

ON THE STRUCTURE OF CALABI-YAU CATEGORIES  
WITH A CLUSTER TILTING SUBCATEGORY

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ABSTRACT. We prove that for  $d \geq 2$ , an algebraic  $d$ -Calabi-Yau triangulated category endowed with a  $d$ -cluster tilting subcategory is the stable category of a DG category which is perfectly  $(d+1)$ -Calabi-Yau and carries a non degenerate  $t$ -structure whose heart has enough projectives.

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## 1. INTRODUCTION

In this article, we propose a description of a class of Calabi-Yau categories using the formalism of DG-categories and the notion of ‘stabilization’, as used for the description of triangulated orbit categories in section 7 of [21]. For  $d \geq 2$ , let  $\mathcal{C}$  be an algebraic  $d$ -Calabi-Yau triangulated category endowed with a  $d$ -cluster tilting subcategory  $\mathcal{T}$ , *cf.* [23] [18] [19], see also [3] [13] [14]. Such categories occur for example,

- in the representation-theoretic approach to Fomin-Zelevinsky’s cluster algebras [12], *cf.* [6] [9] [15] and the references given there,
- in the study of Cohen-Macaulay modules over certain isolated singularities, *cf.* [17] [23] [16], and the study of non commutative crepant resolutions [36], *cf.* [17].

From  $\mathcal{C}$  and  $\mathcal{T}$  we construct an exact dg category  $\mathcal{B}$ , which is perfectly  $(d+1)$ -Calabi-Yau, and a non-degenerate aisle  $\mathcal{U}$ , *cf.* [25], in  $H^0(\mathcal{B})$  whose heart has enough projectives. We prove, in theorem 7.1, how to recover the category  $\mathcal{C}$  from  $\mathcal{B}$  and  $\mathcal{U}$  using a general procedure of stabilization defined in section 7. This extends previous results of [24] to a more general framework.

It follows from [30] that for  $d = 2$ , up to derived equivalence, the category  $\mathcal{B}$  only depends on  $\mathcal{C}$  (with its enhancement) and not on the choice of  $\mathcal{T}$ . In the appendix, we show how to naturally extend a  $t$ -structure, *cf.* [2], on the compact objects of a triangulated category to the whole category.

EXAMPLE Let  $k$  be a field,  $A$  a finite-dimensional hereditary  $k$ -algebra and  $\mathcal{C} = \mathcal{C}_A$  the cluster category of  $A$ , see [7] [8], i.e. the quotient of the bounded derived category of finitely generated modules over  $A$  by the functor  $F = \tau^{-1}[1]$ , where  $\tau$  denotes the AR-translation and  $[1]$  denotes the shift functor. Then  $\mathcal{B}$  is given by the dg algebra, see section 7 of [21],

$$B = A \oplus (DA)[-3]$$

and theorem 7.1 reduces to the equivalence

$$\mathcal{D}^b(\mathcal{B})/\text{per}(\mathcal{B}) \xrightarrow{\sim} \mathcal{C}_A$$

of section 7.1 of [21].

## 2. ACKNOWLEDGMENTS

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## 3. PRELIMINARIES

Let  $k$  be a field. Let  $\mathcal{E}$  be a  $k$ -linear Frobenius category with split idempotents. Suppose that its stable category  $\mathcal{C} = \underline{\mathcal{E}}$ , with suspension functor  $S$ , has finite-dimensional Hom-spaces and admits a Serre functor  $\Sigma$ , see [4]. Let  $d \geq 2$  be

an integer. We suppose that  $\mathcal{C}$  is Calabi-Yau of CY-dimension  $d$ , *i.e.* [27] there is an isomorphism of triangle functors

$$S^d \simeq \Sigma.$$

We fix such an isomorphism once and for all. See section 4 of [23] for several examples of the above situation.

For  $X, Y \in \mathcal{C}$  and  $n \in \mathbb{Z}$ , we put

$$\text{Ext}^n(X, Y) = \text{Hom}_{\mathcal{C}}(X, S^n Y).$$

We suppose that  $\mathcal{C}$  is endowed with a  $d$ -cluster tilting subcategory  $\mathcal{T} \subset \mathcal{C}$ , *i.e.*

- a)  $\mathcal{T}$  is a  $k$ -linear subcategory,
- b)  $\mathcal{T}$  is functorially finite in  $\mathcal{C}$ , *i.e.* the functors  $\text{Hom}_{\mathcal{C}}(?, X)|_{\mathcal{T}}$  and  $\text{Hom}_{\mathcal{C}}(X, ?)|_{\mathcal{T}}$  are finitely generated for all  $X \in \mathcal{C}$ ,
- c) we have  $\text{Ext}^i(T, T') = 0$  for all  $T, T' \in \mathcal{T}$  and all  $0 < i < d$  and
- d) if  $X \in \mathcal{C}$  satisfies  $\text{Ext}^i(T, X) = 0$  for all  $0 < i < d$  and all  $T \in \mathcal{T}$ , then  $X$  belongs to  $\mathcal{T}$ .

Let  $\mathcal{M} \subset \mathcal{E}$  be the preimage of  $\mathcal{T}$  under the projection functor. In particular,  $\mathcal{M}$  contains the subcategory  $\mathcal{P}$  of the projective-injective objects in  $\mathcal{M}$ . Note that  $\mathcal{T}$  equals the quotient  $\underline{\mathcal{M}}$  of  $\mathcal{M}$  by the ideal of morphisms factoring through a projective-injective.

We dispose of the following commutative square:

$$\begin{array}{ccc} \mathcal{M} & \hookrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{T} & \hookrightarrow & \underline{\mathcal{E}} = \mathcal{C}. \end{array}$$

We use the notations of [20]. In particular, for an additive category  $\mathcal{A}$ , we denote by  $\mathcal{C}(\mathcal{A})$  (resp.  $\mathcal{C}^-(\mathcal{A})$ ,  $\mathcal{C}^b(\mathcal{A})$ , ...) the category of unbounded (resp. right bounded, resp. bounded, ...) complexes over  $\mathcal{A}$  and by  $\mathcal{H}(\mathcal{A})$  (resp.  $\mathcal{H}^-(\mathcal{A})$ ,  $\mathcal{H}^b(\mathcal{A})$ , ...) its quotient modulo the ideal of nullhomotopic morphisms. By [26], *cf.* also [31], the projection functor  $\mathcal{E} \rightarrow \underline{\mathcal{E}}$  extends to a canonical triangle functor  $\mathcal{H}^b(\mathcal{E})/\mathcal{H}^b(\mathcal{P}) \rightarrow \underline{\mathcal{E}}$ . This induces a triangle functor  $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \rightarrow \underline{\mathcal{E}}$ . It is shown in [30] that this functor is a localization functor. Moreover, the projection functor  $\mathcal{H}^b(\mathcal{M}) \rightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$  induces an equivalence from the subcategory  $\mathcal{H}_{\mathcal{E}\text{-ac}}^b(\mathcal{M})$  of bounded  $\mathcal{E}$ -acyclic complexes with components in  $\mathcal{M}$  onto its kernel. Thus, we have a short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}_{\mathcal{E}\text{-ac}}^b(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{C} \longrightarrow 0.$$

Let  $\mathcal{B}$  be the dg (=differential graded) subcategory of the category  $\mathcal{C}^b(\mathcal{M})_{dg}$  of bounded complexes over  $\mathcal{M}$  whose objects are the  $\mathcal{E}$ -acyclic complexes. We denote by  $G : \mathcal{H}^-(\mathcal{M}) \rightarrow \mathcal{D}(\mathcal{B}^{op})^{op}$  the functor which takes a right bounded complex  $X$  over  $\mathcal{M}$  to the dg module

$$B \mapsto \text{Hom}_{\mathcal{M}}^{\bullet}(X, B),$$

where  $B$  is in  $\mathcal{B}$ .

*Remark 3.1.* By construction, the functor  $G$  restricted to  $\mathcal{H}_{\mathcal{E}\text{-}ac}^b(\mathcal{M})$  establishes an equivalence

$$G : \mathcal{H}_{\mathcal{E}\text{-}ac}^b(\mathcal{M}) \xrightarrow{\sim} \text{per}(\mathcal{B}^{op})^{op}.$$

Recall that if  $P$  is a right bounded complex of projectives and  $A$  is an acyclic complex, then each morphism from  $P$  to  $A$  is nullhomotopic. In particular, the complex  $\text{Hom}_{\mathcal{M}}^{\bullet}(P, A)$  is nullhomotopic for each  $P$  in  $\mathcal{H}^-(\mathcal{P})$ . Thus  $G$  takes  $\mathcal{H}^-(\mathcal{P})$  to zero, and induces a well defined functor (still denoted by  $G$ )

$$G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{D}(\mathcal{B}^{op})^{op}.$$

4. EMBEDDING

PROPOSITION 4.1. *The functor  $G$  is fully faithful.*

For the proof, we need a number of lemmas.

