PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special Issue ICOMSS'22,Volume 16 · 2023 · Pages 26–37

An Extension of Korovkin Theorem via P-Statistical $\mathcal{A}$-Summation Process

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Communicated by Stefano De Marchi

Abstract

In the present work, we study and prove Korovkin-type approximation theorems for linear operators defined on derivatives of functions by means of $\mathcal{A}$-summation process via statistical convergence with respect to power series method. We give an example that our theorem is stronger. Also, we study the rate of convergence of these operators. Finally, we summarize our results and we show the importance of the study.

1 Introduction

The study of Korovkin-type theory, which is an area of active research, was initiated by Korovkin in 1960 in his pioneering paper [12]. It deals with approximation of continuous functions on a compact interval gives conditions in order to decide whether a sequence of positive linear operators converges to the identity operator. The convergence is guaranteed on the whole space via test functions in these theorems. The list of variations and extensions of Korovkin type results is too much (see e.g. [1, 2, 3, 6, 7, 10, 11, 16, 20, 21, 22]). Also, at points of discontinuity, they often converge to the average of the left and right limits of the function. The main aim of using summability theory has been to make a non-convergent sequence convergent. Hence, if the classical convergence method does not work, then it would be beneficial to use the summability theory. Recently, by relaxing the positivity condition on linear operators, various approximation theorems have also been gotten. Aiming for the improvement of the classical Korovkin theory, Duman and Anastassiou [8] proved the Korovkin-type approximation theorem for linear operators without the condition of positivity via the concept of statistical convergence such that one another interesting convergence method ([9, 24]). More recently, Şahin Bayram and Yıldız proved this approximation theorem via power series method and P-statistical convergence which is a new and interesting statistical type convergence (see [19, 26]).

In this paper, we give some Korovkin-type approximation theorems for linear operators defined on derivatives of functions by using $\mathcal{A}$-summation process via statistical convergence with respect to power series method. We give an example that our theorems make more sense. Also, we study the rates of convergence of linear operators with the help of the modulus of continuity. Finally, we summarize our results.

Let k be a non-negative integer. By $C^k[0,1]$, we denote the space of the k-times continuously differentiable functions on $[0,1]$ endowed with the sup-norm $\|.\|$. Throughout the paper, we use the following function spaces:

$$\begin{array}{ll}\mathcal{D}_+^1=\left\{g\in C^1[0,1]:g^{'}\geq 0\right\}, & \mathcal{D}_+=\left\{g\in C[0,1]:g\geq 0\right\},\\ \mathcal{D}_+^2=\left\{g\in C^2[0,1]:g^{''}\geq 0\right\}, & \mathcal{D}_{+,1}=\left\{g\in C^1[0,1]:g\geq 0\right\},\\ \mathcal{D}_-^1=\left\{g\in C^1[0,1]:g^{'}\leq 0\right\}, & \mathcal{D}_{+,2}=\left\{g\in C^2[0,1]:g\geq 0\right\},\\ \mathcal{D}_-^2=\left\{g\in C^2[0,1]:g^{''}\leq 0\right\}. & \end{array}$$

Let $\left(p_j\right)$ be a non-negative real sequence such that $p_0>0$ and the corresponding power series

$$p\left(s\right):=\sum_{j=0}^{\infty}p_js^j$$

has radius of convergence R with $0<R\leq\infty$. If the limit

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$$\lim_{0<s\to R^-}\frac{1}{p(s)}\sum_{j=0}^{\infty}x_jp_js^j=L$$

exists, then we say that $x=(x_j)$ is convergent in the sense of power series method ([14, 23]). It is worthwhile to point out that the method is regular if and only if $\lim_{0<s\to R^-}\frac{p_js^j}{p(s)}=0$ for every j (see, e.g. [4]).

Prior to introducing the next definitions, Ünver and Orhan [25] have recently introduced P-density of $F\subset\mathbb{N}_0$ and the definition of P-statistical convergence for single sequences. Now, we recall this convergence.

Let $F\subset\mathbb{N}_0$. If the limit

$$\delta_P(F):=\lim_{s\to R^-}\frac{1}{p(s)}\sum_{j\in F}p_js^j$$

exists, then $\delta_P(F)$ is called the P-density of F. Note that, from the definition of a power series method and P-density it is obvious that $0\le\delta_P(F)\le 1$ whenever it exists.

Let $x=(x_j)$ be a real sequence. Then x is said to be statistically convergent with respect to power series method (P-statistically convergent) to L if for any $\varepsilon>0$

$$\lim_{s\to R^-}\frac{1}{p(s)}\sum_{j\in F_\varepsilon}p_js^j=0$$

where $F_\varepsilon=\{j\in\mathbb{N}_0:|x_j-L|\ge\varepsilon\}$, that is $\delta_P(F_\varepsilon)=0$ for any $\varepsilon>0$. In this case we write $st_P-\lim x_j=L$.

Now we give an example such that there exists a sequence that is P-statistically convergent for a particular power series method but is not statistically convergent and vice versa.

Example 1.1. Let

$$p_j=\begin{cases}1, & j \text{ is a square,}\\ 0, & \text{otherwise,}\end{cases}$$

and

$$x_j=\begin{cases}0, & j \text{ is a square,}\\ \sqrt{j}, & \text{otherwise.}\end{cases}$$

Then $x=(x_j)$ is $P-$statistically convergent to zero. Now it is easy to see that x is not statistically convergent. Conversely let $x=(x_j)$ be a sequence defined by

$$x_j=\begin{cases}\sqrt{j}, & j \text{ is a square,}\\ 0, & \text{otherwise.}\end{cases}$$

On the other-hand x is statistically convergent to zero but it is not $P-$statistically convergent.

This example shows that statistical convergence and P-statistical convergence are incompatible (see also, [18, 25]).

Here and throughout the paper, we assume that power series method is regular.

