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Approximation tools for detecting unforeseen sudden events

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Abstract

A splines-based algorithm is devised to detect the onset of unforeseen events, such as the opening of windows in a house, causing a sudden drop of the internal temperature. It allows also the reprogramming of the thermostat control to restore the desired conditions.

1 Introduction

In this paper we consider a practical problem, originating from a real life situation, namely the setting and control of the average temperature of a flat. This is a topic of interest in the engineering literature, see for instance [6, 8]. Also governments in cold countries suggest ways of preserving heat, [1, 2, 4, 5]. We assume that the house is heated and the user can set the thermostat at night to program a desired temperature for the next day. The device is able to start and eventually turn off the heating system, in order to keep the actual average temperature within a given preset tolerance from the desired target. In general, the goal would be to program the device once a day, as mentioned preferably at night, to turn on and off the heating system at suitable times, so that the fuel cost is minimized, but at the same time so that the desired temperature outcome is attained throughout the following day.

In this setting, however, unforeseen situations may occur, such as the opening of windows and outside doors, for which cold air from the outside can enter into the house and rapidly drive the interior temperature to lower values, below the forecast.

The goal of this investigation is to discuss an easy and cheap way of detecting such events and allow the restoring of the desired conditions. It turns out that simple approximation techniques allow to easily assess the occurrence of such situations and correct them.

The main mathematical tool that we use is represented by spline functions, [3], in that we have to assess the continuous change of the temperature based only on some discrete data points coming from the sampled temperature values.

The paper is organized as follows. In the next section we describe the partial differential equation model, mathematically describing the problem. The reduced, ordinary differential equations, model is presented in Section 3, where each and every one of its basic ingredients is suitably discussed. Section 4 addresses instead the onset of unforeseen sudden situations, presenting at first the solution of the detection problem, based on the analysis of historical data, assumed to be available. Once this is solved, the problem of the control of their occurrence in real time is addressed. A final discussion summarizes the findings and concludes the paper.

2 Theoretical background

The internal temperature of a house $u(x,t)$ is a function of two variables, space, meaning the position x in a room of the house, and time t. The open spatial domain Ω of interest is the interior of the house. The edge $\partial\Omega$ of this domain is composed of the perimeter walls, windows and doors, the ceiling or the roof (if present) and the radiant elements of the heating system, typically radiators or floor coils.

We assume that the temperature detected by the thermostat is

$$U(t) = \frac{1}{|\Omega|}\int_{\Omega} u(x,t)\,dx$$

The goal is to study how this average temperature varies over time. This implies to consider

$$T(t) := \frac{dU(t)}{dt}.$$

Applying Reynolds Theorem, [7], the heat equation and the divergence theorem followed by Neumann's boundary condition we get

$$T(t) = \int_{\Omega} u_t(x,t)\,dx = \int_{\Omega} \Delta u(x,t)\,dx = \int_{\Omega} \mathrm{div}(\nabla u)(x,t)\,dx = \int_{\partial\Omega} \nabla u(x,t)\cdot n\,dx = \int_{\partial\Omega} f(x,t)\,dx,$$

where n represents the outer normal to the boundary $\partial\Omega$ and $f(x,t)$ is the boundary flux.

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The calculations described above show that instead of investigating the temperature in the whole room, it is enough to determine how it varies on its walls. This is an advantage because it represents a dimension reduction, entailing therefore also notable computational advantages. Thus the study of the internal temperature is ultimately reduced to the study of the flux through the domain boundary. The model we propose to describe this flux is presented in the next section, equation (1).

3 The model

We now propose an ordinary differential equation model to describe the internal temperature of a house. The model consists of two equations for the two unknowns: the internal temperature T_{in} in the house and the temperature T_w of the heating system water. Additional quantities that play an essential role in the system are the external temperature $T_{out}(t)$, the starting (interior) temperature $T_{start}(t)$, which is recalculated each time the heating system is turned on, and the maximal temperature $T_{\max}$ at which the water is heated. The function $W(t)$ represents the intensity of the solar radiation, which depends on weather conditions. Time is measured in hours and sunrise and sunset are assumed to occur in the winter season at 9AM and 5PM (=17:00). The full equations, to be described below, read

$$\begin{aligned} \frac{dT_{in}}{dt} &= -k_1 \cdot (T_{in} - T_{out}(t)) + K_2 \cdot (T_w - T_{in}) + k_a \cdot W(t) \cdot \max\left\{0, -\frac{(t-9)\cdot(t-17)}{16}\right\} \\ \frac{dT_w}{dt} &= -K_3 \cdot (T_w - T_{in}) + Q(t) \cdot \frac{T_w - T_{start}}{T_{\max} - T_{start}} \cdot \left(1 - \frac{T_w - T_{start}}{T_{\max} - T_{start}}\right) \end{aligned} \tag{1}$$

