Parametrized Braid Groups of Chevalley Groups

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Abstract. We introduce the notion of a braid group parametrized by a ring, which is defined by generators and relations and based on the geometric idea of painted braids. We show that the parametrized braid group is isomorphic to the semi-direct product of the Steinberg group (of the ring) with the classical braid group. The technical heart of the proof is the Pure Braid Lemma, which asserts that certain elements of the parametrized braid group commute with the pure braid group. This first part treats the case of the root system A_n; in the second part we prove a similar theorem for the root system D_n.

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1 Introduction

Suppose that the strands of a braid are painted and that the paint from a strand spills onto the strand beneath it, modifying the color of the lower strand as in the picture below.

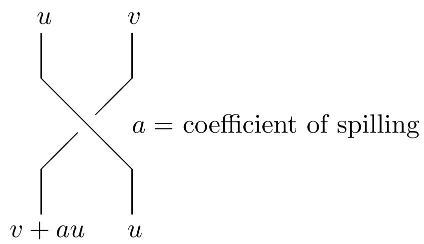

Figure 1

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This gives rise, for any ring A, to the *parametrized braid group* $Br_n(A)$, which is generated by elements y_i^a, where i is an integer, $1 \le i \le n-1$, and a is an element of A, subject to the relations

$$
\begin{array}{rcl}
(A1) & y_i^a y_i^0 y_i^b = y_i^0 y_i^0 y_i^{a+b} & \\
(A1 \times A1) & y_i^a y_j^b = y_j^b y_i^a & \text{if } |i-j| \ge 2, \\
(A2) & y_i^a y_{i+1}^b y_i^c = y_{i+1}^c y_i^{b+ac}\, y_{i+1}^a &
\end{array}
$$

for any $a, b, c \in A$

A variation of this group first appeared in [L]. The choice of names for these relations will be explained in section 2 below. The derivation of these relations from the painted braid model can be seen in Figures 2 and 3 below.)
Observe that when $A = \{0\}$ (the zero ring), one obtains the classical Artin braid group Br_n, whose presentation is by generators y_i, $1 \le i \le n-1$, and relations

$$
\begin{array}{rcl}
(A1 \times A1) & y_i y_j = y_j y_i & \text{if } |i-j| \ge 2, \\
(A2) & y_i y_{i+1} y_i = y_{i+1} y_i\, y_{i+1} &
\end{array}
$$

A question immediately comes to mind: does Figure 1 correctly reflect the elements of the parametrized braid group? Up to equivalence, a picture would be completely determined by a braid and a linear transformation of the set of colors. This linear transformation lies in the subgroup $E_n(A)$ of elementary matrices. Hence if the elements of the parametrized braid group correspond exactly to the pictures, then the group should be the semi-direct product of $E_n(A)$ by Br_n. We will show that this is almost the case: we only need to replace $E_n(A)$ by the Steinberg group $St_n(A)$ (*cf.* [St] [Stb]).

THEOREM. *For any ring A there is an isomorphism*

$$Br_n(A) \cong St_n(A) \rtimes Br_n \ ,$$

where the action of Br_n is via the symmetric group $\mathcal{S}_n$.

The quotient of $Br_n(A)$ by the relation $y_i^0 y_i^0 = 1$ is the group studied by Kassel and Reutenauer [K-R] (in this quotient group, our relation (A1) becomes $y_i^a y_i^0 y_i^b = y_i^{a+b}$, which is exactly the relation used in [K-R] in place of (A1)). They show that this quotient is naturally isomorphic to the semi-direct product $St_n(A) \rtimes \mathcal{S}_n$ of the Steinberg group with the symmetric group. So our theorem is a lifting of theirs.

The proof consists in constructing maps both ways. The key point about their existence is a technical result called the Pure Braid Lemma (cf. 2.2.1). It says that a certain type of parametrized braid commute with the pure braid group.

In the first part of the paper we give a proof of the above theorem which corresponds to the family of Coxeter groups A_n. In the second part we prove a similar theorem for the family D_n, cf. section 3. The Pure Braid Lemma in the D_n case is based on a family of generators of the pure braid group found by Digne and Gomi [D-G]. We expect to prove similar theorems for the other Coxeter groups.
A related result has been announced in [Bon].

2 The parametrized braid group

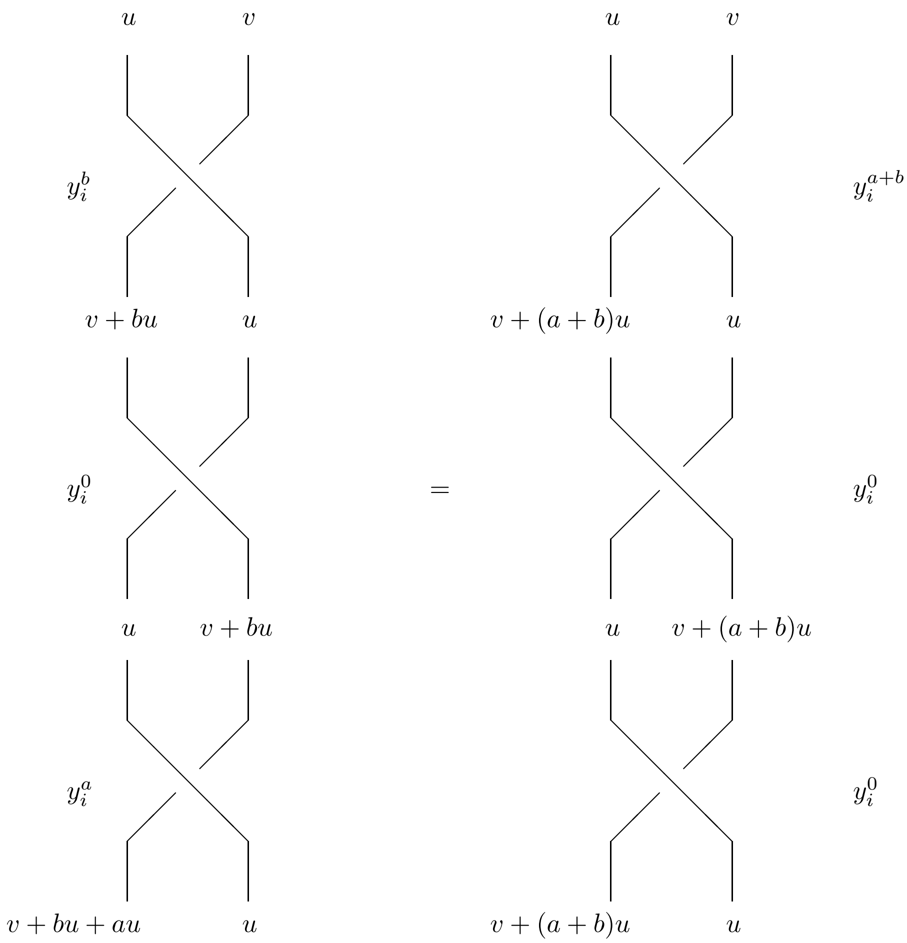
We introduce the parametrized braid group of the family of Coxeter groups A_{n-1}.

Definition 2.0.1. *Let A be a ring (not necessarily unital nor commutative). The parametrized braid group $Br_n(A)$ is generated by the elements y_i^a, where i is an integer, $1 \leq i \leq n-1$, and a is an element of A, subject to the relations*

$$\begin{aligned} &(A1) && y_i^a y_i^0 y_i^b = y_i^0 y_i^0 y_i^{a+b} && \\ &(A1 \times A1) && y_i^a y_j^b = y_j^b y_i^a && \textit{if } |i-j| \geq 2, \\ &(A2) && y_i^a y_{i+1}^b y_i^c = y_{i+1}^c y_i^{b+ac} y_{i+1}^a && \end{aligned}$$

for any $a, b, c \in A$.

The geometric motivation for the defining relations of this group, and its connection with braids, can be seen in the following figures in which u, v, w (the colors) are elements of A, and a, b, c are the *coefficients of spilling.* Relation $(A1)$ comes from Figure 2.
Relation $(A1 \times A1)$ arises because the actions of y_i^a and of y_j^b on the strands of the braid are disjoint when $|i-j| \geq 2$, so that these two elements commute.
Relation $(A2)$ derives from Figure 3.



u
v
y_i^b
$v+bu$
u
y_i^0
u
$v+bu$
y_i^a
$v+bu+au$
u
$=$
u
v
y_i^{a+b}
$v+(a+b)u$
u
y_i^0
u
$v+(a+b)u$
y_i^0
$v+(a+b)u$
u

Figure 2

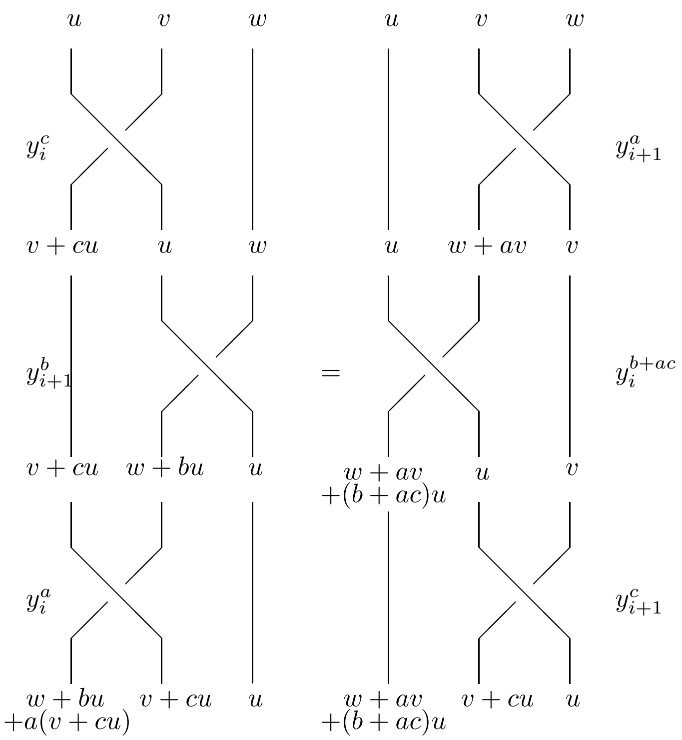


Figure 3

2.1 Technical lemmas

Let Φ be the root system A_n or D_n and let A be a ring (not necessarily unital) which is supposed to be commutative in the D_n case. Let Δ be a simple subsystem of the root system Φ. Elements of Δ are denoted by α or α_i. The image of $b \in Br(\Phi)$ in the Weyl group $W(\Phi)$ is denoted $\overline{b}$. The parametrized braid group $Br(D_n, A)$ is defined in 3.1.1.