Now, we recall the summation process:

Let $\mathcal{A}:=(A^{(n)})=(a_{kj}^{(n)})$ be a sequence of infinite matrices with non-negative real entries. Let (T_j) be a sequence of linear operators from $C^k[0,1]$ into itself. A sequence (T_j) is called an $\mathcal{A}$-summation process on $C^k[0,1]$ if $(T_j(g))$ $\mathcal{A}$-summable to g for every $g\in C^k[0,1]$, i.e.,

$$\lim_k\left\|\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j(g)-g\right\|=0,\ \text{uniformly in } n, \tag{1}$$

where it is assumed that the series in (1) converges for each k, n and g ([1, 17]). Throughout the paper, (T_j) be sequence of linear operators of $C^k[0,1]$ into itself such

$$\sup_{n,k}\sum_{j=1}^{\infty}a_{kj}^{(n)}\left\|T_j(1)\right\|<\infty. \tag{2}$$

Furthermore, for each $k,n\in\mathbb{N}$ and $g\in C^k[0,1]$, let

$$L_k^{(n)}(g;x):=\sum_{j=1}^{\infty}a_{kj}^{(n)}T_j(g;x)$$

which is well defined by (2), and belongs to $C^k[0,1]$.

We assume throughout the paper that the test functions are

$$e_t(x)=x^t,\ t=0,1,2,3,4.$$

2 Approximation Properties via P-Statistical $\mathcal{A}$-summation process

Now we can give a first main theorem:

Theorem 2.1. *Let $\mathcal{A} := (A^{(n)}) = (a_{kj}^{(n)})$ be a sequence of infinite matrices with non-negative real entries satisfying the condition (2). Let $\left(T_j\right)$ be a sequence of linear operators from $C^2[0,1]$ into itself and $T_j\left(\mathcal{D}_{+,2} \cap \mathcal{D}_{+}^2\right) \subset \mathcal{D}_{+,2}$, for all $j \in \mathbb{N}$. Then for every $g \in C^2[0,1]$,*

$$st_P - \lim \left\| L_k^{(n)}(g) - g \right\| = 0, \text{ uniformly in } n, \tag{3}$$

if and only if

$$st_P - \lim \left\| L_k^{(n)}(e_t) - e_t \right\| = 0, \text{ uniformly in } n \ (t = 0,1,2).$$

Proof. First, we suppose that (3) holds for every $g \in C^2[0,1]$. Then, $e_t \in C^2[0,1]$, $t = 0,1,2$, $L_k^{(n)}(e_t)$ is $P-$statistically convergent to e_t for each $t = 0,1,2$. Therefore, we only need to prove the sufficiency part. Let $x \in [0,1]$ be fixed and let $g \in C^2[0,1]$. Since g is bounded and uniformly continuous on $[0,1]$, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$-\varepsilon - \frac{2N_1\beta}{\delta^2}\Psi_x(y) \le g(y) - g(x) \le \varepsilon + \frac{2N_1\beta}{\delta^2}\Psi_x(y) \tag{4}$$

holds for all $y \in [0,1]$ and for any $\beta \ge 1$ where $N_1 = \|g\|$ and $\Psi_x(y) = (y-x)^2$. Then by (4) we have

$$f_{1,\beta}(y) := \varepsilon + \frac{2N_1\beta}{\delta^2}\Psi_x(y) + g(y) - g(x) \ge 0, \tag{5}$$

$$f_{2,\beta}(y) := \varepsilon + \frac{2N_1\beta}{\delta^2}\Psi_x(y) - g(y) + g(x) \ge 0. \tag{6}$$

Also, for all $y \in [0,1]$,

$$f_{1,\beta}^{''}(y) := \frac{4N_1\beta}{\delta^2} + g^{''}(y) \text{ and } f_{2,\beta}^{''}(y) := \frac{4N_1\beta}{\delta^2} - g^{''}(y).$$

Since $g^{''}$ is bounded on $[0,1]$, we can select $\beta \ge 1$ such that $f_{1,\beta}^{''}(y) \ge 0$, $f_{2,\beta}^{''}(y) \ge 0$, for each $y \in [0,1]$. For this reason, $f_{1,\beta}$, $f_{2,\beta} \in \mathcal{D}_{+,2} \cap \mathcal{D}_{+}^2$ and then by the hypothesis

$$T_j\left(f_{t,\beta}; x\right) \ge 0, \text{ for all } j \in \mathbb{N}, \ x \in [0,1] \text{ and } t = 1,2 \tag{7}$$

and thus

$$L_k^{(n)}\left(f_{t,\beta}; x\right) \ge 0, \text{ for } n,k \in \mathbb{N}_0, \ x \in [0,1] \text{ and } t = 1,2.$$

From (4)-(7) and linearity $\left(T_j\right)$, we get

$$\begin{aligned} \varepsilon L_k^{(n)}(e_0;x) + \frac{2N_1\beta}{\delta^2}L_k^{(n)}(\Psi_x;x) + L_k^{(n)}(g;x) - g(x)L_k^{(n)}(e_0;x) &\ge 0, \\ \varepsilon L_k^{(n)}(e_0;x) + \frac{2N_1\beta}{\delta^2}L_k^{(n)}(\Psi_x;x) - L_k^{(n)}(g;x) + g(x)L_k^{(n)}(e_0;x) &\ge 0. \end{aligned}$$

This can be restated as follows:

$$\begin{aligned} & -\varepsilon L_k^{(n)}(e_0;x) - \frac{2N_1\beta}{\delta^2}L_k^{(n)}(\Psi_x;x) + g(x)\left(L_k^{(n)}(e_0;x) - e_0(x)\right) \\ \le\ & L_k^{(n)}(g;x) - g(x) \\ \le\ & \varepsilon L_k^{(n)}(e_0;x) + \frac{2N_1\beta}{\delta^2}L_k^{(n)}(\Psi_x;x) + g(x)\left(L_k^{(n)}(e_0;x) - e_0(x)\right). \end{aligned}$$

Then we have

$$\begin{aligned} \left|L_k^{(n)}(g;x) - g(x)\right| \le\ & \varepsilon + \frac{2N_1\beta}{\delta^2}L_k^{(n)}(\Psi_x;x) \\ & + (\varepsilon + |g(x)|)\left|L_k^{(n)}(e_0;x) - e_0(x)\right|. \end{aligned}$$

We can calculate,

$$\left\|L_k^{(n)}(g) - g\right\| \le \varepsilon + M_1 \sum_{t=0}^{2} \left\|L_k^{(n)}(e_t) - e_t\right\| \tag{8}$$

where $M_1 = \max\left\{\varepsilon + N_1 + \frac{2N_1\beta}{\delta^2}, \frac{4N_1\beta}{\delta^2}\right\}$. Now, for a given $\varepsilon' > 0$, choose an $\varepsilon < \varepsilon'$ and setting

$$\begin{aligned} U &= \left\{k \in \mathbb{N}_0 : \left\|L_k^{(n)}(g) - g\right\| \ge \varepsilon'\right\}, \\ U_t &= \left\{k \in \mathbb{N}_0 : \left\|L_k^{(n)}(e_t) - e_t\right\| \ge \frac{\varepsilon' - \varepsilon}{3M_1}\right\}, \ t = 0,1,2. \end{aligned}$$