It is well-known that the category  $\mathcal{H}^-(\mathcal{E})$  admits a semiorthogonal decomposition, cf. [5], formed by  $\mathcal{H}^-(\mathcal{P})$  and its right orthogonal  $\mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{E})$ , the full subcategory of the right bounded  $\mathcal{E}$ -acyclic complexes. For  $X$  in  $\mathcal{H}^-(\mathcal{E})$ , we write

$$\mathbf{p}X \rightarrow X \rightarrow \mathbf{a}_p X \rightarrow \mathbf{S}pX$$

for the corresponding triangle, where  $\mathbf{p}X$  is in  $\mathcal{H}^-(\mathcal{P})$  and  $\mathbf{a}_p X$  is in  $\mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{E})$ . If  $X$  lies in  $\mathcal{H}^-(\mathcal{M})$ , then clearly  $\mathbf{a}_p X$  lies in  $\mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M})$  so that we have an induced semiorthogonal decomposition of  $\mathcal{H}^-(\mathcal{M})$ .

LEMMA 4.1. *The functor  $\Upsilon : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M})$  which takes  $X$  to  $\mathbf{a}_p X$  is fully faithful.*

*Proof.* By the semiorthogonal decomposition of  $\mathcal{H}^-(\mathcal{M})$ , the functor  $X \mapsto \mathbf{a}_p X$  induces a right adjoint of the localization functor

$$\mathcal{H}^-(\mathcal{M}) \longrightarrow \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P})$$

and an equivalence of the quotient category with the right orthogonal  $\mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M})$ .

$$\begin{array}{ccc}
 & \mathcal{H}^-(\mathcal{P}) & \\
 & \downarrow \uparrow & \\
 & \mathcal{H}^-(\mathcal{M}) & \longleftarrow \mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M}) = \mathcal{H}(\mathcal{P})^\perp \\
 & \downarrow \uparrow & \nearrow \sim \\
 \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \hookrightarrow & \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P}) & .
 \end{array}$$

Moreover, it is easy to see that the canonical functor

$$\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P})$$

is fully faithful so that we obtain a fully faithful functor

$$\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M})$$

taking  $X$  to  $\mathbf{a}_p X$ . ✓

*Remark 4.1.* Since the functor  $G$  is triangulated and takes  $\mathcal{H}^-(\mathcal{P})$  to zero, for  $X$  in  $\mathcal{H}^b(\mathcal{M})$ , the adjunction morphism  $X \rightarrow \mathbf{a}_p X$  yields an isomorphism

$$G(X) \xrightarrow{\sim} G(\mathbf{a}_p X) = G(\Upsilon X).$$

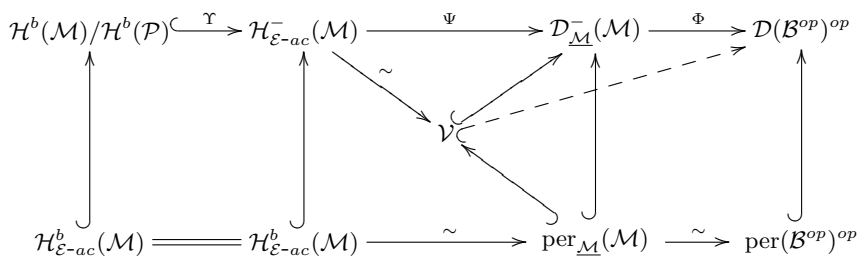
Let  $\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$  be the full subcategory of the derived category  $\mathcal{D}(\mathcal{M})$  formed by the right bounded complexes whose homology modules lie in the subcategory  $\text{Mod } \underline{\mathcal{M}}$  of  $\text{Mod } \mathcal{M}$ . The Yoneda functor  $\mathcal{M} \rightarrow \text{Mod } \mathcal{M}$ ,  $M \mapsto M^\wedge$ , induces a full embedding

$$\Psi : \mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M}) \hookrightarrow \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M}).$$

We write  $\mathcal{V}$  for its essential image. Under  $\Psi$ , the category  $\mathcal{H}_{\mathcal{E}\text{-}ac}^b(\mathcal{M})$  is identified with  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ . Let  $\Phi : \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M}) \rightarrow \mathcal{D}(\mathcal{B}^{op})^{op}$  be the functor which takes  $X$  to the dg module

$$B \mapsto \text{Hom}^\bullet(X_c, \Psi(B)),$$

where  $B$  is in  $\mathcal{H}_{\mathcal{E}\text{-}ac}^b(\mathcal{M})$  and  $X_c$  is a cofibrant replacement of  $X$  for the projective model structure on  $\mathcal{C}(\mathcal{M})$ . Since for each right bounded complex  $M$  with components in  $\mathcal{M}$ , the complex  $M^\wedge$  is cofibrant in  $\mathcal{C}(\mathcal{M})$ , it is clear that the functor  $G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \rightarrow \mathcal{D}(\mathcal{B}^{op})^{op}$  is isomorphic to the composition  $\Phi \circ \Psi \circ \Upsilon$ . We dispose of the following commutative diagram



LEMMA 4.2. *Let  $Y$  be an object of  $\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$ .*

- a)  $Y$  lies in  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$  iff  $\text{H}^p(Y)$  is a finitely presented  $\underline{\mathcal{M}}$ -module for all  $p \in \mathbb{Z}$  and vanishes for all but finitely many  $p$ .
- b)  $Y$  lies in  $\mathcal{V}$  iff  $\text{H}^p(Y)$  is a finitely presented  $\underline{\mathcal{M}}$ -module for all  $p \in \mathbb{Z}$  and vanishes for all  $p \gg 0$ .

*Proof.* a) Clearly the condition is necessary. For the converse, suppose first that  $Y$  is a finitely presented  $\underline{\mathcal{M}}$ -module. Then, as an  $\mathcal{M}$ -module,  $Y$  admits a resolution of length  $d + 1$  by finitely generated projective modules by theorem 5.4 b) of [23]. It follows that  $Y$  belongs to  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ . Since  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$  is triangulated, it also contains all shifts of finitely presented  $\underline{\mathcal{M}}$ -modules and all extensions of shifts. This proves the converse.

b) Clearly the condition is necessary. For the converse, we can suppose without loss of generality that  $Y^n = 0$ , for all  $n \geq 1$  and that  $Y^n$  belongs to  $\text{proj } \mathcal{M}$ , for  $n \leq 0$ . We now construct a sequence

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0$$

of complexes of finitely generated projective  $\mathcal{M}$ -modules such that  $P_n$  is quasi-isomorphic to  $\tau_{\geq -n} Y$  for each  $n$  and that, for each  $p \in \mathbb{Z}$ , the sequence of  $\mathcal{M}$ -modules  $P_n^p$  becomes stationary. By our assumptions, we have  $\tau_{\geq 0} Y \xrightarrow{\sim} H^0(Y)$ . Since  $H^0(Y)$  belongs to  $\text{mod } \underline{\mathcal{M}}$ , we know by theorem 5.4 c) of [23] that it belongs to  $\text{per}(\mathcal{M})$  as an  $\mathcal{M}$ -module. We define  $P_0$  to be a finite resolution of  $H^0(Y)$  by finitely generated  $\mathcal{M}$ -modules. For the induction step, consider the following truncation triangle associated with  $Y$

$$S^{i+1}H^{-i-1}(Y) \rightarrow \tau_{\geq -i-1} Y \rightarrow \tau_{\geq -i} Y \rightarrow S^{i+2}H^{-i-1}(Y),$$

for  $i \geq 0$ . By the induction hypothesis, we have constructed  $P_0, \dots, P_i$  and we dispose of a quasi-isomorphism  $P_i \xrightarrow{\sim} \tau_{\geq -i} Y$ . Let  $Q_{i+1}$  be a finite resolution of  $S^{i+2}H^{-i-1}(Y)$  by finitely presented projective  $\mathcal{M}$ -modules. We dispose of a morphism  $f_i : P_i \rightarrow Q_{i+1}$  and we define  $P_{i+1}$  as the cylinder of  $f_i$ . We define  $P$  as the limit of the  $P_i$  in the category of complexes. We remark that  $Y$  is quasi-isomorphic to  $P$  and that  $P$  belongs to  $\mathcal{V}$ . This proves the converse.  $\checkmark$

Let  $X$  be in  $\mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M})$ .

*Remark 4.2.* Lemma 4.2 shows that the natural  $t$ -structure of  $\mathcal{D}(\mathcal{M})$  restricts to a  $t$ -structure on  $\mathcal{V}$ . This allows us to express  $\Psi(X)$  as

$$\Psi(X) \xrightarrow{\sim} \text{holim}_i \tau_{\geq -i} \Psi(X),$$

where  $\tau_{\geq -i} \Psi(X)$  is in  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ .

