On Suslin Matrices and Their Connection to Spin Groups

Vineeth Reddy Chintala

Received: January 2, 2014
Revised: March 17, 2015

Communicated by Alexander Merkurjev

Abstract. A representation of the Clifford algebra (of a hyperbolic module) is given using what are called Suslin matrices. This explicit construction is used to analyze the corresponding Spin groups. Moreover, some properties of Suslin matrices are better understood via this link to Clifford algebras.

2010 Mathematics Subject Classification: 13C10, 15A63, 15A66, 17C50
Keywords and Phrases: Clifford algebra, Spin groups, Suslin matrices, Unimodular rows, Jordan algebra

1 Introduction

Let R be a commutative ring and V be a free R-module. This article is on the Clifford algebra of the hyperbolic module $H(V) = V \oplus V^*$ equipped with the quadratic form $q(x, f) = f(x)$. These algebras are isomorphic to total matrix rings of size $2^n \times 2^n$, where $\dim V = n$. A representation of these algebras is given using what are called Suslin matrices. The identities followed by these matrices will then be used to study the corresponding Spin groups. Conversely one will now be able to relate some (seemingly) accidental properties of Suslin matrices to the geometry of Clifford algebras.

In the introduction of [JR], the authors shared the insight of the referee (of [JR2]) that there should be a link between Clifford algebras and Suslin matrices such that the Spin groups are isomorphic to matrix groups which stabilize the Suslin matrices (under a suitable action). The present paper describes and analyzes this link. This (half-spin) representation in terms of Suslin matrices is faithful only when $\dim V$ is odd and we will mostly be concerned with this case. With an explicit construction at hand, one can expect to derive some results on the Clifford algebra and its Spin group using simple matrix computations. For example, the identities followed by the Suslin matrices (in Equation 1) give us (as we will see) the involution on the Clifford algebra. Indeed given any element of the Clifford algebra we will be able to explicitly construct its conjugate matrix under the involution. As an application of Suslin matrices, we give a proof of the following exceptional isomorphisms :

$$Spin_4(R) \cong SL_2(R) \times SL_2(R),\ Spin_6(R) \cong SL_4(R).$$

On the other hand, some key properties of Suslin matrices (see Theorem 3.4 and Remark 8.5) are re-derived here (with minimal computation) using the connection to Clifford algebras. These properties play the crucial role in the papers [JR], [JR2] which study an action of the orthogonal group on unimodular rows. In addition, this connection to Clifford algebras explains *why* the size of a Suslin matrix has to be of the form $2^{n-1} \times 2^{n-1}$. One can go on and ask if there are other matrix-constructions, possibly of smaller sizes. Indeed, it turns out that the size chosen by Suslin is the least possible. This is an *embedding* problem of a quadratic space into the algebra of matrices - it belongs to a larger theme of finding natural ways of constructing algebras out of quadratic spaces. It would be interesting to see similar constructions for other types of quadratic forms - and for this reason alone, it might be worthwhile having this connection between Suslin matrices and Clifford algebras.

We begin with a few preliminaries on Clifford algebras and Suslin matrices (to make the article accessible) and the link between them. The involution of the Clifford algebra is then described via some identities followed by the Suslin matrices. It turns out that the behavior of the Suslin matrices (hence that of the Spin groups) depends on whether the dimension of V is odd or even. One realizes this after observing the pattern followed by the involution in the odd and even cases. This leads us to Section 5 where the action of the Spin group on the Suslin space is studied in detail. The last part of the paper contains a few remarks on the Spin and Epin groups.

By a ring, we always mean a ring with unity. In this paper, V will always denote a free R-module where R is any commutative ring and $H(V) = V \oplus V^*$. From now on, we fix the standard basis for $V = R^n$ and identify V with its dual V^*. One can then write the quadratic form on $H(V)$ as

$$q(v, w) = v \cdot w^T = a_1 b_1 + \cdots + a_n b_n.$$

for $v = (a_1, \cdots, a_n)$, $w = (b_1, \cdots, b_n)$.
For general literature on Clifford algebras and Spin groups over a commutative ring, the reader is referred to [B1], [B2] of H. Bass. For a more detailed introduction to Suslin matrices (and their important connection to unimodular

rows), a few references are Suslin's paper [S], the papers [JR], [JR2] of Ravi Rao and Selby Jose, and T. Y. Lam's book [L] (sections III.7, VIII.5).

Acknowledgements. I would like to thank my advisor Prof. Ravi Rao for the many discussions we had while working on this paper. His willingness to give his time so generously has been very much appreciated. The comments given by referees have been very helpful and I would like to thank them for improving the clarity of the paper.

Part of this work was done during a short enjoyable visit to Fields Institute (in Toronto), which funded me to attend its Spring school on Torsors in 2013.

2 The Suslin Construction

The Suslin construction gives a sequence of matrices whose size doubles at every step. Moreover each Suslin matrix S has a conjugate Suslin matrix $\bar{S}$ such that $S\bar{S}$ and $S+\bar{S}$ are scalars (norm and the trace).

Let us pause here to see the recursive process by which the Suslin matrix $S_n(v,w)$ of size $2^n \times 2^n$, and determinant $(v \cdot w^T)^{2^{n-1}}$, is constructed from two vectors v, w in R^{n+1}.

Let $v = (a_0, v_1)$, $w = (b_0, w_1)$ where v_1, w_1 are vectors in R^n. Define

$$S_0(a_0, b_0) = a_0, \quad S_1(v,w) = \begin{pmatrix} a_0 & v_1 \\ -w_1 & b_0 \end{pmatrix}$$

and

$$S_n(v,w) = \begin{pmatrix} a_0 I_{2^{n-1}} & S_{n-1}(v_1,w_1) \\ -S_{n-1}(w_1,v_1)^T & b_0 I_{2^{n-1}} \end{pmatrix}.$$

For $S_n = S_n(v,w)$, define

$$\bar{S_n} := S_n(w,v)^T = \begin{pmatrix} b_0 I_{2^{n-1}} & -S_{n-1}(v_1,w_1) \\ S_{n-1}(w_1,v_1)^T & a_0 I_{2^{n-1}} \end{pmatrix}.$$

Notice that $\overline{S_n(v,w)}$ is also a Suslin matrix. Indeed, $\overline{S_n(v,w)} = S_n(v',w')$ where

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} (b_0, -v_1) \\ (a_0, -w_1) \end{pmatrix}.$$

In particular, we have $\overline{S_0(a_0,b_0)} = b_0$.