The model is composed of four distinct contributions: thermal dispersion, radiant panels, solar radiation and heating system. We now analyze each such contribution.

Thermal dispersion with the exterior through the perimetral walls of the house influences only the internal temperature T_{in}, but not the one of the heating system water. Therefore, letting k_1 denote the thermal dispersion coefficient between the interior and the exterior environments, we can write

$$\frac{dT_{in}}{dt} = -k_1 \cdot (T_{in} - T_{out}(t)), \quad \frac{dT_w}{dt} = 0.$$

In the above equation, the external temperature $T_{out}(t)$ is not constant over time, but its value can be gathered from various weather forecast services and therefore can be assumed to be known.

The radiant panels component allow the heat exchange between the heating system water and the interior of the house. It has the following representation

$$\frac{dT_{in}}{dt} = K_2 \cdot (T_w - T_{in}), \quad \frac{dT_w}{dt} = -K_3 \cdot (T_w - T_{in}),$$

where the parameters K_i, $i = 2, 3$, take two different constant values, depending on the relationship between the water temperature T_w and the internal temperature T_{in}:

$$K_2 = \begin{cases} \dfrac{k_2}{60} & \text{if } T_w \le T_{in} \\ k_2 & \text{if } T_w > T_{in} \end{cases} \qquad K_3 = \begin{cases} \dfrac{k_3}{60} & \text{if } T_w \le T_{in} \\ k_3 & \text{if } T_w > T_{in} \end{cases}$$

Here k_2 measures the impact of the heat lost by radiant panels due to their temperature variation, while the parameter k_3 models the impact of the heat absorbed by the room due to the internal temperature variation. Note that the denominator in the fraction is the number of minutes in a hour; we assume indeed that the heat lost by the radiant panels in one minute corresponds to the one absorbed in an hour. We introduced this denominator to correctly model the heat exchange in the case where $T_w < T_{in}$.

The contribution of solar radiation has specific characteristics. Indeed, it depends on time: obviously at night it vanishes, at sunrise and sunset it is minimum, and around noon it is maximum. Further, it is also affected by weather conditions. We choose this representation:

$$\frac{dT_{in}}{dt} = +k_a \cdot W(t) \cdot \max\left\{0, -\frac{(t-9)\cdot(t-17)}{16}\right\}, \quad \frac{dT_w}{dt} = 0.$$

The parameter k_a is the solar radiation absorption coefficient of the house. It describes the fraction of the radiation that penetrates the house and contributes to its heating. The function $W(t)$ representing the meteorological conditions is obtained from Table 1, which contains hypothetical values of the radiation coming from the different weather conditions that may change daily and even within the day, based on current real-time available information.

Meteo conditions	clear sky	sparse clouds	overcast	cloudy	rain
$W(t)$	1	0.8	0.5	0.2	0

Table 1: Example of correspondence between weather conditions and value of the variable $W(t)$.

To model the intensity of solar radiation along the day, we use this expression

$$F(t) = \max\left\{0, -\frac{(t-9)\cdot(t-17)}{16}\right\}$$

that approximates the relative inclination of solar rays with a parabola, setting sunrise at 9:00 am and sunset at 5:00 pm, with the peak at 1PM. Figure 1 contains its graphic representation.

