mathrm{hf}} = R \circ \mathbb{I}$. The map $F_0T^0 \longrightarrow \mathrm{ev}(R \circ \mathbb{I})$ in degree n is the canonical motivic weak equivalence $T^{\wedge n} \xrightarrow{\sim} R(T^{\wedge n})$, hence a pointwise weak equivalence.

To proceed in the slightly more general case when $E = F_nT^0$, note that $i_*F_nT^0 \cong \mathcal{M}(T^{\wedge n},-)$. Since $T^{\wedge n}$ is cofibrant, we may choose $\mathcal{M}(T^{\wedge n},-)^{\mathrm{hf}} = \mathcal{M}(T^{\wedge n},R(-))$, cp. 3.18. Hence $\mathrm{ev}\mathcal{M}(T^{\wedge n},R(-)) = \mathcal{M}(T^{\wedge n},\mathrm{ev}R(-))$. The map $F_nT^0 \longrightarrow \mathcal{M}(T^{\wedge n},\mathrm{ev}R(-))$ has an adjoint $F_nT^0 \wedge T^{\wedge n} \longrightarrow \mathrm{ev}R(-)$ which is $* \longrightarrow R(T^{\wedge k})$ in degree $k < n$ and the canonical motivic weak equivalence $T^{\wedge m} \xrightarrow{\sim} R(T^{\wedge m})$ in degree $m \geq n$. In particular, it is a stable equivalence. Similarly for

$$F_nT^0 \wedge T^{\wedge n} \xrightarrow{\sim} \mathrm{ev}R(-) \xrightarrow{\sim} \Theta^\infty\mathrm{ev}R(-).$$

¿From the proof of 3.24, one can see that $\mathcal{M}(T^{\wedge n},-)$ applied to the second map is a stable equivalence with a stably fibrant codomain. Since $-\wedge T$ is a Quillen equivalence on $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}},T)$, this proves the slightly more general case.

The case $E = F_nA$, where A is any motivic space, follows since

$$F_nA \longrightarrow \mathrm{ev}((i_*F_nA)^{\mathrm{hf}}) \cong (F_nT^0 \longrightarrow \mathrm{ev}\mathcal{M}(T^{\wedge n},R(-))) \wedge A$$

and tensoring with any motivic space preserves stable equivalences of motivic spectra. The latter follows from 2.20. This includes the domains and codomains of the generating cofibrations in $\mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$.

The general case of any cofibrant motivic spectrum E follows, since E is a retract of a motivic spectrum E' such that $* \rightarrowtail E'$ is obtained by attaching cells. That is, we can assume $E = E'$. We proceed by transfinite induction on the cells, with the successor ordinal case first. Suppose $E_{\alpha+1}$ is the pushout of

$$F_n tj \xleftarrow{F_n j} F_n sj \longrightarrow E_\alpha$$

where j is a generating cofibration in $\mathcal{M}_{\mathrm{mo}}$. Then $(i_* E_{\alpha+1}) \circ R \circ \mathbb{I}$ is the pushout of the diagram

$$\mathcal{M}(T^{\wedge n}, R(-)) \wedge tj \xleftarrow{\mathcal{M}(T^{\wedge n}, R(-)) \wedge j} \mathcal{M}(T^{\wedge n}, R(-)) \wedge sj \longrightarrow i_* E_\alpha \circ R \circ \mathbb{I}.$$

The left horizontal map is pointwise a monomorphism. All the motivic functors in this diagram are homotopy functors, so up to pointwise weak equivalence, they coincide with their fibrant replacement in $\mathbf{MF}_{\mathrm{hf}}$. The induction step follows, since ev preserves pushouts, pointwise weak equivalences and pointwise monomorphisms, by applying the gluing lemma to the diagram:

$$\begin{tikzcd}
F_n tj \arrow[d, "\sim"] & F_n sj \arrow[l, "F_n j"'] \arrow[r] \arrow[d, "\sim"] & E_\alpha \arrow[d, "\sim"] \\
\mathrm{ev}\mathcal{M}(T^{\wedge n}, R(-)) \wedge tj & \mathrm{ev}\mathcal{M}(T^{\wedge n}, R(-)) \arrow[l] \arrow[r] & \mathrm{ev}(i_* E_\alpha \circ R \circ \mathbb{I})
\end{tikzcd}$$

The limit ordinal case follows similarly; we leave the details to the reader. □

For a general S, it is not known whether ev detects stable equivalences of stably fibrant motivic functors. In order to obtain the "correct" homotopy theory of motivic functors we modify the stable model structure.

3.6 The spherewise structure

Definition 3.30. A map $f\colon X \longrightarrow Y$ of motivic functors is a *spherewise equivalence* if the induced map $\mathrm{ev}(f^{\mathrm{hf}})$ is a stable equivalence of motivic spectra. The map f is a *spherewise fibration* if the following three conditions hold for every $A \in \mathbf{f}\mathcal{M}$ such that there exists an acyclic monomorphism $T^{\wedge n} \overset{\sim}{\hookrightarrow} A$ for some $n \geq 0$:

- $f(A)\colon X(A) \longrightarrow Y(A)$ is a motivic fibration.
- For every motivic weak equivalence $A \xrightarrow{\sim} B$ in $\mathbf{f}\mathcal{M}$,

$$\begin{tikzcd}
XA \arrow[r] \arrow[d, "f(A)"'] & XB \arrow[d, "f(B)"] \\
YA \arrow[r] & YB
\end{tikzcd}$$

is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$.

- The diagram

$$\begin{tikzcd} XA \arrow[r] \arrow[d, "f(A)"'] & \mathcal{M}(T, X(T\wedge A)) \arrow[d, "\mathcal{M}(T,f(T\wedge A))"] \\ YA \arrow[r] & \mathcal{M}(T, Y(T\wedge A)) \end{tikzcd}$$

is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$.

A map is a *spherewise cofibration* if it has the left lifting property with respect to the maps which are both spherewise equivalences and spherewise fibrations.

We shall refer to these classes as the *spherewise structure* on $\mathbf{MF}$ and use the notations $\mathbf{MF}_{\mathrm{sph}}$, $X \xrightarrow{\sim\mathrm{sph}} Y$, $X \xrightarrow{\mathrm{sph}} Y$ and $X \xrightarrow{\mathrm{sph}} Y$. Now every stable equivalence is a spherewise equivalence by 3.23, and stable fibrations are spherewise fibrations. Hence the identity is a left Quillen functor $\mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathbf{MF}_{\mathrm{st}}$ provided the spherewise structure is a model structure.

Theorem 3.31. *The spherewise structure is a cofibrantly generated proper monoidal model structure on* $\mathbf{MF}$. *The monoid axiom holds. Furthermore, the evaluation functor*

$$\mathrm{ev}\colon \mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$$

is the right adjoint in a Quillen equivalence.

Proof. Let us denote by $\mathbf{t}\mathcal{M}$ the full sub-$\mathcal{M}$-category given by the finitely presentable motivic spaces A such that there exists an acyclic monomorphism $T^{\wedge n} \xhookrightarrow{\sim} A$ for some $n \geq 0$. It is possible to apply the general machinery from [5] to the category $[\mathbf{t}\mathcal{M}, \mathcal{M}]$ of $\mathcal{M}$-functors from $\mathbf{t}\mathcal{M}$ to $\mathcal{M}$ and get a cofibrantly generated proper model structure. We may then lift this model structure using [6, 11.3.2] from $[\mathbf{t}\mathcal{M}, \mathcal{M}]$ to $\mathbf{MF}$ via the left Kan extension along the full inclusion $\mathbf{t}\mathcal{M} \hookrightarrow \mathbf{f}\mathcal{M}$.

We follow a direct approach. By the proof of 3.21, a spherewise acyclic fibration $f\colon X \xrightarrow{\sim\mathrm{sph}} Y$ is characterized by the property that the map $f(A)\colon XA \longrightarrow YA$ is an acyclic fibration in $\mathcal{M}_{\mathrm{mo}}$ for every $A \in \mathbf{t}\mathcal{M}$. This gives us the set of generating spherewise cofibrations

$$\{\mathcal{M}(A,-) \wedge h_U \wedge (\partial\Delta^n \hookrightarrow \Delta^n)_+\}_{A\in\mathbf{t}\mathcal{M},\, U\in\mathrm{Ob}\,\mathrm{Sm}/S, n\geq 0}.$$

This set is simply the restriction of the set of generating cofibrations for the model structures on the motivic spaces in $\mathbf{t}\mathcal{M}$. Similarly, one can restrict the generating acyclic cofibrations in 3.26 to the motivic spaces in $\mathbf{t}\mathcal{M}$. This gives a set of generating spherewise acyclic cofibrations. Theorem [7, 2.1.19] implies the existence of the cofibrantly generated model structure. In fact, the conditions required to apply this theorem have been checked before without the restriction that A be in $\mathbf{t}\mathcal{M}$. For example, sequential compositions of cobase changes of the generating spherewise acyclic cofibrations are even stable equivalences by 3.24, hence in particular spherewise equivalences.

Note that $\mathbf{t}\mathcal{M}$ is closed under the smash product in $\mathcal{M}$. In fact, if the maps $T^{\wedge m} \xhookrightarrow{\sim} A$ and $T^{\wedge n} \xhookrightarrow{\sim} B$ are acyclic monomorphisms, then their smash product $T^{\wedge m+n} \longrightarrow A \wedge B$ is an acyclic monomorphism. This is the crux observation leading to the conclusion that the model structure is monoidal. We claim that the monoid axiom holds. If X is an arbitrary motivic functor and j is a generating spherewise acyclic cofibration, then j is in particular a generating stable acyclic cofibration. The monoid axiom for the stable model structure 3.28 implies that $X \wedge j$-cell consists of stable equivalences, which are in particular spherewise equivalences. Our claim follows.

Finally, since $T^{\wedge n} \in \operatorname{Ob} \mathbf{t}\mathcal{M}$ for every $n \geq 0$, the evaluation functor ev preserves spherewise fibrations and spherewise acyclic fibrations. Hence ev is a right Quillen functor. By definition, ev reflects spherewise equivalences of motivic homotopy functors. This implies ev also reflects spherewise equivalences of motivic functors which are spherewise fibrant (A spherewise fibrant motivic functor does not necessarily preserve all of the motivic weak equivalences in $\mathbf{f}\mathcal{M}$, only those in $\mathbf{t}\mathcal{M}$. However, this is sufficient.). If E is a cofibrant motivic spectrum and $i_*E \xrightarrow{\sim \mathrm{sph}} (i_*E)^{\mathrm{sph}}$ is a spherewise fibrant replacement, there is a spherewise equivalence $(i_*E)^{\mathrm{sph}} \xrightarrow{\sim \mathrm{sph}} (i_*E)^{\mathrm{st}}$.

Using 3.29 above we conclude that $\mathrm{ev}\colon \mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ is a Quillen equivalence. □

Note that we do not claim that smashing with a spherewise cofibrant motivic functor preserves spherewise equivalences.

3.7 Comparison with motivic symmetric spectra

We extend the result about the Quillen equivalence 3.31 to Jardine's category of motivic symmetric spectra [10]. As mentioned above, if U is the functor induced by the inclusion $T\mathrm{Sph} \hookrightarrow T\mathrm{Sph}^{\Sigma}$, and ev' is the inclusion $T\mathrm{Sph}^{\Sigma} \hookrightarrow \mathbf{f}\mathcal{M}$, then $\mathrm{ev}\colon \mathbf{MF} \longrightarrow \mathrm{Sp}(\mathcal{M}, T)$ allows the factorization

$$\mathbf{MF} \xrightarrow{\mathrm{ev}'} \mathrm{Sp}^{\Sigma}(\mathcal{M}, T) \xrightarrow{U} \mathrm{Sp}(\mathcal{M}, T).$$

The functor ev' is lax symmetric monoidal and has a strict symmetric monoidal left adjoint. Hovey's work [8, 8.7] yields a stable model structure on $\mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{mo}}, T)$, slightly different from the stable model structure on motivic symmetric spectra constructed in [10], that is, $\mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{GJ}}, T)$. The latter uses as input the model category M_{GJ} in 2.17. The right adjoint of the Quillen equivalence $\mathcal{M}_{\mathrm{GJ}} \longrightarrow \mathcal{M}_{\mathrm{mo}}$ given by $\mathrm{Id}_{\mathcal{M}}$ induces the commutative square

$$\begin{array}{ccc} \mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{GJ}}, T) & \xrightarrow{U} & \mathrm{Sp}(\mathcal{M}_{\mathrm{GJ}}, T) \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{mo}}, T) & \xrightarrow{U} & \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T) \end{array}$$

where the vertical functors are Quillen equivalences [8, 5.7, 9.3]. To apply Hovey's results one needs to check that $\mathcal{M}_{\mathrm{GJ}}$ is a cellular model structure. An approach is to apply Smith's work on combinatorial model categories, or one can proceed directly. Indeed, using 2.17 one can show that the stable equivalences coincide in both model structures. The upper forgetful functor in the above displayed diagram is a Quillen equivalence by [10, 4.31], hence so is the lower U. Since the evaluation $\mathrm{ev}\colon \mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathrm{Sp}(\mathcal{M}_{\mathrm{mo}}, T)$ is a Quillen equivalence 3.31, it suffices to prove the following result.

Theorem 3.32. *The lax symmetric monoidal functor*

$$\mathrm{ev}'\colon \mathbf{MF}_{\mathrm{sph}} \longrightarrow \mathrm{Sp}^{\Sigma}(\mathcal{M}_{\mathrm{mo}}, T)$$

is the right adjoint in a Quillen equivalence. Its left adjoint is strict symmetric monoidal. The induced pair on homotopy categories is a monoidal equivalence.

Proof. If ev' is a right Quillen functor, then the monoidality statements follow from [5, 2.16] and [7, 4.3.3]. The Quillen equivalence then follows by 2-out-of-3, as explained prior to the statement of the theorem.

Since the spherewise acyclic fibrations are the maps f such that $f(A)$ is an acyclic fibration in $\mathcal{M}_{\mathrm{mo}}$ for every A weakly equivalent to some T^n, we get that ev' preserves stable acyclic fibrations. Similarly, any spherewise fibration gets mapped to a stable fibration, because its evaluation on some T^n is a fibration and the square

$$\begin{array}{ccc} XT^n & \longrightarrow & \mathcal{M}(T, XT^{n+1}) \\ {\scriptstyle F(T^{\wedge n})}\downarrow & & \downarrow{\scriptstyle \mathcal{M}(T, f(T^{n+1}))} \\ YT^n & \longrightarrow & \mathcal{M}(T, YT^{n+1}) \end{array}$$

is a homotopy pullback square in $\mathcal{M}_{\mathrm{mo}}$ for every $n \geq 0$. ¿From the definition of stable fibrations of symmetric T-spectra [10, 4.2], which also applies to $\mathcal{M}_{\mathrm{mo}}$ instead of $\mathcal{M}_{\mathrm{GJ}}$, it follows that ev' preserves stable fibrations. $\square$

4 Algebraic structure

In the paper so far, we have set up models for doing homotopical algebra over the initial motivic ring $\mathbb{I}$, which was simply the inclusion $\mathbb{I}\colon \mathbf{f}\mathcal{M} \subseteq \mathcal{M}$.

However, the structure we have developed is sufficient to do homotopical algebra in module categories, as well as in categories of algebras over commutative ring functors.

In this section we use the results in [14] (for which many of the previous formulations were custom-built), to outline how this can be done. The spherewise structure $\mathbf{MF}_{\mathrm{sph}}$ is slightly different from the other ones, but deserves special attention due to its Quillen equivalence to motivic symmetric spectra.

The reader's attention should perhaps be drawn to corollary 4.5, where our setup gives less than one should hope for: in order for a map of motivic rings $f\colon A \longrightarrow B$ to induce a Quillen equivalence of module categories in the spherewise structure, we must assume that f is a stable equivalence. We would of course have preferred that our setup immediately gave the conclusion for spherewise equivalences, but apart from this deficiency the section can be summed up by saying that each of the model structures given in the previous section give rise to a natural homotopy theory for modules and algebras satisfying all expected properties, where the weak equivalences and fibrations are the same as in the underlying structure on $\mathbf{MF}$.

4.1 Motivic rings and modules

Recall that a motivic ring is the same as a monoid in $\mathbf{MF}$, i.e. a motivic functor A together with a "unit" $\mathbb{I} \longrightarrow A$ and a unital and associative "multiplication" $A \wedge A \longrightarrow A$. We use the same language for modules and algebras as e.g. [14]. A left A-module is a motivic functor M together with a unital and associative action $A \wedge M \longrightarrow M$. If M is a left A-module and N is a right A-module, then $N \wedge_A M$ is defined as the coequalizer of the two obvious action maps from $N \wedge A \wedge M$ to $N \wedge M$. The category mod_A of left A-modules is enriched over $\mathbf{MF}$ by a similar equalizer.

If k is a commutative motivic ring, then left and right modules can be identified and the category of k-modules becomes a closed symmetric monoidal category. The monoids therein are called k-algebras (which means that we have a third legitimate name – "$\mathbb{I}$-algebra"– for a motivic ring).

Definition 4.1. Let A be a motivic ring and k a commutative motivic ring. Let mod_A be the category of left A-modules and alg_k the category of k-algebras. A map in mod_A or alg_k is called a weak equivalence resp. fibration if it is so when considered in $\mathbf{MF}$. Cofibrations are defined by the left lifting property.

Theorem 4.2. *Let A be a motivic ring, let k be a commutative motivic ring and let* $\mathbf{MF}$ *be equipped with either of the model structures of section 3.*

- *The category* mod_A *of left A-modules is a cofibrantly generated model category.*
- *The category of k-modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.*

- *The category* alg_k *of* k*-algebras is a cofibrantly generated model category.*

Proof. This follows immediately from [14, 4.1] and the results in section 3. □

By the argument for [5, 8.4], we have

Lemma 4.3. *Let* $\mathbf{MF}$ *be equipped with the pointwise structure, the homotopy functor structure or the stable structure. Let* A *be a motivic ring. Then for any cofibrant* A*-module* N*, the functor* $- \wedge_A N$ *takes weak equivalences in* $\mathrm{mod}_{A^{\mathrm{op}}}$ *to weak equivalences in* $\mathbf{MF}$*.*

Corollary 4.4. *Let* $\mathbf{MF}$ *be equipped with the pointwise structure, the homotopy functor structure or the stable structure. Let* $f\colon A \xrightarrow{\sim} B$ *be a weak equivalence of motivic rings. Then extension and restriction of scalars define the Quillen equivalence*

$$\mathrm{mod}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{mod}_B.$$

If A *and* B *are commutative, there is the Quillen equivalence*

$$\mathrm{alg}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{alg}_B.$$

Proof. This is a consequence of [14, 4.3 and 4.4] according to 4.3. □

In the case of the spherewise structure, we have the following result.

Corollary 4.5. *Suppose* $f\colon A \xrightarrow{\sim} B$ *is a stable equivalence of motivic rings and choose* $\mathbf{MF}_{\mathrm{sph}}$ *as our basis for model structures on modules and algebras. Then extension and restriction of scalars define the Quillen equivalence*

$$\mathrm{mod}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{mod}_B.$$

If A *and* B *are commutative, there is the Quillen equivalence*

$$\mathrm{alg}_A \underset{f^*}{\overset{B\wedge_A -}{\rightleftarrows}} \mathrm{alg}_B.$$

Proof. Follows from 4.3, cf. [14, 4.3 and 4.4]. □

4.2 Motivic cohomology

Recall the commutative motivic ring $M\mathbb{Z}$ of example 3.4. We show:

Lemma 4.6. *The evaluation* $\mathrm{ev}(M\mathbb{Z})$ *of* $M\mathbb{Z}$ *represents motivic cohomology with integer coefficients.*

Proof. Let us repeat Voevodsky's construction of the spectrum representing motivic cohomology in [16]. His motivic spaces (simply called *spaces*) are pointed Nisnevich sheaves on Sm/S, equipped with a model structure in which the cofibrations are the monomorphisms. Let us denote this model category by $\mathcal{V}$. Note that $\mathcal{V}$ is closed symmetric monoidal. There is the standard cosimplicial object $\Delta_S\colon \Delta \longrightarrow \mathrm{Sm}/S$ which maps $[n]$ to the scheme $\mathbb{A}^{n+1}_S/(\Sigma_{i=0}^n X_i = 1)$. The right Quillen functor

$$\mathrm{Sing} : \mathcal{V} \longrightarrow \mathcal{M}_{\mathrm{mo}}, \qquad A \longmapsto ((U,n) \longmapsto A(\Delta^n_S \times U))$$

is a Quillen equivalence by [10, B.4, B.6] and 2.17.

Its left adjoint maps a motivic space A to the coend

$$|A|_S = \int^{n\in\Delta} \mathrm{Nis}(A_n \wedge h_{\Delta^n_S})$$

where $\mathrm{Nis}(B)$ is the Nisnevich sheafification of the presheaf B. The functor $|-|_S$ is strict symmetric monoidal. As a special case, if $A \in \mathcal{M}$ is a discrete Nisnevich sheaf (for example $A = h_U$ for some $U \in \mathrm{Sm}/S$), then $|A|_S \cong A$.

The spectrum $H\mathbb{Z}$ defined by Voevodsky is an object in $\mathrm{Sp}(\mathcal{V}, |(\mathbb{P}^1_S,\infty)|_S)$, where $|(\mathbb{P}^1_S,\infty)|_S := |h_{\mathbb{P}^1_S}/h_S|_S$. Here $h_S \longrightarrow h_{\mathbb{P}^1_S}$ corresponds to the rational point $\infty \in \mathbb{P}^1_S(S)$. Its nth term is

$$H\mathbb{Z}_n = |M\mathbb{Z}((\mathbb{P}^1_S,\infty)^{\wedge n})|_S$$

with structure map given by the composition

$$\begin{array}{c} |M\mathbb{Z}((\mathbb{P}^1_S,\infty)^{\wedge n}) \wedge (\mathbb{P}^1_S,\infty)|_S \\ \big\downarrow \\ |M\mathbb{Z}((\mathbb{P}^1_S,\infty)^{\wedge n}) \wedge M\mathbb{Z}(\mathbb{P}^1_S,\infty)|_S \\ \big\downarrow \\ |M\mathbb{Z}((\mathbb{P}^1_S,\infty)^{\wedge n+1})|_S \end{array}$$

which involves the unit and the multiplication of the motivic ring $M\mathbb{Z}$. The lemma follows now, essentially because $(\mathbb{P}^1_S,\infty)$ and T are connected via a zigzag of motivic weak equivalences, which both $|-|_S$ and $M\mathbb{Z}$ respect. For $|-|_S$ this is clear, since it is a left Quillen functor on $\mathcal{M}_{\mathrm{GJ}}$. For $M\mathbb{Z}$ the claim is not so clear, so we discuss this case in some details.

As a motivic functor, $M\mathbb{Z}$ preserves simplicial homotopy equivalences. One can equip the category $\mathcal{M}^{\mathrm{tr}}$ of motivic spaces with transfers with a whole host of model structures. In the motivic model structure on $\mathcal{M}^{\mathrm{tr}}$, a map f of motivic spaces with transfers is a weak equivalence resp. fibration if and only if $u(f) \in \mathcal{M}$ is a motivic weak equivalence resp. motivic fibration [13]. By definition, it follows that u is a right Quillen functor, so that $\mathbb{Z}_{\mathrm{tr}}$ is a left Quillen functor. Consequently, the composition $u \circ \mathbb{Z}_{\mathrm{tr}}$ maps motivic weak equivalences of cofibrant motivic spaces to motivic weak equivalences.

The zig-zag of motivic weak equivalences between $(\mathbb{P}^1_S, \infty)$ and the Tate object T involves only homotopy pushouts of representable motivic spaces and their simplicial suspensions. By repeatedly applying the simplicial mapping cylinder one can replace this zig-zag by a zig-zag of motivic weak equivalences involving only cofibrant motivic spaces, except for the weak equivalence $T' \xrightarrow{\sim} (\mathbb{P}^1_S, \infty)$. Here $T' = C/h_S$ where C denotes the simplicial mapping cylinder of the map $h_S \longrightarrow h_{\mathbb{P}^1_S}$. However, we claim the following map is a weak equivalence

$$\mathbb{Z}_{\mathrm{tr}}(C/h_S) \longrightarrow \mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^1_S, \infty).$$

Our claim holds because the following map of chain complexes of motivic spaces with transfers is schemewise a quasi-isomorphism:

$$\begin{tikzcd}
\mathbb{Z}_{\mathrm{tr}}(h_{\mathbb{P}^1_S}) \arrow[d] & \mathbb{Z}_{\mathrm{tr}}(h_S) \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathbb{Z}_{\mathrm{tr}}(h_{\mathbb{P}^1_S})/\mathbb{Z}_{\mathrm{tr}}(h_S) & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

This finishes the proof. □

References

[1] Benjamin A. Blander. Local projective model structures on simplicial presheaves. *K-Theory*, 24(3):283–301, 2001.

[2] Marcel Bökstedt. Topological Hochschild homology. Preprint, Bielefeld, 1986.

[3] Francis Borceux. *Handbook of categorical algebra. 2*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[4] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar, IV*, pages 1–38. Springer, Berlin, 1970.

[5] Bjørn Ian Dundas, Oliver Röndigs, and Paul Arne Østvær. Enriched functors and stable homotopy theory. *Documenta Math.*, 8:409–488, 2003.

[6] Philip S. Hirschhorn. *Model categories and their localizations*. American Mathematical Society, Providence, RI, 2003.

[7] Mark Hovey. *Model categories*. American Mathematical Society, Providence, RI, 1999.

[8] Mark Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.

[9] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[10] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[11] Manos Lydakis. Simplicial functors and stable homotopy theory. Preprint 98-049, SFB 343, Bielefeld, June 1998.

[12] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[13] Oliver Röndigs and Paul Arne Østvær. Motives and modules over motivic cohomology. In preparation.

[14] Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[15] Andrei Suslin and Vladimir Voevodsky. Relative cycles and Chow sheaves. In *Cycles, transfers, and motivic homology theories*, pages 10–86. Princeton Univ. Press, Princeton, NJ, 2000.

[16] Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, pages 579–604 (electronic), 1998.

[17] Vladimir Voevodsky. Homotopy theory of simplicial sheaves in completely decomposable topologies. Available via http://www.math.uiuc.edu/K-theory/, August 2000.

[18] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

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Severi-Brauer Varieties of Semidirect Product Algebras

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Abstract. A conjecture of Amitsur states that two Severi-Brauer varieties are birationally isomorphic if and only if the underlying algebras are the same degree and generate the same cyclic subgroup of the Brauer group. It is known that generating the same cyclic subgroup is a necessary condition, however it has not yet been shown to be sufficient.

In this paper we examine the case where the algebras have a maximal subfield K/F of degree n with Galois closure E/F whose Galois group is of the form $C_n \rtimes H$, where $E^H = K$ and $|H|$ is prime to n. For such algebras we show that the conjecture is true for certain cases of n and H. In particular we prove the conjecture in the case that G is a dihedral group of order $2p$, where p is prime.

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1 Introduction

Let F be a field. We fix for the entire paper a positive integer n, and we suppose that either n is prime, or that F contains a primitive n'th root of unity. For a field extension L/F, and A a central simple L-algebra, we write $V(A)$ or $V(A/L)$ to denote the Severi-Brauer variety of A, consisting of $(deg\ A)$-dimensional right ideals of A, and denote the function field of this variety by $L(A)$.
We recall the following conjecture:

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Conjecture (Amitsur, 1955 [Ami55]). Given A, B Central Simple algebras over F, $F(A) \cong F(B)$ iff $[A]$ and $[B]$ generate the same cyclic subgroup of the $Br(F)$.

Amitsur showed in [Ami55] that one of these implications hold. Namely if $F(A) \cong F(B)$, then the equivalence classes of A and B generate the same cyclic subgroup of the Brauer Group. The aim of this note is to prove the reverse implication for certain algebras A and B. We will say that the conjecture holds for the pair (A, l), or simply that (A, l) is true to mean that l is prime to $deg(A)$ and $F(A) \cong F(A^l)$. We say that the conjecture is true for A if, for all l prime to $deg(A)$, (A, l) is true.
One important case is when the algebra A has a cyclic Galois maximal splitting field. In this case we know that the conjecture is true for A ([Ami55], [Roq64]). In this paper we extend this result to certain $G - H$ crossed products. We recall the following definitions:

Definition 1. Let G be a finite group, and H a subgroup of G. A field extension K/F is called $G - H$ Galois if there exists a field E containing K such that E/F is G-Galois, and $E^H = K$.

Definition 2. Let A be a central simple F-algebra. A is called a $G - H$ crossed product if A has a maximal subfield K which is $G - H$ Galois. In the case $H = 1$, we call A a G-crossed product.

The main theorem in this note concerns the case of an algebra which is a so-called semidirect product algebra in the sense of [RS96]

Definition 3. A is called a semidirect product algebra if it is a $G-H$ crossed product where $G = N \rtimes H$.

This can be interpreted as meaning that A becomes an N- crossed product after extending scalars by some field K/F which is H-Galois. In the case where N is a cyclic group, we will try to exploit the fact that we know Amitsur's conjecture to be true for N crossed products to prove the conjecture for G crossed products.

Semidirect product theorem 1. *Let A be a semidirect product algebra of degree n as in definition 3, with $N = C_m = \langle \tau \rangle$, $H = C_n = \langle \sigma \rangle$, such that the homomorphism $N \to Aut(H)$ (induced by conjugation) is injective and $|N|$ and $|H|$ are relatively prime. Choose r so that we may write $\tau\sigma\tau^{-1} = \sigma^r$. Let*

$$S = \frac{\mathbb{Z}[\rho]}{1 + \rho + \rho^2 + \cdots + \rho^{n-1}},$$

and define an action of τ on S via $\tau(\rho) = \rho^r$, and a ring homomorphism $\overline{\epsilon} : S \to \mathbb{Z}/n\mathbb{Z}$ via $\overline{\epsilon}(\rho) = \overline{1}$. Then (A, l) is true for all l such that $\overline{l} \in \overline{\epsilon}\big((S^*)^\tau\big)$.

We give the proof of this in section 3. For now, we give the following corollary:

Corollary 1. *Suppose $deg(A) = n$, n odd. If A has a dihedral splitting field of degree $2n$. Then the conjecture is true for A.*

Proof. In this case, we have $\overline{\epsilon}((S^*)^\tau) = (\mathbb{Z}/n\mathbb{Z})^*$. If $l = 2^k$, $\bar{l}$ is the image of $(\rho + \rho^{-1})^k$. If $l = 2k+1$, then $\bar{l}$ is the image of $\rho^{-k} + \rho^{-(k-1)} + \ldots + \rho^{-1} + 1 + \rho + \ldots + \rho^{k-1} + \rho^k$. Any other unit in $(\mathbb{Z}/n\mathbb{Z})^*$ is easily seen to be the image of a product of those above. □

Remark. This theorem is already known when F contains the n'th roots of unity, since by a theorem of Rowen and Saltman [RS82], any such algebra is in fact cyclic, and so the theorem follows from [Ami55] or [Roq64].

It is worth noting that the hypothesis concerning the splitting field E can be stated in weaker terms for the case $n = p$ a prime number. In particular we have:

Proposition 1. *Suppose A is a central simple F-algebra of degree p with a maximal subfield K, and suppose that there is some extension E' of K such that E'/F is Galois with group $G = C_p \rtimes H$ where $(E')^H = K$. Then there is a subfield $E \subset E'$ containing K such that E/F is Galois with group $C_p \rtimes C_m$ where C_m acts faithfully on C_p.*

Proof. We define a homomorphism $\phi : H \to Aut(C_p)$ via the natural conjugation action of H on C_p. Since $Aut(C_p)$ is a cyclic group every subgroup is cyclic, and we may regard ϕ as a surjective map $H \to C_m$. Now we define a map

$$\begin{aligned} G = C_p \rtimes H &\to C_p \rtimes C_m \\ (a, h) &\mapsto (a, \phi(h)) \end{aligned}$$

one may check quickly that this is a homomorphism of groups and its kernel is precisely the kernel of ϕ. Set $H' = ker\phi$, and let $E = (E')^{H'}$. Since H' is normal in G (as the kernel of a homomorphism), we know that E/F is Galois and its Galois group is $G/H' = C_p \rtimes C_m$. By construction, the action of C_m is faithful on C_p. □

Note also that in the case $n = p$ a prime, S is a ring of cyclotomic integers.

2 Preliminaries

To begin, let us fix some notation. Let F be an infinite field, and let F^{alg} be an algebraic closure of F. The symbol $\otimes$ when unadorned will always denote a tensor product over F and $\times$ will denote a fiber product of schemes over $Spec(F)$. For us an F-variety will mean a quasi-projective geometrically integral seperated scheme of finite type over F (note F is not assumed to be algebraically closed). By geometrically integral we mean that the scheme remains integral when fibered up to the algebraic closure of its field of definition. If X is a variety, we denote its function field by $F(X)$. We remark that X being

geometrically integral variety implies that $F(X)$ is a regular field extension of F, that is to say, $F(X) \otimes_F F^{alg}$ is a field.
Let E/F be G-Galois for some group G. If B is an E algebra, then a homomorphism $\alpha : G \to Aut_F(B)$ defines an action of G on B (as an F algebra) which is called semilinear in case

$$\forall x \in E, b \in B, \quad \alpha(\sigma)(xb) = \sigma(x)\alpha(\sigma)(b).$$

We refer to the pair (B, α) as a G-algebra, and a morphism between two G-algebras is simply defined to be an algebra map which commutes with the G-actions. The theory of descent tells us that the category of G-algebras is equivalent to the category of F-algebras by taking a G-algebra (B, α) to its G-invariants. Conversely, an F-algebra C gives us the G-algebra $(C \otimes_F E, \iota_C)$, where $\iota_C(\sigma) = id \otimes \sigma$.
Similarly, if X is an E-variety with structure map $k : X \to Spec(E)$, a homomorphism $\alpha : G \to Aut_{Spec(F)}(X)$ defines an action of G on X (as an F-scheme) which is called semilinear in case $\sigma \circ k = k \circ \alpha(\sigma)$.
Also for B any E-algebra, given $\sigma \in G$, we define ${}^{\sigma}B$ to be the algebra with the same underlying set and ring structure as B, but with the structure map $\sigma^{-1} : E \hookrightarrow {}^{\sigma}B$.
Given A a central simple F-algebra, we recall that the functor of points of the Severi-Brauer variety $V(A)$ is given the following subfunctor of the Grassmannian functor of points (see [Jah00], [VdB88], or [See99], and [EH00] for the definition of the Grassmannian functor):

$$V(A)(R) = \left\{ I \subset A_R \,\middle|\, \begin{array}{l} I \text{ is a left ideal and } A_R/I \\ \text{is } R\text{-projective of rank } n \end{array} \right\}$$

and for a homomorphism of commutative F-Algebras $R \xrightarrow{\psi} S$ we obtain the set map

$$\begin{aligned} V(A)(\psi) : V(A)(R) &\to V_k(A)(S) \\ \text{via } I &\mapsto I \otimes_R S \end{aligned}$$

2.1 Descent and functors of points

Given X an F-variety, we obtain a functor

$$X^{E/F} : \{\text{commutative } F\text{-algebras}\} \to \{\text{sets}\}$$

by $X^{E/F}(R) = Mor_{sch_E}(Spec(R_E), X_E)$. If $f \in Mor(X, Y)$, we abuse notation, and refer to the natural transformation induced by f by f also.
For $\sigma \in G$, we have an action of σ on $X_E = X \times_{Spec(F)} Spec(E)$ induced by σ^{-1} acting on E. With this in mind, we obtain a natural transformation $\sigma : X^{E/F} \to X^{E/F}$ via for $\phi \in X^{E/F}(R)$, $\sigma \centerdot \phi = \sigma \circ \phi \circ \sigma^{-1}$. We denote this action by

$$\iota_X : G \to NatAut(X^{E/F}).$$

PROPOSITION 2. *Let $f \in Mor_E(X_E, Y_E)$. Then $f = g_E$ for $g \in Mor_F(X, Y)$ iff the following diagram commutes:*

$$\begin{tikzcd} X^{E/F} \arrow[r, "f"] \arrow[d, "\sigma"'] & Y^{E/F} \arrow[d, "\sigma"] \\ X^{E/F} \arrow[r, "f"] & Y^{E/F} \end{tikzcd}$$

that is, for every commutative F-algebra R and $\phi \in X^{E/F}(R)$, we have

$$\sigma \centerdot f(\phi) = f(\sigma \centerdot \phi)$$

Proof. To begin, assume the above condition holds. We have (recalling that $f(\phi) = f \circ \phi$),

$$\sigma \centerdot f(\phi) = \sigma \centerdot (f \circ \phi) = \sigma \circ (f \circ \phi) \circ \sigma^{-1}$$
$$f(\sigma \centerdot \phi) = f(\sigma \circ \phi \circ \sigma^{-1}) = f \circ \sigma \circ \phi \circ \sigma^{-1}$$

And setting these two to be equal, we have

$$\sigma \circ f \circ \phi = f \circ \sigma \circ \phi$$

which in turn gives us

$$f \circ \phi = \sigma^{-1} \circ f \circ \sigma \circ \phi$$

Since this must hold for each ϕ, this just says that the elements $f, \sigma^{-1} \circ f \circ \sigma \in Mor_E(X_E, Y_E)$ correspond to the same natural transformation when thought of as elements of $Nat(X^{E/F}, Y^{E/F})$ via the Yoneda embedding, and therefore they must actually be equal - that is to say $f = \sigma^{-1} \circ f \circ \sigma$ or $\sigma \circ f = f \circ \sigma$. But now, by Galois descent of schemes, we know $f = g_E$.
Conversely, assume that $f = g_E$. In this case it is easy to see that we have $\sigma \circ f = f \circ \sigma$. Now we simply make our previous argument backwards and find that the desired diagram does in fact commute. $\square$

We note the following lemma, which can be checked by examining the Grassmannian in terms of its Plüker embedding:

LEMMA 1. *Suppose that V is an F-vector space, and let $X = Gr_k(V)$. Then the natural semilinear action ι_X can be described functorially as the natural transformation from $X^{E/F}$ to itself such that for R a commutative F-algebra, $\sigma \in G$, $M \in X^{E/F}(R)$,*

$$X^{E/F}(\iota_X(\sigma))(M) = \sigma(M) = \{\sigma(m) | m \in M\}$$

where σ acts on the elements of $V_{R \otimes E}$ in the natural way.

