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Pointwise convergence of generalized Kantorovich exponential sampling series

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Abstract

The present paper is a continuation of the recent paper "A. Aral, T. Acar, S. Kursun, Generalized Kantorovich forms of exponential sampling series, Anal. Math. Pyh., 12:50, 1-19 (2022)" in which a new Kantorovich form of generalized exponential sampling series $K_w^{\chi,\mathcal{G}}$ has been introduced by means of Mellin Gauss Weierstrass singular integrals. In this paper, in order to investigate pointwise convergence of the family of operators $K_w^{\chi,\mathcal{G}}$, we first obtain an estimate for the remainder of Mellin-Taylor's formula and by this estimate we give the Voronovskaya theorem in quantitative form by means of Mellin derivatives. Furthermore, we present quantitative Voronovskaya theorem for difference of family of operators $K_w^{\chi,\mathcal{G}}$ and generalized exponential sampling series E_w^{χ}. The results are examined by illustrative numerical examples.

1 INTRODUCTION

The sampling type operators are approximate versions of the classical Whittaker-Kotel'nikov-Shannon sampling theorem (see, e.g., [29]). Such an approximation process aims to exactly reconstruct the target function f from its samples $f(k/w)$ taken on the set $\{k/w : k \in \mathbb{Z}\}$ of equally spaced points on the real line. The operators are given in the form

$$(S_w f)(t) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \operatorname{sinc}(wt-k), \quad t \in \mathbb{R}, w > 0, \tag{1}$$

where the sin$c$ function is defined by $\operatorname{sinc}(x) := \sin(\pi x)/(\pi x)$, for $x \neq 0$ $\operatorname{sinc}(0) = 1$.

The family of operators (1) and their Kantorovich forms are very useful from the theoretical point of view and applications as well. For the most recent results obtained by classical theory based on Fourier transform analysis, we refer the readers to [10, 18, 19, 25, 26] and references therein. In 1980's, researchers consisting of optical physicists and engineers introduced a mathematical method for the study of certain phenomena related to light scattering, diffraction, radio-astronomy (see, e.g., [17, 27, 28, 32]). Roughly speaking, their approach aimed to find a solution of integral equations of the type

$$h(t) = \int_0^\infty K(ts) f(s)\, ds,$$

where h is the data function, K is a kernel and f is the unknown function. The most suitable method for the solution was found in the theory of Mellin transform in which sampling type series was constructed with exponentially spaced samples. A Mellin transform theory completely independent of Fourier analysis was developed by Butzer and they introduced and intensively studied the exponential sampling series

$$\left(\tilde{E}_w f\right)(t) := \sum_{k\in\mathbb{Z}} f\left(e^{k/w}\right) \operatorname{lin}_{c/w}\left(e^{-k} t^w\right), \quad t \in \mathbb{R}^+ = (0, +\infty), w > 0, c \in \mathbb{R},$$

where the lin_c function is defined by $\operatorname{lin}_c(x) := x^{-c} \operatorname{sinc}(\log x)$ for $x \neq 1$ and $\operatorname{lin}_c(1) = 1$.

In [11], Bardaro et al. constructed a family of linear operators as a generalization of the exponential sampling series using instead of lin_c function using by arbitrary $\chi : \mathbb{R}^+ \to \mathbb{R}$ function satisfying certain assumptions of approximate identities given by

$$\left(E_w^{\chi} f\right)(t) := \sum_{k\in\mathbb{Z}} f\left(e^{k/w}\right) \chi\left(e^{-k} t^w\right), \quad t \in \mathbb{R}^+, w > 0 \tag{2}$$

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for any function $f:\mathbb{R}^+\to\mathbb{R}$ for which the series is absolutely convergent. Then, they investigated essential convergence results of them. The family of operators (2) has been studied by considering its different forms: Kantorovich forms in [7], Durrmeyer forms in [14], bivariate forms in [9], multivariate forms in [4, 30]. Another recent study on exponential sampling series is due to Aral et al. [8] in which authors constructed a new family of operators by generalizing Kantorovich type of exponential sampling series by replacing integral means over exponentially spaced intervals with its more general analogue, Mellin Gauss Weierstrass singular integrals. The operators are of the form

