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Asymptotic Approximate Fekete Arrays

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Abstract

The notion of asymptotic Fekete arrays, arrays of points in a compact set $K \subset \mathbb{C}^d$ which behave asymptotically like Fekete arrays, has been well-studied, albeit much more recently in dimensions $d > 1$. Here we show that one can allow a more flexible definition where the points in the array need not lie in K. Our results, which work in the general setting of weighted pluripotential theory, rely heavily, in the multidimensional setting, on the ground-breaking work of Berman, Boucksom and Nystrom from [3].

1 Introduction

Let K be a compact set in $\mathbb{C}$. For $n = 1, 2, ...$

$$\delta_n(K) := \max_{z_0,...,z_n \in K} \prod_{j<k} |z_j - z_k|^{1/\binom{n+1}{2}}$$

is called the $n - th$ order diameter of K. Note that

$$VDM(z_0, ..., z_n) = \det[z_i^j]_{i,j=0,1,...,n} = \prod_{j<k}(z_j - z_k)$$

$$= \det \begin{bmatrix} 1 & z_0 & \dots & z_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_n & \dots & z_n^n \end{bmatrix}$$

is a classical Vandermonde determinant; the basis monomials $1, z, ..., z^n$ for the space of polynomials of degree at most n are evaluated at the points $z_0, ..., z_n$. The sequence of numbers $\{\delta_n(K)\}$ is decreasing and hence the limit

$$\lim_{n\to\infty} \left[\max_{\lambda_i \in K} |VDM(\lambda_0, ..., \lambda_n)\right]^{1/\binom{n+1}{2}} := \delta(K) \tag{1}$$

exists and is called the transfinite diameter of K. Points $z_{n0}, ..., z_{nn} \in K$ for which

$$|VDM(z_{n0}, ...z_{nn})| = |\det \begin{bmatrix} 1 & z_{n0} & \dots & z_{n0}^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_{nn} & \dots & z_{nn}^n \end{bmatrix}|$$

is maximal are called Fekete points of order n for K. For such Fekete arrays $\{z_{nj}\}_{j=0,...,n;\ n=1,2,...}$, it is classical that if K is not polar, then

$$\mu_n := \frac{1}{n+1}\sum_{j=0}^{n} \delta_{z_{nj}} \to \mu_K \text{ weak-*} \tag{2}$$

where μ_K is the equilibrium measure for K; i.e., the probability measure on K of minimal logarithmic energy. In fact, for *asymptotically Fekete arrays* $\{z_{nj}\}_{j=0,...,n;\ n=1,2,...} \subset K$, i.e., such that

$$\lim_{n\to\infty} |VDM(z_{n0}, ..., z_{nn})|^{1/\binom{n+1}{2}} = \delta(K), \tag{3}$$

(2) holds.

There are higher dimensional analogues of asymptotically Fekete arrays and equilibrium measures associated to nonpluripolar compact sets K in $\mathbb{C}^d$, $d > 1$; cf., [13] for the definition and properties of transfinite diameter in this setting. A striking generalization of (2) was achieved by Berman, Boucksom and Nystrom in [3]. There the authors proved a much more general result, one which, in the setting of the current work, requires *weighted* pluripotential theory. In both the univariate and multivariate cases, one regularly encounters situations where "near extremal" arrays contain points lying *outside* of K; cf., [1] and [11]. It is

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the purpose of this note to show that, by suitably modifying the arguments in [3], one still recovers the appropriate generalization of (2), even in the weighted setting.

In the next section, we repeat a standard argument from [7] to deal with the weighted, univariate case. Section 3 gives the necessary definitions and background in weighted pluripotential theory, and section 4 follows the strategy in [3] to prove the main result on asymptotic, approximate weighted Fekete arrays in $\mathbb{C}^d$, $d > 1$. Using the notation and terminology from those sections, here is the statement:

Theorem 1.1. *Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar, let $\{K_n\}$ be a sequence of compact sets which decrease to K; i.e., $K_{n+1} \subset K_n$ for all K and $K = \cap_n K_n$, and let w be an admissible weight function on K_n for $n \geq n_0$ whose restriction to K is admissible. For each n, take points $x_1^{(n)}, x_2^{(n)}, \cdots, x_{m_n}^{(n)} \in K_n$ for which*

$$\lim_{n\to\infty} \left[|VDM(x_1^{(n)}, \cdots, x_{m_n}^{(n)})| w(x_1^{(n)})^n w(x_2^{(n)})^n \cdots w(x_{m_n}^{(n)})^n \right]^{\frac{1}{l_n}} = \delta^w(K) \tag{4}$$

and let $\mu_n := \frac{1}{m_n} \sum_{j=1}^{m_n} \delta_{x_j^{(n)}}$. Then

$$\mu_n \to \mu_{K,Q} := (dd^c V_{K,Q}^*)^d \ weak-*.$$

In fact this is a special case of a general result on a sequence $\{\mu_n\}$ of measures, μ_n supported on K_n, given as Theorem 4.8 in section 4.