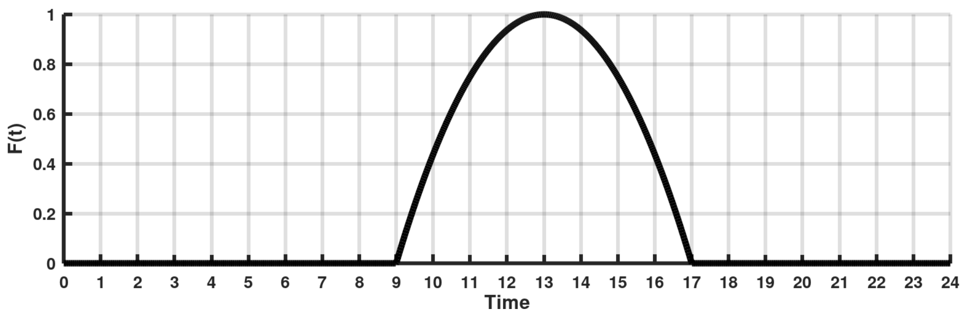

Figure 1: Inclination of solar rays along the day.

Finally, when switched on, the heating system heats the water.

$$\frac{dT_{in}}{dt} = 0, \quad \frac{dT_w}{dt} = +Q(t) \cdot \frac{T_w - T_{start}}{T_{\max} - T_{start}} \cdot \left(1 - \frac{T_w - T_{start}}{T_{\max} - T_{start}}\right), \tag{2}$$

where

$$T_{start} = T_w^{acc} - \frac{T_{\max}}{20}, \qquad Q(t) = \begin{cases} q & \text{if } T_{in} \le T_{target} \\ 0 & \text{if } T_{in} > T_{target} \end{cases} . \tag{3}$$

Note that in the right equation (2), the quantity $T_{\max} - T_{start}$ in view of the position made in the left equation of (3), takes the value $T_{\max}/20$, chosen somewhat arbitrarily. This mimics the fact that initially the heating starts to warm the water slowly. As time goes by, the limit $T_{\max} - T_{start}$ in (2) is finally approached.

The parameters are q, the heat provided by the heating system, and $T_{\max}$, the maximal temperature at which the water is heated. The variable $Q(t)$ controls when the heating system is on. The temperature that the user wishes to maintain in the house is denoted by T_{target}. Note that in (2) we choose to express the contribution of the heating system with a logistic-type growth, borrowing the term from mathematical biology. The fraction before the bracket represents the temperature growth. Indeed, when the heating system is turned on, initially the speed at which temperature changes is slow, then it speeds up to attain a peak and then diminishing settling to zero. Correspondingly, the water temperature increases slowly at first, the function attains an inflection point and then slowly asymptotically approaches the constant value $T_{\max}$ from below. The parameter q controls the speed at which this increase occurs. Note that in the above fraction (3) we introduced T_w^{acc}, which represents the water temperature at the instant when the heating system is turned on. The interval $[T_{start}, T_{\max}]$ is thus rescaled into the interval $[0, 1]$.

4 Sudden unexpected events

The above described model takes into account all the contributions affecting the internal temperature in a normal situation. However, in real situations, from time to time unforeseen sudden events occur. Neither the occurrence, nor the duration, nor the intensity of these events can be known a priori. They disrupt the normal trend of the temperature. The mathematical model (1) fails to describe these events, so that when they occur its simulations give a course different from that real one. There is thus a need to correct the model (1) to possibly incorporate these situations. In order to do so, it is necessary to identify these events, isolate the affected data and restore the model proper functioning. The model could then be applied again and will be able to correctly describe the data not affected by these events.

We propose to interpolate these data by means of splines. The use of splines allows a fast and easy calculation of the geometric characteristics of the data: e.g. the norm of the derivative, the normal vector, and so on. Once these are available, to detect sudden events, it is just necessary to geometrically characterize them. We need to identify the specific features of the spline when one such event occurs. This will be the condition that will then be used to automatically detect unexpected situations.

A good practice to promote air recirculation in the house is to open the windows at least once a day. This represents the sudden unforeseen event to capture, as neither the time when it occurs nor its duration can be predicted in advance. It is an operation performed manually and dependent on the people living in the house. This operation pushes a lot of cold air into the house and consequently produces a fast decrease in the internal temperature, thereby breaking its normal trend.

Figures 2 and 3 compare two similar internal temperature trends. In the former there is no unexpected event, while in the second one the window was opened. The data come from real measurements.