$$\begin{aligned}\left(K_w^{\chi,\mathcal{G}}f\right)(t) &:= \sum_{k\in\mathbb{Z}}\chi\left(e^{-k}t^w\right)\int_0^\infty \mathcal{G}_w(z)f\left(ze^{k/w}\right)\frac{dz}{z}\\ &= \sum_{k\in\mathbb{Z}}\chi\left(e^{-k}t^w\right)\int_0^\infty \mathcal{G}_w\left(ze^{-k/w}\right)f(z)\frac{dz}{z},\quad t\in\mathbb{R}^+, w>0, \end{aligned}\tag{3}$$

where function $f:\mathbb{R}^+\to\mathbb{R}$ is an integrable function on $\mathbb{R}^+$ such that the above series is convergent for every $x\in\mathbb{R}^+$ and $\mathcal{G}_w$ is the Mellin-Gauss-Weierstrass kernel in the following form

$$\mathcal{G}_w(z)=\frac{w}{\sqrt{4\pi}}\exp\left(-\left(\frac{w}{2}\log z\right)^2\right),\quad z\in\mathbb{R}^+, w>0.$$

After investigating basic convergence properties of the above family of operators, new weighted spaces of functions constructed by logarithmic weight were considered and the convergence properties of operators (3) were obtained for functions belonging to logarithmic weighted space of functions. In order to determine the rate of convergence, a new modulus of continuity, called "weighted logarithmic modulus of continuity" was introduced. Here we mention that weighted approximation of sampling type operators are very recent and active research area, for most recent paper on weighted approximation of classical sampling operators, we refer the readers to [1, 2, 5, 6]. However, rate of pointwise convergence and an upper estimate for pointwise convergence were not presented for the operators (3). This paper aims to solve this problem and to present such a results via weighted logarithmic modulus of continuity. To do this, we first obtain an estimate for the remainder of Mellin-Taylor's formula. Using the estimate, we present a quantitative Voronovskaya type result for the operators (3). We also obtain an estimate for the difference of the operators (3) and (2).

2 PRELIMINARIES

Let $C(\mathbb{R}^+)$ be the space of all continuous functions defined on $\mathbb{R}^+$. We will denote that $C_B(\mathbb{R}^+)$ is the space of all bounded functions $f\in C(\mathbb{R}^+)$.

We shall use the symbol $C^{(n)}(\mathbb{R}^+), n\in\mathbb{N}$ for a class of functions locally at the point $t\in\mathbb{R}^+$ if f is $(n-1)$-times differentiable in a neighborhood of t and the derivative $f^{(n)}(t)$ exists.

Moreover, we consider by $L^p(\mathbb{R}^+)$ for $1\le p<\infty$, space of all the Lebesgue measurable and p-integrable functions defined on $\mathbb{R}^+$, endowed with the usual norm $\|f\|_p$.

Let $c\in\mathbb{R}$. We consider the space

$$X_c=\left\{f:\mathbb{R}^+\to\mathbb{C}: f(\cdot)(\cdot)^{c-1}\in L^1\left(\mathbb{R}^+\right)\right\}$$

endowed with the norm

$$\|f\|_{X_c}=\left\|f(\cdot)(\cdot)^{c-1}\right\|_1=\int_0^\infty |f(u)|u^{c-1}du.$$

Then, the Mellin transform of $f\in X_c$ is given by

$$[f]_{\hat{M}}(s):=\int_0^\infty f(u)u^{s-1}du,\quad s=c+it;\ c,t\in\mathbb{R},$$

where i is the complex unit (see [20]).

The Mellin differential operator Θ or the Mellin derivative of $f:\mathbb{R}^+\to\mathbb{R}$ at the point $x\in\mathbb{R}^+$ is defined by

$$\Theta f(x)=xf'(x)$$

provided the usual derivative f' exists at the point x. Subsequently, the Mellin differential operator of order $r\in\mathbb{N}$ can be written as

$$\Theta^r:=\Theta\left(\Theta^{r-1}\right).$$

For more details, we refer the readers to [20].

Throughout the paper a continuous function $\chi:\mathbb{R}^+\to\mathbb{R}$ is called a *kernel*, if the following assumptions are satisfied:

i)

$$\sum_{k\in\mathbb{Z}} \chi\left(e^{-k}u\right) = 1 \text{ for every } u \in \mathbb{R}^+$$

and

$$M_0(\chi) := \sup_{u\in\mathbb{R}^+} \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}u\right)\right| < \infty,$$

ii) for every $r > 0$,

$$\lim_{r\to\infty} \sum_{|k-\log u|>r} \left|\chi\left(e^{-k}u\right)\right| = 0$$

holds uniformly with respect to $u \in \mathbb{R}^+$.