Lemma 2.1.1. *The following relations in $Br(\Phi, A)$ are consequences of relation $(A1)$:*

$$\begin{aligned}(y_\alpha^0 y_\alpha^0) y_\alpha^a &= y_\alpha^a (y_\alpha^0 y_\alpha^0),\\ y_\alpha^a (y_\alpha^0)^{-1} y_\alpha^b &= y_\alpha^{a+b},\\ y_\alpha^{-a} &= y_\alpha^0 (y_\alpha^a)^{-1} y_\alpha^0.\end{aligned}$$

Proof. Replacing b by 0 in $(A1)$ shows that $y_\alpha^0 y_\alpha^0$ commutes with y_α^a. From this follows

$$\begin{aligned}y_\alpha^a (y_\alpha^0)^{-1} y_\alpha^b &= (y_\alpha^0 y_\alpha^0)^{-1} y_\alpha^a (y_\alpha^0 y_\alpha^0)(y_\alpha^0)^{-1} y_\alpha^b\\ &= (y_\alpha^0 y_\alpha^0)^{-1} y_\alpha^a y_\alpha^0 y_\alpha^b\\ &= (y_\alpha^0 y_\alpha^0)^{-1} y_\alpha^0 y_\alpha^0 y_\alpha^{a+b}\\ &= y_\alpha^{a+b}.\end{aligned}$$

Putting $b=-a$ in the second relation yields the third relation. $\square$

Lemma 2.1.2. *Assume that the Pure Braid Lemma holds for the root system Φ. Suppose that $\alpha\in\Delta$ and $b\in Br(\Phi)$ are such that $\bar{b}(\alpha)\in\Delta$ (where $\bar{b}\in W(\Phi)$ denotes the image of b). Then for any $a\in A$ one has*

$$by^a_\alpha(y^0_\alpha)^{-1}b^{-1}=y^a_{\bar{b}(\alpha)}(y^0_{\bar{b}(\alpha)})^{-1}\quad \textit{in } Br(\Phi,A).$$

Proof. First let us show that there exists $b'\in Br(\Phi)$ such that $b'y^a_\alpha b'^{-1}=y^a_{\bar{b}(\alpha)}$. The two roots α and $\bar{b}(\alpha)$ have the same length, hence they are connected, in the Dynkin diagram, by a finite sequence of edges with $m=3$ (cf. [Car, Lemma 3.6.3]). Therefore it is sufficient to prove the existence of b' when α and $\bar{b}(\alpha)$ are adjacent. In that case α and $\bar{b}(\alpha)$ generate a subsystem of type A_2; we may assume $\alpha=\alpha_1$ and $\bar{b}(\alpha)=\alpha_2\in A_2$; and we can use the particular case of relation $(A2)$, namely

$$y^0_{\alpha_1}y^0_{\alpha_2}y^a_{\alpha_1}=y^a_{\alpha_2}y^0_{\alpha_1}y^0_{\alpha_2}$$

to show that

$$y^0_{\alpha_1}y^0_{\alpha_2}y^a_{\alpha_1}(y^0_{\alpha_2})^{-1}(y^0_{\alpha_1})^{-1}=y^a_{\alpha_2}.$$

(Here $b'=y^0_{\alpha_1}y^0_{\alpha_2}$, $\alpha=\alpha_1=-\epsilon_1+\epsilon_2, \bar{b'}(\alpha)=\alpha_2=-\epsilon_2+\epsilon_3$.)
To conclude the proof of the Lemma it is sufficient to show that

$$by^a_\alpha(y^0_\alpha)^{-1}b^{-1}=y^a_\alpha(y^0_\alpha)^{-1}$$

whenever $b(\alpha)=\alpha$. According to [H, Theorem, p. 22], $\bar{b}$ is a product of simple reflections σ_{α_i} for $\alpha_i\in\Delta$ which are not connected to α in the Dynkin diagram of Δ. Hence we can write b as the product of an element in the pure braid group and generators y_{α_i} which commute with y^a_α by relation $(A1\times A1)$. Since we have assumed that the Pure Braid Lemma holds for Φ, we can thus conclude that $by^a_\alpha(y^0_\alpha)^{-1}b^{-1}=y^a_\alpha(y^0_\alpha)^{-1}$ as desired. $\square$

2.2 Braid group and pure braid group

The group $Br_n(0)=Br_n$ is the classical Artin braid group with generators $y_i\,, 1\le i\le n-1$, and relations

$$\begin{aligned} y_i\,y_j &= y_j\,y_i\,, \qquad |i-j|\ge 2,\\ y_i\,y_{i+1}\,y_i &= y_{i+1}\,y_i\,y_{i+1}\,. \end{aligned}$$

The quotient of Br_n by the relations $y_iy_i=1\,, 1\le i\le n-1$ is the symmetric group $\mathcal{S}_n$; the image of $b\in Br_n$ in $\mathcal{S}_n$ is denoted by $\bar{b}$. The kernel of the surjective homomorphism $Br_n\to\mathcal{S}_n$ is the *pure braid group*, denoted PBr_n. It is generated by the elements

$$\mathbf{a}_{j,i}:=y_jy_{j-1}\cdots y_iy_i\cdots y_{j-1}y_j,$$

for $n \geq j \geq i \geq 1$, ([Bir]; see Figure 4 below).

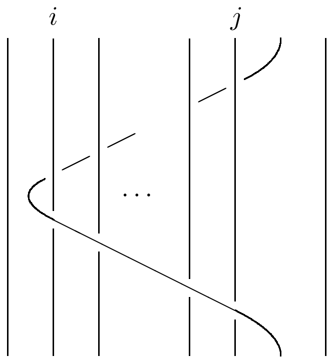


Figure 4: the pure braid $\mathbf{a}_{j,i}$

Lemma 2.2.1 (Pure Braid Lemma for A_{n-1}). *Let y_k^a be a generator of $Br_n(A)$ and let $\omega \in PBr_n = PBr_n(0)$. Then there exists $\omega' \in Br_n$, independent of a, such that*

$$y_k^a\ \omega = \omega'\ y_k^a.$$

Hence for any integer k and any element $a \in A$, the element $y_k^a(y_k^0)^{-1} \in PBr_n(A)$ commutes with every element of the pure braid group PBr_n.

Notation. Before beginning the proof of Lemma 2.2.1, we want to simplify our notation. We will abbreviate

$$\begin{array}{rcl} y_k^0 & = & \mathbf{k} \\ (y_k^0)^{-1} & = & \mathbf{k}^{-1} \\ y_k^a & = & \mathbf{k}^a \end{array}$$

Note that $\mathbf{k}^{-1}$ does *not* mean $\mathbf{k}^a$ for $a = -1$.
If there exist ω' and $\omega'' \in Br_n$ such that $\mathbf{k}^a\omega = \omega'\mathbf{j}^a\omega''$, we will write

$$\mathbf{k}^a\omega \sim \mathbf{j}^a\omega''.$$

Observe that $\sim$ is *not* an equivalence relation, but it is compatible with multiplication on the right by elements of Br_n: if $\eta \in Br_n$,

$$\mathbf{k}^a\omega \sim \mathbf{j}^a\omega'' \Leftrightarrow \mathbf{k}^a\omega = \omega'\mathbf{j}^a\omega'' \Leftrightarrow \mathbf{k}^a\omega\eta = \omega'\mathbf{j}^a\omega''\eta \Leftrightarrow \mathbf{k}^a\omega\eta \sim \mathbf{j}^a\omega''\eta$$

For instance

$$\begin{array}{rcll} \mathbf{k}^a\ \mathbf{k}\ \mathbf{k} & \sim & \mathbf{k}^a & \text{by Lemma 2.1.1,} \\ \mathbf{k}^a\ (\mathbf{k}-\mathbf{1})\ \mathbf{k} & \sim & (\mathbf{k}-\mathbf{1})^a & \text{by relation (A2),} \\ \mathbf{k}^a\ (\mathbf{k}-\mathbf{1}) & \sim & (\mathbf{k}-\mathbf{1})^a\ \mathbf{k} & \text{by relation (A2).} \end{array}$$

Proof of Lemma 2.2.1. It is clear that the first assertion implies the second one: since ω' is independent of a, we can set $a = 0$ to determine that $\omega' = y_k^0\ \omega\ (y_k^0)^{-1}$. Substituting this value of ω' in the expression $y_k^a\ \omega = \omega'\ y_k^a$ completes the proof.

To prove the first assertion, we must show, in the notation just introduced, that $\mathbf{k}^a\omega \sim \mathbf{k}^a$ for every $\omega \in PBr_n$, and it suffices to show this when ω is one of the generators $\mathbf{a}_{j,i} = \mathbf{j}\ (\mathbf{j-1})\ \cdots \mathbf{i}\ \mathbf{i} \cdots\ (\mathbf{j-1})\ \mathbf{j}$ above. Since y_k^a commutes with y_i^0 for all $i \neq k-1, k, k+1$, it commutes with $\mathbf{a}_{j,i}$ whenever $k < i-1$ or whenever $k > j+1$. So we are left with the following 3 cases:

(1a) $\quad (\mathbf{j+1})^a\mathbf{a}_{j,i} \sim (\mathbf{j+1})^a$

(1b) $\quad \mathbf{j}^a\mathbf{a}_{j,i} \sim \mathbf{j}^a$

(1c) $\quad \mathbf{k}^a\mathbf{a}_{j,i} \sim \mathbf{k}^a \quad i-1 \leq k \leq j-1$

Our proof of case (1a) is by induction on the half-length of $\omega = \mathbf{a}_{j,i}$. When $i = k-1$, $\omega = (\mathbf{k-1})\ (\mathbf{k-1})$, and we have

$$\begin{aligned}
\mathbf{k}^a\ (\mathbf{k-1})\ (\mathbf{k-1}) &= \mathbf{k}^a\ (\mathbf{k-1})\ \mathbf{k}^{-1}\,\mathbf{k}\ (\mathbf{k-1}) \\
&\sim (\mathbf{k-1})^a\ \mathbf{k}\ (\mathbf{k-1}) \\
&\sim \mathbf{k}^a\ .
\end{aligned}$$

and more generally,

$$\begin{aligned}
&\mathbf{k}^a\ (\mathbf{k-1})\ (\mathbf{k-2})\cdots(\mathbf{k-2})\ (\mathbf{k-1}) \\
&\qquad \sim (\mathbf{k-1})^a\ \mathbf{k}\ (\mathbf{k-2})\cdots(\mathbf{k-2})(\mathbf{k-1}) \\
&\qquad = (\mathbf{k-1})^a\ (\mathbf{k-2})\cdots(\mathbf{k-2})\ \mathbf{k}\ (\mathbf{k-1}) \\
&\qquad \sim (\mathbf{k-1})^a\ \mathbf{k}\ (\mathbf{k-1}) \qquad \text{(by induction)}, \\
&\qquad \sim \mathbf{k}^a\ .
\end{aligned}$$