One can easily check that a Suslin matrix $S_n = S_n(v,w)$ satisfies the following properties:

(1) $S_n\bar{S_n} = \bar{S_n}S_n = (v \cdot w^T)I_{2^n}$, and
(2) $\det S_n = (v \cdot w^T)^{2^{n-1}}$, for $n \geq 1$.

Unless otherwise specified, we assume that $n > 0$. Then the element (v, w) is determined by its corresponding Suslin matrix $S_n(v, w)$. So we sometimes identify the element (v, w) with $S_n(v, w)$ for $n > 0$. The set of Suslin matrices of size $2^n \times 2^n$ is an R-module under matrix-addition and scalar multiplication given by $rS_n(v, w) = S_n(rv, rw)$ for $r \in R$; Moreover when $n > 0$, the mapping $S_n \to \bar{S_n}$ is an isometry of the quadratic space $H(R^{n+1})$, preserving the quadratic form $q(v, w) = v \cdot w^T$.

In his paper [S], A. Suslin then describes a sequence of matrices $J_n \in M_{2^n}(R)$ by the recurrence formula

$$J_n = \begin{cases} 1 & \text{for } n = 0 \\ \begin{pmatrix} J_{n-1} & 0 \\ 0 & -J_{n-1} \end{pmatrix} & \text{for } n \text{ even} \\ \begin{pmatrix} 0 & J_{n-1} \\ -J_{n-1} & 0 \end{pmatrix} & \text{for } n \text{ odd.} \end{cases}$$

Remark 2.1. We will simply write J and S (or $S(v, w)$) and drop the subscript when there is no confusion. When there is some confusion, remember that the subscript r in S_r is used to indicate that S_r (or J_r) is a $2^r \times 2^r$ matrix.

It is easy to check that $\det J = 1$ and $J^T = J^{-1} = (-1)^{\frac{n(n+1)}{2}} J$.

That means that J is $\begin{cases} \text{skew-symmetric for } n = 4k+1 \text{ and } n = 4k+2, \\ \text{symmetric for } n = 4k \text{ and } n = 4k+3. \end{cases}$

It can be shown inductively that the forms J satisfy the following identities :

$$JS^TJ^T = \begin{cases} S & \text{for } n \text{ even,} \\ \bar{S} & \text{for } n \text{ odd.} \end{cases} \tag{1}$$

The even case in Equation (1) can also be deduced from Lemma 5.3, [S], while the odd case follows from *loc. cit.* only if $v \cdot w^t$ is not a zero-divisor. The above relations will be used to describe the involution on the Clifford algebra. The Suslin matrices will now be used to give a set of generators of the Clifford algebra.

3 The link to Clifford algebras

Let R be any commutative ring. Let (V, q) be a quadratic space where V is a free R-module of dim $V = n$, equipped with a quadratic form q. The Clifford algebra $Cl(V, q)$ is the quotient of the tensor algebra

$$T(V) = R \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

by the two sided ideal $I(V, q)$ generated by all $x \otimes x - q(x)$ with $x \in V$.

Thus $Cl(V, q)$ is an associative algebra (with unity) over R with a linear map $i : V \to Cl(V, q)$ such that $i(x)^2 = q(x)$. The terms $x \otimes x$ and $q(x)$ appearing in the generators of $I(V, q)$ have degrees 0 and 2 in the grading of $T(V)$. By grading $T(V)$ modulo 2 by even and odd degrees, it follows that the Clifford algebra has a Z_2-grading $Cl(V, q) = Cl_0(V, q) \oplus Cl_1(V, q)$.

The Clifford algebra $Cl(V, q)$ has the following universal property : Given any associative algebra A over R and any linear map $j : V \to A$ such that

$$j(x)^2 = q(x) \text{ for all } x \in V,$$

then there is a unique algebra homomorphism $f : Cl(V, q) \to A$ such that $f \circ i = j$.

Let Cl denote the Clifford algebra of $H(R^n)$ equipped with the quadratic form

$$q(v, w) = v \cdot w^T.$$

It can be proved (see [B1], Ch. 5, Theorem 3.9) that

$$Cl \cong M_{2^n}(R).$$

Let $\phi : H(R^n) \to M_{2^n}(R)$ be the linear map defined by

$$\phi(v, w) = \begin{pmatrix} 0 & S_{n-1}(v, w) \\ \overline{S_{n-1}(v, w)} & 0 \end{pmatrix}$$

where $\overline{S_{n-1}(v, w)} = S_{n-1}(w, v)^T$.

Since $\phi(v, w)^2 = q(v, w)I_{2^n}$, the map ϕ uniquely extends to an R-algebra homomorphism

$$\phi : Cl \to M_{2^n}(R)$$

(by the universal property of Clifford algebras). In the next section, we will prove that ϕ is an isomorphism.

3.1 Optimal embedding of Suslin matrices

The Clifford algebra and the Suslin matrix are two different ways of constructing algebras out of the quadratic space $H(R^n)$. One can ask if one can construct similar matrices but of different sizes - possibly smaller? First we need a notion of an embedding of a quadratic space in an algebra to compare two constructions.

Let (V, q) be a quadratic space (i.e. V is a free R-module and q a non-degenerate quadratic form on V). Let A be a faithful R-algebra ($R \hookrightarrow A$).

Definition 3.2. *The quadratic space (V, q) is said to be embedded in A if $V \subseteq A$ and there is an isometry $\alpha : V \to V$ such that*

$$v\alpha(v) = \alpha(v)v = q(v).$$

The object of our interest is the hyperbolic quadratic space $H(R^n)$ with the quadratic form $q(v, w) = v \cdot w^T$. By taking $\alpha : H(R^n) \to H(R^n)$ (for $n > 1$) to be the isometry

$$\alpha : S_{n-1}(v, w) \to \overline{S_{n-1}(v, w)}$$

one sees that the Suslin construction $(v, w) \to S(v, w)$ is an embedding of $H(R^n)$ into $M_{2^{n-1}}(R)$.