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2 The univariate case

Let $K \subset \mathbb{C}$ be compact and let $\mathcal{M}(K)$ denote the convex set of probability measures on K. For $\mu \in \mathcal{M}(K)$ define the logarithmic energy

$$I(\mu) := \int_K \int_K \log \frac{1}{|z-\zeta|} d\mu(z) d\mu(\zeta).$$

Either $\inf_{\mu\in\mathcal{M}(K)} I(\mu) =: I(\mu_K) < +\infty$ for a unique $\mu_K \in \mathcal{M}(K)$ or else $I(\mu) = +\infty$ for all $\mu \in \mathcal{M}(K)$; this latter occurs when K is polar. Suppose K is nonpolar. Let $\{\epsilon_n\}$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty} \epsilon_n = 0$; and set

$$K_n := \{z \in \mathbb{C}^d : \mathrm{dist}(z, K) \leq \epsilon_n\}.$$

These are compact sets decreasing to K (in fact, regular compacta; cf., [9]); thus

$$\lim_{n\to\infty} \delta(K_n) = \delta(K).$$

Given an array of points $\{z_{nj}\}_{j=0,\ldots,n;\ n=1,2,\ldots}$ where $z_{n0}, \ldots, z_{nn} \in K_n$, it follows from modifying the standard proof for asymptotic Fekete arrays in K that under the condition

$$\lim_{n\to\infty} |VDM(z_{n0}, \ldots z_{nn})|^{\frac{1}{\binom{n+1}{2}}} = \delta(K),$$

we have

$$\mu_n := \frac{1}{n+1} \sum_{j=0}^{n} \delta_{z_{nj}} \to \mu_K \text{ weak-*}.$$

The idea, utilized in [4], is simply to show that, if one replaces μ_n by $\widetilde{\mu}_n$ by spreading the point masses to little disks or circles centered at these points with radii $r_n \to 0$ where $r_n \leq \epsilon_n$, then the hypothesis on the VDM's gives that $\widetilde{\mu}_n \to \mu_K$ weak-* (any weak-* limit of $\{\widetilde{\mu}_n\}$ has logarithmic energy equal to that of μ_K, hence it must equal μ_K). Any weak-* limit of $\{\mu_n\}$ is supported on K and coincides with the weak-* limit of $\{\widetilde{\mu}_n\}$. In particular, no condition on ϵ_n other than $\lim_{n\to\infty} \epsilon_n = 0$ is necessary. The same conclusion holds for any sequence $\{K_n\}$ of compact sets which decrease to K; i.e., $K_{n+1} \subset K_n$ for all K and $K = \cap_n K_n$, as follows from Proposition 2.1 below.

A similar result holds for a weighted situation where w is an admissible weight function on K_n for $n \geq n_0$ whose restriction to K is admissible. We refer the reader to [12] for details of this theory. Let $K \subset \mathbb{C}$ be compact and let w be an admissible weight function on K: w is a nonnegative, uppersemicontinuous function with $\{z \in K : w(z) > 0\}$ nonpolar – hence K is not polar. We write $Q := -\log w$. Associated to K, Q is a *weighted energy minimization problem*: for a probability measure τ on K, we consider the weighted energy

$$I^w(\tau) := \int_K \int_K \log \frac{1}{|z-t| w(z) w(t)} d\tau(t) d\tau(z) = I(\tau) + 2\int_K Q d\tau$$

and find $\inf_\tau I^w(\tau)$ where the infimum is taken over all probability measures τ with compact support in K. There exists a unique minimizer which we denote by $\mu_{K,Q}$. The associated discrete problem leads to the *weighted transfinite diameter of K with respect to w*:

$$\delta^w(K) := \lim_{n\to\infty} \left[\max_{\lambda_i \in K} |VDM(\lambda_0, \ldots, \lambda_n)| w(\lambda_0)^n \cdots w(\lambda_n)^n \right]^{1/\binom{n+1}{2}} := \lim_{n\to\infty} \delta_n^w(K). \tag{5}$$

We have (cf., [12])

$$I^w(\mu_{K,Q}) = \inf_{\tau\in\mathcal{M}(K)} I^w(\tau) = -\log \delta^w(K). \tag{6}$$

Proposition 2.1. *Let $K \subset \mathbb{C}$ be compact and nonpluripolar and let $\{K_n\}$ be any sequence of compact sets which decrease to K; i.e., $K_{n+1} \subset K_n$ for all K and $K = \cap_n K_n$. Let $w = e^{-Q}$ be any admissible weight function on K_n for $n \geq n_0$ whose restriction to K is admissible. Given an array of points $\{z_{nj}\}_{j=0,...,n;\ n=1,2,...}$ where $z_{n0},...,z_{nn} \in K_n$, if*

$$\lim_{n\to\infty} [|VDM(z_{n0},...z_{nn})| w(z_{n0})^n \cdots w(z_{nn})^n]^{\frac{1}{\binom{n+1}{2}}} = \delta^w(K),$$

we have

$$\mu_n := \frac{1}{n+1} \sum_{j=0}^{n} \delta_{z_{nj}} \to \mu_{K,Q} \text{ weak-*}.$$

Proof. We follow the argument in [7]. Since $K_n \searrow K$, any weak-* limit μ of $\{\mu_n\}$ will be a probability measure supported on K. Take a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ which converges to μ. It suffices to show $I^w(\mu) = -\log \delta^w(K)$ since $I^w(\mu_{K,Q}) = -\log \delta^w(K)$ and the minimizer $\mu_{K,Q}$ is unique.

