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Almost lacunary strong $(\mathbf{D}, \mu)$- convergence of order α

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Abstract

In this paper we present a new almost strong sequence space of order α generated by real matrix D and also we examine some properties of this sequence space.

1 Introduction and Background

Let s denote the set of all real and complex sequences $\xi = (\xi_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $\xi = (\xi_k)$ normed by $||\xi|| = \sup_n |\xi_n|$, respectively.
We now quote the definition of Banach limits.

A Banach limit is a functional (see, Banach [1]) $L : l_\infty \to R$ which satisfies the following properties:

1. $L(\xi) \geq 0$ if $n \geq 0$ (i.e. $\xi_n \geq 0$ for all n),
2. $L(e) = 1$ where $e = (1, 1, \ldots)$,
3. $L(S\xi) = L(\xi)$,

where S denotes the shift operator on l_∞, that is, $S : l_\infty \to l_\infty$ defined by $S(\xi_n) = \{\xi_{n+1}\}$.
Let B be the set of all Banach limits on l_∞. A sequence $\xi \in \ell_\infty$ is said to be almost convergent if all Banach limits of ξ coincide.
Let $\hat{c}$ denote the space of almost convergent sequences. Lorentz [5] has shown that

$$\hat{c} = \left\{ \xi \in l_\infty : \lim_m t_{m,n}(\xi) \text{ exists uniformly in } n \right\}$$

where

$$t_{m,n}(\xi) = \frac{\xi_n + \xi_{n+1} + \xi_{n+2} + \cdots + \xi_{n+m}}{m+1}.$$

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \ldots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. Freedman at al [4] defined the space of lacunary strongly convergent sequences N_θ as follows:

$$N_\theta = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |\xi_k - L| = 0, \text{ for some } L \right\}.$$

There is a strong connection between N_θ and the space w of strongly Cesàro summable sequences which is defined by, (see, Maddox [8])

$$w = \left\{ \xi = (\xi_k) : \lim_n \frac{1}{n} \sum_{k=0}^{n} |\xi_k - L| = 0, \text{ for some } L \right\}.$$

In the special case where $\theta = (2^r)$, we have $N_\theta = w$.

The space AC_θ of lacunary almost convergent sequences and the space $|AC_\theta|$ of lacunary strongly almost convergent sequences were introduced by Das and Mishra[3] as follows:

$$AC_\theta = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} (\xi_{k+i} - L) = 0, \text{ for some } L \text{ uniformly in } i \right\}.$$

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and

$$|AC_\theta| = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |\xi_{k+i} - L| = 0, \text{ for some } L \text{ uniformly in } i \right\}.$$

Note that in the special case where $\theta = 2^r$, we have $AC_\theta = \hat{c}$ and $|AC_\theta| = [\hat{c}]$. which is defined by Maddox [6].

A modulus function ψ is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $\psi(\xi) = 0$ if and only if $\xi = 0$,

(ii) $\psi(\xi + \rho) \le \psi(\xi) + \psi(\rho)$ for all $\xi, \rho \ge 0$,

(iii) ψ increasing,

(iv) ψ is continuous from the right at zero.

Since $|\psi(\xi) - \psi(\rho)| \le \psi(|\xi - \rho|)$, it follows from condition (iv) that ψ is continuous on $[0, \infty)$.
Ruckle [10] used the idea of a modulus function ψ to construct a class of FK spaces

$$L(\psi) = \left\{ \xi = (\xi_k) : \sum_{k=1}^{\infty} \psi(|\xi_k|) < \infty \right\}$$

The space $L(\psi)$ is closely related to the space l_1 which is an $L(\psi)$ space with $\psi(\xi) = \xi$ for all real $\xi \ge 0$.

Maddox [7] generalized the well-known spaces w_0, w and w_∞ of strongly summable by introducing some properties of the sequence spaces $w_0(\psi)$, $w(\psi)$ and $w_\infty(\psi)$ using a modulus ψ.