We will denote by Φ the class of functions satisfying the conditions *i*) and *ii*).

Let $j \in \mathbb{N}$. The algebraic moments of $\chi \in \Phi$ of order j are defined as

$$m_j(\chi, x) := \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x\right) \log^j\left(e^k x^{-1}\right) = \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x\right)(k - \log x)^j, x \in \mathbb{R}^+. \tag{4}$$

Similarly, the absolute moments of $\chi \in \Phi$ of arbitrary order $\alpha > 0$ can be defined by

$$M_\alpha(\chi, x) := \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x\right)\right| |k - \log x|^\alpha, x \in \mathbb{R}^+.$$

We will put $M_\alpha(\chi) := \sup_{x\in\mathbb{R}^+} M_\alpha(\chi, x)$.

Remark 1. As it is shown in [15], if $M_\alpha(\varphi) < \infty$, then $M_\beta(\varphi) < \infty$ for $0 \le \beta < \alpha$. Moreover if χ is a kernel with compact support, then its all absolute moments of any order are finite.

Now, let us consider the following logarithmic weighted space of continuous functions and its natural subspaces considered in [8]

$$\begin{aligned}
B_2\left(\mathbb{R}^+\right) &: = \left\{ f : \mathbb{R}^+ \to \mathbb{R} : \exists M > 0 \text{ such that } \frac{|f(x)|}{1+\log^2 x} \le M \text{ for every } x \in \mathbb{R}^+ \right\}, \\
C_2\left(\mathbb{R}^+\right) &: = C\left(\mathbb{R}^+\right) \cap B_2\left(\mathbb{R}^+\right) \text{ and} \\
C_2^*\left(\mathbb{R}^+\right) &: = \left\{ f \in C_2\left(\mathbb{R}^+\right) : \lim_{x\to+\infty} \frac{|f(x)|}{1+\log^2 x} \in \mathbb{R} \right\}.
\end{aligned}$$

The linear space of functions $B_2(\mathbb{R}^+)$ is normed linear space with the norm

$$\|f\|_L := \sup_{x>0} \frac{|f(x)|}{1+\log^2 x}.$$

Finally, we recall weighted logarithmic modulus of continuity defined in [8]. For $f \in C_2(\mathbb{R}^+)$ and $\delta > 0$, the weighted logarithmic modulus of continuity is considered as

$$\Omega(f, \delta) := \sup_{|\log t|\le\delta,\, x>0} \frac{|f(tx) - f(x)|}{\left(1+\log^2 x\right)\left(1+\log^2 t\right)}. \tag{5}$$

(5) has the following fundamental properties.

Lemma 2.1. *([8]) Let $\delta > 0$. Then*

a.) for $f \in C_2(\mathbb{R}^+)$, the quantity $\Omega(f, \delta)$ is finite,

b.) for all $f \in C_2(\mathbb{R}^+)$ and each $\lambda \in \mathbb{R}^+$,

$$\Omega(f; \lambda\delta) \le 2(1+\lambda)^3\left(1+\delta^2\right)\Omega(f, \delta), \tag{6}$$

c.) for all $f \in C_2(\mathbb{R}^+)$,

$$|f(h) - f(x)| \le 16\left(1+\delta^2\right)^2\left(1+\log^2 x\right)\left(1 + \frac{|\log h - \log x|^5}{\delta^5}\right)\Omega(f, \delta), \quad h, x > 0, \tag{7}$$

d.) for $f \in C_2^*(\mathbb{R}^+)$,

$$\lim_{\delta\to 0} \Omega(f, \delta) = 0.$$

Remark 2. $\Omega(f, \delta)$ is a monotonically increasing function of δ.