LEMMA 4.3. *We dispose of the following isomorphism*

$$\Phi(\Psi(X)) = \Phi(\text{holim}_i \tau_{\geq -i} \Psi(X)) \xrightarrow{\sim} \text{holim}_i \Phi(\tau_{\geq -i} \Psi(X)).$$

*Proof.* It is enough to show that the canonical morphism induces a quasi-isomorphism when evaluated at any object  $B$  of  $\mathcal{B}$ . We have

$$\Phi(\text{holim}_i \tau_{\geq -i} \Psi(X))(B) = \text{Hom}^\bullet(\text{holim}_i \tau_{\geq -i} \Psi(X), B),$$

but since  $B$  is a bounded complex, for each  $n \in \mathbf{Z}$ , the sequence

$$i \mapsto \text{Hom}^n(\tau_{\geq -i} \Psi(X), B)$$

stabilizes as  $i$  goes to infinity. This implies that

$$\text{Hom}^\bullet(\text{holim}_i \tau_{\geq -i} \Psi(X), B) \xleftarrow{\sim} \text{holim}_i \Phi(\tau_{\geq -i} \Psi(X))(B).$$

$\checkmark$

LEMMA 4.4. *The functor  $\Phi$  restricted to the category  $\mathcal{V}$  is fully faithful.*

*Proof.* Let  $X, Y$  be in  $\mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M})$ . The following are canonically isomorphic :

$$(4.1) \quad \begin{aligned} & \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi\Psi X, \Phi\Psi Y) \\ & \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\Psi Y, \Phi\Psi X) \\ & \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\text{hocolim}_i \Phi\tau_{\geq -i}\Psi Y, \text{hocolim}_j \Phi\tau_{\geq -j}\Psi X) \end{aligned}$$

$$(4.2) \quad \begin{aligned} & \text{holim}_i \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\tau_{\geq -i}\Psi Y, \text{hocolim}_j \Phi\tau_{\geq -j}\Psi X) \\ & \text{holim}_i \text{hocolim}_j \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\tau_{\geq -i}\Psi Y, \Phi\tau_{\geq -j}\Psi X) \end{aligned}$$

$$(4.3) \quad \begin{aligned} & \text{holim}_i \text{hocolim}_j \text{Hom}_{\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(\tau_{\geq -j}\Psi X, \tau_{\geq -i}\Psi Y) \\ & \text{holim}_i \text{Hom}_{\mathcal{V}}(\text{holim}_j \tau_{\geq -j}\Psi X, \tau_{\geq -i}\Psi Y) \\ & \text{Hom}_{\mathcal{V}}(\Psi(X), \Psi(Y)). \end{aligned}$$

Here (4.1) is by the lemma 4.3 seen in  $\mathcal{D}(\mathcal{B}^{op})$ , (4.2) is by the fact that  $\Phi\tau_{\geq -i}\Psi Y$  is compact and (4.3) is by the fact that  $\tau_{\geq -i}\Psi Y$  is bounded. √

It is clear now that lemmas 4.1, 4.3 and 4.4 imply the proposition 4.1.

### 5. DETERMINATION OF THE IMAGE OF $G$

Let  $L_\rho : \mathcal{D}^-(\underline{\mathcal{M}}) \rightarrow \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$  be the restriction functor induced by the projection functor  $\mathcal{M} \rightarrow \underline{\mathcal{M}}$ .  $L_\rho$  admits a left adjoint  $L : \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M}) \rightarrow \mathcal{D}^-(\underline{\mathcal{M}})$  which takes  $Y$  to  $Y \otimes_{\mathcal{M}}^{\mathbb{L}} \underline{\mathcal{M}}$ . Let  $\mathcal{B}^-$  be the dg subcategory of  $\mathcal{C}^-(\text{Mod } \mathcal{M})_{dg}$  formed by the objects of  $\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$  that are in the essential image of the restriction of  $\Psi$  to  $\mathcal{H}_{\mathcal{E}\text{-}ac}^b(\mathcal{M})$ . Let  $\mathcal{B}'$  be the DG quotient, *cf.* [11], of  $\mathcal{B}^-$  by its quasi-isomorphisms. It is clear that the dg categories  $\mathcal{B}'$  and  $\mathcal{B}$  are quasi-equivalent, *cf.* [22], and that the natural dg functor  $\mathcal{M} \rightarrow \mathcal{C}^-(\text{Mod } \mathcal{M})_{dg}$  factors through  $\mathcal{B}^-$ . Let  $R' : \mathcal{D}(\mathcal{B}^{op})^{op} \rightarrow \mathcal{D}(\underline{\mathcal{M}}^{op})^{op}$  be the restriction functor induced by the dg functor  $\underline{\mathcal{M}} \rightarrow \mathcal{B}'$ . Let  $\Phi' : \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M}) \rightarrow \mathcal{D}(\mathcal{B}'^{op})^{op}$  be the functor which takes  $X$  to the dg module

$$B' \mapsto \text{Hom}^\bullet(X_c, B'),$$

where  $B'$  is in  $\mathcal{B}'$  and  $X_c$  is a cofibrant replacement of  $X$  for the projective model structure on  $\mathcal{C}(\text{Mod } \mathcal{M})$ . Finally let  $\Gamma : \mathcal{D}(\underline{\mathcal{M}}) \rightarrow \mathcal{D}(\underline{\mathcal{M}}^{op})^{op}$  be the functor that sends  $Y$  to

$$M \mapsto \text{Hom}^\bullet(Y_c, \underline{\mathcal{M}}(?, M)),$$

where  $Y_c$  is a cofibrant replacement of  $Y$  for the projective model structure on  $\mathcal{C}(\text{Mod } \underline{\mathcal{M}})$  and  $M$  is in  $\underline{\mathcal{M}}$ .

We dispose of the following diagram :

$$\begin{array}{ccccccc}
 & & & & & \mathcal{D}(\mathcal{B}^{op})^{op} & & \mathcal{B} \\
 & & & & & \uparrow \sim & & \downarrow \\
 \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) & \xrightarrow{\Upsilon} & \mathcal{H}_{\mathcal{E}\text{-}ac}^-(\mathcal{M}) & \xrightarrow{\Psi} & \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M}) & \xrightarrow{\Phi'} & \mathcal{D}(\mathcal{B}'^{op})^{op} & \mathcal{B}' \\
 & & & & \downarrow L & & \downarrow R' & \downarrow \\
 & & & & \mathcal{D}^-(\underline{\mathcal{M}}) & \xrightarrow{\Gamma} & \mathcal{D}(\underline{\mathcal{M}}^{op})^{op} & \underline{\mathcal{M}} \\
 & & & & & & & \uparrow
 \end{array}$$

LEMMA 5.1. *The following square*

$$\begin{array}{ccc}
 \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M}) & \xrightarrow{\Phi'} & \mathcal{D}(\mathcal{B}'^{op})^{op} & \mathcal{B}' \\
 \downarrow L & & \downarrow R' & \uparrow \\
 \mathcal{D}^-(\underline{\mathcal{M}}) & \xrightarrow{\Gamma} & \mathcal{D}(\underline{\mathcal{M}}^{op})^{op} & \underline{\mathcal{M}}
 \end{array}$$

is commutative.

*Proof.* By definition  $(R' \circ \Phi')(X)(M)$  equals  $\text{Hom}^\bullet(X_c, \underline{\mathcal{M}}(? , M))$ . Since  $\underline{\mathcal{M}}(? , M)$  identifies with  $L_\rho M^\wedge$  and by adjunction, we have

$$\text{Hom}^\bullet(X_c, \underline{\mathcal{M}}(? , M)) \xrightarrow{\sim} \text{Hom}^\bullet(X_c, L_\rho M^\wedge) \xrightarrow{\sim} \text{Hom}^\bullet((LX)_c, \underline{\mathcal{M}}(? , M)),$$

where the last member equals  $(\Gamma \circ L)(X)(M)$ . ✓

LEMMA 5.2. *The functor L reflects isomorphisms.*

*Proof.* Since  $L$  is a triangulated functor, it is enough to show that if  $L(Y) = 0$ , then  $Y = 0$ . Let  $Y$  be in  $\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$  such that  $L(Y) = 0$ . We can suppose, without loss of generality, that  $H^p(Y) = 0$  for all  $p > 0$ . Let us show that  $H^0(Y) = 0$ . Indeed, since  $H^0(Y)$  is an  $\underline{\mathcal{M}}$ -module, we have  $H^0(Y) \cong L^0 H^0(Y)$ , where  $L^0 : \text{Mod } \mathcal{M} \rightarrow \text{Mod } \underline{\mathcal{M}}$  is the left adjoint of the inclusion  $\text{Mod } \underline{\mathcal{M}} \rightarrow \text{Mod } \mathcal{M}$ . Since  $H^p(Y)$  vanishes in degrees  $p > 0$ , we have

$$L^0 H^0(Y) = H^0(LY).$$

By induction, one concludes that  $H^p(Y) = 0$  for all  $p \leq 0$ . ✓

PROPOSITION 5.1. *An object Y of  $\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$  lies in the essential image of the functor  $\Psi \circ \Upsilon : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \rightarrow \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$  iff  $\tau_{\geq -n} Y$  is in  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ , for all  $n \in \mathbb{Z}$  and  $L(Y)$  belongs to  $\text{per}(\underline{\mathcal{M}})$ .*