The proof of case (1b) is also by induction on the half-length of $\omega = \mathbf{a}_{j,i}$. When $i = k$, $\omega = \mathbf{k}\ \mathbf{k}$, and we have

$$\mathbf{k}^a\mathbf{k}\ \mathbf{k} \sim \mathbf{k}^a \qquad \text{by Lemma 2.1.1.}$$

Then

$$\begin{aligned}
&\mathbf{k}^a\mathbf{k}\ (\mathbf{k-1})\cdots(\mathbf{k-1})\ \mathbf{k} \\
&\qquad = \mathbf{k}^a\mathbf{k}\ (\mathbf{k-1})\ \mathbf{k}\ \mathbf{k}^{-1}\ (\mathbf{k-2})\cdots(\mathbf{k-2})\ (\mathbf{k-1})\ \mathbf{k} \\
&\qquad \sim \mathbf{k}^a(\mathbf{k-1})\ \mathbf{k}\ (\mathbf{k-1})\ (\mathbf{k-2})\cdots(\mathbf{k-2})\ \mathbf{k}^{-1}\,(\mathbf{k-1})\ \mathbf{k} \\
&\qquad \sim (\mathbf{k-1})^a(\mathbf{k-1})\ (\mathbf{k-2})\cdots(\mathbf{k-2})\ (\mathbf{k-1})\ \mathbf{k}\ (\mathbf{k-1})^{-1} \\
&\qquad \sim (\mathbf{k-1})^a\ \mathbf{k}\ (\mathbf{k-1})^{-1} \qquad \text{(by induction)} \\
&\qquad \sim \mathbf{k}^a.
\end{aligned}$$

For case (1c) it is sufficient to check the cases

(2a) $\quad \omega = (\mathbf{k+1})\ (\mathbf{k+1})$

(2b) $\quad \omega = (\mathbf{k+1})\ \mathbf{k}\ \mathbf{k}\ (\mathbf{k+1})$

(2c) $\quad \omega = (\mathbf{k+1})\ \mathbf{k}\ (\mathbf{k-1})\cdots(\mathbf{k-1})\ \mathbf{k}\ (\mathbf{k+1})$

which are proved as follows:

$$\begin{array}{rcl}\mathbf{k}^a(\mathbf{k+1})\ (\mathbf{k+1}) & = & \mathbf{k}^a(\mathbf{k+1})\ \mathbf{k}^{-1}\,\mathbf{k}\ (\mathbf{k+1}) \\ & \sim & (\mathbf{k+1})^a\ \mathbf{k}\ (\mathbf{k+1}) \\ & \sim & \mathbf{k}^a \qquad \text{Case (2a)}\end{array}$$

$$\begin{array}{rcl}\mathbf{k}^a(\mathbf{k+1})\mathbf{kk}(\mathbf{k+1}) & \sim & (\mathbf{k+1})^a\mathbf{k}\ (\mathbf{k+1}) \\ & \sim & \mathbf{k}^a \qquad \text{Case (2b)}\end{array}$$

$$\begin{array}{rl}\multicolumn{2}{l}{\mathbf{k}^a(\mathbf{k+1})\ \mathbf{k}\ (\mathbf{k-1})\cdots(\mathbf{k-1})\ \mathbf{k}\ (\mathbf{k+1})} \\ \sim & (\mathbf{k+1})^a(\mathbf{k-1})\cdots(\mathbf{k-1})\ \mathbf{k}\ (\mathbf{k+1}) \\ \sim & (\mathbf{k+1})^a\mathbf{k}\ (\mathbf{k+1}) \\ \sim & \mathbf{k}^a \qquad \text{Case (2c)}\end{array}$$

□

Proposition 2.2.2. *For any $\omega \in Br_n(0)$ the element $\omega y_k^a (y_k^0)^{-1}\omega^{-1}$ depends only on the class $\bar\omega$ of ω in $\mathcal{S}_n$. Moreover if $\bar\omega(j) = k$ and $\bar\omega(j+1) = k+1$, then $\omega y_k^a (y_k^0)^{-1}\,\omega^{-1} = y_j^a (y_j^0)^{-1}$.*

Proof. The first statement is a consequence of Lemma 2.2.1 since the Weyl group $W(A_{n-1}) = \mathcal{S}_n$ is the quotient of Br_n by PBr_n.
The second part is a consequence of relation $(A1 \times A1)$ and the following computation:

$$\begin{array}{rcl}\mathbf{2}^{-1}\mathbf{1}^{-1}\mathbf{2}^a\mathbf{12} & = & \mathbf{1}^a\mathbf{2}^{-1}\mathbf{1}^{-1}\mathbf{2}^{-1}\mathbf{12} \\ & = & \mathbf{1}^a\mathbf{1}^{-1}\mathbf{2}^{-1}\mathbf{1}^{-1}\mathbf{12} \\ & = & \mathbf{1}^a\mathbf{1}^{-1}\ .\end{array}$$

□

2.3 The Steinberg group and an action of the Weyl group

When $\Phi = A_{n-1}$, the Steinberg group of the ring A is well-known and customarily denoted $St_n(A)$. In that case it is customary to write $x_{ii+1}^a = x_{ii+1}(a)$ for the element $x_\alpha(a), \alpha = \epsilon_i - \epsilon_{i+1} \in \Delta$, and, more generally, x_{ij}^a when $\alpha = \epsilon_i - \epsilon_j \in A_{n-1}$.

Definition 2.3.1 ([Stb]). *The Steinberg group of the ring A, denoted $St_n(A)$, is presented by the generators $x_{ij}^a, 1 \le i,j \le n, i \ne j, a \in A$ subject to the relations*

$$\begin{array}{lll}(St0) & x_{ij}^a x_{ij}^b = x_{ij}^{a+b} & \\ (St1) & x_{ij}^a x_{kl}^b = x_{kl}^b x_{ij}^a, & i \ne l, j \ne k \\ (St2) & x_{ij}^a x_{jk}^b = x_{jk}^b x_{ik}^{ab} x_{ij}^a, & i \ne k.\end{array}$$

We should make two observations about this definition. First, it follows from $(St0)$ that $x_{ij}^0 = 1$. Second, relation $(St2)$ is given in a perhaps unfamiliar form. We have chosen this form, which is easily seen to be equivalent to $(R2)$, because of its geometric significance (*cf.* [K-S] for the relationship with the Stasheff polytope), and for the simplification it brings in computation.

The Weyl group $W(A_{n-1})$ is the symmetric group $\mathcal{S}_n$. Its action on the Steinberg group is induced by the formula

$$(3) \qquad \sigma \cdot x_{ij}^a := x_{\sigma(i)\sigma(j)}^a, \quad \sigma \in \mathcal{S}_n, a \in A$$

2.4 The main result

THEOREM 2.4.1. *For any (not necessarily unital) ring A the map*

$$\phi : Br_n(A) \to St_n(A) \rtimes Br_n$$

from the parametrized braid group to the semi-direct product of the Artin braid group with the Steinberg group induced by $\phi(y_i^a) = x_{i\,i+1}^a\, y_i$ is a group isomorphism.

Proof. Step (a). We show that ϕ is a well-defined group homomorphism.

• Relation (A1):

$$\begin{array}{rcll} \phi(y_i^a y_i^0 y_i^b) & = & x_{i\,i+1}^a y_i\, y_i\, x_{i\,i+1}^b y_i\,, & \text{since} \quad x_{i\,i+1}^0 = 1, \\ & = & y_i\, y_i\, x_{i\,i+1}^a x_{i\,i+1}^b y_i & \text{since} \quad \overline{y_i\, y_i} = 1 \in \mathcal{S}_n, \\ & = & y_i\, y_i\, x_{i\,i+1}^{a+b} y_i & \text{by } (St0), \\ & = & \phi(y_i^0 y_i^0 y_i^{a+b}). & \end{array}$$

• Relation $(A1 \times A1)$ follows immediately from $(St1)$.
• Relation $(A2)$ is proved by using the relations of Br_n and the 3 relations $(St0), (St1), (St2)$ as follows:

$$\begin{array}{rcl} \phi(y_i^a y_{i+1}^b y_i^c) & = & x_{i\,i+1}^a y_i\, x_{i+1\,i+2}^b\, \underbrace{y_{i+1}\, x_{i\,i+1}^c}\, y_i \\ & = & x_{i\,i+1}^a y_i\, \underbrace{x_{i+1\,i+2}^b x_{i\,i+2}^c}\, y_{i+1}\, y_i\,, \\ & = & x_{i\,i+1}^a\, \underbrace{y_i\, x_{i\,i+2}^c}\, x_{i+1\,i+2}^b\, y_{i+1}\, y_i\,, \\ & = & \underbrace{x_{i\,i+1}^a x_{i+1\,i+2}^c}\, \underbrace{y_i\, x_{i+1\,i+2}^b}\, y_{i+1}\, y_i\,, \\ & = & x_{i+1\,i+2}^c x_{i\,i+2}^{ac}\, \underbrace{x_{i\,i+1}^a x_{i\,i+2}^b}\, \underbrace{y_i\, y_{i+1}\, y_i}\,, \\ & = & x_{i+1\,i+2}^c x_{i\,i+2}^{ac+b}\, \underbrace{x_{i\,i+1}^a y_{i+1}}\, y_i\, y_{i+1}\,, \\ & = & x_{i+1\,i+2}^c\, \underbrace{x_{i\,i+2}^{ac+b}\, y_{i+1}}\, \underbrace{x_{i\,i+2}^a\, y_i\, y_{i+1}}\,, \\ & = & \underbrace{x_{i+1\,i+2}^c y_{i+1}}\, \underbrace{x_{i\,i+1}^{ac+b} y_i}\, \underbrace{x_{i+1\,i+2}^a\, y_{i+1}}\,, \\ & = & \phi(y_{i+1}^c\, y_i^{b+ac}\, y_{i+1}^a). \end{array}$$

Step (b). This is the Pure Braid Lemma 2.2.1 for A_n.