Now it is easy to see that

$$U \subseteq \bigcup_{t=0}^{2} U_t.$$

This allowed us to write, thanks to (8), we get that

$$0 \leq \delta_P(U) \leq \sum_{t=0}^{2} \delta_P(U_t). \tag{9}$$

From hypothesis and the inequailty (9), the proof is complete. □

Theorem 2.2. *Let $\mathcal{A} := (A^{(n)}) = (a_{kj}^{(n)})$ be a sequence of infinite matrices with non-negative real entries satisfying the condition (2). Let (T_j) be a sequence of linear operators from $C^2[0,1]$ into itself and $T_j\left(\mathcal{D}_{+,2} \cap \mathcal{D}_{-}^2\right) \subset \mathcal{D}_{-}^2$, for all $j \in \mathbb{N}$. Then for all $g \in C^2[0,1]$*

$$st_P - \lim \left\| \left(L_k^{(n)}\right)''(g) - g'' \right\| = 0, \text{ uniformly in } n, \tag{10}$$

if and only if

$$st_P - \lim \left\| \left(L_k^{(n)}\right)''(e_t) - e_t'' \right\| = 0, \text{ uniformly in } n \ (t = 0,1,2,3,4). \tag{11}$$

Proof. It is clear that (10) implies that (11). Now, let $g \in C^2[0,1]$ and $x \in [0,1]$ be fixed. Similar to the proof of Theorem 2.1, we can write as follows:

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$-\varepsilon - \frac{2N_2\beta}{\delta^2}\alpha_x''(y) \leq g''(y) - g''(x) \leq \varepsilon + \frac{2N_2\beta}{\delta^2}\alpha_x''(y) \tag{12}$$

holds for all $y \in [0,1]$ and for any $\beta \geq 1$ where $N_2 = \left\| g'' \right\|$ and $\alpha_x(y) = 1 - \dfrac{(y-x)^4}{12}$. Take the following functions on $[0,1]$:

$$w_{1,\beta}(y) := \frac{2N_2\beta}{\delta^2}\alpha_x(y) + g(y) - \frac{\varepsilon}{2}y^2 - \frac{g''(y)}{2}y^2 \geq 0,$$

$$w_{2,\beta}(y) := \frac{2N_2\beta}{\delta^2}\alpha_x(y) - g(y) - \frac{\varepsilon}{2}y^2 + \frac{g''(y)}{2}y^2 \geq 0.$$

Also, by (12) and for all $y \in [0,1]$,

$$w_{1,\beta}''(y) \leq 0 \text{ and } w_{2,\beta}''(y) \leq 0,$$

which gives $w_{1,\beta}, w_{2,\beta} \in \mathcal{D}_{-}^2$. Also we can show that $\alpha_x(y) \geq \dfrac{11}{12}$ for all $y \in [0,1]$. Then inequality

$$\frac{\left(\pm g(y) + \frac{\varepsilon}{2} \pm \frac{g''(x)}{2}y^2\right)\delta^2}{2N_2\alpha_x(y)} \leq \frac{(N_1 + N_2 + \varepsilon)\delta^2}{N_2}$$

holds for all $y \in [0,1]$, where $N_1 = \|g\|$ and $N_2 = \left\| g'' \right\|$ as stated before. As we know, we can select $\beta \geq 1$ so that $w_{1,\beta}(y) \geq 0$, $w_{2,\beta}(y) \geq 0$, for each $y \in [0,1]$ and then $w_{1,\beta}, w_{2,\beta} \in \mathcal{D}_{+,2} \cap \mathcal{D}_{-}^2$. Then by the hypothesis

$$T_j''\left(w_{t,\beta}; x\right) \leq 0, \text{ for all } j \in \mathbb{N},\ x \in [0,1] \text{ and } t = 1,2$$

and hence

$$\left(L_k^{(n)}\right)''\left(w_{t,\beta}; x\right) \leq 0 \text{ for } n,k \in \mathbb{N}_0,\ x \in [0,1] \text{ and } t = 1,2.$$

Then we have

$$\frac{2N\beta}{\delta^2}\left(L_k^{(n)}\right)''(\alpha_x; x) + \left(L_k^{(n)}\right)''(g; x) - \frac{\varepsilon}{2}\left(L_k^{(n)}\right)''(e_2; x) - \frac{g''(x)}{2}\left(L_k^{(n)}\right)''(e_2; x) \leq 0,$$

$$\frac{2N\beta}{\delta^2}\left(L_k^{(n)}\right)''(\alpha_x; x) - \left(L_k^{(n)}\right)''(g; x) - \frac{\varepsilon}{2}\left(L_k^{(n)}\right)''(e_2; x) + \frac{g''(x)}{2}\left(L_k^{(n)}\right)''(e_2; x) \leq 0,$$

and thus

$$\begin{aligned}
& \frac{2N\beta}{\delta^2}\left(L_k^{(n)}\right)''(\alpha_x; x) - \frac{\varepsilon}{2}\left(L_k^{(n)}\right)''(e_2; x) + \frac{g''(x)}{2}\left(L_k^{(n)}\right)''(e_2; x) - g''(x) \\
\leq\ & \left(L_k^{(n)}\right)''(g; x) - g''(x) \\
\leq\ & -\frac{2N\beta}{\delta^2}\left(L_k^{(n)}\right)''(\alpha_x; x) + \frac{\varepsilon}{2}\left(L_k^{(n)}\right)''(e_2; x) + \frac{g''(x)}{2}\left(L_k^{(n)}\right)''(e_2; x) - g''(x).
\end{aligned}$$

Since $\alpha_x \in \mathcal{D}_{+,2} \cap \mathcal{D}_{-}^2$, then we can write $\left(L_k^{(n)}\right)''(\alpha_x;x) \leq 0$ and using this

$$\begin{aligned}\left|\left(L_k^{(n)}\right)''(g;x)-g''(x)\right| &\leq -\frac{2N_2\beta}{\delta^2}\left(L_k^{(n)}\right)''(\alpha_x;x)+\frac{\varepsilon}{2}\left(L_k^{(n)}\right)''(e_2;x)\\ &\quad+\frac{\left|g''(x)\right|}{2}\left|\left(L_k^{(n)}\right)''(e_2;x)-2\right|.\end{aligned}$$