3 MAIN RESULTS

First of all, we recall the Mellin-Taylor formula. For any $f \in C_B(\mathbb{R}^+)$ belonging to $C^{(n)}$ locally at the point $x \in \mathbb{R}^+$, the Mellin-Taylor formula with Mellin derivatives is defined by (see [12, 31])

$$f(tx) = f(x) + \Theta f(x)\log t + \frac{\Theta^2 f(x)}{2!}\log^2 t + \dots + \frac{\Theta^n f(x)}{n!}\log^n t + h(t)\log^n t, \quad n \in \mathbb{N};\ t, x > 0, \tag{8}$$

where $h : \mathbb{R}^+ \to \mathbb{R}$ is a bounded function such that $\lim_{t\to 1} h(t) = 0$. The expression (8) can be rewritten with the variable substitution $tx = u$, $u \in \mathbb{R}^+$ as follows:

$$\begin{aligned} f(u) &= f(x) + \Theta f(x)(\log u - \log x) + \frac{\Theta^2 f(x)}{2!}(\log u - \log x)^2 + \dots \\ &+ \frac{\Theta^n f(x)}{n!}(\log u - \log x)^n + h\left(\frac{u}{x}\right)(\log u - \log x)^n . \end{aligned}$$

Then we have

$$f(u) = \sum_{i=0}^{n} \frac{1}{i!}\Theta^i f(x)(\log u - \log x)^i + R_n(u), \tag{9}$$

where

$$R_n(u) = \frac{(\Theta^n f(\xi) - \Theta^n f(x))}{n!}(\log u - \log x)^n$$

is the Lagrange remainder in Mellin-Taylor's formula at the point $x \in \mathbb{R}^+$ and ξ is a suitable number lying between u and x.

Proposition 3.1. *Let $\Theta^n f \in C_2(\mathbb{R}^+)$. Then we have that*

$$|R_n(u)| \le \frac{64}{n!}\left(1 + \log^2 x\right)\Omega(\Theta^n f, \delta)\left(|\log u - \log x|^n + \frac{|\log u - \log x|^{n+5}}{\delta^5}\right) \tag{10}$$

for $u, x > 0$ and $\delta \le 1$.

Proof. Let $\Theta^n f \in C_2(\mathbb{R}^+)$. Similar to operations in [3], using the Remark 2, the inequality $|\log \xi - \log x| \le |\log u - \log x|$ and the inequality (6), from the definition of the weighted logarithmic modulus of continuity, we can easily get the estimates

$$\begin{aligned} \left|\bar{R}_n(u)\right| &:= \frac{1}{n!}|\Theta^n f(\xi) - \Theta^n f(x)| \\ &\le \frac{1}{n!}\Omega(\Theta^n f, |\log \xi - \log x|)\left(1 + \log^2 x\right)\left(1 + (\log \xi - \log x)^2\right) \\ &\le \frac{1}{n!}\Omega(\Theta^n f, |\log u - \log x|)\left(1 + \log^2 x\right)\left(1 + (\log u - \log x)^2\right) \\ &\le \frac{2}{n!}\left(1 + \frac{|\log u - \log x|}{\delta}\right)^3\left(1 + \delta^2\right)\left(1 + \log^2 x\right)\left(1 + (\log u - \log x)^2\right)\Omega(\Theta^n f, \delta). \end{aligned}$$

Now, we obtain

$$\left|\bar{R}_n(u)\right| \le \begin{cases} \frac{16}{n!}\left(1 + \delta^2\right)^2\left(1 + \log^2 x\right)\Omega(\Theta^n f, \delta) & , \quad |\log u - \log x| \le \delta \\ \frac{16}{n!}\left(1 + \delta^2\right)^2\left(1 + \log^2 x\right)\frac{|\log u - \log x|^5}{\delta^5}\Omega(\Theta^n f, \delta) & , \quad |\log u - \log x| > \delta \end{cases}.$$

Then, by combining the two cases with choise of $\delta \le 1$ we get desired result (10). □

Theorem 3.2. *Let $\chi \in \Phi$ and $\Theta f \in C_2^*(\mathbb{R}^+)$. Supposing that $M_6(\chi) < \infty$ and $m_1(\chi, x^w) = m_1(\chi) \ne 0 < \infty$, we have*

$$\begin{aligned} &\left|w\left[\left(K_w^{\chi,\mathcal{G}} f\right)(x) - f(x)\right] - \Theta f(x)\, m_1(\chi)\right| \\ \le\ & 64\left(1 + \log^2 x\right)\Omega\left(\Theta f, \frac{1}{w}\right)\left(M_0(\chi)\left(\frac{2}{\sqrt{\pi}} + 3840\right) + M_1(\chi) + 32M_6(\chi)\right). \end{aligned}$$

Proof. Applying the Mellin-Taylor formula (9) to f at $x \in \mathbb{R}^+$, and considering the equality

$$\int_0^\infty \mathcal{G}_w\left(z e^{-k/w}\right)\frac{dz}{z} = \int_0^\infty \mathcal{G}_w(z)\frac{dz}{z} = 1,$$