The differential equation model presented above can be used for two purposes: first to replicate the trend of historical data and secondly to make predictions. In both cases, the detection of unforeseen events is performed by using interpolating splines, but in a slightly different way. We now proceed to their detailed analysis.

4.1 Historical data

Assume to have a set of historical data $P_0, \ldots, P_N$ collected at discrete time instants $t_0, \ldots, t_N$. The first task is to obtain from this information a continuous function that allows the calculation of the temperature at times other than the sampled ones. To this end, the historical data must be interpolated. We choose to use an interpolating spline curve $S(t)$. In general the choice of the type of curve to be used depends on the features of phenomenon originating the sampled data. For data having a very regular trend in time, a cubic

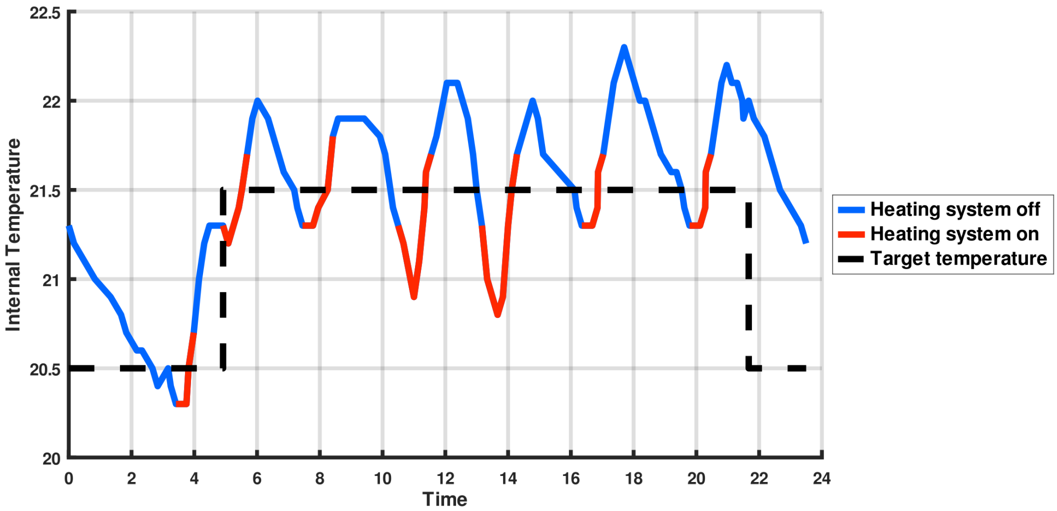

Figure 2: Normal daily trend of the internal temperature in a typical day. The temperature fluctuates around a user-defined target value (black dashed line). The blue line denotes the heating system being off, the red line means heating on.

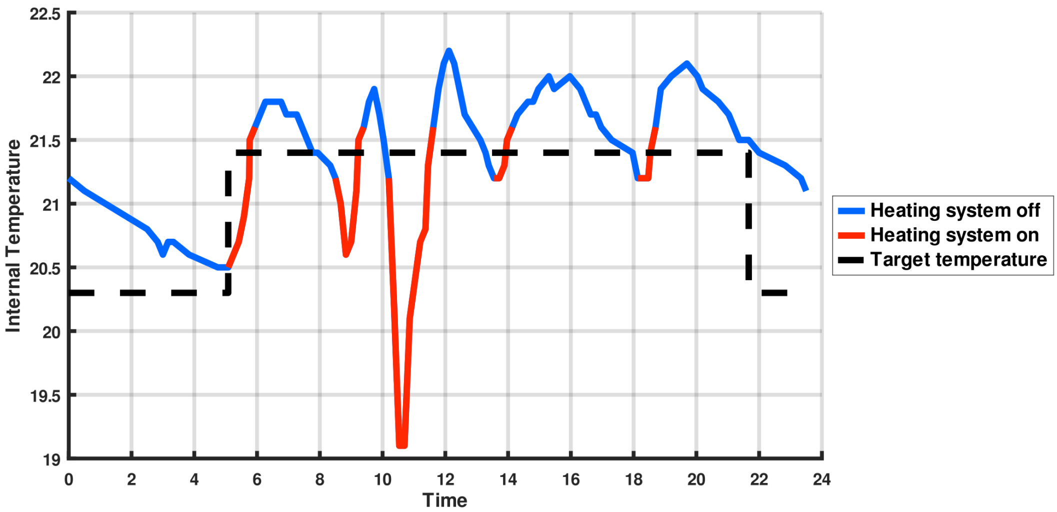

Figure 3: Internal temperature trend influenced by an unforeseen sudden event. Opening the windows around 10 AM causes an unusual rapid decrease of the temperature, breaking the normal trend. The temperature generally fluctuates around the user-defined target threshold, around the target temperature value given by the black dashed line. The blue line denotes the heating system being off, the red line means heating on.