Thus

$$\begin{aligned}\left|\left(L_k^{(n)}\right)''(g;x)-g''(x)\right| &\leq \varepsilon+\frac{\varepsilon+\left|g''(x)\right|}{2}\left|\left(L_k^{(n)}\right)''(e_2;x)-e_2''(x)\right|\\ &\quad+\frac{2N_2\beta}{\delta^2}\left(L_k^{(n)}\right)''(-\alpha_x;x).\end{aligned} \tag{13}$$

Now we compute the quantity $\left(L_k^{(n)}\right)''(-\alpha_x;x)$ in inequality (13). Then we have

$$\begin{aligned}\left(L_k^{(n)}\right)''(-\alpha_x;x) &= \left(L_k^{(n)}\right)''\left(\frac{(y-x)^4}{12}-1;x\right)\\ &\leq \frac{1}{12}\left(L_k^{(n)}\right)''(e_4;x)-\frac{x}{3}\left(L_k^{(n)}\right)''(e_3;x)+\frac{x^2}{2}\left(L_k^{(n)}\right)''(e_2;x)\\ &\quad-\frac{x^3}{3}\left(L_k^{(n)}\right)''(e_1;x)+\left(\frac{x^4}{12}-1\right)\left(L_k^{(n)}\right)''(e_0;x)\\ &= \frac{1}{12}\left\{\left(L_k^{(n)}\right)''(e_4;x)-e_4''(x)\right\}-\frac{x}{3}\left\{\left(L_k^{(n)}\right)''(e_3;x)-e_3''(x)\right\}\\ &\quad+\frac{x^2}{2}\left\{\left(L_k^{(n)}\right)''(e_2;x)-e_2''(x)\right\}-\frac{x^3}{3}\left\{\left(L_k^{(n)}\right)''(e_1;x)-e_1''(x)\right\}\\ &\quad+\left(\frac{x^4}{12}-1\right)\left\{\left(L_k^{(n)}\right)''(e_0;x)-e_0''(x)\right\}.\end{aligned}$$

Combining this with (13), for every $\varepsilon>0$ we get

$$\begin{aligned}\left\|\left(L_k^{(n)}\right)''(g)-g''\right\| &\leq \varepsilon+\left(\frac{\varepsilon+\left|g''(x)\right|}{2}+\frac{2N_2\beta}{\delta^2}\right)\left\|\left(L_k^{(n)}\right)''(e_2)-e_2''\right\|\\ &\quad+\frac{N_2\beta}{6\delta^2}\left\|\left(L_k^{(n)}\right)''(e_4)-e_4''\right\|\\ &\quad+\frac{2N_2\beta}{3\delta^2}\left\|\left(L_k^{(n)}\right)''(e_3)-e_3''\right\|\\ &\quad+\frac{2N_2\beta}{3\delta^2}\left\|\left(L_k^{(n)}\right)''(e_1)-e_1''\right\|\\ &\quad+\frac{2N_2\beta}{3\delta^2}\left(1-\frac{x^4}{12}\right)\left\|\left(L_k^{(n)}\right)''(e_0)-e_0''\right\|.\end{aligned} \tag{14}$$

Therefore we derive, for every $\varepsilon>0$, that

$$\left\|\left(L_k^{(n)}\right)''(g)-g''\right\| \leq \varepsilon+M_2\sum_{t=0}^{4}\left\|\left(L_k^{(n)}\right)''(e_t)-e_t''\right\| \tag{15}$$

where $M_2=\dfrac{\varepsilon+N_2}{2}+\dfrac{N_2\beta}{\delta}$ and $N_2=\left\|g''\right\|$ as mentioned before. Now, for a given $r'>0$, choose an $\varepsilon<r'$ and setting

$$\begin{aligned}K &= \left\{k\in\mathbb{N}_0:\left\|\left(L_k^{(n)}\right)''(g)-g''\right\|\geq r'\right\},\\ K_t &= \left\{k\in\mathbb{N}_0:\left\|\left(L_k^{(n)}\right)''(e_t)-e_t''\right\|\geq\frac{r'-\varepsilon}{5M_2}\right\},\ t=0,1,2,3,4.\end{aligned}$$

Now it is easy to see that

$$K\subseteq\bigcup_{t=0}^{4}K_t$$

which gives, thanks to (15), we have

$$0\leq\delta_P(K)\leq\sum_{t=0}^{4}\delta_P(K_t). \tag{16}$$

From hypothesis and the inequailty (16), this completes the proof. □

Theorem 2.3. *Let $\mathcal{A} := (A^{(n)}) = (a_{kj}^{(n)})$ be a sequence of infinite matrices with non-negative real entries satisfying the condition (2). Let $\left(T_j\right)$ be a sequence of linear operators from $C^1[0,1]$ into itself and $T_j\left(\mathcal{D}_{+,1} \cap \mathcal{D}_+^1\right) \subset \mathcal{D}_+^1$, for all $j \in \mathbb{N}$. Then for all $g \in C^1[0,1]$*

$$st_P - \lim \left\| \left(L_k^{(n)}\right)^{'}(g) - g^{'} \right\| = 0, \text{ uniformly in } n, \tag{17}$$

if and only if

$$st_P - \lim \left\| \left(L_k^{(n)}\right)^{'}(e_t) - e_t^{'} \right\| = 0, \text{ uniformly in } n, \ t = 0,1,2,3. \tag{18}$$

Proof. We only need to prove the implication (18)⇒(17). Let $g \in C^1[0,1]$ and $x \in [0,1]$ be fixed. Then, for every $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that

$$-\varepsilon - \frac{2N_3\beta}{\delta^2}\gamma_x^{'}(y) \le g^{'}(y) - g^{'}(x) \le \varepsilon + \frac{2N_3\beta}{\delta^2}\gamma_x^{'}(y)$$

holds for all $y \in [0,1]$ and for any $\beta \ge 1$ where $N_3 = \left\|g^{'}\right\|$ and $\gamma_x(y) = \dfrac{(y-x)^3}{3} + 1$. Now consider the functions defined by

$$\delta_{1,\beta}(y) := \frac{2N_3\beta}{\delta^2}\gamma_x(y) - g(y) + \varepsilon y + y g^{'}(x),$$
$$\delta_{2,\beta}(y) := \frac{2N_3\beta}{\delta^2}\gamma_x(y) + g(y) + \varepsilon y - y g^{'}(x).$$

