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Approximation by bivariate generalized sampling series in weighted spaces of functions

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Abstract

In this paper, we study weighted approximation by bivariate generalized sampling series. Considering polynomially weighted spaces of bivariate functions we obtain pointwise and uniform convergence of the series. A rate of convergence is also presented by means of weighted modulus of continuity. In order to determine a rate of pointwise convergence, after estimating remainder of Taylor expansion of bivariate function with weighted modulus of continuity, we present a quantitative Voronovskaja theorem. Some numerical examples are also given.

1 Introduction

Generalized sampling operators, which represents an approximate version of the classical Whittaker-Kotelnikov-Shannon sampling theorem, plays a relevant role in approximation theory with its application areas, especially, image processing and signal analysis. The theory of generalized sampling series was introduced by P. L. Butzer and his school ([24, 25, 26, 29]) in Aachen during 1980^{th} by

$$\left(G_w^\chi f\right)(x)=\sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi\left(wx-k\right),\quad x\in\mathbb{R}, w>0, \tag{1}$$

with the aim of reconstruction of a function f with its sampled values at some discrete points and then developed in various directions by many researchers, among the others, we refer the readers to [12, 32, 19, 34].

To expand applications areas, P. L. Butzer et al. [28] introduced a constructive tool motivating from generalized sampling series which is the multivariate case of generalized sampling series and defined by

$$\left(G_{\underline{w}}^\chi f\right)(\underline{t}):=\frac{1}{\left(\sqrt{2\pi}\right)^n}\sum_{\underline{k}\in\mathbb{Z}^n} f\left(\frac{\underline{k}}{\underline{w}}\right)\chi\left(\underline{wt}-\underline{k}\right),\qquad \left(\underline{t}\in\mathbb{R}^n,\quad \underline{w}\in\mathbb{R}_+^n\right)$$

such that $\chi:\mathbb{R}^n\to\mathbb{C}$ is a continuous and bounded kernel function and f is a continuous function. These operators have applications to box splines [27], signal theory [39], image processing [16, 8]. Here, the bivariate case is more prominent than the others, as it includes real-life applications. For example, the mentioned references about image processing consider $n=2$ for applications. Also, there are studies just works bivariate case [13, 14]. No doubtly, the series has a crucial role from theoretical point of view.

There are also other constructive tools motivated by generalized sampling series such as Kantorovich modifications of sampling series, Durrmeyer modifications of sampling series, exponential sampling series; to enlarge class of corresponding functions and to study certain phenomena related to light scattering, diffraction, radioastronomy etc. which are introduced by Bardaro et al. in [12], Bardaro et al. in [15], Bardaro et al. in [18], respectively, and studied by many researchers, see [33, 35, 19, 45, 9], [17, 20],[48, 46, 5, 6, 47].

Weierstrass approximation theorem characterizes that the space of all continuous functions on a compact interval via uniform approximation by algebraic polynomials and its most popular proof is given quarter century after theorem stated by Bernstein [22] based on Bernstein polynomials. Bernstein polynomials sparked off the construction of many other polynomials and operators such as Szász-Mirakyan operators, Baskakov operators, Chlodovsky polynomials, [49, 21, 30]. Since the Weierstrass approximation theorem is given for the continuous functions defined on compact intervals of $\mathbb{R}$, studies on approximation theory by above operators were restricted either on compact intervals of $\mathbb{R}$ or on the space of uniformly continuous functions on $\mathbb{R}$. This problem

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also exists in case of Bohman-Korovkin theorem, which is a method to determine a family of linear positive operators is an approximation process [23, 44]. In order to overcome this restrictivity, Gadjiev [37, 38] introduced weighted spaces of functions and proved Korovkin-type theorem for the functions belonging to these spaces.

The studies of sampling type series generally considered functions belonging to $\mathcal{C}_b(\mathbb{R})$ (the space uniformly continuous and bounded). Since the main aim is to reconstruct a function with its sample values on whole $\mathbb{R}$, uniform continuity and boundedness are very restrictive conditions. Using the same idea of Gadjiev, Acar et al. [3] studied the weighted approximations for generalized sampling series, then modifications of generalized sampling series were studied in papers [4, 7, 10].

In this paper, we study bivariate generalized sampling series defined by

$$\left(G_w^{\chi} f\right)(x,y)=\sum_{(k_1,k_2)\in\mathbb{Z}^2} f\left(\frac{k_1}{w},\frac{k_2}{w}\right)\chi\left(w(x,y)-(k_1,k_2)\right),\quad (x,y)\in\mathbb{R}^2, w>0, \tag{2}$$

for functions belonging to polynomial weighted space of bivariate continuous functions. Rate of convergence of the operators given in (2), was studied in [14] and asymptotic formulae in [13]. We start with some preliminaries in Section 2. Our main results given in Section 3 and consists of pointwise and uniform convergence of $\left(G_w^{\chi}\right)$, rate of convergence via weighted modulus of continuity for bivariate function of and quantitative Voronovskaja type theorem. The last section is devoted to numerical examples of the convergence results belonging to weighted spaces of continuous functions.

2 Preliminaries

Let us denote by $\mathbb{N}^2, \mathbb{N}_0^2$ and $\mathbb{Z}^2$ the sets of vectors $\underline{k}=(k_1,k_2)$ positive integers, non-negative integers and integers, respectively and we set $\left|\underline{k}\right|:=k_1+k_2$. Moreover, by $\mathbb{R}^2$ we will denote the 2-dimensional Euclidean space consisting of all vectors $(x_1,x_2)\in\mathbb{R}^2$.

Let $\underline{x}=(x_1,x_2), \underline{y}=(y_1,y_2)\in\mathbb{R}^2$. We say that $\underline{x}>\underline{y}$ if and only if $x_i>y_i$ for $i=1,2$ and we will denote by $\underline{0}:=(0,0)$ and by $\mathbb{R}_+^2$ the space of all vectors $\underline{x}>\underline{0}$. Given $\underline{x},\underline{y}\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$ the usual operations are given by

$$\underline{x}+\underline{y}:=(x_1+y_1,x_2+y_2),$$
$$\alpha\underline{x}:=(\alpha x_1,\alpha x_2).$$

We will put by $\langle\underline{x}\rangle=x_1x_2$ and we write $\underline{k}!=k_1!\cdot k_2!$. Further, the product and division of two vectors of $\mathbb{R}^2$ are

$$\underline{x}\,\underline{y}:=(x_1y_1,x_2y_2),$$
$$\frac{\underline{x}}{\underline{y}}:=\left(\frac{x_1}{y_1},\frac{x_2}{y_2}\right),\quad (y_i\neq 0 \text{ for all } i=1,2).$$

The norm of a vector $\underline{x}:=(x_1,x_2)\in\mathbb{R}^2$ is given by $\|\underline{x}\|=\|(x_1,x_2)\|:=\sqrt{x_1^2+x_2^2}$, and the Euclidean distance is $d\left(\underline{x},\underline{y}\right):=\|\underline{x}-\underline{y}\|$ for $\underline{x},\underline{y}\in\mathbb{R}^2$.