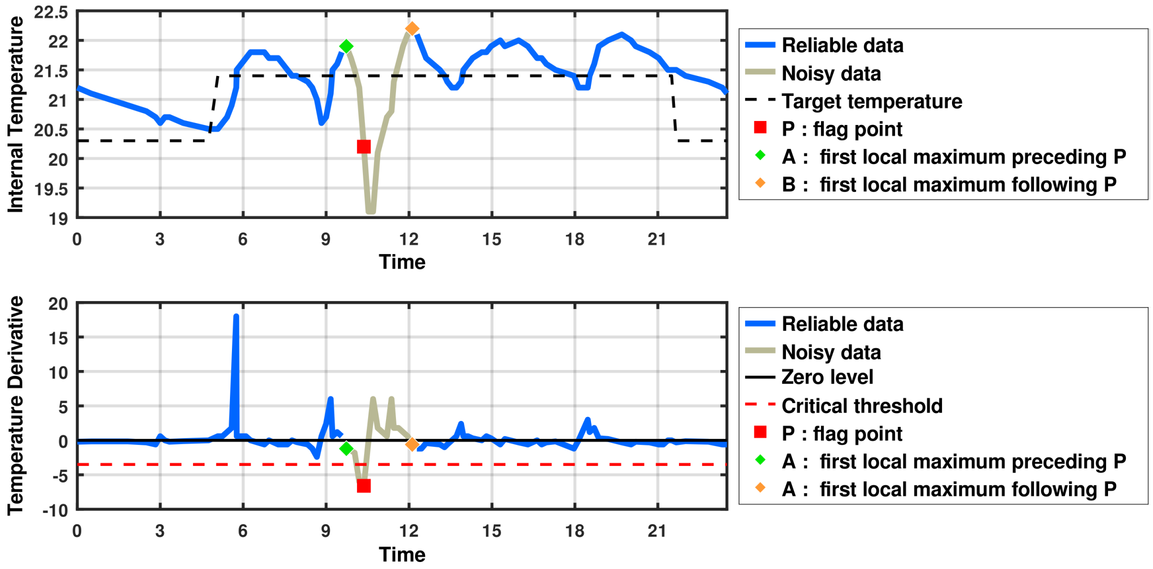

Figure 4: Example of the localization of unexpected events in historical data. Top: temperature data. Bottom: temperature derivative. The flag (red square) corresponds to the point at which the derivative falls below the chosen threshold (red dashed line). The start of the event (green square) corresponds to the first temperature local maximum preceding the flag point. The orange square represents instead the end of the event, corresponding to the first temperature local maximum following the flag point.

spline can be used. On the other hand, if irregularities or noise are present, it may be preferable to use a linear spline or, alternatively, a linear regression line.

To identify unexpected events we need to define a geometrical characteristic, specific to that situation, that can be used as a flag of such occurrence.

When opening windows, the sudden event entails a rapid drop of the room temperature. In the temperature graph, such a fast decrease mathematically corresponds to a negative derivative with a large absolute value. Thus, we are able to identify the occurrence of windows opening simply by checking the derivative of the spline: if it falls below a certain threshold, specified in advance, then the point in question is a flag of the sudden event. Mathematically, we need to verify the following condition

$$S'(t_i) < T < 0 \quad \text{for } i = 1, \dots, N-1 \tag{4}$$

where T represents the specified threshold. Let the point at which it occurs be denoted by t_k. It will be identified as the flag of the event. On the other hand, if the condition is not satisfied for any $i = 1, \dots, N-1$, then the given data are not affected by unexpected events.