Step (c). We construct a homomorphism $\psi : St_n(A) \rtimes Br_n \to Br_n(A)$. We first construct $\psi : St_n(A) \to \mathrm{Ker}\ \pi$, where π is the surjection $Br_n(A) \to Br_n$, by setting

$$\psi(x_{ij}^a) \quad := \quad \omega\ y_k^a (y_k^0)^{-1}\ \omega^{-1}$$

where ω is an element of Br_n such that $\bar{\omega}(k) = i$ and $\bar{\omega}(k+1) = j$ (for instance, $\psi(x_{12}^a) = y_1^a(y_1^0)^{-1}$ and $\psi(x_{13}^a) = y_2^0(y_1^a(y_1^0)^{-1})(y_2^0)^{-1}$). Observe that this definition does not depend on the choice of $\bar{\omega}$ (by Lemma 2.1.2), and does not depend on how we choose a lifting ω of $\bar{\omega}$ (by the Pure Braid Lemma 2.2.1). In order to show that ψ is a homomorphism, we must demonstrate that the Steinberg relations are preserved.

• Relation ($St0$): it suffices to show that $\psi(x_{12}^a x_{12}^b) = \psi(x_{12}^{a+b})$,

$$\begin{aligned}
\psi(x_{12}^a x_{12}^b) &= y_1^a(y_1^0)^{-1} y_1^b (y_1^0)^{-1} = y_1^a (y_1^0)^{-2} y_1^0 y_1^b (y_1^0)^{-1} \\
&= (y_1^0)^{-2} y_1^a y_1^0 y_1^b (y_1^0)^{-1} && \text{by 2.2.1,} \\
&= y_1^{a+b}(y_1^0)^{-1} && \text{by (A1),} \\
&= \psi(x_{12}^{a+b}).
\end{aligned}$$

• Relation ($St1$): it suffices to show that $\psi(x_{12}^a x_{34}^b) = \psi(x_{34}^b x_{12}^a)$ and that $\psi(x_{12}^a x_{13}^b) = \psi(x_{13}^b x_{12}^a)$. The first case is an immediate consequence of the Pure Braid Lemma 2.2.1 and of relation $(A1 \times A1)$. Let us prove the second case, which relies on the Pure Braid Lemma 2.2.1 and relation $(A2)$:

$$\begin{aligned}
\psi(x_{12}^a x_{13}^b) &= \underbrace{y_1^a (y_1^0)^{-1}}\, y_2^0 y_1^b (y_1^0)^{-1} (y_2^0)^{-1} \\
&= (y_1^0)^{-2} y_1^a\, \underbrace{y_1^0 y_2^0 y_1^b} (y_1^0)^{-1} (y_2^0)^{-1} \\
&= (y_1^0)^{-2}\, \underbrace{y_1^a y_2^b y_1^0}\ \underbrace{y_2^0 (y_1^0)^{-1} (y_2^0)^{-1}} \\
&= (y_1^0)^{-2} y_2^0 y_1^b\, \underbrace{y_2^a (y_1^0)^{-1} (y_2^0)^{-1}}\, y_1^0 \\
&= (y_1^0)^{-2} y_2^0 y_1^b (y_1^0)^{-1} (y_2^0)^{-1}\ \underbrace{y_1^a y_1^0} \\
&= \underbrace{(y_1^0)^{-2} y_2^0 y_1^b (y_1^0)^{-1} (y_2^0)^{-1} (y_1^0)^2}\, \underbrace{y_1^a (y_1^0)^{-1}} \\
&= \psi(x_{13}^b)\psi(x_{12}^a).
\end{aligned}$$

• Relation ($St2$): it suffices to show that $\psi(x_{12}^a x_{23}^b) = \psi(x_{23}^b x_{13}^{ab} x_{12}^a)$.

$$\begin{aligned}
\psi(x_{12}^a x_{23}^b) &= y_1^a (y_1^0)^{-1}\ \underbrace{y_2^b}\ (y_2^0)^{-1} \\
&= y_1^a\, \underbrace{(y_1^0)^{-1} y_2^b y_1^0} (y_1^0)^{-1} (y_2^0)^{-1} \\
&= \underbrace{y_1^a y_2^0 y_1^b}\, \underbrace{(y_2^0)^{-1} (y_1^0)^{-1} (y_2^0)^{-1}} \\
&= y_2^b\, \underbrace{\qquad}\, y_1^{ab}\, \underbrace{\qquad}\, y_2^a (y_1^0)^{-1} (y_2^0)^{-1} (y_1^0)^{-1} \\
&= y_2^b (y_2^0)^{-1} y_2^0 y_1^{ab} (y_1^0)^{-1}\, \underbrace{y_1^0 y_2^a (y_1^0)^{-1} (y_2^0)^{-1}} (y_1^0)^{-1} \\
&= \underbrace{y_2^b (y_2^0)^{-1}}\, \underbrace{y_2^0 y_1^{ab} (y_1^0)^{-1} (y_2^0)^{-1}}\, \underbrace{y_1^a (y_1^0)^{-1}} \\
&= \psi(x_{23}^b)\psi(x_{13}^{ab})\psi(x_{12}^a),
\end{aligned}$$

as a consequence of relation $(A2)$.
From 2.2.2 it follows that the action of an element of Br_n by conjugation on Ker π depends only on its class in $\mathcal{S}_n$. The definition of ψ on $St_n(A)$ makes clear that it is an $\mathcal{S}_n$-equivariant map.
Defining ψ on Br_n by $\psi(y_\alpha) = y_\alpha^0 \in Br_n(0)$ yields a group homomorphism

$$\psi : St_n(A) \rtimes Br_n \to \mathrm{Ker}\ \pi \rtimes Br_n = Br_n(A).$$

The group homomorphisms ϕ and ψ are clearly inverse to each other since they interchange y_α^a and $x_\alpha^a y_\alpha$. Hence they are both isomorphisms, as asserted. $\square$

COROLLARY 2.4.2 (KASSEL-REUTENAUER [K-R]). *The group presented by generators y_i^a, $1 \leq i \leq n-1$, $a \in A$, and relations*

$$\begin{aligned} (y_i^0)^2 &= 1 \\ y_i^a (y_i^0)^{-1} y_i^b &= y_i^{a+b} \\ y_i^a y_j^b &= y_j^b y_i^a \qquad \textit{if } |i-j| \geq 2 \\ y_i^a y_{i+1}^b y_i^c &= y_{i+1}^c y_i^{b+ac}\, y_{i+1}^a \end{aligned}$$

$a, b, c \in A$, is isomorphic to the semi-direct product $St_n(A) \rtimes \mathcal{S}_n$.

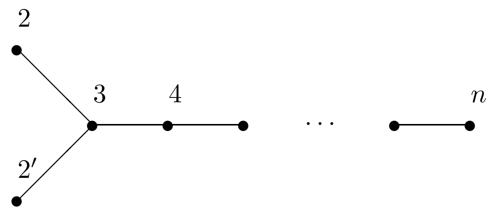
Observe that when the first relation in this Corollary is deleted, the second relation has several possible non-equivalent liftings. The one we have chosen, $(A1)$, is what allows us to prove Theorem 2.4.1.

3 THE PARAMETRIZED BRAID GROUP IN THE D_n CASE

In this section we discuss the parametrized braid group $Br(D_n, A)$ for a commutative ring A and prove that it is isomorphic to the semi-direct product of the Steinberg group $St(D_n, A)$ by the braid group $Br(D_n, 0)$.

3.1 THE BRAID GROUP AND THE PARAMETRIZED BRAID GROUP FOR D_n

Let $\Delta = \{\alpha_2, \alpha_{2'}, \alpha_3, \ldots, \alpha_n\}$ be a fixed simple subsystem of a root system of type $D_n, n \geq 3$. We adopt the notation of [D-G] in which the simple roots on the fork of D_n are labeled $\alpha_2, \alpha_{2'}$. The system D_n contains 2 subsystems of type A_{n-1} generated by the simple subsystems $\{\alpha_2, \alpha_3, \ldots, \alpha_n\}$ and $\{\alpha_{2'}, \alpha_3, \ldots, \alpha_n\}$, and, for $n \geq 4$, a subsystem of type D_{n-1} generated by the simple subsystem $\{\alpha_2, \alpha_{2'}, \alpha_3, \ldots, \alpha_{n-1}\}$.



Dynkin diagram of D_n

The *Weyl group* $W(D_n)$ is generated by the simple reflections $\{\sigma_i = \sigma_{\alpha_i} | \alpha_i \in \Delta\}$, with defining relations

$$\begin{aligned} \sigma_i^2 &= 1 \\ (\sigma_i\sigma_j)^{m(i,j)} &= 1 \end{aligned}$$

for $i, j \in \{2, 2', 3, \ldots, n\}$, where

$$m(i,j) = \begin{cases} 2 & \text{if } \alpha_i, \alpha_j \text{ are not connected in the Dynkin diagram,} \\ 3 & \text{if } \alpha_i, \alpha_j \text{ are connected in the Dynkin diagram} \end{cases}$$

Since the only values for $m(\alpha, \beta)$ are $1, 2$ and 3, the group $Br(D_n, A)$ involves only relations $(A1), (A1 \times A1)$ and $(A2)$.

DEFINITION 3.1.1. *The parametrized braid group of type D_n with parameters in the commutative ring A, denoted $Br(D_n, A)$, is generated by the elements y_α^a, where $\alpha \in \Delta$ and $a \in A$. The relations are, for $a, b \in A$ and $\alpha, \beta \in \Delta$*

$$\begin{array}{rll} (A1) & y_\alpha^a y_\alpha^0 y_\alpha^b = y_\alpha^0 y_\alpha^0 y_\alpha^{a+b} & \\ (A1 \times A1) & y_\alpha^a y_\beta^b = y_\beta^b y_\alpha^a & \textit{if } m(\alpha,\beta) = 2 \\ (A2) & y_\alpha^a y_\beta^b y_\alpha^c = y_\beta^c y_\alpha^{b+ac} y_\beta^a & \textit{if } m(\alpha,\beta) = 3 \textit{ and } \alpha < \beta \end{array}$$

Note that the simple roots in D_n are ordered so that $\alpha_{2'} < \alpha_3$.

3.2 THE STEINBERG GROUP OF D_n AND THE MAIN RESULT

The roots of D_n are $\{\pm\epsilon_i \pm \epsilon_j | 1 \leq i \neq j \leq n\}$, [Car],[H]. The Weyl group $W(D_n) \cong (\mathbb{Z}/2)^{n-1} \rtimes \mathcal{S}_n$ [Bour, p. 257, (X)] acts on the roots by permuting the indices (action of $\mathcal{S}_n$) and changing the signs (action of $(\mathbb{Z}/2)^{n-1}$). For the simple subsystem Δ we take $\alpha_i = -\epsilon_{i-1} + \epsilon_i$ for $i = 2, \cdots, n$, and $\alpha_{2'} = \epsilon_1 + \epsilon_2$. If u and v are positive integers and α, β two roots, the linear combination $u\alpha + v\beta$ is a root if and only if $u = 1 = v$, $\alpha = \pm\epsilon_i \pm \epsilon_j, \beta \mp \epsilon_j \pm \epsilon_k$ and $\pm\epsilon_i \pm \epsilon_k \neq 0$. In this case the definition of the Steinberg group is as follows.