In 1999, E. Savas [11] defined the class of sequences which are strongly almost Cesàro summable with respect to modulus as follows:

$$[\hat{c}(\psi, p)] = \left\{ \xi : lim_n \frac{1}{n} \sum_{k=1}^{n} \psi(|\xi_{k+i} - L|)^{p_k} = 0, \text{ for some L, uniformly in } i \right\}$$

and

$$[\hat{c}(\psi, p)]_0 = \left\{ \xi : lim_n \frac{1}{n} \sum_{k=1}^{n} \psi(|\xi_{k+i}|)^{p_k} = 0, \text{ uniformly in } i \right\}.$$

where $p = (p_k)$ is a sequence of strictly positive real numbers and ψ be a modulus.

By a μ-function we understood a continuous non-decreasing function $\mu(u)$ defined for $u \ge 0$ and such that $\mu(0) = 0, \mu(u) > 0$, for $u > 0$ and $\mu(u) \to \infty$ as $u \to \infty$, (see, [12], [13]).

On the other hand in [2] a different direction was given to the study of lacunary statistical convergence of order α, $0 < \alpha \le 1$ where the notion of lacunary statistical convergence was introduced by replacing h_r by h_r^α in the denominator in the definition of lacunary statistical convergence.

In the present paper, we introduce and study some properties of the following sequence space of order α that is defined using the μ- function and modulus.

2 Main Results

Let μ and ψ be given μ-function and modulus function, respectively and $p = (p_k)$ be a sequence of positive real numbers. Moreover, let $D = (d_{nk})(n, k = 1, 2, ...)$ be a real matrix, a lacunary sequence $\theta = (k_r)$ and $0 < \alpha \le 1$ be given. Then we define the following sequence space,

$$\hat{N}_\theta^\alpha(D, \mu, \psi, p)_0 = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r^\alpha} \sum_{n \in I_r} \psi\Big(\Big| \sum_{k=1}^{\infty} d_{nk} \mu(|\xi_{k+i}|) \Big| \Big)^{p_k} = 0, \text{ uniformly in } i \right\},$$

where h_r^α denote the αth power $(h_r)^\alpha$ of h_r, that is $h^\alpha = (h_r^\alpha) = (h_1^\alpha, h_2^\alpha, h_3^\alpha, ...)$.

If $\xi \in \hat{N}_\theta^\alpha(D, \mu, \psi)_0$, the sequence ξ is said to be lacunary strong (D, μ)- almost convergent of order α to zero with respect to a modulus ψ. When $\mu(\xi) = \xi$ for all ξ, we obtain

$$\hat{N}_\theta^\alpha(D, \psi, p)_0 = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r} \sum_{n \in I_r} \psi(\Big| \sum_{k=1}^{\infty} d_{nk} (|\xi_{k+i}|) \Big|)^{p_k} = 0, \text{ uniformly in } i \right\}.$$

If we take $\psi(\xi) = \xi$, we write

$$\hat{N}_\theta^\alpha(D,\mu,p)_0 = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r} \sum_{n\in I_r} \left| \sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|) \right|^{p_k} = 0, \text{ uniformly in } i \right\}.$$

If we take $p_k = p$, for all k, we have

$$\hat{N}_\theta^\alpha(D,\mu,\psi)_0 = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r^\alpha} \sum_{n\in I_r} f\Big(\left| \sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|) \right| \Big)^{p} = 0, \text{ uniformly in } i \right\}.$$

If we take $D = I$ and $\mu(\xi) = \xi$ respectively, then we have

$$(\hat{N}_\theta^\alpha)_0 = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r^\alpha} \sum_{k\in I_r} \psi(|\xi_{k+i}|)^{p_k} = 0, \text{ uniformly in } i \right\}.$$