One can ask which properties of Suslin matrices are unique to its construction and which ones are true in general for any other embedding into matrices.

The size of a Suslin matrix corresponding to $(v, w) \in H(R^n)$ is $2^{n-1} \times 2^{n-1}$; the Clifford algebra (another embedding of $H(R^n)$) is isomorphic to the algebra of $2^n \times 2^n$ matrices. Are there embeddings of smaller sizes? Theorem 3.3 tells us that the size of the Suslin matrices is optimal.

Let us analyze an embedding $\alpha : H(R^n) \hookrightarrow A$ into an associative algebra A. We will see that this places a strong restriction on the choice of A. Consider the R-linear map $\phi : H(R^n) \to M_2(A)$ defined by $\phi(x) = \left(\begin{smallmatrix} 0 & x \\ \alpha(x) & 0 \end{smallmatrix}\right)$.

Since $\phi(x)^2 = q(x)$, the map ϕ uniquely extends (by the universal property of Clifford algebras) to an R-algebra homomorphism

$$\phi : Cl \to M_2(A).$$

Theorem 3.3. *Let ϕ be defined as above; then ϕ is injective.*

Proof. Recall that we have an isomorphism of R-algebras $Cl \cong M_{2^n}(R)$ [[B1], Chapter 5, theorem 3.9], so it suffices to prove that the composite homomorphism $M_{2^n}(R) \to M_2(A)$ is injective. Now every ideal of Cl is of the form $M_{2^n}(I)$. In particular $\ker(\phi) = M_{2^n}(I)$ for some ideal $I \subseteq R$. So to prove that ϕ is injective it is enough to observe that $I = 0$. But this is obvious since ϕ is R-linear and acts as identity on R. □

For the Suslin embedding α is the isometry $S(v,w) \to \overline{S(v,w)}$ defined for Suslin matrices in Section 2. Since the map ϕ is injective, it follows (by dimension arguments) that ϕ is an isomorphism for the Suslin embedding. We will identify the elements $(v,w) \in H(R^n)$ with their images under the representation ϕ. The following fundamental lemma of Jose-Rao in [JR] is an easy consequence of the basic properties of Clifford algebras.

Theorem 3.4. *Let X and Y be Suslin matrices in $M_{2^{n-1}}(R)$. Then XYX is also a Suslin matrix. Moreover $\overline{XYX} = \bar{X}\bar{Y}\bar{X}$.*

Proof. Let $z_1, z_2 \in H(R^n)$. Then

$$\langle z_1, z_2\rangle := z_1 z_2 + z_2 z_1 = (z_1 + z_2)^2 - z_1^2 - z_2^2$$

is an element in R. Multiplying by z_1 we get

$$z_1\langle z_1, z_2\rangle = z_1^2 z_2 + z_1 z_2 z_1.$$

Since $z_1^2 = q(z_1)$, it follows that $z_1 z_2 z_1 \in H(R^n)$. Take $z_1 = \left(\begin{smallmatrix} 0 & X \\ \bar{X} & 0 \end{smallmatrix}\right)$ and $z_2 = \left(\begin{smallmatrix} 0 & \bar{Y} \\ Y & 0 \end{smallmatrix}\right)$. Then $z_1 z_2 z_1 = \left(\begin{smallmatrix} 0 & XYX \\ \bar{X}\bar{Y}\bar{X} & 0 \end{smallmatrix}\right)$. ☐

3.5 The basic automorphism of Cl

The Clifford algebra $Cl = Cl_0 \oplus Cl_1$ has a 'basic automorphism' given by

$$x = x_0 + x_1 \to x' = x_0 - x_1.$$

Two automorphisms of Cl are the same if they agree on the elements $(v,w) \in H(R^n)$. Under the isomorphism ϕ in Section 3, the basic automorphism corresponds to conjugation by the matrix $\lambda = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. One can infer this by checking that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\phi(v,w)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = -\phi(v,w)$. It is also easy to check that

$$\lambda^* := J\lambda^T J^T = \begin{cases} \lambda & \text{for } n \text{ even,} \\ -\lambda & \text{for } n \text{ odd.} \end{cases}$$

Remark 3.6. One needs this basic automorphism to make a suitable adjustment for the grading of the Clifford algebra - for example when defining the norm or a 'graded' conjugation. We will be concerned mostly with the Spin group which lies in Cl_0 (on which conjugation by λ is the identity map).

4 The involution on Cl

We will identify Cl with $M_{2^n}(R)$ via the isomorphism ϕ. Recall that ϕ is defined on the elements (v,w) as

$$\phi(v,w) = \begin{pmatrix} 0 & S_{n-1}(v,w) \\ \overline{S_{n-1}(v,w)} & 0 \end{pmatrix}.$$

The map $(v, w) \to (-v, -w)$ can also be viewed as an inclusion of $H(R^n)$ in the opposite algebra of Cl. By the universal property of the Clifford algebra, this map extends to an anti-automorphism of Cl. We will call this the standard involution on Cl.

Theorem 4.1. *Let $M \in Cl \cong M_{2^n}(R)$. The standard involution $*$ is given by*

$$M^* = J_n M^T J_n^T.$$

Proof. That $*$ is an involution is clear since $J_n^T = J_n^{-1}$. The elements (v, w) generate the Clifford algebra. Therefore, to prove that the above involution is the correct one, it is enough to check that its action on the matrices

$$\phi(v,w) = \begin{pmatrix} 0 & S_{n-1}(v,w) \\ \overline{S_{n-1}(v,w)} & 0 \end{pmatrix}$$

is multiplication by -1.