A function $\widetilde{w}$ is called a weight function if it is a positive continuous function on the whole $\mathbb{R}^2$. Throughout this paper, we consider the weight function

$$\widetilde{w}(x,y)=\frac{1}{1+x^2+y^2},\ x,y\in\mathbb{R}.$$

We denote by $B_{\widetilde{w}}\left(\mathbb{R}^2\right)$, the space of real functions whose product with the weight function $\widetilde{w}$ on $\mathbb{R}^2$ is bounded, that is

$$B_{\widetilde{w}}\left(\mathbb{R}^2\right)=\left\{f:\mathbb{R}^2\to\mathbb{R}:\sup_{x,y\in\mathbb{R}}\widetilde{w}(x,y)\left|f(x,y)\right|\in\mathbb{R}\right\}.$$

We denote by $C^0(\mathbb{R}^2)$ the space of continuous functions on the whole $\mathbb{R}^2$. We can also consider the following natural subspaces of $B_{\widetilde{w}}\left(\mathbb{R}^2\right)$:

$$C_{\widetilde{w}}\left(\mathbb{R}^2\right):=C^0\left(\mathbb{R}^2\right)\cap B_{\widetilde{w}}\left(\mathbb{R}^2\right),$$
$$C_{\widetilde{w}}^*\left(\mathbb{R}^2\right):=\left\{f\in C_{\widetilde{w}}\left(\mathbb{R}^2\right):\exists\lim_{\|(x,y)\|\to\mp\infty}\widetilde{w}(x,y)f(x,y)\in\mathbb{R}\right\},$$
$$U_{\widetilde{w}}\left(\mathbb{R}^2\right):=\left\{f\in C_{\widetilde{w}}\left(\mathbb{R}^2\right):\widetilde{w}f \text{ is uniformly continuous}\right\}.$$

The linear space of functions $B_{\widetilde{w}}\left(\mathbb{R}^2\right)$, and its above subspaces are normed spaces with the norm

$$\|f\|_{\widetilde{w}}:=\sup_{x,y\in\mathbb{R}}\widetilde{w}(x,y)\left|f(x,y)\right|$$

see [1, 2, 37, 38, 40].

The weighted modulus of continuity for bivariate functions was defined in [42] by

$$\Omega(f;\delta_1,\delta_2)=\sup_{|u|<\delta_1,|v|<\delta_2,(x,y)\in\mathbb{R}^2}\frac{|f(x+u,y+v)-f(x,y)|}{(1+u^2+v^2)(1+x^2+y^2)} \tag{3}$$

for every $f\in C^*_{\tilde{w}}\left(\mathbb{R}^2\right)$. Here we note that

$$\Omega(f;\delta_1,\delta_2)\to 0 \text{ for } \delta_1\to 0,\delta_2\to 0 \tag{4}$$

and for $\lambda_1>0,\lambda_2>0$, the weighted modulus of continuity follows

$$\Omega(f;\lambda_1\delta_1,\lambda_2\delta_2)\le 4(1+\lambda_1)(1+\lambda_2)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\Omega(f,\delta_1,\delta_2). \tag{5}$$

For more details related to weighted modulus of continuity we refer the readers to [41, 42]. Now, we state an auxiliary results which will be used in the next section.

Remark 1. In the inequality (5) if we replace $\lambda_1=\frac{|x_2-x_1|}{\delta_1}$, $\lambda_2=\frac{|y_2-y_1|}{\delta_2}$, $(x_1,y_1)\in\mathbb{R}^2,(x_2,y_2)\in\mathbb{R}^2,\delta_1,\delta_2>0$ and consider the definition of the weighted modulus of continuity, we have

$$\begin{aligned}&|f(x_2,y_2)-f(x_1,y_1)|\\ \le&4\left(1+\frac{|x_2-x_1|}{\delta_1}\right)\left(1+\frac{|y_2-y_1|}{\delta_2}\right)\left(1+(x_2-x_1)^2+(y_2-y_1)^2\right)\left(1+x_1^2+y_1^2\right)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\Omega(f;\delta_1,\delta_2)\\ \le&\begin{cases}16\left(1+\delta_1^2\right)^2\left(1+\delta_2^2\right)^2\left(1+x_1^2+y_1^2\right)\Omega(f;\delta_1,\delta_2), & |x_2-x_1|\le\delta_1 \text{ and } |y_2-y_1|\le\delta_2\\ 16\left(1+\delta_1^2\right)^2\left(1+\delta_2^2\right)^2\left(1+x_1^2+y_1^2\right)\Omega(f;\delta_1,\delta_2)\frac{|y_2-y_1|^3}{\delta_2^3}, & |x_2-x_1|\le\delta_1 \text{ and } |y_2-y_1|>\delta_2\\ 16\left(1+\delta_1^2\right)^2\left(1+\delta_2^2\right)^2\left(1+x_1^2+y_1^2\right)\Omega(f;\delta_1,\delta_2)\frac{|x_2-x_1|^3}{\delta_1^3}, & |x_2-x_1|>\delta_1 \text{ and } |y_2-y_1|\le\delta_2\\ 16\left(1+\delta_1^2\right)^2\left(1+\delta_2^2\right)^2\left(1+x_1^2+y_1^2\right)\Omega(f;\delta_1,\delta_2)\frac{|y_2-y_1|^3}{\delta_2^3}\frac{|x_2-x_1|^3}{\delta_1^3}, & |x_2-x_1|>\delta_1 \text{ and } |y_2-y_1|>\delta_2\end{cases}.\end{aligned}$$

Hence, combining these four cases we get

$$\begin{aligned}&|f(x_2,y_2)-f(x_1,y_1)|\\ \le&16\left(1+\delta_1^2\right)^2\left(1+\delta_2^2\right)^2\left(1+x_1^2+y_1^2\right)\Omega(f;\delta_1,\delta_2)\left[1+\frac{|y_2-y_1|^3}{\delta_2^3}+\frac{|x_2-x_1|^3}{\delta_1^3}+\frac{|y_2-y_1|^3}{\delta_2^3}\frac{|x_2-x_1|^3}{\delta_1^3}\right].\end{aligned}$$

Finally we obtain

$$\begin{aligned}&|f(x_2,y_2)-f(x_1,y_1)|\\ \le&256\left(1+x_1^2+y_1^2\right)\Omega(f;\delta_1,\delta_2)\left[1+\frac{|y_2-y_1|^3}{\delta_2^3}+\frac{|x_2-x_1|^3}{\delta_1^3}+\frac{|y_2-y_1|^3}{\delta_2^3}\frac{|x_2-x_1|^3}{\delta_1^3}\right]\end{aligned} \tag{6}$$

with the choice of $\delta_1\le 1$ and $\delta_2\le 1$.

A function $\chi:\mathbb{R}^2\to\mathbb{R}$ will be called a kernel function if it satisfies the following assumptions:

(χ1) χ is continuous on $\mathbb{R}^2$,

(χ2) the discrete algebraic moment of order 0:

$$m_0(\chi,(u_1,u_2)):=\sum_{(k_1,k_2)\in\mathbb{Z}^2}\chi((u_1,u_2)-(k_1,k_2))=1,$$

for every $(u_1,u_2)\in\mathbb{R}^2$,

(χ3) there exists $\beta>0$, such that the discrete absolute moments of order β:

$$M_\beta(\chi)=\sup_{(u_1,u_2)\in\mathbb{R}^2}\sum_{(k_1,k_2)\in\mathbb{Z}^2}|\chi((u_1,u_2)-(k_1,k_2))|\,\|(u_1,u_2)-(k_1,k_2)\|^\beta,$$

is finite.