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we get

$$\begin{aligned}\left(K_w^{\chi,\mathcal{G}}f\right)(x) &= \sum_{k\in\mathbb{Z}}\chi\left(e^{-k}x^w\right)\int_0^\infty \mathcal{G}_w\left(ze^{-k/w}\right)\left(f(x)+\Theta f(x)(\log z-\log x)\right)\frac{dz}{z}\\ &+ \sum_{k\in\mathbb{Z}}\chi\left(e^{-k}x^w\right)\int_0^\infty \mathcal{G}_w(z)R_1\left(ze^{k/w}\right)\frac{dz}{z}\\ &= f(x)+\frac{\Theta f(x)}{w}m_1(\chi)+\sum_{k\in\mathbb{Z}}\chi\left(e^{-k}x^w\right)\int_0^\infty \mathcal{G}_w(z)R_1\left(ze^{k/w}\right)\frac{dz}{z}.\end{aligned}$$

Now, let $I:=\sum_{k\in\mathbb{Z}}\chi\left(e^{-k}x^w\right)\int_0^\infty \mathcal{G}_w(z)R_1\left(ze^{k/w}\right)\frac{dz}{z}$. Thanks to inequality (10), we can write that

$$\begin{aligned}|I| &\leq \sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\int_0^\infty \mathcal{G}_w(z)\left|R_1\left(ze^{k/w}\right)\right|\frac{dz}{z}\\ &\leq 64\left(1+\log^2 x\right)\Omega(\Theta f,\delta)\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\\ &\times \int_0^\infty \mathcal{G}_w(z)\left(\left|\log\left(ze^{k/w}\right)-\log x\right|+\frac{\left|\log\left(ze^{k/w}\right)-\log x\right|^6}{\delta^5}\right)\frac{dz}{z}\\ &\leq 64\left(1+\log^2 x\right)\Omega(\Theta f,\delta)\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\\ &\times \int_0^\infty \mathcal{G}_w(z)\left(|\log z|+\frac{|k-w\log x|}{w}+\frac{2^5}{\delta^5}\left(|\log z|^6+\frac{|k-w\log x|^6}{w^6}\right)\right)\frac{dz}{z}\\ &= 64\left(1+\log^2 x\right)\Omega(\Theta f,\delta)\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\left\{\frac{2}{\sqrt{\pi}w}+\frac{|k-w\log x|}{w}+\frac{2^5}{\delta^5w^6}\left(120+|k-w\log x|^6\right)\right\}\\ &\leq 64\left(1+\log^2 x\right)\Omega(\Theta f,\delta)\left\{\frac{1}{w}\left(\frac{2}{\sqrt{\pi}}M_0(\chi)+M_1(\chi)\right)+\frac{2^5}{\delta^5w^6}\left(120M_0(\chi)+M_6(\chi)\right)\right\}.\end{aligned}$$

Setting $\delta=w^{-1}$, this completes the proof. □

Theorem 3.3. *Let $\chi\in\Phi$ and $\Theta^2 f\in C_2^*(\mathbb{R}^+)$. Supposing that $M_7(\chi)<\infty$, then we have*

$$\begin{aligned}&\left|w^2\left[\left(K_w^{\chi,\mathcal{G}}-E_w^{\chi}\right)(f)(x)\right]-\Theta^2 f(x)\right|\\ \leq\ & 32\left(1+\log^2 x\right)\Omega\left(\Theta^2 f,\frac{1}{w}\right)\left[M_0(\chi)\left(4+\frac{98304}{\sqrt{\pi}}\right)+3M_2(\chi)+129M_7(\chi)\right].\end{aligned}$$

Proof. Let $\chi \in \Phi$. Since f'' exists at the point $x \in \mathbb{R}^+$, we have