Corollary 2. *Suppose that A is an F-central simple algebra, and let $X = V_k(A)$. Then the natural semilinear action ι_X can be described functorially as the natural transformation from $X^{E/F}$ to itself such that for R a commutative F-algebra, $\sigma \in G$, $I \subset A_{R\otimes E}$ an element of $X^{E/F}(R)$*

$$X^{E/F}(\iota_X(\sigma))(I) = \sigma(I) = \{\sigma(x) | x \in I\}$$

where σ acts on the elements of $A_{R\otimes E} = A \otimes R \otimes E$ as $id \otimes id \otimes \sigma$.

2.2 Severi-Brauer varieties of crossed product algebras

We give here an explicit birational description of the Severi-Brauer Variety of a crossed product algebra. A similar discussion (without the functorial viewpoint) may be found in [Sal99] (Cor. 13.15). Let L/F be a G-galois extension of degree n. Let $A = (L, G, c)$ be a crossed product algebra, where c is taken to be a specific 2-cocycle (not just a cohomology class) normalized so that $c(id, id) = 1$.

We define the "functor of splitting 1-chains for c" via: To begin, define the functor

$$\mathcal{F} : \left\{ \begin{matrix} \text{commutative } L\text{-algebras} \\ \text{with } G\text{-semilinear action} \end{matrix} \right\} \to \{\text{sets}\}$$
$$\mathcal{F}(S) = \{z \in C^1(G, S^*) | \delta z = c\}$$

Where $C^1(G, S^*)$ denotes the set of 1-cochains. From this we define the functor of splitting 1-chains of c as

$$Sp_c : \{\text{commutative } F\text{-algebras}\} \to \{sets\}$$
$$Sp_c(R) = \mathcal{F}(L \otimes R)$$

Proposition 3. *Sp_c is represented by on open subvariety of U of $V(A)$, which is given as an open subfunctor by $U(R) = \{I \in V(A)(R) | I + L_R = A_R\}$. This isomorphism of functors is given by the natural isomorphism $\Lambda : U \to Sp_c$, where $\Lambda(R)(I)$ is the 1-cochain*

$$\sigma \mapsto z(I)_\sigma$$

where $z(I)_\sigma$ is the unique element of L_R such that

$$z(I)_\sigma - u_\sigma \in I.$$

Further, the inverse is given by

$$\Lambda^{-1}(z) = \sum_{\sigma \in G} (L \otimes R)(z(\sigma) - u_\sigma)$$

Proof. First we note that if $I \in U(R)$ then $I \cap L_R = 0$. This is because we have the exact sequence of R-modules

$$0 \to I \cap L_R \to L_R \to A_R/I \to 0$$

which is split since A_R/I is projective. Hence $L_R = A_R/I \oplus (I \cap L_R)$, and since L_R is projective (since L is) we have $I \cap L_R$ is also projective. By additivity of ranks, we have that $rk(A_R/I) = n$, $rk(L_R) = dim_F(L) = n$ and so $rk(I \cap L_R) = 0$. Since $I \cap L_R$ is projective, it must be trivial. Consequently, $I + L_R = A_R$ implies that $A_R = I \oplus L_R$.
To see now that $\Lambda(R)$ is well defined, we just note that $-u_\sigma \in A_R = I \oplus L_R$, and so there is a unique element $z(I)_\sigma \in I$ such that $z(I)_\sigma - u_\sigma \in I$. Next we check that $z(I)$ defines an element of $Sp_c(R)$. Since I is a left ideal,

$$\begin{aligned} & z(I)_\tau - u_\tau, z(I)_{\sigma\tau} - u_{\sigma\tau} \in I \\ \Rightarrow\; & u_\sigma\big(z(I)_\tau - u_\tau\big) - c(\sigma,\tau)\big(z(I)_{\sigma\tau} - u_{\sigma\tau}\big) \in I \end{aligned}$$

And also we have therefore:

$$\begin{gathered} \sigma\big(z(I)_\tau\big)\big(z(I)_\sigma - u_\sigma\big), \\ u_\sigma\big(z(I)_\tau - u_\tau\big) - c(\sigma,\tau)\big(z(I)_{\sigma\tau} - u_{\sigma\tau}\big) + \sigma\big(z(I)_\tau\big)\big(z(I)_\sigma - u_\sigma\big) \in I \end{gathered}$$

But this last expression can be rewritten as:

$$\begin{aligned} & u_\sigma\big(z(I)_\tau - u_\tau\big) - c(\sigma,\tau)\big(z(I)_{\sigma\tau} - u_{\sigma\tau}\big) + \sigma\big(z(I)_\tau\big)\big(z(I)_\sigma - u_\sigma\big) \\ & \qquad = \sigma\big(z(I)_\tau\big)u_\sigma - c(\sigma,\tau)u_{\sigma\tau} - c(\sigma,\tau)z(I)_{\sigma\tau} + \\ & \qquad\quad c(\sigma,\tau)u_{\sigma\tau} + \sigma\big(z(I)_\tau\big)z(I)_\sigma - \sigma\big(z(I)_\tau\big)u_\sigma \\ & \qquad\qquad = \sigma\big(z(I)_\tau\big)z(I)_\sigma - c(\sigma,\tau)z(I)_{\sigma\tau} \in L \otimes R \end{aligned}$$

But since this quantity is also in I and $I \cap (L \otimes R) = 0$, we get $\sigma(z(I)_\tau)z(I)_\sigma = c(\sigma,\tau)z(I)_{\sigma\tau}$ which says that

$$\delta(z(I))(\sigma,\tau) = \sigma(z(I)_\tau)z(I)_\sigma(z(I)_{\sigma\tau})^{-1} = c(\sigma,\tau)$$

and therefore, $z(I) \in Sp_c(R)$.
Next, we check that Λ^{-1} is well defined. Let $z \in Sp_c(R)$ and set $I = \Lambda^{-1}(z)$. It is clear from the definition that $I + (L \otimes R) = A_R$, and $A_R/I = L \otimes R$ is free (and so projective) of rank 1. To check that I is actually a left ideal, since it follows from the definition that $(L \otimes R)I = I$, we need only check that for each $\sigma \in G$, $u_\sigma I \subset I$, and this in turn will follow if we can show $u_\sigma(z(\tau) - u_\tau) \in I$ for each $\tau \in G$. Calculating, we get

$$\begin{aligned} u_\sigma\big(z(\tau) - u_\tau\big) &= \sigma\big(z(\tau)\big)u_\sigma - c(\sigma,\tau)u_{\sigma\tau} \\ &= \sigma\big(z(\tau)\big)u_\sigma - z(\sigma)\sigma\big(z(\tau)\big)z(\sigma\tau)^{-1}u_{\sigma\tau} \\ &= \sigma\big(z(\tau)\big)\big(u_\sigma - z(\sigma)z(\sigma\tau)^{-1}u_{\sigma\tau}\big) \\ &= \sigma\big(z(\tau)\big)\Big(-\big(z(\sigma) - u_\sigma\big) + \big(z(\sigma) - z(\sigma)z(\sigma\tau)^{-1}u_{\sigma\tau}\big)\Big) \\ &= \sigma\big(z(\tau)\big)\Big(-\big(z(\sigma) - u_\sigma\big) + z(\sigma)z(\sigma\tau)^{-1}\big(z(\sigma\tau) - u_{\sigma\tau}\big)\Big) \\ &= -\sigma\big(z(\tau)\big)\big(z(\sigma) - u_\sigma\big) + z(\sigma)\sigma\big(z(\tau)\big)z(\sigma\tau)^{-1}\big(z(\sigma\tau) - u_{\sigma\tau}\big) \\ &\in (L \otimes R)\big(z(\tau) - u_\tau\big) + (L \otimes R)\big(z(\sigma\tau) - u_{\sigma\tau}\big) \subset I \end{aligned}$$

and hence I is a left ideal, and Λ^{-1} makes sense.
It remains to show that Λ and Λ^{-1} are natural transformations and are inverses to one another. It follows fairly easily that if Λ is natural and they are inverses of one another then Λ^{-1} will automatically be natural also.
To see that Λ is natural, we need to check that for $\phi : R \to S$ a ring homomorphism, and $I \in U(R)$, that

$$\Lambda(S)(U(\phi)(I)) = Sp_c(\phi)(\Lambda(R)(I))$$

the right hand side is

$$Sp_c(\phi)(\Lambda(R)(I)) = Sp_c(\phi)(z(I)) = (id_L \otimes \phi)(z(I))$$

and by definition of $z(I)$, we know for $\sigma \in G$, $z(I)_\sigma - u_\sigma \in I$ and for the left hand side we have

$$\Lambda(S)(U(\phi)(I)) = \Lambda(S)(I \otimes_R S)$$

but

$$z(I)_\sigma - u_\sigma \in I \Rightarrow z(I)_\sigma \otimes 1 - u_\sigma \in I \otimes_R S$$

and now, using the identification

$$\begin{aligned} (L \otimes R) \otimes_R S &\xrightarrow{\sim} L \otimes S \\ (l \otimes r) \otimes s &\mapsto l \otimes \phi(r)s \end{aligned}$$

$z(I)_\sigma \otimes 1$ becomes $(id_L \otimes \phi)(z(I)_\sigma)$, and so combining these facts gives

$$Sp_c(\phi)(\Lambda(R)(I))(\sigma) = (id_L \otimes \phi)(z(I)_\sigma) \in I \otimes_R S \cap (L \otimes S - u_\sigma)$$

(here, $L \otimes S - u_\sigma$ denotes translation of $L \otimes S$ by u_σ) and by definition of Λ, this means

$$\Lambda(S)(I \otimes_R S)(\sigma) = (id_L \otimes \phi)(z(I)_\sigma) = Sp_c(\phi)(\Lambda(R)(I))(\sigma)$$

as desired.
Finally, we need to check that transformations are mutually inverse. Choosing $I \in U(R)$, we want to show

$$I = \sum_{\sigma \in G} (L \otimes R)(z(I)_\sigma - u_\sigma)$$

Now, it is easy to see that the right hand side is contained in the left hand side. Furthermore, both of these are direct summands of A_R of corank n. For convenience of notation, let us call the right hand side J.

Claim. *I / J is projective*

We show this by considering the exact sequence

$$0 \to I/J \to A_R/J \to A_R/I \to 0$$

Since A_R/I is projective, this sequence splits and $I/J \oplus A_R/I \cong A_R/J$. But since A_R/J is projective, and I/J is a summand of it, I/J must be projective as well, proving the claim.
Now, from the exact sequence

$$0 \to J \to I \to I/J \to 0$$

we know $rank(I/J) = rank(I) - rank(J) = 0$, and so $I/J = 0$ which says $I = J$ as desired.
Conversely, if $z \in Sp_c(R)$, we need to verify that

$$I = \sum_{\sigma \in G} (L \otimes R)(z(\sigma) - u_\sigma) \Rightarrow I \cap (L \otimes R - u_\sigma) = z(\sigma) - u_\sigma$$

But since $z(\sigma) - u_\sigma \in I$, this immediately follows. □

Remark. This same proof will work for an Azumaya algebra (the case where F is a commutative ring).

This becomes simpler for the case that L/F is a cyclic extension, say $A = (L/F, \sigma, b)$. In this case, choosing c to be the standard 2-cocycle:

$$c(\sigma^i, \sigma^j) = \begin{cases} 1 & i + j < n \\ b & i + j \geq n \end{cases}$$

If $z \in Sp_c(R)$, then z is determined by its value on σ, and $z(\sigma)$ must be an element of $(L \otimes R)$ with "σ-norm" equal to b, and conversely it is easy to check that such an element will determine an element of $Sp_c(R)$. With this in mind, we will write $[N_{L/F} = b]$ for the functor Sp_c. By the above we may write (up to natural isomorphism)

$$[N_{L/F} = b](R) = \{x \in L \otimes R | x\tau(x) \cdots \tau^{m-1}(x) = b\} \tag{1}$$

and for a homomorphism $f : R \to S$, we have:

$$[N_{L/F} = b](f)(x) = (id_L \otimes f)(x)$$

and by proposition 3, this is represented by an open subvariety of $V(A)$.

2.3 Group algebra computations

For convenience of notation, since we will be dealing often with certain elements of the group algebra $R = \mathbb{Z}G$, we define for $\gamma \in G$, and j a positive integer

$$N_\gamma^j = 1 + \gamma + \gamma^2 + \ldots + \gamma^{j-1}$$

which we will call the j'th partial norm of γ.
These satisfy the following useful identity which can be easily verified:

$$(N_{\gamma}^{j})(N_{\gamma^j}^{i}) = N_{\gamma}^{ij}$$

where γ is an element of G.
Now, suppose that u is an element in a arbitrary F-algebra B, and E' is a subfield of B such that for all $x \in E'$, $ux = \gamma(x)u$ for $\gamma \in Aut_F(E')$. Then we have the identity:

$$\left(\sum_{k=0}^{i-1}\left(\prod_{j=1}^{k}\gamma^{i-j}(x)\right)u^{i-k-1}\right)(x-u) = \gamma^{i-1}(x)\gamma^{i-2}(x)\cdots\gamma(x)x - u^i$$

(where we consider the empty product in the case $k=0$ to equal 1).
If we consider the group algebra $\mathbb{Z}\langle\gamma\rangle$ to act on E', then in the above notation, there is an element $a \in B$ such that

$$a(x-u) = N_{\gamma}^{i}x - u^i. \tag{2}$$

2.4 Galois monomial maps

As in (1), let $[N_{E/L} = b]$ be the functor representing elements of norm b. Recall, that for a commutative F-algebra R, $x \in [N_{E/L}](R)$ means that $x \in E \otimes R$ with σ-norm equal to $b^k \otimes 1 \in L \otimes R$),

DEFINITION 4. A Galois monomial in σ is an element of the group algebra $\mathbb{Z}\langle\sigma\rangle$.

Suppose P is a Galois monomial in σ. Let $\epsilon : \mathbb{Z}\langle\sigma\rangle \longrightarrow \mathbb{Z}$ be the augmentation map defined by mapping all group elements to 1. Then if we set $l = \epsilon(P)$, for every integer k, and every commutative F-algebra R, P induces a map of sets:

$$P : [N_{E/L} = b^k](R) \to [N_{E/L} = b^{kl}](R)$$

via for $x \in [N_{E/L} = b^k](R)$, if $P = \sum_{i=0}^{p-1} n_i\sigma^i$,

$$P(x) = x^{n_0+n_1\sigma+n_2\sigma^2+\cdots+n_{p-1}\sigma^{p-1}} \stackrel{def}{=} \prod_{i=0}^{p-1}\sigma^i(x^{n_i}).$$

We refer to this as the Galois monomial map induced by P.

LEMMA 2. *Let P be a Galois monomial as above with $\epsilon(P) = l$. Then the maps induced by P fit together to give a natural transformation*

$$[N_{E/L} = b^k] \to [N_{E/L} = b^{kl}]$$

Proof. This is a routine check. □

We will refer to the induced natural transformation above also by the letter P.

3 PROOF OF THE SEMIDIRECT PRODUCT THEOREM

We begin by fixing notation. Let A be a central simple semidirect product algebra of degree n as in the statement of theorem 1, and fix K/F maximal separable in A so that we have the following diagram of fields:

$$\begin{tikzcd} & E \arrow[dl, dash, "\sigma"'] \arrow[dr, dash, "\tau"] & \\ L \arrow[dr, dash, "\tau"'] & & K \arrow[dl, dash] \\ & F & \end{tikzcd}$$

Now, as was shown in section 2.2, since A_L is a cyclic algebra, the functor $[N_{E/L} = b]$ is represented by an open subvariety of $V(A_L)$. The idea of the proving theorem 1 will be to construct rational maps of Severi-Brauer varieties by constructing natural transformations between the corresponding functors. By the lemma from the previous section, one way to construct these natural transformations is via Galois monomial maps.
Since the functor $[N_{E/L} = b^r]$ is represented by an open subvariety of $V(A_L^r)$, a Galois monomial P with $\epsilon(P) = l$ yields an L-rational map $V(A_L^k) \to V(A_L^{kl})$. Our goal will be to determine when such a map induces an F-rational map $V(A^k) \to V(A^{kl})$. To understand when this will happen, we first remark that $[N_{E/L} = b^k]$ can be written as $U_k \times_F L$ where U_k is the open subfunctor of $V(A^k)$ described in proposition 3, section 2.1. By proposition 2, section 2.1, this will happen when when the τ actions on $U_k^{L/F}$ and $U_{kl}^{L/F}$ commute with the natural transformation induced by P.

Remark. Since $U_k^{L/F}(R)$ is the same as $\big(U^k \times_F L\big)(R \otimes L) = [N_{E/L} = b^k](R \otimes L)$, we will abuse notation and write $[N_{E/L} = b^k]$ in place of $U_k^{L/F}$. This means that for an F-algebra R, the notation $x \in [N_{E/L} = b^k](R)$ means $x \in R \otimes E$ has norm $1 \otimes b^k \in R \otimes L$.

In the notation above, we now need to examine when the action of τ on $[N_{E/L} = b^k]$ and $[N_{E/L} = b^{kl}]$ commute with a given monomial P. We will proceed now to describe the actions of τ, and then to translate these into actions on the "norm set" functors, which will let us answer our question.

3.1 THE ACTION OF τ

We recall the notation of section 2. For an F-algebra C, we define the action ι_C on $C \otimes L$ by $\iota_C(\tau) = id \otimes \tau$.

LEMMA 3. *Let $B = C \otimes L$, where C is a central simple F-algebra of degree n. If α is an arbitrary τ-semilinear action on B, then there is an isomorphism $(B, \alpha) \cong (B, \iota_C)$.*

Proof. Let $D = B^\alpha$. Then by descent, we have an isomorphism $(D \otimes L, \iota_D) \cong (B, \alpha)$
Since $m = [L : F]$ is relatively prime to $n = deg(B)$, the restriction map of Brauer groups:

$$Br_n(F) \xrightarrow{res_{L/F}} Br_n(L)$$

is injective. Therefore, since both D and C restrict to the same element, they are F-isomorphic. We can therefore write $D \cong C$, and again by descent we get an isomorphism of $(D \otimes L, \iota_D) \cong (C \otimes L, \iota_C)$. Combining this with the isomorphism $(D \otimes L, \iota_D) \cong (B, \alpha)$, we have an isomorphism $(C \otimes L, \iota_C) \cong (B, \alpha)$. □

Corollary 3. *Let B be a central simple L-algebra of degree n, α, β τ-semilinear actions on B. Then $(B, \alpha) \cong (B, \beta)$.*

Proof. If B has a semilinear action α, we may write $B \cong C \otimes L$ by descent, where C is defined to the the invariants of B under the action induced by α. Therefore the result follows directly from the lemma above. □

Therefore, to understand the action of τ on A_L via $1 \otimes \tau$ up to an isomorphism of pairs, we need only define any τ-semilinear action on A_L.

3.1.1 An Action of τ on A_L

Since the algebra A_L has a maximal subfield E which is cyclic over L, we may write $A_L = (E, \sigma, b)$ for some element $b \in L$. Our goal in this section will be to define a semilinear action of τ on A_L.
Borrowing some of the ideas of Rowen and Saltman ([RS96]), we first investigate the action of τ on $b \in L$. We first note that since A is an F-algebra, that if we consider the algebra ${}^\tau A_L$, then this is isomorphic to A_L by the map $id_A \otimes \tau$. On the other hand, one may also check that there is an isomorphism

$$\begin{gathered} {}^\tau A_L = {}^\tau(E, \sigma, b) \to (E, \sigma^r, \tau(b)) \\ \text{via } E \xrightarrow{\tau} E \text{ and } u \to u \end{gathered}$$

and extending to make a homomorphism. Consequently, we have an isomorphism of central simple algebras $(E, \sigma, b) \cong (E, \sigma^r, \tau(b))$. In addition, there is also an isomorphism $(E, \sigma, b) \cong (E, \sigma^r, b^r)$ ([Pie82] p.277 Cor.a), which means $(E, \sigma^r, b^r) \cong (E, \sigma^r, \tau(b))$. This implies $\tau(b) = ab^r$ where $a = N_{\sigma^r}(x) = N_\sigma(x)$ for some $x \in E^*$ ([Pie82] p.279 Prop.b).
Now to define an action of τ on A_L, we must first extend the action to the maximal subfield of A_L which is of the form $L(b^{1/n})$. This will be made more tractable by choosing a different b.

Lemma 4. *There exists $b' \in L$ such that $(E, \sigma, b) \cong (E, \sigma, b')$ and such that $\tau(b') = \lambda^n (b')^r$ where $\lambda \in L$.*

Proof. In the case where F contains the n'th roots of unity, this follows directly from [RS96], Lemma 1.2.
For the case where $n = p$ is prime, We consider the exact sequence of $\mathbb{Z}/p\mathbb{Z}[\tau]$ modules:

$$0 \to \frac{N_{E/L}(E^*)}{(L^*)^p} \to \frac{L^*}{(L^*)^p} \xrightarrow{\pi} \frac{L^*}{N_{E/L}(E^*)} \to 0$$

By Maschke's theorem ([Pie82], p.51), $\mathbb{Z}/p\mathbb{Z}[\tau]$ is a semisimple algebra and hence every module is projective and every exact sequence splits. We may therefore choose a splitting map $\phi : \frac{L^*}{N_{E/L}(E^*)} \to \frac{L^*}{(L^*)^p}$. Let b' be a coset representative for $\phi(bN_{E/L}(E^*))$. Since ϕ is a splitting, $\pi(b(L^*)^p) = \pi(b'(L^*)^p)$ implies $\pi(b/b'(L^*)^p) = 1$ which means that b and b' differ by a norm and so $(E, \sigma, b) \cong (E, \sigma, b')$. Further, since ϕ is a τ-morphism,

$$\tau(b')(L^*)^p = \tau(\phi(b)(L^*)^p) = \phi(\tau(b)N_{E/L}(E^*)) = \\ \phi(ab^r N_{E/L}(L^*)) = \phi(b^r N_{E/L}(E^*)) = (b')^r (L^*)^p$$

This gives us $\tau(b') = \lambda^p (b')^r$ for some $\lambda \in L$ as desired. □

Without loss of generality, we now substitute b' for b and assume that $\tau(b) = \lambda^n b^r$.
Now consider the field $L(\beta)$, where β is defined to be a root of the polynomial $x^n - b$. We want to show that we can extend the action of τ to an order m automorphism of $L(\beta)/F$. To this effect we first define an map $\tau' : L(\beta) \to L(\beta)$, where $\tau'|_L = \tau$ and $\tau'(\beta) = \lambda\beta^r$. One may verify this defines an automorphism by considering $L(\beta) = L[x]/(x^n - b)$ and noting that τ' preserves the ideal $(x^n - b)$.

Lemma 5. *We may choose λ above so that τ' has order m in $Aut(L(\beta))$.*

Proof. Since by definition $\tau'|_L = \tau$, we have $(\tau')^m \in Aut(L(\beta)/L)$. We thereby find that $ord(\tau')|mn$, $ord(\tau) = m|ord(\tau')$. Therefore we can write $ord(\tau') = km$, $k|n$, and set $\gamma = (\tau')^k$ and $M = L(\beta)^\gamma$. Since $[M : F] = n$, $[L : F] = m$ have relatively prime degrees and are both subfields of $L(\beta)$, which has degree nm, we find that $L(\beta) = L \otimes_F M$. Hence we may define $\tau'' = \tau \otimes id_M \in Aut(L(\beta))$, which is an order m automorphism. But now

$$\tau''|_L = \tau|_L = \tau'|_L$$

and $\tau(b) = \lambda^n b^r \implies \tau''(\beta) = \rho\lambda\beta^r$ where ρ is an n'th root of unity. But we see τ'' is defined in the same way as τ' except for using $\rho\lambda$ instead of λ as a n'th root of unity. Hence, by changing our choice of λ to $\rho\lambda$ we obtain an order m automorphism. □

For simplicity of notation we denote the extension τ' of τ to $L(\beta)$ also by τ. By the above description, we have

$$\tau\beta = \lambda\beta^r$$

where $\lambda \in L$.
We now use this information to define an action of τ on A. Since $A_L = (E, \sigma, b)$ can be thought of as the free noncommutative F-algebra generated by E and u modulo the relations $ux - \sigma(x)u = 0$ and $u^n = b$, giving an F-homomorphism $A_L \to B$ is equivalent to giving an F-map $\phi : L \to B$ and choosing and element $\phi(u) \in B$ such that $\phi(u)\phi(x) - \phi(\sigma(x))\phi(u) = 0$ and $\phi(u)^n - \phi(b) = 0$. Consequently, since any F-endomorphism of A_L is an automorphism (since A_L is finite dimensional and simple), to define an action of τ on A_L, we need only define τ on E and on u and then check that our relation is preserved.
To begin, we define $\tau|_E : E \to E \subset A_L$ to be the original Galois action, and $\tau(u) = \lambda u^r$. Checking our relations we have:

$$\begin{aligned}\tau(u)\tau(x) = \lambda u^r \tau(x) = \lambda \sigma^r \tau(x) u^r = \lambda \tau\sigma(x) u^r = \tau\sigma(x)\lambda u^r \\ = \tau(\sigma(x))\tau(u)\end{aligned}$$

and

$$\tau(u^n) = \lambda^n u^{rn} = \lambda^n b^r = \tau(b)$$

Since $L(u) \cong L(\beta)$ where $L(\beta)$ is as above, we know that $\tau^m(u) = u$. Since τ has order m on E, together this means that τ as defined above is an order m semilinear automorphism of A_L. We will refer to this action as $\alpha : \langle\tau\rangle \to Aut_F(A_L)$.

3.1.2 An Action of τ on A_L^l

We define $A_L^l \subset \otimes^l A_L = \overbrace{A_L \otimes_L \ldots \otimes_L A_L}^{l\text{ - times}}$ to be the algebra generated by $E \otimes_L 1 \otimes_L \ldots \otimes_L 1$ (which we will identify with just E), and $v = u \otimes_L u \otimes_L \ldots \otimes_L u$.

Lemma 6. *$[A_L^l] = [A_L]^l$, where brackets denote classes in $Br(L)$*

Proof. Since $A_L \cong (E, \sigma, b)$, we simply need to verify that A_L^l is just the symbol algebra (E, σ, b^l). But this follows because we clearly have $A_L^l = \coprod_{i=0}^{p} Ev^i$, and we need only check the two defining identities:

$$\begin{aligned} v^p = (u \otimes u \otimes \ldots \otimes u)^p &= u^p \otimes u^p \otimes \ldots \otimes u^p \\ &= b \otimes b \otimes \ldots \otimes b \\ &= b^l \otimes 1 \otimes \ldots \otimes 1 \end{aligned}$$

and

$$\begin{aligned} vx &= (u \otimes u \otimes \ldots \otimes u)(x \otimes 1 \otimes \ldots \otimes 1) \\ &= (ux \otimes u \otimes \ldots \otimes u) \\ &= (\sigma(x)u \otimes u \otimes \ldots \otimes u) \\ &= (\sigma(x) \otimes 1 \otimes \ldots \otimes 1)(u \otimes u \otimes \ldots \otimes u) \\ &= \sigma(x)v \end{aligned}$$

□

Next we note that we have a τ-semilinear action on $\otimes^l A_L$ which is induced (diagonally) by the τ action on A_L, and further, since it is easy to establish that:

$$\tau v = \lambda^l v^r$$

and the action of τ is the usual one on E, we know that A_L^l is preserved by τ and hence we have an induced action on A_L^l. We call this action α^l

3.1.3 τ-Action on Norm Sets

Our goal now will be to describe an action on the norm sets which is compatible with the above τ-action on ideals. The following lemma assures us that since the actions on the algebras A_L and A_L^l given above are isomorphic to the standard actions, they also induce isomorphic actions on $V(A)^{L/F}$ and $V(A^l)^{L/F}$ respectively. Therefore, we may proceed to find actions on the norm sets compatible with the τ actions given above.
Let (B, β) be an algebra with τ-semilinear action such that B central simple over L. Then by corollary 2, we have an induced action on $V(B)(R \otimes L)$ via for $I \in V(B)(R \otimes L)$, thinking of $I \subset B \otimes_L (L \otimes R)$

$$\beta(\tau)I = \{\beta(\tau)(x) | x \in I\}$$

where $\beta(\tau)$ is acting here on $B \otimes_L (L \otimes R) = B \otimes R$ as $\beta(\tau) \otimes 1$.

Lemma 7. *If $f : (B, \beta) \to (B', \beta')$ is an isomorphism, then the induced isomorphism $V(B)(\_) \to V(B')(\_)$ commutes with the actions of τ*

Proof. This is a simple check:

$$f(\beta(\tau)(I)) = f(\{\beta(\tau)(x) | x \in I\}) = \{f(\beta(\tau)(x)) | x \in I\} = \\ \{\beta'(\tau)(f(x)) | x \in I\} = \beta'(\tau)(f(I))$$

□

As was shown in the previous section, we know τ acts on A_L and hence also on $A_L \otimes R$. This translates to τ acting on $E \otimes R$ via the natural action on E, and by $\tau u = \lambda u^r$ (where we abuse notation by writing u for $u \otimes 1$). Therefore, if $x - u \in I$ then $\tau x - \tau u \in \tau I$, or in other words, $\tau x/\lambda - u^r \in \tau I$. To translate this into an action on norm sets, we recall that our birational identification between $V(A_L)$ and $[N_{E/L} = x]$ is via $I \in V(A_L)^{E/L}(R)$ being identified with $I \cap (E_R - u)$. Therefore to find our τ action on norm sets, we take an ideal I with a given intersection $x - u \in E_R - u$ and find the intersection of the new ideal $\tau(I)$.

Since $GCD(r,n)=1$ (because $r^m \equiv_n 1$), we may select a positive integer t so that $rt = sn+1$. By equation 2 in 2.3, there is an $a \in A$ such that

$$\begin{aligned} a(\tau x/\lambda - u^r) &= N^t_{\sigma^r}(\tau x/\lambda) - u^{rt} \\ &= N^t_{\sigma^r}(\tau x/\lambda) - b^s u \\ &= \tau N^t_\sigma(x)/N^t_{\sigma^r}(\lambda) - b^s u \\ &= \tau N^t_\sigma(x)/\lambda^t - b^s u \end{aligned}$$

where the last step follows from the fact that $\lambda \in L = E^\sigma$.
Now, since τI is a left ideal containing $\tau x/\lambda - u^r$, it must also contain $\tau N^t_\sigma(x)/\lambda^t b^s - u$. Therefore, $\tau N^t_\sigma(x)/\lambda^t b^s - u \in \tau I \cap (E-u)$. This tells us precisely that τI corresponds to $\tau N^t_\sigma(x)/\lambda^t b^s$, and so we get an action of τ on $[N_{E/L} = b]$ via

$$x \overset{\tau}{\mapsto} \tau N^t_\sigma(x)/\lambda^t b^s$$

which makes the following diagram commute:

$$\begin{tikzcd} V(A)^{E/L} \arrow[r, "\tau"] & V(A)^{E/L} \\ {[N_{E/L} = b]} \arrow[u] \arrow[r, "\tau"] & {[N_{E/L} = b]} \arrow[u] \end{tikzcd}$$

Similarly, using the fact that τ acts on A^l via $v \mapsto \lambda^l v^r$, $v^p = b^l$, and v induces σ on E, we get that τ's action on $V(A^l)$ yields an action on $[N_{E/L} = b^l]$ via:

$$x \overset{\tau}{\mapsto} \tau N^t_\sigma(x)/\lambda^{lt} b^{ls}.$$

Now, suppose $P \in \mathbb{Z}\langle\sigma\rangle$, $\epsilon(P) = l$. As P can be considered as a map from $[N_{E/L} = b^k]$ to $[N_{E/L} = b^{kl}]$, it is acted upon by τ via $\tau \bullet P = \tau \circ P \circ \tau^{-1}$. On the other hand, there is a natural action of τ on the group algebra $\mathbb{Z}\langle\sigma\rangle$ given by conjugation by τ, i.e. $\sigma \overset{\tau}{\mapsto} \sigma^r$. We claim that these two actions coincide. Thinking of P as an element of the group algebra $\mathbb{Z}G$, we write $\tau P \tau^{-1}$ as the action of τ by conjugation (as a group element).

Proposition 4. $\tau \bullet P = \tau P \tau^{-1}$

Proof. We aim to show $\tau \circ P = \tau P \tau^{-1} \circ \tau$. To start choose $x \in [N_{E/L} = b^k](R)$. Since $P(x) \in [N_{E/L} = b^{kl}](R)$, we have

$$\begin{aligned} \tau \circ P(x) &= \tau N^t_\sigma(P(x))/\lambda^{klt} b^{kls} \\ &= \tau P N^t_\sigma(x)/\lambda^{klt} b^{kls} \end{aligned}$$

On the other hand,

$$\begin{aligned}\tau P\tau^{-1}\circ\tau(x) &= \tau P\tau^{-1}(\tau N_\sigma^t(x)/\lambda^{kt}b^{ks})\\ &= \tau P N_\sigma^t(x)/\tau P\tau^{-1}(\lambda^{kt}b^{ks})\\ &= \tau P N_\sigma^t(x)/\lambda^{klt}b^{kls}\\ &= \tau\circ P(x)\end{aligned}$$

where the second to last step follows from the fact that $\tau P\tau^{-1}$ is a monomial in σ, and that λ and b are σ-fixed. To finish, we see that by composing on the right by the map τ, we get

$$\tau P\tau^{-1} = \tau\circ P\circ\tau^{-1} = \tau\bullet P$$

as desired. □

Corollary 4. *If $P\in\mathbb{Z}\langle\sigma\rangle^\tau$ then the induced map on norm sets*

$$P:[N_{E/L}=b^k]\to[N_{E/L}=b^{kl}]$$

commutes with the action of τ.

3.2 Proof of the Main Theorem

We recall our earlier definitions of ϵ, $\overline{\epsilon}$, and S.

Lemma 8. *Let $P_1,P_2\in\mathbb{Z}\langle\sigma\rangle$, $\epsilon(P)=l_i$. Then for any $k\in\mathbb{Z}$, P_1P_2 induces a map*

$$P_1P_2:[N_{E/L}=b^k]\to[N_{E/L}=b^{kl_1l_2}]$$

which is the composition of the maps

$$\begin{aligned}P_2&:[N_{E/L}=b^k]\to[N_{E/L}=b^{kl_2}]\\ \textit{and}\ \ P_1&:[N_{E/L}=b^{kl_2}]\to[N_{E/L}=b^{kl_1l_2}]\end{aligned}$$

Proof. This just comes from the fact that the group algebra acts on E^* with composition being identified with multiplication in the group algebra. □

Definition 5. For $i,k\in\mathbb{Z}$, we define a natural transformation (morphism)

$$\phi_k:[N_{E/L}=b^i]\to[N_{E/L}=b^{i+nk}]$$

by the rule: for $x\in[N=b^i](R)$, $\phi_j(x)=xb^k$.

Note that we abuse notation here, and don't specify the domain or range of ϕ_j in the notation. In any case, one may easily verify that $\phi_j\circ\phi_k=\phi_{j+k}$. In particular, these maps are all invertible and hence are birational morphisms.

Lemma 9. *ϕ_k commutes with the action of τ.*

Proof. We consider $\phi_k : [N_{E/L} = b^i] \to [N_{E/L} = b^{i+nk}]$. Using the formulas for the τ actions described earlier, we have:

$$\begin{aligned}
\tau(\phi_k(x)) &= \tau(xb^k) \\
&= \frac{\tau N_\sigma^t(b^k x)}{\lambda^{(i+nk)t} b^{(i+nk)s}} \\
&= \frac{\tau N_\sigma^t(b^k)\tau N_\sigma^t(x)}{\lambda^{it} b^{is} \lambda^{nkt} b^{nks}} \\
&= \frac{\tau(b^{tk})\tau N_\sigma^t(x)}{\lambda^{it} b^{is} \lambda^{nkt} b^{nks}} \\
&= \frac{(\lambda^n b^r)^{tk}\tau N_\sigma^t(x)}{\lambda^{it} b^{is} \lambda^{nkt} b^{nks}} \\
&= \frac{\lambda^{nkt}(b^{sn+1})^k\tau N_\sigma^t(x)}{\lambda^{it} b^{is} \lambda^{nkt} b^{nks}} \\
&= \frac{b^k \tau N_\sigma^t(x)}{\lambda^{it} b^{is}} \\
&= b^k \tau(x) \\
&= \phi_k(\tau(x))
\end{aligned}$$

□

LEMMA 10. *For P any Galois monomial in σ, $P\phi_k = \phi_{\epsilon(P)k} P$*

Proof. If $P = \sum_{i=0}^{n-1} n_i \sigma^i$ then we simply compute:

$$\begin{aligned}
P \circ \phi_k(x) = P(b^k x) = \prod_{i=0}^{n-1} (\sigma^i(b^k x))^{n_i} &= \prod_{i=0}^{n-1} (\sigma^i(b^k))^{n_i} (\sigma^i(x))^{n_i} \\
= \prod_{i=0}^{n-1} (b^k)^{n_i} \prod_{i=0}^{n-1} (\sigma^i(x))^{n_i} = (b^k)^{\sum n_i} P(x) &= b^{\epsilon(P)k} P(x) = \phi_{\epsilon(P)k} P(x)
\end{aligned}$$

□

We now prove the main theorem:

THEOREM 2. *If $l \in \mathbb{Z}$ such that $\bar{l} \in \bar{\epsilon}((S^*)^\tau)$ then there is a birational map*

$$V(A) \to V(A^l)$$

Proof. We note first that $\mathbb{Z}\langle\sigma\rangle^\tau \to S^\tau$ is surjective, since if we consider the short exact sequence:

$$0 \to N\mathbb{Z} \to \mathbb{Z}\langle\sigma\rangle \to S \to 0$$

We get a long exact sequence in group cohomology

$$N\mathbb{Z} \to \mathbb{Z}\langle\sigma\rangle^{\tau} \to S^{\tau} \to H^1(\tau, N\mathbb{Z})$$

and since as a τ-module, $N\mathbb{Z} \cong \mathbb{Z}$ with trivial action, we have

$$H^1(\tau, N\mathbb{Z}) = Hom(\langle\tau\rangle, \mathbb{Z}) = Hom(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = 0$$

giving us a surjective map $\mathbb{Z}\langle\sigma\rangle^{\tau} \to S^{\tau}$ as claimed.
Now, choosing $\alpha \in (S^*)^{\tau}$ with $\overline{\epsilon}(\alpha) = \overline{l}$, we can find $\widetilde{\alpha} \in \mathbb{Z}\langle\sigma\rangle^{\tau}$ mapping to α. If $\epsilon(\widetilde{\alpha}) = l'$, then by the commutativity of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}N \arrow[r] & \mathbb{Z}\langle\sigma\rangle \arrow[r] \arrow[d, "\epsilon"'] & S \arrow[r] \arrow[d, "\overline{\epsilon}"'] & 0 \\
& & \mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z} &
\end{tikzcd} \tag{3}$$

we have $\overline{l'} = \overline{l}$ and so $l' = l + kn$ for some $k \in \mathbb{Z}$. This means that $\widetilde{\alpha} : [N_{E/L} = b] \to [N_{E/L} = b^{l+nk}]$, which commutes with the action of τ by 4. Composing this with the map ϕ_{-k} gives us a natural transformation $\phi_{-k} \circ \widetilde{\alpha} : [N_{E/L} = b] \to [N_{E/L} = b^l]$ which commutes with τ. We will show that this actually induces an natural isomorphism.
Next pick $\beta \in (S^*)^{\tau}$ such that $\alpha\beta = 1 \in S$, and choose $\widetilde{\beta} \in \mathbb{Z}\langle\sigma\rangle^{\tau}$ mapping to β, and let $\big(\epsilon(\widetilde{\beta})\big)l = 1 + ns$. We now have $\widetilde{\beta} : [N_{E/L} = b^l] \to [N_{E/L} = b^{1+ns}]$ and composing with ϕ_{-s} yields a natural transformation $\phi_{-s} \circ \widetilde{\beta} : [N_{E/L} = b^l] \to [N_{E/L} = b]$ which commutes with τ.
Now, we compose $\phi_{-k} \circ \widetilde{\alpha}$ with $\phi_{-s} \circ \widetilde{\beta}$, which by construction is a natural transformation $[N_{E/L} = b] \to [N_{E/L} = b]$. If we write $\widetilde{\alpha}\widetilde{\beta} = 1 + rN$, then using 10, we compute:

$$\phi_{-s} \circ \widetilde{\beta} \circ \phi_{-k} \circ \widetilde{\alpha} = \phi_{-s}\phi_{-\epsilon(\widetilde{\beta})k}\widetilde{\beta}\widetilde{\alpha} = \phi_{-(s+\epsilon(\widetilde{\beta})k)}(1 + rN)$$

But $1 + rN : [N_{E/L} = b] \to [N_{E/L} = b^{1+rn}]$ is simply the map ϕ_r, so

$$\phi_{-s} \circ \widetilde{\beta} \circ \phi_{-k} \circ \widetilde{\alpha} = \phi_{-(s+\epsilon(\widetilde{\beta})k)}\phi_r = \phi_{r-s-\epsilon(\widetilde{\beta})k}$$

which is clearly an isomorphism (one can check in fact that $r - s - \epsilon(\widetilde{\beta})k = 0$ giving that the right hand side above is the identity). This argument shows that $\phi_{-k} \circ \widetilde{\alpha} : [N_{E/L} = b] \to [N_{E/L} = b^l]$ is an also an isomorphism which therefore induces a birational map

$$V(A) \to V(A^l).$$

$\square$

References

[Ami55] S. A. Amitsur. Generic splitting fields of central simple algebras. *Ann. of Math. (2)*, 62:8–43, 1955.