Lemma 2.1 ([31]). *Let χ be a function satisfying (χ1) and (χ3) for some $\beta>0$. For every $\delta>0$ there holds:*

$$\lim_{w\to\infty}\sum_{\|(k_1,k_2)-w(x,y)\|>w\delta}|\chi(w(x,y)-(k_1,k_2))|=0,$$

uniformly with respect to $x\in\mathbb{R}$.

3 Main Results

This section is devoted to main results of the paper. Results consist of well definiteness of the operator $\left(G_w^\chi f\right)$, pointwise and uniform convergence, rate of convergence and at the end a Voronovskaja-type theorem.

3.1 Well definiteness of the operators

We start with the well definiteness of the generalized sampling operators in the weighted spaces of bivariate functions.

Theorem 3.1. *Let χ be a kernel satisfying $(\chi 1),(\chi 2)$, and $(\chi 3)$ for $\beta = 2$. Then for a fixed $w > 0$, the operator G_w^χ is a linear operator from $B_{\widetilde{w}}\left(\mathbb{R}^2\right)$ to $B_{\widetilde{w}}\left(\mathbb{R}^2\right)$ and its operator norm turns out be:*

$$\left\|G_w^\chi f\right\|_{\widetilde{w}} \leq 2\left\|f\right\|_{\widetilde{w}}\left\{M_0(\chi)+\frac{1}{w^2}M_2(\chi)\right\}. \tag{7}$$

Proof. Let us fix $w > 0$. From the definition of the G_w^χ, for a function $f \in B_{\widetilde{w}}\left(\mathbb{R}^2\right)$ and $(x,y)\in\mathbb{R}^2$, we have

$$\begin{aligned}
\left|\left(G_w^\chi f\right)(x,y)\right| &\leq \sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\chi\left(w(x,y)-(k_1,k_2)\right)\right|\left|f\left(\frac{k_1}{w},\frac{k_2}{w}\right)\right| \\
&\leq \left\|f\right\|_{\widetilde{w}}\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\chi\left(w(x,y)-(k_1,k_2)\right)\right|\frac{1}{\widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)} \\
&= \left\|f\right\|_{\widetilde{w}}\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\chi\left(w(x,y)-(k_1,k_2)\right)\right|\left[1+\left(\frac{k_1}{w}\right)^2+\left(\frac{k_2}{w}\right)^2\right] \\
&\leq \left\|f\right\|_{\widetilde{w}}\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\chi\left(w(x,y)-(k_1,k_2)\right)\right|\left\{1+2\left[\left(\frac{k_1}{w}-x\right)^2+x^2\right]+2\left[\left(\frac{k_2}{w}-y\right)^2+y^2\right]\right\} \\
&\leq 2\left\|f\right\|_{\widetilde{w}}\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\chi\left(w(x,y)-(k_1,k_2)\right)\right|\left\{1+x^2+y^2+\left(\frac{k_1}{w}-x\right)^2+\left(\frac{k_2}{w}-y\right)^2\right\} \\
&= 2\left\|f\right\|_{\widetilde{w}}\left(1+x^2+y^2\right)\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\chi\left(w(x,y)-(k_1,k_2)\right)\right| \\
&+ \frac{2\left\|f\right\|_{\widetilde{w}}}{w^2}\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\chi\left(w(x,y)-(k_1,k_2)\right)\right|\left[(k_1-wx)^2+(k_2-wy)^2\right] \\
&\leq 2\left\|f\right\|_{\widetilde{w}}\left(1+x^2+y^2\right)\left\{M_0(\chi)+\frac{1}{w^2}M_2(\chi)\right\}
\end{aligned}$$

which implies

$$\frac{\left|\left(G_w^\chi f\right)(x,y)\right|}{1+x^2+y^2} \leq 2\left\|f\right\|_{\widetilde{w}}\left\{M_0(\chi)+\frac{1}{w^2}M_2(\chi)\right\} \tag{8}$$

for every $(x,y)\in\mathbb{R}^2$. By the assumptions, we have $M_0(\chi)<+\infty$ and $M_2(\chi)<+\infty$; hence we conclude that $\left\|G_w^\chi f\right\|_{\widetilde{w}}<+\infty$, that is, $G_w^\chi f \in B_{\widetilde{w}}\left(\mathbb{R}^2\right)$. Hence, by taking supremum over $(x,y)\in\mathbb{R}^2$ in (8) and the supremum with respect to $f\in B_{\widetilde{w}}\left(\mathbb{R}^2\right)$ with $\left\|f\right\|_{\widetilde{w}}\leq 1$, we have (7). □

3.2 Convergence Results

Theorem 3.2. *Let χ be a kernel satisfying $(\chi 1),(\chi 2)$, and $(\chi 3)$ for $\beta = 2$ and $f \in C_{\widetilde{w}}\left(\mathbb{R}^2\right)$ be fixed. Then*

$$\lim_{w\to+\infty} G_w^\chi(x,y) = f(x,y) \tag{9}$$

holds for every $(x,y)\in\mathbb{R}^2$. Moreover, if $f\in U_{\widetilde{w}}\left(\mathbb{R}^2\right)$, then

$$\lim_{w\to+\infty}\left\|G_w^\chi f - f\right\|_{\widetilde{w}} = 0. \tag{10}$$

Proof. First of all, for $w>0, k_1,k_2\in\mathbb{Z}$ and $x,y\in\mathbb{R}$, by straightforward computations, it can be easily seen that

$$\begin{aligned}
\left|f\left(\frac{k_1}{w},\frac{k_2}{w}\right)-f(x,y)\right| &\leq \widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)\left|f\left(\frac{k_1}{w},\frac{k_2}{w}\right)\right|\left|\frac{1}{\widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)}-\frac{1}{\widetilde{w}(x,y)}\right| \\
&+\frac{1}{\widetilde{w}(x,y)}\left|\widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)f\left(\frac{k_1}{w},\frac{k_2}{w}\right)-\widetilde{w}(x,y)f(x,y)\right|.
\end{aligned}$$

Then, using the definition of $\left(G_w^\chi f\right)$, $(\chi 2)$ and above inequality we get

$$
\begin{aligned}
&\left|\left(G_w^\chi f\right)(x,y)-f(x,y)\right| \\
&\leq \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\chi(w(x,y)-(k_1,k_2))| \\
&\times\left\{\widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)\left|f\left(\frac{k_1}{w},\frac{k_2}{w}\right)\right|\left|\frac{1}{\widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)}-\frac{1}{\widetilde{w}(x,y)}\right|+\frac{1}{\widetilde{w}(x,y)}\left|\widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)f\left(\frac{k_1}{w},\frac{k_2}{w}\right)-\widetilde{w}(x,y)f(x,y)\right|\right\} \\
&=: I_1+I_2.
\end{aligned}
$$

Let us first consider to estimate I_1.

$$
\begin{aligned}
I_1 &\leq \|f\|_{\widetilde{w}} \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\chi(w(x,y)-(k_1,k_2))| \left|\left(\frac{k_1}{w}\right)^2+\left(\frac{k_2}{w}\right)^2-x^2-y^2\right| \\
&\leq \|f\|_{\widetilde{w}} \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\chi(w(x,y)-(k_1,k_2))| \\
&\quad\times\left[\left|\frac{k_1}{w}-x\right|^2+2|x|\left|\frac{k_1}{w}-x\right|+\left|\frac{k_2}{w}-y\right|^2+2|y|\left|\frac{k_2}{w}-y\right|\right] \\
&=\frac{\|f\|_{\widetilde{w}}}{w^2} \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\chi(w(x,y)-(k_1,k_2))|\left[|k_1-wx|^2+|k_2-wy|^2\right] \\
&\quad+\frac{2(|x|+|y|)\|f\|_{\widetilde{w}}}{w}\left[|k_1-wx|+|k_2-wy|\right] \\
&\leq\frac{\|f\|_{\widetilde{w}}}{w^2}M_2(\chi)+\frac{4(|x|+|y|)\|f\|_{\widetilde{w}}}{w}M_1(\chi).
\end{aligned}
\tag{11}
$$