$$\begin{aligned}
&\left(K_w^{\chi,\mathcal{G}} - E_w^{\chi}\right)(f)(x) \\
=\ & \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \int_0^\infty \mathcal{G}_w\left(ze^{-k/w}\right)\left(f(z) - f\left(e^{k/w}\right)\right)\frac{dz}{z} \\
=\ & \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \int_0^\infty \mathcal{G}_w\left(ze^{-k/w}\right)\Big\{ f(x) + \Theta f(x)(\log z - \log x) + \frac{\Theta^2 f(x)}{2!}(\log z - \log x)^2 \\
& - f(x) - \Theta f(x)\left(\frac{k}{w} - \log x\right) - \frac{\Theta^2 f(x)}{2!}\left(\frac{k}{w} - \log x\right)^2 \Big\} \frac{dz}{z} \\
+\ & \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \int_0^\infty \mathcal{G}_w\left(ze^{-k/w}\right)\left[R_2(z) - R_2\left(e^{k/w}\right)\right]\frac{dz}{z} \\
=\ & \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \frac{\Theta^2 f(x)}{2!}\left\{\left(\frac{k}{w} - \log x\right)^2 + \frac{2}{w^2} - \left(\frac{k}{w} - \log x\right)^2\right\} \\
+\ & \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \int_0^\infty \mathcal{G}_w(z)\left[R_2\left(ze^{k/w}\right) - R_2\left(e^{k/w}\right)\right]\frac{dz}{z} \\
=\ & \frac{\Theta^2 f(x)}{w^2} + \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \int_0^\infty \mathcal{G}_w(z)\left[R_2\left(ze^{k/w}\right) - R_2\left(e^{k/w}\right)\right]\frac{dz}{z}.
\end{aligned}$$

Let $I := \sum_{k\in\mathbb{Z}} \chi\left(e^{-k}x^w\right) \int_0^\infty \mathcal{G}_w(z)\left[R_2\left(ze^{k/w}\right) - R_2\left(e^{k/w}\right)\right]\frac{dz}{z}$. Now, we estimate I. We have

$$\begin{aligned}
|I| &\leq \sum_{k\in\mathbb{Z}} \left|\chi\left(e^{-k}x^w\right)\right| \int_0^\infty \mathcal{G}_w(z)\left[\left|R_2\left(ze^{k/w}\right)\right| + \left|R_2\left(e^{k/w}\right)\right|\right]\frac{dz}{z} \\
&:= I_1 + I_2.
\end{aligned}$$

Since $\Theta^2 f \in C_2^*(\mathbb{R}^+)$, using the inequality (10), we obtain

$$\begin{aligned}
I_1 &\leq 32\left(1+\log^2 x\right)\Omega\left(\Theta^2 f, \delta\right)\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right| \\
&\times \int_0^\infty \mathcal{G}_w(z)\left\{\left|\log\left(ze^{k/w}\right) - \log x\right|^2 + \frac{\left|\log\left(ze^{k/w}\right) - \log x\right|^7}{\delta^5}\right\}\frac{dz}{z} \\
&\leq 32\left(1+\log^2 x\right)\Omega\left(\Theta^2 f, \delta\right)\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right| \\
&\times \int_0^\infty \mathcal{G}_w(z)\left\{2\left(|\log z|^2 + \frac{|k - w\log x|^2}{w^2}\right) + \frac{2^7}{\delta^5}\left(|\log z|^7 + \frac{|k - w\log x|^7}{w^7}\right)\right\}\frac{dz}{z} \\
&= 32\left(1+\log^2 x\right)\Omega\left(\Theta^2 f, \delta\right)\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right|\left[2\left(\frac{2 + |k - w\log x|^2}{w^2}\right) + \frac{2^7}{\delta^5 w^7}\left(\frac{768}{\sqrt{\pi}} + |k - w\log x|^7\right)\right] \\
&\leq 32\left(1+\log^2 x\right)\Omega\left(\Theta^2 f, \delta\right)\left\{\frac{1}{w^2}\left(4M_0(\chi) + 2M_2(\chi)\right) + \frac{2^7}{\delta^5 w^7}\left(\frac{768}{\sqrt{\pi}}M_0(\chi) + M_7(\chi)\right)\right\}
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq 32\left(1+\log^2 x\right)\Omega\left(\Theta^2 f, \delta\right)\sum_{k\in\mathbb{Z}}\left|\chi\left(e^{-k}x^w\right)\right| \\
&\times \int_0^\infty \mathcal{G}_w(z)\left\{\frac{|k - w\log x|^2}{w^2} + \frac{|k - w\log x|^7}{\delta^5 w^7}\right\} \\
&\leq 32\left(1+\log^2 x\right)\Omega\left(\Theta^2 f, \delta\right)\left(\frac{M_2(\chi)}{w^2} + \frac{M_7(\chi)}{\delta^5 w^7}\right),
\end{aligned}$$

respectively. Now choosing $\delta = w^{-1}$, this gives the result which is desired. □

4 APPLICATIONS OF CERTAIN KERNELS

In this section, we give numerical applications of certain kernels.