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*. Springer-Verlag, New York, 2000.

[Jah00] Jörg Jahnel. The Brauer-Severi variety associated with a central simple algebra: A survey. Preprint 52, Linear Algebraic Groups and Related Structures Preprint Server, http://www.mathematik.uni-bielefeld.de/LAG/man/052.html, 2000.

[Pie82] Richard S. Pierce. *Associative algebras*. Springer-Verlag, New York, 1982. Studies in the History of Modern Science, 9.

[Roq64] Peter Roquette. Isomorphisms of generic splitting fields of simple algebras. *J. Reine Angew. Math.*, 214/215:207–226, 1964.

[RS82] Louis H. Rowen and David J. Saltman. Dihedral algebras are cyclic. *Proc. Amer. Math. Soc.*, 84(2):162–164, 1982.

[RS96] Louis H. Rowen and David J. Saltman. Semidirect product division algebras. *Israel J. Math.*, 96(, part B):527–552, 1996.

[Sal99] David J. Saltman. *Lectures on division algebras*. Published by American Mathematical Society, Providence, RI, 1999.

[See99] George F. Seelinger. Brauer-Severi schemes of finitely generated algebras. *Israel J. Math.*, 111:321–337, 1999.

[VdB88] Michel Van den Bergh. The Brauer-Severi scheme of the trace ring of generic matrices. In *Perspectives in ring theory (Antwerp, 1987)*, pages 333–338. Kluwer Acad. Publ., Dordrecht, 1988.

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On the scattering theory of the Laplacian with a periodic boundary condition. I. Existence of wave operators

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Abstract. We study spectral and scattering properties of the Laplacian $H^{(\sigma)} = -\Delta$ in $L_2(\mathbb{R}^2_+)$ corresponding to the boundary condition $\frac{\partial u}{\partial \nu} + \sigma u = 0$ for a wide class of periodic functions σ. The Floquet decomposition leads to problems on an unbounded cell which are analyzed in detail. We prove that the wave operators $W_\pm(H^{(\sigma)}, H^{(0)})$ exist.

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Introduction

0.1 Setting of the problem

The present paper studies the Laplacian

$$H^{(\sigma)}u = -\Delta u \qquad \text{on } \mathbb{R}^2_+ \tag{0.1}$$

on the halfplane together with a boundary condition of the third type

$$\frac{\partial u}{\partial \nu} + \sigma u = 0 \qquad \text{on } \mathbb{R} \times \{0\}, \tag{0.2}$$

where ν denotes the exterior unit normal and where the function $\sigma : \mathbb{R} \to \mathbb{R}$ is assumed to be 2π-periodic. Moreover, let

$$\sigma \in L_{q,loc}(\mathbb{R}) \qquad \text{for some } q > 1.$$

Under this condition $H^{(\sigma)}$ can be defined as a self-adjoint operator in $L_2(\mathbb{R}^2_+)$ by means of the closed and lower semibounded quadratic form

$$\int_{\mathbb{R}^2_+} |\nabla u(x)|^2 \, dx + \int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2 \, dx_1, \qquad u \in H^1(\mathbb{R}^2_+).$$

This is the first part of a paper where we analyze the spectrum of $H^{(\sigma)}$ and develop a scattering theory viewing $H^{(\sigma)}$ as a (rather singular) perturbation of $H^{(0)}$, the Neumann Laplacian on $\mathbb{R}^2_+$. (For the abstract mathematical scattering theory see, e.g., [Ya].)
The main result of the present paper is that the wave operators

$$W_\pm^{(\sigma)} := W_\pm(H^{(\sigma)}, H^{(0)})$$

exist.

0.2 Physical interpretation

In the physical interpretation, $H^{(\sigma)}$ is the Hamiltonian of a two-dimensional quantum-mechanical system which consists of a particle in the upper halfplane and a crystal that fills the lower halfplane. The particle can not enter the crystal but interacts non-trivially with the surface of the crystal, described by the function σ. The existence of the wave operators means that every particle which is described by a state $u \in \mathcal{R}\left(W_\pm^{(\sigma)}\right)$ behaves like a free particle in the distant future and the distant past. We emphasize that there may also exist particles which are described by a state $u \in \mathcal{R}\left(W_\pm^{(\sigma)}\right)^\perp$. These are surface states which propagate along the boundary and decay exponentially away from the boundary. Such surface states will be investigated in the second part [FrSh] of the paper.

0.3 Outline of the paper

Let us explain some of the mathematical ideas involved. A precise definition of the operator $H^{(\sigma)}$ in terms of a quadratic form is given in Subsection 1.4. By means of the Bloch-Floquet theory we represent $H^{(\sigma)}$ in Subsection 2.2 as a direct integral

$$\int_{-1/2}^{1/2} \bigoplus H^{(\sigma)}(k) \, dk$$

with fiber operators $H^{(\sigma)}(k)$ acting in $L_2(\Pi)$ where $\Pi := (-\pi,\pi) \times \mathbb{R}_+$ is the halfstrip. Functions in the domain of $H^{(\sigma)}(k)$ satisfy the third type condition (0.2) on $(-\pi,\pi) \times \{0\}$ (at least if σ is smooth), so $H^{(\sigma)}(k)$ differs from $H^{(0)}(k)$ by a relatively compact form perturbation. This makes a rather detailed analysis of the operators $H^{(\sigma)}(k)$ possible.
Our approach leans on a quadratic form version of the resolvent identity which

we present in Subsection 3.2 following [Ya]. A similar approach has been successfully applied to study periodic Schrödinger operators (cf. [BShSu]). In our case it allows to show that the difference of resolvents of $H^{(\sigma)}(k)$ and $H^{(0)}(k)$ belongs to the trace class, and from the Birman-Kreĭn theorem (which is sometimes called Birman-Kuroda theorem, unaware of [BKr]) we deduce in Subsection 3.4 the existence and completeness of the wave operators on the halfstrip. Using the same representation we can prove a limiting absorption principle in Subsection 3.6, which implies the absence of singular continuous spectrum.
The existence of the wave operators $W_\pm^{(\sigma)}$ on the halfplane is derived from the existence of the wave operators on the halfstrip.

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1 Setting of the problem. The main result

1.1 Notation

We introduce the *halfplane*

$$\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\} = \mathbb{R} \times \mathbb{R}_+,$$

and the *halfstrip*

$$\Pi := \{x = (x_1, x_2) \in \mathbb{R}^2_+ : \ -\pi < x_1 < \pi, \ x_2 > 0\} = (-\pi, \pi) \times \mathbb{R}_+,$$

where $\mathbb{R}_+ := (0, +\infty)$. Moreover, we need the lattice $2\pi\mathbb{Z}$. Unless stated otherwise, periodicity conditions are understood with respect to this lattice. We think of the corresponding torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ as the interval $[-\pi, \pi]$ with endpoints identified.
We use the notation $D = (D_1, D_2) = -i\nabla$ in $\mathbb{R}^2$.
For an open set $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, the index in the notation of the norm $\|.\|_{L_2(\Omega)}$ is usually dropped. The space $L_2(\mathbb{T})$ may be formally identified with $L_2(-\pi, \pi)$. We define the (discrete) Fourier transformation $\mathcal{F} : L_2(\mathbb{T}) \to l_2(\mathbb{Z})$ by

$$(\mathcal{F}f)_n = \hat{f}_n := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x_1) e^{-inx_1}\, dx_1. \qquad n \in \mathbb{Z},$$

Next, for an open set $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, $H^s(\Omega)$ is the Sobolev space of order $s \in \mathbb{R}$ (with integrability index 2). By $H^s(\mathbb{T})$ we denote the closure of $C^\infty(\mathbb{T})$ in $H^s(-\pi,\pi)$. Here $C^\infty(\mathbb{T})$ is the space of functions in $C^\infty(-\pi,\pi)$ which can be extended 2π-periodically to functions in $C^\infty(\mathbb{R})$. The space $H^s(\mathbb{T})$ is endowed with the norm

$$\|f\|^2_{H^s(\mathbb{T})} := \sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{f}_n|^2, \qquad f \in H^s(\mathbb{T}).$$

By $\tilde{H}^s(\Pi)$ we denote the closure of $\tilde{C}^\infty(\Pi) \cap H^s(\Pi)$ in $H^s(\Pi)$. Here $\tilde{C}^\infty(\Pi)$ is the space of functions in $C^\infty(\Pi)$ which can be extended 2π-periodically with respect to x_1 to functions in $C^\infty(\mathbb{R}^2_+)$.
Statements and formulae which contain the double index "$\pm$" are understood as two independent assertions.

1.2 Scattering theory

Here we summarize the definitions and basic results on scattering theory. For proofs we refer to [Ya].
Let H_0, H be self-adjoint operators in a Hilbert space $\mathfrak{H}$. The projection onto the absolutely continuous subspace of H_0 and the unitary group of H_0 are denoted by P_0 and $U_0(t) := \exp(-itH_0)$, respectively. We put $\mathfrak{H}_0^{(ac)} := \mathcal{R}(P_0)$. For the similar objects related to the operator H we omit the index "0".
In case of existence, the limit

$$W_\pm(H,H_0) := s - \lim_{t\to\pm\infty} U(-t)U_0(t)P_0$$

is called the *wave operator* for the pair H, H_0 and the sign $\pm$. Thus the elements $u = W_\pm(H,H_0)u_0^\pm \in \mathcal{R}(W_\pm(H,H_0))$, $u_0^\pm \in \mathfrak{H}_0^{(ac)}$, satisfy

$$\lim_{t\to\pm\infty} \|U(t)u - U_0(t)u_0^\pm\| = 0.$$

The wave operators are partial isometries with initial subspace $\mathfrak{H}_0^{(ac)}$. One easily establishes the *intertwining property*

$$W_\pm(H,H_0)\, H_0 = H\, W_\pm(H,H_0).$$

It follows that the subspace $\mathcal{R}(W_\pm(H,H_0))$ and its orthogonal complement are invariant under H and that the wave operator provides a unitary equivalence between the part of H on $\mathcal{R}(W_\pm(H,H_0))$ and the absolutely continuous part of H_0. In particular,

$$\mathcal{R}(W_\pm(H,H_0)) \subset \mathfrak{H}^{(ac)}. \tag{1.1}$$

The wave operator $W_\pm(H,H_0)$ is said to be *complete* if equality holds in (1.1). It is easy to see that the completeness of $W_\pm(H,H_0)$ is equivalent to the existence of $W_\pm(H_0,H)$. Thus, if the wave operator $W_\pm(H,H_0)$ exists and is

complete, then the absolutely continuous parts of H_0 and H are unitarily equivalent.
Let us conclude this brief overview with a convenient sufficient condition for the existence and completeness of the wave operators due to Birman and Kreĭn (cf. [BKr]).

Proposition 1.1. *Let H_0, H be self-adjoint operators on a Hilbert space $\mathfrak{H}$ such that $(H - zI)^{-1} - (H_0 - zI)^{-1}$ belongs to the trace class for some $z \in \rho(H_0) \cap \rho(H)$. Then the wave operators $W_\pm(H, H_0)$ exist and are complete.*

1.3 Multiplication on the boundary

Here we present auxiliary statements related to Sobolev embedding theorems. Let σ be a periodic function satisfying

$$\sigma \in L_q(\mathbb{T}) \qquad \text{for some } q > 1. \tag{1.2}$$

It follows from the compactness of the embedding $H^{1/2}(-\pi, \pi) \subset L_{2q'}(-\pi, \pi)$ (with $\frac{1}{q} + \frac{1}{q'} = 1$) that the form $\int_{-\pi}^{\pi} |\sigma(x_1)||f(x_1)|^2\, dx_1$, $f \in H^{1/2}(-\pi, \pi)$, is compact in $H^{1/2}(-\pi, \pi)$. This implies

Lemma 1.2. *Assume* (1.2) *and let $\epsilon > 0$. Then there exists a constant $C_1(\epsilon, \sigma) > 0$ such that*

$$\int_{-\pi}^{\pi} |\sigma(x_1)||f(x_1)|^2\, dx_1 \le \epsilon \|f\|^2_{H^{1/2}(-\pi,\pi)} + C_1(\epsilon, \sigma)\|f\|^2, \qquad f \in H^{1/2}(-\pi, \pi).$$

Now let us pass to the situation on the halfstrip and on the halfplane. The trace operator $u \mapsto u(.,0)$ is bounded from $H^1(\Pi)$ to $H^{1/2}(-\pi, \pi)$. Hence the form $\int_{-\pi}^{\pi} |\sigma(x_1)||u(x_1, 0)|^2\, dx_1$, $u \in H^1(\Pi)$, is compact in $H^1(\Pi)$ and we obtain

Lemma 1.3. *Assume* (1.2) *and let $\epsilon > 0$. Then there exists a constant $C_2(\epsilon, \sigma) > 0$ such that*

$$\int_{-\pi}^{\pi} |\sigma(x_1)||u(x_1, 0)|^2\, dx_1 \le \epsilon \|u\|^2_{H^1(\Pi)} + C_2(\epsilon, \sigma)\|u\|^2, \qquad u \in H^1(\Pi).$$

Now let $u \in H^1(\mathbb{R}^2_+)$. For each $n \in \mathbb{Z}$ we apply Lemma 1.3 to the function $\Pi \ni x \mapsto u(x_1 + 2\pi n, x_2)$ and then sum over all $n \in \mathbb{Z}$. This yields

Lemma 1.4. *Assume* (1.2) *and let $\epsilon > 0$. Then there exists a constant $C_2(\epsilon, \sigma) > 0$ such that*

$$\int_{\mathbb{R}} |\sigma(x_1)||u(x_1, 0)|^2\, dx_1 \le \epsilon \|u\|^2_{H^1(\mathbb{R}^2_+)} + C_2(\epsilon, \sigma)\|u\|^2, \qquad u \in H^1(\mathbb{R}^2_+).$$

Our treatment in Section 3 needs a more precise, quantitative result on the embedding $H^{1/2}(-\pi,\pi) \subset L_{2q'}(-\pi,\pi)$. We begin by recalling the definition of the weak l_p-spaces

$$l_{p,w}(\mathbb{Z}) = \{(\alpha_n)_{n\in\mathbb{Z}} : \sup_{t>0} t\rho_\alpha^{1/p}(t) < \infty\}, \qquad 0<p<\infty,$$

where $\rho_\alpha(t) := \sharp\{n \in \mathbb{Z} : |\alpha_n| > t\}$ for $t>0$. $l_{p,w}(\mathbb{N})$ is defined in a similar way. Further, recall (cf. Section 11.6 in [BS]) that $\Sigma_p(\mathfrak{H}_1,\mathfrak{H}_2)$, $0<p<\infty$, is the class of compact operators K from a Hilbert space $\mathfrak{H}_1$ to a Hilbert space $\mathfrak{H}_2$ for which $(s_n(K))_{n\in\mathbb{N}} \in l_{p,w}(\mathbb{N})$, where $(s_n(K))_{n\in\mathbb{N}}$ is the sequence of singular numbers of K. One puts $\Sigma_p(\mathfrak{H}_1) := \Sigma_p(\mathfrak{H}_1,\mathfrak{H}_1)$. The dependence on $\mathfrak{H}_1$, $\mathfrak{H}_2$ is usually dropped in the notation if this does not lead to confusion.
The connection with the well-known Schatten class $\mathfrak{S}_p$ of order p (which consists of compact operators K for which $(s_n(K))_{n\in\mathbb{N}} \in l_p(\mathbb{N})$) can be seen from the inclusions

$$\Sigma_r \subset \mathfrak{S}_p \subset \Sigma_p, \qquad r<p.$$

We will often use the fact that $K_1 \in \Sigma_{p_1}$, $K_2 \in \Sigma_{p_2}$ implies

$$K_1K_2 \in \Sigma_{(\frac{1}{p_1}+\frac{1}{p_2})^{-1}}. \tag{1.3}$$

For a more general statement as well as for the proof of the following Cwikel-type estimate we refer to Theorem 4.8 in [BKaS].

Proposition 1.5. *Let $\beta \in L_p(\mathbb{T})$ and $\alpha \in l_{p,w}(\mathbb{Z})$ for some $p>2$. Then $\beta\mathcal{F}^*\alpha \in \Sigma_p(l_2(\mathbb{Z}), L_2(\mathbb{T}))$.*

Let us consider the sequence α given by

$$\alpha_n := (1+n^2)^{-1/4}, \qquad n\in\mathbb{Z}, \tag{1.4}$$

and a function β as above. We write $\beta\mathcal{F}^*\alpha = \left(\beta\mathcal{F}^*\alpha^{2/p}\right)\alpha^{(p-2)/p}$. Clearly, the operator of multiplication by $\alpha^{(p-2)/p}$ belongs to $\Sigma_{2p/(p-2)}(l_2(\mathbb{Z}))$, and by Proposition 1.5 $\beta\mathcal{F}^*\alpha^{2/p} \in \Sigma_p(l_2(\mathbb{Z}), L_2(\mathbb{T}))$. Thus, taking into account (1.3), we obtain

Corollary 1.6. *Let $\beta \in L_p(\mathbb{T})$ for some $p>2$ and α be given by* (1.4). *Then $\beta\mathcal{F}^*\alpha \in \Sigma_2(l_2(\mathbb{Z}), L_2(\mathbb{T}))$.*

This is the desired embedding result. Note that $\mathcal{F}^*\alpha\mathcal{F}$ maps $L_2(\mathbb{T})$ unitarily onto $H^{1/2}(\mathbb{T})$.

1.4 Definition of the operators $H^{(\sigma)}$ on the halfplane

Let σ be a real-valued periodic function satisfying (1.2). In the Hilbert space $L_2(\mathbb{R}^2_+)$ we consider the quadratic form

$$\begin{aligned} &\mathcal{D}[h^{(\sigma)}] := H^1(\mathbb{R}^2_+),\\ &h^{(\sigma)}[u] := \int_{\mathbb{R}^2_+} |Du(x)|^2\,dx + \int_{\mathbb{R}} \sigma(x_1)|u(x_1,0)|^2\,dx_1. \end{aligned} \tag{1.5}$$

According to Lemma 1.4 the form $h^{(\sigma)}$ is lower semibounded and closed, so it generates a self-adjoint operator which will be denoted by $H^{(\sigma)}$. By construction

$$(H^{(\sigma)}u, v) = h^{(\sigma)}[u, v], \qquad u \in \mathcal{D}(H^{(\sigma)}),\ v \in H^1(\mathbb{R}^2_+),$$

so it follows that the distributional Laplacian Δu of $u \in \mathcal{D}(H^{(\sigma)})$ belongs to $L_2(\mathbb{R}^2_+)$ and that

$$H^{(\sigma)}u = -\Delta u.$$

The case $\sigma = 0$ corresponds to the *Neumann Laplacian* on the halfplane, whereas the case $\sigma \neq 0$ implements a (generalized) *boundary condition of the third type*. More precisely, we have

Remark 1.7. Under the condition that σ is absolutely continuous with $\sigma' \in L_q(\mathbb{T})$ for some $q > 1$ it can be proved that

$$\mathcal{D}(H^{(\sigma)}) = \left\{ u \in H^2(\mathbb{R}^2_+) :\ -\frac{\partial u}{\partial x_2} + \sigma u = 0 \text{ on } \mathbb{R} \times \{0\} \right\}.$$

1.5 Main result

First we remark that the spectrum of the "unperturbed" operator $H^{(0)}$ coincides with $[0, +\infty)$ and is purely absolutely continuous of infinite multiplicity. This can be seen easily by applying a Fourier transformation with respect to the variable x_1 and a Fourier cosine transformation with respect to the variable x_2.

We turn now to the "perturbed" operator $H^{(\sigma)}$. We have recalled the abstract definition of the wave operators in Subsection 1.2. In case of existence we will use the notation

$$W_\pm^{(\sigma)} := W_\pm(H^{(\sigma)}, H^{(0)}).$$

The main result of the present paper is

Theorem 1.8. *Assume that σ satisfies* (1.2). *Then the wave operators $W_\pm^{(\sigma)}$ exist and satisfy $\mathcal{R}(W_+^{(\sigma)}) = \mathcal{R}(W_-^{(\sigma)})$.*

We note that the equality $\mathcal{R}(W_+^{(\sigma)}) = \mathcal{R}(W_-^{(\sigma)})$ implies the unitarity of the scattering matrix.

It follows from Theorem 1.8 that the part of $H^{(\sigma)}$ on $\mathcal{R}(W_\pm^{(\sigma)})$ is unitarily equivalent to $H^{(0)}$, and so $\sigma_{ac}\left(H^{(\sigma)}\right) \supset [0, +\infty)$.

In the second part [FrSh] we supplement this theorem with the following results. *The operator $H^{(\sigma)}$ has purely absolutely continuous spectrum.* In general, $\sigma\left(H^{(\sigma)}\right)$ may contain (apart from $[0 + \infty)$) additional bands, so the wave operators $W_\pm^{(\sigma)}$ may be not complete. The spectral subspaces corresponding to the additional bands of $H^{(\sigma)}$ are *additional channels of scattering*. However, *under the additional assumption*

$$\sigma \geq 0 \quad a.e. \tag{1.6}$$

the wave operators $W_\pm^{(\sigma)}$ *turn out to be complete and unitary.*

Remark 1.9. The statement of Theorem 1.8 holds with obvious changes when the role of the comparison operator H_0 is played by the Dirichlet Laplacian on the halfplane or the Laplacian on the whole plane. This follows easily from the chain rule for wave operators and the fact, that the wave operators $W_\pm(H^{(0)}, H_0)$ exist and are complete. Indeed, they can be calculated easily. (Clearly, if H_0 is the Laplacian on the whole plane one has to use an identification operator.)

Remark 1.10. A time-dependent characterization of the range of the wave operators and its orthogonal complement can be established by standard methods (see [DaSi], [Sa]): With the notation $U^{(\sigma)}(t) := \exp(-itH^{(\sigma)})$ for $t \in \mathbb{R}$ it follows that

$$\mathcal{R}(W_\pm^{(\sigma)}) = \{u \in L_2(\mathbb{R}^2_+) : \lim_{t\to\pm\infty} \int_{\mathbb{R}\times(0,a)} |U^{(\sigma)}(t)u(x)|^2\,dx = 0,\ a \in \mathbb{R}_+\},$$

$$\mathcal{R}(W_\pm^{(\sigma)})^\perp = \{u \in L_2(\mathbb{R}^2_+) : \lim_{a\to+\infty} \sup_{t\in\mathbb{R}} \int_{\mathbb{R}\times(a,+\infty)} |U^{(\sigma)}(t)u(x)|^2\,dx = 0\}.$$

Thus, the additional channels of scattering correspond to "surface states", i.e., states concentrated near the boundary for all time.

2 Direct integral decomposition

2.1 Definition of the operators $H^{(\sigma)}(k)$ on the halfstrip

Let σ be a real-valued periodic function satisfying (1.2) and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. In the Hilbert space $L_2(\Pi)$ we consider the quadratic form

$$\begin{aligned} &\mathcal{D}[h^{(\sigma)}(k)] := \tilde{H}^1(\Pi), \\ &h^{(\sigma)}(k)[u] := \int_\Pi \left(|(D_1+k)u(x)|^2 + |D_2u(x)|^2\right)\,dx + \int_{-\pi}^{\pi} \sigma(x_1)|u(x_1,0)|^2\,dx_1. \end{aligned} \tag{2.1}$$

According to Lemma 1.3 the form $h^{(\sigma)}(k)$ is lower semibounded and closed, so it generates a self-adjoint operator which will be denoted by $H^{(\sigma)}(k)$. Similarly as above one finds

$$H^{(\sigma)}u = (D_1+k)^2u + D_2^2u = -\Delta u + 2kD_1u + k^2u, \qquad u \in \mathcal{D}(H^{(\sigma)}).$$

In addition to the Neumann (if $\sigma = 0$) or third type (if $\sigma \neq 0$) boundary condition at $\{x_2 = 0\}$ the functions in $\mathcal{D}(H^{(\sigma)})$ satisfy periodic boundary conditions at $\{x_1 = \pm\pi\}$. A statement analogous to Remark 1.7 holds.

2.2 Direct integral decomposition of the operator $H^{(\sigma)}$

The operator $H^{(\sigma)}$ can be partially diagonalized by means of the *Gelfand transformation* $\mathcal{U}$. This operator is initially defined for $u \in \mathcal{S}(\mathbb{R}^2_+)$, the Schwartz class on $\mathbb{R}^2_+$, by

$$(\mathcal{U}u)(k,x) := \sum_{n\in\mathbb{Z}} e^{-ik(x_1+2\pi n)} u(x_1+2\pi n, x_2), \qquad k \in [-\tfrac{1}{2}, \tfrac{1}{2}],\ x \in \Pi,$$

and extended by continuity to a *unitary* operator

$$\mathcal{U} : L_2(\mathbb{R}^2_+) \to \int_{-1/2}^{1/2} \oplus L_2(\Pi)\, dk.$$

Moreover, it turns out that $u \in H^1(\mathbb{R}^2_+)$ iff $(\mathcal{U}u)(k,.) \in \tilde{H}^1(\Pi)$ for a.e. $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $\int_{-1/2}^{1/2} \|(\mathcal{U}u)(k,.)\|^2_{H^1(\Pi)}\, dk < \infty$, and in this case

$$(\mathcal{U}D_1 u)(k,.) = (D_1 + k)(\mathcal{U}u)(k,.), \qquad (\mathcal{U}D_2 u)(k,.) = D_2(\mathcal{U}u)(k,.).$$

Concerning the multiplication on the boundary by a periodic function σ satisfying (1.2), one finds for $u \in H^1(\mathbb{R}^2_+)$

$$\int_{-1/2}^{1/2} \int_{-\pi}^{\pi} \sigma(x_1) |(\mathcal{U}u)(k, x_1, 0)|^2\, dx_1\, dk = \int_{\mathbb{R}} \sigma(x_1) |u(x_1, 0)|^2\, dx_1.$$

To summarize, the Gelfand transformation satisfies

$$\begin{aligned} \mathcal{U}\left(\mathcal{D}[h^{(\sigma)}]\right) &= \Bigg\{ F \in \int_{-1/2}^{1/2} \oplus L_2(\Pi)\, dk :\ F(k) \in \tilde{H}^1(\Pi) \text{ for a.e. } k \in [-\tfrac{1}{2}, \tfrac{1}{2}], \\ &\qquad \int_{-1/2}^{1/2} \left| h^{(\sigma)}(k)[F(k)] \right| dk < \infty \Bigg\}, \\ h^{(\sigma)}[u] &= \int_{-1/2}^{1/2} h^{(\sigma)}(k)[(\mathcal{U}u)(k,.)]\, dk, \qquad u \in H^1(\mathbb{R}^2_+), \end{aligned}$$

which implies

$$\mathcal{U}\, H^{(\sigma)}\, \mathcal{U}^* = \int_{-1/2}^{1/2} \oplus H^{(\sigma)}(k)\, dk. \tag{2.2}$$

This relation allows us to investigate the operator $H^{(\sigma)}$ by studying the fibers $H^{(\sigma)}(k)$.

2.3 Main result for the operators $H^{(\sigma)}(k)$ on the halfstrip

In case of existence we will use the notation

$$W_\pm^{(\sigma)}(k) := W_\pm(H^{(\sigma)}(k), H^{(0)}(k)), \qquad k \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

Theorem 2.1. *Assume that σ satisfies* (1.2) *and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the wave operators $W_\pm^{(\sigma)}(k)$ exist and are complete.*

The spectrum of the "unperturbed" operator $H^{(0)}(k)$ (see Subsection 3.1) coincides with $[k^2, +\infty)$ and is purely absolutely continuous. By the remarks in Subsection 1.2 Theorem 2.1 implies that the absolutely continuous part of $H^{(\sigma)}(k)$ is unitarily equivalent to $H^{(0)}(k)$, in particular

$$\sigma_{ac}\left(H^{(\sigma)}(k)\right) = [k^2, +\infty).$$

Concerning the singular continuous spectrum of $H^{(\sigma)}(k)$ we prove

Theorem 2.2. *Assume that σ satisfies* (1.2) *and let $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then*

$$\sigma_{sc}\left(H^{(\sigma)}(k)\right) = \emptyset.$$

The point spectrum of $H^{(\sigma)}(k)$ is investigated in the second part [FrSh]. We prove there that $\sigma_p\left(H^{(\sigma)}(k)\right)$ consists of eigenvalues of finite multiplicities which may accumulate at $+\infty$ only. The situation of infinitely many (embedded) eigenvalues does actually occur. The discrete eigenvalues of $H^{(\sigma)}(k)$ produce bands in the spectrum of $H^{(\sigma)}$. In general, the same is true for the embedded eigenvalues of $H^{(\sigma)}(k)$. However, under the additional assumption (1.6) we prove $\sigma_p\left(H^{(\sigma)}(k)\right) = \emptyset$, which implies the completeness and even unitarity of the wave operators $W_\pm^{(\sigma)}$.

2.4 Reduction of Theorem 1.8 to Theorem 2.1

Assuming Theorem 2.1, the proof of Theorem 1.8 is easy.

Proof of Theorem 1.8. For $t \in \mathbb{R}$, $k \in [-\frac{1}{2}, \frac{1}{2}]$ put $U^{(\sigma)}(t) := \exp(-itH^{(\sigma)})$ and $U^{(\sigma)}(t,k) := \exp(-itH^{(\sigma)}(k))$, and similarly with σ replaced by 0. The wave operators $W_\pm^{(\sigma)}(k)$ exist by Theorem 2.1 and are measurable with respect to k, so $\int_{-1/2}^{1/2} \oplus W_\pm^{(\sigma)}(k)\, dk$ is well defined. Moreover, by (2.2) together with Theorem XIII.85 in [ReSi4] one has

$$\mathcal{U}\, U^{(\sigma)}(t)\, \mathcal{U}^* = \exp(-it\mathcal{U}\, H^{(\sigma)}\, \mathcal{U}^*) = \int_{-1/2}^{1/2} \oplus U^{(\sigma)}(t,k)\, dk,$$

where $\mathcal{U}$ is the Gelfand transformation from Subsection 2.2, and similarly with σ replaced by 0. It follows that for all $u \in L_2(\mathbb{R}^2_+)$

$$\begin{aligned} &\left\| \mathcal{U}\, U^{(\sigma)}(-t) U^{(0)}(t) u - \left(\int_{-1/2}^{1/2} \oplus W_\pm^{(\sigma)}(k)\, dk \right) \mathcal{U} u \right\|^2 = \\ &= \int_{-1/2}^{1/2} \left\| U^{(\sigma)}(-t,k) U^{(0)}(t,k)\, (\mathcal{U}u)(k,.) - W_\pm^{(\sigma)}(k)\, (\mathcal{U}u)(k,.) \right\|^2 dk \to 0 \\ &\qquad (t \to \pm\infty) \end{aligned}$$

by Lebesgue's theorem. This means that the strong limits of $U^{(\sigma)}(-t)U^{(0)}(t)$ for $t \to \pm\infty$ exist and coincide with $W_\pm^{(\sigma)} = \mathcal{U}^* \,(\int_{-1/2}^{1/2} \oplus W_\pm^{(\sigma)}(k)\, dk)\,\mathcal{U}$. In particular, because of the completeness of $W_\pm^{(\sigma)}(k)$,

$$\mathcal{R}(W_\pm^{(\sigma)}) = \mathcal{U}^* \left(\int_{-1/2}^{1/2} \oplus \mathcal{R}(P_{ac}^{(\sigma)}(k))\, dk \right) \mathcal{U}.$$

□

3 The operators $H^{(\sigma)}(k)$ on the halfstrip

3.1 The unperturbed operator $H^{(0)}(k)$ on the halfstrip

We start our investigation by summarizing results on the "unperturbed" operator $H^{(0)}(k)$.
Let $k \in [-\frac{1}{2}, \frac{1}{2}]$. By separation of variables one easily finds that the spectrum of the operator $H^{(0)}(k)$ coincides with $[k^2, +\infty)$, is purely absolutely continuous and that the spectral multiplicity of $\lambda \in [k^2, +\infty)$ is $\sharp\{n \in \mathbb{Z} : (n+k)^2 \le \lambda\}$. Note that the spectral multiplicity changes at the "threshold points" $(n+k)^2$, $n \in \mathbb{Z}$.
It is also easy to verify that the resolvent

$$R^{(0)}(z,k) := \left(H^{(0)}(k) - zI \right)^{-1}, \qquad z \in \rho\left(H^{(0)}(k) \right) = \mathbb{C} \setminus [k^2, +\infty),$$

is an integral operator with kernel

$$r^{(0)}(x,y;z,k) := \frac{1}{4\pi} \sum_{n\in\mathbb{Z}} \frac{e^{in(x_1-y_1)}}{\beta_n(z,k)} \left(e^{-\beta_n(z,k)\,(x_2+y_2)} + e^{-\beta_n(z,k)\,|x_2-y_2|} \right), \qquad x,\, y \in \Pi,\; x_2 \ne y_2. \tag{3.1}$$

where

$$\beta_n(z,k) := \sqrt{(n+k)^2 - z}, \qquad n \in \mathbb{Z},\; z \in \mathbb{C} \setminus [k^2, +\infty). \tag{3.2}$$

Here and in the following we choose the canonical branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ satisfying $\operatorname{Re} \sqrt{\cdot} > 0$.
Note that the RHS of (3.1) converges absolutely and uniformly on compact subsets of $\{(x,y) \in \Pi \times \Pi : \; x_2 \ne y_2\}$.

3.2 A general approach to the inversion of a perturbed operator

To investigate the "perturbed" operators $H^{(\sigma)}(k)$ we use a version of the resolvent identity. The "classical" resolvent identity $(H - zI)^{-1} - (H_0 - zI)^{-1} = -(H_0 - zI)^{-1}(H - H_0)(H - zI)^{-1}$ involves the difference of H and H_0, which may be not well-defined if the operators are defined via quadratic forms. Here

we present a version of the resolvent identity that works also in the quadratic form case. The proof may be found in Section 1.9 of [Ya].
We work in a general setting: Let $\mathfrak{H}$ be a Hilbert space and H, H_0 be self-adjoint operators satisfying

$$\mathcal{D}(|H|^{1/2}) = \mathcal{D}(|H_0|^{1/2}). \tag{3.3}$$

We note that H and H_0 are not assumed to be semibounded (but they will be so in our application). Denote their resolvents by

$$R_0(z) := (H_0 - zI)^{-1},\ z \in \rho(H_0), \qquad R(z) := (H - zI)^{-1},\ z \in \rho(H).$$

Suppose that there is an "auxiliary" Hilbert space $\mathfrak{G}$ and operators

$$G_0 : \mathfrak{H} \supset \mathcal{D}(G_0) \to \mathfrak{G}, \qquad G : \mathfrak{H} \supset \mathcal{D}(G) \to \mathfrak{G},$$

such that the following is true.

