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Differentiated Bernstein Type Operators

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Abstract

The present paper deals with the derivatives of Bernstein type operators preserving some exponential functions. We investigate the uniform convergence of the differentiated operators. The rate of convergence by means of a modulus of continuity is studied, an upper estimate theorem for the difference of new constructed differentiated Bernstein type operators is presented.

1 Introduction

The study of the simultaneous approximation of a given function by linear positive operators is one of the main purposes in Approximation Theory.

In the present paper, we investigate the above problems for the sequence of the Bernstein type operators introduced by Aral et al. [7] in which quantitative and qualitative type theorems, in particular, an asymptotic formula and a saturation result were studied for Bernstein type operators in the form

$$G_nf(x) = G_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) e^{-\mu k/n} e^{\mu x} p_{n,k}(a_n(x)),\ x \in [0,1],\ n \in \mathbb{N}, \tag{1}$$

where $\mu > 0$ is a real number, f is a continuous function on $[0,1]$ and

$$a_n(x) = \frac{e^{\mu x/n} - 1}{e^{\mu/n} - 1},\ p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1],\ k,n \in \mathbb{N}. \tag{2}$$

The inspiration in the construction of the operators (1) depends on the preservation of increasing exponential functions being in the form $\exp_\mu(x) := e^{\mu x}$, $\mu > 0$. These operators are special case of a modification introduced by Morigi and Neamtu in [11] and they are a generalization of the classical Bernstein operators with the connection

$$G_nf(x) = \exp_\mu(x) B_n\left(f_\mu; a_n(x)\right), \tag{3}$$

where $B_n(\cdot;x)$ is classical Bernstein operator, $f_\mu := f / \exp_\mu$, (say).For the sake of a convenient notation we shall use the notation $\mathcal{G}_{n,\mu} := G_n / \exp_\mu$ throughout the rest of paper.

It is shown in [7] that the operators G_n present better (less error of approximation) approximation than the classical Bernstein operator under certain assumptions on the whole interval $[0,1]$. The operators G_n both preserve $\exp_\mu$ and $\exp_\mu^2$ and satisfy some shape preserving properties depending on generalized convexity which is defined via the function $\exp_\mu$.

The effectiveness of linear positive operators preserving some exponential functions takes much attention from the researchers and corresponding modifications of the other operators have been extensively studied nowadays, among the others, we refer the readers to [1, 2, 4, 5, 8, 10, 12]. In a more general case, positive linear operators preserving function τ and τ^2 were studied in [3].

Here we shall focus on simultaneous approximation behaviors of the operators G_n, that is, we shall investigate the approximation $D^k \mathcal{G}_{n,\mu}(f;x) \to D^k f_\mu(x)$ as $n \to \infty$ for functions which are k ($k \in \mathbb{N}$) times continuously differentiable. The results will be given in qualitative and quantitative form. An upper bound for difference $D^k \mathcal{G}_{n,\mu}(f) - \mathcal{G}_{n-k,\mu}\left(D^k f_\mu\right)$ will be presented.

2 Auxiliary Results

We first give some lemmas which will be necessary to prove the main results. In what follows, we denote by $C[0,1]$ the space of all continuous functions $f : [0,1] \to \mathbb{R}$, endowed with the supremum norm $\|f\| = \max_{x\in[0,1]} |f(x)|$. Also we denote by $C^k[0,1]$, $k \in \mathbb{N}$, the subspace of $C[0,1]$ for which the derivatives $f^{(m)}$ exists for every $m \le k$, $m \in \mathbb{N}$, and each $f^{(m)} \in C[0,1]$.

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In the light of the definition given in (2), the following equality occurs:

$$a_n^{(k)}(x)=\left(\frac{\mu}{n}\right)^k b_n(x), \tag{4}$$

where $b_n(x)=\frac{e^{\mu x/n}}{e^{\mu/n}-1}$ for any $k,n\in\mathbb{N}$ and for all $x\in[0,1]$, which allow us to deduce the following first lemma which states k-th order derivative of the operators $\mathcal{G}_{n,\mu}$ in terms of forward differences of corresponding functions.

Let D be the differential operator and $\Delta^i_{\frac{1}{n}}$ denotes the i-th order forward difference operator with step size $1/n$, $(n)_i$ is Pochhammer symbol defined by $(n)_i=n(n-1)\dots(n-i+1)$. Hence we have following lemma.