Now, we consider I_2. Since f is continuous at $(x,y)\in\mathbb{R}^2$, there exists a $\delta>0$ such that

$$
\left|\widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)f\left(\frac{k_1}{w},\frac{k_2}{w}\right)-\widetilde{w}(x,y)f(x,y)\right|<\varepsilon \tag{12}
$$

whenever

$$
\left\|\left(\frac{k_1}{w},\frac{k_2}{w}\right)-(x,y)\right\|\leq\delta.
$$

Hence, we can write

$$
\begin{aligned}
I_2 &=\frac{1}{\widetilde{w}(x,y)}\left\{\sum_{\left\|\left(\frac{k_1}{w},\frac{k_2}{w}\right)-(x,y)\right\|\leq w\delta}+\sum_{\left\|\left(\frac{k_1}{w},\frac{k_2}{w}\right)-(x,y)\right\|>w\delta}\right\}\sum_{(k_1,k_2)\in\mathbb{Z}^2}|\chi(w(x,y)-(k_1,k_2))| \\
&\quad\times\left|\widetilde{w}\left(\frac{k_1}{w},\frac{k_2}{w}\right)f\left(\frac{k_1}{w},\frac{k_2}{w}\right)-\widetilde{w}(x,y)f(x,y)\right| \\
&=: I_{2,1}+I_{2,2}.
\end{aligned}
$$

Using the inequality (12), we immediately have

$$
\begin{aligned}
I_{2,1} &\leq \frac{\varepsilon_1}{\widetilde{w}(x,y)}\sum_{\left\|\left(\frac{k_1}{w},\frac{k_2}{w}\right)-(x,y)\right\|\leq w\delta}|\chi(w(x,y)-(k_1,k_2))| \\
&\leq\frac{\varepsilon}{\widetilde{w}(x,y)}M_0(\chi).
\end{aligned}
\tag{13}
$$

On the other hand, by Lemma 2.1 we have for sufficiently large $w>0$ that

$$
I_{2,2}\leq\frac{2\|f\|_{\widetilde{w}}\,\varepsilon_2}{\widetilde{w}(x,y)}. \tag{14}
$$

Hence by combining the inequalities (11), (13) and (14) we conclude

$$
\begin{aligned}
&\left|\left(G_w^\chi f\right)(x,y)-f(x,y)\right| \\
&\leq\frac{\|f\|_{\widetilde{w}}}{w^2}M_2(\chi)+\frac{2(|x|+|y|)\|f\|_{\widetilde{w}}}{w}M_1(\chi)+\frac{\varepsilon}{\widetilde{w}(x,y)}M_0(\chi)+\frac{2\|f\|_{\widetilde{w}}\,\varepsilon_2}{\widetilde{w}(x,y)}.
\end{aligned}
\tag{15}
$$

Finally, taking the limit of both sides as $w\to+\infty$ we have assertion (9).

At the end, we have to show that the assertion (10). To do this, let $f \in U_{\widetilde{w}}\left(\mathbb{R}^2\right)$. Let us follow the proof of the assertion (9) and replace the δ with the corresponding parameter of the uniform continuity of $\widetilde{w}f$. From the inequality (15) we can write:

$$\begin{aligned}
&\widetilde{w}(x,y)\left|\left(G_w^\chi f\right)(x,y)-f(x,y)\right| \\
&\leq \frac{\widetilde{w}(x,y)\|f\|_{\widetilde{w}}}{w^2}M_2(\chi)+\frac{2\widetilde{w}(x,y)(|x|+|y|)\|f\|_{\widetilde{w}}}{w}M_1(\chi)+\varepsilon M_0(\chi)+2\|f\|_{\widetilde{w}}\varepsilon_2 \\
&\leq \frac{\|f\|_{\widetilde{w}}}{w^2}M_2(\chi)+\frac{2\|f\|_{\widetilde{w}}}{w}M_1(\chi)+\varepsilon M_0(\chi)+2\|f\|_{\widetilde{w}}\varepsilon_2. \qquad (16)
\end{aligned}$$

Now, passing supremum in (16) over $(x,y)\in\mathbb{R}^2$ we have (10) for $w\to+\infty$, which completes the proof. □

3.3 Rate of Convergence

This subsection is devoted to determine rate of convergence the operators G_w^χ in the weighted spaces of bivariate functions via weighted modulus of continuity given in (3).

Theorem 3.3. *Let χ be a kernel satisfying the assumptions $(\chi 1),(\chi 2)$ and $(\chi 3)$ for $\beta=6$. Then, for $f\in C_{\widetilde{w}}(\mathbb{R})$ the inequality*

$$\left\|G_w^\chi f-f\right\|_{\widetilde{w}}\leq 256\Omega\left(f;w^{-1},w^{-1}\right)\left[M_0(\chi)+2M_3(\chi)+M_6(\chi)\right]$$

holds for $w\geq 1$.

Proof. For every $(x,y)\in\mathbb{R}^2$, using the assumption $(\chi 2)$ and (6) we have

$$\begin{aligned}
\left|\left(G_w^\chi f\right)(x,y)-f(x,y)\right| &\leq \sum_{(k_1,k_2)\in\mathbb{Z}^2}|\chi(w(x,y)-(k_1,k_2))|\left|f\left(\frac{k_1}{w},\frac{k_2}{w}\right)-f(x,y)\right| \\
&\leq 256\left(1+x^2+y^2\right)\Omega(f;\delta_1,\delta_2)\sum_{(k_1,k_2)\in\mathbb{Z}^2}|\chi(w(x,y)-(k_1,k_2))| \\
&\times\left[1+\frac{1}{\delta_1^3}\left|\frac{k_1}{w}-x\right|^3+\frac{1}{\delta_2^3}\left|\frac{k_2}{w}-y\right|^3+\frac{1}{\delta_1^3\delta_2^3}\left|\frac{k_1}{w}-x\right|^3\left|\frac{k_2}{w}-y\right|^3\right] \\
&\leq 256\left(1+x^2+y^2\right)\Omega(f;\delta_1,\delta_2)\sum_{(k_1,k_2)\in\mathbb{Z}^2}|\chi(w(x,y)-(k_1,k_2))| \\
&\times\left\{1+\left(\frac{1}{\delta_1^3w^3}+\frac{1}{\delta_2^3w^3}\right)\left[2\left(|k_1-wx|^2+|k_2-wy|^2\right)\right]^{\frac{3}{2}}\right. \\
&\left.+\frac{1}{\delta_1^3\delta_2^3w^3}\left[2\left(|k_1-wx|^2+|k_2-wy|^2\right)\right]^3\right\} \\
&\leq 256\left(1+x^2+y^2\right)\Omega(f;\delta_1,\delta_2) \\
&\times\left[M_0(\chi)+\left(\frac{1}{\delta_1^3w^3}+\frac{1}{\delta_2^3w^3}\right)2\sqrt{2}M_3(\chi)+8M_6(\chi)\right].
\end{aligned}$$

By choosing $\delta_1=w^{-1},\delta_2=w^{-1},w\geq 1$, and taking the supremum for $(x,y)\in\mathbb{R}^2$ we get

$$\left\|G_w^\chi f-f\right\|_{\widetilde{w}}\leq 256\Omega\left(f;w^{-1},w^{-1}\right)\left[M_0(\chi)+2\sqrt{2}M_3(\chi)+8M_6(\chi)\right],$$

which is the desired. □

Corollary 3.4. *If we assume $f\in C_{\widetilde{w}}^*$ in Theorem 3.3, by (4) we have*

$$\lim_{w\to\infty}\left\|G_w^\chi f-f\right\|_{\widetilde{w}}=0.$$

3.4 Voronovskaja Type Theorem

In this section, we state a quantitative Voronovskaja-type theorem for the operators G_w^χ in the weighted spaces of bivariate functions.