We take continuous weight functions $\{w_m\}$ on K_{n_0} with $w_m \searrow w$ and $w_m \geq a_m > 0$. For $M \in \mathbb{R}$ let

$$h_{M,m}(z,t) := \min[M, \log \frac{1}{|z-t| w_m(z) w_m(t)}] \leq \log \frac{1}{|z-t| w_m(z) w_m(t)} \text{ and}$$

$$h_M(z,t) := \min[M, \log \frac{1}{|z-t| w(z) w(t)}] \leq \log \frac{1}{|z-t| w(z) w(t)}.$$

Then $h_{M,m} \leq h_M$. Every continuous function $F(z,t)$ on $K_{n_0} \times K_{n_0}$ can be uniformly approximated by finite sums of the form $\sum_j f_j(z) g_j(t)$ where f_j, g_j are continuous on K_{n_0}. Thus $\mu_{n_j} \times \mu_{n_j} \to \mu \times \mu$ and hence

$$\begin{aligned}
I^w(\mu) &= \lim_{M\to\infty} \lim_{m\to\infty} \int_K \int_K h_{M,m}(z,t) d\mu(z) d\mu(t) \\
&= \lim_{M\to\infty} \lim_{m\to\infty} \int_{K_{n_0}} \int_{K_{n_0}} h_{M,m}(z,t) d\mu(z) d\mu(t) \\
&= \lim_{M\to\infty} \lim_{m\to\infty} \lim_{j\to\infty} \int_{K_{n_0}} \int_{K_{n_0}} h_{M,m}(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t) \\
&\leq \lim_{M\to\infty} \limsup_{j\to\infty} \int_{K_{n_0}} \int_{K_{n_0}} h_M(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t),
\end{aligned}$$

the last inequality arising since $h_{M,m} \leq h_M$.

For convenience in notation, we write $z_k^{(n_j)} := z_{n_j k}$. Then for $k \neq l$,

$$h_M(z_k^{(n_j)}, z_l^{(n_j)}) \leq \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}| w(z_k^{(n_j)}) w(z_l^{(n_j)})}$$

so that

$$\int_{K_{n_0}} h_M(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t) \leq \frac{1}{n_j} M + (\frac{1}{n_j^2 - n_j})[\sum_{k\neq l} \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}| w(z_k^{(n_j)}) w(z_l^{(n_j)})}].$$

By hypothesis, given $\epsilon > 0$,

$$(\frac{1}{n_j^2 - n_j})[\sum_{k\neq l} \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}| w(z_k^{(n_j)}) w(z_l^{(n_j)})}] \leq -\log[\delta^w(K) - \epsilon]$$

for $n_j \geq n_j(\epsilon)$. Thus we can assume $w(z_k^{(n_j)}) > 0$ and hence

$$I^w(\mu) \leq \lim_{M\to\infty} \limsup_{j\to\infty} \frac{1}{n_j} M - \log[\delta^w(K) - \epsilon] = -\log[\delta^w(K) - \epsilon].$$

This holds for all $\epsilon > 0$ and hence $I^w(\mu) \leq -\log \delta^w(K)$. Since $\mu \in \mathcal{M}(K)$ implies $I^w(\mu) \geq -\log \delta^w(K)$, equality holds. □

We will call arrays as in Proposition 2.1 *asymptotic approximate (weighted) Fekete arrays* (AAF or AAWF for short).

3 Weighted pluripotential theory in $\mathbb{C}^d,\ d>1$

Let $e_1(z),...,e_j(z),...$ be a listing of the monomials $\{e_i(z)=z^{\alpha(i)}=z_1^{\alpha_1}\cdots z_d^{\alpha_d}\}$ in $\mathbb{C}^d$ indexed using a lexicographic ordering on the multiindices $\alpha=\alpha(i)=(\alpha_1,...,\alpha_d)\in\mathbb{N}^d$, but with $\deg e_i=|\alpha(i)|$ nondecreasing. We write $|\alpha|:=\sum_{j=1}^d\alpha_j$. For $\zeta_1,...,\zeta_m\in\mathbb{C}^d$, let

$$VDM(\zeta_1,...,\zeta_m)=\det[e_i(\zeta_j)]_{i,j=1,...,m} \tag{7}$$

$$=\det\begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_m)\\ \vdots & \vdots & \ddots & \vdots\\ e_m(\zeta_1) & e_m(\zeta_2) & \dots & e_m(\zeta_m)\end{bmatrix}$$

be a generalized Vandermonde determinant. In analogy with the univariate case, for a compact subset $K\subset\mathbb{C}^d$ let

$$V_m=V_m(K):=\max_{\zeta_1,...,\zeta_m\in K}|VDM(\zeta_1,...,\zeta_m)|.$$

Let m_n be the number of monomials $e_i(z)$ of degree at most n in d variables, i.e., the dimension of $\mathcal{P}_n$, the space of holomorphic polynomials of degree at most n, and let $l_n:=\sum_{j=1}^{m_n}\deg e_j$. Define $\delta_n(K):=V_{m_n}^{1/l_n}$. Zaharjuta [13] showed that the limit

$$\delta(K):=\lim_{n\to\infty}\delta_n(K) \tag{8}$$

exists; this is the *transfinite diameter* of K. We remark that $l_n=\frac{d}{d+1}nm_n$.