Let $S_n' = S_n(v', w') = \begin{pmatrix} 0 & S_{n-1}(v,w) \\ -\overline{S_{n-1}(v,w)} & 0 \end{pmatrix}$ where $v' = (0, v)$ and $w' = (0, w)$. Then

$$\phi(v,w) = \lambda S_n' = -S_n' \lambda.$$

Moreover observe that $\bar{S_n'} = -S_n'$. It follows by the identities in Equation (1) that

$$S_n'^{\,*} = \begin{cases} S_n' & \text{for } n \text{ even,} \\ -S_n' & \text{for } n \text{ odd.} \end{cases}$$

Therefore for n even, we have

$$\phi(v,w)^* = S_n'^{\,*} \lambda^* = S_n' \lambda = -\lambda S_n'.$$

And for n odd, we have

$$\phi(v,w)^* = S_n'^{\,*} \lambda^* = (-S_n')(-\lambda) = -\lambda S_n'.$$

Hence for any n,

$$\phi(v,w)^* = -\phi(v,w).$$

$\square$

Given an element of the Clifford algebra, one can now compute its conjugate under the involution $*$. One can spot two different patterns of the involution depending on the parity of n. To see this, let us write $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ as a 2×2 matrix and analyze its conjugate in terms of its blocks. Then one can compute

M^* inductively as follows: (For $n = 0$, the involution is the identity map on R.)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{cases} \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix} & \text{for } n \text{ odd}, \\ \\ \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix} & \text{for } n \text{ even}. \end{cases} \tag{2}$$

With this the setup is complete for us to analyze and compute the Spin groups.

4.2 SPIN GROUP

The Clifford algebra is a Z_2-graded algebra $Cl = Cl_0 \oplus Cl_1$. Under the isomorphism ϕ, the elements of Cl_0 correspond to matrices of the form $\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right)$.

The following groups are relevant to our discussion :

$$U_{2n}^0(R) := \{x \in Cl_0 \mid xx^* = 1\}.$$

$$Spin_{2n}(R) := \{x \in U_{2n}^0(R) \mid xH(R^n)x^{-1} = H(R^n)\}.$$

Let $(g_1, g_2) \in Spin_{2n}(R)$. We have by the identities in Equation (2),

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}^* = \begin{cases} \begin{pmatrix} g_1^* & 0 \\ 0 & g_2^* \end{pmatrix} & \text{for } n \text{ even}, \\ \\ \begin{pmatrix} g_2^* & 0 \\ 0 & g_1^* \end{pmatrix} & \text{for } n \text{ odd}. \end{cases}$$

Where there is no confusion possible, we will write (g_1, g_2) for the diagonal matrix $\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right)$.

5 THE ACTION OF THE SPIN GROUP ON THE SPACE OF SUSLIN MATRICES : WHEN n IS ODD

In the introduction of [JR], the authors shared the insight of their referee (of [JR2]) that one should be able to construct a subgroup $G_r(R)$ of $GL_{2^r}(R)$ which is defined by the property that $g \in G_r(R)$ if $gS(v,w)g^*$ is a Suslin matrix for all Suslin matrices $S \in M_{2^r}(R)$. From this action of $G_r(R)$ on the space of Suslin matrices $S(v,w)$, consider the subgroup $SG_r(R)$ consisting of $g \in G_r(R)$ preserving the norm $v \cdot w^T$ for all pairs (v,w). The referee guessed that the group $SG_r(R)$ is isomorphic to the Spin group.

In this section, we construct such a group $G_r(R)$ and prove that there is such an isomorphism between $SG_r(R)$ and the corresponding Spin group when $n = \dim V$ is odd (i.e. $r = n-1$ is even). This is achieved by taking $g \to g^*$ to be the restriction of the involution on $M_{2^r}(R)$ defined earlier. In the rest of the section, $n = \dim V$ is odd.

Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2^n}(R)$. When n is odd, we have $M^* = \left(\begin{smallmatrix} D^* & -B^* \\ -C^* & A^* \end{smallmatrix}\right)$. Therefore for $(g_1, g_2) \in U^0_{2n}(R)$, we have

$$(g_1, g_2)^* = (g_2^*, g_1^*).$$

In addition, since $(g_1, g_2) \in U^0_{2n}(R)$ has unit norm, i.e. $(g_1, g_2)(g_1, g_2)^* = 1$, it follows that

$$g_2 = g_1^{*^{-1}}.$$

Now $Spin_{2n}(R)$ is precisely the subgroup of $U^0_{2n}(R)$ which stabilizes $H(R^n)$ under conjugation. This means that if $(g, g^{*^{-1}}) \in Spin_{2n}(R)$ and $S \in M_{2^{n-1}}(R)$ is any Suslin matrix, then there exists a Suslin matrix $T \in M_{2^{n-1}}(R)$ such that

$$(g, g^{*^{-1}}) \begin{pmatrix} 0 & S \\ \bar{S} & 0 \end{pmatrix} (g^{-1}, g^*) = \begin{pmatrix} 0 & T \\ \bar{T} & 0 \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} 0 & gSg^* \\ g^{*^{-1}} \bar{S} g^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & T \\ \bar{T} & 0 \end{pmatrix}.$$

Hence if $S \in M_{2^{n-1}}(R)$ is a Suslin matrix and $(g, g^{*^{-1}}) \in Spin_{2n}(R)$, then gSg^* is also a Suslin matrix.

Remark 5.1. The space of Suslin matrices is nothing but the quadratic space $H(R^n)$. For simplicity, we will write S instead of S_{n-1} and when we say "for all Suslin matrices S", we really mean "for all Suslin matrices $S \in M_{2^{n-1}}(R)$".

Let $g \bullet S = gSg^*$.

Consider

$$G_{n-1}(R) = \{\, g \in GL_{2^{n-1}}(R) \mid g \bullet S \text{ is a Suslin matrix } \forall \text{ Suslin matrices } S\}.$$

One has the homomorphism

$$\chi : Spin_{2n}(R) \to G_{n-1}(R)$$

given by $(g, g^{*-1}) \to g$.

In general, this homomorphism is not surjective, but one can expect that it is the case on the subgroup of $G_{n-1}(R)$ which preserves the quadratic form $v \cdot w^T$.

For this we introduce a length function on the space of Suslin matrices : for $S = S_{n-1}(v, w)$, define
$$l(S) := S\bar{S} = v \cdot w^T.$$
Let
$$SG_{n-1}(R) := \{\ g \in G_{n-1}(R) \mid l(g \bullet S) = l(S)\ \forall \text{ Suslin matrices } S\}.$$
Suppose $g \in SG_{n-1}(R)$. Then $l(gg^*) = 1$. One expects $(g, {g^*}^{-1})$ to be an element of the Spin group.