Lemma 2.1. *Let $x\in[0,1]$, $k,n\in\mathbb{N}$, $\mu>0$. For any $f\in C[0,1]$ it holds that:*

$$D^k\mathcal{G}_{n,\mu}(f;x)=\sum_{i=1}^{k}(n)_i\left(\frac{\mu}{n}\right)^k C_i^k b_n^i(x)\sum_{s=0}^{n-i}p_{n-i,s}(a_n(x))\Delta^i_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right), \tag{5}$$

where C_i^k, ($i=1,2,..,k$) are constants satisfying the following relations:

$$\begin{aligned} C_1^k &= C_k^k=1,\ C_0^k=0,\ C_n^k=0 \text{ if } n>k \\ C_i^k &= iC_i^{k-1}+C_{i-1}^{k-1},\ \ i=2,..,k-1. \end{aligned}$$

Proof. The proof will be given by induction on k. For $k=1$, taking into account that $C_1^1=1$, we have

$$D\mathcal{G}_{n,\mu}(f;x)=\sum_{s=0}^{n}f\left(\frac{s}{n}\right)e^{-\mu s/n}p'_{n,s}(a_n(x)).$$

Indeed, using (4) the above equality can be rewritten as follows:

$$\begin{aligned} D\mathcal{G}_{n,\mu}(f;x) &= na'_n(x)\sum_{s=0}^{n}f\left(\frac{s}{n}\right)e^{-\mu s/n}\left[p_{n-1,s-1}(a_n(x))-p_{n-1,s}(a_n(x))\right] \\ &= na'_n(x)\sum_{s=0}^{n-1}p_{n-1,s}(a_n(x))\Delta_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &= (n)_1\left(\frac{\mu}{n}\right)b_n(x)\sum_{s=0}^{n-1}p_{n-1,s}(a_n(x))\Delta_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right). \end{aligned}$$

Let us now assume that (5) is true for $k-1$. That is, the equality

$$D^{k-1}\mathcal{G}_{n,\mu}(f;x)=\sum_{i=1}^{k-1}(n)_i\left(\frac{\mu}{n}\right)^{k-1}C_i^{k-1}b_n^i(x)\sum_{s=0}^{n-i}p_{n-i,s}(a_n(x))\Delta^i_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right)$$

holds. Then we can write

$$\begin{aligned} D^{k-1}\mathcal{G}_{n,\mu}(f;x) &= n\left(\frac{\mu}{n}\right)^{k-1}b_n(x)\sum_{s=0}^{n-1}p_{n-1,s}(a_n(x))\Delta_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad+\sum_{i=2}^{k-2}(n)_i\left(\frac{\mu}{n}\right)^{k-1}C_i^{k-1}b_n^i(x)\sum_{s=0}^{n-i}p_{n-i,s}(a_n(x))\Delta^i_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad+(n)_{k-1}\left(\frac{\mu}{n}\right)^{k-1}b_n^{k-1}(x)\sum_{s=0}^{n-k+1}p_{n-k+1,s}(a_n(x))\Delta^{k-1}_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right). \end{aligned}$$

In the last line of the above equality we use the facts $C_1^{k-1}=C_k^{k-1}=1$. Taking one more derivative of $D^{k-1}G_{n,\mu}(f;x)$ and using (4), we obtain

$$\begin{aligned} D\left(D^{k-1}\mathcal{G}_{n,\mu}(f;x)\right) &= n\left(\frac{\mu}{n}\right)^k b_n(x)\sum_{s=0}^{n-1}p_{n-1,s}(a_n(x))\Delta_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad+n(n-1)\left(\frac{\mu}{n}\right)^k b_n^2(x)\sum_{s=0}^{n-2}p_{n-2,s}(a_n(x))\Delta^2_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad+\sum_{i=2}^{k-2}(n)_i\left(\frac{\mu}{n}\right)^k i b_n^i(x)C_i^{k-1}\sum_{s=0}^{n-i}p_{n-i,s}(a_n(x))\Delta^i_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad+\sum_{i=2}^{k-2}(n)_{i+1}\left(\frac{\mu}{n}\right)^k C_i^{k-1}b_n^{i+1}(x)\sum_{s=0}^{n-i-1}p_{n-i-1,k}(a_n(x))\Delta^{i+1}_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad+(n)_{k-1}\left(\frac{\mu}{n}\right)^k(k-1)b_n^{k-1}(x)\sum_{s=0}^{n-k+1}p_{n-k+1,s}(a_n(x))\Delta^{k-1}_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad+(n)_{k-1}\left(\frac{\mu}{n}\right)^k b_n^k(x)(n-k+1)\sum_{s=0}^{n-k}\left(p_{n-k,s}(a_n(x))\right)\Delta^k_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \end{aligned}$$

which allows us to write

$$\begin{aligned} D^k \mathcal{G}_{n,\mu}(f;x) &= \sum_{i=1}^{k-1} (n)_i \left(\frac{\mu}{n}\right)^k i b_n^i(x) C_i^{k-1} \sum_{s=0}^{n-i} p_{n-i,s}(a_n(x)) \Delta_{\frac{1}{n}}^i f_\mu\left(\frac{s}{n}\right) \\ &+ \sum_{i=1}^{k-1} (n)_{i+1} \left(\frac{\mu}{n}\right)^k C_i^{k-1} b_n^{i+1}(x) \sum_{s=0}^{n-i-1} p_{n-i-1,s}(a_n(x)) \Delta_{\frac{1}{n}}^{i+1} f_\mu\left(\frac{s}{n}\right). \end{aligned}$$