Suppose now that t_k is the flag point. We know that at this instant an unforeseen event occurs. This situation must be time-limited, i.e. we have to determine its start and end points. To identify these points we use again a geometric method. There is a need to define the geometric characteristic of these instants.

Let us begin with the starting point for our event. Because opening the windows causes a decrease in temperature, it is plausible that the phenomenon indeed began when the temperature started the decrease. Therefore, we define the start of the unforeseen event to be the instant corresponding to the first local maximum of the temperature preceding the flag point. Thus, geometrically, the onset of the event must satisfy both the following conditions

$$S(t_{i-1}) \le S(t_i), \quad S(t_i) > S(t_{i+1}). \tag{5}$$

These inequalities must be evaluated for every i from $k-1$ and backtracking down possibly to 1. We stop when (5) turns out to be true for the first time, say at step j, or if 1 is reached. By its very definition, all data belonging to $[t_j, t_k]$ are already affected by the unexpected event.

For the point at which the phenomenon stops, the procedure is the same. We need again to identify a numerically implementable geometric characteristic for the endpoint and to calculate when it is verified. For our case, in view of the fact that the temperature will eventually increase again after its rapid decrease, we choose the first local maximum of the room temperature following the flag point. Indeed, this point indicates that the situation in the house has returned to normal conditions. The procedure is the same as for the starting point, (5), but in this case it must be evaluated forward in time, i.e. for every i from $k+1$ and to $N-1$. We stop as soon as (5) is verified, say at step ℓ, or else when $N-1$ is reached. The data in $[t_k, t_\ell]$ are all affected by the unforeseen event.

In summary, we have identified all the data affected by the unexpected phenomenon, just by verifying whether three geometric conditions hold. Figure 4 shows an example of unforeseen event identification in a set of historical data. The flag point corresponds to the point at which the derivative falls below the chosen threshold. After that, the starting and ending points respectively correspond to the first local maxima before and after the flag point.

The timespan used in the model does not influence the detection of the sudden event, as long as the latter falls entirely within the time interval considered. If this does not occur, if it would instead belong to two different time intervals, there is the possibility that the anomaly would not be detected at all. The time window frame is relevant for the model parameters assessment, but this topic is not addressed in this paper.

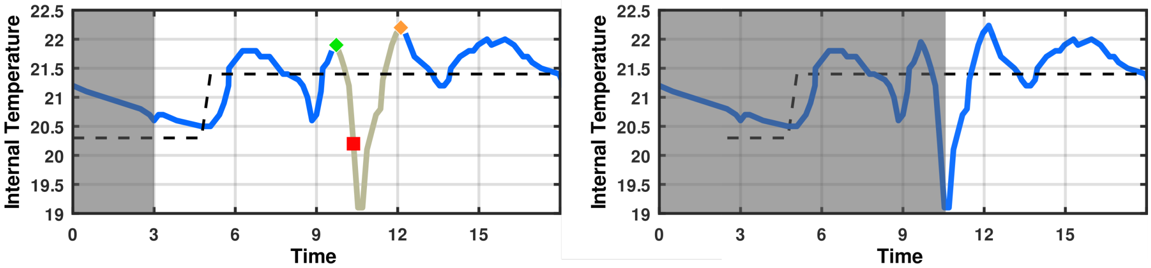

Figure 5: Left: example of detection where the unexpected event falls entirely within the considered time window (3:00 AM - 6:00 PM). The event is detected correctly. Right: the same example, but with the unexpected event falling only partially within the time window (10:30 AM - 6:00 PM). The event is not detected at all because the flag point does not fall in the time window. See Figure 4 for the legend.