Theorem 5.2. *The homomorphism $\chi : Spin_{2n}(R) \cong SG_{n-1}(R)$ is an isomorphism.*

Proof. We first prove that if $g \in SG_{n-1}(R)$ then ${g^*}^{-1} \in G_{n-1}(R)$. Let $T = g \bullet S$. If $l(S) = l(T) = 1$, then $\bar{T} = T^{-1} = {g^*}^{-1} \bullet \bar{S}$. Write a general Suslin matrix as a linear combination of unit-length Suslin matrices. By linearity of the action $\bullet$ it follows that $\bar{T} = {g^*}^{-1} \bullet \bar{S}$ for a general Suslin matrix S. It follows that we can define a homomorphism $SG_{n-1}(R) \to Spin_{2n}(R)$ by $g \to (g, {g^*}^{-1})$ and one checks easily that this is an inverse of χ. $\square$

The assumption $l(g \bullet S) = l(S)$ is simply a translation of the definition of the Spin group in terms of Suslin matrices. It does not give us much insight into the action $\bullet$. A simpler equivalent criterion is the following: for $g \in G_{n-1}(R)$, we have $l(g \bullet S) = l(S)$ for all Suslin matrices S if and only if $l(gg^*) = 1$.

This will be proved by replacing the length function which is defined on Suslin matrices with a 'norm' which makes sense for any element $g \in G_{r-1}(R)$. The Spin group then corresponds to the subgroup consisting of elements which have unit norm. In particular, the following theorem implies that
$$SG_{n-1}(R) = \{\ g \in G_{n-1}(R) \mid l(gg^*) = 1\}.$$

Theorem 5.3. *Let $g \in G_{n-1}(R)$. Then $l(g \bullet S) = l(gg^*)l(S)$ for all Suslin matrices S.*

Proof. Case 1 : $l(gg^*) = 1$.

It is enough to show that $\overline{g \bullet S} = {g^*}^{-1} \bullet \bar{S}$ which in turn implies that $l(g \bullet S) = l(S)$. We will first prove this for a subset of Suslin matrices and then show that our chosen set spans the total space.

If X, Y are Suslin matrices then it follows by Theorem 3.4 that
$$l(XYX) = l(X)^2 l(Y).$$
Take $X = g \bullet S$ and $Y = (gg^*)^{-1}$. Then $XYX = g \bullet S^2$. In addition we have
$$gg^* \bullet (gg^*)^{-1} = gg^*.$$

Since gg^* is a Suslin matrix and $l(gg^*) = 1$ (by the hypothesis) it follows that its inverse Y is also a Suslin matrix with $l(Y) = 1$. Therefore $l(XYX) = l(X)^2$, i.e.

$$l(g \bullet S^2) = l(g \bullet S)^2.$$

By induction we get that

$$l(g \bullet S^{2^k}) = l(g \bullet S)^{2^k} \text{ for all } k \geq 1.$$

Suppose $l(S) = 1$. From the definition of the length function, we have

$$l(g \bullet S)^{2^{n-2}} = \det(g \bullet S) = \det(gg^*)\det(S) = 1.$$

Therefore

$$l(g \bullet S^{2^k}) = 1, \text{ for all } k \geq n-2. \tag{3}$$

Let

$$W = \{\ r_iT_i + \cdots + r_kT_k \ |r_i \in R \text{ and } T_i = S_i^{2^{m_i}} \text{ for some } S_i \text{ with } l(S_i) = 1 \text{ and } m_i > n.\}$$

If $T = S^{2^m}$ with $m > n$ and $l(S) = 1$, then we have by Equation (3) that

$$\overline{g \bullet T} = (g \bullet T)^{-1} = {g^*}^{-1} \bullet \bar{T}.$$

For any $T \in W$ it follows by the linearity of the action $\bullet$ that

$${g^*}^{-1} \bullet \bar{T} = {g^*}^{-1} \bullet (r_i\bar{T_i} + \cdots + r_k\bar{T_k}) = \overline{g \bullet T}.$$

Therefore $l(g \bullet T) = l(T)$ for all $T \in W$.

Let $\{e_i\}$ be the standard basis of R^n and $\{f_i\}$ be its dual basis. We will now show that W contains the generators $S_{n-1}(e_i, 0)$ and $S_{n-1}(0, f_i)$, hence the whole space. Let

$$E_i = S_{n-1}(e_i, 0), \quad F_i = S_{n-1}(0, f_i).$$

Let $X = \left(\begin{smallmatrix} 1 & A \\ -\bar{A} & 0 \end{smallmatrix}\right)$ for some Suslin matrix A with $A\bar{A} = I$. Then X is also a Suslin matrix and $X^2 = \left(\begin{smallmatrix} 0 & A \\ -\bar{A} & -1 \end{smallmatrix}\right)$ and $X^3 = -I$. Therefore $X^{2^k} = X^2$ or $X^{2^k} = -X$, depending on whether k is odd or even. It follows by the above discussion that $X \in W$. For example, one can take $X = E_i + F_i + E_1$ with $i \neq 1$. Also, if $i \neq j$ and $i, j \neq 1$, then $E_i + E_j + F_j + E_1$ is an example.

Let $i \neq j$ and $i, j \neq 1$. Then $E_i \in W$ since

$$E_i = (E_i + E_j + F_j + E_1) - (E_j + F_j + E_1)$$

is a linear combination of matrices in W. Similarly $F_i \in W$ since

$$F_i = (F_i + E_j + F_j + E_1) - (E_j + F_j + E_1).$$

And finally $E_1 = (E_i + F_i + E_1) - (E_i + F_i)$ and $F_1 = I - E_1$ also lie in W.

Case 2 : $l(gg^*) = a$.

Clearly a has to be invertible since $g \in G_{n-1}(R)$. Suppose there is an $x \in R$ such that $x^2 = a^{-1}$.