First of all, we recall the Taylor's formula. Let $\underline{x}=(x_1,x_2),\underline{k}=(k_1,k_2)\in\mathbb{R}_+^2$, $\left|\underline{k}\right|=r$, for a function $f:\mathbb{R}^2\to\mathbb{R}$, we denote the r-th order derivatives of f by

$$D^rf:=\frac{\partial^r}{\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}}f.$$

For $r \in \mathbb{N}$, by $C^{(r)}(J)$ we denote the subspace of $C^0(J)$ which consists of all functions f with the derivatives up to the order r in $C^0(J)$. By the Taylor expansion of f (see [43]),

$$\begin{aligned} f(t_1,t_2) =& f(x,y) + (t_1-x)\frac{\partial f}{\partial x}(x,y) + (t_2-y)\frac{\partial f}{\partial y}(x,y) \\ &+ \frac{1}{2}\left((t_1-x)^2\frac{\partial^2 f}{\partial x^2}(x,y) + 2(t_1-x)(t_2-y)\frac{\partial^2 f}{\partial x\partial y}(x,y) + (t_2-y)^2\frac{\partial^2 f}{\partial y^2}(x,y)\right) \\ &+ R_2(t_1,t_2), \end{aligned} \tag{17}$$

where

$$\begin{aligned} R(t_1,t_2) =& \frac{1}{2}\left\{(t_1-x)^2\left[\frac{\partial^2 f}{\partial x^2}(\xi_x,\xi_y) - \frac{\partial^2 f}{\partial x^2}(x,y)\right] + 2(t_1-x)(t_2-y)\left[\frac{\partial^2 f}{\partial x\partial y}(\xi_x,\xi_y) - \frac{\partial^2 f}{\partial x\partial y}(x,y)\right]\right. \\ &\left. + (t_2-y)^2\left[\frac{\partial^2 f}{\partial y^2}(\xi_x,\xi_y) - \frac{\partial^2 f}{\partial y^2}(x,y)\right]\right\} \end{aligned} \tag{18}$$

such that $\xi_x = x + \theta(t_1-x)$, $\xi_y = y + \theta(t_2-y)$, and $0 < \theta < 1$.

According to inequality (6), with similar method presented in [1], we have the estimate

$$\begin{aligned} &|R_2(t_1,t_2)| \\ \le& \frac{1}{2}\left\{(t_1-x)^2\left|\frac{\partial^2 f}{\partial x^2}(t_1,t_2) - \frac{\partial^2 f}{\partial x^2}(x,y)\right| + 2|t_1-x||t_2-y|\left|\frac{\partial^2 f}{\partial x\partial y}(t_1,t_2) - \frac{\partial^2 f}{\partial x\partial y}(x,y)\right| + (t_2-y)^2\left|\frac{\partial^2 f}{\partial y^2}(t_1,t_2) - \frac{\partial^2 f}{\partial y^2}(x,y)\right|\right\} \\ \le& \frac{1}{2}\left\{256\left(1+x^2+y^2\right)\Omega(f_{xx};\delta_1,\delta_2)(t_1-x)^2\left[1 + \frac{|t_2-y|^3}{\delta_2^3} + \frac{|t_1-x|^3}{\delta_1^3} + \frac{|t_2-y|^3}{\delta_2^3}\frac{|t_1-x|^3}{\delta_1^3}\right]\right. \\ &+ 512\left(1+x^2+y^2\right)\Omega\left(f_{xy};\delta_1,\delta_2\right)|t_1-x||t_2-y|\left[1 + \frac{|t_2-y|^3}{\delta_2^3} + \frac{|t_1-x|^3}{\delta_1^3} + \frac{|t_2-y|^3}{\delta_2^3}\frac{|t_1-x|^3}{\delta_1^3}\right] \\ &\left. + 256\left(1+x^2+y^2\right)\Omega\left(f_{yy};\delta_1,\delta_2\right)(t_2-y)^2\left[1 + \frac{|t_2-y|^3}{\delta_2^3} + \frac{|t_1-x|^3}{\delta_1^3} + \frac{|t_2-y|^3}{\delta_2^3}\frac{|t_1-x|^3}{\delta_1^3}\right]\right\} \\ \le& 128\left(1+x^2+y^2\right)\left\{\Omega(f_{xx};\delta_1,\delta_2)\left[(t_1-x)^2 + \frac{1}{\delta_2^3}|t_2-y|^3(t_1-x)^2 + \frac{1}{\delta_1^3}|t_1-x|^5 + \frac{1}{\delta_1^3\delta_2^3}|t_1-x|^5|t_2-y|^3\right]\right. \\ &+ 2\Omega\left(f_{xy};\delta_1,\delta_2\right)\left[|t_1-x||t_2-y| + \frac{1}{\delta_2^3}|t_2-y|^4|t_1-x| + \frac{1}{\delta_1^3}|t_1-x|^4|t_2-y| + \frac{1}{\delta_1^3\delta_2^3}|t_2-y|^4|t_1-x|^4\right] \\ &\left. + \Omega\left(f_{yy};\delta_1,\delta_2\right)\left[(t_2-y)^2 + \frac{1}{\delta_2^3}|t_2-y|^5 + \frac{1}{\delta_1^3}|t_1-x|^3(t_2-y)^2 + \frac{1}{\delta_1^3\delta_2^3}|t_2-y|^5|t_1-x|^3\right]\right\}. \end{aligned} \tag{19}$$

Now, let $\underline{h} = (h_1,h_2) \in \mathbb{N}_0^2$ and let $\nu = |\underline{h}|$. For $(u_1,u_2) \in \mathbb{R}_+^2$ we define the discrete algebraic moments of order $\underline{h}$ of χ as

$$\begin{aligned} m_{\underline{h}}^{\nu}(\chi,\underline{u}) :=& \sum_{(k_1,k_2)\in\mathbb{Z}^2} \chi\left((u_1,u_2)-(k_1,k_2)\right)\langle(k_1,k_2)-(u_1,u_2)\rangle^{\underline{h}} \\ =& \sum_{(k_1,k_2)\in\mathbb{Z}^2} \chi\left((u_1,u_2)-(k_1,k_2)\right)(k_1-u_1)^{h_1}(k_2-u_2)^{h_2}. \end{aligned}$$

In order to obtain an estimation of the order of approximation under some local regularity assumption on the function f we need more assumptions on the kernel function χ, i.e., there exists $l \in \mathbb{N}$ such that $\underline{h} \in \mathbb{N}_0^2$, $|\underline{h}| \le l$

(χ4) $m_{\underline{h}}^{|\underline{h}|}(\chi,\underline{u}) := m_{\underline{h}}^{|\underline{h}|}(\chi)$ is independent of $\underline{u}$.