We can easily show that $\delta_{1,\beta}$ and $\delta_{2,\beta}$ belong to $\mathcal{D}_+^1$ for any $\beta \ge 1$ such that $\delta_{1,\beta}(y) \ge 0$, $\delta_{2,\beta}(y) \ge 0$. Also we compute that $\gamma_x(y) \ge \dfrac{2}{3}$ for all $y \in [0,1]$, then inequality

$$\frac{\left(\pm g(y) - \varepsilon y \pm g^{'}(x) y\right)\delta^2}{2N_3\gamma_x(y)} \le \frac{(N_1 + N_3 + \varepsilon)\delta^2}{N_3}$$

holds for all $y \in [0,1]$, where $N_1 = \|g\|$ as stated before. Now we can choose $\beta \ge 1$ such that $\delta_{1,\beta}(y) \ge 0$, $\delta_{2,\beta}(y) \ge 0$, for each $y \in [0,1]$ and hence $\delta_{1,\beta}$, $\delta_{2,\beta} \in \mathcal{D}_{+,1} \cap \mathcal{D}_+^1$. Then by the hypothesis

$$T_j^{\prime}\left(\delta_{t,\beta}; x\right) \ge 0, \text{ for all } j \in \mathbb{N}, \ x \in [0,1] \text{ and } t = 1,2$$

and

$$\left(L_k^{(n)}\right)^{'}\left(\delta_{t,\beta}; x\right) \ge 0, \text{ for all } k \in \mathbb{N}_0, \ x \in [0,1] \text{ and } t = 1,2.$$

Then we have

$$\frac{2N_3\beta}{\delta^2}\left(L_k^{(n)}\right)^{'}(\gamma_x;x) - \left(L_k^{(n)}\right)^{'}(g;x) + \varepsilon\left(L_k^{(n)}\right)^{'}(e_1;x) + g^{'}(x)\left(L_k^{(n)}\right)^{'}(e_1;x) \ge 0,$$
$$\frac{2N_3\beta}{\delta^2}\left(L_k^{(n)}\right)^{'}(\gamma_x;x) + \left(L_k^{(n)}\right)^{'}(g;x) + \varepsilon\left(L_k^{(n)}\right)^{'}(e_2;x) - g^{'}(x)\left(L_k^{(n)}\right)^{'}(e_1;x) \ge 0,$$

and thus

$$\begin{aligned} &\frac{2N_3\beta}{\delta^2}\left(L_k^{(n)}\right)^{'}(\gamma_x;x) - \varepsilon\left(L_k^{(n)}\right)^{'}(e_1;x) + g^{'}(x)\left(L_k^{(n)}\right)^{'}(e_1;x) - g^{'}(x) \\ \le\ & \left(L_k^{(n)}\right)^{'}(g;x) - g^{'}(x) \\ \le\ & -\frac{2N_3\beta}{\delta^2}\left(L_k^{(n)}\right)^{'}(\gamma_x;x) + \varepsilon\left(L_k^{(n)}\right)^{'}(g_1;x) + g^{'}(x)\left(L_k^{(n)}\right)^{'}(e_1;x) - g^{'}(x). \end{aligned}$$

Since the function $\gamma_x \in \mathcal{D}_{+,1} \cap \mathcal{D}_+^1$, we have $\left(L_k^{(n)}\right)^{'}(\gamma_x) \in \mathcal{D}_+^1$

$$\begin{aligned} \left|\left(L_k^{(n)}\right)^{'}(g;x) - g^{'}(x)\right| \ \le\ & \varepsilon + \left(\varepsilon + \left|g^{'}(x)\right|\right)\left|\left(L_k^{(n)}\right)^{'}(e_1;x) - e_1^{'}(x)\right| \\ & + \frac{2N_3\beta}{\delta^2}\left(L_k^{(n)}\right)^{'}(\gamma_x;x) \end{aligned} \tag{19}$$

holds. Also, we can calculate

$$\begin{aligned} \left(L_k^{(n)}\right)^{'}(\gamma_x;x) &= \left(L_k^{(n)}\right)^{'}\left(\frac{(y-x)^3}{3} + 1; x\right) \\ &\le \frac{1}{3}\left(L_k^{(n)}\right)^{'}(e_3;x) - x\left(L_k^{(n)}\right)^{'}(e_2;x) + x^2\left(L_k^{(n)}\right)^{'}(e_1;x) \\ &\quad + \left(1 - \frac{x^3}{3}\right)\left(L_k^{(n)}\right)^{'}(e_0;x) \\ &= \frac{1}{3}\left\{\left(L_k^{(n)}\right)^{'}(e_3;x) - e_3^{'}(x)\right\} - x\left\{\left(L_k^{(n)}\right)^{'}(e_2;x) - e_2^{'}(x)\right\} \\ &\quad + x^2\left\{\left(L_k^{(n)}\right)^{'}(e_1;x) - e_1^{'}(x)\right\} + \left(1 - \frac{x^3}{3}\right)\left\{\left(L_k^{(n)}\right)^{'}(e_0;x) - e_0^{'}(x)\right\}, \end{aligned}$$

combining this with (19), for every $\varepsilon>0$, we get

$$\begin{aligned}\left|\left(L_k^{(n)}\right)'(g;x)-g'(x)\right| &\leq \varepsilon+\left(\varepsilon+\left|g'(x)\right|+\frac{2N_3\beta x^2}{\delta^2}\right)\left|\left(L_k^{(n)}\right)'(e_1;x)-e_1'(x)\right| \\ &+\frac{2N_3\beta}{3\delta^2}\left|\left(L_k^{(n)}\right)'(e_3;x)-e_3'(x)\right| \\ &+\frac{2N_3\beta x}{3\delta^2}\left|\left(L_k^{(n)}\right)'(e_2;x)-e_2'(x)\right| \\ &+\frac{2N_3\beta}{3\delta^2}\left(1-\frac{x^3}{3}\right)\left|\left(L_k^{(n)}\right)'(e_0;x)-e_0'(x)\right|.\end{aligned}$$