Take $h = xg$. Then $l(hh^*) = 1$ and by Case 1, we have

$$l(h \bullet S) = l(hh^*)l(S)$$

for any Suslin matrix S.

We also have $l(h \bullet S) = x^2 \cdot l(g \bullet S)$, hence $l(g \bullet S) = l(gg^*)l(S)$.

Now suppose $x^2 = a^{-1}$ has no solutions in R. Then one has the identity

$$l(g \bullet S) = l(gg^*)l(S)$$

over the ring $\frac{R[x]}{(x^2 - a^{-1})}$; since each term of the equation lies in R, the theorem is proved in this case too. □

Remark 5.4. Let $R^\bullet$ denote the group of units in R. Define $d : G_{n-1}(R) \to R^\bullet$ as

$$d(g) := l(gg^*).$$

Let $g, h \in G_{n-1}(R)$. As a consequence of Theorem 5.3, we have

$$d(gh) = l(ghh^*g^*) = l(gg^*)l(hh^*) = d(g)d(h).$$

Thus d is a group homomorphism and $\ker(d) = SG_{n-1}(R) \cong Spin_{2n}(R)$.

6 When n is even:

Let $(g_1, g_2) \in Spin_{2n}(R)$. By definition if $S \in M_{2^{n-1}}(R)$ is a Suslin matrix, then there exists a Suslin matrix T such that

$$(g_1, g_2)\begin{pmatrix} 0 & S \\ \bar{S} & 0 \end{pmatrix}(g_1^{-1}, g_2^{-1}) = \begin{pmatrix} 0 & T \\ \bar{T} & 0 \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} 0 & g_1 S g_2^{-1} \\ g_2 \bar{S} g_1^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & T \\ \bar{T} & 0 \end{pmatrix}.$$

Let us now consider the group homomorphism $\chi : Spin_{2n}(R) \to GL_{2^{n-1}}(R)$ given by

$$\chi(g_1, g_2) = g_1.$$

In this case the projection χ has a nontrivial kernel. Let $(1,g) \in \ker(\chi)$. Taking $S = 1$, one sees that g is also a Suslin matrix. Now both g and Sg^{-1} are Suslin matrices for any Suslin matrix S. In the case when $n > 2$, one can then conclude by a simple computation that $g = uI$ where $u \in R$ and $u^2 = 1$ (see [JR], Lemma 3.1).

Define $\mu_2 := \{u \in R \mid u^2 = 1\}$. Then one has an following exact sequence when $n > 2$ and n even :

$$1 \to \mu_2 \to Spin_{2n}(R) \to GL_{2^{n-1}}(R). \tag{4}$$

Question : Let $SG'_{n-1}(R) = \chi(Spin_{2n}(R))$. For what conditions on R is there a homomorphsim $f : SG'_{n-1}(R) \to Spin_{2n}(R)$ such that $f \circ \chi = Id$?

Recall that when n is even, $(g_1, g_2)^* = (g_1^*, g_2^*)$. From this one can immediately compute $Spin_4(R)$. Firstly any 2×2 matrix is of the form $S(v,w) = \left(\begin{smallmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{smallmatrix}\right)$ for $v = (a_1, a_2)$, $w = (b_1, b_2)$. Therefore $g_1 S g_2^{-1}$ is a Suslin matrix for any pair (g_1, g_2) and $S \in M_2(R)$. Moreover it is clear from the definition of the involution that the norm is precisely the determinant for 2×2 matrices.

$$(g_1, g_2)(g_1^*, g_2^*) = (\det g_1, \det g_2).$$

Therefore

$$Spin_4(R) = SL_2(R) \times SL_2(R).$$

For another proof, see [[K], Chapter V, § 4.5].

Remark 6.1. An immediate corollary is that $Spin_4(R)$ acts transitively on the set of 2×2 Suslin matrices $S(v,w)$ of unit length (i.e. $v \cdot w^T = 1$). The set of unit length Suslin matrices is nothing but $SL_2(R)$ and the action is given by $S \to g_1 S g_2^{-1}$ for $(g_1, g_2) \in Spin_4(R)$ and $S \in SL_2(R)$.

7 Computing $Spin_6(R)$

In this section, we compute $Spin_6(R)$ using Suslin matrices. For another proof of the following result, see [[K], Chapter V, § 5.6].

Theorem 7.1. *$Spin_6(R) \cong SL_4(R)$.*

Proof. We will prove that $SG_2(R) \cong SL_4(R)$.

First it will be shown that given any 4×4 matrix M, the product MSM^* is a Suslin matrix whenever S is a Suslin matrix.

Write $S = \left(\begin{smallmatrix} a & S_1 \\ -\bar{S}_1 & b \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Since M is a 4×4 matrix, we have

$$M^* = \left(\begin{smallmatrix} A^* & -C^* \\ -B^* & D^* \end{smallmatrix}\right).$$

To conclude that MSM^* is a Suslin matrix, it is enough to check that both $M\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)M^*$ and $M\left(\begin{smallmatrix} 0 & S_1 \\ -\bar{S}_1 & 0 \end{smallmatrix}\right)M^*$ are Suslin matrices.

We have

$$M\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}M^* = \begin{pmatrix} aAA^* - bBB^* & -aAC^* + bBD^* \\ aCA^* - bDB^* & -aCC^* + bDD^* \end{pmatrix}$$

and

$$M\begin{pmatrix} 0 & S_1 \\ -\bar{S}_1 & 0 \end{pmatrix}M^* = \begin{pmatrix} -B\bar{S}_1A^* - AS_1B^* & B\bar{S}_1C^* + AS_1D^* \\ -D\bar{S}_1A^* - CS_1B^* & D\bar{S}_1C^* + CS_1D^* \end{pmatrix}.$$

Recall that any 2×2 matrix $X = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ is a Suslin matrix and $X^* = \bar{X} = \left(\begin{smallmatrix} w & -y \\ -z & x \end{smallmatrix}\right)$. Also, $X^* + X$ are XX^* are scalar matrices.

A 4×4 matrix $\left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)$ is a Suslin matrix if and only if X, W are scalar matrices and $Y = -Z^*$. Therefore MSM^* is a Suslin matrix.