Theorem 3.5. *Let χ satisfies the assumptions $(\chi1),(\chi2),(\chi3)$ for $\beta = 8$ and $(\chi4)$ for $l = 2$. Moreover, we also assume that χ be a kernel such that, for every $\gamma > 0$*

$$\lim_{w\to+\infty} \sum_{\|w\underline{t}-\underline{k}\|>\gamma w} |\chi(w\underline{t}-\underline{k})| \cdot \|w\underline{t}-\underline{k}\|^2 = 0$$

uniformly with respect to $\underline{t} \in \mathbb{R}^2$. Then, for $f \in C^2\left(\mathbb{R}^2\right)$ such that $f^{(2)} \in C_{\tilde{w}}^{*}\left(\mathbb{R}^2\right)$, there holds

$$\begin{aligned} &\left| w\left[\left(G_w^{\chi} f\right)(x,y) - f(x,y)\right] - m_{(1,0)}^1(\chi)\left(\frac{\partial f}{\partial x}(x,y)\right) - m_{(0,1)}^1(\chi)\left(\frac{\partial f}{\partial y}(x,y)\right)\right. \\ &\left. - \frac{1}{2w}\left[m_{(2,0)}^2(\chi)\frac{\partial^2 f}{\partial x^2}(x,y) + m_{(1,1)}^2(\chi)\frac{\partial^2 f}{\partial x\partial y}(x,y) + m_{(0,2)}^2(\chi)\frac{\partial^2 f}{\partial y^2}(x,y)\right]\right| \\ \le& 256\left(1+x^2+y^2\right)\left[\Omega(f_{xx};\delta_1,\delta_2) + 2\Omega\left(f_{xy};\delta_1,\delta_2\right) + \Omega\left(f_{yy};\delta_1,\delta_2\right)\right] \\ &\times \left[\frac{1}{w}\left(M_2(\chi) + 8M_5(\chi) + 8M_8(\chi)\right)\right] \end{aligned}$$

Proof. Let us first consider the Taylor expansion of $f\in C^2\left(\mathbb{R}^2\right)$ at any point $(x,y)\in\mathbb{R}^2$. That is,

$$\begin{aligned} f(t_1,t_2) &= f(x,y)+(t_1-x)\frac{\partial f}{\partial x}(x,y)+(t_2-y)\frac{\partial f}{\partial y}(x,y)\\ &+\frac{1}{2}\left((t_1-x)^2\frac{\partial^2 f}{\partial x^2}(x,y)+2(t_1-x)(t_2-y)\frac{\partial^2 f}{\partial x\partial y}(x,y)+(t_2-y)^2\frac{\partial^2 f}{\partial y^2}(x,y)\right)\\ &+R_2(t_1,t_2), \end{aligned} \tag{20}$$

with the remainder $R_2(t_1,t_2)$ given as in (18). By using the (20) in the definition of the operator $\left(G_w^\chi f\right)$, we can write

$$\left(G_w^\chi f\right)(x,y)=\sum_{(k_1,k_2)\in\mathbb{Z}^2}\chi\left(w(x,y)-(k_1,k_2)\right)f\left(\frac{k_1}{w},\frac{k_2}{w}\right)$$

$$\begin{aligned} &=\Bigg\{\sum_{(k_1,k_2)\in\mathbb{Z}^2}\chi\left(w(x,y)-(k_1,k_2)\right)\Bigg[f(x,y)+\frac{1}{w}\left((k_1-wx)\frac{\partial f}{\partial x}(x,y)+(k_2-wy)\frac{\partial f}{\partial y}(x,y)\right)\\ &+\frac{1}{2w^2}\left((k_1-wx)^2\frac{\partial^2 f}{\partial x^2}(x,y)+2(k_1-wx)(k_2-wy)\frac{\partial^2 f}{\partial x\partial y}(x,y)+(k_2-wy)^2\frac{\partial^2 f}{\partial y^2}(x,y)\right)\Bigg]\Bigg\}\\ &+\left\{\sum_{(k_1,k_2)\in\mathbb{Z}^2}\chi\left(w(x,y)-(k_1,k_2)\right)R_2\left(\frac{k_1}{w},\frac{k_2}{w}\right)\right\}\\ &:=I_1+I_2. \end{aligned}$$

Let us first take into account that I_1. Easily, by the definition of the discrete algebraic moments and $(\chi 2)$ we have

$$\begin{aligned} I_1 &= f(x,y)+\frac{1}{w}\left[m^1_{(1,0)}(\chi)\left(\frac{\partial f}{\partial x}(x,y)\right)+m^1_{(0,1)}(\chi)\left(\frac{\partial f}{\partial y}(x,y)\right)\right]\\ &+\frac{1}{2w^2}\left[m^2_{(2,0)}(\chi)\frac{\partial^2 f}{\partial x^2}(x,y)+m^2_{(1,1)}(\chi)\frac{\partial^2 f}{\partial x\partial y}(x,y)+m^2_{(0,2)}(\chi)\frac{\partial^2 f}{\partial y^2}(x,y)\right]. \end{aligned}$$

Now, we estimate I_2. By using (19), we have

$$\begin{aligned} |I_2| \leq 128\left(1+x^2+y^2\right)\Bigg\{&\Omega(f_{xx};\delta_1,\delta_2)\left[\left(\frac{k_1}{w}-x\right)^2+\frac{1}{\delta_2^3}\left|\frac{k_2}{w}-y\right|^3\left(\frac{k_1}{w}-x\right)^2+\frac{1}{\delta_1^3}\left|\frac{k_1}{w}-x\right|^5+\frac{1}{\delta_1^3\delta_2^3}\left|\frac{k_2}{w}-y\right|^3\left|\frac{k_1}{w}-x\right|^5\right]\\ &+2\Omega(f_{xy};\delta_1,\delta_2)\left[\left|\frac{k_1}{w}-x\right|\left|\frac{k_2}{w}-y\right|+\frac{1}{\delta_2^3}\left|\frac{k_2}{w}-y\right|^4\left|\frac{k_1}{w}-x\right|+\frac{1}{\delta_1^3}\left|\frac{k_1}{w}-x\right|^4\left|\frac{k_2}{w}-y\right|+\frac{1}{\delta_1^3\delta_2^3}\left|\frac{k_2}{w}-y\right|^4\left|\frac{k_1}{w}-x\right|^4\right]\\ &+\Omega(f_{yy};\delta_1,\delta_2)\left[\left(\frac{k_2}{w}-y\right)^2+\frac{1}{\delta_2^3}\left|\frac{k_2}{w}-y\right|^5+\frac{1}{\delta_1^3}\left|\frac{k_1}{w}-x\right|^3\left(\frac{k_2}{w}-y\right)^2+\frac{1}{\delta_1^3\delta_2^3}\left|\frac{k_2}{w}-y\right|^5\left|\frac{k_1}{w}-x\right|^3\right]\Bigg\}, \end{aligned}$$