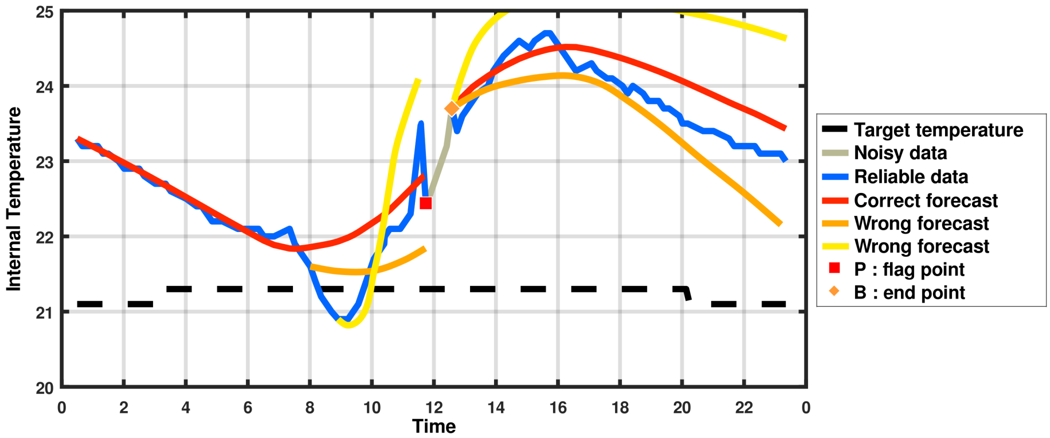

Figure 6: Example showing the importance of making the forecast at a time when the indoor temperature is not affected by human activities. The orange and yellow forecasts are carried out at times (respectively 8:00 AM and 9:00 AM) when the normal pattern of indoor temperature is changed. This erroneous initial forecast causes a considerable gap between the recorded data and the forecast. On the other hand, the red curve shows the behavior for the initial forecast being made shortly after midnight, when the indoor temperature is not affected by human activities.

We show in Figure 5 what happens in case the initial instant occurs during an unforeseen event.

4.2 Control in real time

Suppose now that we have obtained a forecast from the differential equation model. This forecast describes the future room temperature trend under normal conditions. If an unexpected events occurs, the forecast becomes however useless. To restore a reliable forecast, we need to incorporate the new information coming from the unforeseen event and use it to predict anew the temperature evolution at the end of the unforeseen situation. Thus, we need to be able to determine in real time the occurrence and termination of unexpected events.

To assess the reliability of the forecast, it is necessary to check for unexpected events each time a new temperature sampling is performed. In this case, the full set of recorded data is not available, but the dataset is augmented by new measurements as time goes by. The check for unexpected events is therefore performed in real time. If $\{P_0,\ldots,P_N\}$ represents the set of the last collected data, P_N always indicates the last collected datum. This means that when a new detection is made, the data set formally becomes $\{P_0,\ldots,P_{N+1}\}$, but the points are still denoted by $P_0,\ldots,P_N$, because N always denotes the last recording. At first, we need to interpolate a suitable set of the last obtained data with a spline, in which the number of datapoints being considered depends on the degree of the curve and its shape (e.g. Bézier, B-spline,...). To check if an unexpected event is occurring, we need to define a geometric characteristic flag, usually the same one used for historical data, given by (4). This equation must be tested for $i = N-1$, i.e. for the next to last available datum. It is not tested for $i = N$ because the geometric characteristics of the spline curves at the extremes of the interval depend on the type of the curve chosen and may not reflect the true trend in the data. The geometric feature testing must be repeated with each new measurement. When it holds, an unexpected event is occuring. Therefore the prediction on the temperature evolution carried out in advance, i.e. at the start of the day or at the end of another occurrence of a previous sudden event, fails to hold and eventually must be corrected.

Having detected the presence of an unforeseen situation, we must determine its termination. Again, we need to check at each new measurement the geometric end feature, generally coinciding with the one used for historical data, namely (5) for $i = N-2$. As in the previous case the point where the spline curve ends is excluded. When (5) holds, the data are no longer affected by the unexpected event and the situation has returned back to normal. Then a new valid prediction for the future temperature can be made; we go back to the normal routine, proceeding then to check the flag condition with each new measurement, to control that no new unexpected event occurs.