*Proof.* Let  $X$  be in  $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ . By lemma 4.2 a),  $\tau_{\geq -n} \Psi \Upsilon(X)$  is in  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ , for all  $n \in \mathbb{Z}$ . Since  $X$  is a bounded complex, there exists an  $s \ll 0$  such that for all  $m < s$  the  $m$ -components of  $\Upsilon(X)$  are in  $\mathcal{P}$ , which implies that  $L \Psi \Upsilon(X)$  belongs to  $\text{per}(\underline{\mathcal{M}})$ .

Conversely, suppose that  $Y$  is an object of  $\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$  which satisfies the conditions. By lemma 4.2,  $Y$  belongs to  $\mathcal{V}$ . Thus we have  $Y = \Psi(Y')$  for some  $Y'$



in  $\mathcal{H}_{\mathcal{E}\text{-ac}}^-(\mathcal{M})$ . We now consider  $Y'$  as an object of  $\mathcal{H}^-(\mathcal{M})$  and also write  $\Psi$  for the functor  $\mathcal{H}^-(\mathcal{M}) \rightarrow \mathcal{D}^-(\mathcal{M})$  induced by the Yoneda functor. We can express  $Y'$  as

$$Y' \xleftarrow{\sim} \operatorname{hocolim}_i \sigma_{\geq -i} Y',$$

where the  $\sigma_{\geq -i}$  are the naive truncations. By our assumptions on  $Y'$ ,  $\sigma_{\geq -i} Y'$  belongs to  $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ , for all  $i \in \mathbb{Z}$ . The functors  $\Psi$  and  $L$  clearly commute with the naive truncations  $\sigma_{\geq -i}$  and so we have

$$L(Y) = L(\Psi Y') \xleftarrow{\sim} \operatorname{hocolim}_i L(\sigma_{\geq -i} \Psi Y') \xrightarrow{\sim} \operatorname{hocolim}_i \sigma_{\geq -i} L(\Psi Y').$$

By our hypotheses,  $L(Y)$  belongs to  $\operatorname{per}(\underline{\mathcal{M}})$  and so there exists an  $m \gg 0$  such that

$$L(Y) = L(\Psi Y') \xleftarrow{\sim} \sigma_{\geq -m} L(\Psi Y') = L(\sigma_{\geq -m} \Psi Y').$$

By lemma 5.2, the inclusion

$$\Psi(\sigma_{\geq -m} Y') = \sigma_{\geq -m} \Psi Y' \longrightarrow \Psi(Y') = Y$$

is an isomorphism. But since  $\sigma_{\geq -m} Y'$  belongs to  $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ ,  $Y$  identifies with  $\Psi(\sigma_{\geq -m} Y')$ . √

*Remark 5.1.* It is clear that if  $X$  belongs to  $\operatorname{per}(\underline{\mathcal{M}})$ , then  $\Gamma(X)$  belongs to  $\operatorname{per}(\underline{\mathcal{M}}^{op})^{op}$ . We also have the following partial converse.

**LEMMA 5.3.** *Let  $X$  be in  $\mathcal{D}_{\operatorname{mod} \underline{\mathcal{M}}}^-(\underline{\mathcal{M}})$  such that  $\Gamma(X)$  belongs to  $\operatorname{per}(\underline{\mathcal{M}}^{op})^{op}$ . Then  $X$  is in  $\operatorname{per}(\underline{\mathcal{M}})$ .*

*Proof.* By lemma 4.2 b) we can suppose, without loss of generality, that  $X$  is a right bounded complex with finitely generated projective components. Applying  $\Gamma$ , we get a perfect complex  $\Gamma(X)$ . In particular  $\Gamma(X)$  is homotopic to zero in high degrees. But since  $\Gamma$  is an equivalence

$$\operatorname{proj} \underline{\mathcal{M}} \xrightarrow{\sim} (\operatorname{proj} \underline{\mathcal{M}}^{op})^{op},$$

it follows that  $X$  is already homotopic to zero in high degrees. √

*Remark 5.2.* The natural right aisle on  $\mathcal{D}(\mathcal{M})$  is the full subcategory of the objects  $X$  such that  $H^n(X) = 0$  for all  $n < 0$ . The associated truncation functor  $\tau_{\geq 0}$  takes  $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$  to itself. Therefore, the natural right aisle on  $\mathcal{D}(\mathcal{M})$  restricts to a natural right aisle  $\mathcal{U}^{op}$  on  $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ .

**DEFINITION 5.1.** *Let  $\mathcal{U}$  be the natural left aisle in  $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$  associated with  $\mathcal{U}^{op}$ .*

**LEMMA 5.4.** *The natural left aisle  $\mathcal{U}$  on  $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op} \xrightarrow{\sim} \operatorname{per}(\mathcal{B}^{op})$  satisfies the conditions of proposition A.1 b).*

*Proof.* Clearly the natural left aisle  $\mathcal{U}$  in  $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$  is non-degenerate. We need to show that for each  $C \in \operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$ , there is an integer  $N$  such that  $\operatorname{Hom}(C, S^N U) = 0$  for each  $U \in \mathcal{U}$ . We dispose of the following isomorphism

$$\operatorname{Hom}_{\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}}(C, S^N U) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(S^{-N} \mathcal{U}^{op}, C),$$

where  $\mathcal{U}^{op}$  denotes the natural right aisle on  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ . Since by theorem 5.4 c) of [23] an  $\underline{\mathcal{M}}$ -module admits a projective resolution of length  $d + 1$  as an  $\mathcal{M}$ -module and  $C$  is a bounded complex, we conclude that for  $N \gg 0$

$$\text{Hom}_{\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(S^{-N}\mathcal{U}^{op}, C) = 0.$$

This proves the lemma. ✓

We denote by  $\tau_{\leq n}$  and  $\tau_{\geq n}$ ,  $n \in \mathbb{Z}$ , the associated truncation functors on  $\mathcal{D}(\mathcal{B}^{op})^{op}$ .

LEMMA 5.5. *The functor  $\Phi : \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M}) \rightarrow \mathcal{D}(\mathcal{B}^{op})^{op}$  restricted to the category  $\mathcal{V}$  is exact with respect to the given  $t$ -structures.*

*Proof.* We first prove that  $\Phi(\mathcal{V}_{\leq 0}) \subset \mathcal{D}(\mathcal{B}^{op})_{\leq 0}^{op}$ . Let  $X$  be in  $\mathcal{V}_{\leq 0}$ . We need to show that  $\Phi(X)$  belongs to  $\mathcal{D}(\mathcal{B}^{op})_{\leq 0}^{op}$ . The following have the same classes of objects :

$$(5.1) \quad \begin{aligned} & \mathcal{D}(\mathcal{B}^{op})_{\leq 0}^{op} \\ & \mathcal{D}(\mathcal{B}^{op})_{> 0} \\ & (\text{per}(\mathcal{B}^{op})_{\leq 0})^\perp \end{aligned}$$

$$(5.2) \quad \perp(\text{per}(\mathcal{B}^{op})^{op})_{> 0},$$

where in (5.1) we consider the right orthogonal in  $\mathcal{D}(\mathcal{B}^{op})$  and in (5.2) we consider the left orthogonal in  $\mathcal{D}(\mathcal{B}^{op})^{op}$ . These isomorphisms show us that  $\Phi(X)$  belongs to  $\mathcal{D}(\mathcal{B}^{op})_{\leq 0}^{op}$  iff

$$\text{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X), \Phi(P)) = 0,$$

for all  $P \in \text{per}_{\underline{\mathcal{M}}}(\mathcal{M})_{> 0}$ . Now, by lemma 4.4 the functor  $\Phi$  is fully faithful and so

$$\text{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X), \Phi(P)) \xrightarrow{\sim} \text{Hom}_{\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(X, P).$$