4.1 Mellin-Spline Kernel

For $u \in \mathbb{R}^+$ and $n \in \mathbb{N}$, we recall the Mellin-Spline kernel of order n which is the Mellin version of classical B-Spline given by

$$B_n(u) := \frac{1}{(n-1)!}\sum_{i=0}^{n}(-1)^i \binom{n}{i}\left(\frac{n}{2}+\log u - i\right)_+^{n-1}, \quad u \in \mathbb{R}^+,$$

where r_+ denotes the positive part of the numbers r. Note that, B_n is a kernel with compact support. For $c \in \mathbb{R}$, the Mellin transform of B_n is given by

$$[B_n]_{\hat{M}}(c+iv) = \left(\frac{\sin(v/2)}{v/2}\right)^n, \quad v \in \mathbb{R}\setminus\{0\}$$

(see [11]). Using the Mellin-Poisson summation formula (see [22, 23]) which is the form

$$(i)^j \sum_{k\in\mathbb{Z}} B_n\left(e^k t\right)(k-\log t)^j = \sum_{k\in\mathbb{Z}} [B_n]_M^{(j)}(2k\pi i)\, t^{-2k\pi i}, \quad t \in \mathbb{R}^+,$$

we can show

$$\sum_{k\in\mathbb{Z}} B_n\left(e^k t\right) = 1 \text{ for every } t \in \mathbb{R}^+,$$

since

$$[B_n]_{\hat{M}}(2k\pi i) = \begin{cases} 1, & k=0 \\ 0, & \text{otherwise} \end{cases}.$$

Since B_n has the compact support, all the algebraic and absolute moments are finite. $m_j(B_n, x)$ are independent of x for $j = 1,\dots,n$ (see [13]). For a particular case, we consider the Mellin-Spline kernel of order 3 given by

$$B_3(x) = \begin{cases} \frac{1}{2}\left(\frac{3}{2}+\log x\right)^2 & , \quad e^{-3/2} < x \le e^{-1/2} \\ \frac{3}{4} - \log^2 x & , \quad e^{-1/2} < x \le e^{1/2} \\ \frac{1}{2}\left(\frac{3}{2}-\log x\right)^2 & , \quad e^{1/2} < x < e^{3/2} \\ 0 & , \quad \text{otherwise} \end{cases}.$$

It can be shown the equalities

$$m_1(B_3) = 0 \text{ and } m_2(B_3) = \frac{1}{4}$$

hold using the Mellin-Poisson summation formula.

Now, we consider the function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log^3 x$. Then $\Theta^2 f$ belongs to space $C_2^*(\mathbb{R}^+)$. Thus supposing that $m_1(\chi) = 0$, $m_2(\chi) \neq 0 < \infty$ and applying similar operations to Theorem 3.2, we obtain

$$\lim_{w\to\infty} w^2\left[\left(K_w^{B_3,\mathcal{G}} f\right)(x) - f(x)\right] = \frac{\Theta^2 f(x)}{4}$$

and we have

$$\lim_{w\to\infty} w^2\left[\left(K_w^{B_3,\mathcal{G}} - E_w^{B_3}\right)(f)(x)\right] = \Theta^2 f(x).$$

We present numerical results for Theorem 3.2 and Theorem 3.3 in the following tables, respectively.

TABLE 1

w	$\left\lvert\left(K_w^{B_3,\mathcal{G}} f\right)(2) - f(2)\right\rvert$	$\left\lvert\left(K_w^{B_3,\mathcal{G}} f\right)(3) - f(3)\right\rvert$	$\left\lvert\left(K_w^{B_3,\mathcal{G}} f\right)(5) - f(5)\right\rvert$
5	0.1872730286158768	0.2966524971165803	0.4346417748071320
10	0.0467706244553701	0.0741528628791428	0.1086598131959710
50	0.0018711339217417	0.0029661170806456	0.0043455854803997
100	0.0004679218543695	0.0007415312744427	0.0010863567162908
150	0.0002079420922546	0.0003295709399606	0.0004828408827284
300	0.0000519855279017	0.0000823947392714	0.0001207064596655