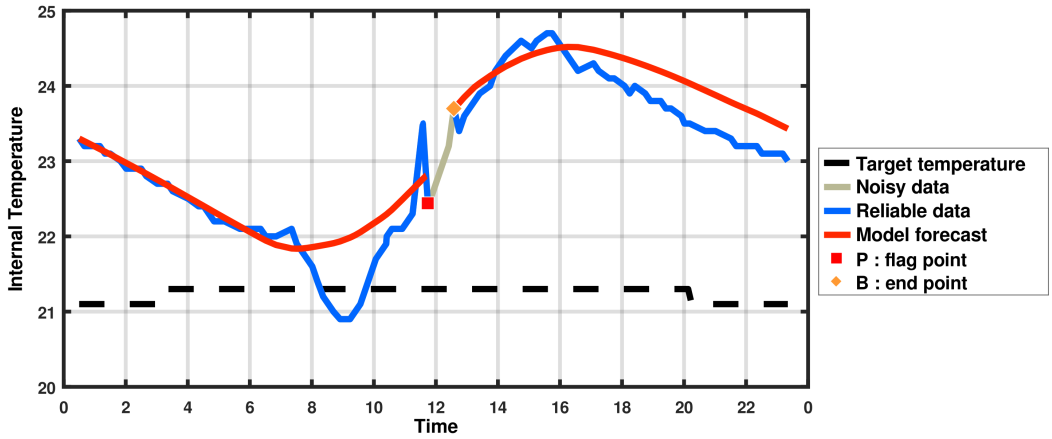

Figure 7: Example of real time localization of unforeseen events. The red square represents the flag point where the unforeseen situation is detected. The orange square represents the end of the event. It corresponds to the first local maximum following the flag point. The blue line shows the measurements in time of the internal temperature. The left-hand portion of the red line is the forecast trend predicted at the beginning of the day. The right-hand portion of the red line is the forecast trend prediction at the end of the unexpected situation.

Note that in our simulation we want a forecast for the temperature along the entire day. In the historical data, we consider one day at a time and typically set $t_N = t_0 + 24$ for the end of the day, measuring time in hours. The starting time t_0 is chosen typically at 3 : 00 AM, in the very early morning, when the activities in an average household are minimal and have no effect on the internal temperature.

Figure 6 shows the results of the forecast when the indoor temperature is and is not affected by human activities. The importance of the former appears from the gaps in the forecast made when the initial instant is not taken at a quite time.

In summary, to have a real time room temperature control, one of the flag or the end geometric conditions must be verified with each new measurement. This simple test allows the assessment of unforeseen events. Figure 7 shows an example of real time unexpected situations identification. The red line, which corresponds to the forecast trend of the internal temperature, is broken. In fact, the unforeseen event renders useless the forecast made at the beginning of the day, i.e. the left-hand portion of the red curve. A new forecast is then made at the end of the unforeseen event, represented by the right-hand portion of the red curve.

4.3 Pseudocode

We provide here the pseudocode of the program that was used in the simulations. Table 2 contains the case of the analysis of historical data, while Table 3 considers the analysis in real time.

```
# T is the vector of time instants to analyze

for (each instant t_i in T) do
    if (the spline first derivative evaluated in t_i is less than the fixed thresold) do
        exit the for loop
   if (the end of T is reached) do
        the entire set of data is reliable
   else
# delimit the noisy data
    for (each instant t_j in T from t_i going backwards) do
        if (in t_j there is a local maximum or t_j is the first element of T) do
            set t_start = t_j
            exit the for loop
    for (each instant t_j in T from t_i going forward) do
        if (in t_j there is a local maximum or t_j is the last element of T) do
            set t_end = t_j
            exit the for loop
        set noisy data as the ones in [t_start,t_end]
        repeat the same process with the data in [t_end,last element of T]
```

Table 2: Pseudocode for the analysis of historical data.

```
# p_i is the last detection made

at the start a reliable forecast is made
set forecast_status = reliable
for (each new detection p_i) do
    if (forecast_status = reliable) do
        if (the spline first derivative in p_i is less than the fixed thesold) do
            set forecast_status = redo
    else do # forecast_status = redo
        if (p_i is a local maximum) do
            make a new reliable forecast
    if (24h have passed from the start of the process) do
        repeat the same process from the start
```

Table 3: Pseudocode for the analysis in real time.

5 Conclusions

In this investigation, a splines-based algorithm has been proposed to automatically assess the occurrence of unexpected situations. In particular we focused on the opening of windows or outside doors in a heated house, operations that cause a sudden drop of the internal temperature, driving it outside the desired range.

In addition, a way of reprogramming the behavior of the thermostat control so as to restore the temperature set by the user is discussed. Overall, the employed approximation theoretic tools allow a reliable and cheap solution of this relevant problem in practical applications.

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