Definition 3.2.1 ([Stb][St]). *The Steinberg group of type D_n with parameters in the commutative ring A, denoted $St(D_n, A)$, is generated by elements x_α^a, where $\alpha \in \Phi$ and $a \in A$, subject to the relations (for $a, b \in A$ and $\alpha, \beta \in \Phi$)*

$$\begin{array}{lll} (St0) & x_\alpha^a x_\alpha^b = x_\alpha^{a+b} & \\ (St1) & x_\alpha^a x_\beta^b = x_\beta^b x_\alpha^a & \text{if } \alpha+\beta \notin D_n \text{ and } \alpha+\beta \neq 0, \\ (St2) & x_\alpha^a x_\beta^b = x_\beta^b x_{\alpha+\beta}^{ab} y_\alpha^a & \text{if } \alpha+\beta \in D_n. \end{array}$$

The Weyl group $W(D_n)$ acts on $St(D_n, A)$ by $\sigma \cdot x_\alpha^a = x_{\sigma(\alpha)}^a$, and we can construct the semi-direct product $St(D_n, A) \rtimes Br(D_n)$ with respect to this action.

Theorem 3.2.2. *For any commutative ring A the map*

$$\phi : Br(D_n, A) \to St(D_n, A) \rtimes Br(D_n)$$

induced by $\phi(y_\alpha^a) = x_\alpha^a \, y_\alpha$ is a group isomorphism.

Corollary 3.2.3. *The group presented by generators y_i^a, $i = 2', 2, 3, \cdots, n$, $a \in A$ and relations*

$$\begin{array}{rll} (y_i^0)^2 &= 1 & \\ y_i^a (y_i^0)^{-1} y_i^b &= y_i^{a+b} & \\ y_i^a y_j^b &= y_j^b y_i^a & \text{if } |i-j| \geq 2, \text{ or } i=2, j=2', \\ y_i^a y_{i+1}^b y_i^c &= y_{i+1}^c y_i^{b+ac} \, y_{i+1}^a & \text{where } i+1 = 3 \text{ when } i = 2' \end{array}$$

for $a, b, c \in A$, is isomorphic to the semi-direct product $St(D_n, A) \rtimes W(D_n)$.

Proof of Corollary. For each simple root $\alpha_i \in D_n$, write y_i^a for $y_{\alpha_i}^a$. □

Proof of Theorem 3.2.2.
Step (a). Since the relations involved in the definitions of $Br(D_n, A)$ and $St(D_n, A)$ are the same as the relations in the case of A_{n-1}, the map ϕ is well-defined (*cf.* Theorem 2.4.1).
Step (b). The proof of the Pure Braid Lemma in the D_n case will be given below in 3.3.
Step (c). Let $\pi : Br(D_n, A) \to Br(D_n)$ be the projection which sends each $a \in A$ to 0 (as usual we identify $Br(D_n, 0)$ with $Br(D_n)$). We define

$$\psi : St(D_n, A) \rtimes Br(D_n) \to Br(D_n, A) \cong \mathrm{Ker}\ \pi \rtimes Br(D_n)$$

on the first component by $\psi(x_\alpha^a) = y_\alpha^a (y_\alpha^0)^{-1} \in \mathrm{Ker}\ \pi$ for $\alpha \in \Delta$. For any $\alpha \in D_n$ there exists $\sigma \in W(D_n)$ such that $\sigma(\alpha) \in \Delta$. Let $\tilde{\sigma} \in Br(D_n)$ be a lifting of σ, and define $\psi(x^a_\alpha) = \tilde{\sigma}^{-1}\psi(x^a_{\sigma(\alpha)})\tilde{\sigma} \in \operatorname{Ker} \pi$. This element is well-defined since it does not depend on the lifting of σ by the Pure Braid Lemma for D_n (Lemma 3.3.2), and does not depend on the choice of σ by Lemma 2.1.2. In order to show that ψ is a well-defined group homomorphism, it suffices to show that the Steinberg relations are preserved. But this is the same verification as in the A_{n-1} case, (*cf.* Theorem 2.4.1.)

The group homomorphisms ϕ and ψ are inverse to each other since they interchange y^a_α and $x^a_\alpha y_\alpha$. Hence they are both isomorphisms. $\square$

3.3 The Pure Braid Lemma for D_n

3.3.1 Generators for the Pure Braid Group of D_n

In principle, the method of Reidemeister-Schreier [M-K-S] is available to deduce a presentation of $PBr(D_n)$ from that of $Br(D_n)$. The details have been worked out by Digne and Gomi [D-G], although not in the specificity we need here. From their work we can deduce that the group $PBr(D_n)$ is generated by the elements $y^2_\alpha, \alpha \in \Delta$, together with a very small set of their conjugates. For example, $PBr(D_4)$ is generated by the 12 elements

$$\mathbf{2}^2, \mathbf{2'}^2, \mathbf{3}^2, {}^{\mathbf{3}}\mathbf{2}^2, {}^{\mathbf{3}}\mathbf{2'}^2, {}^{\mathbf{322'}}\mathbf{3}^2, \mathbf{4}^2, {}^{\mathbf{4}}\mathbf{3}^2, {}^{\mathbf{43}}\mathbf{2'}^2, {}^{\mathbf{43}}\mathbf{2}^2, {}^{\mathbf{4322'}}\mathbf{3}^2, {}^{\mathbf{4322'3}}\mathbf{4}^2$$

where a prefixed exponent indicates conjugation: ${}^h g = hgh^{-1}$. Here (and throughout) we use the simplified notations $\mathbf{k}^a = y^a_{\alpha_k}$ and $\mathbf{k} = y^0_{\alpha_k}$ similar to those of 2.2.

Proposition 3.3.1. *For $n \geq 4$, $PBr(D_n)$ is generated by the elements*
- $\mathbf{a}_{j,i} = \mathbf{j(j-1)\dots(i+1)ii(i+1)\dots(j-1)j}$, *$n \geq j \geq i \geq 2$, and*
- $\mathbf{b}_{j,i} = \mathbf{j(j-1)\dots 322'3\dots(i-1)ii(i-1)\dots 32'23(j-1)j}$, *$n \geq j \geq i \geq 3$,*

where $i+1 = 3$ when $i = 2'$.

Note. Since the notation can be confusing, let us be clear about the definition of these generators in certain special cases:
- When $i = j, \mathbf{a}_{j,i} = \mathbf{i}^2$.
- When $i = 3, \mathbf{b}_{j,3} = \mathbf{j(j-1)\dots 322'332'2\dots(j-1)j}$.

Proof of (3.3.1). We work in the case where $W = W(D_n)$ in the notation of [D-G]. In the proof of [D-G, Corollary 2.7], we see that $P_W = U_n \rtimes P_{W_{I_{n-1}}}$; taking $I_n = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_{2'}, \dots, \mathbf{s}_{n-1}\}$, $n \geq 4$, as on [D-G, p. 10], we see that their P_W is equal to (our) $PBr(D_n)$ and their $P_{W_{I_{n-1}}}$ is equal to (our) $PBr(D_{n-1})$. It follows that a set of generators for $PBr(D_n)$ can be obtained as the union of a set of generators for $PBr(D_{n-1})$ with a set of generators for U_n. This sets the stage for an inductive argument, since $D_3 = A_3$ (with $\{\alpha_2, \alpha_3, \alpha_{2'}\} \subset D_3$ identified with $\{\alpha_1, \alpha_2, \alpha_3\} \subset A_3$). Because $W(D_n)$ is a finite Weyl group, it follows from [D-G, Proposition 3.6], that U_n is generated (not just normally generated) by the elements $\mathbf{a_{b,s}}$, and a list of these generators in our case is given on [D-G, p. 10].

The calculations necessary to prove the Pure Braid Lemma for D_n are simpler if we replace the Digne-Gomi generators by the equivalent set in which conjugation is replaced by *reflection*; that is, we replace a generator ${}^hg = hgh^{-1}$ by hgh', where if $h = y_{i_1} \dots y_{i_k}$, $h' = y_{i_k} \dots y_{i_1}$. (We already used this trick in the case of A_{n-1}.) For D_4, this procedure yields as generators of $PBr(D_4)$ the set

$$\mathbf{2}^2, \mathbf{2}'^2, \mathbf{3}^2, \mathbf{32^23}, \mathbf{32'^23}, \mathbf{322'3^22'23}, \mathbf{4}^2, \mathbf{43^24}, \mathbf{432^234}, \mathbf{432'^234},$$
$$\mathbf{4322'3^22'234}, \mathbf{4322'34^232'234},$$

and, more generally, $PBr(D_n), n \geq 4$ is generated by the elements stated in the Proposition. □

LEMMA 3.3.2 (PURE BRAID LEMMA FOR D_n). *Let $\alpha_k \in D_n$, let y_k^a be a generator of $Br(D_n, A)$, and let $\omega \in PBr(D_n)$. Then there exists $\omega' \in Br(D_n)$, independent of a, such that*

$$y_k^a\ \omega = \omega'\ y_k^a.$$

Hence for any integer k and any element $a \in A$, the element $y_k^a(y_k^0)^{-1} \in Br(D_n, A)$ commutes with every element of the pure braid group $PBr(D_n)$.

Proof. Let us show that the first assertion implies the second one. Let $\omega \in PBr(D_n) \subset PBr(D_n, A)$. By the first assertion of the Lemma we have

$$y_k^a \omega = \omega' y_k^a$$

for some $\omega' \in Br(D_n, 0)$, independent of a. Setting $a = 0$ tells us that $\omega' = y_k^0 \omega (y_k^0)^{-1}$. Thus

$$y_k^a (y_k^0)^{-1} \omega = \omega y_k^a (y_k^0)^{-1}$$

as desired.