where we consider the following inequalities

$$\begin{array}{rlcrl} \left|\frac{k_1}{w}-x\right|^2 & \leq\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2 & , & \left|\frac{k_1}{w}-x\right|^2 & \leq\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\\ \left|\frac{k_1}{w}-x\right|^5 & \leq\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^{5/2} & , & \left|\frac{k_2}{w}-y\right|^5 & \leq\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^{5/2}\\ \left|\frac{k_1}{w}-x\right|\left|\frac{k_2}{w}-y\right| & \leq 2\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right) & , & \left|\frac{k_1}{w}-x\right|^4\left|\frac{k_2}{w}-y\right|^4 & \leq 2^4\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^4\\ \left|\frac{k_1}{w}-x\right|^3\left|\frac{k_2}{w}-y\right|^2 & \leq 2^{5/2}\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^{5/2} & , & \left|\frac{k_1}{w}-x\right|^2\left|\frac{k_2}{w}-y\right|^3 & \leq 2^{5/2}\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^{5/2}\\ \left|\frac{k_1}{w}-x\right|^3\left|\frac{k_2}{w}-y\right|^5 & \leq 2^3\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^4 & , & \left|\frac{k_1}{w}-x\right|^5\left|\frac{k_2}{w}-y\right|^3 & \leq 2^3\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^4\\ \left|\frac{k_1}{w}-x\right|^4\left|\frac{k_2}{w}-y\right|^1 & \leq 2^3\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^{5/2} & , & \left|\frac{k_1}{w}-x\right|^1\left|\frac{k_2}{w}-y\right|^4 & \leq 2^3\left(\left|\frac{k_1}{w}-x\right|^2+\left|\frac{k_2}{w}-y\right|^2\right)^{5/2} \end{array}.$$

So, we get

$$
\begin{aligned}
|I_2| \leq 128\left(1+x^2+y^2\right)\Big\{&\Omega\left(f_{xx};\delta_1,\delta_2\right)\Big[\frac{1}{w^2}\left(|k_1-wx|^2+|k_2-wy|^2\right)+\frac{2^{5/2}}{\delta_2^3 w^5}\left(|k_1-wx|^2+|k_2-wy|^2\right)^{5/2} \\
&+\frac{1}{\delta_1^3 w^5}\left(|k_1-wx|^2+|k_2-wy|^2\right)^{5/2}+\frac{2^3}{\delta_1^3\delta_2^3 w^8}\left(|k_1-wx|^2+|k_2-wy|^2\right)^4\Big] \\
&+2\Omega\left(f_{xy};\delta_1,\delta_2\right)\Big[\frac{2}{w^2}\left(|k_1-wx|^2+|k_2-wy|^2\right)+\frac{2^3}{\delta_2^3 w^5}\left(|k_1-wx|^2+|k_2-wy|^2\right)^{5/2} \\
&+\frac{2^3}{\delta_1^3 w^5}\left(|k_1-wx|^2+|k_2-wy|^2\right)^{5/2}+\frac{2^4}{\delta_1^3\delta_2^3 w^8}\left(|k_1-wx|^2+|k_2-wy|^2\right)^4\Big] \\
&+\Omega\left(f_{yy};\delta_1,\delta_2\right)\Big[\frac{1}{w^2}\left(|k_1-wx|^2+|k_2-wy|^2\right)+\frac{1}{\delta_2^3 w^5}\left(|k_1-wx|^2+|k_2-wy|^2\right)^{5/2} \\
&+\frac{2^{5/2}}{\delta_1^3 w^5}\left(|k_1-wx|^2+|k_2-wy|^2\right)^{5/2}+\frac{2^3}{\delta_1^3\delta_2^3 w^8}\left(|k_1-wx|^2+|k_2-wy|^2\right)^4\Big]\Big\} \\
=128\left(1+x^2+y^2\right)\Big\{&\Omega\left(f_{xx};\delta_1,\delta_2\right)\Big[\frac{1}{w^2}M_2(\chi)+\frac{2^{5/2}}{\delta_2^3 w^5}M_5(\chi)+\frac{2^{5/2}}{\delta_1^3 w^5}M_5(\chi)+\frac{2^3}{\delta_1^3\delta_2^3 w^8}M_8(\chi)\Big] \\
&2\Omega\left(f_{xy};\delta_1,\delta_2\right)\Big[\frac{2}{w^2}M_2(\chi)+\frac{2^3}{\delta_2^3 w^5}M_5(\chi)+\frac{2^3}{\delta_1^3 w^5}M_5(\chi)+\frac{2^4}{\delta_1^3\delta_2^3 w^8}\Big]M_8(\chi) \\
&\Omega\left(f_{yy};\delta_1,\delta_2\right)\Big[\frac{1}{w^2}M_2(\chi)+\frac{2^{5/2}}{\delta_1^3 w^5}M_5(\chi)+\frac{2^{5/2}}{\delta_2^3 w^5}M_5(\chi)+\frac{2^3}{\delta_1^3\delta_2^3 w^8}M_8(\chi)\Big]\Big\}.
\end{aligned}
$$

Hence choosing $\delta_1=\delta_2=w^{-1}$ we have

$$
\begin{aligned}
&\Big|w\left[\left(G_w^\chi f\right)(x,y)-f(x,y)\right]-m_{(1,0)}^1(\chi)\left(\frac{\partial f}{\partial x}(x,y)\right)-m_{(0,1)}^1(\chi)\left(\frac{\partial f}{\partial y}(x,y)\right) \\
&\quad-\frac{1}{2w}\left[m_{(2,0)}^2(\chi)\frac{\partial^2 f}{\partial x^2}(x,y)+m_{(1,1)}^2(\chi)\frac{\partial^2 f}{\partial x\partial y}(x,y)+m_{(0,2)}^2(\chi)\frac{\partial^2 f}{\partial y^2}(x,y)\right]\Big| \\
&\leq 256\left(1+x^2+y^2\right)\left[\Omega\left(f_{xx};w^{-1},w^{-1}\right)+2\Omega\left(f_{xy};w^{-1},w^{-1}\right)+\Omega\left(f_{yy};w^{-1},w^{-1}\right)\right] \\
&\qquad\times\left[\frac{1}{w}\left(M_2(\chi)+8M_5(\chi)+8M_8(\chi)\right)\right]
\end{aligned}
$$

which is desired. □

Corollary 3.6.

1. *By similar consideration of Theorem 3.5, if we assume $f\in C^1\left(\mathbb{R}^2\right)$ such that $f'\in C_{\tilde{w}}^*\left(\mathbb{R}^2\right)$ and χ satisfies the assumptions $(\chi 1),(\chi 2),(\chi 3)$ for $\beta=7$ and $(\chi 4)$ for $l=1$, there holds*

$$
\begin{aligned}
&\Big|w\left[\left(G_w^\chi f\right)(x,y)-f(x,y)\right]-m_{(1,0)}^1(\chi)\left(\frac{\partial f}{\partial x}(x,y)\right)-m_{(0,1)}^1(\chi)\left(\frac{\partial f}{\partial y}(x,y)\right)\Big| \\
&\leq 128\left(1+x^2+y^2\right)\left(\Omega\left(f_x;w^{-1},w^{-1}\right)+\Omega\left(f_y;w^{-1},w^{-1}\right)\right)\left[\sqrt{2}M_1(\chi)+6M_4(\chi)+2^{7/2}M_7(\chi)\right].
\end{aligned}
$$

2. *In addition to the assumptions of Theorem 3.5, if we also assume $m_{(1,0)}^1(\chi)=m_{(0,1)}^1(\chi)=0$ then we have*

$$
\begin{aligned}
&\Big|w^2\left[\left(G_w^\chi f\right)(x,y)-f(x,y)\right]-\frac{1}{2}\left[m_{(2,0)}^2(\chi)\frac{\partial^2 f}{\partial x^2}(x,y)+m_{(1,1)}^2(\chi)\frac{\partial^2 f}{\partial x\partial y}(x,y)+m_{(0,2)}^2(\chi)\frac{\partial^2 f}{\partial y^2}(x,y)\right]\Big| \\
&\leq 256\left(1+x^2+y^2\right)\left[\Omega\left(f_{xx};w^{-1},w^{-1}\right)+2\Omega\left(f_{xy};w^{-1},w^{-1}\right)+\Omega\left(f_{yy};w^{-1},w^{-1}\right)\right]\left[M_2(\chi)+8M_5(\chi)+8M_8(\chi)\right].
\end{aligned}
$$

Corollary 3.7.