If we define the matrix $D = (d_{nk})$ as follows:

$$d_{nk} := \begin{cases} \frac{1}{n}, & \text{if } \quad n \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

then we have,

$$\hat{N}_\theta^\alpha(C,\mu,\psi)_0 = \left\{ \xi = (\xi_k) : \lim_r \frac{1}{h_r^\alpha} \sum_{n\in I_r} \psi\Big(\left| \frac{1}{n} \sum_{k=1}^{n} \mu(|\xi_{k+i}|) \right| \Big)^{p_k} = 0, \text{ uniformly in } i \right\}.$$

In the next theorem we establish inclusion relations between $\hat{w}^\alpha(D,\mu,\psi,p)$ and $\hat{N}_\theta^\alpha(D,\mu,\psi,p)_0$. We now have

Theorem 2.1. *Let ψ be a any modulus function and let μ-function μ, $p = (p_k)$ be a sequence of positive real numbers, a real matrix D and the sequence θ be given. If*

$$\hat{w}^\alpha(D,\mu,\psi,p)_0 = \left\{ \xi = (\xi_k) : \lim_m \frac{1}{m} \sum_{n=1}^{m} \psi\Big(\left| \sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|) \right| \Big)^{p_k} = 0, \textit{ uniformly in } i \right\}.$$

then the following relations are true :
(a) *If* $\liminf_r q_r > 1$ *then we have* $\hat{w}^\alpha(D,\mu,\psi,p)_0 \subseteq \hat{N}_\theta^\alpha(D,\mu,\psi,p)_0$,
(b) *If* $\sup_r q_r < \infty$, *then we have* $\hat{N}_\theta^\alpha(D,\mu,\psi,p)_0 \subseteq \hat{w}^\alpha(D,\mu,\psi,p)_0$,
(c) $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, *then we have* $\hat{N}_\theta^\alpha(D,\mu,\psi,p)_0 = \hat{w}^\alpha(D,\mu,\psi,p)_0$.

Proof. (a) Let us suppose that $\xi \in \hat{w}^\alpha(D,\mu,\psi,p)$. There exists $\delta > 0$ such that $q_r > 1+\delta$ for all $r \geq 1$ and we have $h_r/k_r \geq \delta/(1+\delta)$ for sufficiently large r. Then, for all i,

$$\begin{aligned} \frac{1}{k_r^\alpha} \sum_{n=1}^{k_r} \psi\Big(\left| \sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|) \right| \Big)^{p_n} & \\ &\geq \frac{1}{k_r^\alpha} \sum_{n\in I_r} \psi \left| \sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|) \right| \Big)^{p_n} \\ &= \frac{h_r^\alpha}{k_r^\alpha} \frac{1}{h_r^\alpha} \sum_{n\in I_r} \psi\Big(\left| \sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|) \right| \Big)^{p_n} \\ &\geq \frac{\delta^\alpha}{(1+\delta)^\alpha} \frac{1}{h_r^\alpha} \sum_{n\in I_r} \psi\Big(\left| \sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|) \right| \Big)^{p_n}. \end{aligned}$$

Hence, $\xi \in \hat{N}_\theta^\alpha(D,\mu,\psi,p)_0$. (b) If $\limsup_r q_r < \infty$ then there exist $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $\xi \in \hat{N}_\theta^\alpha(D,\mu,\psi,p)_0$ and ε is an arbitrary positive number, then there exists an index j_0 such that for every $j \geq j_0$ and all i,

$$R_j = \frac{1}{h_j^\alpha} \sum_{n\in I_r} \psi\Big(\left| \sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|) \right| \Big)^{p_n} < \varepsilon.$$