(H1) The operators G_0, G are $|H_0|^{1/2}$-bounded, i.e.,

$$\begin{array}{ll} \mathcal{D}(|H_0|^{1/2}) \subset \mathcal{D}(G_0), & \mathcal{D}(|H_0|^{1/2}) \subset \mathcal{D}(G), \\ G_0(|H_0|^{1/2} + I)^{-1} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{G}), & G(|H_0|^{1/2} + I)^{-1} \in \mathfrak{B}(\mathfrak{H}, \mathfrak{G}). \end{array}$$

(H2) The operators G_0, G satisfy

$$(G_0 f, Gg) = (Gf, G_0 g), \qquad f,\ g \in \mathcal{D}(|H_0|^{1/2}).$$

(H3) The relation $H = H_0 + G^* G_0$ holds in the sense of forms, i.e.,

$$(Hf, f_0) = (f, H_0 f_0) + (Gf, G_0 f_0), \qquad f_0 \in \mathcal{D}(H_0),\ f \in \mathcal{D}(H).$$

The assumption (H1) guarantees that the operators $GR_0(z) : \mathfrak{H} \to \mathfrak{G}$ and $G_0(GR_0(\overline{z}))^* : \mathfrak{G} \to \mathfrak{G}$ are well-defined and bounded for $z \in \rho(H_0)$. With slight abuse of notation we put

$$R_0(z)G^* := (GR_0(\overline{z}))^*, \qquad G_0R_0(z)G^* := G_0(GR_0(\overline{z}))^*.$$

Proposition 3.1. *Let H_0, H be self-adjoint operators satisfying* (3.3) *and assume that the operators G_0, G satisfy* (H1)-(H3)*. Let $z \in \rho(H_0)$, then $z \in \rho(H)$ iff $I + G_0R_0(z)G^*$ is boundedly invertible, and in this case*

$$R(z) - R_0(z) = -R_0(z)G^* \left(I + G_0R_0(z)G^*\right)^{-1} G_0R_0(z). \tag{3.4}$$

3.3 Some auxiliary operators

For $k \in [-\frac{1}{2}, \frac{1}{2}]$, $z \in \mathbb{C} \setminus [k^2, +\infty)$ we consider in $L_2(\mathbb{T})$ the operator

$$\begin{aligned} &\mathcal{D}(B(z,k)) := H^1(\mathbb{T}), \\ &(B(z,k)f)(x_1) := \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \beta_n(z,k)\, \hat{f}_n\, e^{inx_1}, \qquad x_1 \in \mathbb{T}, \end{aligned} \tag{3.5}$$

with $\beta_n(z,k)$ defined in (3.2). The operator $B(z,k)$ is invertible for $z \in \mathbb{C} \setminus [k^2, +\infty)$ and its square root is well-defined (it may be considered as the square root of an m-accretive operator). Now, let σ be a periodic function satisfying (1.2) and define operators on $L_2(\mathbb{T})$ by

$$T_0^{(\sigma)}(z,k) := (\operatorname{sgn}\sigma)|\sigma|^{1/2}B(z,k)^{-1/2}, \quad T^{(\sigma)}(z,k) := |\sigma|^{1/2}B(z,k)^{-1/2}.$$

It follows from Corollary 1.6 that these are compact operators of class $\Sigma_2(L_2(\mathbb{T}))$.
Finally, for $z \in \mathbb{C} \setminus [k^2, +\infty)$ we consider the integral operator $Y(z,k)$ acting from $L_2(\Pi)$ to $L_2(\mathbb{T})$ whose kernel is given by

$$Y(x_1, y; z, k) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{\beta_n(z,k)}} e^{in(x_1 - y_1)} e^{-\beta_n(z,k)\, y_2}, \quad x_1 \in \mathbb{T},\, y \in \Pi. \tag{3.6}$$

Writing down the singular value expansion explicitly we find that $Y(z,k)$ is a compact operator of class $\Sigma_1(L_2(\Pi), L_2(\mathbb{T}))$.

3.4 The resolvent difference

We are now ready to apply the general results from Subsection 3.2 to our situation. Denote the resolvent of the operator $H^{(\sigma)}(k)$ by

$$R^{(\sigma)}(z,k) := \left(H^{(\sigma)}(k) - zI\right)^{-1}, \qquad z \in \rho\left(H^{(\sigma)}(k)\right).$$

The following statement is of crucial importance.

Proposition 3.2. *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$ and $z \in \mathbb{C} \setminus [k^2, +\infty)$. Then $z \in \rho\left(H^{(\sigma)}(k)\right)$ iff the operator $I + T_0^{(\sigma)}(z,k) T^{(\sigma)}(\overline{z},k)^*$ is boundedly invertible, and in this case*

$$\begin{aligned} &R^{(\sigma)}(z,k) - R^{(0)}(z,k) = \\ &\quad = -Y(\overline{z},k)^*\, T^{(\sigma)}(\overline{z},k)^* \left(I + T_0^{(\sigma)}(z,k)\, T^{(\sigma)}(\overline{z},k)^*\right)^{-1} T_0^{(\sigma)}(z,k)\, Y(z,k). \end{aligned} \tag{3.7}$$

Proof. We want to apply the results of Subsection 3.2 to the case $\mathfrak{H} = L_2(\Pi)$, $\mathfrak{G} = L_2(\mathbb{T})$, $H_0 = H^{(0)}(k)$, $H = H^{(\sigma)}(k)$ and

$$\begin{aligned} \mathcal{D}(G_0) = \mathcal{D}(G) &:= \tilde{H}^1(\Pi), \\ (G_0 u)(x_1) &:= (\operatorname{sgn}\sigma(x_1))\sqrt{|\sigma(x_1)|}\, u(x_1, 0), \qquad x_1 \in \mathbb{T}, \\ (Gu)(x_1) &:= \sqrt{|\sigma(x_1)|}\, u(x_1, 0), \qquad x_1 \in \mathbb{T}. \end{aligned}$$

According to Lemma 1.3, the operators G_0, G are well-defined and bounded from $\tilde{H}^1(\Pi)$ to $L_2(\mathbb{T})$. Since $(|H_0|^{1/2} + I)^{-1}$ is a bounded operator from $\mathfrak{H}$ to $\tilde{H}^1(\Pi)$, the assumption (H1) in Subsection 3.2 holds. The remaining (3.3),

(H2) and (H3) are obvious, so that the statement of Proposition 3.1 holds. Let us determine the products $G_0R_0(z)$, $GR_0(z)$ and $G_0R_0(z)G^*$.
The explicit form (3.1) of the free resolvent implies that for $u \in L_2(\Pi)$

$$(R_0(z)u)(x_1,0) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{1}{\beta_n(z,k)} \int_\Pi u(y)\, e^{in(x_1-y_1)}\, e^{-\beta_n(z,k)\,y_2}\, dy = \\ = \Big(B(z,k)^{-1/2}\, Y(z,k)u\Big)(x_1), \qquad x_1 \in \mathbb{T},$$

and hence

$$G_0R_0(z) = T_0^{(\sigma)}(z,k)\,Y(z,k), \qquad GR_0(z) = T^{(\sigma)}(z,k)\,Y(z,k).$$

Moreover, for $f \in L_2(\mathbb{T})$ we have

$$(Y(\overline{z},k)^* f)(x) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \frac{\hat{f}_n}{\sqrt{\beta_n(z,k)}}\, e^{inx_1}\, e^{-\beta_n(z,k)\;x_2}, \quad x \in \Pi,$$

so that $Y(\overline{z},k)^* f \in \tilde{H}^1(\Pi)$ and $(Y(\overline{z},k)^* f)(.,0) = B(z,k)^{-1/2} f$. It follows that $G_0Y(\overline{z},k)^* = T_0^{(\sigma)}(z,k)$ and

$$G_0R_0(z)G^* = G_0\left(GR_0(\overline{z})\right)^* = T_0^{(\sigma)}(z,k)\,T^{(\sigma)}(\overline{z},k)^*.$$

This concludes the proof of the Proposition. □

As an easy consequence of (3.7) we obtain

COROLLARY 3.3. *Let $k \in [-\frac{1}{2},\frac{1}{2}]$ and $z \in \rho\left(H^{(\sigma)}(k)\right)$, then*

$$R^{(\sigma)}(z,k) - R^{(0)}(z,k) \in \Sigma_{1/3}(L_2(\Pi)).$$

Proof. We have remarked in Subsection 3.3 that $T_0^{(\sigma)}(z,k), T^{(\sigma)}(z,k) \in \Sigma_2(L_2(\mathbb{T}))$ and $Y(z,k) \in \Sigma_1(L_2(\Pi), L_2(\mathbb{T}))$, so the statement follows from (1.3). □

The proof of Theorem 2.1 is now immediate.

Proof of Theorem 2.1. Combine Corollary 3.3 with Proposition 1.1. □

3.5 THE LIMITING ABSORPTION PRINCIPLE FOR THE UNPERTURBED OPERATOR

It remains to prove Theorem 2.2, the absence of singular continuous spectrum. This will be achieved by controlling the behavior of the resolvent $R^{(\sigma)}(z,k)$ as the spectral parameter z tends to the real axis. We start with the unperturbed case $\sigma \equiv 0$.

We introduce the function $\Lambda_2(x) := (1+x_2^2)^{1/2}$, $x \in \Pi$, and note that for $k \in [-\frac{1}{2}, \frac{1}{2}]$, $z \in \mathbb{C} \setminus [k^2, +\infty)$ and $s > \frac{1}{2}$ the operator

$$\Lambda_2^{-s} R^{(0)}(z,k) \Lambda_2^{-s}$$

belongs to the Hilbert-Schmidt class. The following result is called the *limiting absorption principle* for the operator $H^{(0)}(k)$.

Proposition 3.4. *Let $s > \frac{1}{2}$ and $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the limits*

$$\lim_{\epsilon \to 0+} \Lambda_2^{-s} R^{(0)}(\lambda \pm i\epsilon, k) \Lambda_2^{-s}, \qquad \lambda \neq (n+k)^2,\, n \in \mathbb{Z},$$

exist in the Hilbert-Schmidt norm and are uniform for λ from compact intervals of $\mathbb{R} \setminus \{(n+k)^2 :\, n \in \mathbb{Z}\}$.

Proof. Fix $\lambda \in \mathbb{R} \setminus \{(n+k)^2 :\, n \in \mathbb{Z}\}$, and define

$$r^{(0)}(x,y;\lambda \pm i0, k) := \lim_{\epsilon \to 0+} r^{(0)}(x,y;\lambda \pm i\epsilon, k), \qquad x \neq y \in \Pi,$$

as pointwise limit using formula (3.1). We have to prove that

$$\int_\Pi \int_\Pi \left| r^{(0)}(x,y;\lambda \pm i\epsilon,k) - r^{(0)}(x,y;\lambda \pm i0,k) \right|^2 \frac{dx}{(1+x_2^2)^s} \frac{dy}{(1+y_2^2)^s} \longrightarrow 0 \quad (\epsilon \to 0+). \tag{3.8}$$

We restrict ourselves to the "+"-case, the other being similar, and for simplicity of notation, we put

$$c_n(\epsilon) := \beta_n(\lambda + i\epsilon, k), \qquad c_n := \lim_{\epsilon \to 0+} c_n(\epsilon).$$

Using Parseval's identity and the triangle inequality we find

$$\begin{aligned}
&\int_\Pi \int_\Pi \left| r^{(0)}(x,y;\lambda \pm i\epsilon,k) - r^{(0)}(x,y;\lambda \pm i0,k) \right|^2 \frac{dx}{(1+x_2^2)^s} \frac{dy}{(1+y_2^2)^s} = \\
&\quad = \sum_{n \in \mathbb{Z}} \int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \int_0^\infty \frac{dy_2}{(1+y_2^2)^s} \\
&\qquad \left| \frac{e^{-c_n(\epsilon)(x_2+y_2)} + e^{-c_n(\epsilon)|x_2-y_2|}}{2c_n(\epsilon)} - \frac{e^{-c_n(x_2+y_2)} + e^{-c_n|x_2-y_2|}}{2c_n} \right|^2 \leq \\
&\quad \leq 2 \sum_{n \in \mathbb{Z}} \int_0^\infty \int_0^\infty \left(t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2) \right) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s},
\end{aligned}$$

where

$$t_{n,\epsilon}(a) := \left| \frac{e^{-c_n(\epsilon)|a|}}{2c_n(\epsilon)} - \frac{e^{-c_n|a|}}{2c_n} \right|^2, \qquad a \in \mathbb{R}.$$

For $(n+k)^2 < \lambda$ it follows from Lebesgue's theorem that

$$\sum_{(n+k)^2<\lambda} \int_0^\infty \int_0^\infty (t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2)) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s} \longrightarrow 0,$$
$$(\epsilon \to 0+).$$

Suppose now that $(n+k)^2 > \lambda$. To control the convergence of the $t_{n,\epsilon}$ in terms of n we need the elementary estimate

$$t_{n,\epsilon}(a) \le \epsilon^2 \frac{C}{|c_n|^6}, \qquad a \in \mathbb{R},$$

with a constant C independent of a, ϵ, λ and n. It follows that

$$\sum_{(n+k)^2>\lambda} \int_0^\infty \int_0^\infty (t_{n,\epsilon}(x_2+y_2) + t_{n,\epsilon}(x_2-y_2)) \frac{dx_2}{(1+x_2^2)^s} \frac{dy_2}{(1+y_2^2)^s} \le$$
$$\le 2C\epsilon^2 \left(\int_0^\infty \frac{dx_2}{(1+x_2^2)^s} \right)^2 \sum_{n\in\mathbb{Z}} \frac{1}{|c_n|^6}.$$

The RHS converges to 0 as $\epsilon \to 0+$, which completes the proof of (3.8).
Finally, we remark that the limit in (3.8) is uniform in λ for λ from a compact intervall not containing any of the points $(n+k)^2$, $n \in \mathbb{Z}$. This follows from the fact, that c_n depends continuously on λ. $\square$

3.6 The limiting absorption principle for the perturbed operator

Using the Analytic Fredholm Alternative and the resolvent identity (3.7) we derive the limiting absorption principle for the operator $H^{(\sigma)}(k)$ from Proposition 3.4.

Lemma 3.5. *Let $k \in [-\frac{1}{2}, \frac{1}{2}]$, then the operator families*

$$T_0^{(\sigma)}(z,k),\, T^{(\sigma)}(z,k), \qquad z \in \mathbb{C} \setminus [k^2, \infty),$$

can be extended norm-continuously to the cut from above and from below with the exception of the points $z = (n+k)^2$, $n \in \mathbb{Z}$. Denoting the (upper and lower) boundary values by

$$T_0^{(\sigma)}(\lambda \pm i0, k), \qquad T^{(\sigma)}(\lambda \pm i0, k), \qquad \lambda \ne (n+k)^2,$$

the sets

$$\mathcal{N}_\pm(k) := \{\lambda \in \mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\} : \mathcal{N}(I + T_0^{(\sigma)}(\lambda \pm i0, k)\, T^{(\sigma)}(\lambda \mp i0, k)^*) \ne \{0\}\}$$

are discrete in $\mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$.

Remark 3.6. One can show that the sets $\mathcal{N}_+(k)$ and $\mathcal{N}_-(k)$ coincide. Moreover, in the second part [FrSh] we will prove that the sets $\mathcal{N}_\pm(k)$ can accumulate at $+\infty$ only.

Proof. Let

$$\tilde{B}(z,k), \qquad z \in \mathbb{C} \setminus \{(n+k)^2 + iy :\ n \in \mathbb{Z},\ y \le 0\} =: D_+(k)$$

be the analytic family of operators given by the same formal expression (3.5) as the operators $B(z,k)$, but where we choose in the definition (3.2) the branch of the square root on $\mathbb{C} \setminus \{iy :\ y \ge 0\}$ which coincides with the canonical branch on the lower halfplane. In particular,

$$\tilde{B}(z,k) = B(z,k), \qquad z \in \mathbb{C}_+. \tag{3.9}$$

It follows from Corollary 1.6 that

$$\tilde{T}^{(\sigma)}(z,k) := |\sigma|^{1/2} \tilde{B}(z,k)^{-1/2}, \qquad z \in D_+(k),$$

is an analytic family of compact operators. Because of (3.9) it is a bounded analytic (and hence norm-continuous) extension of the family $T^{(\sigma)}(z,k)$ across the cut from above. We put

$$T^{(\sigma)}(\lambda + i0, k) := \tilde{T}^{(\sigma)}(\lambda, k), \qquad \lambda \ne (n+k)^2,\ n \in \mathbb{Z}.$$

The construction of the operators $T^{(\sigma)}(\lambda - i0, k)$ is similar, replacing $D_+(k)$ by $D_-(k) := \mathbb{C} \setminus \{(n+k)^2 + iy :\ n \in \mathbb{Z},\ y \ge 0\}$, and the statement about the operators $T_0^{(\sigma)}(z,k)$ follows by multiplying $T^{(\sigma)}(z,k)$ with $\operatorname{sgn}\sigma$.
Let us prove the statement about the sets $\mathcal{N}_\pm(k)$. It follows easily that $\|\tilde{T}^{(\sigma)}(z,k)\| < 1$ if $|\operatorname{Im} z|$ is large. Now the Analytic Fredholm Alternative (cf. Theorem VII.1.9 in [K]) applied to the operators $\tilde{T}_0^{(\sigma)}(z,k)\, \tilde{T}^{(\sigma)}(\overline{z},k)^*$ yields the discreteness of the sets $\mathcal{N}_\pm(k)$. This concludes the proof. □

Proposition 3.7. *Let $s > \frac{1}{2}$ and $k \in [-\frac{1}{2}, \frac{1}{2}]$. Then the limits*

$$\lim_{\epsilon \to 0+} \Lambda_2^{-s} R^{(\sigma)}(\lambda \pm i\epsilon, k) \Lambda_2^{-s}, \qquad \lambda \notin \{(n+k)^2 :\ n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k),$$

exist in the Hilbert-Schmidt norm and are uniform for λ from compact intervals of $\mathbb{R} \setminus \big(\{(n+k)^2 :\ n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k)\big)$.

Proof. We consider the resolvent identity (3.7). Because of Proposition 3.4 and Lemma 3.5 it suffices to prove that the limits

$$\lim_{\epsilon \to 0+} Y(\lambda \pm i\epsilon, k) \Lambda_2^{-s}, \qquad \lambda \ne (n+k)^2,\ n \in \mathbb{Z},$$

exist in the Hilbert-Schmidt norm and are uniform for λ from compact intervals of $\mathbb{R} \setminus \{(n+k)^2 : n \in \mathbb{Z}\}$.
Considering the integral kernel (3.6) of $Y(z,k)$ one can procede similar to the proof of Proposition 3.4. □

As an easy consequence of Proposition 3.7 we obtain now Theorem 2.2.

Proof of Theorem 2.2. Let $[a,b] \subset \mathbb{R} \setminus \big(\{(n+k)^2 :\ n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k)\big)$ be a compact interval. Then for every u from the dense set $\mathcal{R}(\Lambda_2^{-s})$ (with $s > \frac{1}{2}$ arbitrary) we have

$$\sup_{0<\epsilon<1} \int_a^b |(R^{(\sigma)}(\lambda \pm i\epsilon, k)u, u)|^2 \, d\lambda < \infty$$

by Proposition 3.7. It follows (cf. Proposition 1.5.2 in [Ya]) that the spectrum of $H^{(\sigma)}(k)$ is purely absolutely continuous on $[a,b]$. Therefore $\sigma_{sing}\left(H^{(\sigma)}(k)\right) \subset \{(n+k)^2 : n \in \mathbb{Z}\} \cup \mathcal{N}_\pm(k)$. Since the latter set accumulates only at $\{(n+k)^2 : n \in \mathbb{Z}\}$ and $+\infty$, we conclude $\sigma_{sc}\left(H^{(\sigma)}(k)\right) = \emptyset$, as claimed. □

REFERENCES

[BKaS] M. Sh. Birman, G. E. Karadzhov, M. Z. Solomyak, *Boundedness conditions and spectrum estimates for the operators b(X)a(D) and their analogs*, Estimates and asymptotics for discrete spectra of integral and differential equations, Adv. Soviet. Math., vol. 7, Amer. Math. Soc., 1991, 85-106.

[BKr] M. Sh. Birman, M. G. Kreĭn, *On the theory of wave operators and scattering operators*, Soviet. Math. Dokl. 3 (1962), 740-744.

[BShSu] M. Sh. Birman, R. G. Shterenberg, T. A. Suslina, *Absolute continuity of the spectrum of a two-dimensional Schrödinger operator with potential supported on a periodic system of curves*, St. Petersburg Math. J. 12, no. 6 (2001), 983-1012.

[BS] M. Sh. Birman, M. Z. Solomyak, *Spectral theory of self-adjoint operators in Hilbert space*, Reidel, 1987.

[DaSi] E. B. Davies, B. Simon, *Scattering Theory for Systems with Different Spatial Asymptotics on the Left and Right*, Commun. Math. Phys. 63 (1978), 277-301.

[FrSh] R. L. Frank, R. G. Shterenberg, *On the scattering theory of the Laplacian with a periodic boundary condition. II. Additional channels of scattering*, in preparation.

[K] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1976.

[ReSi4] M. Reed, B. Simon, *Methods of Modern Mathematical Physics*, vol. 4, Academic Press, 1978.

[Sa] A. W. Saenz, *Quantum-mechanical scattering by impenetrable periodic surfaces*, J. Math. Phys. 22, no. 12 (1981), 2872-2884.

[Ya] D. R. Yafaev, *Mathematical Scattering Theory*, Amer. Math. Soc., 1992.

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Separatedness in Constructive Topology

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Abstract. We discuss three natural, classically equivalent, Hausdorff separation properties for topological spaces in constructive mathematics. Using Brouwerian examples, we show that our results are the best possible in our constructive framework.

1 Introduction

A typical feature of constructive mathematics—that is, mathematics with intuitionistic logic [1, 2, 3, 4, 10]—is that a classical property may have several constructively inequivalent counterparts. In this paper we describe such counterparts of the notion of a Hausdorff space, examine their interconnections, and, by means of Brouwerian examples, show that our results cannot be improved without some additional, nonconstructive principles.

The original impetus for our work came from the constructive theory of apartness (point–set [5, 11] and set–set [9]). However, in order to make the work below accessible to anyone familiar with only the most basic notions of topology, we have chosen to work with the usual notion of topological space. Note, however, that we require a topological space to be equipped from the outset with an inequality relation $\neq$ satisfying the following properties for all x and y:

$$x \neq y \Rightarrow \neg\,(x = y)\,,$$
$$x \neq y \Rightarrow y \neq x.$$

We then denote the complement of a subset S of X by

$$\sim S = \{x \in X : \forall y \in S\,(x \neq y)\}\,.$$

On the other hand, the apartness complement of S defined to be

$$-S = (\sim S)^{\circ}\,.$$

If

$$\forall x, y \, (\neg \, (x \neq y) \Rightarrow x = y) \, ,$$

then we say that the inequality is TIGHT.

We note here, for future reference, that a topological space X is

- TOPOLOGICALLY COTRANSITIVE if

$$(x \in U \wedge U \in \tau) \Rightarrow \forall y \in X \, (x \neq y \vee y \in U) \, ;$$

- LOCALLY DECOMPOSABLE if

$$(x \in U \wedge U \in \tau) \Rightarrow \exists V \in \tau \, (x \in V \wedge X = U \cup \sim V) \, .$$

Note that local decomposability implies topological cotransitivity. In the constructive theory of point–set apartness spaces, topological cotransitivity is postulated, and local decomposability is an extremely valuable property to have [6, 7]. For example, local decomposability ensures that if an apartness relation is induced by a topology, then the natural topology produced by the apartness relation coincides with the original topology. So important is local decomposability that it is actually *postulated* as a property of an apartness relation between sets [6].

We will need some basic facts about nets in constructive topology. By a DIRECTED SET we mean a nonempty set D with a preorder[1] $\succcurlyeq$ such that for all $m, n \in D$ there exists $p \in D$ with $p \succcurlyeq m$ and $p \succcurlyeq n$. If (X, τ) is a topological space, then to each x in X there corresponds a special net defined as follows. Let

$$D_x = \{(\xi, U) \in X \times \tau : x \in U \wedge \xi \in U\} \, ,$$

with equality defined by

$$(\xi, U) = (\xi', U') \Leftrightarrow (\xi = \xi' \wedge U = U') \, ,$$

and for each $n = (\xi, U)$ in D_x define $x_n = \xi$. It is easy to see that D is a directed set under the INCLUSION PREORDER defined by

$$(\xi, U) \succcurlyeq (\xi', U') \Leftrightarrow U \subset U',$$

so that $\mathcal{N}_x = (x_n)_{n \in D_x}$ is a net—the BASIC NEIGHBOURHOOD NET of x.

We say that a net $(x_n)_{n \in D}$ in X CONVERGES to a LIMIT x in X if

$$\forall U \in \tau \; (x \in U \Rightarrow \exists \, n_0 \in D \; \forall n \succcurlyeq n_0 \; (x_n \in U)) \, .$$

[1]The classical theory of nets requires a partial order. If we used a partial order in our constructive theory, we would run into difficulties which the classical theory avoids by applications of the axiom of choice.

2 HAUSDORFF AND UNIQUE LIMIT PROPERTIES

Let (X, τ) be a topological space. More or less as in classical topology, we say that X is HAUSDORFF, or SEPARATED, if it satisfies the following condition:

H *If $x, y \in X$ and $x \neq y$, then there exist $U, V \in \tau$ such that $x \in U, y \in V$, and $U \subset\sim V$*

In that case, $V \subset\sim U$.
Classically, being Hausdorff is equivalent to having the UNIQUE LIMITS PROPERTY:

ULP *If $(x_n)_{n\in D}$ is a net converging to limits x and y in X, then $x = y$.*

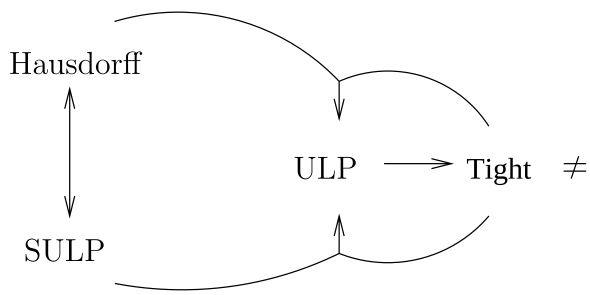
We say that a point y in X is EVENTUALLY BOUNDED AWAY FROM A NET $(x_n)_{n\in D}$ in X if there exists $n_0 \in D$ such that

$$y \in -\{x_n : n \succcurlyeq n_0\}.$$

From a constructive viewpoint, the unique limits property appears rather weak; of more likely interest is the (classically equivalent) STRONG UNIQUE LIMITS PROPERTY:

SULP *If $(x_n)_{n\in D}$ is a net in X that converges to a limit x, and if $x \neq y \in X$, then $(x_n)_{n\in D}$ is eventually bounded away from y.*

In this section we investigate constructively the connection between these two uniqueness properties and condition H. Specifically, we prove that the following diagram of implications occurs:



We first have an elementary, but useful, lemma.

LEMMA 1 *Let X be a topological space, x a point of X, and $\nu = (\xi, U) \in D_x$. Then*

$$U = \{x_n : n \in D_x, n \succcurlyeq \nu\}. \tag{1}$$

Proof. Consider $n \in D_x$. If $n = (x_n, W) \succcurlyeq \nu$, then $x_n \in W \subset U$. Hence $\{x_n : n \succcurlyeq \nu\} \subset U$. On the other hand, for each $y \in U$ we have $(y, U) \succcurlyeq \nu$, so $y \in \{x_n : n \succcurlyeq \nu\}$. ■

Proposition 2 *A topological space is Hausdorff if and only if it has the strong unique limits property.*

Proof. Let (X, τ) be a topological space. Assume first that X is Hausdorff, let $(x_n)_{n \in D}$ be a net converging to a limit x in X, and let $x \neq y$ in X. Choose $U, V \in \tau$ such that $x \in U, y \in V$, and $U \subset \sim V$. There exists n_0 such that $x_n \in U$ for all $n \succcurlyeq n_0$. Then

$$y \in V \subset \sim U \subset \sim \{x_n : n \succcurlyeq n_0\},$$

so

$$y \in (\sim \{x_n : n \succcurlyeq n_0\})^\circ = -\{x_n : n \succcurlyeq n_0\}.$$

Now suppose that X has the strong unique limits property, and let x, y be points of X with $x \neq y$. Since the net $\mathcal{N}_x$ converges to x, there exist $n_0 = (\xi, U) \in D_x$ and $V \in \tau$ such that

$$y \in V \subset \sim \{x_n : n \in D_x,\ n \succcurlyeq n_0\}.$$

By Lemma 1,

$$U = \{x_n : n \in D_x,\ n \succcurlyeq n_0\}.$$

Thus $x \in U,\ y \in V$, and $V \subset \sim U$; so X is Hausdorff. ■

Corollary 3 *A Hausdorff space with tight inequality has the unique limits property.*

Proof. Let $(x_n)_{n \in D}$ be a net converging to limits x, y in a Hausdorff space X with tight inequality. If $x \neq y$, then we obtain a contradiction from Proposition 2. Hence $\neg (x \neq y)$, and so, by tightness, $x = y$. ■

By a T_1–space we mean a topological space (X, τ) with the property

$$x \neq y \Rightarrow \exists U \in \tau \, (x \in U \subset \sim \{y\}).$$

The following lemma enables us to prove a partial converse to Corollary 3.

Lemma 4 *Let X be a topological space. If X is a T_1–space with tight inequality, then*

$$\forall x, y \in X \left(y \in \overline{\{x\}} \Rightarrow x = y \right). \tag{2}$$

Conversely, if (2) holds and X is topologically cotransitive, then the inequality on X is tight.

Proof. Suppose that X is a T_1–space with tight inequality, let $y \in \overline{\{x\}}$, and assume that $x \neq y$. Then there exists $U \in \tau$ such that $x \in U \subset \sim\{y\}$; whence $y \notin \overline{\{x\}}$, which is absurd. Thus $\neg(x \neq y)$ and therefore, by tightness, $x = y$. Conversely, suppose that (2) holds and that X is topologically cotransitive. Let $\neg(x \neq y)$. For each $U \in \tau$ with $x \in U$, the topological cotransitivity of X implies that either $x \neq y$ or else $y \in U$; the former alternative is ruled out, so we must have $y \in U$. Hence $y \in \overline{\{x\}}$ and so, by (2), $x = y$. ■

Proposition 5 *In a topological space with the unique limits property the inequality is tight.*

Proof. Let X be a topological space with the unique limits property, and suppose that $y \in \overline{\{x\}}$. Then every open set containing x contains y. Let

$$L = \{(z, U, V) : U, V \in \tau,\ x \in U,\ y \in V,\ z \in U \cap V\},$$

where

$$(z, U, V) = (z', U', V') \Leftrightarrow z = z' \wedge U = U' \wedge V = V'.$$

Define a binary relation $\succcurlyeq$ on L by

$$(z_1, U_1, V_1) \succcurlyeq (z_2, U_2, V_2) \Longleftrightarrow U_1 \subset U_2.$$

It is easy to show that L is directed with respect to this binary relation. For each $n = (z, U, V)$ in L define $x_n = z$. Then the net $(x_n)_{n \in L}$ converges to both x and y in X; whence $x = y$. It follows from Lemma 4 that the inequality on X is tight. ■

3 Limiting examples

In this section we show that the connections (summarised in the diagram presented earlier) we have established between the Hausdorff condition, the unique limits property, and the strong unique limits property are the best possible within our constructive framework. We begin by showing that Hausdorff is not enough to establish tightness.

Proposition 6 *If every topologically cotransitive topological space with the unique limits property has tight inequality, then the law of excluded middle holds in the weak form* $(\neg\neg P \Rightarrow P)$.

Proof. Let P be any syntactically correct statement such that $\neg\neg P$ holds, and take $X = \{0, 1, 2\}$ with equality satisfying

$$0 = 1 \Leftrightarrow P$$

and inequality given by

$$0 \neq 2,\ 1 \neq 2, \text{ and } (0 \neq 1 \Leftrightarrow \neg P)\,.$$

Define a topology τ on X by taking the basic open sets to be the complements of subsets of X. To see that X has the topological cotransitivity property, consider all possible cases that arise when $x \in U$ and $U \in \tau$. We may assume that $U = {\sim}S$ for some $S \subset X$. If $x = 0$, then $0 \in {\sim}S$. It follows that $S \subset \{2\}$: for if $s \in S$, then either $s = 1$ or $s = 2$; in the former case, $0 \neq 1$ and therefore $\neg P$, which contradicts our hypotheses. Since $1 \neq 2$, we have $1 \in {\sim}S$; since also $0 \neq 2$, we conclude that

$$\forall y \in X \, (0 \neq y \vee y \in {\sim}S = U) \, .$$

The case $x = 1$ is similar, and the case $x = 2$ is even easier to handle.
We claim that X is a Hausdorff apartness space. If $x \neq y$, then without loss of generality, either $x = 0$ and $y = 2$ or else $x = 1$ and $y = 2$. Taking, for illustration, the former case, we have $0 \in \{0, 1\} = \sim \{2\}$, $2 \in \{2\} = \sim \{0\}$, and $\sim \{2\} = \sim\sim \{0\}$ .Thus there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \subset {\sim}V$.
Finally, if the inequality on X is tight, then as $\neg \, (0 \neq 1)$, we have $0 = 1$ and therefore P. ■

Our final proposition shows that, even when the inequality is tight and certain additional hypotheses hold, the strong unique limits property does not entail being Hausdorff. For the proof we introduce a strange lemma and a general construction. The lemma may seem obvious, but in fact we have to be careful to avoid the axiom of choice, which implies the law of excluded middle [8].

Lemma 7 *Let $\mathcal{C}$ be a class of subsets of a set X, and let $(S_i)_{i \in I}$ be a family of subsets of X such that for each i, if $S_i \neq \emptyset$, then S_i is a union of sets in $\mathcal{C}$. If $S = \bigcup_{i \in I} S_i \neq \emptyset$, then S is also a union of sets in $\mathcal{C}$.*

Proof. For each $x \in X$ define

$$I_x = \{i \in I : x \in S_i\} \, .$$

Then

$$S = \bigcup_{x \in X} \bigcup_{i \in I} S_i.$$

If $x \in S$ and $i \in I_x$, then $S_i \neq \emptyset$ and so is a union of sets in $\mathcal{C}$. Hence S itself is such a union. ■

Let X be a set with a nontrivial inequality $\neq$. We say that a subset S of X is

- finitely enumerable (respectively, finite) if there exist a natural number n and a mapping (respectively, one–one mapping) f of $\{1, \ldots, n\}$ onto S;

- cofinite if it is the complement of a finitely enumerable subset.

Note that the empty set is finitely enumerable, so X is cofinite. Also, if S is finitely enumerable, then either $S = \emptyset$ or else $S \neq \emptyset$ (that is, there exists an element of S).

We define the COFINITE TOPOLOGY on X to be

$$\tau = \{S \subset X : S \neq \emptyset \Rightarrow S \text{ is a union of cofinite sets}\} .$$

To see that this is a topology, note that (as above) $X \in \tau$ and, by *ex falso quodlibet*, $\emptyset \in \tau$. The unions axiom for a topology is an immediate consequence of Lemma 7 with $\mathcal{C} = \tau$. To verify the intersections axiom, let $(S_i)_{i\in I}$ and $(T_j)_{j\in J}$ be families of sets in τ. For all i, j choose finitely enumerable subsets A_i, B_j of X such that $S_i = \sim A_i$ and $T_j = \sim B_j$. Then

$$\begin{aligned}\left(\bigcup\nolimits_{i\in I} S_i\right) \cap \left(\bigcup\nolimits_{j\in J} T_j\right) &= \left(\bigcup\nolimits_{i\in I} \sim A_i\right) \cap \left(\bigcup\nolimits_{j\in J} \sim B_j\right) \\ &= \bigcup\nolimits_{i\in I}\bigcup\nolimits_{j\in J} (\sim A_i \cap \sim B_j) \\ &= \bigcup\nolimits_{i\in I}\bigcup\nolimits_{j\in J} \sim (A_i \cup B_j) ,\end{aligned}$$

where each $A_i \cup B_j$ is cofinite.

We now recall MARKOV'S PRINCIPLE,

> *For every binary sequence* $(a_n)_{n=1}^{\infty}$ *such that* $\neg\forall n\,(a_n = 1)$, *there exists* n *such that* $a_n = 0$,

a form of unbounded search that is well–known to be independent of Heyting arithmetic (Peano arithmetic with intuitionistic logic) and is therefore generally regarded as essentially nonconstructive.

PROPOSITION 8 *If every locally decomposable* T_1*–space with the unique limits property and tight inequality is Hausdorff, then Markov's Principle holds.*

PROOF. We take a specific case of the foregoing construction. Let $(a_n)_{n=1}^{\infty}$ be a decreasing binary sequence such that $a_1 = 1$ and $\neg\forall n\,(a_n = 1)$. Take

$$X = \{0\} \cup \left\{\frac{a_n}{n} : n = 1, 2, 3, \ldots\right\}$$

with the discrete inequality, and let τ be the cofinite topology on X. To show that X is a T_1–space, let $x \neq y$ in X. Either one of x, y is 0 or else both are nonzero. If, for example, $x = 0$, then $y = 1/n$ for some n with $a_n = 1$. Writing

$$U = \{0\} \cup \left\{\frac{a_k}{k} : k > n\right\} = \sim\left\{\frac{a_k}{k} : k \leq n\right\},$$

we see that

$$U \in \tau \text{ and } x \in U \subset \sim\{y\} . \tag{3}$$

So we are left with the case where $x = 1/m$ and $y = 1/n$, with $a_m = a_n = 1$. In this case, without loss of generality taking $m > n$, we obtain (3) by defining

$$U = \{0\} \cup \left\{ \frac{a_k}{k} : k \geq m \right\}.$$

To show that X is locally decomposable, again consider $x \in X$ and $U \in \tau$ with $x \in U$. We may assume that $U = \sim A$ for some finitely enumerable set $A \subset X$; without loss of generality, $A \neq \emptyset$. Consider first the case $x = 0$. Let

$$K = \max \left\{ k : \frac{1}{k} \in A \right\}$$

and

$$V = \{0\} \cup \left\{ \frac{a_k}{k} : k > K \right\} = \sim \left\{ \frac{a_k}{k} : k \leq K \right\}.$$

Then V is a neighbourhood of 0. For each $y \in X$, either $y = 0 \in U$ or else $y = 1/k$ for some k with $a_k = 1$. In the latter case, if $k > K$, then $y \in \sim A = U$; whereas if $k \leq K$, then $y \in \sim V$. This deals with the case $x = 0$. Now consider the case where $x = 1/m$ for some m with $a_m = 1$. Again let $U = \sim A$ be an open neighbourhood of x, where A is finitely enumerable. If $a_{m+1} = 0$, then X is finite and hence locally decomposable; so we may assume that $a_{m+1} = 1$. Without loss of generality we may further assume that $1/(m+1) \in A$. Thus

$$L = \max \left\{ n : \frac{1}{n} \in A \right\} > m.$$

Set

$$V = \left\{ \frac{a_k}{k} : (k > L \wedge a_k = 1) \vee k = m \right\} = \sim \left(\{0\} \cup \left\{ \frac{a_k}{k} : k \leq L, k \neq m \right\} \right).$$

Then V is a neighbourhood of x. For each $y \in X$, either $y = 0$ and hence $y \in \sim V$, or else $y = a_k/k$ for some k with $a_k = 1$. If $k > L$, then $y \in \sim A = U$; if $k = m$, then $y \in U$; if $k \leq L$ and $k \neq m$, then $y \in \sim V$.