Since  $X$  belongs to  $\mathcal{V}_{\leq 0}$  and  $P$  belongs to  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})_{> 0}$ , we conclude that

$$\text{Hom}_{\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(X, P) = 0,$$

which implies that  $\Phi(X) \in \mathcal{D}(\mathcal{B}^{op})_{\leq 0}^{op}$ . Let us now consider  $X$  in  $\mathcal{V}$ . We dispose of the truncation triangle

$$\tau_{\leq 0}X \rightarrow X \rightarrow \tau_{> 0}X \rightarrow S\tau_{\leq 0}X.$$

The functor  $\Phi$  is triangulated and so we dispose of the triangle

$$\Phi\tau_{\leq 0}X \rightarrow X \rightarrow \Phi\tau_{> 0}X \rightarrow S\Phi\tau_{\leq 0}X,$$

where  $\Phi\tau_{\leq 0}X$  belongs to  $\mathcal{D}(\mathcal{B}^{op})_{\leq 0}^{op}$ . Since  $\Phi$  induces an equivalence between  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$  and  $\text{per}(\mathcal{B}^{op})^{op}$  and  $\text{Hom}(P, \tau_{> 0}X) = 0$ , for all  $P$  in  $\mathcal{V}_{\leq 0}$ , we conclude that  $\Phi\tau_{> 0}X$  belongs to  $\mathcal{D}(\mathcal{B}^{op})_{> 0}^{op}$ . This implies the lemma. ✓

DEFINITION 5.2. *Let  $\mathcal{D}(\mathcal{B}^{op})_f^{op}$  denote the full triangulated subcategory of  $\mathcal{D}(\mathcal{B}^{op})^{op}$  formed by the objects  $Y$  such that  $\tau_{\geq -n}Y$  is in  $\text{per}(\mathcal{B}^{op})^{op}$ , for all  $n \in \mathbb{Z}$ , and  $R(Y)$  belongs to  $\text{per}(\underline{\mathcal{M}}^{op})^{op}$ .*

PROPOSITION 5.2. *An object  $Y$  of  $\mathcal{D}(\mathcal{B}^{op})^{op}$  lies in the essential image of the functor  $G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \rightarrow \mathcal{D}(\mathcal{B}^{op})^{op}$  iff it belongs to  $\mathcal{D}(\mathcal{B}^{op})_f^{op}$ .*

*Proof.* Let  $X$  be in  $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ . It is clear that the  $\tau_{\geq -n}G(X)$  are in  $\text{per}(\mathcal{B}^{op})^{op}$  for all  $n \in \mathbb{Z}$ . By proposition 5.1 we know that  $L\Psi\Upsilon(X)$  belongs to  $\text{per}(\underline{\mathcal{M}})$ . By lemma 5.1 and remark 5.1 we conclude that  $RG(X)$  belongs to  $\text{per}(\underline{\mathcal{M}}^{op})^{op}$ . Let now  $Y$  be in  $\mathcal{D}(\mathcal{B}^{op})_f^{op}$ . We can express it, by the dual of lemma A.2 as the homotopy limit of the following diagram

$$\cdots \rightarrow \tau_{\geq -n-1}Y \rightarrow \tau_{\geq -n}Y \rightarrow \tau_{\geq -n+1}Y \rightarrow \cdots,$$

where  $\tau_{\geq -n}Y$  belongs to  $\text{per}(\mathcal{B}^{op})^{op}$ , for all  $n \in \mathbb{Z}$ . But since  $\Phi$  induces an equivalence between  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$  and  $\text{per}(\mathcal{B}^{op})^{op}$ , this last diagram corresponds to a diagram

$$\cdots \rightarrow M_{-n-1} \rightarrow M_{-n} \rightarrow M_{-n+1} \rightarrow \cdots$$

in  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ . Let  $p \in \mathbb{Z}$ . The relations among the truncation functors imply that the image of the above diagram under each homology functor  $H^p$ ,  $p \in \mathbb{Z}$ , is stationary as  $n$  goes to  $+\infty$ . This implies that

$$H^p \text{holim}_n M_{-n} \xrightarrow{\sim} \lim_n H^p M_{-n} \cong H^p M_j,$$

for all  $j < p$ . We dispose of the following commutative diagram

$$\begin{array}{ccc} \text{holim}_n M_{-n} & \longrightarrow & \text{holim}_n \tau_{\geq -i} M_{-n} \cong M_{-i} \\ \downarrow & \nearrow \sim & \\ \tau_{\geq -i} \text{holim}_n M_{-n} & & \end{array}$$

which implies that

$$\tau_{\geq -i} \text{holim}_n M_{-n} \xrightarrow{\sim} M_{-i},$$

for all  $i \in \mathbb{Z}$ . Since  $\text{holim}_n M_{-n}$  belongs to  $\mathcal{V}$ , lemma 4.3 allows us to conclude that  $\Phi(\text{holim}_n M_{-n}) \cong Y$ . We now show that  $\text{holim}_n M_{-n}$  satisfies the conditions of proposition 5.1. We know that  $\tau_{\geq -i} \text{holim}_n M_{-n}$  belongs to  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ , for all  $i \in \mathbb{Z}$ . By lemma 5.1  $(\Gamma \circ L)(\text{holim}_n M_{-n})$  identifies with  $R(Y)$ , which is in  $\text{per}(\underline{\mathcal{M}}^{op})^{op}$ . Since  $\text{holim}_n M_{-n}$  belongs to  $\mathcal{V}$ , its homologies lie in  $\text{mod } \underline{\mathcal{M}}$  and so we are in the conditions of lemma 5.1, which implies that  $L(\text{holim}_n M_{-n})$  belongs to  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ . This finishes the proof. √

### 6. ALTERNATIVE DESCRIPTION

In this section, we present another characterization of the image of  $G$ , which was identified as  $\mathcal{D}(\mathcal{B}^{op})_f^{op}$  in proposition 5.2. Let  $M$  denote an object of  $\mathcal{M}$  and also the naturally associated complex in  $\mathcal{H}^b(\mathcal{M})$ . Since the category  $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$  is generated by the objects  $M \in \mathcal{M}$  and the functor  $G$  is fully faithful, we remark that  $\mathcal{D}(\mathcal{B}^{op})_f^{op}$  equals the triangulated subcategory of  $\mathcal{D}(\mathcal{B}^{op})^{op}$  generated by the objects  $G(M)$ ,  $M \in \mathcal{M}$ . The rest of this section is concerned with the

problem of characterizing the objects  $G(M)$ ,  $M \in \mathcal{M}$ . We denote by  $P_M$  the projective  $\underline{\mathcal{M}}$ -module  $\underline{\mathcal{M}}(?, M)$  associated with  $M \in \mathcal{M}$  and by  $X_M$  the image of  $M$  under  $\Psi \circ \Upsilon$ .

LEMMA 6.1. *We dispose of the following isomorphism*

$$\mathrm{Hom}_{\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})}(X_M, Y) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{mod}\ \underline{\mathcal{M}}}(P_M, \mathrm{H}^0(Y)),$$

for all  $Y \in \mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})$ .

*Proof.* Clearly  $X_M$  belongs to  $\mathcal{D}_{\underline{\mathcal{M}}}(\mathcal{M})_{\leq 0}$  and is of the form

$$\dots \rightarrow P_n^\wedge \rightarrow \dots \rightarrow P_1^\wedge \rightarrow P_0^\wedge \rightarrow M^\wedge \rightarrow 0,$$

where  $P_n \in \mathcal{P}$ ,  $n \geq 0$ . Now Yoneda's lemma and the fact that  $\mathrm{H}^m(Y)(P_n) = 0$ , for all  $m \in \mathbb{Z}$ ,  $n \geq 0$ , imply the lemma. √

*Remark 6.1.* Since the functor  $\Phi$  restricted to  $\mathcal{V}$  is fully faithful and exact, we have

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(G(M), \Phi(Y)) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})^{op}}(\Phi(P_M), \mathrm{H}^0(\Phi(Y))),$$

for all  $Y \in \mathcal{V}$ .

We now characterize the objects  $G(M) = \Phi(X_M)$ ,  $M \in \mathcal{M}$ , in the triangulated category  $\mathcal{D}(\mathcal{B}^{op})$ . More precisely, we give a description of the functor

$$R_M := \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(?, \Phi(X_M)) : \mathcal{D}(\mathcal{B}^{op})^{op} \rightarrow \mathrm{Mod}\ k$$

using an idea of M. Van den Bergh, cf. lemma 2.13 of [10]. Consider the following functor

$$F_M := \mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\mathrm{H}^0(?), \Phi(P_M)) : \mathrm{per}(\mathcal{B}^{op})^{op} \rightarrow \mathrm{mod}\ k.$$

*Remark 6.2.* Remark 6.1 shows that the functor  $R_M$  when restricted to  $\mathrm{per}(\mathcal{B}^{op})$  coincides with  $F_M$ .