Rearranging the second term of the right hand side of the above equality we get

$$\begin{aligned} D^k \mathcal{G}_{n,\mu}(f;x) &= \sum_{i=1}^{k-1} (n)_i \left(\frac{\mu}{n}\right)^k i b_n^i(x) C_i^{k-1} \sum_{s=0}^{n-i} p_{n-i,s}(a_n(x)) \Delta_{\frac{1}{n}}^i f_\mu\left(\frac{s}{n}\right) \\ &+ \sum_{i=2}^{k} (n)_i \left(\frac{\mu}{n}\right)^k C_{i-1}^{k-1} b_n^i(x) \sum_{s=0}^{n-i} p_{n-i,s}(a_n(x)) \Delta_{\frac{1}{n}}^i f_\mu\left(\frac{s}{n}\right). \end{aligned}$$

Taking into account that $C_k^{k-1} = 0$ and $C_0^{k-1} = 0$ we can deduce that

$$\begin{aligned} D^k \mathcal{G}_{n,\mu}(f;x) &= \sum_{i=1}^{k} (n)_i \left(\frac{\mu}{n}\right)^k i b_n^i(x) C_i^{k-1} \sum_{s=0}^{n-i} p_{n-i,s}(a_n(x)) \Delta_{\frac{1}{n}}^i f_\mu\left(\frac{s}{n}\right) \\ &+ \sum_{i=1}^{k} (n)_i \left(\frac{\mu}{n}\right)^k C_{i-1}^{k-1} b_n^i(x) \sum_{s=0}^{n-i} p_{n-i,s}(a_n(x)) \Delta_{\frac{1}{n}}^i f_\mu\left(\frac{s}{n}\right) \\ &= \sum_{i=1}^{k} (n)_i \left(\frac{\mu}{n}\right)^k b_n^i(x) \left(i C_i^{k-1} + C_{i-1}^{k-1}\right) \sum_{s=0}^{n-i} p_{n-i,s}(a_n(x)) \Delta_{\frac{1}{n}}^i f_\mu\left(\frac{s}{n}\right). \end{aligned}$$

Then in view of the equality

$$C_i^k = i C_i^{k-1} + C_{i-1}^{k-1},$$

we get the desired result. □

On the other hand, it is also possible to have a representation for k-th order derivatives of the operators $\mathcal{G}_{n,\mu}$ for functions belonging to $C^k[0,1]$. The following representation which is based on a certain relation between forward difference and an auxiliary operator on $C^k[0,1]$ which will have crucial role in next section.

Lemma 2.2. *For $f \in C^k[0,1]$, $n, k \in \mathbb{N}$ and $\mu > 0$ we get*

$$D^k \mathcal{G}_{n,\mu}(f;x) = \sum_{i=1}^{k} \left(\frac{\mu}{n}\right)^k C_i^k b_n^i(x) \mathcal{L}_{n,i}\left(D^i f_\mu; x\right),$$

where

$$\mathcal{L}_{n,i}\left(D^i f_\mu; x\right) = (n)_i \sum_{s=0}^{n-i} p_{n-i,s}(a_n(x)) \int_0^{\frac{1}{n}} \cdots \int_0^{\frac{1}{n}} D^i f_\mu\left(\frac{s}{n} + \nu_1 + \cdots + \nu_i\right) d\overline{\nu} \tag{6}$$

and $d\overline{\nu} = d\nu_1 \ldots d\nu_i$.

Proof. If we consider the equality for $g \in C^i[0,1]$ and $i \in \mathbb{N}$ (see [9, Page 91]) is

$$\Delta_{\frac{1}{n}}^i g(x) = \int_0^{1/n} \cdots \int_0^{1/n} D^i g(x + \nu_1 + \cdots + \nu_i) d\overline{\nu}, \tag{7}$$

in view of (5) we have

$$\begin{aligned} D^k \mathcal{G}_{n,\mu}(f;x) &= \sum_{i=1}^{k} \left(\frac{\mu}{n}\right)^k C_i^k b_n^i(x) (n)_i \sum_{s=0}^{n-i} p_{n-i,s}(a_n(x)) \int_0^{1/n} \cdots \int_0^{1/n} D^i f_\mu\left(\frac{s}{n} + \nu_1 + \cdots + \nu_i\right) d\overline{\nu} \\ &= \sum_{i=1}^{k} \left(\frac{\mu}{n}\right)^k C_i^k b_n^i(x) \mathcal{L}_{n,i}\left(D^i f_\mu; x\right). \end{aligned}$$