Hence we can write,

$$\left\|\left(L_k^{(n)}\right)'(g)-g'\right\|\leq\varepsilon+M_3\sum_{k=0}^{3}\left\|\left(L_k^{(n)}\right)'(e_k)-e_k'\right\| \tag{20}$$

where $M_3=\varepsilon+N_3+\dfrac{2N_3\beta}{\delta}$. Now, for a given $r>0$, choose an $\varepsilon<r$ and setting

$$\begin{aligned}R &= \left\{k\in\mathbb{N}_0:\left\|\left(L_k^{(n)}\right)'(g)-g'\right\|\geq r\right\}, \\ R_t &= \left\{k\in\mathbb{N}_0:\left\|\left(L_k^{(n)}\right)'(e_t)-e_t'\right\|\geq\frac{r-\varepsilon}{4M_3}\right\},\ t=0,1,2,3.\end{aligned}$$

Now we see that

$$R\subseteq\bigcup_{t=0}^{3}R_t$$

which gives, thanks to (20), we get

$$0\leq\delta_P(R)\leq\sum_{t=0}^{3}\delta_P(R_t). \tag{21}$$

From hypothesis and the inequailty (21), the proof is complete. □

3 An Application

In this section, we give an interesting application showing that in general, our results are stronger than classical ones.

Example 3.1. Assume that $\mathcal{A}:=(A^{(n)})=(a_{kj}^{(n)})$ is a sequence of infinite matrices defined by $a_{kj}^{(n)}=\frac{1}{k}$ if $n\leq j\leq n+k-1$ and $a_{kj}^{(n)}=0$ otherwise. In this case $\mathcal{A}$–summability method reduces to almost convergence of sequences introduced by Lorentz [13, 15]. Let $g[x_0,x_1,...,x_i]$ denote the divided difference of the function $g\in C[0,1]$ in the points $x_0,x_1,...,x_i\in[0,1]$ where $i=0,1,2,....$ Also, let $G=\{g\in C[0,1]:\exists M>0$ such that $g[x_0,x_1,x_2]\leq M\ \ \forall x_0,x_1,x_2\in[0,1]\}$ and $\left(T_j\right)$, $T_j:G\to C[0,1]$, be the sequence of linear operators defined by

$$T_j(g;x)=\begin{cases} g[0]+g\left[0,\frac{1}{n}\right]x+g\left[0,\frac{1}{n},\frac{2}{n}\right]x^2, & x\in\left[0,\frac{1}{n}\right],\\ T_j(g;\frac{i}{n})+DT_j(g;\frac{i}{n})\left(x-\frac{i}{n}\right) & x\in\left(\frac{i}{n},\frac{i+1}{n}\right],\\ +g\left[\frac{1}{n},\frac{i+1}{n},\frac{i+2}{n}\right]\left(x-\frac{i}{n}\right)^2, & i=1,...,n-3,\\ T_j(g;\frac{n-2}{n})+DT_j(g;\frac{n-2}{n})\left(x-\frac{n-2}{n}\right) & \\ +g\left[\frac{n-2}{n},\frac{n-1}{n},1\right]\left(x-\frac{n-2}{n}\right)^2, & x\in\left(\frac{n-2}{n},1\right],\end{cases}$$

where D denote the differential operator ([5]). Then, we know from [5] that T_j is not positive operator and $T_j(e_0;x)=e_0(x)$, $T_j(e_1;x)=e_1(x)$, $T_j(e_2;x)=\frac{x}{j}+e_2(x)$ $\forall x\in[0,1]$. Hence, $T_j(e_t)$ converges uniformly to e_t for $t=0,1,2$ and thus $T_j(g)$ converges uniformly to g as $n\to\infty$ for all $g\in G$. Now, using this operator T_j, we introduce the following linear operators

$$T_j^*(g;x)=a_j(x)T_j(g;x) \tag{22}$$

where $a_j(x)=1+(-1)^j\frac{x}{2}$. Observe that

$$\left\|L_k^{(n)}(.)\right\|\leq\sup_{n,k}\sum_{j=1}^{\infty}a_{kj}^{(n)}\left\|T_j^*(1)\right\|<\infty.$$

Now, it is clear that

$$
\begin{aligned}
\left|L_k^{(n)}(e_0;x)-e_0(x)\right| &= \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} a_j(x)\, T_j^*(e_0;x)-e_0(x)\right| \\
&\leq \left|\frac{1}{k}\sum_{j=n}^{n+k-1}\left(1+(-1)^j\frac{x}{2}\right)-1\right| \\
&= \left|\frac{1}{k}\sum_{j=n}^{n+k-1}(-1)^j\frac{x}{2}\right| \leq \left|\frac{1}{k}\sum_{j=n}^{n+k-1}(-1)^j\right|.
\end{aligned}
$$

Since $((-1)^j)$ is almost convergent to zero, we have,

$$
st_P-\lim\left\|L_k^{(n)}(e_0)-e_0\right\|=0, \text{ uniformly in } n.
$$

Similarly, we obtain,

$$
\begin{aligned}
\left|L_k^{(n)}(e_1;x)-e_1(x)\right| &\leq \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} a_j(x)\, x-x\right| \\
&= \left|x\frac{1}{k}\sum_{j=n}^{n+k-1} a_j(x)-x\right| \\
&\leq \left|\frac{1}{k}\sum_{j=n}^{n+k-1}\left(1+(-1)^j\frac{x}{2}\right)-1\right| \\
&= \left|\frac{1}{k}\sum_{j=n}^{n+k-1}(-1)^j\frac{x}{2}\right| \leq \left|\frac{1}{k}\sum_{j=n}^{n+k-1}(-1)^j\right|,
\end{aligned}
$$

which gives

$$
st_P-\lim\left\|L_k^{(n)}(e_1)-e_1\right\|=0, \text{ uniformly in } n.
$$

Finally, we get,

$$
\begin{aligned}
\left|L_k^{(n)}(e_2;x)-e_2(x)\right| &= \left|\sum_{j=1}^{\infty} a_{kj}^{(n)} a_j(x)\, T_j^*(e_2;x)-e_2(x)\right| \\
&= \left|\sum_{j=1}^{\infty} a_{kj}^{(n)}\left(1+(-1)^j\frac{x}{2}\right)\left(\frac{x}{j}+e_2(x)\right)-e_2(x)\right| \\
&\leq x^2\left|\sum_{j=1}^{\infty} a_{kj}^{(n)}\left(1+(-1)^j\frac{x}{2}\right)-1\right| \\
&\quad +\left(x-x^2\right)\left|\sum_{j=1}^{\infty} a_{kj}^{(n)}\left(1+(-1)^j\frac{x}{2}\right)\frac{1}{j}-1\right| \\
&\leq \left|\frac{1}{k}\sum_{j=n}^{n+k-1}(-1)^j\right|+\frac{1}{4}\left|\frac{1}{k}\sum_{j=n}^{n+k-1}\frac{(-1)^j}{j}\right|.
\end{aligned}
$$