Let  $DF_M$  be the composition of  $F_M$  with the duality functor  $D = \mathrm{Hom}(?, k)$ . Note that  $DF_M$  is homological.

LEMMA 6.2. *We dispose of the following isomorphism of functors on  $\mathrm{per}(\mathcal{B}^{op})$*

$$DF_M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(X_M), ?[d + 1]).$$

*Proof.* The following functors are canonically isomorphic to  $DF\Phi$  :

$$(6.1) \quad D\mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\mathrm{H}^0\Phi(?), \Phi(P_M))$$

$$(6.2) \quad D\mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\Phi\mathrm{H}^0(?), \Phi(P_M))$$

$$(6.3) \quad D\mathrm{Hom}_{\mathrm{per}\ \underline{\mathcal{M}}(\mathcal{M})}(P_M, \mathrm{H}^0(?))$$

$$(6.4) \quad D\mathrm{Hom}_{\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})}(X_M, ?)$$

$$(6.5) \quad \mathrm{Hom}_{\mathcal{D}_{\underline{\mathcal{M}}}^-(\mathcal{M})}(?[-d - 1], X_M)$$

$$(6.6) \quad \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(?)[-d - 1], \Phi(X_M))$$

$$(6.6) \quad \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X_M), \Phi(?)[d + 1])$$

Step (6.1) follows from the fact that  $\Phi$  is exact. Step (6.2) follows from the fact that  $\Phi$  is fully faithful and we are considering the opposite category. Step (6.3) is a consequence of lemma 6.1. Step (6.4) follows from the  $(d + 1)$ -Calabi-Yau property and remark 4.2. Step (6.5) is a consequence of  $\Phi$  being fully faithful and step (6.6) is a consequence of working in the opposite category. Since the functor  $\Phi^{op}$  establish an equivalence between  $\text{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$  and  $\text{per}(\mathcal{B}^{op})$  the lemma is proven.  $\checkmark$

Now, since the category  $\text{Mod } k$  is cocomplete, we can consider the left Kan extension, cf. [28],  $E_M$  of  $DF_M$  along the inclusion  $\text{per}(\mathcal{B}^{op}) \hookrightarrow \mathcal{D}(\mathcal{B}^{op})$ . We dispose of the following commutative square :

$$\begin{array}{ccc} \text{per}(\mathcal{B}^{op}) & \xrightarrow{DF_M} & \text{mod } k \\ \downarrow & & \downarrow \\ \mathcal{D}(\mathcal{B}^{op}) & \xrightarrow{E_M} & \text{Mod } k. \end{array}$$

For each  $X$  of  $\mathcal{D}(\mathcal{B}^{op})$ , the comma-category of morphisms  $P \rightarrow X$  from a perfect object  $P$  to  $X$  is filtered. Therefore, the functor  $E_M$  is homological. Moreover, it preserves coproducts and so  $DE_M$  is cohomological and transforms coproducts into products. Since  $\mathcal{D}(\mathcal{B}^{op})$  is a compactly generated triangulated category, the Brown representability theorem, cf. [29], implies that there is a  $Z_M \in \mathcal{D}(\mathcal{B}^{op})$  such that

$$DE_M \xrightarrow{\sim} \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\?, Z_M).$$

*Remark 6.3.* Since the duality functor  $D$  establishes an anti-equivalence in  $\text{mod } k$ , the functor  $DE_M$  restricted to  $\text{per}(\mathcal{B}^{op})$  is isomorphic to  $F_M$ .

**THEOREM 6.1.** *We dispose of an isomorphism*

$$G(M) \xrightarrow{\sim} Z_M.$$

*Proof.* We now construct a morphism of functors from  $R_M$  to  $DE_M$ . Since  $R_M$  is representable, by Yoneda's lemma it is enough to construct an element in  $DE_M(\Phi(X_M))$ . Let  $\mathcal{C}$  be the category  $\text{per}(\mathcal{B}^{op}) \downarrow \Phi(X_M)$ , whose objects are the morphisms  $Y' \rightarrow \Phi(X_M)$  and let  $\mathcal{C}'$  be the category  $X_M \downarrow \text{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ , whose objects are the morphisms  $X_M \rightarrow X'$ . The following are canonically isomorphic :

$$(6.7) \quad DE_M(\Phi(X_M)) \cong D \text{colim}_{\mathcal{C}} \text{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(X_M), Y'[d + 1])$$

$$(6.8) \quad D \text{colim}_{\mathcal{C}'} \text{Hom}_{\mathcal{D}_{\underline{\mathcal{M}}}(\mathcal{M})}(X'[-d - 1], X_M)$$

$$(6.9) \quad D \text{colim}_i \text{Hom}_{\mathcal{D}_{\underline{\mathcal{M}}}(\mathcal{M})}((\tau_{\geq -i} X_M)[-d - 1], X_M)$$

$$(6.10) \quad \lim_i \text{DHom}_{\mathcal{D}_{\underline{\mathcal{M}}}(\mathcal{M})}((\tau_{\geq -i} X_M)[-d - 1], X_M) \cong \lim_i \text{Hom}_{\mathcal{D}_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, \tau_{\geq -i} X_M)$$

Step (6.7) is a consequence of the definition of the left Kan extension and lemma 6.2. Step (6.8) is obtained by considering the opposite category. Step (6.9) follows from the fact that the system  $(\tau_{\geq -i} X_M)_{i \in \mathbb{Z}}$  forms a cofinal system for the index system of the colimit. Step (6.10) follows from the  $(d+1)$ -Calabi-Yau property. Now, the image of the identity by the canonical morphism

$$\mathrm{Hom}_{\mathcal{D}_{\mathcal{M}}^-(\mathcal{M})}(X_M, X_M) \longrightarrow \lim_i \mathrm{Hom}_{\mathcal{D}_{\mathcal{M}}^-(\mathcal{M})}(X_M, \tau_{\geq -i} X_M),$$

gives us an element of  $(DE_M)(\Phi(X_M))$  and so a morphism of functors from  $R_M$  to  $DE_M$ . We remark that this morphism is an isomorphism when evaluated at the objects of  $\mathrm{per}(\mathcal{B}^{op})$ . Since both functors  $R_M$  and  $DE_M$  are cohomological, transform coproducts into products and  $\mathcal{D}(\mathcal{B}^{op})$  is compactly generated, we conclude that we dispose of an isomorphism

$$G(M) \xrightarrow{\sim} Z_M.$$

✓

## 7. THE MAIN THEOREM

Consider the following commutative square as in section 3:

$$\begin{array}{ccc} \mathcal{M} & \hookrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{T} & \hookrightarrow & \underline{\mathcal{E}} = \mathcal{C}. \end{array}$$

In the previous sections we have constructed, from the above data, a dg category  $\mathcal{B}$  and a left aisle  $\mathcal{U} \subset \mathrm{H}^0(\mathcal{B})$ , see [25], satisfying the following conditions :

- $\mathcal{B}$  is an exact dg category over  $k$  such that  $\mathrm{H}^0(\mathcal{B})$  has finite-dimensional Hom-spaces and is Calabi-Yau of CY-dimension  $d+1$ ,
- $\mathcal{U} \subset \mathrm{H}^0(\mathcal{B})$  is a non-degenerate left aisle such that :
  - for all  $B \in \mathcal{B}$ , there is an integer  $N$  such that  $\mathrm{Hom}_{\mathrm{H}^0(\mathcal{B})}(B, S^N U) = 0$  for each  $U \in \mathcal{U}$ ,
  - the heart  $\mathcal{H}$  of the  $t$ -structure on  $\mathrm{H}^0(\mathcal{B})$  associated with  $\mathcal{U}$  has enough projectives.