This completes the proof. □

Let us point out that, as a conclusion of Lemma 2.2, we immediately have the following:

Remark 1. For $x \in [0,1]$, $k \in \mathbb{N}$ and $\mu > 0$, we have

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\mu}{n}\right)^k b_n^i(x) &= \lim_{n\to\infty}\left(\frac{\mu}{n}\right)^{k-i}\left(\frac{\mu}{n}b_n(x)\right)^i \\ &= \lim_{n\to\infty}\left(\frac{\mu}{n}\right)^{k-i}\left(\frac{\mu}{n}\frac{e^{\mu x/n}}{e^{\mu/n}-1}\right)^i \\ &= 0,\end{aligned}$$

for $i = 1,2,...,k-1$, in view of Lemma 2.2 we can write

$$\begin{aligned}D^k\mathcal{G}_{n,\mu}(f;x) &= \sum_{i=1}^{k}\left(\frac{\mu}{n}\right)^k C_i^k b_n^i(x)\mathcal{L}_{n,i}\left(D^i f_\mu;x\right) \\ &= \sum_{i=1}^{k-1}\left(\frac{\mu}{n}\right)^k C_i^k b_n^i(x)\mathcal{L}_{n,i}\left(D^i f_\mu;x\right)+\left(\frac{\mu}{n}\right)^k b_n^k(x)\mathcal{L}_{n,k}\left(D^k f_\mu;x\right) \\ &= o(1)+\left(\frac{\mu}{n}\right)^k b_n^k(x)\mathcal{L}_{n,k}\left(D^k f_\mu;x\right),\end{aligned}$$

where

$$\lim_{n\to\infty}\left(\frac{\mu}{n}\right)^k b_n^k(x) = 1.$$

Thus for the approximation properties of the operator $D^k\mathcal{G}_{n,\mu}$ it is enough to consider the auxiliary operator $\mathcal{L}_{n,k}\left(D^k\right)$.

Now we calculate the exponential moments of the auxiliary operator $\mathcal{L}_{n,k}$.

Lemma 2.3. *For $n,k \in \mathbb{N}$, $x \in [0,1]$ and $\mu > 0$, the followings hold:*

i) $\mathcal{L}_{n,k}(e_0;x) = \frac{(n)_k}{n^k}$,

ii) $\mathcal{L}_{n,k}\left(\exp_\mu;x\right) = \frac{(n)_k}{(\mu)^k}\left(e^{\frac{\mu}{n}}-1\right)^k e^{\frac{\mu x(n-k)}{n}}$,

iii) $\mathcal{L}_{n,k}\left(\exp_\mu^2;x\right) = \frac{(n)_k}{(2\mu)^k}\left(e^{\frac{2\mu}{n}}-1\right)^k\left(\left(e^{\frac{\mu x}{n}}-1\right)\left(e^{\frac{\mu}{n}}+1\right)+1\right)^{n-k}$.

Proof. By the equality (6), we get

$$\begin{aligned}\mathcal{L}_{n,k}(e_0;x) &= (n)_k\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x))\int_0^{\frac{1}{n}}\cdots\int_0^{\frac{1}{n}}d\overline{z} \\ &= \frac{(n)_k}{n^k}\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x)) \\ &= \frac{(n)_k}{n^k}\end{aligned}$$

and we have

$$\begin{aligned}\mathcal{L}_{n,k}\left(\exp_\mu;x\right) &= (n)_k\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x))\int_0^{\frac{1}{n}}\cdots\int_0^{\frac{1}{n}}e^{\mu\left(\frac{s}{n}+z_1+\cdots+z_k\right)}d\overline{z} \\ &= \frac{(n)_k}{(\mu)^k}\left(e^{\frac{\mu}{n}}-1\right)^k\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x))\,e^{\frac{\mu s}{n}} \\ &= \frac{(n)_k}{(\mu)^k}\left(e^{\frac{\mu}{n}}-1\right)^k e^{\frac{\mu x(n-k)}{n}}.\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{L}_{n,k}\left(\exp_\mu^2;x\right) &= (n)_k\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x))\int_0^{\frac{1}{n}}\cdots\int_0^{\frac{1}{n}}e^{2\mu\left(\frac{s}{n}+z_1+\cdots+z_k\right)}d\overline{z} \\ &= \frac{(n)_k}{(2\mu)^k}\left(e^{\frac{2\mu}{n}}-1\right)^k\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x))\,e^{\frac{2\mu s}{n}} \\ &= \frac{(n)_k}{(2\mu)^k}\left(e^{\frac{2\mu}{n}}-1\right)^k\left[\left(e^{\frac{\mu x}{n}}-1\right)\left(e^{\frac{\mu}{n}}+1\right)+1\right]^{n-k}.\end{aligned}$$

□

Further, in order to prove uniform convergence of the sequence of derivative operators of order k, we need the following limits for auxiliary operators.