Before beginning the proof of the first assertion, we recall some notation introduced in 2.2. We abbreviate $y_{\alpha_k}^a$ by $\mathbf{k}^a$ and $y_{\alpha_k}^0$ by $\mathbf{k}$. Whenever there exist ω' and $\omega'' \in Br(D_n)$ such that $\mathbf{k}^a\omega = \omega'\mathbf{j}^a\omega''$, we will write $\mathbf{k}^a\omega \sim \mathbf{j}^a\omega''$. This is *not* an equivalence relation, but it is compatible with multiplication on the right by elements of $Br(D_n)$: if $\eta \in Br(D_n)$,

$$\mathbf{k}^a\omega \sim \mathbf{j}^a\omega'' \Leftrightarrow \mathbf{k}^a\omega = \omega'\mathbf{j}^a\omega'' \Leftrightarrow \mathbf{k}^a\omega\eta = \omega'\mathbf{j}^a\omega''\eta \Leftrightarrow \mathbf{k}^a\omega\eta \sim \mathbf{j}^a\omega''\eta$$

From defining relations $(A1), (A1 \times A1)$, and $(A2)$ of 2, we can deduce the following:

$$\begin{array}{lrcl}
\text{(4a)} & \mathbf{k}^a\mathbf{k}\mathbf{k} & \sim & \mathbf{k}^a \\
\text{(4b)} & \mathbf{k}^a\ (\mathbf{k}-\mathbf{1})\ \mathbf{k} & \sim & (\mathbf{k}-\mathbf{1})^a \\
\text{(4c)} & \mathbf{k}^a\ (\mathbf{k}-\mathbf{1})\ \mathbf{k}^{-1} & \sim & (\mathbf{k}-\mathbf{1})^a \\
\text{(4d)} & \mathbf{k}^a(\mathbf{k}+\mathbf{1})\mathbf{k} & \sim & (\mathbf{k}+\mathbf{1})^a \\
\text{(4e)} & (\mathbf{k}+\mathbf{1})^a\mathbf{k}(\mathbf{k}+\mathbf{1}) & \sim & \mathbf{k}^a \\
\text{(4f)} & (\mathbf{k}+\mathbf{1})^a\mathbf{k}(\mathbf{k}+\mathbf{1})^{-1} & \sim & \mathbf{k}^a \\
\text{(4g)} & \mathbf{k}^a(\mathbf{k}+\mathbf{1})\mathbf{k}^{-1} & \sim & (\mathbf{k}+\mathbf{1})^a \\
\text{(4h)} & \mathbf{k}^a(\mathbf{k}+\mathbf{1})^{-1}\mathbf{k}^{-1} & \sim & (\mathbf{k}+\mathbf{1})^a \\
\text{(4i)} & (\mathbf{k}+\mathbf{1})^a\mathbf{k}^{-1}(\mathbf{k}+\mathbf{1})^{-1} & \sim & \mathbf{k}^a \\
\text{(4j)} & \mathbf{k}^a\mathbf{k}(\mathbf{k}-\mathbf{1}) & \sim & (\mathbf{k}-\mathbf{1})^a(\mathbf{k}-\mathbf{1})\mathbf{k}^{-1}
\end{array}$$

Proof, continued. In the notation just introduced, we must show, for every $\omega \in PBr(D_n)$, that $\mathbf{k}^a\omega \sim \mathbf{k}^a$, and it suffices to show this when ω is one of the generators $\mathbf{a}_{j,i}$ or $\mathbf{b}_{j,i}$ of 3.3.1. That is, we must show

$$\begin{array}{lrcll}
\text{(5a)} & \mathbf{k}^a\mathbf{a}_{j,i} & \sim & \mathbf{k}^a & n \geq j \geq i \geq 2, \quad 1 \leq k \leq n \\
\text{(5b)} & \mathbf{k}^a\mathbf{b}_{j,i} & \sim & \mathbf{k}^a & n \geq j \geq i \geq 3, \quad 1 \leq k \leq n
\end{array}$$

The proofs of (5a) for $i \geq 3$ are exactly the same as the corresponding proofs for A_{n-1} (see section 2); the additional case $i = 2'$ presents no new issues. Thus we shall concentrate on proving (5b); the proof proceeds by induction on n.

The case $n = 3$ is the case of the root system $D_3 = A_3$, which is part of the Pure Braid Lemma 2.2.1 for A_{n-1}. Hence we may assume $n \geq 4$, and that (5b) holds whenever $j, k \leq n-1$. That is, we must prove (5b) in these cases:

$$k = n,\ j \leq n-1; \quad k \leq n-1,\ j = n; \quad k = n,\ j = n$$

which further subdivide into the cases

$$\begin{array}{lll}
(6) & k = n, & j \leq n-2 \\
(7) & k = n, & j = n-1 \\
(8) & k \leq n-2, & j = n \\
(9) & k = n-1, & j = n \\
(10) & k = n, & j = n
\end{array}$$

• *Case* (6) $k = n$ *and* $j \leq n-2$. Since $i \leq j \leq n-2$, it follows from relation $(A1 \times A1)$ that $\mathbf{n}^a$ commutes with every generator which occurs in the expression for $\mathbf{b}_{j,i}$; hence

$$\begin{aligned}
\mathbf{n}^a\mathbf{b}_{j,i} &= \mathbf{n}^a\mathbf{j}(\mathbf{j}-\mathbf{1})\ldots\mathbf{3}\mathbf{2}\mathbf{2}'\mathbf{3}\ldots(\mathbf{i}-\mathbf{1})\mathbf{i}\mathbf{i}(\mathbf{i}-\mathbf{1})\ldots\mathbf{3}\mathbf{2}'\mathbf{2}\mathbf{3}(\mathbf{j}-\mathbf{1})\mathbf{j} \\
&= \mathbf{b}_{j,i}\mathbf{n}^a \\
&\sim \mathbf{n}^a
\end{aligned}$$

as desired.

• *Case* (7) $k = n$ *and* $j = n-1$.

If $i \leq n-2$, then

$$\begin{aligned}
&\mathbf{n}^a\mathbf{b}_{n-1,i} \\
&= \mathbf{n}^a\mathbf{(n-1)(n-2)\dots 322'3\dots(i-1)ii(i-1)\dots 32'23\dots(n-2)(n-1)} \\
&= \underbrace{\mathbf{n}^a\mathbf{(n-1)n}^{-1}}\mathbf{n(n-2)\dots 322'3\dots(i-1)ii(i-1)\dots 32'23\dots(n-2)(n-1)} \\
&\sim \mathbf{(n-1)}^a\mathbf{n(n-2)\dots 322'3\dots(i-1)ii(i-1)\dots 32'23\dots(n-2)(n-1)} \\
&= \underbrace{\mathbf{(n-1)}^a\mathbf{(n-2)\dots 322'3\dots(i-1)ii(i-1)\dots 32'23\dots(n-2)}}\mathbf{n(n-1)} \\
&\sim \mathbf{(n-1)}^a\mathbf{n(n-1)} \\
&\sim \mathbf{n}^a
\end{aligned}$$

as desired.
The case $k = n, i = j = n-1$, is considerably more complicated. We first prove some preliminary lemmas.

LEMMA 3.3.3.

$$\mathbf{n}^a\mathbf{(n-1)(n-2)\dots 322'} \sim \mathbf{2}^a\mathbf{32'4\dots(n-2)(n-1)n}$$

Proof.

$$\begin{aligned}
\mathbf{n}^a\mathbf{(n-1)(n-2)\dots 322'} &= \mathbf{n}^a\mathbf{(n-1)n}^{-1}\mathbf{n(n-2)\dots 322'} \\
&\sim \mathbf{(n-1)}^a\mathbf{n(n-2)\dots 322'} \\
&= \mathbf{(n-1)}^a\mathbf{(n-2)\dots 322'n} \\
&\vdots \\
&\sim \mathbf{3}^a\mathbf{22'4\dots(n-2)(n-1)n} \\
&= \mathbf{3}^a\mathbf{23}^{-1}\mathbf{32'4\dots(n-2)(n-1)n} \\
&\sim \mathbf{2}^a\mathbf{32'4\dots(n-2)(n-1)n}
\end{aligned}$$

□

LEMMA 3.3.4.

$$\mathbf{32'4354\dots(n-3)(n-1)(n-2)nn} = \mathbf{345\dots(n-1)nn2'34\dots(n-2)}$$

Proof.

$$\begin{aligned}
&\mathbf{32'4354\dots(n-3)(n-1)(n-2)nn} \\
&\qquad = \mathbf{32'4354\dots(n-3)(n-1)nn(n-2)} \\
&\qquad = \mathbf{32'4354\dots(n-1)nn(n-3)(n-2)} \\
&\qquad \vdots \\
&\qquad = \mathbf{345\dots(n-1)nn2'34\dots(n-2)}
\end{aligned}$$

□

We now complete the case $k = n, i = j = n-1$.

$$
\begin{aligned}
&\mathbf{n}^a\mathbf{b}_{n-1,n-1}\\
&= \mathbf{n}^a(\mathbf{n-1})(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{n-2})(\mathbf{n-1})(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\sim \mathbf{2}^a\mathbf{32'4}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n3}\dots(\mathbf{n-2})(\mathbf{n-1})(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1}) \quad \text{(by Lemma 3.3.3)}\\
&= \mathbf{2}^a\mathbf{32'4}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{3}\dots(\mathbf{n-2})\underbrace{\mathbf{n}(\mathbf{n-1})\mathbf{n}^{-1}}\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&= \mathbf{2}^a\mathbf{32'4}\dots(\mathbf{n-2})\underbrace{(\mathbf{n-1})\mathbf{3}\dots}(\mathbf{n-2})(\mathbf{n-1})^{-1}\mathbf{n}(\mathbf{n-1})\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&= \mathbf{2}^a\mathbf{32'4}\dots(\mathbf{n-2})\mathbf{3}\dots\underbrace{(\mathbf{n-1})(\mathbf{n-2})(\mathbf{n-1})^{-1}}\mathbf{n}(\mathbf{n-1})\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&= \mathbf{2}^a\mathbf{32'4}\dots(\mathbf{n-2})\mathbf{3}\dots\underbrace{(\mathbf{n-2})^{-1}(\mathbf{n-1})(\mathbf{n-2})}\mathbf{n}(\mathbf{n-1})\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad \vdots\\
&= \mathbf{2}^a\mathbf{32'}\underbrace{\mathbf{434}^{-1}}\mathbf{54}\dots(\mathbf{n-1})(\mathbf{n-2})\mathbf{n}(\mathbf{n-1})\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&= \mathbf{2}^a\underbrace{\mathbf{32'3}^{-1}}\mathbf{4354}\dots(\mathbf{n-1})(\mathbf{n-2})\mathbf{n}(\mathbf{n-1})\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&= \mathbf{2}^a\underbrace{\mathbf{2'}^{-1}\mathbf{32'}}\mathbf{4354}\dots(\mathbf{n-1})(\mathbf{n-2})\mathbf{n}(\mathbf{n-1})\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\sim \mathbf{2}^a\mathbf{32'4354}\dots(\mathbf{n-1})(\mathbf{n-2})\mathbf{n}\underbrace{(\mathbf{n-1})\mathbf{n}(\mathbf{n-1})}(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&= \mathbf{2}^a\mathbf{32'4354}\dots(\mathbf{n-1})(\mathbf{n-2})\mathbf{n}\underbrace{\mathbf{n}(\mathbf{n-1})\mathbf{n}}(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&= \mathbf{2}^a\mathbf{345}\dots(\mathbf{n-1})\mathbf{nn2'34}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n}(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1}) \quad \text{(by Lemma 3.3.4)}
\end{aligned}
$$