The same definition of admissible weight function w is used for $K\subset\mathbb{C}^d$ compact (of course now $\{z\in K: w(z)>0\}$ should be nonpluripolar). For K compact, let $w=e^{-Q}$ be an admissible weight function on K. Given $\zeta_1,...,\zeta_{m_n}\in K$, let

$$W(\zeta_1,...,\zeta_{m_n}):=VDM(\zeta_1,...,\zeta_{m_n})w(\zeta_1)^n\cdots w(\zeta_{m_n})^n$$

$$=\det\begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{m_n})\\ \vdots & \vdots & \ddots & \vdots\\ e_{m_d}(\zeta_1) & e_{m_d}(\zeta_2) & \dots & e_{m_n}(\zeta_{m_n})\end{bmatrix}\cdot w(\zeta_1)^n\cdots w(\zeta_{m_n})^n$$

be a *weighted Vandermonde determinant*. Define an *n–th order weighted Fekete set for K and w* to be a set of m_n points $\zeta_1,...,\zeta_{m_n}\in K$ with the property that

$$W_{m_n}=W_{m_n}(K):=|W(\zeta_1,...,\zeta_{m_n})|=\sup_{\xi_1,...,\xi_{m_n}\in K}|W(\xi_1,...,\xi_{m_n})|.$$

In analogy with the univariate notation, we also set $\delta_n^w(K):=W_{m_n}^{1/l_n}$. Then the limit

$$\delta^w(K):=\lim_{n\to\infty}\delta_n^w(K) \tag{9}$$

exists [8]; this is the weighted transfinite diameter of K with respect to w.

We define the weighted extremal function or weighted pluricomplex Green function $V_{K,Q}^*(z):=\limsup_{\zeta\to z}V_{K,Q}(\zeta)$ where

$$V_{K,Q}(z):=\sup\{u(z): u\in L(\mathbb{C}^d),\ u\le Q \text{ on } K\}.$$

Here, $L(\mathbb{C}^d):=\{u\in PSH(\mathbb{C}^d): u(z)-\log|z|=0(1),\ |z|\to\infty\}$ are the psh functions in $\mathbb{C}^d$ of minimal growth. We have $V_{K,Q}^*\in L(\mathbb{C}^d)$ and $\mu_{K,Q}:=(dd^cV_{K,Q}^*)^d$ is a well-defined positive measure with support in K. If $w\equiv 1$; i.e., $Q=-\log w\equiv 0$, we simply write V_K, V_K^* and $\mu_K:=(dd^cV_K^*)^d$. In this setting, we say K is *regular* if V_K is continuous. Here we are normalizing our definition of dd^c so that $\mu_{K,Q},\mu_K$ are probability measures.

4 AAF and AAWF in $\mathbb{C}^d,\ d>1$

The analogue of Proposition 2.1 in $\mathbb{C}^d,\ d>1$ holds but the proof is much more difficult. For $E\subset\mathbb{C}^d$, a measure ν on E, and a weight w on E, we use the notation

$$G_n^{\nu,w}:=\left[\int_E\overline{e_i(z)}e_j(z)w(z)^{2n}d\nu\right]\in\mathbb{C}^{m_n\times m_n} \tag{10}$$

for the weighted Gram matrix of ν of order n with respect to the standard basis monomials $\{e_1,...,e_{m_n}\}$ in $\mathcal{P}_n$. We let

$$Z_n:=\int_E\cdots\int_E|VDM(z_1,...,z_{m_n})|^2w(z_1)^{2n}\cdots w(z_{m_n})^{2n}d\nu(z_1)\cdots d\nu(z_{m_n})$$

and we have

$$B_n^{\nu,w}(z):=\sum_{j=1}^{m_n}|q_j^{(n)}(z)|^2w(z)^{2n}, \tag{11}$$

the *$n-th$ Bergman function* of E,w,ν. Here, $\{q_j^{(n)}\}_{j=1,...,m_n}$ is an orthonormal basis for $\mathcal{P}_n$ with respect to the weighted L^2–norm $p\to||w^np||_{L^2(\nu)}$. The following calculations are straightforward.

Lemma 4.1. *Suppose that $\nu \in \mathcal{M}(E)$ and that w is an admissible weight on E. Then*

$$\det(G_n^{\nu,w}) = \tag{12}$$

$$\frac{1}{m_n!}\int_{E^{m_n}} |VDM(z_1,\cdots,z_{m_n})|^2 w(z_1)^{2n}\cdots w(z_{m_n})^{2n} d\,\nu(z_1)\cdots d\,\nu(z_{m_n}) = \frac{Z_n}{m_n!}$$

and

$$B_n^{\nu,w}(z) = \tag{13}$$

$$\frac{m_n}{Z_n}\int_{E^{m_n-1}} |VDM(z,z_2,\cdots,z_{m_n})|^2 w(z)^{2n} w(z_2)^{2n}\cdots w(z_{m_n})^{2n} d\,\nu(z_2)\cdots d\,\nu(z_{m_n}).$$

The notion of *optimal measure* will be useful; see [5] for more information.