Lemma 2.4. *Let* $\exp_{\mu,x} := (e^{\mu t} - e^{\mu x})$ *for* $\mu > 0$ *and* $x, t \in [0,1]$. *For* $n, k \in \mathbb{N}$, *we get*

$$\lim_{n\to\infty} \left\| \mathcal{L}_{n,k}\left(\exp_{\mu,x};\cdot\right)\right\| = 0$$

and

$$\lim_{n\to\infty} \left\| \mathcal{L}_{n,k}\left(\exp^2_{\mu,x};\cdot\right)\right\| = 0.$$

Proof. Using Lemma 2.3, we have

$$\begin{aligned}
\left\| \mathcal{L}_{n,k}\left(\exp_{\mu,x};\cdot\right)\right\| &= \max_{x\in[0,1]} \left| \left(\frac{(n)_k}{(\mu)^k}\left(e^{\frac{\mu}{n}}-1\right)^k e^{-\frac{\mu x k}{n}} - \frac{(n)_k}{n^k}\right) e^{\mu x}\right| \\
&= \frac{(n)_k}{(\mu)^k}\left(e^{\frac{\mu}{n}}-1\right)^k - 1,
\end{aligned}$$

which tends to 0 as $n \to \infty$. Using serial expansion of exponential functions and elementary calculations, we have

$$\begin{aligned}
&\frac{(n)_k}{(2\mu)^k}\left(e^{\frac{2\mu}{n}}-1\right)^k \left(\left(e^{\frac{\mu x}{n}}-1\right)\left(e^{\frac{\mu}{n}}+1\right)+1\right)^{n-k} \\
=\ & e^{2\mu x} + e^{2\mu x}\left(\frac{-\mu^2x^2 - 2k\mu x + \mu^2 x}{n}\right) \\
&+ \frac{e^{2\mu x}}{n^2}\left(\frac{1}{2}\mu^4x^4 + 2k\mu x^3 - \mu^4x^3 + 2k^2\mu^2x^2 - 2k\mu^3x^2\right. \\
&\left. + \frac{1}{2}\mu^4x^2 + \mu^3x^3 + k\mu^2x^2 - \frac{3}{2}\mu^3x^2 + \mu^2x + \frac{1}{2}\mu^3 x\right) + \mathcal{O}\left(\frac{1}{n^3}\right).
\end{aligned}$$

Thus from Lemma 2.3, we can write

$$\begin{aligned}
\mathcal{L}_{n,k}\left(\exp^2_{\mu,x};x\right) &= \frac{(n)_k}{(2\mu)^k}\left(e^{\frac{2\mu}{n}}-1\right)^k \left[\left(e^{\frac{\mu x}{n}}-1\right)\left(e^{\frac{\mu}{n}}+1\right)+1\right]^{n-k} \\
&\quad -2e^{2\mu x}\frac{(n)_k}{(\mu)^k}\left(e^{\frac{\mu}{n}}-1\right)^k e^{-\frac{\mu x k}{n}} + e^{2\mu x}\frac{(n)_k}{n^k} \\
&= e^{2\mu x}\left(\frac{-\mu^2x^2 - 2k\mu x + \mu^2 x}{n}\right) \\
&\quad + \frac{e^{2\mu x}}{n^2}\left(\frac{1}{2}\mu^4x^4 + 2k\mu x^3 - \mu^4x^3 + 2k^2\mu^2x^2 - 2k\mu^3x^2\right. \\
&\quad \left. + \frac{1}{2}\mu^4x^2 + \mu^3x^3 + k\mu^2x^2 - \frac{3}{2}\mu^3x^2 + \mu^2x + \frac{1}{2}\mu^3 x\right) + \mathcal{O}\left(\frac{1}{n^3}\right) \\
&\quad + e^{2\mu x}\left(\frac{(n)_k}{n^k} - 1\right) + 2e^{2\mu x}\left(\frac{(n)_k}{(\mu)^k}\left(e^{\frac{\mu}{n}}-1\right)^k e^{-\frac{\mu x k}{n}} - 1\right)
\end{aligned}$$

and

$$\begin{aligned}
\sup_{x\in[0,1]}\left|\mathcal{L}_{n,k}\left(\exp^2_{\mu,x};x\right)\right| &\leq e^{2\mu}\left(\frac{\mu^2 x}{n} + \frac{2\mu^4 + 2k\mu + 2k^2\mu^2 + 2\mu^3 + k\mu^2 + \mu^2}{n^2}\right) + \mathcal{O}\left(\frac{1}{n^3}\right). \\
: &= \beta^{\mu}_{n,k}(x).\ \text{(say)}
\end{aligned} \tag{8}$$