We now manipulate part of this expression so that we can apply induction.

$$
\begin{aligned}
&\mathbf{2}^a\mathbf{345}\dots(\mathbf{n-1})\mathbf{nn2'}\\
&\qquad = \mathbf{2}^a\mathbf{345}\dots(\mathbf{n-1})\mathbf{nn}\underbrace{(\mathbf{n-1})\dots\mathbf{5433}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots(\mathbf{n-1})^{-1}}\mathbf{2'}\\
&\qquad \sim \mathbf{2}^a\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots(\mathbf{n-1})^{-1}\mathbf{2'}
\end{aligned}
$$

(by the case of $A_{n-1} = \{\alpha_2, \alpha_3, \dots, \alpha_n\}$). Hence

$$\begin{aligned}
&\mathbf{n}^a\mathbf{b}_{n-1,n-1}\\
&\quad= \mathbf{2}^a\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots(\mathbf{n-1})^{-1}\mathbf{2'34}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n}(\mathbf{n-2})(\mathbf{n-3})\dots\\
&\qquad\dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad= \mathbf{2}^a\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots(\mathbf{n-1})^{-1}\mathbf{2'34}\dots(\mathbf{n-2})(\mathbf{n-1})(\mathbf{n-2})\mathbf{n}(\mathbf{n-3})\dots\\
&\qquad\dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad= \mathbf{2}^a\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots(\mathbf{n-1})^{-1}\mathbf{2'34}\dots(\mathbf{n-1})(\mathbf{n-2})(\mathbf{n-1})\mathbf{n}(\mathbf{n-3})\dots\\
&\qquad\dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad= \mathbf{2}^a\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots(\mathbf{n-2})^{-1}\mathbf{2'34}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n}(\mathbf{n-3})\dots\\
&\qquad\dots\mathbf{32'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\qquad\vdots\\
&\quad= \mathbf{2}^a\mathbf{3}^{-1}\mathbf{2'34}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n2'23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad= \mathbf{2}^a\mathbf{3}^{-1}\mathbf{2'32'4}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad= \mathbf{2}^a\mathbf{3}^{-1}\mathbf{32'34}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad= \mathbf{2}^a\mathbf{2'34}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad\sim \mathbf{2}^a\mathbf{34}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n23}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad= \mathbf{2}^a\mathbf{324}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n3}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\quad\sim \mathbf{3}^a\mathbf{4}\dots(\mathbf{n-2})(\mathbf{n-1})\mathbf{n3}\dots(\mathbf{n-2})(\mathbf{n-1})\\
&\qquad\vdots\\
&\quad\sim (\mathbf{n-1})^a\mathbf{n}(\mathbf{n-1})\\
&\quad\sim \mathbf{n}^a
\end{aligned}$$

as desired. □

- *Case* (8) $k \le n-2$ *and* $j = n$.

$$\begin{aligned}
&\mathbf{k}^a\mathbf{b}_{n,i}\\
&\quad= \mathbf{k}^a\mathbf{n}(\mathbf{n-1})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&\quad\sim \underbrace{\mathbf{k}^a(\mathbf{k+1})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{k+1})}(\mathbf{k+2})\dots(\mathbf{n-1})\mathbf{n}\\
&\quad\sim \mathbf{k}^a(\mathbf{k+2})\dots\mathbf{n} \quad \text{(by induction)}\\
&\quad\sim \mathbf{k}^a
\end{aligned}$$

as desired.

- *Case* (9) $k = n-1$ *and* $j = n$.

$$\begin{aligned}
&(\mathbf{n-1})^a\mathbf{b}_{n,i}\\
&\quad= (\mathbf{n-1})^a\mathbf{n}(\mathbf{n-1})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&\quad= \underbrace{(\mathbf{n-1})^a\mathbf{n}(\mathbf{n-1})}\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'}\,\mathbf{223}\dots(\mathbf{n-1})\mathbf{n}\\
(\dagger)\quad&\quad\sim \mathbf{n}^a(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}
\end{aligned}$$

Suppose first that $i \le n-2$. Then

$$\begin{aligned}
(\dagger) &= \mathbf{n}^a(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&= (\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{n-2})\mathbf{n}^a(\mathbf{n-1})\mathbf{n}\\
&\sim \mathbf{n}^a(\mathbf{n-1})\mathbf{n}\\
&\sim (\mathbf{n-1})^a
\end{aligned}$$

as desired.

If $i = n-1$

$$\begin{aligned}
(\dagger) &= \mathbf{n}^a(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{n-2})(\mathbf{n-1})(\mathbf{n-1})(\mathbf{n-2})\dots\\
&\quad \dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&\sim \underbrace{\mathbf{n}^a(\mathbf{n-1})(\mathbf{n-1})}(\mathbf{n-2})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&\sim \mathbf{n}^a(\mathbf{n-2})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n} \quad \text{(by the case } A_2=\{\alpha_{n-1},\alpha_n\})\\
&\sim \mathbf{n}^a(\mathbf{n-1})\mathbf{n} \quad \text{(by relation } (A1\times A1))\\
&\sim (\mathbf{n-1})^a
\end{aligned}$$

as desired.

If $i = n$

$$\begin{aligned}
(\dagger) &= \mathbf{n}^a(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{n-1})\mathbf{nn}(\mathbf{n-1})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&\sim \mathbf{n}^a(\mathbf{n-1})\mathbf{nn}(\mathbf{n-1})(\mathbf{n-2})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&\sim (\mathbf{n-1})^a\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&\sim \mathbf{n}^a(\mathbf{n-2})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
&\sim \mathbf{n}^a(\mathbf{n-1})\mathbf{n}\\
&\sim (\mathbf{n-1})^a
\end{aligned}$$

as desired.

- *Case* (10) $k = n$ *and* $j = n$.

$$\begin{aligned}
&\mathbf{n}^a\mathbf{b}_{n,i}\\
&= \mathbf{n}^a\mathbf{n}(\mathbf{n-1})(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}\\
(\dagger)\ &\sim (\mathbf{n-1})^a(\mathbf{n-1})\mathbf{n}^{-1}(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n}
\end{aligned}$$

If $i \le n-2$, then

$$\begin{aligned}
&(\dagger)\\
&= (\mathbf{n-1})^a(\mathbf{n-1})(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots\underbrace{\mathbf{n}^{-1}(\mathbf{n-1})\mathbf{n}}\\
&= \underbrace{(\mathbf{n-1})^a(\mathbf{n-1})(\mathbf{n-2})\dots\mathbf{322'3}\dots(\mathbf{i-1})\mathbf{ii}(\mathbf{i-1})\dots\mathbf{32'23}\dots(\mathbf{n-1})}\mathbf{n}(\mathbf{n-1})^{-1}\\
&\qquad \sim (\mathbf{n-1})^a\mathbf{n}(\mathbf{n-1})^{-1} \quad \text{(by induction)}\\
&\qquad \sim \mathbf{n}^a
\end{aligned}$$

as desired.

If $i = n-1$, then

$$
\begin{aligned}
(\dagger) \quad = \quad & \mathbf{(n-1)^{\mathit{a}}(n-1)(n-2)\ldots} \\
& \mathbf{\ldots 322'3\ldots(n-2)n^{-1}(n-1)(n-1)(n-2)\ldots 32'23\ldots(n-1)n} \\
& \vdots \\
\sim \quad & \mathbf{3^{\mathit{a}}322'4^{-1}35^{-1}4\ldots} \\
& \mathbf{\ldots(n-1)^{-1}(n-2)n^{-1}(n-1)(n-1)(n-2)\ldots 32'23\ldots(n-1)n} \\
= \quad & \mathbf{3^{\mathit{a}}322'4^{-1}35^{-1}4\ldots(n-1)^{-1}(n-2)}\underbrace{\mathbf{n^{-1}(n-1)n}}\mathbf{n^{-1}(n-1)(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n} \\
= \quad & \mathbf{3^{\mathit{a}}322'4^{-1}35^{-1}4\ldots} \\
& \mathbf{\ldots}\underbrace{\mathbf{(n-1)^{-1}(n-2)(n-1)}}\mathbf{n}\underbrace{\mathbf{(n-1)^{-1}n^{-1}(n-1)}}\mathbf{(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n} \\
= \quad & \mathbf{3^{\mathit{a}}322'4^{-1}35^{-1}4\ldots} \\
& \mathbf{\ldots(n-3)}\underbrace{\mathbf{(n-2)(n-1)(n-2)^{-1}}}\mathbf{n}\underbrace{\mathbf{n(n-1)^{-1}}}\mathbf{n^{-1}(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n} \\
= \quad & \mathbf{3^{\mathit{a}}322'4^{-1}35^{-1}4\ldots} \\
& \mathbf{\ldots(n-3)(n-2)(n-1)nn(n-2)^{-1}(n-1)^{-1}n^{-1}(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n} \\
& \vdots \\
= \quad & \mathbf{3^{\mathit{a}}322'}\underbrace{\mathbf{4^{-1}34}}\mathbf{\ldots(n-2)(n-1)nn4^{-1}5^{-1}\ldots} \\
& \mathbf{\ldots(n-1)^{-1}n^{-1}(n-2)\ldots 32'23\ldots(n-1)n} \\
\sim \quad & \mathbf{2^{\mathit{a}}2}\underbrace{\mathbf{3^{-1}2'3}}\mathbf{43^{-1}5\ldots(n-1)nn4^{-1}5^{-1}\ldots(n-1)^{-1}n^{-1}(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n} \\
= \quad & \mathbf{2^{\mathit{a}}22'32'^{-1}4\ldots(n-1)nn3^{-1}4^{-1}5^{-1}\ldots(n-1)^{-1}n^{-1}(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n} \\
= \quad & \mathbf{2^{\mathit{a}}22'34\ldots(n-1)nn2'^{-1}3^{-1}4^{-1}5^{-1}\ldots(n-1)^{-1}n^{-1}(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n} \\
\sim \quad & \mathbf{2^{\mathit{a}}234\ldots(n-1)nn2'^{-1}3^{-1}4^{-1}5^{-1}\ldots(n-1)^{-1}n^{-1}(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n} \\
= \quad & \mathbf{2^{\mathit{a}}234\ldots} \\
& \mathbf{\ldots(n-1)nn}\underbrace{\mathbf{(n-1)\ldots 4322^{-1}3^{-1}4^{-1}\ldots(n-1)^{-1}}}\mathbf{2'^{-1}3^{-1}4^{-1}5^{-1}\ldots} \\
& \mathbf{\ldots(n-1)^{-1}n^{-1}(n-2)\ldots} \\
& \mathbf{\ldots 32'23\ldots(n-1)n}
\end{aligned}
$$