Definition 4.1. If a probability measure μ on E has the property that

$$\det(G_n^{\mu',w}) \le \det(G_n^{\mu,w}) \tag{14}$$

for all other probability measures μ' on E then μ is said to be an *optimal measure* of order n for E and w.

Let K be a nonpluripolar compact set in $\mathbb{C}^d$, and let $\{K_n\}$ be any sequence of compact sets which decrease to K; i.e., $K_{n+1} \subset K_n$ for all K and $K = \cap_n K_n$. Let w be an admissible weight function on K_n for $n \ge n_0$ whose restriction to K is admissible. In this setting, an elementary but crucial result is a modification of Proposition 2.10 of [10].

Proposition 4.2. *Suppose that, for the diagonal sequence $\{\delta_n^w(K_n)\}_n$,*

$$\lim_{n\to\infty} \delta_n^w(K_n) = \delta^w(K). \tag{15}$$

For $n = 1,2,\ldots$, let μ_n be an optimal measure of order n for K_n and w. Then

$$\lim_{n\to\infty} \det(G_n^{\mu_n,w})^{\frac{1}{2l_n}} = \delta^w(K).$$

Proof. Since μ_n is a probability measure, it follows from (12) that

$$\det(G_n^{\mu_n,w}) \le \frac{1}{m_n!}(\delta_n^w(K_n))^{2l_n}. \tag{16}$$

Now if $f_1, f_2, \cdots, f_{m_n} \in K_n$ are weighted Fekete points of order n for K_n and w, i.e., points in K_n for which

$$|VDM(z_1,\cdots,z_{m_n})| w^n(z_1) w^n(z_2)\cdots w^n(z_{m_n})$$

is maximal, then the discrete measure

$$\nu_n = \frac{1}{m_n}\sum_{k=1}^{m_n} \delta_{f_k} \tag{17}$$

is a candidate for an optimal measure of order n for K_n and w; hence

$$\det(G_n^{\nu_n,w}) \le \det(G_n^{\mu_n,w}).$$

But from (17)

$$\begin{aligned}
\det(G_n^{\nu_n,w}) &= \frac{1}{{m_n}^{m_n}} |VDM(f_1,\cdots,f_{m_n})|^2 w(f_1)^{2n} w(f_2)^{2n}\cdots w(f_{m_n})^{2n} \\
&= \left(\max_{z_i\in K} |VDM(z_1,\cdots,z_{m_n})| w^n(z_1) w^n(z_2)\cdots w^n(z_{m_n})\right)^2 \\
&= \frac{1}{{m_n}^{m_n}} (\delta_n^w(K_n))^{2l_n}
\end{aligned}$$

so that

$$\frac{1}{{m_n}^{m_n}} (\delta_n^w(K_n))^{2l_n} \le \det(G_n^{\mu_n,w})$$

and the result follows from this, (16), and the hypothesis (15). □

The bulk of the proof of the analogue of Proposition 2.1 in $\mathbb{C}^d$, $d > 1$ is to modify the arguments in [3] to show that if (15) holds; i.e., for the diagonal sequence,

$$\lim_{n\to\infty} \delta_n^w(K_n) = \delta^w(K),$$

then for $\{\mu_n\}$ a sequence of probability measures on K_n with the property that

$$\lim_{n\to\infty} \det(G_n^{\mu_n,w})^{\frac{1}{2l_n}} = \delta^w(K),$$

we have $\frac{1}{m_n} B_n^{\mu_n,w} d\mu_n \to \mu_{K,Q}$ (Theorem 4.8).

We first show that (15) holds in a very general setting, beginning with the unweighted case.

Proposition 4.3. *Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar and let $\{K_n\}$ be any sequence of compact sets which decrease to K; i.e., $K_{n+1} \subset K_n$ for all K and $K = \cap_n K_n$. Then*

$$\lim_{n\to\infty} \delta_n(K_n) = \delta(K).$$

Proof. It is standard that δ is continuous under decreasing limits; i.e.,

$$\lim_{n\to\infty} \delta(K_n) = \delta(K);$$

and, by Zaharjuta [13], for each compact set E,

$$\lim_{n\to\infty} \delta_n(E) = \delta(E).$$

We will use these facts.

First, for each n, $K \subset K_n$ so that $\delta_n(K) \le \delta_n(K_n)$. Thus

$$\delta(K) = \liminf_{n\to\infty} \delta_n(K) \le \liminf_{n\to\infty} \delta_n(K_n).$$

On the other hand, fixing n_0, for all $n > n_0$ we have $K_n \subset K_{n_0}$ so that $\delta_n(K_n) \le \delta_n(K_{n_0})$. Thus

$$\limsup_{n\to\infty} \delta_n(K_n) \le \limsup_{n\to\infty} \delta_n(K_{n_0}) = \delta(K_{n_0}).$$

This holds for each n_0; hence

$$\limsup_{n\to\infty} \delta_n(K_n) \le \lim_{n_0\to\infty} \delta(K_{n_0}) = \delta(K).$$

□

We turn to the weighted case, which requires a few more ingredients.