On the other hand, since

$$\lim_{n\to\infty}\frac{(n)_k}{n^k} = 1 \text{ and } \lim_{n\to\infty}\frac{(n)_k}{(\mu)^k}\left(e^{\frac{\mu}{n}}-1\right)^k e^{-\frac{\mu x k}{n}} = 1,$$

we have

$$\lim_{n\to\infty} \left\| \mathcal{L}_{n,k}\left(\exp^2_{\mu,x};\cdot\right)\right\| = 0.$$

This completes the proof. □

3 Main Results

Now we are ready to give our main results which consist of uniform convergence of the sequence $\left\{D^k\mathcal{G}_{n,\mu}(f)\right\}$ for functions belonging to $C^k[0,1]$, rate of simultaneous approximation for the sequence of $\left\{D^k\mathcal{G}_{n,\mu}(f)\right\}$ to k-th order derivatives of corresponding function. As a last result we will investigate the rate of norm convergence for difference $D^k\mathcal{G}_{n,\mu}(f)-\mathcal{G}_{n-k,\mu}\left(D^kf_\mu\right)$ via modulus of continuity.

Theorem 3.1. *If $f\in C^k[0,1]$, then sequence $\left\{D^k\mathcal{G}_{n,\mu}(f)\right\}$ converges to D^kf_μ uniformly in x on $[0,1]$.*

Proof. The set $\left\{e_0,\exp_\mu,\exp_\mu^2\right\}$ is an extended complete Tchebychev system, it is easy to prove that $\left\{\mathcal{L}_{n,k}\left(\exp_\mu^i\right)\right\}$ converges to $\exp_\mu^i(x)$ for each $i=0,1,2$ uniformly in x on $[0,1]$. As an immediate consequence of Lemma 2.3, the followings hold:

$$\begin{aligned}\lim_{n\to\infty}\mathcal{L}_{n,k}(e_0;x) &= 1,\\ \lim_{n\to\infty}\mathcal{L}_{n,k}\left(\exp_\mu;x\right) &= e^{\mu x},\\ \lim_{n\to\infty}\mathcal{L}_{n,k}\left(\exp_\mu^2;x\right) &= e^{2\mu x}\end{aligned}$$

for each $x\in[0,1]$. For the uniform convergence, we have from Lemma 2.4 that

$$\left\|\mathcal{L}_{n,k}\left(\exp_\mu\right)-\exp_\mu\right\|\to 0$$

and

$$\left\|\mathcal{L}_{n,k}\left(\exp_\mu^2\right)-\exp_\mu^2\right\|\to 0,$$

as $n\to\infty$. Thus from Korovkin theorem we have

$$\lim_{n\to\infty}\left\|\mathcal{L}_{n,k}(g)-g\right\|=0 \tag{9}$$

for all $g\in C[0,1]$. As in given Remark 1, since

$$D^k\mathcal{G}_{n,\mu}(f)=o(1)+\left(\frac{\mu}{n}\right)^k b_n^k(x)\mathcal{L}_{n,k}\left(D^kf_\mu;x\right)$$

and

$$\lim_{n\to\infty}\left\|\left(\frac{\mu}{n}\right)^k b_n^k(x)-1\right\|=0,\ \text{for all } k\in\mathbb{N},$$

using (9) for $g=f_\mu^{(k)}$ we have

$$\begin{aligned}\lim_{n\to\infty}\left\|D^k\mathcal{G}_{n,\mu}(f)-D^kf_\mu\right\| &= \lim_{n\to\infty}\left\|\left(\left(\frac{\mu}{n}\right)^k b_n^k(\cdot)-1\right)\mathcal{L}_{n,k}\left(D^kf_\mu\right)\right\|\\ &\quad+\lim_{n\to\infty}\left\|\mathcal{L}_{n,k}\left(D^kf_\mu\right)-D^kf_\mu\right\|\\ &= 0.\end{aligned}$$

Hence, we have the desired result. □

The following theorem presents an estimate for the above convergence by means of the first order modulus of continuity ω which is defined for functions $f\in C[a,b]$ ($a,b\in\mathbb{R}$, $a<b$) as $\omega(f;\delta)=\sup\{|f(x)-f(y)|:x,y\in[a,b],|x-y|<\delta\}$ with $\delta>0$. The modulus of continuity is increasing with respect to $\delta>0$ and also satisfies the inequality $\omega(f;\lambda\delta)\le(1+\lambda)\omega(f;\delta)$, $\delta,\lambda>0$.