$$
\begin{aligned}
&\sim && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots \\
& && \dots(\mathbf{n-1})^{-1}\mathbf{2}'^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots(\mathbf{n-1})^{-1}\mathbf{n}^{-1}(\mathbf{n-2})\dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n} \\
& && \text{(by the case of } A_{n-1}=\{\alpha_2,\alpha_3,\dots,\alpha_n\}) \\
&\sim && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots \\
& && \dots(\mathbf{n-1})^{-1}\mathbf{2}'^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots \\
& && \dots(\mathbf{n-3})^{-1}\underbrace{(\mathbf{n-2})^{-1}(\mathbf{n-1})^{-1}(\mathbf{n-2})}\mathbf{n}^{-1}(\mathbf{n-3})\dots \\
& && \dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n} \\
&= && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots(\mathbf{n-1})^{-1}\mathbf{2}'^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots(\mathbf{n-3})^{-1} \\
& && \underbrace{(\mathbf{n-1})(\mathbf{n-2})^{-1}(\mathbf{n-1})^{-1}}\mathbf{n}^{-1}(\mathbf{n-3})\dots \\
& && \dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n} \\
&= && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots \\
& && \dots(\mathbf{n-1})^{-1}(\mathbf{n-1})\mathbf{2}'^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots \\
& && \dots(\mathbf{n-3})^{-1}(\mathbf{n-2})^{-1}(\mathbf{n-1})^{-1}\mathbf{n}^{-1}(\mathbf{n-3})\dots \\
& && \dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n} \\
&= && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots(\mathbf{n-2})\mathbf{2}'^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\mathbf{5}^{-1}\dots \\
& && \dots(\mathbf{n-3})^{-1}(\mathbf{n-2})^{-1}(\mathbf{n-1})^{-1}\mathbf{n}^{-1}(\mathbf{n-3})\dots \\
& && \dots\mathbf{32'23}\dots(\mathbf{n-1})\mathbf{n} \\
& && \vdots \\
&= && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{2}'^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots\mathbf{n}^{-1}\mathbf{2'23}\dots(\mathbf{n-1})\mathbf{n} \\
&= && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\underbrace{\mathbf{2}'^{-1}\mathbf{3}^{-1}\mathbf{2}'}\mathbf{4}^{-1}\dots\mathbf{n}^{-1}\mathbf{23}\dots(\mathbf{n-1})\mathbf{n} \\
&= && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{3}\mathbf{2}'^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots\mathbf{n}^{-1}\mathbf{23}\dots(\mathbf{n-1})\mathbf{n} \\
&\sim && \mathbf{2}^{a}\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots\mathbf{n}^{-1}\mathbf{23}\dots(\mathbf{n-1})\mathbf{n} \\
&= && \mathbf{2}^{a}\underbrace{\mathbf{2}^{-1}\mathbf{3}^{-1}\mathbf{2}}\mathbf{4}^{-1}\dots\mathbf{n}^{-1}\mathbf{3}\dots(\mathbf{n-1})\mathbf{n} \\
&= && \underbrace{\mathbf{2}^{a}\mathbf{3}\mathbf{2}^{-1}}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots\mathbf{n}^{-1}\mathbf{3}\dots(\mathbf{n-1})\mathbf{n} \\
&\sim && \mathbf{3}^{a}\mathbf{3}^{-1}\mathbf{4}^{-1}\dots\mathbf{n}^{-1}\mathbf{3}\dots(\mathbf{n-1})\mathbf{n} \quad \text{(by (4g))} \\
& && \vdots \\
&\sim && (\mathbf{n-1})^{a}(\mathbf{n-1})^{-1}\underbrace{\mathbf{n}^{-1}(\mathbf{n-1})\mathbf{n}} \quad \text{(by (4g))} \\
&= && (\mathbf{n-1})^{a}(\mathbf{n-1})^{-1}(\mathbf{n-1})\mathbf{n}(\mathbf{n-1})^{-1} \\
&= && (\mathbf{n-1})^{a}\mathbf{n}(\mathbf{n-1})^{-1} \\
&\sim && \mathbf{n}^{a}
\end{aligned}
$$

as desired.

If $i = n$, then

$$
\begin{aligned}
(\dagger) \quad = \quad & \mathbf{(n-1)}^a\mathbf{(n-1)n}^{-1}\mathbf{(n-2)}\ldots\mathbf{322'3}\ldots\mathbf{(n-1)nn(n-1)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n} \\
= \quad & \mathbf{(n-1)}^a\mathbf{(n-1)(n-2)}\ldots\mathbf{322'3}\ldots\underbrace{\mathbf{n}^{-1}\mathbf{(n-1)n}}\,\mathbf{n(n-1)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n} \\
= \quad & \mathbf{(n-1)}^a\mathbf{(n-1)(n-2)}\ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)n}\,\underbrace{\mathbf{(n-1)}^{-1}\mathbf{n(n-1)}}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n} \\
= \quad & \mathbf{(n-1)}^a\mathbf{(n-1)(n-2)}\ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-1)n}^{-1}\mathbf{(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n} \\
= \quad & \mathbf{(n-1)}^a\mathbf{(n-1)(n-2)}\ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{n}^{-1}\mathbf{(n-1)n} \\
= \quad & \mathbf{(n-1)}^a\mathbf{(n-1)(n-2)}\ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n(n-1)}^{-1} \\
\sim \quad & \mathbf{(n-2)}^a\mathbf{(n-2)(n-1)}^{-1}\mathbf{(n-3)}\ldots \\
& \ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n(n-1)}^{-1} \qquad \text{(by (4g))} \\
= \quad & \mathbf{(n-2)}^a\mathbf{(n-2)(n-3)}\ldots \\
& \ldots\mathbf{322'3}\ldots\mathbf{(n-1)}^{-1}\mathbf{(n-2)(n-1)nn(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n(n-1)}^{-1} \\
= \quad & \mathbf{(n-2)}^a\mathbf{(n-2)(n-3)}\ldots \\
& \ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)(n-2)}^{-1}\mathbf{nn(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n(n-1)}^{-1} \\
= \quad & \mathbf{(n-2)}^a\mathbf{(n-2)(n-3)}\ldots \\
& \ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-2)}^{-1}\mathbf{(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n(n-1)}^{-1} \\
= \quad & \mathbf{(n-2)}^a\mathbf{(n-2)(n-3)}\ldots \\
& \ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-1)(n-2)(n-1)}^{-1}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-1)n(n-1)}^{-1} \\
= \quad & \mathbf{(n-2)}^a\mathbf{(n-2)(n-3)}\ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-3)(n-1)}^{-1}\mathbf{(n-2)(n-1)n(n-1)}^{-1} \\
= \quad & \mathbf{(n-2)}^a\mathbf{(n-2)(n-3)}\ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-3)(n-2)(n-1)(n-2)}^{-1}\mathbf{n(n-1)}^{-1} \\
= \quad & \mathbf{(n-2)}^a\mathbf{(n-2)(n-3)}\ldots\mathbf{322'3}\ldots\mathbf{(n-2)(n-1)nn(n-1)(n-2)}\ldots \\
& \ldots\mathbf{32'23}\ldots\mathbf{(n-3)(n-2)(n-1)n(n-2)}^{-1}\mathbf{(n-1)}^{-1} \\
& \vdots
\end{aligned}
$$

$$\begin{aligned}
&\sim \mathbf{2}^a\mathbf{2}\mathbf{2}'\mathbf{3}\ldots(\mathbf{n}-\mathbf{1})\mathbf{n}\mathbf{n}(\mathbf{n}-\mathbf{1})\ldots\\
&\quad\ldots\mathbf{3}\mathbf{2}'\mathbf{2}\mathbf{3}\ldots(\mathbf{n}-\mathbf{1})\mathbf{n}\mathbf{2}^{-1}\mathbf{3}^{-1}\ldots(\mathbf{n}-\mathbf{1})^{-1} \qquad \text{(by (4g))}\\
&\sim \underbrace{\mathbf{2}^a\mathbf{2}\mathbf{3}\ldots(\mathbf{n}-\mathbf{1})\mathbf{n}\mathbf{n}(\mathbf{n}-\mathbf{1})\ldots\mathbf{3}\mathbf{2}}\,\mathbf{2}'\mathbf{3}\ldots(\mathbf{n}-\mathbf{1})\mathbf{n}\mathbf{2}^{-1}\mathbf{3}^{-1}\ldots(\mathbf{n}-\mathbf{1})^{-1}\\
&\sim \mathbf{2}^a\mathbf{2}'\mathbf{3}\ldots(\mathbf{n}-\mathbf{1})\mathbf{n}\mathbf{2}^{-1}\mathbf{3}^{-1}\ldots(\mathbf{n}-\mathbf{1})^{-1}\\
&\quad \text{(by the case of } A_{n-1}=\{\alpha_2,\alpha_3,\ldots,\alpha_n\})\\
&\sim \mathbf{2}^a\mathbf{3}\ldots(\mathbf{n}-\mathbf{1})\mathbf{n}\mathbf{2}^{-1}\mathbf{3}^{-1}\ldots(\mathbf{n}-\mathbf{1})^{-1}\\
&= \mathbf{2}^a\mathbf{3}\mathbf{2}^{-1}\ldots(\mathbf{n}-\mathbf{1})\mathbf{n}\mathbf{3}^{-1}\ldots(\mathbf{n}-\mathbf{1})^{-1}\\
&= \mathbf{3}^a\mathbf{4}\ldots(\mathbf{n}-\mathbf{1})\mathbf{n}\mathbf{3}^{-1}\ldots(\mathbf{n}-\mathbf{1})^{-1}\\
&\quad\vdots\\
&= (\mathbf{n}-\mathbf{1})^a\mathbf{n}(\mathbf{n}-\mathbf{1})^{-1}\\
&= \mathbf{n}^a
\end{aligned}$$

as desired. □

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