Let now  $\mathcal{A}$  be a dg category and  $\mathcal{W} \subset \mathrm{H}^0(\mathcal{A})$  a left aisle satisfying the above conditions. We can consider the following general construction : Let  $\mathcal{Q}$  denote the category of projectives of the heart  $\mathcal{H}$  of the  $t$ -structure on  $\mathrm{H}^0(\mathcal{A})$  associated with  $\mathcal{W}$ . We claim that the following inclusion

$$\mathcal{Q} \hookrightarrow \mathcal{H} \hookrightarrow \mathrm{H}^0(\mathcal{A}),$$

lifts to a morphism  $\mathcal{Q} \xrightarrow{j} \mathcal{A}$  in the homotopy category of small dg categories, cf. [20] [32] [33] [34] [35]. Indeed, recall the following argument from section 7 of [22]: Let  $\tilde{\mathcal{Q}}$  be the full dg subcategory of  $\mathcal{A}$  whose objects are the same as those of  $\mathcal{Q}$ . Let  $\tau_{\leq 0} \tilde{\mathcal{Q}}$  denote the dg category obtained from  $\tilde{\mathcal{Q}}$  by applying

the truncation functor  $\tau_{\leq 0}$  of complexes to each Hom-space. We dispose of the following diagram in the category of small dg categories

$$\begin{array}{ccc}
 \tilde{\mathcal{Q}} & \hookrightarrow & \mathcal{A} \\
 \uparrow & & \\
 \tau_{\leq 0} \tilde{\mathcal{Q}} & & \\
 \downarrow & & \\
 \mathcal{Q} \simeq \mathbf{H}^0(\tilde{\mathcal{Q}}) & & .
 \end{array}$$

Let  $X, Y$  be objects of  $\mathcal{Q}$ . Since  $X$  and  $Y$  belong to the heart of a  $t$ -structure in  $\mathbf{H}^0(\mathcal{A})$ , we have

$$\mathrm{Hom}_{\mathbf{H}^0(\mathcal{A})}(X, Y[-n]) = 0,$$

for  $n \geq 1$ . The dg category  $\mathcal{A}$  is exact, which implies that

$$\mathbf{H}^{-n} \mathrm{Hom}_{\tilde{\mathcal{Q}}}^{\bullet}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{H}^0(\mathcal{A})}(X, Y[-n]) = 0,$$

for  $n \geq 1$ . This shows that the dg functor  $\tau_{\leq 0} \tilde{\mathcal{Q}} \rightarrow \mathbf{H}^0(\tilde{\mathcal{Q}})$  is a quasi-equivalence and so we dispose of a morphism  $\mathcal{Q} \xrightarrow{j} \mathcal{A}$  in the homotopy category of small dg categories. We dispose of a triangle functor  $j^* : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{Q})$  given by restriction. By proposition A.1, the left aisle  $\mathcal{W} \subset \mathbf{H}^0(\mathcal{A})$  admits a smallest extension to a left aisle  $\mathcal{D}(\mathcal{A}^{op})_{\leq 0}^{op}$  on  $\mathcal{D}(\mathcal{A}^{op})^{op}$ . Let  $\mathcal{D}(\mathcal{A}^{op})_f^{op}$  denote the full triangulated subcategory of  $\mathcal{D}(\mathcal{A}^{op})^{op}$  formed by the objects  $Y$  such that  $\tau_{\geq -n} Y$  is in  $\mathrm{per}(\mathcal{A}^{op})^{op}$ , for all  $n \in \mathbb{Z}$ , and  $j^*(Y)$  belongs to  $\mathrm{per}(\mathcal{Q}^{op})^{op}$ .

DEFINITION 7.1. *The stable category of  $\mathcal{A}$  with respect to  $\mathcal{W}$  is the triangle quotient*

$$\mathrm{stab}(\mathcal{A}, \mathcal{W}) = \mathcal{D}(\mathcal{A}^{op})_f^{op} / \mathrm{per}(\mathcal{A}^{op})^{op}.$$

We are now able to formulate the main theorem. Let  $\mathcal{B}$  be the dg category and  $\mathcal{U} \subset \mathbf{H}^0(\mathcal{B})$  the left aisle constructed in sections 1 to 5.

THEOREM 7.1. *The functor  $G$  induces an equivalence of categories*

$$\tilde{G} : \mathcal{C} \xrightarrow{\sim} \mathrm{stab}(\mathcal{B}, \mathcal{U}).$$

*Proof.* We dispose of the following commutative diagram :

$$\begin{array}{ccc}
 \mathcal{C} & \overset{\tilde{G}}{\dashrightarrow} & \mathrm{stab}(\mathcal{B}, \mathcal{U}) \\
 \uparrow & & \uparrow \\
 \mathcal{H}^b(\mathcal{M}) / \mathcal{H}^b(\mathcal{P}) & \xrightarrow[\sim]{G} & \mathcal{D}(\mathcal{B}^{op})_f^{op} \\
 \uparrow & & \uparrow \\
 \mathcal{H}_{\mathcal{E}^{-ac}}^b(\mathcal{M}) & \xrightarrow{\sim} & \mathrm{per}(\mathcal{B}^{op})^{op}.
 \end{array}$$

The functor  $G$  is an equivalence since it is fully faithful by proposition 4.1 and essentially surjective by proposition 5.2. Since we dispose of an equivalence  $\mathcal{H}_{\mathcal{E}\text{-ac}}^b(\mathcal{M}) \xrightarrow{\sim} \text{per}(\mathcal{B}^{op})^{op}$  by construction of  $\mathcal{B}$  and the columns of the above diagram are short exact sequences of triangulated categories, the theorem is proved.  $\checkmark$

APPENDIX A. EXTENSION OF  $t$ -STRUCTURES

Let  $\mathcal{T}$  be a compactly generated triangulated category with suspension functor  $S$ . We denote by  $\mathcal{T}_c$  the full triangulated sub-category of  $\mathcal{T}$  formed by the compact objects, see [29]. We use the terminology of [25]. Let  $\mathcal{U} \subseteq \mathcal{T}_c$  be a left aisle on  $\mathcal{T}_c$ , i.e. a full additive subcategory  $\mathcal{U}$  of  $\mathcal{T}_c$  which satisfies:

- a)  $S\mathcal{U} \subset \mathcal{U}$ ,
- b)  $\mathcal{U}$  is stable under extensions, i.e. for each triangle

$$X \rightarrow Y \rightarrow Z \rightarrow SX$$

of  $\mathcal{T}_c$ , we have  $Y \in \mathcal{U}$  whenever  $X, Z \in \mathcal{U}$  and

- c) the inclusion functor  $\mathcal{U} \hookrightarrow \mathcal{T}_c$  admits a right adjoint.

As shown in [25], the concept of aisle is equivalent to that of  $t$ -structure.

- PROPOSITION A.1.      a) *The left aisle  $\mathcal{U}$  admits a smallest extension to a left aisle  $\mathcal{T}_{\leq 0}$  on  $\mathcal{T}$ .*  
 b) *If  $\mathcal{U} \subseteq \mathcal{T}_c$  is non-degenerate (i.e.,  $f : X \rightarrow Y$  is invertible iff  $\text{HP}(f)$  is invertible for all  $p \in \mathbb{Z}$ ) and for each  $X \in \mathcal{T}_c$ , there is an integer  $N$  such that  $\text{Hom}(X, S^N U) = 0$  for each  $U \in \mathcal{U}$ , then  $\mathcal{T}_{\leq 0}$  is also non-degenerate.*

*Proof.* a) Let  $\mathcal{T}_{\leq 0}$  be the smallest full subcategory of  $\mathcal{T}$  that contains  $\mathcal{U}$  and is stable under infinite sums and extensions. It is clear that  $\mathcal{T}_{\leq 0}$  is stable by  $S$  since  $\mathcal{U}$  is. We need to show that the inclusion functor  $\mathcal{T}_{\leq 0} \hookrightarrow \mathcal{T}$  admits a right adjoint. For completeness, we include the following proof, which is a variant of the ‘small object argument’, cf. also [1]. We dispose of the following recursive procedure. Let  $X = X_0$  be an object in  $\mathcal{T}$ . For the initial step consider all morphisms from any object  $P$  in  $\mathcal{U}$  to  $X_0$ . This forms a set  $I_0$  since  $\mathcal{T}$  is compactly generated and so we dispose of the following triangle

$$\coprod_{f \in I_0} P \longrightarrow X_0 \longrightarrow X_1 \rightsquigarrow \coprod_{f \in I_0} P.$$

For the induction step consider the above construction with  $X_n, n \geq 1$ , in the place of  $X_{n-1}$  and  $I_n$  in the place of  $I_{n-1}$ . We dispose of the following diagram

$$\begin{array}{ccccccccccc} X = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \cdots & \longrightarrow & X' \\ \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & & & & \\ \coprod_{f \in I_0} P & & \coprod_{f \in I_1} P & & \coprod_{f \in I_2} P & & \coprod_{f \in I_3} P & & & & \end{array},$$



where  $X'$  denotes the homotopy colimit of the diagram  $(X_i)_{i \in \mathbb{Z}}$ . Consider now the following triangle

$$S^{-1}X' \rightarrow X'' \rightarrow X \rightarrow X',$$

where the morphism  $X \rightarrow X'$  is the transfinite composition in our diagram. Let  $P$  be in  $\mathcal{U}$ . We remark that since  $P$  is compact,  $\text{Hom}_{\mathcal{T}}(P, X') = 0$ . This also implies, by construction of  $\mathcal{T}_{\leq 0}$ , that  $\text{Hom}_{\mathcal{T}}(R, X') = 0$ , for all  $R$  in  $\mathcal{T}_{\leq 0}$ . The long exact sequence obtained by applying the functor  $\text{Hom}_{\mathcal{T}}(R, ?)$  to the triangle above shows that