Theorem 3.2. *If $f\in C^k[0,1]$, then we have*

$$\begin{aligned}\left|D^k\mathcal{G}_{n,\mu}(f;x)-D^kf_\mu(x)\right| &\le o(1)+\left|\left(\frac{\mu}{n}\right)^k b_n^k(x)-1\right|\frac{(n)_k}{n^k}\left\|D^kf_\mu\right\|+\left\|D^kf_\mu\right\|\left(1-\frac{(n)_k}{n^k}\right)\\ &\quad+\left(1+\frac{(n)_k}{n^k}\right)\omega\left(D^kf_\mu\circ\log_\mu;\sqrt{\beta_{n,k}^\mu(x)}\right)\end{aligned}$$

for $x\in[0,1]$, where $\beta_{n,k}^\mu(x)$ is defined as in (8) and $\omega(f;\cdot)$ is the modulus of continuity of f.

Proof. By applying classical Shisha and Mond technique, for any $g\in C[0,1]$ we have

$$\begin{aligned}|g(t)-g(x)| &= \left|\left(g\circ\log_\mu\right)\left(e^{\mu t}\right)-\left(g\circ\log_\mu\right)\left(e^{\mu x}\right)\right|\\ &\le \omega\left(g\circ\log_\mu;\left|e^{\mu t}-e^{\mu x}\right|\right)\\ &\le \left(1+\frac{\left(e^{\mu t}-e^{\mu x}\right)^2}{\delta^2}\right)\omega\left(g\circ\log_\mu;\delta\right),\ \delta>0.\end{aligned} \tag{10}$$

On the other hand, with the facts $\mathcal{L}_{n,k}(e_0;x) \leq 1$ and $\mathcal{L}_{n,k}$ are linear and positive operator, we get

$$\begin{aligned} \left|\mathcal{L}_{n,k}(g;x)-g(x)\right| &= \left|\mathcal{L}_{n,k}(g(t)-g(x);x)+g(x)\left(\mathcal{L}_{n,k}(e_0;x)-1\right)\right| \\ &\leq \mathcal{L}_{n,k}(|g(t)-g(x)|;x)+|g(x)|\left(1-\mathcal{L}_{n,k}(e_0;x)\right) \\ &\leq |g(x)|\left(1-\mathcal{L}_{n,k}(e_0;x)\right) \\ &\quad +\omega\left(g\circ\log_\mu;\delta\right)\left[\mathcal{L}_{n,k}(e_0;x)+\frac{1}{\delta^2}\mathcal{L}_{n,k}\left(\exp^2_{\mu,x};x\right)\right]. \end{aligned}$$

Choosing $\delta=\sqrt{\beta^{\mu}_{n,k}(x)}$ which is given in (8), from Lemma 2.3 we deduce

$$\begin{aligned} \left|\mathcal{L}_{n,k}(g;x)-g(x)\right| &\leq |g(x)|\left(1-\frac{(n)_k}{n^k}\right) \\ &\quad +\omega\left(g\circ\log_\mu;\sqrt{\beta^{\mu}_{n,k}(x)}\right)\left(1+\frac{(n)_k}{n^k}\right). \end{aligned} \tag{11}$$

Thus, from Remark 1, we can write

$$\begin{aligned} D^k\mathcal{G}_{n,\mu}(f;x) &= o(1)+\left(\frac{\mu}{n}\right)^k b_n^k(x)\mathcal{L}_{n,k}\left(D^kf_\mu;x\right) \\ &= o(1)+\left(\left(\frac{\mu}{n}\right)^k b_n^k(x)-1\right)\mathcal{L}_{n,k}\left(D^kf_\mu;x\right) \\ &\quad +\mathcal{L}_{n,k}\left(D^kf_\mu;x\right). \end{aligned}$$

As a conclusion we obtain

$$\begin{aligned} \left|D^kG_{n,\mu}(f;x)-D^kf_\mu(x)\right| &\leq o(1)+\left|\left(\frac{\mu}{n}\right)^k b_n^k(x)-1\right|\left|\mathcal{L}_{n,k}\left(D^kf_\mu;x\right)\right| \\ &\quad +\left(\frac{\mu}{n}\right)^k b_n^k(x)\left|\mathcal{L}_{n,k}\left(D^kf_\mu;x\right)-D^kf_\mu(x)\right| \\ &\leq o(1)+\left|\left(\frac{\mu}{n}\right)^k b_n^k(x)-1\right|\left\|D^kf_\mu\right\|\left|\mathcal{L}_{n,k}(e_0;x)\right| \\ &\quad +\left(\frac{\mu}{n}\right)^k b_n^k(x)\left|\mathcal{L}_{n,k}\left(D^kf_\mu;x\right)-D^kf_\mu(x)\right|. \end{aligned}$$

Using (11) for $g=D^kf_\mu$, we have the desired result. □

Now we estimate the difference $D^k\mathcal{G}_{n,\mu}(f)-\mathcal{G}_{n-k,\mu}\left(D^kf_\mu\right)$ in terms of moduli of continuity.