1. *Under the assumption of Corollary 3.6, in view of (4), we have a qualitative form of the asymptotic formula for G_w^χ, i.e.,*

$$
\lim_{w\to+\infty} w\left[\left(G_w^\chi f\right)(x,y)-f(x,y)\right]=m_{(1,0)}^1(\chi)\left(\frac{\partial f}{\partial x}(x,y)\right)-m_{(0,1)}^1(\chi)\left(\frac{\partial f}{\partial y}(x,y)\right)
$$

2. *In addition to the assumptions of Theorem 3.5, if we also assume $m_{(1,0)}^1(\chi)=m_{(0,1)}^1(\chi)=0$ in view of (4), then we have a qualitative form of the asymptotic formula for G_w^χ by f'',*

$$
\lim_{w\to+\infty} w^2\left[\left(G_w^\chi f\right)(x,y)-f(x,y)\right]=\frac{1}{2}\left[m_{(2,0)}^2(\chi)\frac{\partial^2 f}{\partial x^2}(x,y)+m_{(1,1)}^2(\chi)\frac{\partial^2 f}{\partial x\partial y}(x,y)+m_{(0,2)}^2(\chi)\frac{\partial^2 f}{\partial y^2}(x,y)\right].
$$

4 Numerical Examples

In this section, we provide numerical examples of bivariate generalized sampling series for functions belonging to weighted spaces. We give an example to show that the family of bivariate generalized sampling operator converges to the function which generates it. To do this, first, we need a kernel which satisfies the assumptions given Theorem 3.2.

2-dimensional kernels are built up as products of one dimensional ones (see, [28]). By going out this aim, let χ_1, χ_2 be one-dimensional kernels, i.e., satisfies the followings:

i. χ_1 and χ_2 are continuous on $\mathbb{R}$,

ii. the discrete algebraic moments of order 0

$$m_0(\chi_i, u) = 1,$$

for every $u \in \mathbb{R}$ and $i = 1, 2$.

Now, set $\chi(\underline{u}) = \chi_1(u_1)\chi_2(u_2)$. Then χ is a kernel satisfying the conditions $(\chi 1)$ and $(\chi 2)$, since the product of two continuous function is continuous and

$$m_0(\chi) = m_0(\chi_1)m_0(\chi_2) = 1.$$

As concerns is there any finite absolute moment of kernel χ, one can consider β_1 and β_2 for which makes the absolute moment of order β_1 and β_2 of χ_1 and χ_2, respectively, are finite. Then the $\beta > 0$, which makes the absolute moment of order β of χ is finite, can be found easily.

Now, we recall central B-spline function as the univariate kernel. The central B-spline of order $n \in \mathbb{N}$ is defined by:

$$B_n(t) := \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^i\binom{n}{j}\left(\frac{n}{2}+t-j\right)_+^{n-1}, \qquad t \in \mathbb{R},$$

where $(t)_+ := max\{t, 0\}, t \in \mathbb{R}$. To make everything easier, we take $n = 3$. Then, applying above procedure, bivariate central B-splines of order 3 comes out:

$$B_3(t_1,t_2) = B_3(t_1)B_3(t_2) = \begin{cases} \left(\frac{3}{4}-|t_1|^2\right)\left(\frac{3}{4}-|t_2|^2\right), & |t_1| \le \frac{1}{2}, |t_2| \le \frac{1}{2} \\ \left(\frac{3}{4}-|t_1|^2\right)\left(\frac{1}{2}\left[\frac{3}{2}-|t_2|\right]^2\right), & |t_1| \le \frac{1}{2}, \frac{1}{2} < |t_2| \le \frac{3}{2} \\ \left(\frac{1}{2}\left[\frac{3}{2}-|t_1|\right]^2\right)\left(\frac{3}{4}-t_2^2\right), & \frac{1}{2} < |t_1| \le \frac{3}{2}, |t_2| \le \frac{1}{2} \\ 0, & |t_1| > \frac{3}{2}, |t_2| > \frac{3}{2} \end{cases},$$

see, for more, [26, 27, 50].

There are several kernel functions other than B-spline (see [11, 25, 36]), but we do not state them since they are not related to following examples.

The bivariate generalized sampling series with the kernel bivariate B-spline of order 3, $G_w^{B_3} f$, arises as

$$\left(G_w^{B_3} f\right)(x,y) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} f\left(\frac{k_1}{w},\frac{k_2}{w}\right)B_3(w(x,y)-(k_1,k_2))$$

for every $(x, y) \in \mathbb{R}^2$ and $w > 0$. For example, we apply now $G_w^{B_3}$ to the special functions f and g belonging to $C_{\widetilde{w}}\left(\mathbb{R}^2\right)$, defined by

$$f(x,y) := \begin{cases} \cos\left(\frac{x^2+y^2}{\pi}\right), & |x| < 3 \text{ and } |y| < 2 \\ x^2-\sqrt{|y|}, & |x| < 3 \text{ and } |y| \ge 2 \\ \log x^3 + y^2, & |x| \ge 3 \text{ and } |y| < 2 \\ \frac{x}{y}, & |x| \ge 3 \text{ and } |y| \ge 2 \end{cases}$$

and

$$g(x,y) := x\sin(\pi y).$$

We present the numerical evaluations of difference of bivariate sampling operators and functions f and g at random variables in the Table 1 and Table 2, respectively:

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Table 1: $|(G_w^{B_3} f)(x) - f(x)|$ at some random values

w	$\|(G_w^{B_3} f)(-1.2, 1.7) - f(-1.2, 1.7)\|$	$\|(G_w^{B_3} f)(-1.9, 2.9) - f(1.4, 2.1)\|$
5	0,00784666	0,0102532
20	0,00049493	0,000640821
45	0,0000978565	0,000126582
100	0,0000198176	0,0000256328

w	$\|(G_w^{B_3} f)(3.5, -1.75) - f(3.5, -1.75)\|$	$\|(G_w^{B_3} f)(3.1, 2.55) - f(3.1, 2.55)\|$
5	0,288321	0,00190473
20	0,000548462	0,000116893
45	0,0000877549	0,00002312
100	0,0000219388	0,00000467399

Table 2: $|(G_w^{B_3} g)(x) - g(x)|$ at some random values

w	$\|(G_w^{B_3} g)(2, 1.55) - g(2, 1.55)\|$	$\|(G_w^{B_3} g)(0.2, 4.15) - g(0.2, 4.15)\|$
5	0, 0959681	0, 00403352
100	0, 000243682	0, 0000112008
500	9.74806×10^{-6}	4.48069×10^{-7}
1000	2.43702×10^{-6}	1.12018×10^{-7}

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