So, we have

$$
st_P-\lim\left\|L_k^{(n)}(e_2)-e_2\right\|=0, \text{ uniformly in } n.
$$

Since $\left(a_j(x)\right)$ is neither classically uniform convergent nor statistically uniform convergent to 1, this example shows that the classical and statistical Korovkin theorems for the sequence of non-positive operators (see [5, 8]) do not work. Then, using P-statistical $\mathcal{A}$-summation process, we get Korovkin type approximation result. Hence, our Theorem 2.1 works for the operators T_j defined by (22) by using this method.

4 Rates of P-Statistical Convergence

The main aim of this section is giving us the degree of approximation by means of linear operators.

The classical modulus of continuity, denoted by $\omega(g,\delta)$ is defined by

$$
\omega(g,\delta)=\sup_{|y-x|\leq\delta}|g(y)-g(x)|
$$

where δ is a positive constant, $g \in C[a,b]$ and we will use the following inequality:

for any $\eta > 0$

$$\omega(g;\eta\delta) \leq (1+[\eta])\omega(g;\delta)$$

where $[\eta]$ is represents the greatest integer less than or equal to η.

Now we prove some estimates rate of convergence for Korovkin-type theorems via P-statistical $\mathcal{A}$-summation process.

Theorem 4.1. *Let $\mathcal{A} := (A^{(n)}) = (a_{kj}^{(n)})$ be a sequence of infinite matrices with non-negative real entries satisfying the condition (2). Let (T_j) be a sequence of linear operators from $C^2[0,1]$ into itself and $T_j\left(\mathcal{D}_{+,2} \cap \mathcal{D}_+^2\right) \subset \mathcal{D}_{+,2}$, for all $j \in \mathbb{N}$. Suppose that the following conditions hold:*

$$st_P - \lim \left\| L_k^{(n)}(e_0) - e_0 \right\| = 0, \text{ uniformly in } n, \tag{23}$$

and

$$st_P - \lim \omega\left(g, \delta_k^{(n)}\right) = 0, \text{ uniformly in } n, \tag{24}$$

where $\delta_k^{(n)} := \sqrt{\left\| L_k^{(n)}(\Psi_x) \right\|}$ and $\Psi_x(y) = (y-x)^2$, then we have, for all $g \in C^2[0,1]$

$$st_P - \lim \left\| L_k^{(n)}(g) - g \right\| = 0, \text{ uniformly in } n.$$

Proof. Let $x \in [0,1]$ be fixed and let $g \in C^2[0,1]$. We can write that

$$-\left(1+\frac{\beta}{\delta^2}\Psi_x(y)\right)\omega(g,\delta) \tag{25}$$

$$\leq \quad g(y)-g(x) \leq \left(1+\frac{\beta}{\delta^2}\Psi_x(y)\right)\omega(g,\delta) \tag{26}$$

for all $y \in [0,1]$ and for any $\beta \geq 1$ where $\Psi_x(y) = (y-x)^2$. Then by (25) we get that

$$f_{1,\beta}(y) := \left(1+\frac{\beta}{\delta^2}\Psi_x(y)\right)\omega(g,\delta) + g(y) - g(x) \geq 0, \tag{27}$$

$$f_{2,\beta}(y) := \left(1+\frac{\beta}{\delta^2}\Psi_x(y)\right)\omega(g,\delta) - g(y) + g(x) \geq 0. \tag{28}$$

Also for all $y \in [0,1]$,

$$f_{1,\beta}''(y) := \frac{2\beta}{\delta^2}\omega(g,\delta) + g''(y) \text{ and } f_{2,\beta}''(y) := \frac{2\beta}{\delta^2}\omega(g,\delta) - g''(y).$$

Since g'' is bounded on $[0,1]$ we can choose $\beta \geq 1$ such a way that $f_{1,\beta}''(y) \geq 0$, $f_{2,\beta}''(y) \geq 0$, for each $y \in [0,1]$. Hence $f_{1,\beta}$, $f_{2,\beta} \in \mathcal{D}_{+,2} \cap \mathcal{D}_+^2$ and then by the hypothesis

$$T_j\left(f_{t,\beta};x\right) \geq 0, \text{ for all } j \in \mathbb{N}, \ x \in [0,1] \text{ and } t = 1,2 \tag{29}$$

and hence

$$L_k^{(n)}\left(f_{t,\beta};x\right) \geq 0, \text{ for } k \in \mathbb{N}_0, \ x \in [0,1] \text{ and } t = 1,2.$$

Thanks to (27)–(29) and the linearity of (T_j) we get

$$L_k^{(n)}(e_0;x)\,\omega(g,\delta) + \frac{\beta\omega(g,\delta)}{\delta^2}L_k^{(n)}(\Psi_x;x) + L_k^{(n)}(g;x) - g(x)\,L_k^{(n)}(e_0;x) \geq 0,$$

$$L_k^{(n)}(e_0;x)\,\omega(g,\delta) + \frac{\beta\omega(g,\delta)}{\delta^2}L_k^{(n)}(\Psi_x;x) - L_k^{(n)}(g;x) + g(x)\,L_k^{(n)}(e_0;x) \geq 0,$$

thus

$$\begin{aligned} & -L_k^{(n)}(e_0;x)\,\omega(g,\delta) - \frac{\beta\omega(g,\delta)}{\delta^2}L_k^{(n)}(\Psi_x;x) \\ \leq \quad & g(x)\,L_k^{(n)}(e_0;x) - L_k^{(n)}(g;x) \\ \leq \quad & L_k^{(n)}(e_0;x)\,\omega(g,\delta) + \frac{\beta\omega(g,\delta)}{\delta^2}L_k^{(n)}(\Psi_x;x). \end{aligned}$$