Theorem 3.3. *If $f\in C^k[0,1]$, then we have*

$$\begin{aligned} \left\|D^k\mathcal{G}_{n,\mu}(f)-\mathcal{G}_{n-k,\mu}\left(D^kf_\mu\right)\right\| &\leq o(1)+\frac{k(k-1)}{2n}\left\|D^kf_\mu\right\| \\ &\quad +\frac{(n)_k}{n^k}\left(\left(\frac{\mu}{n}\right)^k\left(\frac{e^{\mu/n}}{e^{\mu/n}-1}\right)^k-1\right)\left\|D^kf_\mu\right\| \\ &\quad +\omega\left(D^kf_\mu;\frac{k}{n}\right). \end{aligned}$$

Proof. Using the representation given in (5), we can write

$$\begin{aligned} &D^k\mathcal{G}_{n,\mu}(f;x)-\mathcal{G}_{n-k,\mu}\left(D^kf_\mu;x\right) \\ &= \sum_{i=1}^{k-1}(n)_i\left(\frac{\mu}{n}\right)^k C_i^k b_n^i(x)\sum_{s=0}^{n-i}p_{n-i,s}(a_n(x))\Delta^i_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad +(n)_k\left(\frac{\mu}{n}\right)^k b_n^k(x)\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x))\Delta^k_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right) \\ &\quad -\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x))D^kf_\mu\left(\frac{s}{n-k}\right). \end{aligned}$$

On the other hand, by Remark 1, we reach

$$\begin{aligned} &D^k\mathcal{G}_{n,\mu}(f;x)-\mathcal{G}_{n-k,\mu}\left(D^kf_\mu;x\right) \\ &= o(1)+\sum_{s=0}^{n-k}p_{n-k,s}(a_n(x))\left[(n)_k\left(\frac{\mu}{n}\right)^k b_n^k(x)\Delta^k_{\frac{1}{n}}f_\mu\left(\frac{s}{n}\right)-D^kf_\mu\left(\frac{s}{n-k}\right)\right]. \end{aligned}$$

By mean value theorem, we can write

$$\Delta^k_{\frac{1}{n}} f_\mu\left(\frac{s}{n}\right) = \frac{1}{n^k} D^k f_\mu(\xi_s),$$

where $\frac{s}{n} \leq \xi_s \leq \frac{s+k}{n}$. Hence, we have

$$\begin{aligned} & D^k \mathcal{G}_{n,\mu}(f;x) - \mathcal{G}_{n-k,\mu}\left(D^k f_\mu; x\right) \\ = \; & o(1) + \sum_{s=0}^{n-k} p_{n-k,s}(a_n(x)) \left[\left(\frac{\mu}{n}\right)^k b_n^k(x) \frac{(n)_k}{n^k} D^k f_\mu(\xi_s) - D^k f_\mu\left(\frac{s}{n-k}\right)\right] \end{aligned}$$

which direct us to

$$\begin{aligned} & D^k \mathcal{G}_{n,\mu}(f;x) - \mathcal{G}_{n-k,\mu}\left(D^k f_\mu; x\right) \\ = \; & o(1) + \sum_{s=0}^{n-k} p_{n-k,s}(a_n(x)) \left\{\left(\frac{(n)_k}{n^k} - 1\right) D^k f_\mu(\xi_s) + \right. \\ & \left. + \frac{(n)_k}{n^k}\left(\left(\frac{\mu}{n}\right)^k b_n^k(x) - 1\right) D^k f_\mu(\xi_i) + \left(D^k f_\mu(\xi_s) - D^k f_\mu\left(\frac{s}{n-k}\right)\right)\right\}. \end{aligned}$$

Since

$$0 < 1 - \frac{(n)_k}{n^k} \leq \frac{k(k-1)}{2n} \quad \text{and} \quad \frac{s}{n} \leq \frac{s}{n-k} \leq \frac{s+k}{n} \text{ for } 0 \leq n-k \leq s$$

we have the desired result. □

Here we note that similar result of Theorem 3.3 was obtained for Bernstein type operators in [6].

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