Proposition 4.4. *Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar and let $\{K_n\}$ be any sequence of compact sets which decrease to K; i.e., $K_{n+1} \subset K_n$ for all K and $K = \cap_n K_n$. Let $w = e^{-Q}$ be any admissible weight function on K_n for $n \ge n_0$ whose restriction to K is admissible. Then*

$$\lim_{n\to\infty} \delta_n^w(K_n) = \delta(K).$$

Proof. We claim that

$$\lim_{n\to\infty} \delta^w(K_n) = \delta^w(K). \tag{18}$$

Since, as in the unweighted case, for each compact set E and admissible weight w on E

$$\lim_{n\to\infty} \delta_n^w(E) = \delta^w(E),$$

given (18), we can repeat the proof of Proposition 4.3 in this weighted setting.

To verify (18), we first observe that since $K \subset K_n$, $\delta^w(K_n) \ge \delta^w(K)$ so that

$$\liminf_{n\to\infty} \delta^w(K_n) \ge \delta^w(K).$$

For the reverse inequality with limsup, we note that one can define a slightly different weighted transfinite diameter, in the spirit of Zaharjuta, via

$$d^w(K) := \exp\Big[\frac{1}{|\Sigma|}\int_{\Sigma^0} \log \tau^w(K,\theta)d\theta\Big]$$

(cf., [8] for the appropriate definitions and results). There is a relationship between $\delta^w(K)$ and $d^w(K)$:

$$\delta^w(K) = \Big(\exp[-\int_K Q(dd^cV_{K,Q}^*)^d]\Big)^{1/d} d^w(K). \tag{19}$$

Now it is straightforward that for *any* decreasing family of compact sets $\{K_n\}$ decreasing to K and $w = e^{-Q}$ any admissible weight function on K_n for $n \ge n_0$ whose restriction to K is admissible, we have

$$\lim_{n\to\infty} \tau^w(K_n,\theta) = \tau^w(K,\theta)$$

for $\theta \in \Sigma^0$ and hence

$$\lim_{n\to\infty} d^w(K_n) = d^w(K).$$

But we also have $V_{K_n,Q}^* \nearrow V_{K,Q}^*$ pointwise on $\mathbb{C}^d$ except perhaps a pluripolar set so that $(dd^cV_{K_n,Q}^*)^d \to (dd^cV_{K,Q}^*)^d$ as measures. Since Q is lowersemicontinuous,

$$\liminf_{n\to\infty} \int_{K_n} Q(dd^cV_{K_n,Q}^*)^d \ge \int_K Q(dd^cV_{K,Q}^*)^d.$$

Hence

$$\limsup_{n\to\infty} \delta^w(K_n) = \exp[-\liminf_{n\to\infty} \int_{K_n} Q(dd^cV_{K,Q}^*)^d]\Big)^{1/d} d^w(K)$$

$$\leq \left(\exp[-\int_K Q(dd^c V^*_{K,Q})^d]\right)^{1/d} d^w(K) = \delta^w(K).$$

This shows that

$$\lim_{n\to\infty} \int_{K_n} Q(dd^c V^*_{K_n,Q})^d = \int_K Q(dd^c V^*_{K,Q})^d$$

so that, from (19), we have (18).

□

Given a compact set K, let w be an admissible weight function on K_n for $n \geq n_0$ whose restriction to K is admissible. For a real-valued, continuous function u on K_{n_0}, we consider the weight $w_t(z) := w(z)\exp(-tu(z))$, $t \in \mathbb{R}$, and we let $\{\mu_n\}$ be a sequence of probability measures with μ_n supported on K_n. Define

$$f_n(t) := -\frac{1}{2l_n} \log\det(G_n^{\mu_n,w_t}). \tag{20}$$

Note that only the values of u on K_n are needed to define $G_n^{\mu_n,w_t}$ and hence $f_n(t)$. Also note that $f_n(0) = -\frac{1}{2l_n}\log\det(G_n^{\mu_n,w})$. We have the following (see Lemma 6.4 in [2]).

Lemma 4.5. *We have*

$$f_n'(t) = \frac{d+1}{dm_n}\int_{K_n} u(z)B_n^{\mu_n,w_t}(z)d\mu_n.$$

In particular,

$$f_n'(0) = \frac{d+1}{dm_n}\int_{K_n} u(z)B_n^{\mu_n,w}(z)d\mu_n$$

and if $B_n^{\mu_n,w} = m_n$ a.e. μ_n,

$$f_n'(0) = \frac{d+1}{d}\int_{K_n} u(z)d\mu_n. \tag{21}$$

Before we give the proof, an illustrative example can be given if $\mu_n := \frac{1}{m_n}\sum_{j=1}^{m_n}\delta_{x_j}$. It is easy to see that $B_n^{\mu_n,w}(x_j) = m_n$ for $j = 1,\dots,m_n$ so

$$\begin{aligned}&\log\det(G_n^{\mu_n,w_t})\\ &= \log\left(|W(x_1,\dots,x_{m_n})|^2 e^{-2ntu(x_1)}\cdots e^{-2ntu(x_{m_n})}\right)\end{aligned}$$

implies

$$\frac{d}{dt}\log\det(G_n^{\mu_n,w_t}) = \frac{d}{dt}\left(-2tn\sum_{j=1}^{m_n}u(x_j)\right)$$

$$= -2n\sum_{j=1}^{m_n}u(x_j) = -2nm_n\int_{K_n} u(z)\frac{1}{m_n}B_n^{\mu_n,w}(z)d\mu_n.$$