It remains to show that if $g \in SG_2(R)$, then $\det g = 1$. By definition $l(S)^2 = \det(S)$ for any Suslin matrix $S \in M_4(R)$. One should be able to prove from this that $\det g = l(gg^*) = 1$.

One can also check by hand that for 4×4 matrices, $l(MM^*) = \det M$. We have

$$MM^* = \begin{pmatrix} AA^* - BB^* & -AC^* + BD^* \\ CA^* - DB^* & -CC^* + DD^* \end{pmatrix}$$

and

$$l(MM^*) = AA^*DD^* + BB^*CC^* + AC^*DB^* + BD^*CA^*.$$

□

Remark 7.2. Suppose 2 is not a zero divisor in R. Then a matrix $M \in M_4(R)$ is a Suslin matrix if and only if $M = M^*$, i.e. the set of Suslin matrices is the Jordan algebra consisting of self-adjoint elements in $M_4(R)$. In particular MSM^* is a Suslin matrix for any $M \in M_4(R)$. Observe that the set $\{gg^* | \ g \in SL_4(R)\}$ is precisely the orbit of the identity element under the action of the Spin group. Therefore $Spin_6(R)$ acts transitively on the set of elements of length 1 if and only if every self-adjoint matrix in $SL_4(R)$ can be factorized as $S = gg^*$ for some $g \in SL_4(R)$.

We have already seen that the Spin group acts transitively in the case $n = 2$. A related open question is whether the Orthogonal group acts transitively on the unit sphere (set of (v, w) such that $v \cdot w^T = 1$) for any commutative ring R. In [S] (see Lemma 5.4) it has been proved that this is indeed the case when $n = 4$. The question is open for other dimensions.

8 The group $Epin_{2n}(R)$

Let ∂ denote the permutation $(1\ n+1)...(n\ 2n)$ corresponding to the form $\left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$. We define for $1 \leq i \neq j \leq 2n$, $z \in R$,

$$oe_{ij}(a) = I_{2n} + ae_{ij} - ae_{\partial(j)\partial(i)}.$$

It is clear that when $a \in R$ all these matrices belong to $O_{2n}(R)$. We call them the elementary orthogonal matrices over R and the group generated is called the elementary orthogonal group $EO_{2n}(R)$.

By definition there is a map $\pi : Spin_{2n}(R) \to O_{2n}(R)$ given by

$$\pi(g) : v \to gvg^{-1} \text{ for } g \in Spin_{2n}(R).$$

We denote by $Epin_{2n}(R)$ the inverse image of $EO_{2n}(R)$ under the map π.

Let $V = R^n$ with standard basis $e_1, \cdots, e_n$ and dual basis $f_1, \cdots, f_n$ for V^*. In terms of Suslin matrices, we have

$$e_i = \begin{pmatrix} 0 & S_{n-1}(e_i, 0) \\ \overline{S_{n-1}(e_i, 0)} & 0 \end{pmatrix}$$

and

$$f_i = \begin{pmatrix} 0 & S_{n-1}(0, f_i) \\ \overline{S_{n-1}(0, f_i)} & 0 \end{pmatrix}$$

Remark 8.1. Observe that if $\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_2 \end{smallmatrix}\right)$ is an element of the Spin group, then so is $\left(\begin{smallmatrix} g_2 & 0 \\ 0 & g_1 \end{smallmatrix}\right)$. We have

$$(g_2, g_1) = u \cdot (g_1, g_2) \cdot u$$

where $u = \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$ Below, we get a different basis of $Epin_{2n}(R)$ by conjugating its elements with u.

Remark 8.2. It can be proved (see [B2], section 4.3) that $Epin_{2n}(R)$ is generated by elements of the form $1 + ae_ie_j$ and $1 + af_if_j$ with $a \in R$, $1 \leq i,\ j \leq n$, $i \neq j$. Then it follows from the above symmetry of the Spin group that elements $u(1 + ae_ie_j)u$ and $u(1 + af_if_j)u$ also generate $Epin_{2n}(R)$ for $a \in R$. Therefore $Epin_{2n}(R)$ is generated by elements of the type

$$1 + ae'_ie'_j, \ \ 1 + af'_if'_j$$

where

$$e'_i = e_iu = \begin{pmatrix} S_{n-1}(e_i, 0) & 0 \\ 0 & \overline{S_{n-1}(e_i, 0)} \end{pmatrix}$$

and

$$f_i' = f_i u = \begin{pmatrix} S_{n-1}(0, f_i) & 0 \\ 0 & \overline{S_{n-1}(0, f_i)} \end{pmatrix}.$$

In the rest of the section we will work with the case where n is odd. Recalling Remark 5.4 we have

$$Spin_{2n}(R) \cong SG_{n-1}(R) := \{\ g \in G_{n-1}(R) |\ l(gg^*) = 1\},$$

where the isomorphism is the projection $\chi : (g, {g^*}^{-1}) \to g$.
Let $EG_{n-1}(R)$ denote the image of $Epin_{2n}(R)$ under the above isomorphism. Remark 8.2 tells us that the group $EG_{n-1}(R)$ is generated by elements of the type

$$1 + aE_iE_j,\ \ 1 + aF_iF_j$$

where

$$E_i = S_{n-1}(e_i, 0),\ \ F_i = S_{n-1}(0, f_i).$$

Notice that $E_1 = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $F_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$. For $i \neq 1$, the element E_i is of the form $\left(\begin{smallmatrix} 0 & X_i \\ -\bar{X}_i & 0 \end{smallmatrix}\right)$ for some Suslin matrix X_i and $l(E_i) = X_i\bar{X}_i = 0$.

The elements E_i, F_i satisfy the following properties :

i) $\bar{E}_1 = F_1$ and $E_1 + F_1 = I_{2^{n-1}}$.

i') $\bar{E}_i = -E_i$ and $\bar{F}_i = -F_i$ for $i \neq 1$.

ii) $E_1^2 = E_1$, $E_i^2 = 0$ for $i \neq 1$.

ii') $F_1^2 = F_1$, $F_i^2 = 0$ for $i \neq 1$.

iii) $E_iE_1 + E_1E_i = E_i$ for $i \neq 1$.

iii') $F_iE_1 + E_1F_i = F_i$ for $i \neq 1$.

iv) $E_iE_1 = F_1E_i$ and $F_iE_1 = F_1F_i$ for $i \neq 1$.

iv') $E_1E_i = E_iF_1$ and $E_1F_i = F_iF_1$ for $i \neq 1$.