TABLE 2

w	$\left\|\left(K_w^{B_3,\mathcal{G}} - E_w^{B_3}\right)(f)(2)\right\|$	$\left\|\left(K_w^{B_3,\mathcal{G}} - E_w^{B_3}\right)(f)(3)\right\|$	$\left\|\left(K_w^{B_3,\mathcal{G}} - E_w^{B_3}\right)(f)(5)\right\|$
5	0.1663553233343869	0.2636669492803463	0.3862650989841841
10	0.0415888308335967	0.0659167373200866	0.0965662747460460
50	0.0016635532333439	0.0026366694928035	0.0038626509898418
100	0.0004158883083360	0.0006591673732009	0.0009656627474605
150	0.0001848392481493	0.0002929632769782	0.0004291834433158
300	0.0000462098120373	0.0000732408192445	0.0001072958608289

4.2 Translates of Mellin-Spline Kernel

For

$$c_1 + c_2 = 1 \text{ and } c_1 \log a + c_2 \log b = 0,$$

taking into account that the linear combinations of translates of classical central B-Splines of order n, translates of Mellin-Splines can be considered as follows (see [16]):

$$\bar{B}_n(u) = c_1 B_n(au) + c_2 B_n(bu), \quad \forall u \in \mathbb{R}^+, \ a, b \in \mathbb{R}.$$

Particularly, for $a = e^{-2}, b = e^{-3}$ and $n = 2$, we obtain the combination of the Mellin-Spline of order 2 as the following:

$$\bar{B}_2(u) = 3B_2\left(e^{-2}u\right) - 2B_2\left(e^{-3}u\right).$$

Using the Mellin-Poisson summation formula, we obtain $m_0\left(\bar{B}_2\right) = 1, m_1\left(\bar{B}_2\right) = 0$ and $m_2\left(\bar{B}_2\right) = \frac{17}{3}$.

Finally, let us consider the function $g : \mathbb{R}^+ \to \mathbb{R}, g(x) = \cos(4 \log t)$. Then, we have $\Theta^2 f \in C_2^*(\mathbb{R}^+)$. Thus supposing that $m_1(\chi) = 0$, $m_2(\chi) \neq 0 < \infty$ and applying similar operations to Theorem 3.2, we obtain

$$\lim_{w \to \infty} w^2 \left[\left(K_w^{\bar{B}_2,\mathcal{G}} g\right)(x) - g(x)\right] = \frac{17}{3}\Theta^2 g(x)$$

and we have

$$\lim_{w \to \infty} w^2 \left[\left(K_w^{\bar{B}_2,\mathcal{G}} - E_w^{\bar{B}_2}\right)(g)(x)\right] = \Theta^2 g(x).$$

We present numerical results for Theorem 3.2 and Theorem 3.3 in the following tables, respectively.

TABLE 3

w	$\left\|\left(K_w^{\bar{B}_2,\mathcal{G}} g\right)(1.8) - g(1.8)\right\|$	$\left\|\left(K_w^{\bar{B}_2,\mathcal{G}} g\right)(2.9) - g(2.9)\right\|$	$\left\|\left(K_w^{\bar{B}_2,\mathcal{G}} g\right)(4.4) - g(4.4)\right\|$
10	0.0639793630169120	0.2782884244376554	0.0561806829306111
40	0.0091801760237817	0.0123172422229221	0.0154190978988109
90	0.0023870235083690	0.0020476067499788	0.0033124971943863
150	0.0008954759183915	0.0006740868530503	0.0012350835854185
250	0.0003407971636637	0.0002343881464958	0.0004433267727974
300	0.0002275847543150	0.0001569067489326	0.0003080113499111

TABLE 4

w	$\left\|\left(K_w^{\bar{B}_2,\mathcal{G}} - E_w^{\bar{B}_2}\right)(g)(1.8)\right\|$	$\left\|\left(K_w^{\bar{B}_2,\mathcal{G}} - E_w^{\bar{B}_2}\right)(g)(2.9)\right\|$	$\left\|\left(K_w^{\bar{B}_2,\mathcal{G}} - E_w^{\bar{B}_2}\right)(g)(4.4)\right\|$
10	0.1109687160667529	0.1243108626713763	0.1723329865415991
40	0.0071628405967575	0.0045273294149397	0.0095722804428460
90	0.0013957789938270	0.0008703969763847	0.0018593039529166
150	0.0005011018165399	0.0003121677934811	0.0006674485698626
250	0.0001802135891177	0.0001122422564383	0.0002400240938882
300	0.0001251233025652	0.0000779291871146	0.0001666528210387

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