Next, we prove that X has the unique limits property. To this end, suppose that $(x_n)_{n \in D}$ is a net in X that converges to both x and y. Suppose also that $x \neq y$. For each n, if $a_n = 0$, then X is finitely enumerable and so has the unique limits property; whence $x = y$, a contradiction. Thus $a_n = 1$ for all n, which is also a contradiction. We conclude that $\neg(x \neq y)$; since we are dealing with a discrete inequality, it follows that $x = y$.

Finally, noting that $0 \neq 1$, suppose there exist U, V in τ such that $0 \in U, 1 \in V$, and $U \cap V = \emptyset$. There exist finitely enumerable sets $A, B \subset X$ such that $0 \in \sim A \subset U$ and $1 \in \sim B \subset V$. Let

$$N = \max \left\{ n : \frac{1}{n} \in A \cup B \right\}.$$

Then

$$\left\{ \frac{a_k}{k} : k > N \right\} \subset \sim A \cap \sim B \subset U \cap V.$$

If $a_{N+1} = 1$, then $U \cap V \neq \emptyset$, a contradiction. Hence $a_{N+1} = 0$. ■

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References

[1] Errett Bishop, *Foundations of Constructive Analysis*, McGraw–Hill, New York, 1967.

[2] Errett Bishop and Douglas Bridges, *Constructive Analysis*, Grundlehren der Math. Wissenschaften 279, Springer–Verlag, Heidelberg–Berlin–New York, 1985.

[3] Douglas Bridges, 'Constructive mathematics: a foundation for computable analysis', Theoretical Computer Science 219 (1–2), 95-109, 1999.

[4] Douglas Bridges and Fred Richman, *Varieties of Constructive Mathematics*, London Math. Soc. Lecture Notes 97, Cambridge Univ. Press, 1987.

[5] Douglas Bridges and Luminiţa Vîţă, 'Apartness spaces as a foundation for constructive topology', to appear in Ann. Pure Appl. Logic.

[6] Douglas Bridges and Luminiţa Vîţă, *Foundations of Constructive Topology*, preprint, University of Canterbury, Christchurch, New Zealand, 2002.

[7] Douglas Bridges, Peter Schuster, and Luminiţa Vîţă, 'Apartness, topology, and uniformity: a constructive view', to appear in *Computability and Complexity in Analysis* (Proc. Dagstuhl Seminar 01461, 11–16 November 2001), special issue of Math. Logic Quarterly, 2002.

[8] N. D. Goodman and J. Myhill, "Choice Implies Excluded Middle", Zeit. Logik und Grundlagen der Math. 24, 461.

[9] Peter Schuster, Luminiţa Vîţă, and Douglas Bridges, 'Apartness as a relation between subsets', in: *Combinatorics, Computability and Logic* (Proceedings of DMTCS'01, Constanţa, Romania, 2–6 July 2001; C.S. Calude, M.J. Dinneen, S. Sburlan, eds.), 203–214, DMTCS Series 17, Springer–Verlag, London, 2001.

[10] A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics: An Introduction* (two volumes), North Holland, Amsterdam, 1988.

[11] Luminiţa Vîţă and Douglas Bridges, 'A constructive theory of point–set nearness', to appear in *Topology in Computer Science: Constructivity; Asymmetry and Partiality; Digitization* (Proc. Dagstuhl Seminar 00231, 4-9 June 2000); R. Kopperman, M. Smyth, D. Spreen, eds.), special issue of Theoretical Computer Science.

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Curvature Properties of the Calabi-Yau Moduli

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Abstract. A curvature formula for the Weil-Petersson metric on the Calabi-Yau moduli spaces is given. Its relations to the Hodge metrics and the Bryant-Griffiths cubic form are obtained in the threefold case. Asymptotic behavior of the curvature near the boundary of moduli is also discussed via the theory of variations of Hodge structures.

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Introduction

In a former paper [15], the incompleteness phenomenon of the Weil-Petersson metric on Calabi-Yau moduli spaces was studied. In this note, I shall discuss some curvature properties of it. The first result is a simple explicit formula (Theorem 2.1) for the Riemann curvature tensor. While this problem was treated before in [6] and [10], the approach taken here is more elementary. Two simple proofs of Theorem 2.1 are offered in §2 and both are based on the Hodge-theoretic description of the Weil-Petersson metric [13]. The first one uses a trick to select suitable coordinate system and line bundle section to reduce the computation. The second proof uses Griffiths' curvature formula for Hodge bundles [2].

Direct consequences of Theorem 2.1 are relations between the Weil-Petersson metric and the Hodge metric for Calabi-Yau threefolds and various positivity results on the curvature tensor (see §3). §4 is devoted to the asymptotic analysis of the curvature near the boundary of moduli spaces. The method is modelled on the first proof and uses Schmid's theory on the degenerations of Hodge structures [7]. The final section §5 contains some remarks toward the completion and compactification problems of Calabi-Yau moduli spaces.

1. The Weil-Petersson metric

A Calabi-Yau manifold is a compact Kähler manifold with trivial canonical bundle. The local Kuranish family of polarized Calabi-Yau manifolds $\mathcal{X} \to S$ is smooth (unobstructed) by the Bogomolov-Tian-Todorov theorem [13]. One can assign the unique (Ricci-flat) Yau metric $g(s)$ on $\mathcal{X}_s$ in the polarization Kähler class [17]. Then, on a fiber $\mathcal{X}_s =: X$, the Kodaira-Spencer theory gives rise to an injective map $\rho : T_s(S) \to H^1(X, T_X) \cong \mathcal{H}^{0,1}_{\bar\partial}(T_X)$ (harmonic representatives). The metric $g(s)$ induces a metric on $\Lambda^{0,1}(T_X)$. For v, $w \in T_s(S)$, one then defines the *Weil-Petersson metric* on S by

$$g_{WP}(v,w) := \int_X \langle \rho(v), \rho(w)\rangle_{g(s)}. \tag{1.1}$$

Let $\dim X = n$. Using the fact that the global holomorphic n-form $\Omega(s)$ is flat with respect to $g(s)$, it can be shown [13] that

$$g_{WP}(v,w) = -\frac{\tilde{Q}(i(v)\Omega, \overline{i(w)\Omega})}{\tilde{Q}(\Omega, \bar\Omega)}. \tag{1.2}$$

Here, for convenience, we write $\tilde{Q} = \sqrt{-1}^n Q(\cdot,\cdot)$, where Q is the intersection product. Therefore, $\tilde{Q}$ has alternating signs in the successive primitive cohomology groups $P^{p,q} \subset H^{p,q}$, $p+q=n$.
(1.2) implies that the natural map $H^1(X, T_X) \to \mathrm{Hom}(H^{n,0}, H^{n-1,1})$ via the interior product $v \mapsto i(v)\Omega$ is an isometry from the tangent space $T_s(S)$ to $(H^{n,0})^* \otimes P^{n-1,1}$. So the Weil-Petersson metric is precisely the metric induced from the first piece of the Hodge metric on the horizontal tangent bundle over the period domain. A simple calculation in formal Hodge theory shows that

$$\omega_{WP} = \mathrm{Ric}_{\tilde{Q}}(\mathcal{H}^{n,0}) = -\partial\bar\partial \log \tilde{Q}(\Omega, \bar\Omega), \tag{1.3}$$

where ω_{WP} is the 2-form associated to g_{WP}. In particular, g_{WP} is Kähler and is independent of the choice of Ω. In fact, g_{WP} is also independent of the choice of the polarization.
With this background, one can abstract the discussion by considering a polarized variations of Hodge structures $\mathcal{H} \to S$ of weight n with $h^{n,0} = 1$ and a smooth base S. In this note, I always assume that it is effectively parametrized in the sense that the infinitesimal period map (also called the second fundamental form [2])

$$\sigma : T_s(S) \to \mathrm{Hom}(H^{n,0}, H^{n-1,1}) \oplus \mathrm{Hom}(H^{n-1,1}, H^{n-2,2}) \oplus \cdots \tag{1.4}$$

is bijective in the first piece. Then the *Weil-Petersson metric* g_{WP} on S is defined by formula (1.2) (or equivalently, (1.3)).
One advantage to work with the abstract setting is that, instead of using $P^{p,q}$ in the geometric case, we may write $H^{p,q}$ directly in our presentation.

2. The Riemann curvature tensor formula

Here is the basic formula (compare with [6], [10] and [12]):

Theorem 2.1. *For a given effectively parametrized polarized variations of Hodge structures $\mathcal{H} \to S$ of weight n with $h^{n,0} = 1$, $h^{n-1,1} = d$ and smooth S, in terms of any holomorphic section Ω of $\mathcal{H}^{n,0}$ and the infinitesimal period map σ, the Riemann curvature tensor of the Weil-Petersson metric $g_{WP} = g_{i\bar{j}} dt_i \otimes d\bar{t_j}$ on S is given by*

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}}) + \frac{\tilde{Q}(\sigma_i \sigma_k \Omega, \overline{\sigma_j \sigma_\ell \Omega})}{\tilde{Q}(\Omega, \bar{\Omega})}. \tag{2.1}$$

2.1. The first proof. The main trick in the proof is a nice choice of the holomorphic section Ω and special coordinate system on the base S. Since the problem is local, we may assume that S is a disk in $\mathbb{C}^d$ around $t = 0$. Specifically, we have

Lemma 2.2. *For any $k \in \mathbb{N}$, there is a local holomorphic section Ω of $\mathcal{H}^{n,0}$ such that in the power series expansion at $t = 0$*

$$\Omega(t) = a_0 + \sum\nolimits_i a_i t_i + \cdots + \sum\nolimits_{|I|=k} \frac{1}{I!} a_I t^I + \cdots \tag{2.2}$$

we have $a_0 \in H^{n,0}$, $\tilde{Q}(a_0, \bar{a_0}) = 1$ and $\tilde{Q}(a_0, \bar{a_I}) = 0$ for any multi-index $I \neq 0$ and $|I| \leq k$. (We always assume that $a_I = a_J$ if $I = J$ as unordered sets.)

Proof. Only the last statement needs a proof. Let

$$\begin{aligned} \tilde{\Omega} &= \sum\nolimits_I \tilde{a}_I t^I = \Big(1 + \sum\nolimits_i \lambda_i t_i + \cdots + \sum\nolimits_{|I|=k} \lambda_I t^I\Big)\Omega \\ &= a_0 + \sum\nolimits_i (\lambda_i a_0 + a_i) t_i + \cdots + \sum\nolimits_{|I|=k} (\lambda_I a_0 + a_I) t^I + \cdots \end{aligned}$$

Set $\lambda_I = -\tilde{Q}(a_0, \bar{a_I})$, then clearly $\tilde{Q}(\tilde{a}_0, \overline{\tilde{a}_I}) = 0$ for $I \neq 0$ and $|I| \leq k$. □

Lemma 2.3. *Pick Ω as in Lemma 2.2. For any $k' \in \mathbb{N}$ with $2 \leq k' \leq k$, there is a holomorphic coordinate system t such that a_i form an orthonormal basis of $H^{n-1,1}$, i.e. $\tilde{Q}(a_i, \bar{a_i}) = -\delta_{ij}$. Moreover, $\tilde{Q}(a_i, \bar{a_I}) = 0$ for all i and I with $2 \leq |I| \leq k'$.*

Proof. The Griffiths transversality says that

$$a_i = \frac{\partial}{\partial t_i} \Omega \Big|_{t=0} \in H^{n,0} \oplus H^{n-1,1}.$$

Lemma 2.2 then implies that $a_i \in H^{n-1,1}$. It is also clear that by a linear change of coordinates of t we can make a_i to form an orthonormal basis of $H^{n-1,1}$.

For the second statement, consider the following coordinates transformation:

$$t_i = s_i + \sum\nolimits_{1 \leq j,k \leq d} c_i^{jk} s_j s_k + \cdots + \sum\nolimits_{|I|=k'} c_i^I s^I, \quad 1 \leq i \leq d$$

with $c_i^I = c_i^J$ when $I = J$ as unordered sets.

It's easy to see that the number of coefficients to be determined is the same as the number of equations $\tilde{Q}(a_i, \bar{a_{jk}}) = 0$, hence the lemma. □

Proof. (of Theorem 2.1) Let Ω and t_i be as in the above lemmas. For multi-indices I and J, we set $q_{I,J} := \tilde{Q}(a_I, \bar{a_J})$. By Lemma 2.2 and 2.3,

$$\begin{aligned} q(t) &:= \tilde{Q}(\Omega(t), \overline{\Omega(t)}) \\ &= 1 - \sum_i t_i \bar{t_i} + \cdots + \sum_{i,j,k,\ell} \frac{1}{(ik)!(j\ell)!} q_{ik,j\ell}\, t_i t_k \bar{t_j} \bar{t_\ell} + O(t^5). \end{aligned}$$

To calculate $R_{i\bar{j}k\bar{\ell}}$, we only need to calculate $g_{k\bar{\ell}}$ up to degree 2 terms:

$$\begin{aligned} g_{k\bar{\ell}} &= -\partial_k \partial_{\bar{\ell}} \log q = q^{-2}(\partial_k q \partial_{\bar{\ell}} q - q \partial_k \partial_{\bar{\ell}} q) \\ &= \big(1 + 2\sum_i t_i \bar{t_i} + \cdots\big) \times \\ &\qquad \Big[t_\ell \bar{t_k} - \big(1 - \sum_i t_i \bar{t_i}\big)\big(- \delta_{k\ell} + \sum_{i,j} q_{ik,j\ell} t_i \bar{t_j}\big) + \cdots \Big] \\ &= \delta_{k\ell} - \delta_{k\ell} \sum_i t_i \bar{t_i} + t_\ell \bar{t_k} + 2\delta_{k\ell} \sum_i t_i \bar{t_i} - \sum_{i,j} q_{ik,j\ell} t_i \bar{t_j} + \cdots \\ &= \delta_{k\ell} + \delta_{k\ell} \sum_i t_i \bar{t_i} + t_\ell \bar{t_k} - \sum_{i,j} q_{ik,j\ell} t_i \bar{t_j} + \cdots . \end{aligned}$$

Here we have used the fact that degree 3 terms of mixed type (contain $t_k \bar{t_\ell}$) must be 0 by our choice of Ω.
As a result, we find that the Weil-Petersson metric g is already in its geodesic normal form, so the full curvature tensor at $t = 0$ is given by

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial t_i \partial \bar{t_j}} = -\delta_{ij}\delta_{k\ell} - \delta_{i\ell}\delta_{kj} + q_{ik,j\ell}.$$

Rewriting this in its tensor form then gives the formula. □

Remark 2.4. The proof does not require the full condition that $\mathcal{H} \to S$ is a variation of Hodge structures. The essential part used is the polarization structure on the indefinite metric $\tilde{Q}$ on $\mathcal{H}$. It has a fixed sign on $\mathcal{H}^{n-1,1}$ makes possible the definition of the Weil-Petersson metric. For such cases, in terms of the second fundamental form σ, Lemma 2.2 and 2.3 say that under this choice of Ω and t, ordinary differentiations approximate σ up to second order at $t = 0$. In particular, $a_0 = \Omega(0)$, $a_i = \sigma_i \Omega(0)$ and $a_{ij} = a_{ji} = \sigma_i \sigma_j \Omega(0) = \sigma_j \sigma_i \Omega(0)$.

2.2. The second proof.

Now we give another proof of Theorem 2.1 via Griffiths' curvature formula for Hodge bundles.

Proof. Recall the isometry in §1:

$$T_s(S) \cong (H^{n,0})^* \otimes H^{n-1,1} \tag{2.3}$$

and Griffiths' curvature formula ([2], Ch.II Prop.4):

$$\langle R(e), e' \rangle = \langle \sigma e, \sigma e' \rangle + \langle \sigma^* e, \sigma^* e' \rangle. \tag{2.4}$$

Where R is the matrix valued curvature 2-form of $\mathcal{H}^{p,q}$, e and e' are any two elements of $H^{p,q}$ and $\langle , \rangle$ is the Hodge metric.

Let Ω be a holomorphic section of $\mathcal{H}^{n,0}$ and consider the basis of $H^{n-1,1}$ given by $\sigma_i\Omega$, then T has a basis $e_i = \Omega^* \otimes \sigma_i\Omega$ from (2.3). In this basis, the Weil-Petersson metric takes the form

$$g_{i\bar{j}} = \frac{\langle \sigma_i\Omega, \sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle}. \tag{2.5}$$

Let K, R_1 and R_2 be the curvature of T, $(\mathcal{H}^{n,0})^*$ and $\mathcal{H}^{n-1,1}$ respectively. Using the standard curvature formulae for tensor bundle and dual bundle, we find

$$\begin{aligned} K(e_i) &= (R_1 \otimes I_2 + I_1 \otimes R_2)(\Omega^* \otimes \sigma_i\Omega) \\ &= R_1(\Omega^*) \otimes \sigma_i\Omega + \Omega^* \otimes R_2(\sigma_i\Omega). \end{aligned}$$

By taking scalar product with $e_j = \Omega^* \otimes \sigma_j\Omega$ and using the definition of dual metric, we get

$$\begin{aligned} \langle K(e_i), e_j\rangle_{WP} &= \langle R_1(\Omega^*), \Omega^*\rangle\langle \sigma_i\Omega, \sigma_j\Omega\rangle + \langle \Omega^*, \Omega^*\rangle\langle R_2(\sigma_i\Omega), \sigma_j\Omega\rangle \\ &= -\langle \Omega, \Omega\rangle^{-2}\langle R(\Omega), \Omega\rangle\langle \sigma_i\Omega, \sigma_j\Omega\rangle + \langle \Omega, \Omega\rangle^{-1}\langle R(\sigma_i\Omega), \sigma_j\Omega\rangle. \end{aligned}$$

Now we evaluate this 2-form on $e_k \wedge \bar{e_\ell}$ and apply (2.4), we get (notice the order of ℓ, k and the sign)

$$-\frac{\langle \sigma_k\Omega, \sigma_\ell\Omega\rangle}{\langle \Omega, \Omega\rangle}\frac{\langle \sigma_i\Omega, \sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle} + \frac{\langle \sigma_k\sigma_i\Omega, \sigma_\ell\sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle} - \frac{\langle \sigma_\ell^*\sigma_i\Omega, \sigma_k^*\sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle}. \tag{2.6}$$

Since $h^{n,0} = 1$, $\sigma_p^*\sigma_q$ acts as a scalar operator on $H^{n,0}$:

$$\sigma_p^*\sigma_q = \frac{\langle \sigma_p^*\sigma_q\Omega, \Omega\rangle}{\langle \Omega, \Omega\rangle} = \frac{\langle \sigma_q\Omega, \sigma_p\Omega\rangle}{\langle \Omega, \Omega\rangle}.$$

Hence the last term in (2.6) becomes

$$-\frac{\langle \sigma_i\Omega, \sigma_\ell\Omega\rangle}{\langle \Omega, \Omega\rangle}\frac{\overline{\langle \sigma_j\Omega, \sigma_k\Omega\rangle}}{\langle \Omega, \Omega\rangle} = -\frac{\langle \sigma_i\Omega, \sigma_\ell\Omega\rangle}{\langle \Omega, \Omega\rangle}\frac{\langle \sigma_k\Omega, \sigma_j\Omega\rangle}{\langle \Omega, \Omega\rangle}. \tag{2.7}$$

Using (2.5) and (2.7), then (2.6) gives the formula (2.1). □

3. Some simple consequences of the curvature formula

3.1. Lower bounds of curvature. The immediate consequences of the general curvature formula are various positivity results of different types of curvature. We mention some of them here.

Theorem 3.1. *For the Weil-Petersson metric g_{WP}, we have*

(1) *The holomorphic sectional curvature $\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}}\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell \geq -2|\xi|^4$.*
(2) *The Ricci curvature $R_{i\bar{j}} \geq -(d+1)g_{i\bar{j}}$.*
(3) *The second term in (2.1) is "Nakano semi-positive".*

Proof. This is a pointwise question. For simplicity, let's use the normal coordinate system given by Lemma 2.2. (1) is obvious since $\sum_{i,j,k,\ell} \tilde{Q}(a_{ik}, \overline{a_{j\ell}})\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell = \tilde{Q}(A, \bar{A}) \geq 0$ for $A = \sum_{i,k} a_{ik}\xi^i\xi^k$.

For (2), we need to show that $\big(\sum_{k,\ell} g^{k\bar\ell}\tilde{Q}(a_{ik},\overline{a_{jl}})\big)_{i,j}$ is semi-positive. For any vector $\xi=(\xi_i)$, let A_k be the vector $\sum_i a_{ik}\xi_i$. Then

$$\sum\nolimits_{i,j,k,\ell} g^{k\bar\ell}\tilde{Q}(a_{ik},\bar{a_{j\ell}})\xi_i\underline{\mathrm{a}}r\xi_j=\sum\nolimits_k \tilde{Q}(A_k,\bar{A}_k)\geq 0.$$

For (3), it simply means that for any vector $u=(u_{pq})$ with double indices,

$$\sum\nolimits_{i,j,k,\ell}\tilde{Q}(a_{ik},\overline{a_{j\ell}})u_{ik}\overline{u_{j\ell}}=\tilde{Q}(A,\bar{A})\geq 0$$

where $A:=\sum_{p,q}a_{pq}u_{pq}$. □

3.2. Relation to the Hodge metric. The period domain has a natural invariant metric induced from the Killing form. The horizontal tangent bundle also has a natural metric induced from the metrics on the Hodge bundles. These two metrics are in fact the same ([2], p.18) and we call it the Hodge metric. The *Hodge metric* g_H on S is defined to be the metric induced from the Hodge metric of the full horizontal tangent bundle.
In dimension three, e.g. the moduli spaces of Calabi-Yau threefolds, we can reconstruct the Hodge metric from the Weil-Petersson metric. This result was first deduced by Lu in 1996 through different method, see e.g. [4].

Theorem 3.2. *In the case $n=3$, we have*

$$g_H=(d+3)g_{WP}+\mathrm{Ric}(g_{WP}).$$

In particular, the Hodge metric g_H is Kähler.

Proof. The horizontal tangent bundle is

$$\mathrm{Hom}(H^{3,0},H^{2,1})\oplus\mathrm{Hom}(H^{2,1},H^{1,2})\oplus\mathrm{Hom}(H^{1,2},H^{0,3})$$

The first piece gives the Weil-Petersson metric on S. The third piece is dual to the first one, hence, as one can check easily, gives the same metric. Now

$$(d+3)g_{i\bar j}+R_{i\bar j}=2g_{i\bar j}+\sum\nolimits_{k,\ell}g^{k\bar\ell}\,\frac{\tilde{Q}(\sigma_i\sigma_k\Omega,\overline{\sigma_j\sigma_\ell\Omega})}{\tilde{Q}(\Omega,\bar\Omega)}.$$

The last term gives the Hodge metric of the middle part of the horizontal tangent bundle since $\sigma_k\Omega$ form a basis of $H^{2,1}$ and the Hodge metric is defined to be the metric of linear mappings, which are exactly the infinitesimal period maps σ_i's. □

Remark 3.3. Moduli spaces of polarized complex tori (resp. hyperkähler manifolds) correspond to variations of polarized weight one (resp. weight two) Hodge structures. Their universal covering spaces are Hermitian bounded symmetric domains and the invariant (Bergman) metrics are Kähler-Einstein of negative Ricci curvature. In these cases, the weight n polarized VHS are completely determined by the weight one (resp. weight two) polarized VHS. Based on this observation, one can show that g_{WP} and g_H both coincide with the Bergman metric up to a positive constant (cf. [9] for the case of g_{WP}). However, as we will see in Theorem 4.4, the negativity of Ricci curvature fails for moduli spaces of general Calabi-Yau manifolds. In fact, the Hodge theory of Calabi-Yau threefolds with $h^1(\mathcal{O}) = 0$ may be regarded as the first nontrivial instance of Hodge theory of weight three.

3.3. Relation to the Bryant-Griffiths cubic form. In the case $n = 3$, Bryant and Griffiths [1] has defined a symmetric cubic form on the parameter space S:

$$F_{ijk} := \frac{\tilde{Q}(\sigma_i\sigma_j\sigma_k\Omega, \Omega)}{\tilde{Q}(\Omega, \bar{\Omega})}.$$

Strominger [12] has obtained a formula for the Riemann curvature tensor through this cubic form F_{ijk} (in physics literature it is called the Yukawa coupling), and it has played important role in the study of Mirror Symmetry. We may derive it from our formula (2.1):

Theorem 3.4. *For an effectively parametrized polarized variations of Hodge structures $\mathcal{H} \to S$ of weight 3, the curvature tensor of g_{WP} is given by*

$$R_{i\bar{j}k\bar{\ell}} = -(g_{i\bar{j}}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{k\bar{j}}) + \sum\nolimits_{p,q} g^{p\bar{q}}\, F_{pik}\overline{F_{qj\ell}}.$$

Proof. Since $\tilde{Q}(\sigma_i\sigma_j\Omega, \Omega) = 0$ by the consideration of types, the metric compatibility implies that

$$0 = \partial_k\tilde{Q}(\sigma_i\sigma_j\Omega, \Omega) = \tilde{Q}(\sigma_k\sigma_i\sigma_j\Omega, \Omega) - \tilde{Q}(\sigma_i\sigma_j\Omega, \sigma_k\Omega).$$

Let us write $\overline{\sigma_j\sigma_\ell\Omega} = \sum_p a^p\, \sigma_p\Omega$, then

$$\sum\nolimits_p a^p\, g_{p\bar{q}} = -\sum\nolimits_p a^p\, \frac{\tilde{Q}(\sigma_p\Omega, \overline{\sigma_q\Omega})}{\tilde{Q}(\Omega, \bar{\Omega})} = -\frac{\tilde{Q}(\overline{\sigma_j\sigma_\ell\Omega}, \overline{\sigma_q\Omega})}{\tilde{Q}(\Omega, \bar{\Omega})} = \overline{F_{qj\ell}}.$$

So $a^p = \sum_q g^{p\bar{q}}\overline{F_{qj\ell}}$ and the second term in (2.1) becomes

$$\frac{\tilde{Q}(\sigma_i\sigma_k\Omega, \overline{\sigma_j\sigma_\ell\Omega})}{\tilde{Q}(\Omega, \bar{\Omega})} = \sum\nolimits_{p,q} g^{p\bar{q}}\, \overline{F_{qj\ell}}\, \tilde{Q}(\sigma_i\sigma_k\Omega, \sigma_p\Omega) = \sum\nolimits_{p,q} g^{p\bar{q}}\, F_{pik}\, \overline{F_{qj\ell}}.$$

□

Remark 3.5. In the geometric case, namely moduli of Calabi-Yau threefolds, the cubic form is usually written as

$$F_{ijk} = e^{-K}\int_X \partial_i\partial_j\partial_k\, \Omega \wedge \Omega,$$

where $K = \log\tilde{Q}$ and Ω is a relative holomorphic three-form over S.

4. Asymptotic behavior of the curvature along degenerations

To study the asymptotic behavior of the curvature, we may localize the problem and study degenerations of polarized Hodge structures. By taking a holomorphic curve transversal to the degenerating loci, or equivalently we study the limiting behavior of the holomorphic sectional curvature, we may consider the following situation (consult [2], [7] for more details): a period mapping

$$\phi : \Delta^\times \to \langle T\rangle\backslash D \to \langle T\rangle\backslash\mathbb{P}(V)$$

which corresponds to the degeneration. Here $V = H^n$ is a reference vector space with a quadratic form Q as in §1, and with $T \in \mathrm{Aut}(V, Q)$ the Picard-Lefschetz monodromy. Assume that T is unipotent and let $N = \log T$. There is an uniquely defined weight filtration $W : 0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2n} = V$ such that

$$NW_i \subset W_{i-2} \quad \text{and} \quad N^k : Gr^W_{n+k} \cong Gr^W_{n-k}$$

where $Gr^W_i := W_i/W_{i-1}$. This W, together with the limiting Hodge filtration $F_\infty := \lim_{t\to 0} e^{-zN} F_t$ ($z = \log t/2\pi\sqrt{-1}$ is the coordinates on the upper half plane, $t \in \Delta^\times$) constitute Schmid's polarized limiting mixed Hodge structures. This means that Gr^W_i admits a polarized Hodge structure $\bigoplus_{p+q=i} H^{p,q}_\infty$ of weight i induced from F_∞ and Q such that for $k \geq 0$, the primitive part $P^W_{n+k} := \mathrm{Ker} N^{k+1} \subset Gr^W_{n+k}$ is polarized by $Q(\cdot, N^k \bar{\cdot})$. Notice that N is a morphism of type $(-1, -1)$ in the sense that $N(H^{p,q}_\infty) \subset H^{p-1,q-1}_\infty$. This allows one to view the mixed Hodge structure in terms of a Hodge diamond and view N as the operator analogous to "contraction by the Kähler form".
By Schmid's nilpotent orbit theorem ([7], cf. [15], §0-§1), we can pick the (multi-valued) holomorphic section Ω of $\mathcal{F}^n$ over $\Delta^\times$ by

$$\Omega(t) = \Omega(z) := e^{zN}\mathbf{a}(t) = e^{\log t \tilde{N}}\mathbf{a}(t) \in F^n_t,$$

where $\mathbf{a}(t) = \sum a_i t^i$ is holomorphic over Δ with value in V and $\tilde{N} := N/2\pi\sqrt{-1}$. Also $0 \neq \mathbf{a}(0) = a_0 \in F^n_\infty$. (Notice that while $\Omega(z)$ is single-valued, $\Omega(t)$ is well-defined only locally or with its value mod T.)
Now we may summarize the computations done in [15], §1 in the following form:

Theorem 4.1. *The induced Weil-Petersson metric g_{WP} on $\Delta^\times$ is incomplete at $t = 0$ if and only if $F^n_\infty \subset \mathrm{Ker}\, N$.*
In the complete case, i.e. $Na_0 \neq 0$, let $k := \max\{\, i \mid N^i a_0 \neq 0 \,\}$. Then $\tilde{Q}(\Omega, \bar{\Omega})$ blows up to $+\infty$ with order $c|\log|t|^2|^k$ and the metric g_{WP} blows up to $+\infty$ with order

$$\frac{k dt \otimes d\bar{t}}{|t|^2 \big|\log|t|^2\big|^2},$$

i.e. it is asymptotic to the Poincaré metric, where $c = (k!)^{-1}|\tilde{Q}(\tilde{N}^k a_0, \bar{a_0})| > 0$.
In the incomplete case, i.e. $Na_0 = 0$, the holomorphic section $\Omega(t)$ extends continuously over $t = 0$.

Idea of proof. We have the following well-known calculation: for any $k \in \mathbb{R}$,

$$-\partial\bar{\partial} \log|\log|t|^2|^k = \frac{k dt \wedge d\bar{t}}{|t|^2\big|\log|t|^2\big|^2}, \tag{4.1}$$

which is also true asymptotically if t is a holomorphic section of a Hermitian line bundle as the defining section of certain divisor, in a general smooth base S of arbitrary dimension. The main point is to prove that lower order terms are still of lower order whenever we take derivatives. This is done in [15] when $S = \Delta^\times$. This is also the main point of the remaining discussion in this section. □

To achieve the goal, we define operators $S_k = k + \tilde{N}$ for any $k \in \mathbb{Z}$. Then all S_k commute with each other and S_k is invertible if $k \neq 0$. By Theorem 4.1, we only need to study the incomplete case, i.e. $Na_0 = 0$. As in Lemma 2.2., we may assume that $\tilde{Q}(a_0, \bar{a}_0) = 1$ and $\tilde{Q}(a_0, \bar{a}_i) = 0$ for all $i \geq 1$.

Lemma 4.2. *If a_k is the first nonzero term other than a_0, then $S_k a_k \in F_\infty^{n-1}$. If moreover $Na_k = 0$ then $a_k \in H_\infty^{n-1,1}$, and for the next nonzero term $a_{k+\ell}$ we have $S_\ell S_{k+\ell} a_{k+\ell} \in F_\infty^{n-2}$.*

Proof. By the Griffiths transversality, we have

$$\Omega'(t) = e^{zN}\Big[\frac{1}{t}\tilde{N}\mathbf{a} + \mathbf{a}'\Big] \in F_t^{n-1}. \tag{4.2}$$

Since $Na_0 = 0$, this implies $\tilde{N}a_k t^{k-1} + k a_k t^{k-1} + \cdots \in e^{-zN} F_t^{n-1}$. Take out the factor t^{k-1} and let $t \to 0$, we get

$$S_k a_k \in F_\infty^{n-1}.$$

In fact, we have from (4.2), $e^{zN}(S_k a_k + S_{k+1} a_{k+1} t + \cdots) \in F_t^{n-1}$. Taking derivative and by transversality again, we get

$$e^{zN}\Big[\frac{1}{t}\tilde{N} S_k a_k + S_1 S_{k+1} a_{k+1} + S_2 S_{k+2} a_{k+2} t + \cdots\Big] \in F_t^{n-2}.$$

So, if $Na_k = 0$, then for the next nonzero term $a_{k+\ell}$, we have by the same way

$$S_\ell S_{k+\ell} a_{k+\ell} \in F_\infty^{n-2}.$$

We also know the following equivalence: $Na_k = 0$ iff $NS_k a_k = 0$ iff $S_k a_k \in H_\infty^{n-1,1}$ (because $S_k a_k \in F_\infty^{n-1}$). Since N is a morphism of type $(-1,-1)$ and the only nontrivial part of F_∞^n is in $H_\infty^{n,0}$, this is equivalent to $a_k \in H_\infty^{n-1,1}$. (This follows easily from the Hodge diamond.) □

Define $q_{ij} = \tilde{Q}(e^{zN} a_i, \overline{e^{zN} a_j}) \equiv \tilde{Q}(e^{\log|t|^2 \tilde{N}} a_i, \bar{a}_j)$, which are functions of $\log|t|$. The following basic lemma is the key for all the computations, which explains "lower order terms are stable under differentiations".

Lemma 4.3. *Let $Q^{S_k}(a, \bar{b}) := Q(S_k a, \bar{b}) \equiv Q(a, \overline{S_k b})$, then for all $k, \ell \in \mathbb{Z}$,*

$$t\frac{\partial}{\partial t}(q_{ij} t^k \bar{t}^\ell) = q_{ij}^{S_k} t^k \bar{t}^\ell \quad \textit{and} \quad \bar{t}\frac{\partial}{\partial \bar{t}}(q_{ij} t^k \bar{t}^\ell) = q_{ij}^{S_\ell} t^k \bar{t}^\ell.$$

Proof. Straightforward. □

Now we state the main result of this section:

Theorem 4.4. *For any degeneration of polarized Hodge structures of weight n with $h^{n,0} = 1$, the induced Weil-Petersson metric g_{WP} has finite volume.*
For one parameter degenerations with finite Weil-Petersson distance, if $Na_1 \neq 0$ then g_{WP} blows up to $+\infty$ with order $c|\log|t|^2|^k$ and the curvature form $K_{WP} = K dt \wedge d\bar{t}$ blows up to $+\infty$ with order

$$\frac{k dt \wedge d\bar{t}}{|t|^2 |\log|t|^2|^2},$$

where $k = \max\{\, i \mid N^i S_1 a_1 \neq 0 \,\} \leq n-1$, $c = (k!)^{-1} |\tilde{Q}(\tilde{N}^k S_1 a_1, \overline{S_1 a_1})| > 0$.

Remark 4.5. The statement about finite volume is a standard fact in Hodge theory. By the nilpotent orbit theorem, the Hodge metrics on the Hodge bundles degenerate at most logarithmically. So g_H, and hence g_{WP}, has finite volume. In fact, in view of (4.1), the method of the following proof implies that g_H is asymptotic to the Poincaré metric of the punctured disk along transversal directions toward boundary divisors of the base space.

Proof. Pick $\mathbf{a}(t)$ and $\Omega(t)$ as before, (i.e. $Na_0 = 0$, $q_{00} = 1$, $q_{0i} = 0$ for $i \geq 1$ and all other q_{ij} are functions of $\log|t|$). We have

$$\begin{aligned} q(t) &= \tilde{Q}(\Omega(t), \bar{\Omega}(t)) \\ &= 1 + q_{11}t\bar{t} + (q_{12}t\bar{t}^2 + q_{21}t^2\bar{t}) + (q_{22}t^2\bar{t}^2 + q_{31}t^3\bar{t} + q_{13}t\bar{t}^3) + \cdots \end{aligned}$$

Applying Lemma 4.3, we can compute

$$\begin{aligned} \frac{\partial q}{\partial t}\frac{\partial q}{\partial \bar{t}} - q\frac{\partial^2 q}{\partial t \partial \bar{t}} = q_{11}^{S_1}\bar{t}q_{11}^{S_1}t \;-\; (1 + q_{11}t\bar{t})(&q_{11}^{S_1S_1} + q_{12}^{S_1S_2}\bar{t} + q_{21}^{S_2S_1}t \\ &+ q_{22}^{S_2S_2}t\bar{t} + q_{31}^{S_3S_1}t^2 + q_{13}^{S_1S_3}\bar{t}^2) + \cdots \end{aligned}$$

(Since we will assume that $Na_1 \neq 0$, q_{11} will be the only term needed. However we have calculated more terms in order for later use.) So the metric $ds^2 = g|dt|^2$ is given by

$$\begin{aligned} g &= -\partial_t\partial_{\bar{t}}\log q = q^{-2}(\partial_t q\partial_{\bar{t}}q - q\partial_t\partial_{\bar{t}}q) \\ &= -q_{11}^{S_1S_1} - (q_{12}^{S_1S_2}t + q_{21}^{S_2S_1}\bar{t}) \\ &\qquad + (q_{11}q_{11}^{S_1S_1} + |q_{11}^{S_1}|^2 - q_{22}^{S_2S_2})t\bar{t} - q_{31}^{S_3S_1}t^2 - q_{13}^{S_1S_3}\bar{t}^2 + \cdots \end{aligned} \tag{4.3}$$

As $t \to 0$, the first term determines the behavior of g. If $Na_1 \neq 0$ (so $a_1 \neq 0$), let $\xi = S_1a_1 \in F_\infty^{n-1}$ (by Lemma 4.2) and let $k = \max\{\, i \mid N^i\xi \neq 0\,\}$. ($k \leq n-1$ simply because $Gr_{2n}^W = 0$.) Then the highest order term of $-q_{11}^{S_1S_1}$, with respect to $\log|t|^2$, is given by (for this purpose we can ignore all operators S_i with $i \neq 0$)

$$-\frac{1}{k!}\big(\log|t|^2\big)^k\tilde{Q}(\tilde{N}^k\xi, \bar{\xi}). \tag{4.4}$$

This term has nontrivial coefficient and is in fact positive. This follows from the fact that $\tilde{Q}$ polarizes the limiting mixed Hodge structures. Hence g blows up with the expected order, with $c = (k!)^{-1}|\tilde{Q}(\tilde{N}^kS_1a_1, \overline{S_1a_1})|$.