Then we obtain

$$\begin{aligned} \left|L_k^{(n)}(g;x) - g(x)\right| \quad \leq \quad & \omega(g,\delta) + (\omega(g,\delta) + |g(x)|)\left|L_k^{(n)}(e_0;x) - e_0(x)\right| \\ & + \frac{\beta\omega(g,\delta)}{\delta^2}L_k^{(n)}(\Psi_x;x). \end{aligned}$$

If we take $\delta := \delta_k^{(n)} := \sqrt{\left\| L_k^{(n)}(\Psi_x) \right\|}$ and taking supremum $x \in [0,1]$, then we get

$$\begin{aligned} \left\| L_k^{(n)}(g) - g \right\| &\leq (1+\beta)\,\omega\left(g, \delta_k^{(n)}\right) \\ &\quad + \left(\omega\left(g, \delta_k^{(n)}\right) + \|g\|\right) \left\| L_k^{(n)}(e_0) - e_0 \right\|. \end{aligned} \tag{30}$$

Now, for a given $\varepsilon' > 0$, setting

$$\begin{aligned} W &= \left\{ k \in \mathbb{N}_0 : \left\| L_k^{(n)}(g) - g \right\| \geq \varepsilon' \right\}, \\ W_1 &= \left\{ k \in \mathbb{N}_0 : \omega\left(g, \delta_k^{(n)}\right) \geq \frac{\varepsilon'}{3(1+\beta)} \right\}, \\ W_2 &= \left\{ k \in \mathbb{N}_0 : \omega\left(g, \delta_k^{(n)}\right) \geq \sqrt{\frac{\varepsilon'}{3}} \right\}, \\ W_3 &= \left\{ k \in \mathbb{N}_0 : \left\| L_k^{(n)}(e_0) - e_0 \right\| \geq \sqrt{\frac{\varepsilon'}{3\|g\|}} \right\}. \end{aligned}$$

Now it is easy to see that

$$W \subseteq \bigcup_{t=1}^{3} W_t$$

which gives, thanks to (20), we have

$$0 \leq \delta_P(W) \leq \sum_{t=1}^{3} \delta_P(W_t). \tag{31}$$

From hypothesis and the inequailty (31), the proof is complete. □

It is worth noting that the proofs of following theorems may be given as in Theorem 4.1. So we omit them.

Theorem 4.2. *Let $\mathcal{A} := (A^{(n)}) = (a_{kj}^{(n)})$ be a sequence of infinite matrices with non-negative real entries satisfying the condition (2). Let (T_j) be a sequence of linear operators from $C^2[0,1]$ into itself and $T_j\left(\mathcal{D}_{+,2} \cap \mathcal{D}_-^2\right) \subset \mathcal{D}_-^2$, for all $j \in \mathbb{N}$. Assume that the following conditions hold:*

$$st_P - \lim \left\| \left(L_k^{(n)}\right)''(e_0) - e_0'' \right\| = 0, \text{ uniformly in } n,$$

and

$$st_P - \lim \omega\left(g'', \delta_k^{(n)}\right) = 0, \text{uniformly in } n,$$

where $\delta_k^{(n)} := \sqrt{\left\| \left(L_k^{(n)}\right)''(-\alpha_x) \right\|}$ *and* $\alpha_x(y) = -\frac{(y-x)^4}{12} + 1$, *then we have, for all* $g \in C^2[0,1]$

$$st_P - \lim \left\| \left(L_k^{(n)}\right)'' g - g'' \right\| = 0, \text{uniformly in } n.$$

Theorem 4.3. *Let $\mathcal{A} := (A^{(n)}) = (a_{kj}^{(n)})$ be a sequence of infinite matrices with non-negative real entries satisfying the condition (2). Let (T_j) be a sequence of linear operators from $C^1[0,1]$ into itself and $T_j\left(\mathcal{D}_{+,1} \cap \mathcal{D}_+^1\right) \subset \mathcal{D}_+^1$, for all $j \in \mathbb{N}$. Assume that the following conditions hold:*

$$st_P - \lim \left\| \left(L_k^{(n)}\right)'(e_0) - e_0' \right\| = 0, \text{ uniformly in } n,$$

and

$$st_P - \lim \omega\left(g', \delta_k^{(n)}\right) = 0, \text{uniformly in } n,$$

where $\delta_k^{(n)} := \sqrt{\left\| \left(L_k^{(n)}\right)'(\gamma_x) \right\|}$ *and* $\gamma_x(y) = \frac{(y-x)^3}{3} + 1$, *then we have, for all* $g \in C^1[0,1]$

$$st_P - \lim \left\| \left(L_k^{(n)}\right)'(g) - g' \right\| = 0, \text{uniformly in } n.$$

5 Conclusion

Now, we can give some reduced results emphasizing the importance of Theorem 2.1, Theorem 2.2 and Theorem 2.3 in approximation theory with choices:

► If we take $A^{(n)}$ by the identity matrix, then from our Theorem 2.1, Theorem 2.2 and Theorem 2.3, we immediately get the Korovkin type theorems given by Şahin Bayram and Yıldız in [19] by means of P-statistical convergence.

► Let $\mathcal{A} := (A^{(n)}) = (a_{kj}^{(n)})$ be a sequence of infinite matrices with non-negative real entries satisfying the condition (2). Let (T_j) be a sequence of linear operators from $C[0,1]$ into itself and $T_j(\mathcal{D}_+) \subset \mathcal{D}_+$, for all $j \in \mathbb{N}$. Then for all $g \in C[0,1]$,

$$st_P - \lim \left\| L_k^{(n)}(g) - g \right\| = 0, \text{ uniformly in } n,$$

if and only if

$$st_P - \lim \left\| L_k^{(n)}(e_t) - e_t \right\| = 0, \text{ uniformly in } n \ (t = 0, 1, 2).$$

► We remark that all our theorems also work on any compact subset of $\mathbb{R}$ instead of the unit interval $[0,1]$.

► Theorem 2.3 works if we replace the condition $T_j\left(\mathcal{D}_{+,1} \cap \mathcal{D}_+^1\right) \subset \mathcal{D}_+^1$ by $T_j\left(\mathcal{D}_{+,1} \cap \mathcal{D}_-^1\right) \subset \mathcal{D}_-^1$. To prove this, it is enough to consider the function $\mu_x(y) = 1 - \frac{(y-x)^3}{3}$ instead of $\gamma_x(y)$ defined in the proof of Theorem 2.3.

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