Recalling that $l_n = \frac{d}{d+1}nm_n$, this gives

$$-\frac{1}{2l_n}\log\det(G_n^{\mu_n,w_t}) = \frac{d+1}{dm_n}\int_{K_n} u(z)B_n^{\mu_n,w_t}(z)d\mu_n.$$

In this case, $\frac{d}{dt}\log\det(G_n^{\mu_n,w_t})$ is a constant, independent of t; hence $\frac{d^2}{dt^2}\log\det(G_n^{\mu_n,w_t}) \equiv 0$ – see Lemma 4.6.

Proof. The proof we offer here is modified from [6] and is based on the integral formulas of Lemma 4.1.

By (12) we may write

$$f_n(t) = -\frac{1}{2l_n}\log(F_n) + \frac{1}{2l_n}\log(m_n!)$$

where

$$F_n(t) := \int_{K_n^{m_n}} V\exp(-tU)d\mu$$

and

$$V := V(z_1,z_2,\cdots,z_{m_n}) = |VDM(z_1,\cdots,z_{m_n})|^2 w(z_1)^{2n}\cdots w(z_{m_n})^{2n},$$

$$U := U(z_1,z_2,\cdots,z_{m_n}) = 2n(u(z_1)+\cdots+u(z_{m_n})),$$

$$d\mu := d\mu_n(z_1)d\mu_n(z_2)\cdots d\mu_n(z_{m_n}).$$

Further, by (13) for $w = w_t$ and $\mu = \mu_n$, we have

$$B_n^{\mu_n,w_t}(z)$$

$$= \frac{m_n}{Z_n} \int_{K_n^{m_n-1}} V(z, z_2, z_3, \cdots, z_{m_n}) \exp(-tU) d\mu_n(z_2) \cdots d\mu_n(z_{m_n})$$

where

$$Z_n = Z_n(t) := m_n! \det(G_n^{\mu_n, w_t}) = \int_{K_n^{m_n}} V \exp(-tU) d\mu.$$

Note that $Z_n(t) = F_n(t)$. Now

$$f_n'(t) = -\frac{1}{2l_n} \frac{F_n'(t)}{F_n(t)}$$

and we may compute

$$F_n'(t) = \int_{K_n^{m_n}} V(-U) \exp(-tU) d\mu_n(z_1) \cdots d\mu_n(z_{m_n})$$

$$= -2n \int_{K_n^{m_n}} (u(z_1) + \cdots + u(z_{m_n})) V \exp(-tU) d\mu_n(z_1) \cdots d\mu_n(z_{m_n}).$$

Notice that the integrand is symmetric in the variables and hence we may "de-symmetrize" to obtain

$$F_n'(t)$$

$$= -2nm_n \int_{K_n^{m_n}} u(z_1) V(z_1, \cdots, z_{m_n}) \exp(-tU) d\mu_n(z_1) \cdots d\mu_n(z_{m_n})$$

so that, integrating in all but the z_1 variable, we obtain

$$F_n'(t) = -2nm_n \int_{K_n} u(z) B_n^{\mu_n, w_t}(z) \frac{Z_n}{n} d\mu_n(z).$$

Thus, using the fact that $Z_n(t) = F_n(t)$, we obtain

$$f_n'(t) = \frac{d+1}{dm_n} \int_{K_n} u(z) B_n^{\mu_n, w_t}(z) d\mu_n(z)$$

as claimed. In particular,

$$f_n'(0) = \frac{d+1}{dm_n} \int_{K_n} u(z) B_n^{\mu_n, w}(z) d\mu_n$$

and if $B_n^{\mu_n, w} = m_n$ a.e. μ_n, we recover (21):

$$f_n'(0) = \frac{d+1}{d} \int_{K_n} u(z) d\mu_n.$$

□

The next result was proved in a different way in [3], Lemma 2.2, and also in [5], Lemma 3.6. We follow [6].

Lemma 4.6. *The functions $f_n(t)$ are concave.*

Proof. We show that $f_n''(t) \le 0$. With the notation used in the proof of Lemma 4.5,

$$f_n''(t) = \frac{1}{2l_n} \frac{(F_n'(t))^2 - F_n''(t)}{F_n^2(t)}$$

and

$$\begin{aligned} F_n'(t) &= -\frac{1}{m_n!} \int_{K_n^{m_n}} UV \exp(-tU) d\mu, \\ F_n''(t) &= \frac{1}{m_n!} \int_{K_n^{m_n}} U^2 V \exp(-tU) d\mu. \end{aligned}$$