It follows from properties ii) and iii) that $E_1E_iE_1 = 0 = E_iE_1E_i$. Similarly we have $F_1F_iF_1 = 0 = F_iF_1F_i$.

Using this one can prove the following commutator relations :

$$1 + aE_iE_j = [1 + aE_iE_1, 1 + E_1E_j]\ \ i \neq 1, j \neq 1,$$

$$1 + aF_iF_j = [1 + aF_iF_1, 1 + F_1F_j]\ \ i \neq 1, j \neq 1.$$

It follows from Remark 8.2 and the above commutator relations that $EG_{n-1}(R)$ is generated by elements of the type

$$1 + aE_1E_i,\ \ 1 + aE_iE_1,\ \ 1 + aF_1F_i,\ \ 1 + aF_iF_1,\ \ i \neq 1.$$

Definition 8.3. *An elementary matrix has 1's on the diagonal and at most one other entry is nonzero. Let $E_{ij}(x)$ denote the elementary matrix with x in the (i,j) position. The group generated by $n\times n$ elementary matrices is denoted by $E_n(R)$.*

Theorem 8.4. *$Epin_6(R) = E_4(R)$.*

Proof. Let $v = (a_1, a_2, a_3)$ and $w = (b_1, b_2, b_3)$. Then

$$S(v,w) = \begin{pmatrix} a_1 & 0 & a_2 & a_3 \\ 0 & a_1 & -b_3 & b_2 \\ -b_2 & a_3 & b_1 & 0 \\ -b_3 & -a_2 & 0 & b_1 \end{pmatrix}.$$

The group $EG_2(R)$ contains following matrices and their transposes :

$E_{13}(x) = 1 + xE_1E_2$,
$E_{14}(x) = 1 + xE_1E_3$,
$E_{24}(x) = 1 + xE_1F_2$,
$E_{23}(x) = 1 - xE_1F_3$.

Observe that the generators of $EG_2(R)$

$$1 + xE_1E_i,\ \ 1 + xE_iE_1,\ \ 1 + xF_1F_i,\ \ 1 + xF_iF_1,\ \ i \neq 1$$

are all elementary matrices. Therefore $EG_2(R) \subseteq E_4(R)$.

We also have

$$E_1^T = E_1,\ \ E_i^T = -F_i.$$

Thus if $g \in EG_2(R)$, then $g^T \in EG_2(R)$. Therefore to conclude that $E_4(R) \subseteq EG_2(R)$ it suffices to check that every elementary matrix with a nonzero entry above the diagonal falls into $EG_2(R)$.

It remains to be checked that $E_{12}(x)$ and $E_{34}(x)$ also fall in $EG_2(R)$. Indeed we have

$$E_{12}(x) = [E_{13}(x), E_{32}(1)]$$

and

$$E_{34}(x) = [E_{31}(x), E_{14}(1)].$$

□

Remark 8.5. The generators of $EG_{n-1}(R)$ mentioned above, i.e.

$$1 + xE_1E_i,\ \ 1 + xE_iE_1,\ \ 1 + xF_1F_i,\ \ 1 + xF_iF_1,\ \ i \neq 1$$

are nothing but the "top and bottom" matrices $E(e_i)(x)^{tb}$, $E(e_i^*)(x)^{tb}$ (defined in [JR]). These "top and bottom" matrices are another set of generators of the group $EUm_{n-1}(R)$ generated by the Suslin matrices $S(e_1\epsilon, e_1\epsilon^*)$ where $\epsilon \in E_n(R)$ and $\epsilon^* = (\epsilon^T)^{-1}$ (see [JR3], Proposition 2.6). Hence for odd n, we have

$$EUm_{n-1}(R) = EG_{n-1}(R) \cong Epin_{2n}(R).$$

Moreover since $Epin_{2n}(R)$ maps onto $EO_{2n}(R)$ it follows that $EUm_{n-1}(R)$ maps onto $EO_{2n}(R)$, the key result in developing the Quillen-Suslin theory for the pair $(SUm_{n-1}(R), EUm_{n-1}(R))$ where $SUm_{n-1}(R)$ is the group generated by Suslin matrices of unit length.

References

[B1] H. Bass. Lectures on topics in algebraic K-theory, Notes by Amit Roy, Tata Institute of Fundamental Research Lectures on Mathematics, 41, Tata Institute of Fundamental Research, Bombay, 1967.

[B2] H. Bass. Clifford algebras and spinor norms over a commutative ring. Amer. J. Math. 96:156206, 1974.

[JR] Selby Jose, Ravi A. Rao. A fundamental property of Suslin matrices, Journal of K -theory: K -theory and its Applications to Algebra, Geometry, and Topology 5 (2010), no. 3, 407436.

[JR2] Selby Jose, Ravi A. Rao. A Structure theorem for the Elementary Unimodular Vector group, Trans. Amer. Math. Soc. 358 (2005), no.7, 30973112.

[JR3] Selby Jose, Ravi A. Rao, A local global principle for the elementary unimodular vector group, Commutative Algebra and Algebraic Geometry (Bangalore, India, 2003), 119125, Contemp. Math. 390 (2005), Amer. Math. Soc., Providence, RI.

[K] M.-A. Knus (1991). Quadratic and Hermitian Forms over Rings. Grundlehren der Mathematischen Wissenschaften. Vol. 294. Berlin: Springer-Verlag.

[L] T. Y. Lam. Serre's problem on projective modules, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.

[S] A.A. Suslin. Stably Free Modules. (Russian) Math. USSR Sbornik 102 (144) (1977), no. 4, 537550. Mat. Inst. Steklov. (LOMI) 114 (1982), 187195.

Vineeth Reddy Chintala
Department of Mathematics
Tata Institute of Fundamental Research
Mumbai 400005
India
vineethreddy90@gmail.com