Thus, we can also find $K > 0$ such that $R_j \leq K$ for all $j = 1, 2, \ldots$. Now let m be any integer with $k_{r-1} \leq m \leq k_r$, then we obtain, for all i

$$I = \frac{1}{m^\alpha}\sum_{n=1}^{m}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_k|)\Big|\Big)^{p_n} \leq \frac{1}{k_{r-1}^\alpha}\sum_{n=1}^{k_r}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p_n} = I_1 + I_2$$

where

$$I_1 = \frac{1}{k_{r-1}^\alpha}\sum_{j=1}^{j_0}\sum_{n\in I_j}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p_n}$$

$$I_2 = \frac{1}{k_{r-1}^\alpha}\sum_{j=j_{0+1}}^{m}\sum_{n\in I_j}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p_n}$$

It is easy to see that,

$$\begin{aligned}
I_1 &= \frac{1}{k_{r-1}^\alpha}\sum_{j=1}^{j_0}\sum_{n\in I_j}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p_n}\\
&= \frac{1}{k_{r-1}^\alpha}\Big(\sum_{n\in I_1}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p_n} + \ldots + \sum_{n\in I_{j_0}}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p_n}\\
&\leq \frac{1}{k_{r-1}^\alpha}(h_1R_1 + \ldots + h_{j_0}R_{j_0}),\\
&\leq \frac{1}{k_{r-1}^\alpha} j_0 k_{j_0}^\alpha sup_{1\leq i\leq j_0} R_i,\\
&\leq \frac{j_0 k_{j_0}^\alpha}{k_{r-1}^\alpha} K.
\end{aligned}$$

Moreover, we have for all i

$$\begin{aligned}
I_2 &= \frac{1}{k_{r-1}^\alpha}\sum_{j=j_0+1}^{m}\sum_{n\in I_j}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p_n}\\
&= \frac{1}{k_{r-1}^\alpha}\sum_{j=j_0+1}^{m}\Big(\frac{1}{h_j}\sum_{n\in I_j}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p_n} h_j\\
&\leq \varepsilon\frac{1}{k_{r-1}^\alpha}\sum_{j=j_0+1}^{m} h_j,\\
&\leq \varepsilon\frac{k_r^\alpha}{k_{r-1}^\alpha},\\
&= \varepsilon q_r^\alpha < \varepsilon.M.
\end{aligned}$$

Thus $I \leq \frac{j_0 k_{j_0}^\alpha}{k_{r-1}^\alpha}K + \varepsilon.M$. Finally, $\xi \in \hat{w}^\alpha(D,\mu,\psi,p)$.
The proof of (c) follows from (a) and (b). This completes the proof. □

Theorem 2.2. *Let $0 < \alpha \leq \beta \leq 1$ and p be a positive real number, then $\hat{N}_\theta^\alpha(D,\mu,\psi)_0 \subseteq \hat{N}_\theta^\beta(D,\mu,\psi)_0$.*

Proof. Let $\xi = (\xi_k) \in \hat{N}_\theta^\alpha(D,\mu,\psi)_0$. Then given α and β such that $< \alpha \leq \beta \leq 1$ and a positive real number p, we write

$$\frac{1}{h_r^\beta}\sum_{n\in I_r}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p} \leq \frac{1}{h_r^\alpha}\sum_{n\in I_r}\psi\Big(\Big|\sum_{k=1}^{\infty} d_{nk}\mu(|\xi_{k+i}|)\Big|\Big)^{p}$$

and we get that $\hat{N}_\theta^\alpha(D,\mu,\psi)_0 \subseteq \hat{N}_\theta^\beta(D,\mu,\psi)_0$. □

The proof of the following result is a consequence of Theorem 2.2.

Corollary 2.3. *Let $0 < \alpha \leq \beta \leq 1$ and p be a positive real number. Then*

i) *If $\alpha = \beta$, then $\hat{N}_\theta^\alpha(D,\mu,\psi)_0 = \hat{N}_\theta^\beta(D,\mu,\psi)_0$.*

ii) *$\hat{N}_\theta^\alpha(D,\mu,\psi)_0 \subseteq \hat{N}_\theta(D,\mu,\psi)_0$ for each $\alpha \in (0,1]$ and $0 < p < \infty$.*

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