For the curvature form $K dt \wedge d\bar{t} = -\partial\bar{\partial}\log g$, the most singular term of K is given by

$$\begin{aligned} &\big(-q_{11}^{S_1S_1}\big)^{-2}\Big[\big(-q_{11}^{S_1S_1S_0}t^{-1}\big)\big(q_{11}^{S_1S_1S_0}\bar{t}^{-1}\big) - \big(-q_{11}^{S_1S_1}\big)\big(-q_{11}^{S_1S_1S_0S_0}t^{-1}\bar{t}^{-1}\big)\Big] \\ =\; &\big(-q_{11}^{S_1S_1}\big)^{-2}\Big[|q_{11}^{S_1S_1S_0}|^2 - q_{11}^{S_1S_1}q_{11}^{S_1S_1S_0S_0}\Big]|t|^{-2}. \end{aligned}$$

We need to show that this term is nontrivial. As before, we may ignore all S_i with $i \neq 0$. So the highest order terms of $\big(|q_{11}^{S_1S_1S_0}|^2 - q_{11}^{S_1S_1}q_{11}^{S_1S_1S_0S_0}\big)$ are

$$\frac{1}{(k-1)!}\big(\log|t|^2\big)^{2(k-1)}\big|\tilde{Q}(\tilde{N}^k\xi, \bar{\xi})\big|^2 - \frac{1}{k!(k-2)!}\big(\log|t|^2\big)^{k+(k-2)}\big|\tilde{Q}(\tilde{N}^k\xi, \bar{\xi})\big|^2.$$

It is clear that the coefficient $\frac{1}{(k-1)!} - \frac{1}{k!(k-2)!} = \frac{1}{k!(k-1)!} > 0$. Taking into account the order of $(-q_{11}^{S_1 S_1})^{-2}$ given by (4.4) shows that K blows up to $+\infty$ with the expected order $k|t|^{-2}\big|\log|t|^2\big|^{-2}$. $\square$

Question 4.6. In the original smooth case in §2, a_1 corresponds to the Kodaira-Spencer class of the variation, so we know that $a_1 \neq 0$. For the degenerate case, what is the geometric meaning of a_1? Is it some kind of Kodaira-Spencer class for singular varieties?

Remark 4.7. Assume that $a_1 \neq 0$. If $Na_1 = 0$, then by Lemma 4.2, $a_1 \in H_\infty^{n-1,1}$ and

$$-q_{11}^{S_1 S_1} = -\tilde{Q}(e^{\log|t|^2 \tilde{N}} S_1 a_1, \overline{S_1 a_1}) \equiv -\tilde{Q}(a_1, \bar{a_1}) > 0.$$

From (4.3), g_{WP} has a non-degenerate continuous extension over $t = 0$.
In the case $n = 3$, if the moduli is one-dimensional ($h^{2,1} = 1$), then $a_1 \neq 0$ and $Na_1 = 0$ imply that $N \equiv 0$. That is, the variation of Hodge structures does not degenerate at all. More generally, if we have a positive answer to Question 4.6, then we also have a similar statement for multi-dimensional moduli. Namely, $N_i a_0 = 0$, $N_i a_{1,j} = 0$ for all i, j implies that $N_i \equiv 0$ for all i, where N_i's are the local monodromies, $a_{1,j}$'s are coefficients of the linear terms in $\mathbf{a}(t)$.

We conclude this section by a partial result:

Proposition 4.8. *Assume that $a_1 \neq 0$, $Na_1 = 0$ (so g_{WP} is continuous over $t = 0$). If $Na_2 = 0$ then the curvature tensor has a continuous extension over $t = 0$. The converse is true for $n = 3$. If the curvature tensor does not extend continuously over $t = 0$, it has a logarithmic blowing-up.*

Proof. Following Remark 4.7 and Lemma 2.3, we may assume that $-q_{11} = 1$, $q_{12} = 0$ and q_{1j}'s are constants for $j \geq 2$. So among the degree two terms of g, only the $t\bar{t}$ term contributes to the curvature. Namely from (4.3),

$$g = 1 + (2 - q_{22}^{S_2 S_2}) t\bar{t} + \cdots.$$

Again we are in the "normal coordinates", so

$$K = -2 + q_{22}^{S_2 S_2 S_1 S_1} + \cdots.$$

It is clear that $Na_2 = 0$ implies that $q_{22}^{S_2 S_2 S_1 S_1}$ is a constant ($= 4\tilde{Q}(a_2, \bar{a_2})$). We will show that the converse is true if $n = 3$. We may assume that $a_2 \neq 0$. By Lemma 4.2 we have $a_1 \in H_\infty^{2,1}$ and $S_1 S_2 a_2 \in F_\infty^1$, so we may write (from the Hodge diamond) $S_1 S_2 a_2 = \lambda a_1 + \alpha + \beta$ with $\alpha \in H_\infty^{2,2}$ and $\beta \in H_\infty^{1,1}$. Now the constancy of $q_{22}^{S_2 S_2 S_1 S_1}$ implies that

$$\tilde{Q}(N(\lambda a_1 + \alpha + \beta), \overline{\lambda a_1 + \alpha + \beta}) = 0$$

as it is the highest order term ($N^2 \equiv 0$). So the fact $Na_1 = 0$ and $N\beta = 0$ implies that $\tilde{Q}(N\alpha, \bar{\alpha}) = 0$. This in turn forces $\alpha = 0$ by the polarization condition. So $S_1 S_2 a_2 = \lambda a_1 + \beta$, and $S_1 S_2 N a_2 = \lambda N a_1 + N\beta = 0$. That is, $Na_2 = 0$. The remaining statement about the logarithmic blowing-up is clear. $\square$

5. Concluding Remarks

Based on Yau's solution to the Calabi Conjecture [17], the existence of the coarse moduli spaces of polarized Calabi-Yau manifolds in the category of separated analytic spaces was proved by Schumacher [8] in the 80's. Combined with the Bogomolov-Tian-Todorov theorem [13], these moduli spaces are smooth Kähler orbifolds equipped with the Weil-Petersson metrics.

In the algebraic category, the coarse moduli spaces can also be constructed in the category of Moishezon spaces. Moreover, for polarized (projective) Calabi-Yau manifolds, the quasi-projectivity of such moduli spaces has been proved by Viehweg [14] in late 80's.

From the analytic viewpoint, there is also a theory originated from Siu and Yau [11] dealing with the projective compactification problem for complete Kähler manifolds with finite volume. Results of Mok, Zhong [5] and Yeung [18] say that a sufficient condition is the negativity of Ricci curvature and the boundedness of sectional curvature. In general, the Weil-Petersson metric does not satisfy these conditions. This leads to some puzzles since the ample line bundle constructed by Viehweg ([14], Corollary 7.22) seems to indicate that ω_{WP} will play important role in the compactification problem. Since the curvature tends to be negative in the infinite distance boundaries, the puzzle occurs only at the finite distance part. This leads to two different aspects:

The first one is the geometrical metric completion problem. In [15], it is proposed that degenerations of Calabi-Yau manifolds with finite Weil-Petersson distance should correspond to degenerations with at most canonical singularities in a suitable birational model. Now this is known to follow from the minimal model conjecture in higher dimensions [16]. With this admitted, one may then go ahead to analyze the structure of these completed spaces. Are they quasi-affine varieties?

Another aspect is the usage of Hodge metric. Näively, since the Ricci curvature of the Weil-Petersson metric has a lower bound $-(d+1)g_{WP}$ and blows up to ∞ at some finite distance boundary points, for any $k > 0$ one may pull these points out to infinity by considering the following new Kähler metric

$$\tilde{g}_{i\bar{j}} = (d+1+k)g_{i\bar{j}} + R_{i\bar{j}} = kg_{i\bar{j}} + \sum\nolimits_{k,\ell} g^{k\bar{\ell}} \frac{\tilde{Q}(\sigma_i\sigma_k\Omega, \overline{\sigma_j\sigma_\ell\Omega})}{\tilde{Q}(\Omega,\bar{\Omega})}.$$

When the blowing-up is faster than logarithmic growth (e.g. $Na_1 \neq 0$), $\tilde{g}$ is then complete (at these boundaries). Otherwise one may need to repeat this process. This inductive structure is implicit in Theorem 4.1 and 4.4 and is explicitly expressed in Theorem 3.2 in the case $n = 3$, $k = 2$. It suggests that the resulting metric will be quasi-isometric to the Hodge metric.

If we start with the Hodge metric directly, the coarse moduli spaces being Moishezon allows us to assume that the boundary has a local model as normal crossing divisors. Then a similar asymptotic analysis as in §4 implies that the metric behaves like the Poincaré metric in the transversal direction toward the codimension one boundaries (i.e. points with $N \neq 0$ where N is the local monodromy). In particular, it admits bounded sectional curvature. The main problem here is the higher codimensional boundaries. In this direction, we should mention that the negativity of the Ricci curvature for Hodge metrics has recently been proved by Lu [3]. We expect that the Hodge metric would eventually provide projective compactifications of the moduli spaces through the recipe of [5], [18].

References

1. R. Bryant, P. Griffiths, *Some Observations on the Infinitesimal Period Relations for Regular Threefolds with Trivial Canonical Bundle*, in Arithmetic and Geometry, vol.II, Birkhauser, Boston, 1983, pp. 77-102.
2. P. Griffiths ed, *Topics in Transcendental Algebraic Geometry*, Annal of Math Study 106 1984.
3. Z. Lu, *Geometry of horizontal slices*, Amer. J. Math. 121, 177-198 (1999).
4. ——, *On the Hodge metric of the universal deformation space of Calabi-Yau threefolds*, J. Geom. Anal. 11, 103-118 (2001).
5. N. Mok, J.-Q. Zhong, *Compactifying complete Kähler manifolds of finite topological type*, Ann. Math. 129, 417-470 (1989).
6. A. Nanichinni, *Weil-Petersson metric in the space of compact polarized Kähler Einstein manifolds with* $c_1 = 0$, Manuscripta Math. 54, 405-438 (1986).
7. W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. 22, 211-319 (1973).
8. G. Schumacher, *Moduli of polarized Kähler manifolds*, Math. Ann. 269, 137-144 (1984).
9. ——, *On the geometry of Moduli spaces*, Manuscripta Math. 50, 229-267 (1985).
10. ——, *The Curvature of the Petersson-Weil Metric on the Moduli Space of Kähler-Einstein Manifolds*, Complex Analysis and Geometry, Vincenzo Ancona ed., Plenum Press, New York, 1993, pp. 339-354.
11. Y.T. Siu, S.T. Yau, *Compactification of Negatively Curved Complete Kähler Manifolds of Finite Volume*, Seminar in Differential Geometry, (ed. S.T. Yau), pp. 363-380 (1982).
12. A. Strominger, *Special geometry*, Comm. Math. Phy. 133 163-180 (1990).
13. G. Tian, *Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and Its Peterson-Weil Metric*, Mathematical Aspects of String Theory (ed. S.T. Yau), 629-646.
14. E. Viehweg, *Quasiprojective Moduli of Polarized Manifolds*, Springer-Verlag, New York 1995.
15. C.L. Wang, *On the incompleteness of the Weil-Petersson metric along degenerations of Calabi-Yau manifolds*, Math. Res. Let. 4, 157-171 (1997).
16. ——, *Quasi-Hodge metrics and canonical singularities*, Math. Res. Let. 10, 57-70 (2003).

17. S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Apere equation I*, Comm. Pure Appl. Math. 31 339-441 (1978).
18. S.K. Yeung, *Compactification of Kähler manifolds with negative Ricci curvature*, Invent. Math. 106 13-25 (1993).

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Calculation of Rozansky-Witten Invariants on the Hilbert Schemes of Points on a K3 Surface and the Generalised Kummer Varieties

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Abstract. For any holomorphic symplectic manifold (X, σ), a closed Jacobi diagram with $2k$ trivalent vertices gives rise to a Rozansky-Witten class

$$\mathrm{RW}_{X,\sigma}(\Gamma) \in \mathrm{H}^{2k}(X, \mathcal{O}_X).$$

If X is irreducible, this defines a number $\beta_\Gamma(X, \sigma)$ by $\mathrm{RW}_{X,\sigma}(\Gamma) = \beta_\Gamma(X, \sigma)[\bar\sigma]^k$.

Let $(X^{[n]}, \sigma^{[n]})$ be the Hilbert scheme of n points on a K3 surface together with a symplectic form $\sigma^{[n]}$ such that $\int_{X^{[n]}}(\sigma^{[n]}\bar\sigma^{[n]})^n = n!$. Further, let $(A^{[[n]]}, \sigma^{[[n]]})$ be the generalised Kummer variety of dimension $2n-2$ together with a symplectic form $\sigma^{[[n]]}$ such that $\int_{A^{[[n]]}}(\sigma^{[[n]]}\bar\sigma^{[[n]]})^n = n!$. J. Sawon conjectured in his doctoral thesis that for every connected Jacobi diagram, the functions $\beta_\Gamma(X^{[n]}, \sigma^{[n]})$ and $\beta_\Gamma(A^{[[n]]}, \sigma^{[[n]]})$ are linear in n.

We prove that this conjecture is true for Γ being a connected Jacobi diagram homologous to a polynomial of closed polywheels. We further show how this enables one to calculate all Rozansky-Witten invariants of $X^{[n]}$ and $A^{[[n]]}$ for closed Jacobi diagrams that are homologous to a polynomial of closed polywheels. It seems to be unknown whether every Jacobi diagram is homologous to a polynomial of closed polywheels. If indeed the closed polywheels generate the whole graph homology space as an algebrea, our methods will thus enable us to compute *all* Rozansky-Witten invariants for the Hilbert schemes and the generalised Kummer varieties using these methods.

Also discussed in this article are the definitions of the various graph homology spaces, certain operators acting on these spaces and their relations, some general facts about holomorphic symplectic manifolds and facts about the special geometry of the Hilbert schemes of points on surfaces.

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Contents

1. Introduction

A compact *hyperkähler manifold* (X, g) is a compact Riemannian manifold whose holonomy is contained in $\mathrm{Sp}(n)$. An example of such a manifold is the K3 surface together with a Ricci-flat Kähler metric (which exists by S. Yau's theorem [18]). In [15], L. Rozansky and E. Witten described how one can associate to every vertex-oriented trivalent graph Γ an invariant $b_\Gamma(X)$ to X, henceforth called a *Rozansky-Witten invariant of X associated to* Γ. In fact, this invariant only depends on the homology class of the graph, so the invariants are already defined on the level of the graph homology space $\mathcal{B}$ (see e.g. [1] and this paper for more information about graph homology).

Every hyperkähler manifold (X, g) can be given the structure of a Kähler manifold X (which is, however, not uniquely defined) whose Kähler metric is just given by g. X happens to carry a holomorphic symplectic two-form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$, whereas we shall call X a *holomorphic symplectic manifold.* Now M. Kapranov showed in [8] that one can in fact calculate $b_\Gamma(X)$ from (X, σ) by purely holomorphic methods.
The basic idea is the following: We can identify the holomorphic tangent bundle $\mathcal{T}_X$ of X with its cotangent bundle Ω_X by means of σ. Doing this, the Atiyah class α_X (see [8]) of X lies in $\mathrm{H}^1(X, \mathrm{S}_3\mathcal{T}_X)$. Now we place a copy of α_X at each trivalent vertex of the graph, take the $\cup$-product of all these copies (which gives us an element in $\mathrm{H}^{2k}(X, (\mathrm{S}_3\mathcal{T}_X)^{\otimes 2k})$ if $2k$ is the number of trivalent vertices), and finally contract $(\mathrm{S}_3\mathcal{T}_X)^{\otimes 2k})$ along the edges of the graph by means of the holomorphic symplectic form σ. Let us call the resulting element $\mathrm{RW}_{X,\sigma}(\Gamma) \in \mathrm{H}^{2k}(X, \mathcal{O}_X)$. In case $2k$ is the complex dimension of X, we can integrate this element over X after we have multiplied it with $[\sigma]^{2k}$. This gives us more or less $b_\Gamma(X)$. The orientation at the vertices of the graph is needed in the process to get a number which is not only defined up to sign.
There are two main example series of holomorphic symplectic manifolds, the Hilbert schemes $X^{[n]}$ of points on a K3 surface X and the generalised Kummer varieties $A^{[[n]]}$ (see [2]). Besides two further manifolds constructed by K. O'Grady in [13] and [12], these are the only known examples of *irreducible* holomorphic symplectic manifolds up to deformation.
Not much work was done on actual calculations of these invariants on the example series. The first extensive calculations were carried out by J. Sawon in his doctoral thesis [16]. All Chern numbers are in fact Rozansky-Witten invariants associated to certain Jacobi diagrams, called *closed polywheels*. Let $\mathcal{W}$ be the subspace spanned by these polywheels in $\mathcal{B}$. All Rozansky-Witten invariants associated to graphs lying in $\mathcal{W}$ can thus be calculated from the knowledge of the Chern numbers (which are computable in the case of $X^{[n]}$ ([3]) or $A^{[[n]]}$ ([11]). However, from complex dimension four on, there are graph homology classes that do not lie in $\mathcal{W}$. J. Sawon showed that for some of these graphs the Rozansky-Witten invariants can still be calculated from knowledge of the Chern numbers, which enables one to calculate all Rozansky-Witten invariants up to dimension five. His calculations would work for all irreducible holomorphic manifolds whose Chern numbers are known.
In this article, we will make use of the special geometry of $X^{[n]}$ and $A^{[[n]]}$. Doing this, we are able to give a method which enables us to calculate all Rozansky-Witten invariants for graphs homology classes that lie in the *algebra* $\mathcal{C}$ generated by *closed polywheels* in $\mathcal{B}$. The closed polywheels form the subspace $\mathcal{W}$ of the algebra $\mathcal{B}$ of graph homology. This is really a proper subspace. However, $\mathcal{C}$, the algebra generated by this subspace, is much larger, and, as far as the author knows, it is unknown whether $\mathcal{C} = \mathcal{B}$, i.e. whether this work enables us to calculate *all* Rozanky-Witten invariants for the main example series.
The idea to carry out this computations is the following: Let (Y, τ) be any irreducible holomorphic symplectic manifold. Then $\mathrm{H}^{2k}(Y, \mathcal{O}_Y)$ is spanned by

$[\bar\tau]^k$. Therefore, every graph Γ with $2k$ trivalent vertices defines a number $\beta_\Gamma(Y,\tau)$ by $\mathrm{RW}_{Y,\tau}(\Gamma) = \beta_\Gamma(Y,\tau)[\bar\tau]^k$. J. Sawon has already discussed how knowledge of these numbers for connected graphs is enough to deduce the values of all Rozansky-Witten invariants.

For the example series, let us fix holomorphic symplectic forms $\sigma^{[n]}$, respective $\sigma^{[[n]]}$ with $\int_{X^{[n]}}(\sigma^{[n]}\bar\sigma^{[n]})^n = n!$ respective $\int_{A^{[[n]]}}(\sigma^{[[n]]}\bar\sigma^{[[n]]})^n = n!$. J. Sawon conjectured the following:

> The functions $\beta_\Gamma(X^{[n]},\sigma^{[n]})$ and $\beta_\Gamma(X^{[[n]]},\sigma^{[[n]]})$ are linear in n for Γ being a connected graph.

The main result of this work is the proof of this conjecture for the class of connected graphs lying in $\mathcal{C}$ (see Theorem 3). We further show how one can calculate these linear functions from the knowledge of the Chern numbers and thus how to calculate all Rozansky-Witten invariants for graphs in $\mathcal{C}$.

We should note that we don't make any use of the IHX relation in our derivations, and so we could equally have worked on the level of Jacobi diagrams.

Let us finally give a short description of each section. In section 2 we collect some definitions and results which will be used later on. The next section is concerned with defining the algebra of graph homology and certain operations on this space. We define *connected polywheels* and show how they are related with the usual closed polywheels in graph homology. We further exhibit a natural $\mathfrak{sl}_2$-action on an extended graph homology space. In section 4, we first look at general holomorphic symplectic manifolds. Then we study the two example series more deeply. Section 5 defines Rozansky-Witten invariants while the last section is dedicated to the proof of our main theorem and explicit calculations.

2. Preliminaries

2.1. Some multilinear algebra. Let $\mathcal{T}$ be a tensor category (commutative and with unit). For any object V in $\mathcal{T}$, we denote by S^kV the coinvariants of $V^{\otimes k}$ with respect to the natural action of the symmetric group and by Λ^kV the coinvariants with respect to the alternating action. Further, let us denote by S_kV and Λ_kV the invariants of both actions.

Proposition 1. *Let I be a cyclicly ordered set of three elements. Let V be an object in $\mathcal{T}$. Then there exists a unique map $\Lambda_3V \to V^{\otimes I}$ such that for every bijection $\phi : \{1,2,3\} \to I$ respecting the canonical cyclic ordering of $\{1,2,3\}$ and the given cyclic ordering of I the following diagram*

$$\begin{tikzcd} \Lambda_3 V \arrow[r, equal] \arrow[d] & \Lambda_3 V \arrow[d] \\ V^{\otimes 3} \arrow[r, "\phi_*"'] & V^{\otimes I} \end{tikzcd} \tag{1}$$

commutes, where the map ϕ_* is the canonical one induced by ϕ.

Proof. Let $\phi, \phi' : \{1,2,3\} \to I$ be two bijections respecting the cyclic ordering. Then there exist an even permutation $\alpha \in \mathfrak{A}_3$ such that the lower square of the following diagram commutes:

$$\begin{tikzcd}
\Lambda_3 V \arrow[r, equal] \arrow[d] & \Lambda_3 V \arrow[d] \\
V^{\otimes 3} \arrow[r, "\alpha_*"'] \arrow[d, "\phi"'] & V^{\otimes 3} \arrow[d, "\phi'"] \\
V^{\otimes I} \arrow[r, equal] & V^{\otimes I}.
\end{tikzcd}$$

We have to show that the outer rectangle commutes. For this it suffices to show that the upper square commutes. In fact, since α is an even permutation, every element of $\Lambda_3 V$ is by definition invariant under α_*. □

2.2. Partitions. A partition λ of a non-negative integer $n \in \mathbb{N}_0$ is a sequence $\lambda_1, \lambda_2, \ldots$ of non-negative integers such that

$$\|\lambda\| := \sum_{i=1}^{\infty} i\lambda_i = n. \tag{2}$$

Therefore almost all λ_i have to vanish. In the literature, λ is often notated by $1^{\lambda_1} 2^{\lambda_2} \ldots$. The set of all partitions of n is denoted by $\mathrm{P}(n)$. The union of all $\mathrm{P}(n)$ is denoted by $\mathrm{P} := \bigcup_{n=0}^{\infty} \mathrm{P}(n)$. For every partition $\lambda \in \mathrm{P}$, we set

$$|\lambda| := \sum_{i=1}^{\infty} \lambda_i \tag{3}$$

and

$$\lambda! := \prod_{i=1}^{\infty} \lambda_i!. \tag{4}$$

Let $a_1, a_2, \ldots$ be any sequence of elements of a commutative unitary ring. We set

$$a_\lambda := \prod_{i=1}^{\infty} a_i^{\lambda_i} \tag{5}$$

for any partition $\lambda \in \mathrm{P}$.

With these definitions, we can formulate the following proposition in a nice way:

Proposition 2. *In $\mathbb{Q}[[a_1, a_2, \ldots]]$ we have*

$$\exp\left(\sum_{i=1}^{\infty} a_i\right) = \sum_{\lambda \in \mathrm{P}} \frac{a_\lambda}{\lambda!}. \tag{6}$$

Proof. We calculate

$$\exp\left(\sum_{i=1}^{\infty} a_i\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{i=1}^{\infty} a_i\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\lambda\in \mathrm{P},|\lambda|=n} n! \prod_{i=1}^{\infty} \frac{a_i^{\lambda_i}}{\lambda_i!} = \sum_{\lambda\in\mathrm{P}} \frac{a_\lambda}{\lambda!}. \tag{7}$$

□

If we set

$$\frac{\partial}{\partial a_\lambda} := \prod_{i=1}^{\infty} \frac{\partial^{\lambda_i}}{\partial a_i^{\lambda_i}}\bigg|_{a_i=0}, \tag{8}$$

we have due to Proposition 2:

Proposition 3. *In* $\mathbb{Q}[[s_1, s_2, \dots]][a_1, a_2, \dots]$ *we have*

$$\frac{\partial}{\partial a_\lambda} \exp\left(\sum_{i=1}^{\infty} a_i s_i\right) = s_\lambda. \tag{9}$$

2.3. A lemma from umbral calculus.

Lemma 1. *Let R be any $\mathbb{Q}$-algebra (commutative and with unit) and $A(t) \in R[[t]]$ and $B(t) \in tR[[t]]$ be two power series. Let the polynomial sequences $(p_n(x))$ and $(s_n(x))$ be defined by*

$$\sum_{k=0}^{\infty} p_k(x)\frac{t^k}{k!} = \exp(xB(t)) \tag{10}$$

and

$$\sum_{k=0}^{\infty} s_k(x)\frac{t^k}{k!} = A(t)\exp(xB(t)). \tag{11}$$

Let $W_B(t) \in tR[[t]]$ be defined by $W_B(t\exp(B(t))) = t$. Then we have

$$\sum_{k=0}^{\infty} \frac{xp_k(x-k)}{(x-k)}\frac{t^k}{k!} = \exp(xB(W_B(t))) \tag{12}$$

and

$$\sum_{k=0}^{\infty} \frac{s_k(x-k)}{k!}t^k = \frac{A(W_B(t))}{1 + W_B(t)B'(W_B(t))}\exp(xB(W_B(t))). \tag{13}$$

Proof. It suffices to prove the result for the field $R = \mathbb{Q}(a_0, a_1, \dots, b_1, b_2, \dots)$ and $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=1}^{\infty} b_k t^k$.
So let us assume this special case for the rest of the proof. Let us denote by $f(t)$ the compositional inverse of $B(t)$, i.e. $f(B(t)) = t$. We set $g(t) := A^{-1}(f(t))$. For the following we will make use of the terminology and the statements in [14]. Using this terminology, (10) states that $(p_n(x))$ is the associated sequence to $f(t)$ and (11) states that $(s_n(x))$ is the Sheffer sequence to the pair $(g(t), f(t))$ (see Theorem 2.3.4 in [14]).

Theorem 3.8.3 in [14] tells us that $(s_n(x-n))$ is the Sheffer sequence to the pair $(\tilde{g}(t), \tilde{f}(t))$ with

$$\tilde{g}(t) = g(t)(1 + f(t)/f'(t))$$

and

$$\tilde{f}(t) = f(t)\exp(t).$$

The compositional inverse of $\tilde{f}(t)$ is given by $\tilde{B}(t) := B(W_B(t))$:

$$B(W_B(\tilde{f}(t))) = B(W_B(f(t)\exp(t))) = B(W_B(f(t)\exp(B(f(t))))) = B(f(t)) = t.$$

Further, we have

$$\begin{aligned}\tilde{A}(t) &:= \tilde{g}^{-1}(\tilde{B}(t)) \\ &= (g(B(t))(1 + f(B(t))/f'(B(t))))^{-1} \circ W_B(t) = \frac{A(t)}{1 + tB'(t)} \circ W_B(t),\end{aligned}$$

which proves (13) again due to Theorem 2.3.4 in [14].

It remains to prove (12), i.e. that $(\frac{xp_n(x)}{x-n})$ is the associated sequence to $\tilde{f}(t)$. We already know that $(p_n(x-n))$ is the Sheffer sequence to the pair $(1 + f(t)/f'(t), \tilde{f}(t))$. By Theorem 2.3.6 of [14] it follows that the associated sequence to $\tilde{f}(t)$ is given by $(1+f(\mathrm{d}/\mathrm{d}x)/f'(\mathrm{d}/\mathrm{d}x))p_n(x-n)$. By Theorem 2.3.7 and Corollary 3.6.6 in [14], we have

$$\begin{aligned}\left(1 + \frac{f(\mathrm{d}/\mathrm{d}x)}{f'(\mathrm{d}/\mathrm{d}x)}\right) p_n(x-n) &= p_n(x-n) + \frac{1}{f'(\mathrm{d}/\mathrm{d}x)} n p_{n-1}(x-n) \\ &= p_n(x-n) + \frac{np_n(x-n)}{x-n} = \frac{xp_n(x-n)}{x-n},\end{aligned}$$

which proves the rest of the lemma. □

3. Graph homology

This section is concerned with the space of graph homology classes of unitrivalent graphs. A very detailed discussion of this space and other graph homology spaces can be found in [1]. Further aspects of graph homology can be found in [17], and, with respect to Rozansky-Witten invariant, in [7].

3.1. The graph homology space. In this article, *graph* means a collection of vertices connected by edges, i.e. every edge connects two vertices. We want to call a half-edge (i.e. an edge together with an adjacent vertex) of a graph a *flag*. So, every edge consists of exactly two flags. Every flag belongs to exactly one vertex of the graph. On the other hand, a vertex is given by the set of its flags. It is called *univalent* if there is only one flag belonging to it, and it is called *trivalent* if there are exactly three flags belonging to it. We shall identify edges and vertices with the set of their flags. We shall also call univalent vertices *legs*. A graph is called *vertex-oriented* if, for every vertex, a cyclic ordering of its flags is fixed.

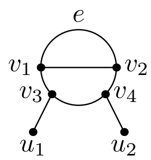

FIGURE 1. This Jacobi diagram has four trivalent vertices $v_1, \ldots, v_4$, and two univalent vertices u_1 and u_2, and e is one of its 7 edges.

DEFINITION 1. A *Jacobi diagram* is a vertex-oriented graph with only uni- and trivalent vertices. A *connected Jacobi diagram* is a Jacobi diagram which is connected as a graph. A *trivalent Jacobi diagram* is a Jacobi diagram with no univalent vertices.
We define the *degree of a Jacobi diagram* to be the number of its vertices. It is always an even number.
We identify two graphs if they are isomorphic as vertex-oriented graphs in the obvious sense.

Example 1. The empty graph is a Jacobi diagram, denoted by 1. The unique Jacobi diagram consisting of two univalent vertices (which are connected by an edge) is denoted by ℓ.

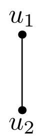

FIGURE 2. The Jacobi diagram ℓ with its two univalent vertices u_1 and u_2.

Remark 1. There are different names in the literature for what we call a "Jacobi diagram", e.g. unitrivalent graphs, chord diagrams, Chinese characters, Feynman diagrams. The name chosen here is also used by D. Thurston in [17]. The name comes from the fact that the IHX relation in graph homology defined later is essentially the well-known Jacobi identity for Lie algebras.
With our definition of the degree of a Jacobi diagram, the algebra of graph homology defined later will be commutative in the graded sense. Further, the map RW that will associate to each Jacobi diagram a Rozansky-Witten class will respect this grading. But note that often the degree is defined to be *half* of the number of vertices, which still is an integer.

We can always draw a Jacobi diagram in a planar drawing so that it looks like a planar graph with vertices of valence 1, 3 or 4. Each 4-valent vertex has to be interpreted as a crossing of two non-connected edges of the drawn graph and not as one of its vertices. Further, we want the counter-clockwise ordering of the flags at each trivalent vertex in the drawing to be the same as the given cyclic ordering.

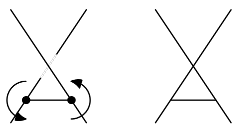

Figure 3. These two graphs depict the same one.

In drawn Jacobi diagrams, we also use a notation like $\cdots \overset{n}{-} \cdots$ for a part of a graph which looks like a long line with n univalent vertices ("legs") attached to it, for example $\ldots \perp\!\perp\!\perp \ldots$ for $n = 3$. The position of n indicates the placement of the legs relative to the "long line".

Definition 2. Let $\mathcal{T}$ be any tensor category (commutative and with unit). Every Jacobi diagram Γ with k trivalent and l univalent vertices induces a natural transformation Ψ^Γ between the functors

$$\mathcal{T} \to \mathcal{T}, V \mapsto \mathrm{S}_k\Lambda_3 V \otimes \mathrm{S}_l V \tag{14}$$

and

$$\mathcal{T} \to \mathcal{T}, V \mapsto \mathrm{S}^e\mathrm{S}^2 V, \tag{15}$$

where $e := \frac{3k+l}{2}$ which is given by

$$\begin{aligned}\Psi^\Gamma : \mathrm{S}_k\Lambda_3 V \otimes \mathrm{S}_l V &\xrightarrow{(1)} \bigotimes_{t\in T} \Lambda_3 V \otimes \bigotimes_{f\in U} V \xrightarrow{(2)} \bigotimes_{t\in T}\bigotimes_{f\in t} V \otimes \bigotimes_{f\in U} V \\ &\xrightarrow{(3)} \bigotimes_{f\in F} V \xrightarrow{(4)} \bigotimes_{e\in E}\bigotimes_{f\in e} V \xrightarrow{(5)} \mathrm{S}^e\mathrm{S}^2 V,\end{aligned} \tag{16}$$

where T is the set of the trivalent vertices, U the set of the univalent vertices, F the set of flags, and E the set of edges of Γ. Further,

(1) is given by the natural inclusions of the invariants in the tensor products,
(2) is given by the canonical maps (see Proposition 1 and recall that the sets t are cyclicly ordered),
(3) is given by the associativity of the tensor product,
(4) is given again by the associativity of the tensor product, and finally
(5) is given by the canonical projections onto the coinvariants.

Definition 3. We define $\mathcal{B}$ to be the $\mathbb{Q}$-vector space spanned by all Jacobi diagrams modulo the IHX relation

$$\mathrm{I} = \mathrm{H} - \mathrm{X} \tag{17}$$

and the anti-symmetry (AS) relation

$$\curlyvee + \curlyvee = 0, \tag{18}$$

which can be applied anywhere within a diagram. (For this definition see also [1] and [17].) Two Jacobi diagrams are said to be *homologous* if they are in the same class modulo the IHX and AS relation.

Furthermore, let $\mathcal{B}'$ be the subspace of $\mathcal{B}$ spanned by all Jacobi diagrams not containing ℓ as a component, and let ${}^{\mathrm{t}}\mathcal{B}$ be the subspace of $\mathcal{B}'$ spanned by all trivalent Jacobi diagrams. All these are graded and double-graded. The grading is induced by the degree of Jacobi diagrams, the double-grading by the number of univalent and trivalent vertices.

The completion of $\mathcal{B}$ (resp. $\mathcal{B}'$, resp. ${}^{\mathrm{t}}\mathcal{B}$) with respect to the grading will be denoted by $\hat{\mathcal{B}}$ (resp. $\hat{\mathcal{B}}'$, resp. ${}^{\mathrm{t}}\hat{\mathcal{B}}$).

We define $\mathcal{B}_{k,l}$ to be the subspace of $\hat{\mathcal{B}}$ generated by graphs with k trivalent and l univalent vertices. $\mathcal{B}'_{k,l}$ and ${}^{\mathrm{t}}\mathcal{B}_k := {}^{\mathrm{t}}\mathcal{B}_{k,0}$ are defined similarly.

All these spaces are called *graph homology spaces* and their elements are called *graph homology classes* or *graphs* for short.

Remark 2. The subspaces $\mathcal{B}_k$ of $\hat{\mathcal{B}}$ spanned by the Jacobi diagrams of degree k are always of finite dimension. The subspace $\mathcal{B}_0$ is one-dimensional and spanned by the graph homology class 1 of the empty diagram 1.

Remark 3. We have $\hat{\mathcal{B}} = \prod_{k,l\geq 0} B_{k,l}$. In view of the following Definition 4, $\hat{\mathcal{B}}'$ and $\hat{\mathcal{B}}$ are naturally $\mathbb{Q}$-algebras. As $\mathbb{Q}$-algebras, we have $\hat{\mathcal{B}} = \hat{\mathcal{B}}'[[\ell]]$. Due to the AS relation, the spaces $\mathcal{B}'_{k,l}$ are zero for $l > k$. Therefore, $\hat{\mathcal{B}}' = \prod_{k=0}^{\infty} \bigoplus_{l=0}^{k} \mathcal{B}'_{k,l}$.

Example 2. If γ is a graph which has a part looking like $\cdots \overset{n}{-} \cdots$, it will become $(-1)^n\gamma$ if we substitute the part $\cdots \overset{n}{-} \cdots$ by $\cdots \underset{n}{-} \cdots$ due to the anti-symmetry relation.

3.2. Operations with graphs.

Definition 4. Disjoint union of Jacobi diagrams induces a bilinear map

$$\hat{\mathcal{B}} \times \hat{\mathcal{B}} \to \hat{\mathcal{B}}, (\gamma, \gamma') \mapsto \gamma \cup \gamma'. \tag{19}$$

By mapping $1 \in \mathbb{Q}$ to $1 \in \hat{\mathcal{B}}$, the space $\hat{\mathcal{B}}$ becomes a graded $\mathbb{Q}$-algebra, which has no components in odd degrees. Often, we omit the product sign "$\cup$". $\mathcal{B}$, $\mathcal{B}'$, ${}^{\mathrm{t}}\mathcal{B}$, and so on are subalgebras.

Definition 5. Let $k \in \mathbb{N}$. We call the graph homology class of the Jacobi diagram $\overset{2k}{\bigcirc}$ the $2k$*-wheel* w_{2k}, i.e. $w_2 = \multimap\!\!\circ\!\!-$, $w_4 = \boxtimes$, and so on. It has $2k$ univalent and $2k$ trivalent vertices. The expression w_0 will be given a meaning later, see section 3.3.

Remark 4. The wheels w_k with k odd vanish in $\hat{\mathcal{B}}$ due to the AS relation.

Let Γ be a Jacobi diagram and u, u' be two different univalent vertices of Γ. These two should not be the two vertices of a component ℓ of Γ. Let v (resp. v') be the vertex u (resp. u') is attached to. The process of *gluing the vertices u and u'* means to remove u and u' together with the edges connecting them to v resp. v' and to add a new edge between v and v'. Thus, we arrive at a new graph $\Gamma/(u, u')$, whose number of trivalent vertices is the number of trivalent vertices of Γ and whose number of univalent vertices is the number of univalent vertices of Γ minus two. To make it a Jacobi diagram we define the cyclic orientation of the flags at v (resp. v') to be the cyclic orientation of the flags at v (resp. v') in Γ with the flag belonging to the edge connecting v (resp. v') with u (resp. u') replaced by the flag belonging to the added edge. For example,

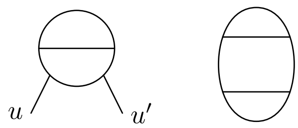

Figure 4. Gluing the two univalent vertices u and u' of the left graph produces the right one, denoted by $\ominus_2$.

gluing the two univalent vertices of w_2 leads to the graph $\ominus$.
If $\pi = \{\{u_1, u'_1\}, \ldots, \{u_k, u'_k\}\}$ is a set of two-element sets of legs that are pairwise disjoint and such that each pair u_k, u'_k fulfills the assumptions of the previous construction, we set

$$\Gamma/\pi := \Gamma/(u_1, u'_1)/\ldots/(u_k, u'_k). \tag{20}$$

Of course, the process of gluing two univalent vertices given above does not work if u and u' are the two univalent vertices of ℓ, thus our assumption on Γ.