$$\text{Hom}(R, X'') \xrightarrow{\sim} \text{Hom}(R, X).$$

Let  $X''_{n-1}$ ,  $n \geq 1$ , be an object as in the following triangle

$$X = X_0 \rightarrow X_n \rightarrow X''_{n-1} \rightarrow S(X).$$

A recursive application of the octahedron axiom implies that  $X''_{n-1}$  belongs to  $S(\mathcal{T}_{\leq 0})$ , for all  $n \geq 1$ . We dispose of the isomorphism

$$\text{hocolim}_n X''_{n-1} \xrightarrow{\sim} S(X'').$$

Since  $\text{hocolim}_n X''_{n-1}$  belongs to  $S(\mathcal{T}_{\leq 0})$ , we conclude that  $X''$  belongs to  $\mathcal{T}_{\leq 0}$ . This shows that the functor that sends  $X$  to  $X''$  is the right adjoint of the inclusion functor  $\mathcal{T}_{\leq 0} \hookrightarrow \mathcal{T}$ . This proves that  $\mathcal{T}_{\leq 0}$  is a left aisle on  $\mathcal{T}$ . We now show that the  $t$ -structure associated to  $\mathcal{T}_{\leq 0}$ , cf. [25], extends, from  $\mathcal{T}_c$  to  $\mathcal{T}$ , the one associated with  $\mathcal{U}$ . Let  $X$  be in  $\mathcal{T}_c$ . We dispose of the following truncation triangle associated with  $\mathcal{U}$

$$X_{\mathcal{U}} \rightarrow X \rightarrow X^{\mathcal{U}^\perp} \rightarrow SX_{\mathcal{U}}.$$

Clearly  $X_{\mathcal{U}}$  belongs to  $\mathcal{T}_{\leq 0}$ . We remark that  $\mathcal{U}^\perp = \mathcal{T}_{\leq 0}^\perp$ , and so  $X^{\mathcal{U}^\perp}$  belongs to  $\mathcal{T}_{>0} := \mathcal{T}_{\leq 0}^\perp$ .

We now show that  $\mathcal{T}_{\leq 0}$  is the smallest extension of the left aisle  $\mathcal{U}$ . Let  $\mathcal{V}$  be an aisle containing  $\mathcal{U}$ . The inclusion functor  $\mathcal{V} \hookrightarrow \mathcal{T}$  commutes with sums, because it admits a right adjoint. Since  $\mathcal{V}$  is stable under extensions and suspensions, it contains  $\mathcal{T}_{\leq 0}$ .

b) Let  $X$  be in  $\mathcal{T}$ . We need to show that  $X = 0$  iff  $H^p(X) = 0$  for all  $p \in \mathbb{Z}$ . Clearly the condition is necessary. For the converse, suppose that  $H^p(X) = 0$  for all  $p \in \mathbb{Z}$ . Let  $n$  be an integer. Consider the following truncation triangle

$$H^{n+1}(X) \rightarrow \tau_{>n}X \rightarrow \tau_{>n+1}X \rightarrow SH^{n+1}(X).$$

Since  $H^{n+1}(X) = 0$  we conclude that

$$\tau_{>n}X \in \bigcap_{m \in \mathbb{Z}} \mathcal{T}_{>m},$$

for all  $n \in \mathbb{Z}$ . Now, let  $C$  be a compact object of  $\mathcal{T}$ . We know that there is a  $k \in \mathbb{Z}$  such that  $C \in \mathcal{T}_{\leq k}$ . This implies that

$$\text{Hom}_{\mathcal{T}}(C, \tau_{>n}X) = 0$$

for all  $n \in \mathbb{Z}$ , since  $\tau_{>n}X$  belongs to  $(\mathcal{T}_{\leq k})^\perp$ . The category  $\mathcal{T}$  is compactly generated and so we conclude that  $\tau_{>n}X = 0$ , for all  $n \in \mathbb{Z}$ . The following truncation triangle

$$\tau_{\leq n}X \rightarrow X \rightarrow \tau_{>n}X \rightarrow S\tau_{\leq n}X,$$

implies that  $\tau_{\leq n}X$  is isomorphic to  $X$  for all  $n \in \mathbb{Z}$ . This can be rephrased as saying that

$$X \in \bigcap_{n \in \mathbb{N}} \mathcal{T}_{\leq -n}.$$

Now by our hypothesis there is an integer  $N$  such that

$$\mathrm{Hom}_{\mathcal{T}}(C, \mathcal{U}_{\leq -N}) = 0.$$

Since  $C$  is compact and by construction of  $\mathcal{T}_{\leq -N}$ , we have

$$\mathrm{Hom}_{\mathcal{T}}(C, \mathcal{T}_{\leq -N}) = 0.$$

This implies that  $\mathrm{Hom}_{\mathcal{T}}(C, X) = 0$ , for all compact objects  $C$  of  $\mathcal{T}$ . Since  $\mathcal{T}$  is compactly generated, we conclude that  $X = 0$ . This proves the converse.  $\checkmark$

LEMMA A.1. *Let  $(Y_p)_{p \in \mathbb{Z}}$  be in  $\mathcal{T}$ . We dispose of the following isomorphism*

$$\mathrm{H}^n\left(\prod_p Y_p\right) \xleftarrow{\sim} \prod_p \mathrm{H}^n(Y_p),$$

for all  $n \in \mathbb{Z}$ .

*Proof.* By definition  $\mathrm{H}^n := \tau_{\geq n} \tau_{\leq n}$ ,  $n \in \mathbb{Z}$ . Since  $\tau_{\geq n}$  admits a right adjoint, it is enough to show that  $\tau_{\leq n}$  commute with infinite sums. We consider the following triangle

$$\prod_p \tau_{\leq n} Y_p \rightarrow \prod_p Y_p \rightarrow \prod_p \tau_{>n} Y_p \rightarrow S\left(\prod_p \tau_{\leq n} Y_p\right).$$

Here  $\prod_p \tau_{\leq n} Y_p$  belongs to  $\mathcal{T}_{\leq n}$  since  $\mathcal{T}_{\leq n}$  is stable under infinite sums. Let  $P$  be an object of  $S^n \mathcal{U}$ . Since  $P$  is compact, we have

$$\mathrm{Hom}_{\mathcal{T}}\left(P, \prod_p \tau_{>n} Y_p\right) \xleftarrow{\sim} \prod_p \mathrm{Hom}_{\mathcal{T}}(P, \tau_{>n} Y_p) = 0.$$

Since  $\mathcal{T}_{\leq n}$  is generated by  $S^n \mathcal{U}$ ,  $\prod_p \tau_{>n} Y_p$  belongs to  $\mathcal{T}_{>n}$ . Since the truncation triangle of  $\prod_p Y_p$  is unique, this implies the following isomorphism

$$\prod_p \tau_{\leq n} Y_p \xrightarrow{\sim} \tau_{\leq n}\left(\prod_p Y_p\right).$$

This proves the lemma.  $\checkmark$

PROPOSITION A.2. *Let  $X$  be an object of  $\mathcal{T}$ . Suppose that we are in the conditions of proposition A.1 b). We dispose of the following isomorphism*

$$\mathrm{hocolim}_i \tau_{\leq i} X \xrightarrow{\sim} X.$$

*Proof.* We need only show that

$$H^n(\operatorname{hocolim}_i \tau_{\leq i} X) \xrightarrow{\sim} H^n(X),$$

for all  $n \in \mathbb{Z}$ . We dispose of the following triangle, cf. [29],

$$\coprod_p \tau_{\leq p} X \rightarrow \coprod_q \tau_{\leq q} X \rightarrow \operatorname{hocolim}_i \tau_{\leq i} X \rightarrow S(\coprod_p \tau_{\leq p} X).$$

Since the functor  $H^n$  is homological, for all  $n \in \mathbb{Z}$  and it commutes with infinite sums by lemma A.1, we obtain a long exact sequence

$$\begin{aligned} \cdots \rightarrow \coprod_p H^n(\tau_{\leq p} X) &\rightarrow \coprod_q H^n(\tau_{\leq q} X) \rightarrow H^n(\operatorname{hocolim}_i \tau_{\leq i} X) \rightarrow \\ &\rightarrow \coprod_p H^n S(\tau_{\leq p} X) \rightarrow \coprod_q H^n S(\tau_{\leq q} X) \rightarrow \cdots \end{aligned}$$

We remark that the morphism  $\coprod_p H^n S(\tau_{\leq p} X) \rightarrow \coprod_q H^n S(\tau_{\leq q} X)$  is a split monomorphism and so we obtain

$$H^n(X) = \operatorname{colim}_i H^n(\tau_{\leq i} X) \xrightarrow{\sim} H^n(\operatorname{hocolim}_i \tau_{\leq i} X).$$

✓

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