We must show that $(F_n'(t))^2 - F_n''(t) \le 0$. Now, for a fixed t, we may mulitply V by a constant so that

$$\int_{K_n^{m_n}} V \exp(-tU) d\mu = 1.$$

Let $d\gamma := V \exp(-tU) d\mu$. Then by the above formulas for F_n' and F_n'', we must show that

$$\int_{K_n^{m_n}} U^2 d\gamma \ge \left(\int_{K_n^{m_n}} U \, d\gamma \right)^2,$$

but this is a simple consequence of the Cauchy-Schwarz inequality. □

Define

$$g(t) = -\log(\delta^{w_t}(K))$$

so that $g(0) = -\log(\delta^w(K))$. From the Berman-Boucksom differentiability result in [2] and their Rumely-type formula (cf. Theorem 5.1 in [10]), we have

$$g'(0) = \frac{d+1}{d}\int_K u(z)(dd^c V_{K,Q}^*)^d \tag{22}$$

(cf., p. 61 of [10]). Note that for each n, μ_n is a candidate to be an optimal measure of order n for K_n and w_t. Thus, if μ_n^t is an optimal measure of order n for K_n and w_t, we have

$$\det G_n^{\mu_n, w_t} \leq \det G_n^{\mu_n^t, w_t}$$

and, from Proposition 4.2, using (15) for the weight w_t,

$$\lim_{n\to\infty} \frac{1}{2l_n} \cdot \log\det G_n^{\mu_n^t, w_t} = \log \delta^{w_t}(K).$$

Thus with

$$f_n(t) := -\frac{1}{2l_n} \log \det(G_n^{\mu_n, w_t})$$

as in (20),

$$\liminf f_n(t) \geq g(t) \text{ for all } t. \tag{23}$$

From Lemma 4.5, we have

$$f_n'(0) = \frac{d+1}{dm_n}\int_{K_n} u(z) B_n^{\mu_n, w}(z) d\mu_n. \tag{24}$$

We state a calculus lemma, Lemma 3.1 from [3].

Lemma 4.7. *Let f_n be a sequence of concave functions on $\mathbb{R}$ and let g be a function on $\mathbb{R}$. Suppose*

$$\liminf f_n(t) \geq g(t) \text{ for all } t \text{ and } \lim f_n(0) = g(0)$$

and that f_n and g are differentiable at 0. Then $\lim f_n'(0) = g'(0)$.

Using Lemma 4.7 along with equations (23), (24) and (22), we have the following general result.

Theorem 4.8. *Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar; let $\{K_n\}$ be any sequence of compact sets which decrease to K; i.e., $K_{n+1} \subset K_n$ for all K and $K = \cap_n K_n$; and let w be an admissible weight function on K_n for $n \geq n_0$ whose restriction to K is admissible. Let $\{\mu_n\}$ be a sequence of probability measures, μ_n supported on K_n, with the property that*

$$\lim_{n\to\infty} \frac{1}{2l_n} \log \det(G_n^{\mu_n, w}) = \log(\delta^w(K)) \tag{25}$$

i.e., $\lim_{n\to\infty} f_n(0) = g(0)$. Then

$$\frac{1}{m_n} B_n^{\mu_n, w} d\mu_n \to \mu_{K,Q} = (dd^c V_{K,Q}^*)^d \text{ weak-}*. \tag{26}$$

In particular, we get Theorem 1.1 on AAWF arrays.

Theorem 1.1 *Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar, let $\{K_n\}$ be a sequence of compact sets which decrease to K; i.e., $K_{n+1} \subset K_n$ for all K and $K = \cap_n K_n$, and let w be an admissible weight function on K_n for $n \geq n_0$ whose restriction to K is admissible. For each n, take points $x_1^{(n)}, x_2^{(n)}, \cdots, x_{m_n}^{(n)} \in K_n$ for which*

$$\lim_{n\to\infty} \left[|VDM(x_1^{(n)}, \cdots, x_{m_n}^{(n)})| w(x_1^{(n)})^n w(x_2^{(n)})^n \cdots w(x_{m_n}^{(n)})^n \right]^{\frac{1}{l_n}} = \delta^w(K) \tag{27}$$

and let $\mu_n := \frac{1}{m_n}\sum_{j=1}^{m_n} \delta_{x_j^{(n)}}$. Then

$$\mu_n \to \mu_{K,Q} \text{ weak} - *.$$

Proof. As observed before the proof of Lemma 4.5, we have $B_n^{\mu_n, w}(x_j^{(n)}) = m_n$ for $j = 1, \ldots, m_n$ and hence a.e. μ_n on K_n. Hence the result follows immediately from Theorem 4.8, specifically, equation (26). Alternately, if μ_n^t is an optimal measure of order n for K_n and w_t, we have

$$\det G_n^{\mu_n, w_t} \leq \det G_n^{\mu_n^t, w_t}$$

and hence

$$\liminf f_n(t) \geq g(t) \text{ for all } t;$$

finally, by hypothesis,

$$\lim_{n\to\infty} f_n(0) = -\log(\delta^w(K)) = g(0).$$

Thus Lemma 4.7 is valid to show $\mu_n \to \mu_{K,Q}$ weak-*. Indeed, in this case, as observed earlier, the functions $f_n(t)$ are affine in t so that $f_n''(t) = 0$ is immediate and Lemma 4.6 is unnecessary. □

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