Definition 6. Let Γ, Γ' be two Jacobi diagrams, at least one of them without ℓ as a component and $U = \{u_1, \ldots, u_n\}$ resp. U' the sets of their univalent vertices. We define

$$\hat{\Gamma}(\Gamma') := \sum_{\substack{f: U \hookrightarrow U' \\ \text{injective}}} (\Gamma \cup \Gamma')/(u_1, f(u_1))/\ldots/(u_n, f(u_n)), \tag{21}$$

viewed as an element in $\hat{\mathcal{B}}$.
This induces for every $\gamma \in \hat{\mathcal{B}}$ a ${}^{\mathrm{t}}\hat{\mathcal{B}}$-linear map

$$\hat{\gamma} : \hat{\mathcal{B}}' \to \hat{\mathcal{B}}', \gamma' \mapsto \hat{\gamma}(\gamma'). \tag{22}$$

Example 3. Set $\partial := \frac{1}{2}\hat{\ell}$. It is is an endomorphism of $\hat{\mathcal{B}}'$ of degree -2. For example, $\partial {-}\!\circ\!{-} = \ominus$. By setting

$$\partial(\gamma, \gamma') := \partial(\gamma \cup \gamma') - \partial(\gamma) \cup \gamma' - \gamma \cup \partial(\gamma') \tag{23}$$

for $\gamma, \gamma' \in \hat{\mathcal{B}}'$, we have the following formula for all $\gamma \in \hat{\mathcal{B}}'$:

$$\partial(\gamma^n) = \binom{n}{1}\partial(\gamma)\gamma^{n-1} + \binom{n}{2}\partial(\gamma,\gamma)\gamma^{n-2}. \tag{24}$$

This shows that ∂ is a differential operator of order two acting on $\hat{\mathcal{B}}'$.
Acting by ∂ on a Jacobi diagram means to glue two of its univalent vertices in all possible ways, acting by $\partial(\cdot,\cdot)$ on two Jacobi diagrams means to connect them by gluing a univalent vertex of the first with a univalent vertex of the second in all possible ways.

Definition 7. Let Γ, Γ' be two Jacobi diagrams, at least one of them without ℓ as a component, and $U = \{u_1, \ldots, u_n\}$ resp. U' the sets of their univalent vertices. We define

$$\langle \Gamma, \Gamma' \rangle := \sum_{\substack{f: U \to U' \\ \text{bijective}}} (\Gamma \cup \Gamma')/(u_1, f(u_1))/\ldots/(u_n, f(u_n)), \tag{25}$$

viewed as an element in ${}^{\mathrm{t}}\hat{\mathcal{B}}$.
This induces a ${}^{\mathrm{t}}\hat{\mathcal{B}}$-bilinear map

$$\langle \cdot, \cdot \rangle : \hat{\mathcal{B}}' \times \hat{\mathcal{B}} \to {}^{\mathrm{t}}\hat{\mathcal{B}}, \tag{26}$$

which is symmetric on $\hat{\mathcal{B}}' \times \hat{\mathcal{B}}'$.

Note that $\langle \Gamma, \Gamma' \rangle$ is zero unless Γ and Γ' have equal numbers of univalent vertices. In this case, the expression is the sum over all possibilities to glue the univalent vertices of Γ with univalent vertices of Γ'.
Note that $\langle \Gamma, \Gamma' \rangle$ is zero unless Γ and Γ' have equal numbers of univalent vertices. In this case, the expression is the sum over all possibilities to glue the univalent vertices of Γ with univalent vertices of Γ'.

Proposition 4. *The map $\langle 1, \cdot \rangle : \hat{\mathcal{B}} \to {}^{\mathrm{t}}\hat{\mathcal{B}}$ is the canonical projection map, i.e. it removes all non-trivalent components from a graph. Furthermore, for $\gamma \in \hat{\mathcal{B}}'$ and $\gamma' \in \hat{\mathcal{B}}$, we have*

$$\left\langle \gamma, \frac{\ell}{2}\gamma' \right\rangle = \langle \partial\gamma, \gamma' \rangle . \tag{27}$$

For $\gamma, \gamma' \in \hat{\mathcal{B}}'$, we have the following (combinatorial) formula:

$$\langle \exp(\partial)(\gamma\gamma'), 1 \rangle = \langle \exp(\partial)\gamma, \exp(\partial)\gamma' \rangle . \tag{28}$$

Proof. The formula (27) should be clear from the definitions.
Let us investigate (28) a bit more. We can assume that γ and γ' are Jacobi diagrams with l resp. l' univalent vertices and $l + l' = 2n$ with $n \in \mathbb{N}_0$. So we have to prove

$$\frac{\partial^n}{n!}(\gamma\gamma') = \sum_{\substack{m,m'=0 \\ l-2m=l'-2m'}}^{\infty} \left\langle \frac{\partial^m}{m!}\gamma, \frac{\partial^{m'}}{m'!}\gamma' \right\rangle,$$

since $\langle\cdot,1\rangle : \hat{\mathcal{B}} \to {}^{\mathrm{t}}\hat{\mathcal{B}}$ means to remove the components with at least one univalent vertex. Recalling the meaning of $\langle\cdot,\cdot\rangle$, it should be clear that (28) follows from the fact that applying $\frac{\partial^k}{k!}$ on a Jacobi diagram means to glue all subsets of $2k$ of its univalent vertices to k pairs in all possible ways. $\square$

3.3. An $\mathfrak{sl}_2$-action on the space of graph homology. In this short section we want to extend the space of graph homology slightly. This is mainly due to two reasons: When we defined the expression $\hat{\Gamma}(\Gamma)$ for two Jacobi diagrams Γ and Γ', we restricted ourselves to the case that Γ or Γ' does not contain a component with an ℓ. Secondly, we have not given the *zero-wheel* w_0 a meaning yet.
We do this by adding an element $\bigcirc$ to the various spaces of graph homology.

Definition 8. *The extended space of graph homology* is the space $\hat{\mathcal{B}}[[\bigcirc]]$. Further, we set $w_0 := \bigcirc$, which, at least pictorially, is in accordance with the definition of w_k for $k > 0$.

Note that this element is not depicting a Jacobi diagram as we have defined it. Nevertheless, we want to use the notion that $\bigcirc$ has no univalent and no trivalent vertices, i.e. the homogeneous component of degree zero of $\hat{\mathcal{B}}[[\bigcirc]]$ is $\mathbb{Q}[[\bigcirc]]$.
When defining $\Gamma/(u,u')$ for a Jacobi diagram Γ with two univalent vertices u and u', i.e. gluing u to u', we assumed that u and u' are not the vertices of one component ℓ of Γ. Now we extend this definition by defining $\Gamma/(u,u')$ to be the extended graph homology class we get by replacing ℓ with $\bigcirc$, whenever u and u' are the two univalent vertices of a component ℓ of Γ.
Doing so, we can give the expression $\hat{\gamma}(\gamma') \in \hat{\mathcal{B}}[[\bigcirc]]$ a meaning with no restrictions on the two graph homology classes $\gamma, \gamma' \in \hat{\mathcal{B}}$, i.e. every $\gamma \in \hat{\mathcal{B}}[[\bigcirc]]$ defines a ${}^{\mathrm{t}}\hat{\mathcal{B}}[[\bigcirc]]$-linear map

$$\hat{\gamma} : \hat{\mathcal{B}}[[\bigcirc]] \to \hat{\mathcal{B}}[[\bigcirc]]. \tag{29}$$

Example 4. We have

$$\partial\ell = \bigcirc. \tag{30}$$

Remark 5. We can similarly extend $\langle\cdot,\cdot\rangle : \hat{\mathcal{B}}' \times \hat{\mathcal{B}} \to {}^{\mathrm{t}}\hat{\mathcal{B}}$ to a ${}^{\mathrm{t}}\hat{\mathcal{B}}[[\bigcirc]]$-bilinear form

$$\langle\cdot,\cdot\rangle : \hat{\mathcal{B}}[[\bigcirc]] \times \hat{\mathcal{B}}[[\bigcirc]] \to {}^{\mathrm{t}}\hat{\mathcal{B}}[[\bigcirc]]. \tag{31}$$

Both $\ell/2$ and ∂ are two operators acting on the extended space of graph homology, the first one just multiplication with $\ell/2$. By calculating their commutator, we show that they induce a natural structure of an $\mathfrak{sl}_2$-module on $\hat{\mathcal{B}}[[\bigcirc]]$.

Proposition 5. *Let $H : \hat{\mathcal{B}}[[\bigcirc]] \to \hat{\mathcal{B}}[[\bigcirc]]$ be the linear operator which acts on $\gamma \in \hat{\mathcal{B}}_{k,l}[[\bigcirc]]$ by*

$$H\gamma = \left(\frac{1}{2}\bigcirc + l\right)\gamma. \tag{32}$$

We have the following commutator relations in $\operatorname{End}\hat{\mathcal{B}}[[\bigcirc]]$*:*

$$[\ell/2, \partial] = -H, \tag{33}$$

$$[H, \ell/2] = 2 \cdot \ell/2, \tag{34}$$

and

$$[H, \partial] = -2\partial, \tag{35}$$

i.e. the triple $(\ell/2, -\partial, H)$ *defines a* $\mathfrak{sl}_2$*-operation on* $\hat{\mathcal{B}}[[\bigcirc]]$.

Proof. Equations (34) and (35) follow from the fact that multiplying by $\bigcirc$ commutes with $\ell/2$ and ∂, and from the fact that $\ell/2$ is an operator of degree 2 with respect to the grading given by the number of univalent vertices, whereas ∂ is an operator of degree -2 with respect to the same grading.
It remains to look at (33). For $\gamma \in \hat{\mathcal{B}}_{k,l}[[\bigcirc]]$, we calculate

$$[\ell, \partial]\gamma = \ell\partial(\gamma) - \partial(\ell\gamma) = \ell\partial(\gamma) - \partial(\ell)\gamma - \ell\partial(\gamma) - \partial(\ell, \gamma) = -\bigcirc\gamma - 2l\gamma = -2H\gamma. \tag{36}$$

$\square$

Remark 6. Since $\hat{\mathcal{B}}[[\bigcirc]]$ is infinite-dimensional, we have unfortunately difficulties to apply the standard theory of $\mathfrak{sl}_2$-representations to this $\mathfrak{sl}_2$-module. For example, there are no eigenvectors for the operator H.

3.4. Closed and connected graphs, the closure of a graph. As the number of connected components of a Jacobi diagram is preserved by the IHX- and AS-relations each graph homology space inherits a grading by the number of connected components. For any $k \in \mathbb{N}_0$ we define $\mathcal{B}^k$ to be the subspace of $\mathcal{B}$ spanned by all Jacobi diagrams with exactly k connected components. Similarly, we define ${}^{\mathrm{t}}\mathcal{B}^k$, $\hat{\mathcal{B}}^k$, ${}^{\mathrm{t}}\hat{\mathcal{B}}^k$.
We have $\mathcal{B} = \bigoplus_{k=0}^\infty \mathcal{B}^k$ with $\mathcal{B}^0 = \mathbb{Q} \cdot 1$. Analogous results hold for ${}^{\mathrm{t}}\mathcal{B}$, $\hat{\mathcal{B}}$, ${}^{\mathrm{t}}\hat{\mathcal{B}}$.

Definition 9. A graph homology class γ is called *closed* if $\gamma \in {}^{\mathrm{t}}\hat{\mathcal{B}}$. The class γ is called *connected* if $\gamma \in \hat{\mathcal{B}}^1$. The *connected component of* γ is defined to be $\mathrm{pr}^1(\gamma)$ where $\mathrm{pr}^1 : \hat{\mathcal{B}} = \prod_{i=0}^\infty \hat{\mathcal{B}}^i \to \hat{\mathcal{B}}^1$ is the canonical projection. The *closure* $\langle\gamma\rangle$ *of* γ is defined by $\langle\gamma\rangle := \langle\gamma, \exp(\ell/2)\rangle$. The *connected closure* $\langle\langle\gamma\rangle\rangle$ *of* γ is defined to be the connected component of the closure $\langle\gamma\rangle$ of γ.

For every finite set L, we define $\mathrm{P}_2(L)$ to be the set of partitions of L into subsets of two elements. With this definition, we can express the closure of a Jacobi diagram Γ as

$$\langle\Gamma\rangle = \sum_{\pi \in \mathrm{P}_2(L)} \Gamma/\pi. \tag{37}$$

Example 5. We have $\langle w_2\rangle = \ominus$, $\langle\langle w_2\rangle\rangle = \ominus$, $\langle w_2^2\rangle = 2\ominus_2 + \ominus^2$, $\langle\langle w_2^2\rangle\rangle = 2\ominus_2$.

Let $L_1, \dots, L_n$ be finite and pairwise disjoint sets. We set $L := \bigsqcup_{i=1}^n L_i$. Let $\pi \in \mathrm{P}_2(L)$ be a partition of L in 2-element-subsets. We say that a pair $l, l' \in L$ *is linked by* π if there is an $i \in \{1, \dots n\}$ such that $l, l' \in L_i$ or $\{l, l'\} \in \pi$. We say that π *connects the sets* $L_1,\dots,L_n$ if and only if for each pair $l,l'\in L$ there is a chain of elements $l_1,\dots,l_k$ such that l is linked to l_1, l_i is linked to l_{i+1} for $i\in\{1,\dots,k-1\}$ and l_k is linked to l'. The subset of $\mathrm{P}_2(L)$ of partitions π connecting $L_1,\dots,L_n$ is denoted by $\mathrm{P}_2(\{L_1,\dots,L_n\})$. We have

$$\mathrm{P}_2(L)=\bigsqcup_{\bigsqcup\mathfrak{I}=\{1,\dots,n\}}\left\{\bigsqcup_{I\in\mathfrak{I}}\pi_I:\pi_I\in P_2(\{L_i:i\in I\})\right\}. \tag{38}$$

Here, $\bigsqcup\mathfrak{I}=\{1,\dots,n\}$ means that $\mathfrak{I}$ is a partition of $\{1,\dots,n\}$ in disjoint subsets.

Let $\Gamma_1,\dots,\Gamma_n$ be connected Jacobi diagrams. We denote by $\Gamma:=\prod_{i=1}^n\Gamma_i$ the product over all these Jacobi diagrams. Let L_i be the set of legs of Γ_i and denote by $L:=\bigsqcup_{i=1}^n L_i$ the set of all legs of Γ.

For every partition $\pi\in\mathrm{P}_2(L)$ the graph Γ/π is connected if and only if $\pi\in\mathrm{P}_2(\{L_1,\dots,L_n\})$.

Using (38) we have

$$\begin{aligned}\langle\Gamma\rangle=\sum_{\pi\in\mathrm{P}_2(L)}\Gamma/\pi&=\sum_{\bigsqcup\mathfrak{I}=\{1,\dots,n\}}\prod_{I\in\mathfrak{I}}\sum_{\pi\in\mathrm{P}_2(\{L_i:i\in I\})}\left(\prod_{i\in I}\Gamma_i\right)/\pi\\&=\sum_{\bigsqcup\mathfrak{I}=\{1,\dots,n\}}\prod_{I\in\mathfrak{I}}\left\langle\left\langle\prod_{i\in I}\Gamma_i\right\rangle\right\rangle.\end{aligned}\tag{39}$$

With this result we can prove the following Proposition:

Proposition 6. *For any connected graph homology class γ we have*

$$\exp\langle\langle\exp\gamma\rangle\rangle=\langle\exp\gamma\rangle\,. \tag{40}$$

Note that both sides are well-defined in $\hat{\mathcal{B}}$ since γ and $\langle\langle\cdots\rangle\rangle$ as connected graphs have no component in degree zero.

Proof. Let Γ be any connected Jacobi diagram. By (39) we have

$$\langle\Gamma^n\rangle=\sum_{\bigsqcup\mathfrak{I}=\{1,\dots,n\}}\prod_{I\in\mathfrak{I}}\langle\langle\Gamma^{\#I}\rangle\rangle=\sum_{\lambda\in\mathrm{P}(n)}n!\prod_{i=1}^\infty\frac{1}{\lambda_i!}\left(\langle\langle\Gamma^i\rangle\rangle/i!\right)^{\lambda_i}.$$

By linearity this result holds also if we substitute Γ by the connected graph homology class γ.

Using this,

$$\begin{aligned}\langle\exp\gamma\rangle&=\sum_{n=0}^\infty\frac{1}{n!}\langle\gamma^n\rangle=\sum_{n=0}^\infty\frac{1}{n!}\sum_{\lambda\in\mathrm{P}(n)}n!\prod_{i=1}^\infty\frac{1}{\lambda_i!}\left(\langle\langle\gamma^i\rangle\rangle/i!\right)^{\lambda_i}\\&=\prod_{i=1}^\infty\sum_{\lambda=0}^\infty\frac{1}{\lambda!}\left(\langle\langle\gamma^i\rangle\rangle/i!\right)^{\lambda}=\prod_{i=1}^\infty\exp\left(\langle\langle\gamma^i\rangle\rangle/i!\right)=\exp\langle\langle\exp\gamma\rangle\rangle\,.\end{aligned}$$

□

3.5. Polywheels.

Definition 10. For each $n \in \mathbb{N}_0$ we set $\tilde{w}_{2n} := -w_{2n}$. Let λ be a partition of n. We set

$$\tilde{w}_{2\lambda} := \prod_{i=1}^{\infty} \tilde{w}_{2i}^{\lambda_i}. \tag{41}$$

The closure $\langle \tilde{w}_{2\lambda} \rangle$ of $\tilde{w}_{2\lambda}$ is called a *polywheel*. The subspace in ${}^{\mathrm{t}}\mathcal{B}$ spanned by all polywheels is denoted by $\mathcal{W}$ and called the *polywheel subspace*. The subalgebra in ${}^{\mathrm{t}}\mathcal{B}$ spanned by all polywheels is denoted by $\mathcal{C}$ and called the *algebra of polywheels*.
The connected closure $\langle\langle \tilde{w}_{2\lambda} \rangle\rangle$ of $\tilde{w}_{2\lambda}$ is called a *connected polywheel*.

Remark 7. As discussed by J. Sawon in his thesis [16], $\mathcal{W}$ is proper graded subspace of ${}^{\mathrm{t}}\mathcal{B}$. From degree eight on, ${}^{\mathrm{t}}\mathcal{B}_k$ is considerably larger than $\mathcal{W}_k$. On the other hand it is unknown (at least to the author) if the inclusion $\mathcal{C} \subseteq {}^{\mathrm{t}}\mathcal{B}$ is proper.

Remark 8. The subalgebra $\mathcal{C}'$ in ${}^{\mathrm{t}}\mathcal{B}$ spanned by all connected polywheels equals $\mathcal{C}$. This is since we can use (40) to express every polywheel as a polynomial of connected polywheels and vice versa.

Example 6. Using Proposition 6 we calculated the following expansions of the connected polywheels in terms of wheels:

$$\begin{aligned}
\langle\langle \tilde{w}_2 \rangle\rangle &= \langle \tilde{w}_2 \rangle \\
\left\langle\left\langle \tilde{w}_2^2 \right\rangle\right\rangle &= \left\langle \tilde{w}_2^2 \right\rangle - \langle \tilde{w}_2 \rangle^2 \\
\langle\langle \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_4 \rangle \\
\left\langle\left\langle \tilde{w}_2^3 \right\rangle\right\rangle &= \left\langle \tilde{w}_2^3 \right\rangle - 3 \langle \tilde{w}_2 \rangle \left\langle \tilde{w}_2^2 \right\rangle + 2 \langle \tilde{w}_2 \rangle^3 \\
\langle\langle \tilde{w}_2 \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_2 \tilde{w}_4 \rangle - \langle \tilde{w}_2 \rangle \langle \tilde{w}_4 \rangle \\
\langle\langle \tilde{w}_6 \rangle\rangle &= \langle \tilde{w}_6 \rangle \\
\left\langle\left\langle \tilde{w}_2^4 \right\rangle\right\rangle &= \left\langle \tilde{w}_2^4 \right\rangle - 4 \langle \tilde{w}_2 \rangle \left\langle \tilde{w}_2^3 \right\rangle - 3 \left\langle \tilde{w}_2^2 \right\rangle^2 + 12 \langle \tilde{w}_2 \rangle^2 - 6 \langle \tilde{w}_2 \rangle^4 \\
\left\langle\left\langle \tilde{w}_2^2 \tilde{w}_4 \right\rangle\right\rangle &= \left\langle \tilde{w}_2^2 \tilde{w}_4 \right\rangle - 2 \langle \tilde{w}_2 \rangle \langle \tilde{w}_2 \tilde{w}_4 \rangle - \left\langle \tilde{w}_2^2 \right\rangle \langle \tilde{w}_4 \rangle + 2 \langle \tilde{w}_2 \rangle^2 \langle \tilde{w}_4 \rangle \\
\langle\langle \tilde{w}_2 \tilde{w}_6 \rangle\rangle &= \langle \tilde{w}_2 \tilde{w}_6 \rangle - \langle \tilde{w}_2 \rangle \langle \tilde{w}_6 \rangle \\
\left\langle\left\langle \tilde{w}_4^2 \right\rangle\right\rangle &= \left\langle \tilde{w}_4^2 \right\rangle - \langle \tilde{w}_4 \rangle^2 \\
\langle\langle \tilde{w}_8 \rangle\rangle &= \langle \tilde{w}_8 \rangle .
\end{aligned} \tag{42}$$

4. Holomorphic symplectic manifolds

4.1. Definition and general properties.

Definition 11. A *holomorphic symplectic manifold* (X, σ) is a compact complex manifold X together with an everywhere non-degenerate holomorphic two-form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$. Here, we call σ *everywhere non-degenerate* if σ induces an isomorphism $\mathcal{T}_X \to \Omega_X$.

The holomorphic symplectic manifold (X, σ) is called *irreducible* if it is simply-connected and $\mathrm{H}^0(X, \Omega_X^2)$ is one-dimensional, i.e. spanned by σ.

It follows immediately that every holomorphic symplectic manifold X has trivial canonical bundle whose sections are multiples of σ^n, and, therefore, vanishing first Chern class. In fact, all odd Chern classes vanish:

Proposition 7. *Let X be a complex manifold and E a complex vector bundle on X. If E admits a symplectic two-form, i.e. there exists a section $\sigma \in \mathrm{H}^0(X, \Lambda^2 E^*)$ such that the induced morphism $E \to E^*$ is an isomorphism, all odd Chern classes of E vanish.*

Remark 9. That the odd Chern classes of E vanish up to two-torsion follows immediately from the fact $c_{2k+1}(E) = -c_{2k+1}(E^*)$ for $k \in \mathbb{N}_0$.

The following proof using the splitting principle has been suggested to me by Manfred Lehn.

Proof. We prove the proposition by induction over the rank of E. For $\operatorname{rk} E = 0$, the claim is obvious.
By the splitting principle (see e.g. [5]), we can assume that E has a subbundle L of rank one. Let $L^\perp$ be the σ-orthogonal subbundle to L of E. Since σ is symplectic, $L^\perp$ is of rank $n-1$ and L is a subbundle of $L^\perp$. We have the following short exact sequences of bundles on X:

$$0 \longrightarrow L \longrightarrow E \longrightarrow E/L \longrightarrow 0$$

and

$$0 \longrightarrow L^\perp/L \longrightarrow E/L \longrightarrow E/L^\perp \longrightarrow 0.$$

Since σ induces a symplectic form on $L^\perp/L$, by induction, all odd Chern classes of this bundle of rank $\operatorname{rk} E - 2$ vanish. Furthermore, note that σ induces an isomorphism between L and $(E/L^\perp)^*$, so all odd Chern classes of $L \oplus E/L^\perp$ vanish.
Now, the two exact sequences give us $c(E) = c(L \oplus E/L^\perp) \cdot c(L^\perp/L)$. Therefore, we can conclude that all odd Chern classes of E vanish. □

Proposition 8. *For any irreducible holomorphic symplectic manifold (X, σ) of dimension $2n$ and $k \in 0, \dots, n$ the space $\mathrm{H}^{2k}(X, \mathcal{O}_X)$ is one-dimensional and spanned by the cohomology class $[\bar\sigma]^k$.*

Proof. See [2]. □

4.2. A pairing on the cohomology of a holomorphic symplectic manifold. Let (X, σ) be a holomorphic symplectic manifold. There is a natural pairing of coherent sheafs

$$\Lambda_* \mathcal{T}_X \otimes \Lambda^* \Omega_X \to \mathcal{O}_X. \tag{43}$$

As the natural morphism from $\Lambda_* \mathcal{T}_X$ to $\Lambda^* \mathcal{T}_X$ is an isomorphism and $\Lambda^* \mathcal{T}_X$ can be identified with $\Lambda^* \Omega_X$ by means of the symplectic form, we therefore have a natural map

$$\Lambda^* \Omega \otimes \Lambda^* \Omega_X \to \mathcal{O}_X. \tag{44}$$

We write

$$\langle \cdot, \cdot \rangle : \mathrm{H}^p(X, \Omega^*) \otimes \mathrm{H}^q(X, \Omega^*) \to \mathrm{H}^{p+q}(X, \mathcal{O}_X), (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle \tag{45}$$

for the induced map for any $p, q \in \mathbb{N}_0$.

In [10] we proved the following proposition:

Proposition 9. *For any $\alpha \in \mathrm{H}^*(X, \Omega^*)$ we have*

$$\int_X \alpha \exp \sigma = \int_X \langle \alpha, \exp \sigma \rangle \exp \sigma. \tag{46}$$

4.3. Example series. There are two main series of examples of irreducible holomorphic symplectic manifolds. Both of them are based on the Hilbert schemes of points on a surface:

Let X be any smooth projective surface over $\mathbb{C}$ and $n \in \mathbb{N}_0$. By $X^{[n]}$ we denote the Hilbert scheme of zero-dimensional subschemes of length n of X. By a result of Fogarty ([4]), $X^{[n]}$ is a smooth projective variety of dimension $2n$. The Hilbert scheme can be viewed as a resolution $\rho : X^{[n]} \to X^{(n)}$ of the n-fold symmetric product $X^{(n)} := X^n / \mathfrak{S}_n$. The morphism ρ, sending closed points, i.e. subspaces of X, to their support counting multiplicities, is called the Hilbert-Chow morphism.

Let $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ be any class. The class $\sum_{i=1}^n \mathrm{pr}_i^* \alpha \in \mathrm{H}^2(X^n, \mathbb{C})$ is invariant under the action of $\mathfrak{S}_n$, where $\mathrm{pr}_i : X^n \to X$ denotes the projection on the i^{th} factor. Therefore, there exists a class $\alpha^{(n)} \in \mathrm{H}^2(X^{(n)}, \mathbb{C})$ with $\pi^* \alpha^{(n)} = \sum_{i=1}^n \mathrm{pr}_i^* \alpha$, where $\pi : X^n \to X^{(n)}$ is the canonical projection. Using ρ this induces a class $\alpha^{[n]}$ in $\mathrm{H}^2(X^{[n]}, \mathbb{C})$.

If X is a K3 surface or an abelian surface, there exists a holomorphic symplectic form $\sigma \in \mathrm{H}^{2,0}(X) \subseteq \mathrm{H}^2(X, \mathbb{C})$. It was shown by Beauville in [2] that $\sigma^{[n]}$ is again symplectic, so $(X^{[n]}, \sigma^{[n]})$ is a holomorphic symplectic manifold.

Example 7. For any K3 surface X and holomorphic symplectic form $\sigma \in \mathrm{H}^{2,0}(X)$, the pair $(X^{[n]}, \sigma^{[n]})$ is in fact an irreducible holomorphic symplectic manifold.

This has also been proven by Beauville. In the case of an abelian surface A, we have to work a little bit more as $A^{[n]}$ is not irreducible in this case:

Let A be an abelian surface and let us denote by $s : A^{[n]} \to A$ the composition of the summation morphism $A^{(n)} \to A$ with the Hilbert-Chow morphism $\rho : A^{[n]} \to A^{(n)}$.

Definition 12. For any $n \in \mathbb{N}$, the n^{th} *generalised Kummer variety* $A^{[[n]]}$ is the fibre of s over $0 \in A$. For any class $\alpha \in \mathrm{H}^2(A, \mathbb{C})$, we set $\alpha^{[[n]]} := \alpha^{[n]}|_{A^{[[n]]}}$.

Remark 10. For $n = 2$ the generalised Kummer variety coincides with the Kummer model of a K3 surface (therefore the name).

Example 8. For every abelian surface A and holomorphic symplectic form $\sigma \in \mathrm{H}^{2,0}(A)$, the pair $(A^{[[n]]}, \sigma^{[[n]]})$ is an irreducible holomorphic symplectic manifold of dimension $2n-2$.

The proof can also be found in [2].

4.4. About $\alpha^{[n]}$ and $\alpha^{[[n]]}$. Let X be any smooth projective surface and $n \in \mathbb{N}_0$.

Let $X^{[n,n+1]}$ denote the incidence variety of all pairs $(\xi, \xi') \in X^{[n]} \times X^{[n+1]}$ with $\xi \subseteq \xi'$ (see [3]). We denote by $\psi : X^{[n,n+1]} \to X^{[n+1]}$ and by $\phi : X^{[n,n+1]} \to X^{[n]}$ the canonical maps. There is a third canonical map $\chi : X^{[n,n+1]} \to X$ mapping $(\xi, \xi') \mapsto x$ if ξ' is obtained by extending ξ at the closed point $x \in X$.

Proposition 10. *For any $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ we have*

$$\psi^* \alpha^{[n+1]} = \phi^* \alpha^{[n]} + \chi^* \alpha. \tag{47}$$

Proof. Let $p : X^{(n)} \times X \to X^{(n)}$ and $q : X^{(n)} \times X \to X$ denote the canonical projections. Let $\tau : X^{(n)} \times X \to X^{(n+1)}$ the obvious symmetrising map. The following diagram

$$\begin{array}{ccc}
X^{[n,n+1]} & = & X^{[n,n+1]} \\
{\scriptstyle (\phi,\chi)}\downarrow & & \downarrow{\scriptstyle \psi} \\
X^{[n]} \times X & & X^{[n+1]} \\
{\scriptstyle \rho \times \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \rho'} \\
X^{(n)} \times X & \xrightarrow{\tau} & X^{(n+1)} \\
{\scriptstyle \pi \times \mathrm{id}_X}\uparrow & & \uparrow{\scriptstyle \pi'} \\
X^{n+1} & = & X^{n+1}
\end{array}$$

is commutative. (Note that we have primed some maps to avoid name clashes.) We claim that $\tau^* \alpha^{(n+1)} = p^* \alpha^{(n)} + q^* \alpha$. In fact, since

$$(\pi \times \mathrm{id}_X)^* \tau^* \alpha^{(n+1)} = \pi'^* \alpha^{(n+1)} = \sum_{i=1}^{n+1} \mathrm{pr}_i^* \alpha,$$

this follows from the definition of $\alpha^{(n)}$. Finally, we can read off the diagram that

$$\begin{aligned}
\psi^* \alpha^{[n+1]} = \psi^* \rho'^* \alpha^{(n+1)} &= (\phi, \chi)^* (\rho \times \mathrm{id}_X)^* \tau^* \alpha^{(n+1)} \\
&= (\phi, \chi)^* (\rho \times \mathrm{id}_X)^* (p^* \alpha^{(n)} + q^* \alpha) = \phi^* \alpha^{[n]} + \chi^* \alpha.
\end{aligned}$$

□

Proposition 11. *Let $X = X_1 \sqcup X_2$ be the disjoint union of two projective smooth surfaces X_1 and X_2. We then have*

$$X^{[n]} = \coprod_{n_1+n_2=n} X_1^{[n_1]} \times X_2^{[n_2]}. \tag{48}$$

If $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ decomposes as $\alpha|_{X_1} = \alpha_1$ and $\alpha|_{X_2} = \alpha_2$, then $\alpha^{[n]}$ decomposes as

$$\alpha^{[n]}|_{X_!^{[n_1]} \times X_2^{[n_2]}} = \mathrm{pr}_1^* \alpha_1^{[n_1]} + \mathrm{pr}_2^* \alpha_2^{[n_2]}. \tag{49}$$

Proof. The splitting of $X^{[n]}$ follows from the universal property of the Hilbert scheme and is a well-known fact. The statement on $\alpha^{[n]}$ is easy to prove and so we shall only give a sketch: Let us denote by $i : X_1^{[n_1]} \times X_2^{[n_2]} \to X^{[n]}$ the natural inclusion. Furthermore let $j : X_1^{(n_1)} \times X_2^{(n_2)} \to X^{(n)}$ denote the natural symmetrising map. The following diagram is commutative:

$$\begin{tikzcd} X_1^{[n_1]} \times X_2^{[n_2]} \arrow[r, "i"] \arrow[d, "\rho_1 \times \rho_2"'] & X^{[n]} \arrow[d, "\rho"] \\ X_1^{(n_1)} \times X_2^{(n_2)} \arrow[r, "j"'] & X^{(n)}, \end{tikzcd} \tag{50}$$

where the $\rho_i : X_i^{[n_i]} \to X_i^{(n_i)}$ are the Hilbert-Chow morphisms. Since $j^* \alpha^{(n)} = \mathrm{pr}_1^* \alpha_1^{(n_1)} + \mathrm{pr}_2^* \alpha_2^{(n_2)}$, the commutativity of the diagram proves the statement on $\alpha^{[n]}$. ☐

Let A be again an abelian surface and $n \in \mathbb{N}$. Since A acts on itself by translation, there is also an induced operation of A on the Hilbert scheme $A^{[n]}$. Let us denote the restriction of this operation to the generalised Kummer variety $A^{[[n]]}$ by $\nu : A \times A^{[[n]]} \to A^{[n]}$. It fits into the following cartesian square:

$$\begin{tikzcd} A \times A^{[[n]]} \arrow[r, "\nu"] \arrow[d, "\mathrm{pr}_1"'] & A^{[n]} \arrow[d, "s"] \\ A \arrow[r, "n"'] & A, \end{tikzcd} \tag{51}$$

where s is the summation map as having been defined above and $n : A \to A, a \mapsto na$ is the (multiplication-by-n)-morphism. Since n is a Galois cover of degree n^4, the same holds true for ν.

Proposition 12. *For any $\alpha \in \mathrm{H}^2(A, \mathbb{C})$, we have*

$$\nu^* \alpha^{[n]} = n \, \mathrm{pr}_1^* \alpha + \mathrm{pr}_2^* \alpha^{[[n]]}. \tag{52}$$

Proof. By the Künneth decomposition theorem, we know that $\nu^* \alpha^{[n]}$ splits:

$$\nu^* \alpha^{[n]} = \mathrm{pr}_1^* \alpha_1 + \mathrm{pr}_2^* \alpha_2.$$

Set $\iota_1 : A \to A \times A^{[[n]]}, a \mapsto (a, \xi_0)$ and $\iota_2 : A^{[[n]]} \to A \times A^{[[n]]}, \xi \mapsto (0, \xi)$, where ξ_0 is any subscheme of length n concentrated in 0. We have

$$\alpha_1 = \iota_1^* \nu^* \alpha^{[n]} = (\rho \circ \nu \circ \iota_1)^* \alpha^{(n)} = (a \mapsto (\underbrace{a, \dots a}_{n}))^* \alpha^{(n)} = n\alpha \tag{53}$$

and

$$\alpha_2 = \iota_2^* \nu^* \alpha^{[n]} = i^* \alpha^{[n]} = \alpha^{[[n]]}, \tag{54}$$

where $i : A^{[[n]]} \to A^{[n]}$ is the natural inclusion map, thus proving the proposition. $\square$

4.5. Complex genera of Hilbert schemes of points on surfaces. The following theorem is an adaption of Theorem 4.1 of [3] to our context.

Theorem 1. *Let P be a polynomial in the variables $c_1, c_2, \dots$ and α over $\mathbb{Q}$. There exists a polynomial $\tilde{P} \in \mathbb{Q}[z_1, z_2, z_3, z_4]$ such that for every smooth projective surface X, $\alpha \in \mathrm{H}^2(X, \mathbb{Q})$ and $n \in \mathbb{N}_0$ we have:*

$$\int_{X^{[n]}} P(c_*(X^{[n]}), \alpha^{[n]}) = \tilde{P}\left(\int_X \alpha^2/2, \int_X c_1(X)\alpha, \int_X c_1(X)^2/2, \int_X c_2(X)\right). \tag{55}$$

Proof. The proof goes along the very same lines as the proof of Proposition 0.5 in [3] (see there). The only new thing we need is Proposition 10 of this paper to be used in the induction step of the adapted proof of Proposition 3.1 of [3] to our situation. $\square$

Let R be any $\mathbb{Q}$-algebra (commutative and with unit) and let $\phi \in R[[c_1, c_2, \dots]]$ be a non-vanishing power series in the universal Chern classes such that ϕ is multiplicative with respect to the Whitney sum of vector bundles, i.e.

$$\phi(E \oplus F) = \phi(E)\phi(F) \tag{56}$$

for all complex manifolds and complex vector bundles E and F on X. Any ϕ with this property induces a complex genus, also denoted by ϕ, by setting $\phi(X) := \int_X \phi(\mathcal{T}_X)$ for X a compact complex manifold. Let us call such a ϕ *multiplicative*.

Remark 11. By Hirzebruch's theory of multiplicative sequences and complex genera ([6]), we know that

(1) each complex genus is induced by a unique multiplicative ϕ, and
(2) the multiplicative elements in $R[[c_1, c_2, \dots]]$ are exactly those of the form $\exp(\sum_{k=1}^\infty a_k s_k)$ with $a_k \in R$.

More or less formally the following theorem follows from Theorem 1.

Theorem 2. *For each multiplicative $\phi \in R[[c_1, c_2, \dots]]$, there exist unique power series $A_\phi(p), B_\phi(p), C_\phi(p), D_\phi(p) \in pR[[p]]$ with vanishing constant coefficient such that for all smooth projective surfaces X and $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ we* *have:*

$$\sum_{n=0}^{\infty}\left(\int_{X^{[n]}}\phi(X^{[n]})\exp(\alpha^{[n]})\right)p^n \tag{57}$$
$$= \exp\left(A_\phi(p)\int_X \alpha^2/2 + B_\phi(p)\int_X c_1(X)\alpha + C_\phi(p)\int_X c_1^2(X)/2 + D_\phi(p)\int_X c_2(X)\right).$$

The first terms of $A_\phi(p), B_\phi(p), C_\phi(p), D_\phi(p)$ are given by

(58)
$$A_\phi(p) = p + O(p^2),\quad B_\phi(p) = \phi_1 p + O(p^2),\quad C_\phi(p) = \phi_{11}p + O(p^2),\text{ and } D_\phi(p) = \phi_2 p + O(p^2),$$

where ϕ_1 is the coefficient of c_1 in ϕ, ϕ_{11} the coefficient of $c_1^2/2$ and ϕ_2 the coefficient of c_2.

Proof. This theorem is again an adaption of a theorem (Theorem 4.2) of [3] to our context. Nevertheless, let us give the proof here:
Set $K := \left\{(X,\alpha) : X \text{ is a smooth projective surface and } \alpha \in \mathrm{H}^2(X,\mathbb{C})\right\}$ and let $\gamma : K \to \mathbb{Q}^4$ be the map $(X,\alpha) \mapsto (\alpha^2/2, c_1(X)\alpha, c_1(X)^2/2, c_2(X))$. Here, we have supressed the integral signs $\int_X$ and interpret the expressions α^2, etc. as intersection numbers on X. The image of K spans the whole $\mathbb{Q}^4$ (for explicit generators, we refer to [3]).
Now let us assume that a $(X,\alpha) \in K$ decomposes as $(X,\alpha) = (X_1,\alpha_1) \sqcup (X_2,\alpha_2)$. By the multiplicative behaviour of ϕ and exp we see that

$$\int_{X^{[n]}}\phi(c_*(X^{[n]}))\exp(\alpha^{[n]})$$
$$= \sum_{n_1+n_2=n}\left(\int_{X_1^{[n_1]}}\phi(c_*(X^{[n_1]}))\exp(\alpha_2^{[n_1]})\right)\left(\int_{X_2^{[n_2]}}\phi(c_*(X^{[n_2]}))\exp(\alpha_2^{[n_2]})\right),$$

whereas $H_\phi(p)(X,\alpha) := \sum_{n=0}^\infty\left(\int_{X^{[n]}}\phi(X^{[n]})\exp(\alpha^{[n]})\right)p^n$ fulfills

$$H_\phi(p)(X,\alpha) = H_\phi(p)(X_1,\alpha_1)H_\phi(p)(X_2,\alpha_2). \tag{$*$}$$

Since $H_\phi(p) : K \to \mathbb{Q}^4$ factors through γ and a map $h : \mathbb{Q}^4 \to R[[p]]$ by Theorem 1 and as the image of γ is Zariski dense in $\mathbb{Q}^4$, we conclude from $(*)$ that $\log h$ is a linear function which proves the first part of the theorem.
To get the first terms of the power series, we expand both sides of (57). The left hand side expands as

$$1 + (\alpha^2/2 + \phi_1 c_1(X)\alpha + \frac{\phi_{11}}{2}c_1^2(X) + \phi_2 c_2(X))p + O(p^2), \tag{59}$$

while the right hand side expands as

$$1 + (A_1\alpha^2/2 + B_1 c_1(X)\alpha + C_1 c_1(X)^2 + D_1 c_2(X))p + O(p^2), \tag{60}$$

where A_1, B_1, C_1, D_1 are the linear coefficients of A_ϕ, B_ϕ, C_ϕ, and D_ϕ, which can therefore be read off by comparing the expansions. □

Corollary 1. *Let X be any smooth projective surface, $\alpha \in \mathrm{H}^2(X, \mathbb{C})$, and $n \in \mathbb{N}_0$. Then*

$$\int_{X^{[n]}} \exp(\alpha^{[n]} + \bar{\alpha}^{[n]}) = \frac{1}{n!} \left(\int_X \alpha\bar{\alpha} \right)^n. \tag{61}$$

For $X = A$ an abelian surface and $n \in \mathbb{N}$, we get

$$\int_{A^{[[n]]}} \exp(\alpha^{[[n]]} + \bar{\alpha}^{[[n]]}) = \frac{n}{(n-1)!} \left(\int_X \alpha\bar{\alpha} \right)^{n-1}. \tag{62}$$

Proof. By Theorem 2, in $\mathbb{C}[[q]]$:

$$\sum_{n=0}^{\infty} \left(\int_{X^{[n]}} \exp(q^{\frac{1}{2}} (\alpha^{[n]} + \bar{\alpha}^{[n]})) \right) p^n = \exp(pq \int_X \alpha\bar{\alpha} + O(p^2)), \tag{63}$$

which proves the first part of the corollary by comparing coefficients of q.
For the Kummer case, we calculate

$$\begin{aligned}\int_{A^{[[n]]}} \exp(\alpha^{[[n]]} + \bar{\alpha}^{[[n]]}) &= \frac{\int_{A^{[[n]]}} \exp(\alpha^{[[n]]} + \bar{\alpha}^{[[n]]}) \int_A \exp(n\alpha + n\bar{\alpha})}{\int_A \exp(n\alpha + n\bar{\alpha})} \\ &= n^2 \frac{\int_{A^{[n]}} \exp(\alpha^{[n]} + \bar{\alpha}^{[n]})}{\int_A \exp(\alpha + \bar{\alpha})},\end{aligned}$$

which proves the rest of the corollary. □

Let ch be the universal Chern character. By $s_k = (2k)!\mathrm{ch}_{2k}$ we denote its components. They span the whole algebra of characteristic classes, i.e. we have $\mathbb{Q}[s_1, s_2, \dots] = \mathbb{Q}[c_1, c_2, \dots]$.
Let us fix the power series

$$\phi := \exp(\sum_{k=1}^{\infty} a_{2k} s_{2k} t^k) \in \mathbb{Q}[a_2, a_4, \dots][t][[c_1, c_2, \dots]].$$

This multiplicative series gives rise to four power series

$$A_\phi(p), B_\phi(p), C_\phi(p), D_\phi(p) \in pR[[p]]$$

according to the previous Theorem 2. We shall set for the rest of this article

$$A(t) := A_\phi(1), \quad \text{and} \quad D(t) := D_\phi(1) \tag{64}$$

The constant terms of these power series in t are given by

$$A(t) = 1 + O(t), \quad \text{and} \quad D(t) = O(t). \tag{65}$$

5. Rozansky-Witten classes and invariants

The idea to associate to every graph Γ and every hyperkähler manifold X a cohomology class $\mathrm{RW}_X(\Gamma)$ is due to L. Rozansky and E. Witten (c.f. [15]). M. Kapranov showed in [8] that the metric structure of a hyperkähler manifold is not nessessary to define these classes. It was his idea to build the whole theory upon the Atiyah class and the symplectic structure of an irreducible holomorphic symplectic manifold. We will make use of his definition of Rozansky-Witten classes in this section. A very detailed text on defining Rozansky-Witten invariants is the thesis by J. Sawon [16].

5.1. Definition. Let (X, σ) be a holomorphic symplectic manifold. Let us work in the category of complexes of coherent sheaves on X. In this category, we have for every $n \in \mathbb{Z}$ a functor $V \mapsto V[n]$ that shifts a complex V by n to the left. Due to the Koszul sign rule (i.e. the natural map $(V[m]) \otimes (W[n]) \to (W[n]) \otimes (V[m])$ for sheaves V and W and integers n and m incorporates a sign $(-1)^{mn}$), we have $\mathrm{S}^n(V[1]) = (\Lambda^n V)[n]$ and $\mathrm{S}_n(V[1]) = (\Lambda_n V)[n]$.
Every Jacobi diagram Γ with k trivalent and l univalent vertices defines in the category of complexes of coherent sheaves on X a morphism

$$\Phi^\Gamma : \mathrm{S}_k\Lambda_3(\mathcal{T}_X[-1]) \otimes \mathrm{S}_l(\mathcal{T}_X[-1]) \to \mathrm{S}^e\mathrm{S}^2(\mathcal{T}_X[-1]), \tag{66}$$

where $\mathcal{T}_X[-1]$ is the tangent sheaf of X shifted by one and $2e = 3k + l$. By the sign rule above, this is equivalent to being given a map:

$$(\Lambda_k\mathrm{S}_3\mathcal{T}_X \otimes \Lambda_l\mathcal{T}_X)[-3k - l] \to (\mathrm{S}^e\Lambda^2\mathcal{T}_X)[-2e], \tag{67}$$

which is induced by a map

$$\Lambda_k\mathrm{S}_3\mathcal{T}_X \otimes \Lambda_l\mathcal{T}_X \to \mathrm{S}^e\Lambda^2\mathcal{T}_X \tag{68}$$

in the category of coherent sheaves on X. This gives rise to a map

$$\Psi^\Gamma : \Lambda_k\mathrm{S}_3\mathcal{T}_X \otimes \mathrm{S}_e\Lambda_2\Omega_X \to \Lambda^l\Omega_X. \tag{69}$$

Let $\tilde{\alpha} \in \mathrm{H}^1(X, \Omega \otimes \operatorname{End}\mathcal{T}_X)$ be the Atiyah class of X, i.e. $\tilde{\alpha}$ represents the extension class of the sequence

$$0 \longrightarrow \Omega_X \otimes \mathcal{T}_X \longrightarrow \mathrm{J}^1\mathcal{T}_X \longrightarrow \mathcal{T}_X \longrightarrow 0 \tag{70}$$

in $\operatorname{Ext}^1_X(\mathcal{T}_X, \Omega_X \otimes \mathcal{T}_X) = \mathrm{H}^1(X, \Omega_X \otimes \operatorname{End}\mathcal{T}_X)$. Here, $\mathrm{J}^1\mathcal{T}_X$ is the bundle of one-jets of sections of $\mathcal{T}_X$ (for more on this, see [8]). The Atiyah class can also be viewed as the obstruction for a global holomorphic connection to exist on $\mathcal{T}_X$. We set $\alpha := i/(2\pi)\tilde{\alpha}$.
We use σ to identify the tangent bundle $\mathcal{T}_X$ of X with its cotangent bundle Ω_X. Doing this, α can be viewed as an element of $\mathrm{H}^1(X, \mathcal{T}_X^{\otimes 3})$. Now the point is that α is not any such element. The following proposition was proven by Kapranov in [8]:

Proposition 13.

$$\alpha \in \mathrm{H}^1(X, \mathrm{S}_3\mathcal{T}_X) \subseteq \mathrm{H}^1(X, \mathcal{T}_X^{\otimes 3}). \tag{71}$$

Therefore, $\alpha^{\cup k} \cup \sigma^{\cup l} \in \mathrm{H}^k(X, \Lambda_k \mathrm{S}_3 \mathcal{T}_X \otimes \mathrm{S}_l \Lambda_2 \Omega_X)$. Applying the map Ψ^Γ on the level of cohomology eventually leads to an element

$$\mathrm{RW}_{X,\sigma}(\Gamma) := \Psi_*^\Gamma(\alpha^{\cup k} \cup \sigma^{\cup l}) \in \mathrm{H}^k(X, \Omega_X^l). \tag{72}$$

We call $\mathrm{RW}_{X,\sigma}(\Gamma)$ the *Rozansky-Witten class of* (X, σ) *associated to* Γ.
For a $\mathbb{C}$-linear combination γ of Jacobi diagrams, $\mathrm{RW}_{X,\sigma}(\gamma)$ is defined by linear extension.
In [8], Kapranov also showed the following proposition, which is crucial for the next definition. It follows from a Bianchi-identity for the Atiyah class.

Proposition 14. *If γ is a $\mathbb{Q}$-linear combination of Jacobi diagrams that is zero modulo the anti-symmetry and IHX relations, then* $\mathrm{RW}_{X,\sigma}(\gamma) = 0$.

Definition 13. We define a double-graded linear map

$$\mathrm{RW}_{X,\sigma} : \hat{\mathcal{B}} \to \mathrm{H}^*(X, \Omega_X^*), \tag{73}$$

which maps $\mathcal{B}_{k,l}$ into $\mathrm{H}^k(X, \Omega_X^l)$ by mapping a homology class of a Jacobi diagram Γ to $\mathrm{RW}_{X,\sigma}(\Gamma)$.

Definition 14. Let $\gamma \in \hat{\mathcal{B}}$ be any graph. The integral

$$b_\gamma(X, \sigma) := \int_X \mathrm{RW}_{X,\sigma}(\gamma) \exp(\sigma + \bar{\sigma}) \tag{74}$$

is called the *Rozansky-Witten invariant of* (X, σ) *associated to* γ.

5.2. Examples and properties of Rozansky-Witten classes. We summarise in this subsection the properties of the Rozansky-Witten classes that will be of use for us. For proofs take a look at [10], please.
Let (X, σ) again be a holomorphic symplectic manifold.

Proposition 15. *The map* $\mathrm{RW}_{X,\sigma} : \hat{\mathcal{B}} \to \mathrm{H}^{*,*}(X)$ *is a morphism of graded algebras.*

Proposition 16. *For all $\gamma \in \hat{\mathcal{B}}'$ and $\gamma' \in \hat{\mathcal{B}}$ we have*

$$\mathrm{RW}_{X,\sigma}(\langle \gamma, \gamma' \rangle) = \langle \mathrm{RW}_{X,\sigma}(\gamma), \mathrm{RW}_{X,\sigma}(\gamma') \rangle . \tag{75}$$

Example 9. The cohomology class $[\sigma] \in \mathrm{H}^{2,0}(X)$ is a Rozansky-Witten class; more precisely, we have

$$\mathrm{RW}_{X,\sigma}(\ell) = 2[\sigma]. \tag{76}$$

Example 10. The components of the Chern charakter are Rozansky-Witten invariants:

$$-\mathrm{RW}_{X,\sigma}(w_{2k}) = \mathrm{RW}_{X,\sigma}(\tilde{w}_{2k}) = s_{2k}. \tag{77}$$

The next two proposition actually aren't stated in [10], so we shall give ideas of their proofs here.

Proposition 17. *Let $\nu : (X, \nu^*\sigma) \to (Y, \sigma)$ be a Galois cover of holomorphic symplectic manifolds. For every graph homology class $\gamma \in \hat{\mathcal{B}}$,*

$$\mathrm{RW}_{X,\nu^*\sigma}(\gamma) = \nu^* \mathrm{RW}_{Y,\sigma}(\gamma). \tag{78}$$

Proof. As ν is a Galois cover, we can identify $\mathcal{T}_X$ with $\nu^*\mathcal{T}_Y$ and so $\tilde{\alpha}_X$ with $\nu^*\tilde{\alpha}_Y$ where $\tilde{\alpha}_X$ and $\tilde{\alpha}_Y$ are the Atiyah classes of X and Y. By definition of the Rozansky-Witten classes, (78) follows. □

Lemma 2. *Let (X, σ) and (Y, τ) be two holomorphic symplectic manifolds. If the tangent bundle of Y is trivial,*

$$\mathrm{RW}_{X\times Y, p^*\sigma + q^*\tau}(\gamma) = p^* \mathrm{RW}_{X,\sigma}(\gamma) \tag{79}$$

for all graphs $\gamma \in \hat{\mathcal{B}}'$. Here $p : X \times Y \to X$ and $q : X \times Y \to Y$ denote the canonical projections.

Proof. This lemma is a special case of the more general proposition in [16] that relates the coproduct in graph homology with the product of holomorphic symplectic manifolds. Since all Rozansky-Witten classes for graphs with at least one trivalent vertex vanish on Y, our lemma follows easily from J. Sawon's statement. □

5.3. Rozansky-Witten classes of closed graphs. Let γ be a homogeneous closed graph of degree $2k$. For every compact holomorphic symplectic manifold (X, σ), we have $\mathrm{RW}_{X,\sigma}(\gamma) \in \mathrm{H}^{0,2k}(X)$. If X is irreducible, we therefore have $\mathrm{RW}_{X,\sigma}(\gamma) = \beta_\gamma \cdot [\bar{\sigma}]^k$ for a certain $\beta_\gamma \in \mathbb{C}$. We can express β_γ as

$$\beta_\gamma = \frac{\int_X \mathrm{RW}_{X,\sigma}(\gamma)\bar{\sigma}^{n-k}\sigma^n}{\int_X (\sigma\bar{\sigma})^n} = \frac{(n-k)!}{n!} \frac{\int_X \mathrm{RW}_{X,\sigma}(\gamma)\exp(\sigma+\bar{\sigma})}{\int_X \exp(\sigma+\bar{\sigma})} \tag{80}$$

where $2n$ is the dimension of X.
This formula makes also sense for non-irreducible X, which leads us to the following definition:

Definition 15. Let (X, σ) be a compact holomorphic symplectic manifold (X, σ) of dimension $2n$. For any homogeneous closed graph homology class γ of degree $2k$ with $k \leq n$ we set

$$\beta_\gamma(X, \sigma) := \frac{(n-k)!}{n!} \frac{\int_X \mathrm{RW}_{X,\sigma}(\gamma)\exp(\sigma+\bar{\sigma})}{\int_X \exp(\sigma+\bar{\sigma})} \tag{81}$$

By linear extension, we can define $\beta_\gamma(X, \sigma)$ also for non-homogeneous closed graph homology classes γ.

Remark 12. The map ${}^{\mathrm{t}}\hat{\mathcal{B}} \to \mathbb{C}, \gamma \mapsto \beta_\gamma(X, \sigma)$ is linear. If X is irreducible, it is also a homomorphism of rings.

For polywheels $\tilde{w}_{2\lambda}$, we can express $\beta_{\langle\tilde{w}_{2\lambda}\rangle}$ in terms of characteristic classes:

Proposition 18. *Let (X, σ) be a compact holomorphic symplectic manifold of dimension $2n$ and $k \in \{1, \ldots, n\}$. Let $\lambda \in P(k)$ be any partition of k. Then*

$$\int_X \mathrm{RW}_{X,\sigma}(\langle\tilde{w}_{2\lambda}\rangle)\exp(\sigma+\bar{\sigma}) = \int_X s_{2\lambda}(X)\exp(\sigma+\bar{\sigma}). \tag{82}$$

Proof. We calculate

$$\int_X \mathrm{RW}_{X,\sigma}(\langle \tilde{w}_{2\lambda}\rangle)\exp(\sigma+\bar\sigma) = \int_X \mathrm{RW}_{X,\sigma}(\langle \tilde{w}_{2\lambda}, \exp(\ell/2)\rangle)\exp(\sigma+\bar\sigma)$$
$$= \int_X \langle s_{2\lambda}, \exp\sigma\rangle \exp(\sigma+\bar\sigma) = \int_X s_{2\lambda}\exp(\sigma+\bar\sigma). \tag{83}$$

□

6. Calculation for the example series

6.1. Proof of the main theorem. Let X be a smooth projective surface that admits a holomorphic symplectic form (e.g. a K3 surface or an abelian surface). Let us fix a holomorphic symplectic form $\sigma \in H^{2,0}(X)$ that is normalised such that $\int_X \sigma\bar\sigma = 1$. It is known ([9]) that $X^{[n]}$ for all $n \in \mathbb{N}_0$ is a compact holomorphic symplectic manifold.

For every homogeneous closed graph homology class γ of degree $2k$ and every $n \in \mathbb{N}_0$, we set

$$h^X_\gamma(n) := \beta_\gamma(X^{[k+n]}, \sigma^{[k+n]}). \tag{84}$$

By linear extension, we define $h^X_\gamma(n)$ for non-homogeneous closed graph homology classes γ.

Proposition 19. *For all closed graph homology classes γ, we have*

$$\sum_{n=0}^\infty \frac{q^n}{n!} h^X_\gamma(n) = \sum_{l=0}^\infty \int_{X^{[l]}} \mathrm{RW}_{X^{[l]},\sigma^{[l]}}(\gamma)\exp(q^{\frac12}(\sigma^{[l]}+\bar\sigma^{[l]})) \tag{85}$$

in $\mathbb{C}[[q]]$.

Proof. Let us assume that γ is homogeneous of degree $2k$. Then

$$h^X_\gamma(n) = \beta_\gamma(X^{[k+n]}, \sigma^{[k+n]}) =$$
$$\frac{n!}{(n+k)!}\frac{\int_{X^{[k+n]}} \mathrm{RW}_{X^{[k+n]},\sigma^{[k+n]}}(\gamma)\exp(\sigma^{[k+n]}+\bar\sigma^{[k+n]})}{\int_{X^{[k+n]}}\exp(\sigma^{[k+n]}+\bar\sigma^{[k+n]})}$$
$$= n!\int_{X^{[k+n]}} \mathrm{RW}_{X^{[k+n]},\sigma^{[k+n]}}(\gamma)\exp(\sigma+\bar\sigma).$$

In the last equation we have used Corollary 1. Summing up and introducing the counting parameter q yields the claim. □

Proposition 20. *Let $a_2, a_4, \dots$ be formal parameters. We set*

$$\omega(t) := \sum_{k=1}^\infty a_{2k} t^k \tilde{w}_{2k} \in \hat{\mathcal{B}}^1[a_2, a_4, \dots][t] \tag{86}$$

and call ω the universal wheel. *Further, we set* $W(t) := \exp(\omega(t))$ *and* $W := W(1)$. *The Rozansky-Witten classes of the universal wheel are encoded by*

$$\sum_{n=0}^{\infty} \frac{q^n}{n!} h^X_{\langle W(t)\rangle}(n) = \exp(qA(t))\exp(c_2(X)D(t)). \tag{87}$$

Proof. Using Proposition 19 and Proposition 18 yields:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^n}{n!} h^X_{\langle W(t)\rangle}(n) &= \sum_{l=0}^{\infty} \int_{X^{[l]}} \mathrm{RW}_{X^{[l]},\sigma^{[l]}}(\langle W(t)\rangle)\exp(q^{\frac{1}{2}}(\sigma^{[l]}+\bar\sigma^{[l]}))\\ &= \sum_{l=0}^{\infty} \int_{X^{[l]}} \exp\left(\sum_{k=1}^{\infty} a_{2k}s_{2k}(X^{[l]})t^k\right)\exp(q^{\frac{1}{2}}(\sigma^{[l]}+\bar\sigma^{[l]}))\\ &= \exp(qA(t))\exp(c_2(X)D(t)).\end{aligned}$$

□

Corollary 2. *For every* $n \in \mathbb{N}_0$ *we have*

$$h^X_{\langle W(t)\rangle}(n) = \exp(c_2(X)D(t))\exp(n\log A(t)) \tag{88}$$

Proof. Comparision of coefficients in (87) gives

$$h^X_{\langle W(t)\rangle}(n) = A(t)^n \exp(c_2(X)D(t)).$$

Lastly, note that A is a power series in t that has constant coefficient one. □

Remark 13. By equation (88) we shall extend the definition of $h^X_{\langle W(t)\rangle}(n)$ to all $n \in \mathbb{Z}$.

Proposition 21. *Let* A *be an abelian surface. Let us fix a holomorphic symplectic form* $\sigma \in \mathrm{H}^{2,0}(A)$ *that is normalised such that* $\int_A \sigma\bar\sigma = 1$.
Let γ *be a homogeneous connected closed graph of degree* $2k$. *Then we have*

$$\beta_\gamma(A^{[[n]]},\sigma^{[[n]]}) = \frac{n}{n-k}\beta_\gamma(A^{[n]},\sigma^{[n]}) \tag{89}$$

for any $n > k$.

Proof. The proof is a straight-forward calculation:

$$\begin{aligned}\beta_\gamma(A^{[[n]]},\sigma^{[[n]]}) &= \frac{(n-1-k)!}{(n-1)!}\frac{\int_{A^{[[n]]}} \mathrm{RW}_{A^{[[n]]},\sigma^{[[n]]}}(\gamma)\exp(\sigma^{[[n]]}+\bar\sigma^{[[n]]})}{\int_{A^{[[n]]}}\exp(\sigma^{[[n]]}+\bar\sigma^{[[n]]})}\\ &= \frac{(n-1-k)!}{(n-1)!}\frac{\int_{A^{[[n]]}} \mathrm{RW}_{A^{[[n]]},\sigma^{[[n]]}}(\gamma)\exp(\sigma^{[[n]]}+\bar\sigma^{[[n]]})}{\int_{A^{[[n]]}}\exp(\sigma^{[[n]]}+\bar\sigma^{[[n]]}}\frac{\int_A \exp(n\sigma+n\bar\sigma)}{\int_A\exp(n\sigma+n\bar\sigma)}\\ &= \frac{(n-1-k)!}{(n-1)!}\frac{\int_{A^{[n]}} \mathrm{RW}_{A^{[n]},\sigma^{[n]}}(\gamma)\exp(\sigma^{[n]}+\bar\sigma^{[n]})}{\int_{A^{[n]}}\exp(\sigma^{[n]}+\bar\sigma^{[n]})} = \frac{n}{n-k}\beta_\gamma(A^{[n]},\sigma^{[n]}),\end{aligned}$$

where we have used Proposition 12, Proposition 17 and Lemma 2. □

Theorem 3. *For any homogenous connected closed graph of degree $2k$ lying in the algebra $\mathcal{C}$ of polywheels there exist two rational numbers a_γ, c_γ such that for each K3 surface X together with a symplectic form $\sigma \in \mathrm{H}^{2,0}(X)$ with $\int_X \sigma\bar{\sigma} = 1$ and $n \geq k$ we have*

$$\beta_\gamma(X^{[n]}, \sigma^{[n]}) = a_\gamma n + c_\gamma \tag{90}$$

and that for each abelian surface A together with a symplectic form $\sigma \in \mathrm{H}^{2,0}(X)$ with $\int_X \sigma\bar{\sigma} = 1$ and $n > k$ we have

$$\beta_\gamma(A^{[[n]]}, \sigma^{[[n]]}) = a_\gamma n. \tag{91}$$

Proof. Let (X, σ) be a K3 surface or an abelian surface together with a symplectic form with $\int_X \sigma\bar{\sigma} = 1$. Let W_{2k} be the homogeneous component of degree $2k$ of $W(1)$. Then $W(t) = \sum_{k=0}^\infty W_{2k} t^k$. Thus we have by (88):

$$h^X_{\langle W(t)\rangle}(n) = \sum_{k=0}^\infty h^X_{\langle W_{2k}\rangle}(n) t^k = U_{c_2(X)}(t)\exp(nV(t)) \tag{92}$$

with $U_{c_2(X)}(t) := \exp(c_2(X)D(t))$ and $V(t) := \log A(t)$.
Let us consider the case of a K3 surface X first. Note that $c_2(X) = 24$. By definition of $h^X_\gamma(n)$ we have

$$\beta_{\langle W_{2k}\rangle}(X^{[n]}, \sigma^{[n]}) = h^X_{\langle W_{2k}\rangle}(n-k) \tag{93}$$

for all $n \geq k$. For $n < k$ we take this equation as a definition for its left hand side. Let the power series $T(t) \in \mathbb{Q}[a_2, a_4, \dots][[t]]$ be defined by $T(t\exp(V(t))) = t$, and set $\tilde{V}(t) := V(T(t))$ and $\tilde{U} := \frac{U_{24}(T(t))}{1+T(t)V'(T(t))}$. By Lemma 1, we have

$$\beta_{\langle W(t)\rangle}(X^{[n]}, \sigma^{[n]}) = \sum_{k=0}^\infty h^X_{\langle W_{2k}\rangle}(n-k)t^k = \tilde{U}(t)\exp(n\tilde{V}(t)).$$

Note that $W(t)$ is of the form $\exp(\gamma)$ where γ is a connected graph. By Proposition 6 and Remark 12 we therefore have

$$\beta_{\langle\langle W(t)\rangle\rangle}(X^{[n]}, \sigma^{[n]}) = \beta_{\log\langle W(t)\rangle}(X^{[n]}, \sigma^{[n]}) = \log \beta_{\langle W(t)\rangle} = n\tilde{V}(t) + \log U(t).$$

Finally, let λ be any partition. Setting

$$\partial_{2\lambda} := \left.\left(\prod_{i=1}^\infty \left.\frac{\partial^{\lambda_i}}{\partial a_i^{\lambda_i}}\right|_{a_i=0}\right)\right|_{t=0}.$$

It is

$$\beta_{\langle\langle \tilde{w}_{2\lambda}\rangle\rangle} = \partial_{2\lambda}\beta_{\langle\langle W(t)\rangle\rangle} = n\partial_{2\lambda}\tilde{V}(t) + \partial_{2\lambda}\log\tilde{U}(t),$$

so the theorem is proven for K3 surfaces and all connected graph homology classes of the form $\langle\langle \tilde{w}_{2\lambda}\rangle\rangle$ and thus for all connected graph homology classes in $\mathcal{C}$.

Let us now turn to the case of a generalised Kummer variety, i.e. let $X = A$ be an abelian surface and $n \geq 1$. Note that $c_2(A) = 0$. Here, we have due to Proposition 21:

$$\beta_{\langle W_{2k}\rangle}(A^{[[n]]}, \sigma^{[[n]]}) = \frac{n}{n-k} h^A_{\langle W_{2k}\rangle}(n-k)$$

for $n > k$. For $n \leq k$ we take this equation as a definition for its left hand side. As $U_0(t) = 1$, Lemma 1 yields in this case that

$$\beta_{\langle W(t)\rangle}(A^{[[n]]}, \sigma^{[[n]]}) = \sum_{k=0}^{\infty} \frac{n}{n-k} h^X_{\langle W_{2k}\rangle}(n-k) t^k = \exp n\tilde{V}(t).$$

We can then proceed as in the case of the Hilbert scheme of a K3 surface to finally get

$$\beta_{\langle\langle \tilde{w}_{2\lambda}\rangle\rangle} = n\partial_{2\lambda}(\tilde{V}(t)).$$

□

6.2. Some explicit calculations. Now, we'd like to calculate the constants a_γ and c_γ for any homogeneous connected closed graph homology class γ of degree $2k$ lying in $\mathcal{C}$. By the previous theorem, we can do this by calculating β_γ on (X, σ) for (X, σ) being the $2k$-dimensional Hilbert scheme of points on a K3 surface and the $2k$-dimensional generalised Kummer variety.
We can do this by recursion over k: Let the calculation having been done for homogeneous connected closed graph homology classes γ of degree less than $2k$ in $\mathcal{C}$ and both example series.
Let λ be any partition of k. We can express $\langle\langle \tilde{w}_{2\lambda}\rangle\rangle$ as

$$\langle\langle \tilde{w}_{2\lambda}\rangle\rangle = \langle \tilde{w}_{2\lambda}\rangle + P, \tag{94}$$

where P is a polynomial in homogeneous connected closed graph homology classes γ of degree less than $2k$ in $\mathcal{C}$ (for this see Proposition 6). Therefore, $\beta_{\langle\langle \tilde{w}_{2\lambda}\rangle\rangle}(X, \sigma)$ is given by

$$\beta_{\langle\langle \tilde{w}_{2\lambda}\rangle\rangle}(X, \sigma) = \beta_{\langle \tilde{w}_{2\lambda}\rangle}(X, \sigma) + P', \tag{95}$$

where P' is a polynomial in terms like $\beta_{\gamma'}(X, \sigma)$ with $\gamma' \in \mathcal{C}$ and $\deg \gamma' < 2k$. However, these terms have been calculated in previous recursion steps. Therefore, the only thing new we have to calculate in this recursion step is $\beta_{\langle \tilde{w}_{2\lambda}\rangle}(X, \sigma)$. We have:

$$\beta_{\langle \tilde{w}_{2\lambda}\rangle}(X, \sigma) = \frac{1}{k!} \frac{\int_X \mathrm{RW}_{X,\sigma}(\tilde{w}_{2\lambda}) \exp(\sigma + \bar{\sigma})}{\int_X \exp(\sigma + \bar{\sigma})} = \frac{\int_X s_{2\lambda}(X)}{\int_X \exp(\sigma + \bar{\sigma})}. \tag{96}$$

As all the Chern numbers of X can be computed with the help of Bott's residue formula (see [3] for the case of the Hilbert scheme and [11] for the case of the generalised Kummer variety), we therefore are able to calculate $\beta_{\langle \tilde{w}_{2\lambda}\rangle}(X, \sigma)$. This ends the recursion step as we have given an algorithm to compute a_γ and c_γ for any homogeneous connected closed graph homology class γ of degree $2k$ in $\mathcal{C}$.

We worked through the recursion for $k = 1, 2, 3$. Firstly, we have

$$
\begin{aligned}
\langle\langle \tilde{w}_2 \rangle\rangle &= \langle \tilde{w}_2 \rangle \\
\langle\langle \tilde{w}_2^2 \rangle\rangle &= \langle \tilde{w}_4 \rangle - \langle\langle \tilde{w}_2 \rangle\rangle^2 \\
\langle\langle \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_4 \rangle \\
\langle\langle \tilde{w}_2^3 \rangle\rangle &= \langle \tilde{w}_2^3 \rangle - 3 \langle\langle \tilde{w}_2 \rangle\rangle \langle\langle \tilde{w}_2^2 \rangle\rangle - \langle\langle \tilde{w}_2 \rangle\rangle^3 \\
\langle\langle \tilde{w}_2 \tilde{w}_4 \rangle\rangle &= \langle \tilde{w}_2 \tilde{w}_4 \rangle - \langle\langle \tilde{w}_2 \rangle\rangle \langle\langle \tilde{w}_4 \rangle\rangle \\
\langle\langle \tilde{w}_6 \rangle\rangle &= \langle \tilde{w}_6 \rangle .
\end{aligned}
\tag{97}
$$

Not let X be a K3 surface and A an abelian surface. Let us denote by σ either a holomorphic symplectic two-form on X with $\int_X \sigma\bar{\sigma} = 1$ or on A with $\int_A \sigma\bar{\sigma} = 1$. We use the following table of Chern numbers for the Hilbert scheme of points on a K3 surface:

k	s	$s[X^{[k]}]$	$s[A^{[[k+1]]}]$
1	s_2	-48	-48
2	s_2^2	3312	3024
	s_4	360	1080
3	s_2^3	-294400	-241664
	$s_2 s_4$	-29440	-66560
	s_6	-4480	-22400

Going through the recursion, we arrive at the following table:

k	γ	$\beta_\gamma(A^{[[k+1]]})$	$\beta_\gamma(X^{[k]})$	a_γ	c_γ
1	$\langle\langle \tilde{w}_2 \rangle\rangle$	-24	-48	12	-36
2	$\langle\langle \tilde{w}_2^2 \rangle\rangle$	-288	-288	-96	-96
	$\langle\langle \tilde{w}_4 \rangle\rangle$	360	360	120	120
3	$\langle\langle \tilde{w}_2^3 \rangle\rangle$	-5120	-4096	-1280	-256
	$\langle\langle \tilde{w}_2 \tilde{w}_4 \rangle\rangle$	6400	5120	1600	320
	$\langle\langle \tilde{w}_6 \rangle\rangle$	-5600	-4480	-1400	-280

Now, we would like to turn to Rozansky-Witten *invariants*: Let γ be any homogeneous closed graph homology class of degree $2k$. For any holomorphic symplectic manifold (X, σ) of dimension $2n$, the associated Rozansky-Witten invariant is given by

$$
\begin{aligned}
b_\gamma(X, \sigma) = \int_X \mathrm{RW}_{X,\sigma}(\gamma) \exp(\sigma + \bar{\sigma}) &= \frac{1}{n!(n-k)!} \beta_\gamma(X, \sigma) \int_X (\sigma\bar{\sigma})^n \\
&= \frac{n!}{(n-k)!} \beta_\gamma(X, \sigma) \int_X \exp(\sigma + \bar{\sigma}).
\end{aligned}
\tag{98}
$$

To know the Rozansky-Witten invariant associated to closed graph homology classes, we therefore have just to calculate the value of β_γ. On an irreducible holomorphic symplectic manifold, $\gamma \mapsto \beta_\gamma$ is multiplicative with respect to the disjoint union of graphs, so it is enough to calculate β_γ for connected closed graph homology classes. However, we have just done this for the Hilbert schemes of points on a K3 surface and the generalised Kummer varieties — as long as γ is spanned by the connected polywheels.

By the procedure outlined above, Theorem 3 therefore enables us to compute all Rozansky-Witten invariants of the two example series associated to closed graph homology classes lying in $\mathcal{C}$.

References

[1] Dror Bar-Natan. On the Vassiliev knot invariants. *Topology*, 34(2):423–472, 1995.

[2] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782, 1983.

[3] Geir Ellingsrud, Lothar Göttsche, and Manfred Lehn. On the cobordism class of the Hilbert scheme of a surface. *J. Algebraic Geom.*, 10(1):81–100, 2001.

[4] John Fogarty. Algebraic families on an algebraic surface. *Amer. J. Math*, 90:511–521, 1968.

[5] William Fulton. *Intersection Theory*, volume 2 of *3*. Springer-Verlag, Berlin, 1998.

[6] Friedrich Hirzebruch. *Neue topologische Methoden in der Algebraischen Geometrie*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, 1962.

[7] Nigel Hitchin and Justin Sawon. Curvature and characteristic numbers of hyper-Kähler manifolds. *Duke Math. J.*, 106(3):599–615, 2001.

[8] Mikhail Kapranov. Rozansky-Witten invariants via Atiyah classes. *Compos. Math.*, 115(1):71–113, 1999.

[9] Shigeru Mukai. Symplectic structure on the moduli space of sheaves on an abelian or K3 surface. *Invent. Math.*, 77:101–116, 1984.

[10] Marc Nieper-Wißkirchen. Hirzebruch-Riemann-Roch formulae on irreducible symplectic Kähler manifolds. *To appear in: Journal of Algebraic Geometry.*

[11] Marc Nieper-Wißkirchen. On the Chern numbers of Generalised Kummer Varieties. *Math. Res. Lett.*, 9(5–6):597–606, 2002.

[12] Kieran O'Grady. A new six-dimensional irreducible symplectic manifold. *arXiv:math.AG/0010187*.

[13] Kieran O'Grady. Desingularized moduli spaces of sheaves on a K3. *J. Reine Angew. Math.*, 512:49–117, 1999.

[14] Steven Roman. *The Umbral Calculus*. Pure and Applied Mathematics. Academic Press, Inc., Orlando, 1984.

[15] Lev Rozansky and Edward Witten. Hyper-Kähler geometry and invariants of three-manifolds. *Selecta Math. (N.S.)*, 3(3):401–458, 1997.

[16] Justin Sawon. *Rozansky-Witten invariants of hyperkähler manifolds*. PhD thesis, University of Cambridge, October 1999.

[17] Dylan P. Thurston. *Wheeling: A Diagrammatic Analogue of the Duflo Isomorphism*. PhD thesis, University of California at Berkeley, arXiv:math.QA/0006083, Spring 2000.

[18] Shing-Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equations. I. *Comm. Pure Appl. Math.*, 31:339–411, 1978.

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